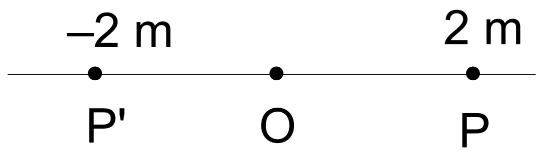
GEOMETRY AND VECTORS

Distinguishing Between Points in Space

- One Approach – Names: (“Fred”, “Steve”, “Alice”...)
  - Problem: distance & direction must be defined point-by-point

- More elegant – take advantage of geometry
  - Label points in organized fashion with numbers (“coordinates”)
  - Use the coordinates to calculate distance & direction
  - Example: Instead of “Chicago” → “42º N, 88º W”

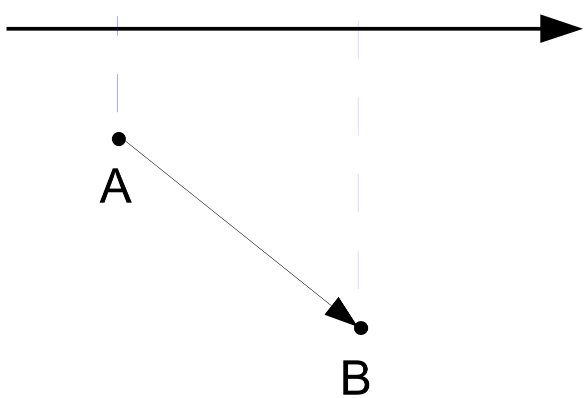
- How to choose coordinates for each point?
  - Common approach: pick a reference point (the “origin”)
  - Label each point P with:
  - distance(origin, P) and direction(origin, P)



Measuring Direction – Projection

- To make a "coordinate system" for a given space:
  - Must quantitatively define direction(A, B)

- Approach: Pick some direction to act as reference
  - Quantitatively compare direction(A, B) to reference direction
  - This comparison is called a "projection"
  - Measures how much of distance(A, B) is parallel to reference
  - Expressed as an angle or a number (between -1.0 and 1.0)



Reference direction (also called a coordinate axis)

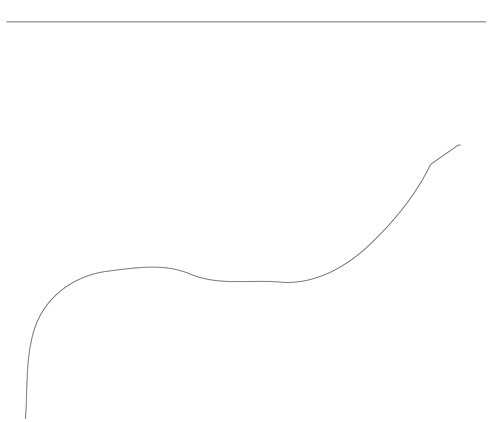
In trigonometry, projection is represented by the cosine of the angle between direction(A, B) and the reference direction

Dimensions

- Points in a space can be organized a variety of ways
  - Depending on how they are “connected” to each other

- Dimension of a space
  - Smallest # of coordinates necessary to specify each point
  - In defining “direction” → 1 “reference direction” per dimension

1-Dimension

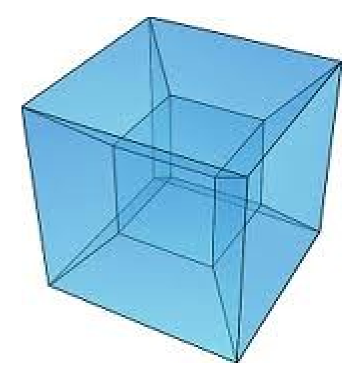


2-Dimensions

3-Dimensions

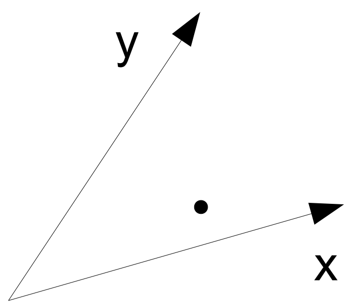
N-Dimensional

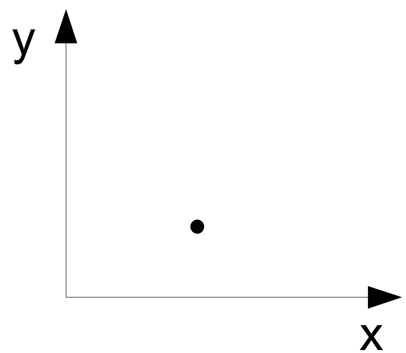
???

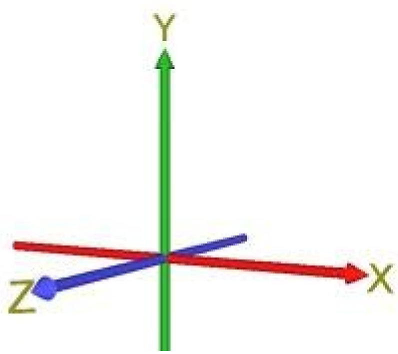


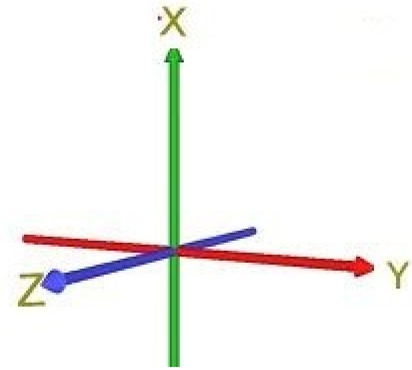
Reference Frames

- Ref. Frame – specific origin and reference directions
  - "Conventions" – man-made rules; convenient but not mathematically necessary:
  - Reference directions (called "coordinate axes") are orthogonal
  - 3-Dimensions: coordinate axes xyz named by "right-hand rule"





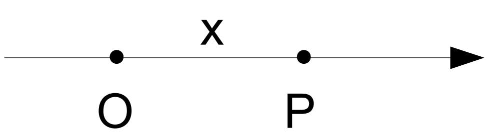




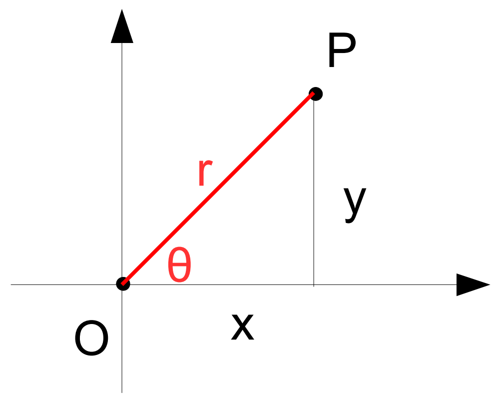
- Two conventions for "naming" points in D dimensions:
  - 1) project Dist(O, P) onto D coordinate axes (Example: x, y, z)
  - 2) Dist(O,P) and projection onto D-1 axes (Example: r, θ, ϕ)

Common Coordinate System Conventions

1-Dimension



Coordinate: x

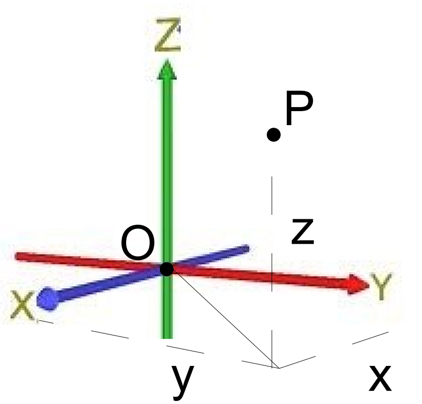


2-Dimensions

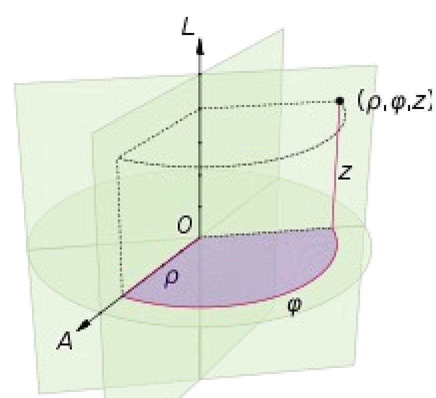
"Cartesian" Coordinates: x, y

"Polar" Coordinates: r, θ

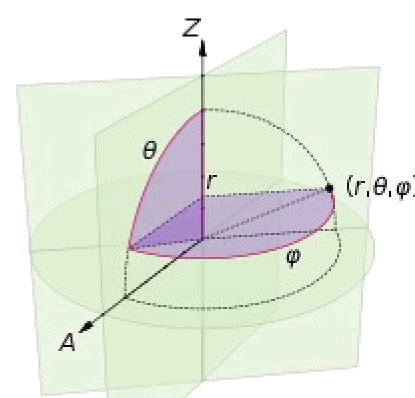
3-Dimensions



"Cartesian" Coordinates:
(x, y, z)



"Cylindrical" Coordinates:
(ρ, ϕ, z) or (r, θ, z)



"Spherical" Coordinates:
(r, θ, ϕ) or (ρ, θ, ϕ)

Note: Math and Physics use different conventions for spherical coordinates

Coordinate System Consistency

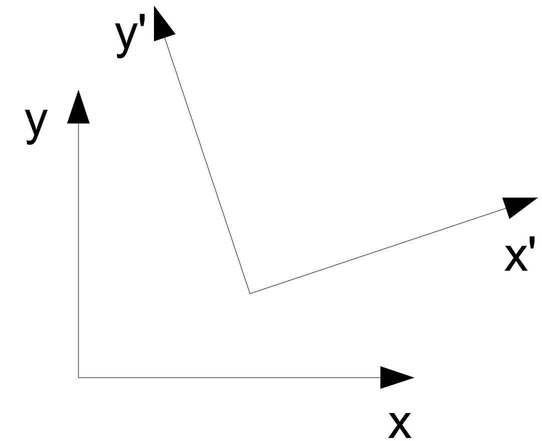
- Geometry → can "translate" coordinates between systems

2-Dimensions	3-Dimensions	
Cartesian / Polar	Cartesian / Cylindrical	Cartesian / Spherical
$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $(\mathit{actually}\ \text{arctan2})$	$\rho = \sqrt{x^2+y^2}$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$ $z = z$	$r = \sqrt{x^2+y^2+z^2}$ $\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$
$x = r\cos(\theta)$ $y = r\sin(\theta)$	$x = \rho\cos(\phi)$ $y = \rho\sin(\phi)$ $z = z$	$x = r\sin(\theta)\cos(\phi)$ $y = r\sin(\theta)\sin(\phi)$ $z = r\cos(\theta)$

<u>Spatial "Transformations"</u>

- Space itself is isotropic → symmetric in all directions
  - There is no universal up, down, left, right – it's all <u>convention</u>
  - No point in space is "special" or distinguishable from others

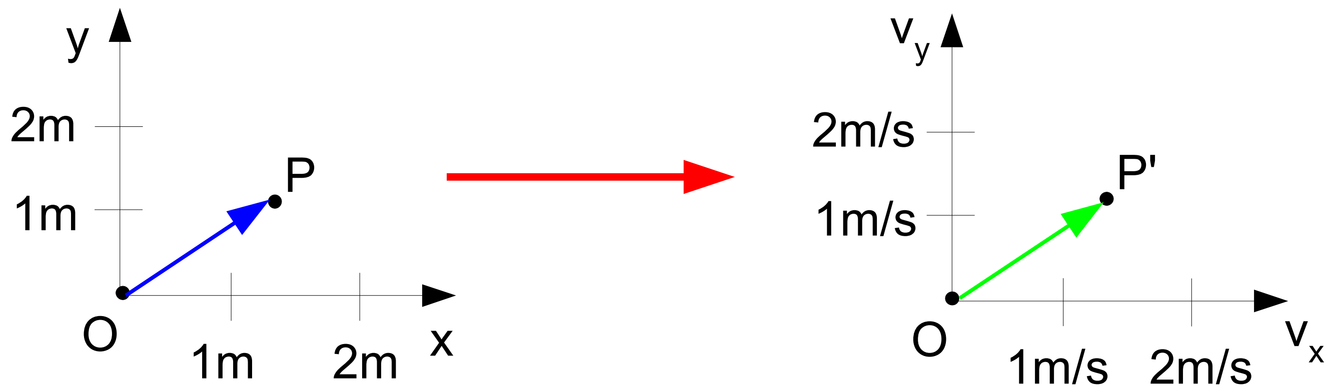
- So any choice for <u>origin</u> and <u>coordinate axes</u> is valid
  - Physics needs to work for all reference frames!
  - <u>Tricky</u>: In different reference frames...
  - ...same point has different coordinates!



- How can reference frames differ?
  - <u>Translation</u> – different origins
  - <u>Rotation</u> – different coordinate axes
  - <u>Scaling</u> – coordinates are multiplied by a constant

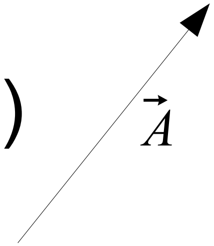
Vector Spaces

- Can scale reference frame using any constant quantity
  - Even one with units! (kg, m, sec, or any multiplicative mix)
  - Creates a new "space" with same directions but different units



- Vector space – mathematical generalization of "space"
  - Which may or may not represent actual physical space
  - Generalized term for points in a vector space: "Vectors"
  - Generalized term for coordinates: "Components"
  - Generalized term for Distance(O, P'): "Magnitude"

Vector Spaces – Conventions



- Vector symbol: letter with arrow ($\vec{A}$) or boldface (**A**)
  - Drawn graphically as an arrow directed from tail to tip
  - Magnitude is denoted by absolute value ($|\vec{A}|$) or letter only (A)

- Can use usual coordinate systems (cartesian, polar...):
  - "Magnitude form" of a vector:
  - Magnitude (in any units) and direction – usually angle(s)
  - Example: $a = 40.3\ m/s^2$ and $\theta = 73.2°$

  - "Component form" of a vector:
  - One component for each dimension
  - Example: ($v_x = 3.0$ m/s, $v_y = 4.1$ m/s, $v_z = 2.2$ m/s)

“Adding” Vectors

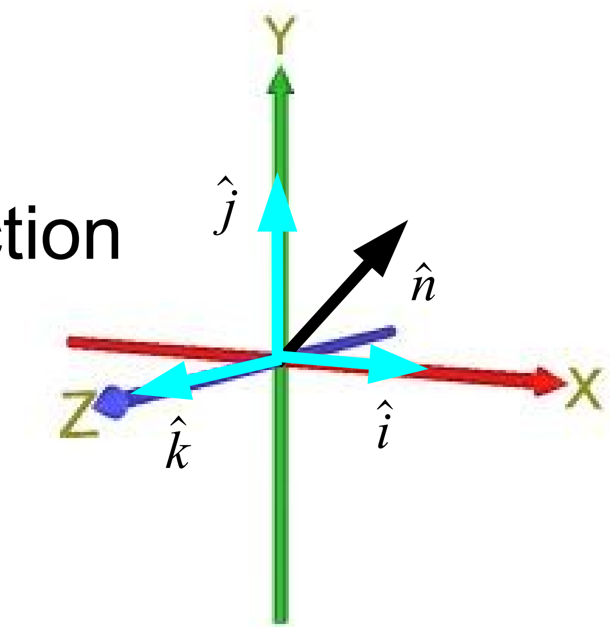
- In physical space:
  - Every point is associated with a position vector
  - 2 different points are connected by a displacement vector
  - Conventional notation: $\vec{r}_B = \vec{r}_A + \vec{d}_{A \to B}$
- “+” operation can be generalized to any vector space
- For vectors in component form: $\vec{A} = \vec{B} + \vec{C} \longrightarrow$ $A_x = B_x + C_x$, $A_y = B_y + C_y$, $A_z = B_z + C_z$
  - Note: Impossible to add vectors from different vector spaces
  - (components would have different units → makes no sense)
  - Ex: Cannot add a displacement vector to a velocity vector!

Unit Vectors

- Vectors which are used only to define direction
  - Magnitude: dimensionless and equal to 1

- Convention: Unit vectors in the x, y, z directions
  - Are called $\hat{i}\,, \hat{j}\,, \hat{k}$ or $\hat{x}\,, \hat{y}\,, \hat{z}$

- Can construct a unit vector in any direction
  - With combinations of $\hat{i}\,, \hat{j}\,, \hat{k}$

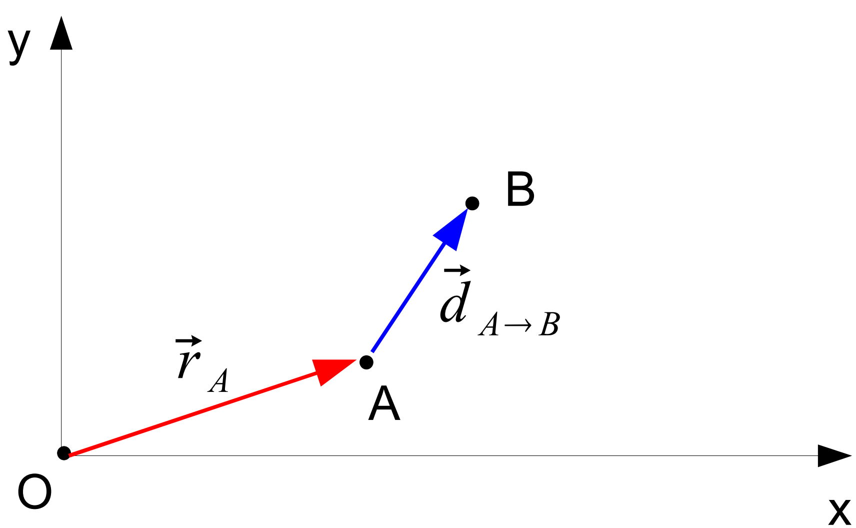


$$\hat{n} = \left(\frac{1}{\sqrt{2}}\right)\hat{i} + \left(\frac{1}{\sqrt{2}}\right)\hat{j} + 0\,\hat{k}$$

Common vector notations:	For any vector **v** :
$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ $\vec{v} = \left(v_x, v_y, v_z\right)$ $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$	$\hat{v} \equiv \dfrac{\vec{v}}{v}$

Actual Physical Space – Conventions

- Displacement Vector
  - Vector from any point A to any point B
- Position Vector (denoted by $\vec{r}$ or $\vec{x}$)
  - From origin to any point P → Components: x, y, z



$$\vec{r}_A = x_A\hat{i} + y_A\hat{j} + z_A\hat{k}$$

$$\vec{d} = (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}$$

Magnitudes of Vectors in "position space":

Measured in units of length

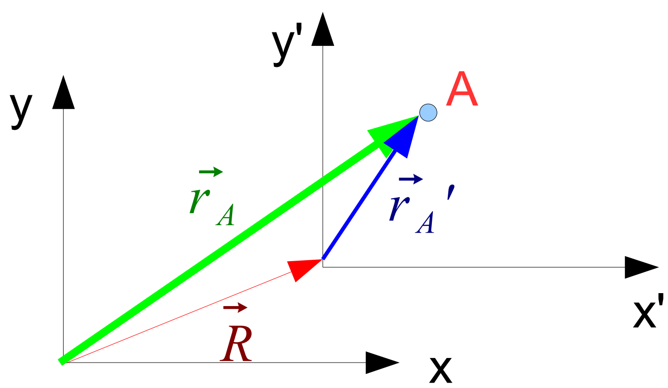
$$|\vec{r}_A| \equiv r_A = \sqrt{x_A^{\,2} + y_A^{\,2} + z_A^{\,2}}$$

$$|\vec{d}| \equiv d = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

Coordinate Transformations

- To describe the same point in 2 reference frames:
  - Need to “transform” coordinates between frames

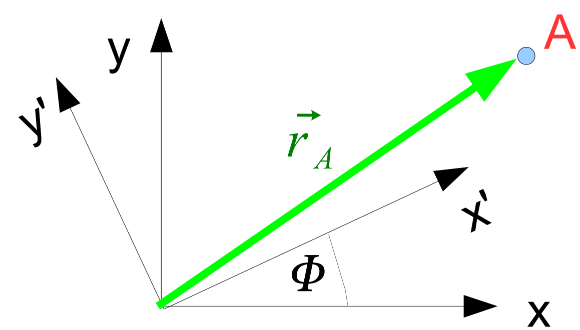
Translating a reference frame



$$\vec{r}_A{}' = \vec{r}_A - \vec{R}$$

$$\begin{pmatrix} x_A{}' \\ y_A{}' \end{pmatrix} = \begin{pmatrix} x_A - R_x \\ y_A - R_y \end{pmatrix}$$

Rotating a reference frame about the z-axis



$$\begin{pmatrix} x_A{}' \\ y_A{}' \end{pmatrix} = \begin{pmatrix} x_A \cos\Phi + y_A \sin\Phi \\ -x_A \sin\Phi + y_A \cos\Phi \end{pmatrix}$$

To rotate about an axis other than z:
Similar concept with more complicated geometry

Rotation Matrix

- Matrix – structure for organizing numbers or functions
  - Matrices can "operate" on a vector (making a new vector)
  - Operations → carried out in specific order (rows and columns)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \equiv \begin{pmatrix} M_{11} v_1 + M_{12} v_2 \\ M_{21} v_1 + M_{22} v_2 \end{pmatrix}$$

In index notation:

$$a_i \equiv M_{ij} v_j$$

- Rotation Matrix – defines coordinate transformation
  - To a frame rotated about a particular axis by angle Φ
  - For rotation about the z-axis:

$$R(\Phi) = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \boxed{\vec{r}\,' = R(\Phi)\,\vec{r}}$$

Vectors and Unit Vectors – Examples

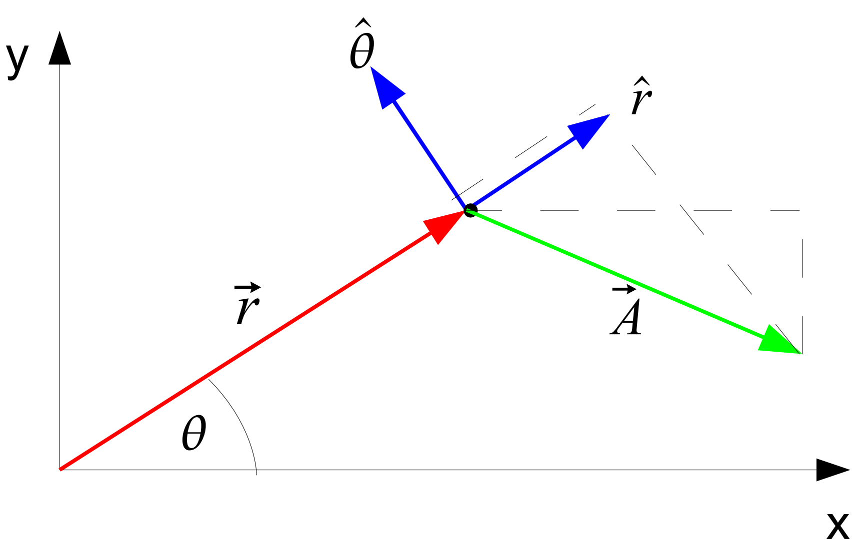
- Let $\vec{A} = (2\ m)\,\hat{i} + (3\ m)\,\hat{j} - (1\ m)\,\hat{k}$ and $\vec{B} = (5\ m)\,\hat{i} - (2\ m)\,\hat{j} - (3\ m)\,\hat{k}$
  - In some particular reference frame S

- Consider a new reference frame S'
  - With the same origin as S, but rotated 45º about the z-axis

- In both reference frames:
  - Calculate the components of $\hat{A}$ and $\hat{B}$
  - Calculate $\left|\vec{A} + \vec{B}\right|$ and $\left|\vec{A} - \vec{B}\right|$

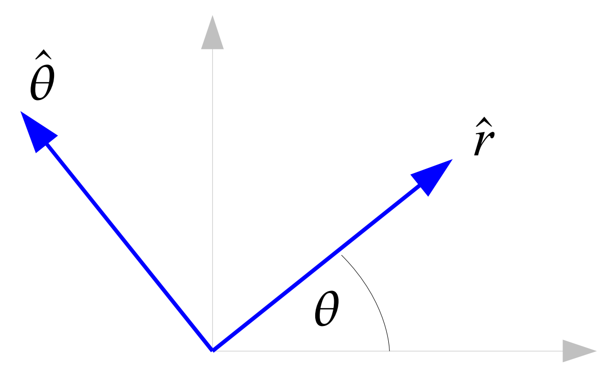
Unit Vectors in Polar Coordinates

- Vector components in Cartesian coordinates:
  - Are projections onto fixed directions xyz
- An alternate method for defining components:
  - Use projections parallel and perpendicular to position vector
  - Unit vectors in these directions are called $\hat{r}$ and $\hat{\theta}$
  - Note: These unit vectors depend on position (not fixed!)



$$\vec{A} = A_r\,\hat{r} + A_\theta\,\hat{\theta}$$

$$\vec{r} = r\,\hat{r}$$

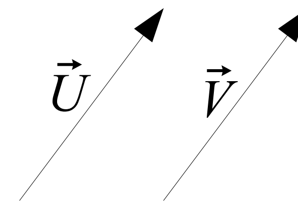


$$\hat{r} = \cos(\theta)\,\hat{i} + \sin(\theta)\,\hat{j}$$

$$\hat{\theta} = -\sin(\theta)\,\hat{i} + \cos(\theta)\,\hat{j}$$

Graphical Representation of Vectors

- Vectors → defined by direction and magnitude only
  - Their “location” in the vector space is arbitrary

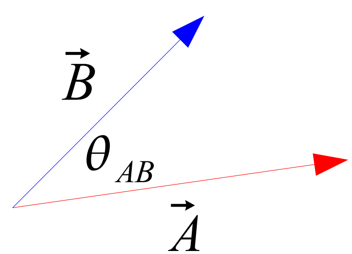


$\vec{U} = \vec{V}$

- Can move vectors around to use geometry
  - With the role of distance replaced by vector magnitudes

Comparing the directions of 2 vectors (i.e. measuring angle between them)

“Tail-to-tail” convention:



Note: It is possible to compare directions of 2 vectors in different vector spaces

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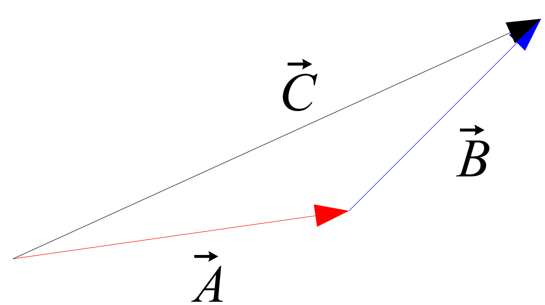
$\vec{A} + \vec{B} = \vec{C}$

“Tail-to-tip” convention:

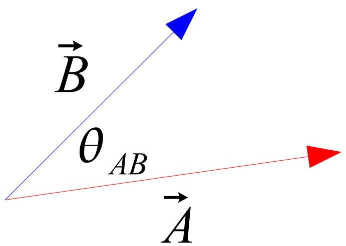


Geometry: These 3 vectors form a triangle in their vector space

Is this true? $\theta_{AB} + \theta_{BC} + \theta_{AC} = 180$

Dot Product

- Angle measurement compares direction of 2 vectors
  - Can be tricky to do with vectors in component form
- Useful tool: the "dot product"
  - Measures how one vector projects onto another
  - Can be defined in either magnitude form or component form



$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos(\theta_{AB})$$

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Prove it using Law of Cosines!

  - Dot product can be positive, negative, or zero
  - Units of dot product: multiply units of individual vectors
  - Also called "scalar product" or "inner product"

Dot Product – Important Features

- Dot product is "invariant"
  - Has the same value in all reference frames

    $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$
  - A_x, B_x, A_y, etc. depend on frame but dot product does not

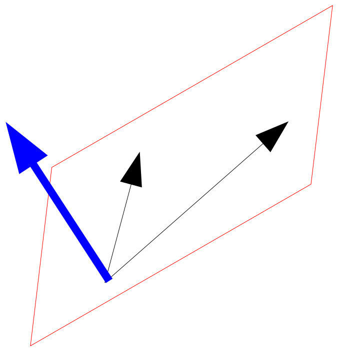
- Dot product is commutative: $\vec{U}\cdot\vec{V} = \vec{V}\cdot\vec{U}$

- Can take dot product of a vector with itself ($\vec{A}\cdot\vec{A}$)
  - Result: "magnitude squared" of the vector (A^2)

- Dot products of unit vectors:

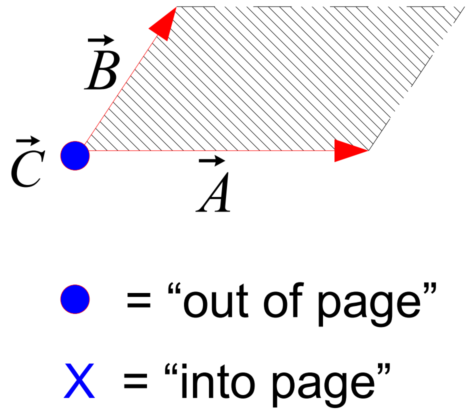
$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \qquad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

Cross Product



- Any 2 vector directions define a plane
  - Ways to mathematically describe the plane:
  - 1) "Equation of constraint" governing coordinates of points
  - 2) Direction which is perpendicular to the plane

- "Cross Product" of 2 vectors $\vec{A}$ and $\vec{B}$
  - Produces a 3rd vector $\vec{C}$ with the properties:
  - Direction: perpendicular to both $\vec{A}$ and $\vec{B}$
  - Magnitude: "area enclosed" by $\vec{A}$ and $\vec{B}$



$$|\vec{A}\times\vec{B}| = |\vec{A}|\,|\vec{B}|\,\sin(\theta_{AB})$$

$$\vec{A}\times\vec{B} = (A_y B_z - A_z B_y)\,\hat{i} - (A_x B_z - A_z B_x)\,\hat{j} + (A_x B_y - A_y B_x)\,\hat{k}$$

Convention: Direction of cross product decided by "right-hand rule"

Cross Product – Important Features

- Cross product is a vector → frame-dependent
  - Components depend on reference frame – magnitude doesn't
- Cross product is anti-commutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- Cross product of a vector with itself ($\vec{A} \times \vec{A}$) is zero
- Cross products of unit vectors:

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{j} = -\hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{j}$$

These relationships are a result of the “right-hand rule” convention.

The 3 equations on the left are an example of a “cyclic permutation”

Cross Product – Determinant Form

- Concise way to remember order and sign of terms:
  - Use the determinant of a matrix!
  - Matrices and determinants to be described in detail later
  - For now, just a tool for getting terms and signs right

2 x 2 matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3 x 3 matrix: $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Determinant: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\left(det\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right) - b\left(det\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right) + c\left(det\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right)$

Determinant form of cross product $\vec{A} \times \vec{B}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$(A_y B_z - A_z B_y)\,\hat{i} - (A_x B_z - A_z B_x)\,\hat{j} + (A_x B_y - A_y B_x)\,\hat{k}$$

Dot/Cross Product Examples

- Which of the following makes sense?
  - And in each case, are parentheses necessary?

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) \qquad \vec{A}\times\left(\vec{B}\cdot\vec{C}\right) \qquad \vec{A}\cdot\vec{B}\cdot\vec{C} \qquad \vec{A}\times\vec{B}\times\vec{C}$$

- Imagine a set of N unit vectors such that:
  - 1) the sum of all N vectors is zero
  - 2) the angle between any 2 unit vectors is constant
  - Draw some examples for different N in 2-D and 3-D space
  - Calculate the angle between two vectors in each case