# GEOMETRY INSTITUTE - DAY 5

## Geometry on Various Surfaces

| Time | Topic | TEKS | Approach |
|---|---|---|---|
| 30 | *The Cable Guy* | 6b | |
| 10 | Reflection on *The Cable Guy* | | |
| 20 | Distance on the Euclidean Plane | 7a, b, c, 8c | |
| 30 | Plane vs. Spherical Geometry | 1c, 6c | |
| 60 | Geometry on a Cube | 1c | |
| 30 | Texas Bullet Train | 2a, 9b | |
| 15 | Connect the Dots | 1a, 3e | |
| 60 | Fractal Tetrahedron | 5b, 8 | |
| 15 | Reflection & Post it notes in texts | | |
| 30 | Closure - Where do we go from here?<br>Evaluations | | |
| 300 | | | |

# Day 5: Materials Needed:

☐ *Sketchpad*

*Sketches*

☐ bullet train

*Scripts*

☐

☐ Models of regular polyhedra

☐ Models of other polyhedra with regular faces

☐ Rubber bands

☐ Large sphere for each group

☐ String for each group

☐ A large cardboard cube for each group.

☐ Several colored straight pins for each group

☐ Large map of Texas with Dallas, Houston, San Antonio highlighted

☐ One cardstock regular tetrahedron (6” edge), 6 card stock equilateral triangles for each group

# THE CABLE GUY

**CUBICAL ROOM**

A cable installer wants to use the least amount of wire through the wall, floor or ceiling of a cubical room to connect the television and cable outlet which are located on two walls of the room. Where should he run the wire? Give a geometric description of this path.

Could the television and cable box be located so that there is more than one way to join them using the least amount of wire?

After solving the above problems, consider other possible room shapes.

**CYLINDRICAL ROOM**

Suppose the room were a turret in a castle (cylindrical). What would be the shortest path for the wire to follow if it had to wrap around the room once. What if it had to wrap around twice before connecting the television and the cable box?

**PYRAMID SHAPED ROOM**

The lobby of the Luxor Hotel in Las Vegas is shaped like a square pyramid. If the television and cable box were on adjacent walls, what path would use the least amount of wire? What path would the Cable Guy use if they were on opposite walls?

**GEODESIC DOME**

Suppose the room were the geodesic dome of Epcot Center. What is the shortest path in that case?

**HEMISPHERICAL ROOM**

Suppose the room were an igloo (a hemisphere). If the television and cable box are at any two points on the floor next to the wall of the igloo, what should be the wire's path? If the television and cable box are not on the floor, what is the shortest path for the wire to follow?

# THE CABLE GUY

## Solutions

This problem is a variation of the Spider and the Fly problem. After solving the problem, most people will realize that building a model and/or drawing a net is an effective problem solving tool. Reducing the rooms to their plane nets makes the problem much simpler.

The problem leads to the definition of line segments and lines on various polyhedra and on the sphere.

**CUBICAL ROOM**

To obtain the shortest path between two points on two adjacent faces, think of the faces as being hinged along the common edge and then lay the faces flat on a table. When the faces are lying flat, the shortest path between A and Bis the line segment joining A to B as shown in figure 2. If C is the point at which the line segment crosses the edge common to the two faces, then m∠1 = m∠2 in figure 2. Since rotating the two faces about the hinge does not change ∠1 and ∠2, nor change the length of AC and BC, the shortest path between A and B is the one shown in figure 1. It can be described geometrically as the unbroken path formed by straight lines on each face that meet the common edge in supplementary angles.

When, as in figure 3, A and B do not lie on adjacent faces the shortest path is the one such that m∠1 = m∠2. Again the shortest path can be described as the unbroken path formed by straight lines on each face that meet the common edge between adjacent faces in supplementary angles. This description remains true for any polyhedral surface.

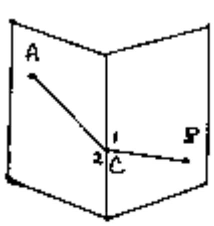



Figure 1

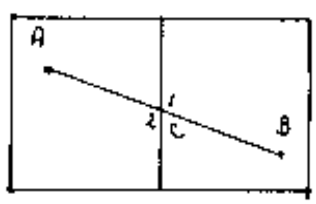



Figure 2

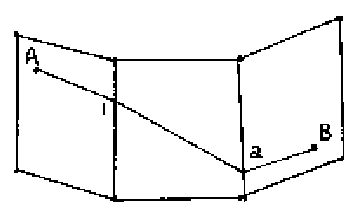



Figure 3

The solution is not unique if the television and cable box are located at the same height on opposite walls.

## CYLINDRICAL ROOM

The shortest path is found by unwrapping the cylinder and drawing the straight line from A to B However when the wire must wrap around the room twice before connecting to the television and cable box, the shortest paths are parallel lines.

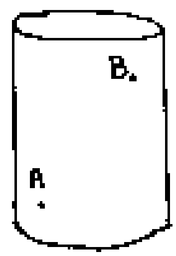
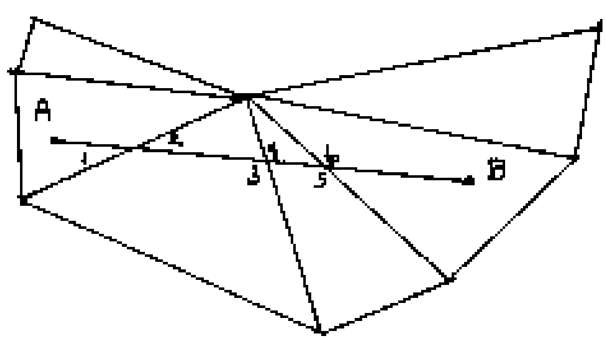



Figure 4

## PYRAMID SHAPED ROOM

The shortest path is once again a straight line when the pyramid is flattened. Note that this path can be found without a net by ensuring that the angles the path makes with the edges (which are vertical angles on the net) are equal.




## GEODESIC DOME

The shortest path between two points on the walls of the dome is the straight line on the net of the dome, as in the case of the pyramid shaped room.

**HEMISPHERICAL ROOM**

The minimal path between the two points on a sphere or hemisphere lies on the great circle joining the points.

# THE CABLE GUY

## Reflections

1. Discuss the presentation styles used in this problem.
   - Hands on
   - Modeling
   - Cooperative learning

2. Discuss the geometry learned from this problem.

3. Discuss the geometric concepts used in the solution to this problem.
   - Prisms
   - Cylinders
   - Pyramids
   - Model building
   - Nets
   - Shortest Path (Geodesics)
   - Parallel lines
   - Optimization

4. How could this problem be adapted to the classroom?

5. Direction of the day. Where could this problem lead?

- Parallel and perpendicular lines
- Distance on the Euclidean plane and other surfaces
- Angles
- Properties of solids and spheres
- Optimization problems
- Polyhedra

**NOTES**

# DISTANCE IN THE EUCLIDEAN PLANE

## Institute Notes

| Lesson | Pages | Time | TEKS |
|---|---|---|---|
| Distance in the Euclidean Plane | | 20 min | 7a, b, c, 8c |

**MATERIALS AND SETUP**

Transparencies

**LEARNING STYLES**

Cooperative learning

**MATH LINKS**

Algebra, measurement, coordinate geometry

**OTHER LINKS**

Architecture, surveying

**LESSON OVERVIEW**

"The shortest distance between two points is a straight line." Young children instinctively know that it is shorter to cut across the lawn than to go around it. This is axiomatic for the student of geometry as well as for the general population. What many fail to realize is that this axiom is valid only in the Euclidean plane. Although the earth is spherical, within our immediate neighborhood we can think of it as flat - a Euclidean plane.

A study of the distance formula and its applications shows the power of the Cartesian plane in making the important connection between algebra and geometry. The Pythagorean Theorem is the key to understanding the formulas. Linear equations and systems of equations are also used to solve problems involving distance.

These lessons derive methods for computing distance in the plane and help develop the student's understanding of what is meant by the distance between a point and a line and the distance between two lines.

**Distance Between Points**

Finding the distance in a plane is an important concept that can be easily derived from the Pythagorean Theorem. It is important for students to understand how formulas are derived and how they work.

Have the participants work through parts I, III, IV and V. Discuss the importance of having students derive the formula.

**Distance Between Points and Lines**

This activity incorporates the distance formula, linear equations and slopes of perpendicular lines to solve problems involving the distance between a point and a line.

**Distance Between Parallel Lines**

This activity uses the distance formula, linear equations and slopes of perpendicular lines.

## NOTES

# DISTANCE IN THE EUCLIDEAN PLANE

## Teacher Guide

**GOAL:**

Develop the distance formula using the Pythagorean Theorem, find the shortest distance from a point to a line, find the distance between parallel lines

**PREREQUISITES:**

The Pythagorean Theorem, distance formula, parallel lines, perpendicular lines, slopes of perpendicular lines

**MATERIALS:**

The Distance Formula worksheets, dot grid overlays, transparencies of problems

**PROCEDURE (Between Points):**

Students can either work individually or in groups.

- Work I, II, III.
- Discuss I, II, and III.
- Work IV.
- Discuss IV.
- Work V (individually or as a class).
- Work VI — examples.

**EXTENSION:**

Show the video: <u>The Theorem of Pythagoras</u> by Project MATHEMATICS!

Use the distance formula to:

- prove that perpendicular lines have negative reciprocal slopes
- find the distance from a point to a line
- find the distance between two parallel lines.

**PROCEDURE (Between Points and Lines):**

- Show students the Sea Turtle problem.
- Class discussion: What is the shortest distance from a point to a line?
  *Perpendicular distance.*

- Suggest solving the Sea Turtle problem using a dot grid overlay. Place the dot grid with dots parallel to edges of overhead. Students may suggest rotating the grid so that dots are parallel to the ocean to make the problem easier to solve. Point out that grids may pre-exist on maps, city plats, etc., or there may be many such distance probems on one map. Therefore, it is useful to be able to solve problems no matter the orientation of the grid.
- Work through the problem with the class to come up with a solution.
- Have students complete the Electric Lines problem in groups. Discuss the processes the students used.

**PROCEDURE (Between Parallel Lines):**

- Show parallel sidewalks overhead to class and discuss what is meant by the distance between two parallel lines (perpendicular distance).
- Suggest solving parallel sidewalks problem using dot grid overlay. Place dot grid with dots parallel to edges of overhead. Students may suggest rotating grid so that dots are parallel to sidewalks to make problem easier to solve. Point out that grids may pre-exist on maps, city plats, etc., or there may be many such distance problems on one map. Therefore it is useful to be able to solve problem no matter the orientation of the grid.
- Either demonstrate solution or have students suggest method.
- Have students work in groups to complete solutions and solve chicken problem.
- After completion discuss processes used and results.

---

**SOLUTIONS (Between Points)**

I. 1. 3 blocks
   2. 4 blocks
   3. 7 blocks
   4. No; A right triangle
   5. 5 blocks

II. A(-3, 3), B(3, -1), C(-3, -1)
   AC = 4; BC = 6; AB = $2\sqrt{13} \approx 7.2$
   A(5, -2), B(3, 3), C(5, 3)

AC = 5; BC = 8; AB = $\sqrt{89} \approx 9.4$

III. C(-3, -2) or C(0, 3)

AC = 5, BC = 3, AB = $\sqrt{34} \approx 5.8$;

or AC = 3, BC = 5; AB = $\sqrt{34} \approx 5.8$

C(-5, 3) or C(4, 1)

AC = 9, BC = 2, AB = $\sqrt{85} \approx 9.2$;

or AC = 2, BC = 9, AB = $\sqrt{85} \approx 9.2$

VI. (11, 3) or (3, 7)

8 or 4 found the difference between the different coordinates

4 or 8 found the difference between the different coordinates

AB = $4\sqrt{5} \approx 8.9$

**SOLUTIONS (Between Points and Lines):**

**Sea Turtles**

To find the distance, drop a perpendicular line (L) from the turtle to the ocean shore. Find the slope of the ocean shore. The slope of L is the negative reciprocal of the slope of the ocean shore. Using the turtle as a point and slope of L, write the point slope equation of L.

Find the coordinates of the point where line L intersects the shore by solving the equations simultaneously.

The distance from the turtle to the ocean shore is the distance from the turtle to the point of intersection.

Use the distance formula to find the distance.

**Electric Lines** Repeat the above procedure.

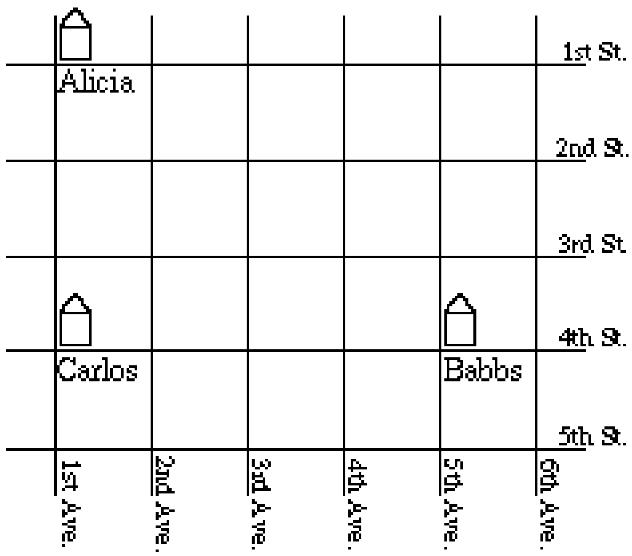
**SOLUTIONS (Between Parallel Lines):**

Find the slope of each line. Since the lines are parallel, they will have the same slope. Take the slope of the line perpendicular to the parallel lines (negative reciprocal). Take any convenient point on the line. Using the point and new slope, write an equation of the line perpendicular to the parallel lines. Find the intersection of the perpendicular line to the other parallel line. Find the distance between the two points. The perpendicular distance from that point to the other line is the distance between the two parallel lines.

# DISTANCE IN THE EUCLIDEAN PLANE

## Between Points

I. Alicia, Babbs and Carlos live in Parallel and Perpendicular City. The city was named so because all the streets are either parallel or perpendicular to each other. Pictured below is a map of the city. Alicia, Babbs and Carlos' houses are all on the corners of the block as shown.



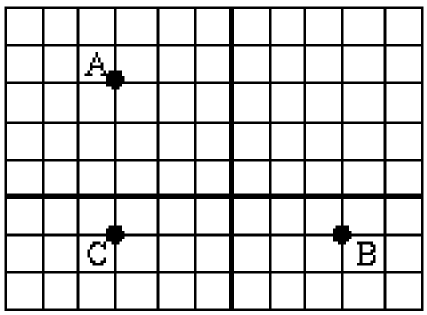

Answer the following questions:

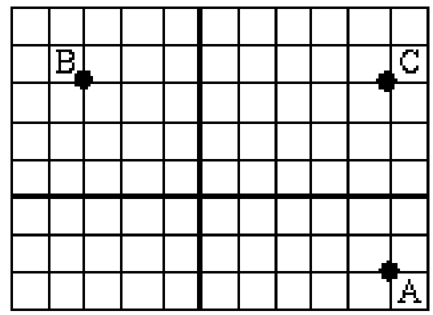
1. Draw the shortest path from Alicia's to Carlos' house. If Alicia is to walk to Carlos' house, how many blocks would she walk? _________
2. On the map, draw the shortest path from Carlos' house to Babb's house. If Carlos wants to ride his bike to Babb's house, how many blocks will he ride? __________
3. If Babbs wants to go to Alicia's house, she must walk along the streets. How far must Babbs walk to go to Alicia's house? __________
4. Is the path that Babbs walked to Alicia's house the shortest distance between the houses? ______ On the map, draw the shortest distance—as "the crow flies." What type of triangle did you draw on the map? _______

5. What is the shortest distance between Babb’s house and Alicia’s house? (Write and equation and determine the distance.) _________

II. For each grid below:

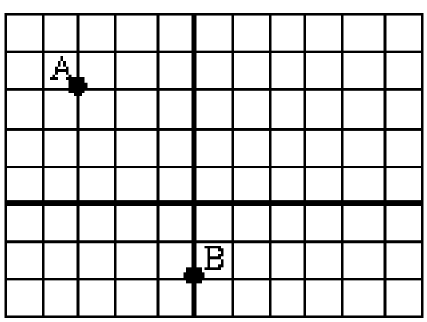
- Give the coordinates for points A, B, and C.
- Determine the distance AC, BC, and AB. Leave your answers in simplest radical form and then as a decimal to the nearest tenth.

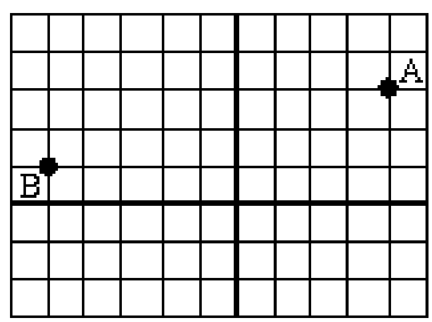



III. For each grid below:

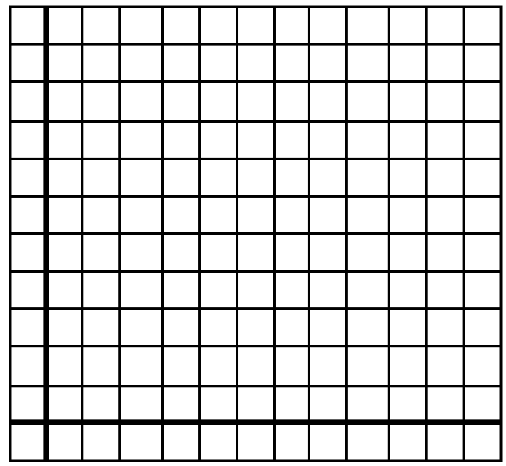
- Draw a right triangle with the right angle at point C so that either $\overline{AC}$ or $\overline{BC}$ is parallel to the x-axis and the other is parallel to the y-axis.
- Determine the coordinates for C.
- Find AC, BC, and AB. Leave answers in simplest radical form and then leave answers as decimals rounded to the nearest tenth.



IV.

- Plot the points A(3, 3) and B(11, 7).
- Complete right triangle ABC by finding right angle C.
- What are the coordinates of C? (_______, ________)
- AC = ___________
- How did the coordinates of points A and C help to determine the distance AC?



- BC = ___________
- How did the coordinates of points B and C help to determine the distance BC?

- AB = ___________

  (Leave answer in simplest radical form and then as a decimal rounded to the nearest tenth.)

We can find the distance from A to B without having to find the coordinates of C.

- Look back at the length of $\overline{AC}$.

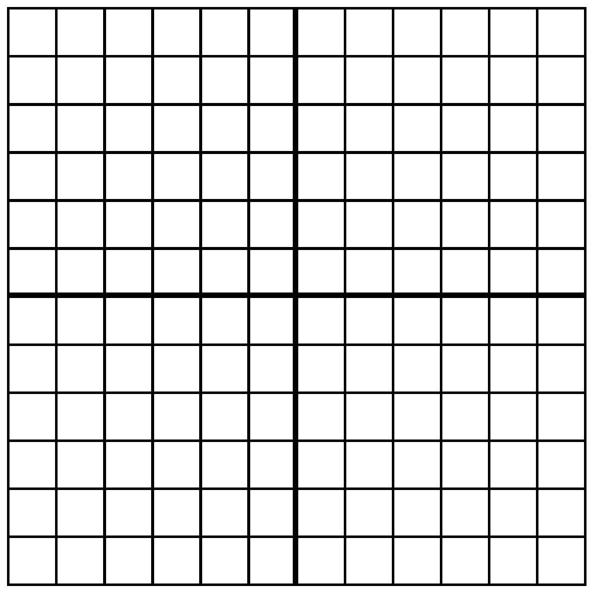
  It was determined by finding the distance between the x-coordinates: $|11 - 3| = 8$.
- Now look at the length of $\overline{BC}$.

  It was determined by finding the distance between the y-coordinates: $|3 - 7| = 4$.
- Find the distance from A to B using the Pythagorean theorem and the differences between the x-coordinates and the y-coordinates.

V. Develop a formula to find the distance between two points A and B.

A has the coordinates $(x_1, y_1)$ and B has the coordinates $(x_2, y_2)$.

- Mark the points A$(x_1, y_1)$ and B$(x_2, y_2)$ anywhere on the grid except in a vertical or horizontal line.



- Create a right triangle ABC with the right angle at C so that either $\overline{AC}$ or $\overline{BC}$ is parallel to the x-axis and the other is parallel to the y-axis.
- Find the coordinates of C. (________, ________)
- Find the distance from A to C. _____________
- Find the distance from B to C. _____________
- Use the Pythagorean Theorem to find the distance AB.

| **The Distance Formula:** |
| --- |
| |

VI. Now use the distance formula that you developed to find the distance between points A and B. Leave answers in simplest radical form and as a decimal rounded to the nearest tenth.

1. A(5, 13) B(–7, 4)

2. A(–3, –10) B(–6, –1)

3. A(5, 2) B(2, 1)

4. A(–6, –6) B(6, 0)

# DISTANCE IN THE EUCLIDEAN PLANE

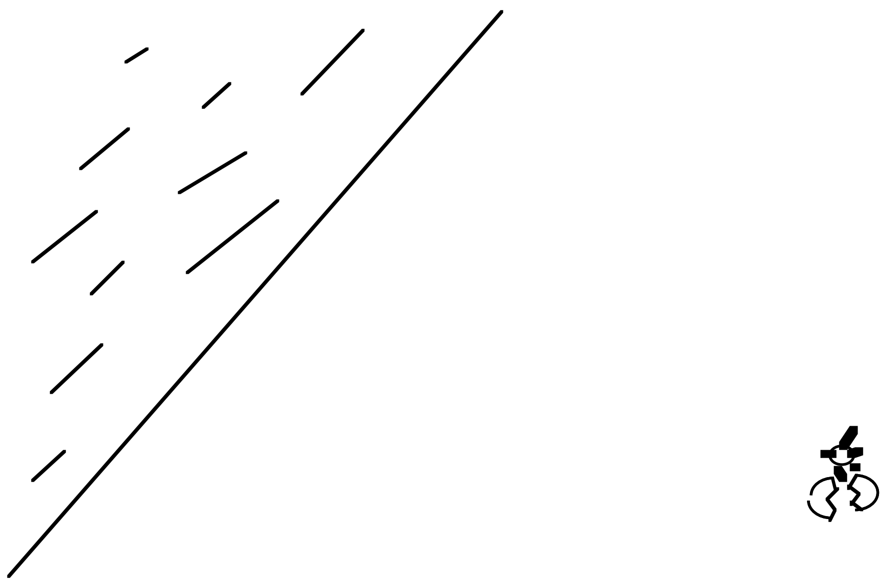
## Between Points and Lines

**SEA TURTLES**

Sea Turtles lay their eggs on sandy beaches. When the eggs hatch, the tiny turtles instinctively crawl in the shortest straight line to the ocean.

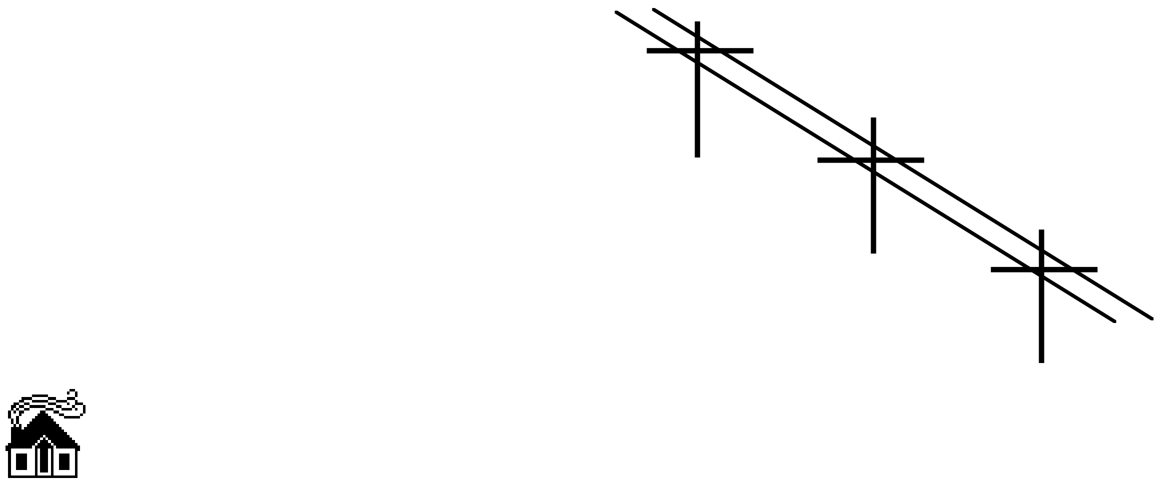
Find the shortest distance the turtle must crawl to get to the ocean.

(Assume the ocean shore is a straight line and the turtle is a point.)



**ELECTRIC LINES**

Your family is building a house out in the country. You need to run an electric line from your house to the main line. What is the shortest length of wire that is needed?

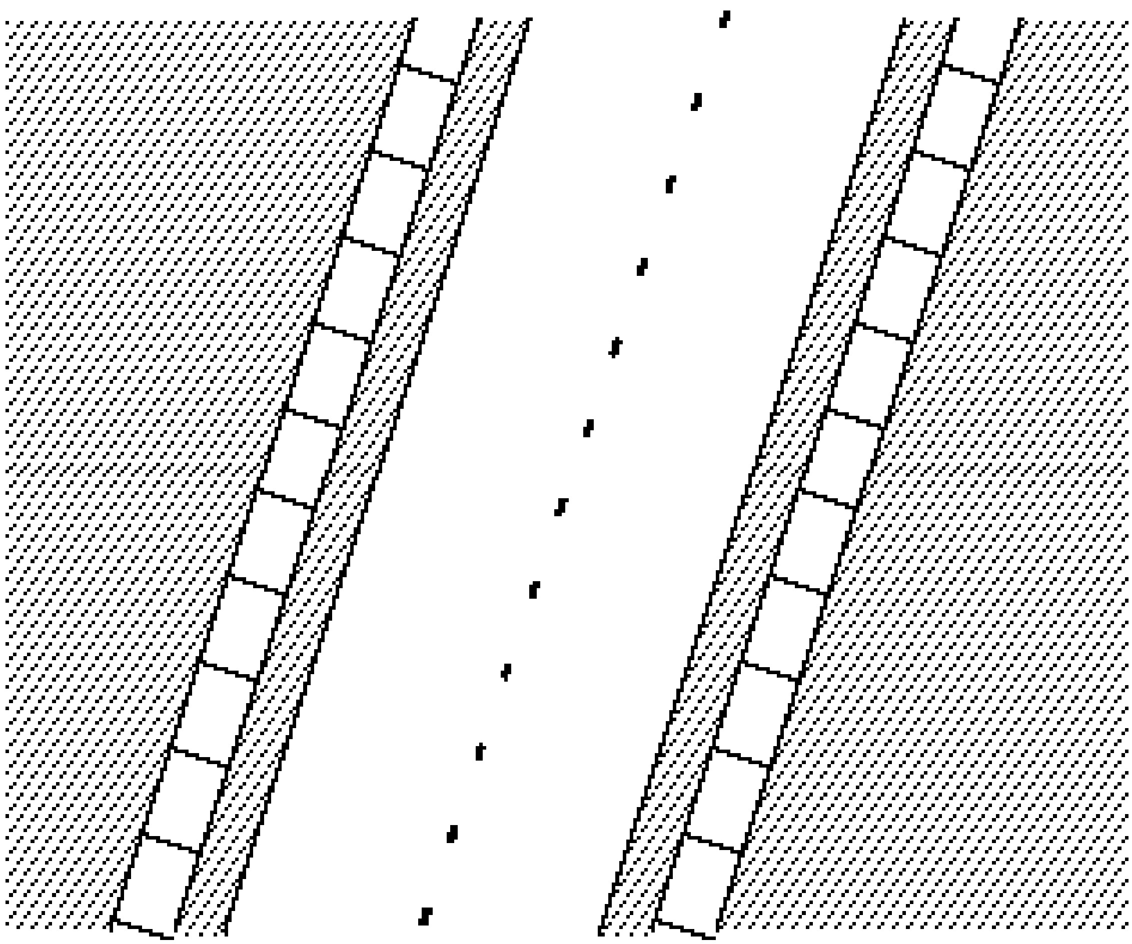


# DISTANCE IN THE EUCLIDEAN PLANE

## Between Parallel Lines

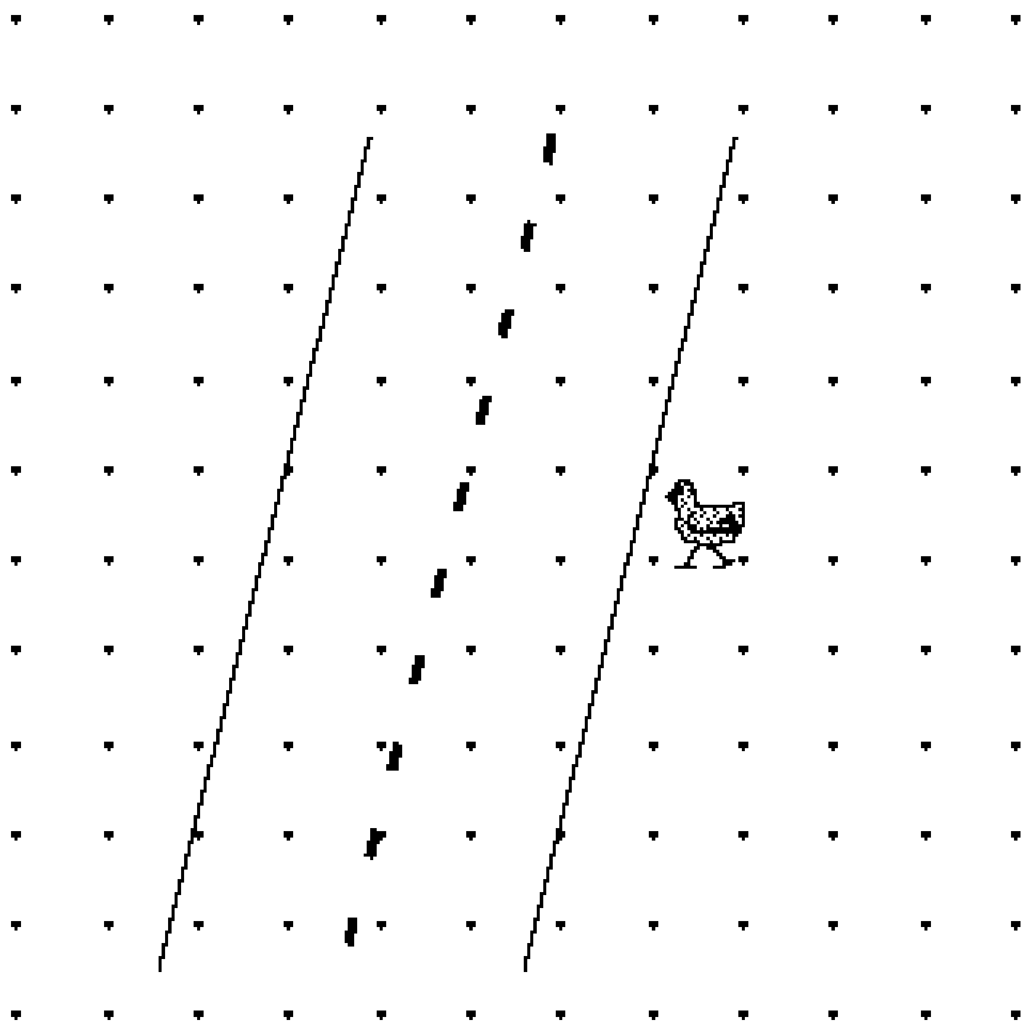
### PARALLEL SIDEWALKS

Pictured below is a street with sidewalks. Find the shortest distance between the sidewalks.



**THE CHICKEN AND THE ROAD**

How far did the chicken have to walk to cross the road? Find the shortest distance.



# PLANE VS. SPHERICAL GEOMETRY

## Teacher Guide

**GOAL:**

Use the parallel postulate, to understand a fundamental difference between Euclidean geometry (plane geometry) and a non-Euclidean geometry (spherical geometry).

**PREREQUISITES:**

The line and parallel postulates for Euclidean plane geometry, lines of longitude and latitude on the sphere.

**MATERIALS:**

Large plastic sphere, string, ruler, large globe.

**PROCEDURE:**

Working alone or in groups, students compare and contrast geometry on the plane and on the sphere by examining two of the postulates for Euclidean plane geometry. A class discussion should follow the exploration of each postulate.

**Postulates**

**Line Postulate:**

***"For any two distinct points, there is one and only one line containing them".***

**Parallel Postulate:**

***"Through a point not on a given line, one and only one line is parallel to the given line."***

**EXTENSIONS:**

- Compare triangles and their angle sums on the plane and on the sphere.
- Investigate airplane routes - an important practical application of spherical geometry.
- Research other non-Euclidean geometries and their discoverers.

**SOLUTIONS:**

**Line Postulate**

2. Straight line segment
8. Answers will vary
10. Yes, only if the points are antipodal. There are infinitely many great circles through the North and South Poles, for example.
11. It fails.
12. All lines of longitude are lines, the equator is the only line of latitude that is a line in spherical geometry.
13. All lines of longitude are great circles, the equator is the only line of latitude that is a great circle.
14. Similarities: Shortest path between points lies on the lines.
    Differences: In a plane, a line has infinite length; on a sphere a line has finite length.

**Parallel Postulate**

1. Non-intersecting lines - this is the best definition because it uses only the notion of lines.
   Equidistant lines - this requires a notion of distance on the sphere to apply to spherical geometry.
   Two line perpendicular to the same line - this requires a notion of angles in spherical geometry.
3. Exactly one line contains point P and is parallel to $\overline{AB}$.
4. Two non-intersecting lines.
6. It is not possible to draw a line on a sphere that is parallel to $\overline{AB}$ because on a sphere, a line is a great circle and all great circles intersect in exactly two points. Parallel lines do not intersect.
7. Thus the parallel postulate, is not valid in spherical geometry.

Geometry on a sphere is considered to be non-Euclidean because spherical geometry does not satisfy of the basic postulates of Euclidean geometry.

# PLANE VS. SPHERICAL GEOMETRY

**GOAL:**

Use the line and parallel postulates to understand a fundamental difference between plane geometry and spherical geometry.

**PROCEDURE:**

Compare and contrast geometry on the plane and geometry on the sphere by examining two postulates.

## Line Postulate.

***"For any two distinct points, there is one and only one line containing them".***

**Line postulate on the plane.**

1. Draw 2 points on a sheet of paper and label them A and B.
   (A sheet of paper is a model for the plane.)

2. Draw at least 3 different paths containing points A and B.
   Of all the possible paths between A and B, which is the shortest?

3. Highlight the shortest path - called the line segment between A and B.
   If this line segment is extended indefinitely in both directions the result is the line containing A and B in plane geometry.

4. Draw a representation of this line containing points A and B.
   This is the path referred to in the Line Postulate,
   "For any two distinct points, there is one and only one line containing them."

**Line postulate on the sphere.**

5. Draw 2 points on the sphere and label them A and B.

6. Draw at least 3 different paths containing points A and B.

7. Find the shortest path between A and B. Use the string to follow the paths and measure the lengths of the paths with the ruler.

8. Record the path length in the following spaces.
   Path 1 ____________ Path 2 ___________ Path 3 _____________

9. Use the string to extend the shortest path between A and B to a path around the sphere. Trace this path on the sphere with a pen. This is a line containing A and B in spherical geometry.

10. Can there be more than one line in spherical geometry which contains points A and B?
    Hint: Think of the North and South Pole on the globe.

11. What does this say about the line postulate for spherical geometry?

12. A globe of the earth is a good example of a sphere. On a globe, when are lines of latitude and longitude lines in spherical geometry?

13. A great circle is the intersection of the sphere and a plane passing through the center of the sphere. On the sphere a great circle is the path that contains the shortest distance between any two points.

    When is a line of latitude or longitude on the globe a great circle?

14. In what ways are the line in the plane and the line on the sphere similar?

    In what ways are they different?

## Parallel Postulate:

***"Through a point not on a given line, one and only one line is parallel to the given line."***

**Parallel postulate on the plane.**

1. What is meant by parallel lines in the plane?

2. On the plane draw a point P not contained on line $\overline{AB}$.

   Through point P draw a line parallel to line $\overline{AB}$.
3. How many lines contain point P and are parallel to $\overline{AB}$?

**Parallel postulate on the sphere.**

4. What might be meant by parallel lines on a sphere?
5. On the sphere, draw a point P not contained on line $\overline{AB}$.

   Try to draw a line through point P parallel to $\overline{AB}$. (Remember - a line on a sphere is a great circle.)

6. Can you draw a line on the sphere that is parallel to $\overline{AB}$?

   Why or why not?

7. What does this day about the parallel postulate in spherical geometry?

**Non-Euclidean geometry.**

Geometry on a sphere is said to be non-Euclidean. Is this reasonable?

# GEOMETRY ON OTHER SURFACES

## Institute Notes

| Lesson | Pages | Time | TEKS |
|---|---|---|---|
| Plane vs. Spherical Geometry | | 30 min. | 1c, 6c |
| Geometry on a Cube | | 60 min. | 1c |

**MATERIALS AND SETUP**

Large (4”) cardboard cubes (2 per group), rubber bands, colored straight pins, models of regular solids and other convex polyhedra with regular faces, large sphere and string for each group

**LEARNING STYLES**

Hands on, deductive reasoning

**MATH LINKS**

Topology, differential geometry, mechanics

**OTHER LINKS**

Navigation, cartography, building construction

**LESSON OVERVIEW**

To study geometry on a surface we need to know how to specify the ‘line segments’ joining pairs of points on the surface. On the plane a line segment joining two points is the shortest path between those two points and a ‘line’ in plane geometry is the usual straight line extension of this shortest path. The Cable Guy problem investigates the shortest path between two points on the surface of a cube as well as other polyhedral surfaces and curved surfaces. This shows how to define line segments on such surfaces.

But what is a line segment on a sphere? The shortest path between two points on the earth is the route a plane would fly between any two airports or the sea route of a ship sailing between, say, San Francisco and Hawaii. It is well known that the shortest route on the earth is always an arc of a great circle, so a line segment between two points on the earth is the arc of a great circle passing through those two points. This makes sense on all spheres because great circles are exactly the cross-sections produced when we slice the sphere with a plane through the center of the sphere. If we slice a sphere with a plane we always get a circle, but the circle will have smallest radius when the plane passes though the center of the sphere - this is why arcs of great circles are the shortest paths on a sphere. This is not obvious because any plane passing through two points A and B on a sphere will always intersect the sphere in a circle. However, an arc AB of the circle will be shortest when the radius of the circle is smallest. This occurs when the plane passes throughthe center of the sphere as well as through A and B.

Thus a sphere is a model for spherical geometry in which the lines are the great circles and the line segments are arcs of great circles just as a plane is a model for Euclidean plane geometry in which the lines are the usual straight lines.

**Plane vs. Spherical Geometry**

Ever since Euclid, one way of studying plane geometry is to try to establish all of its properties from a set of axioms or postulates. In the plane one of the most obvious facts is that exactly one straight line passes through two points. This is often called the **Line Postulate** of Euclidean geometry.

Another fact is that there is exactly one straight line parallel to a given line and passing through a given point not on the given line. This is the famous **Parallel Postulate**. Many of the most important properties of Euclidean geometry, such as the fact that the sum of the angles of a triangle is always 180°, follow from this postulate. Because of this, any geometry in which the Parallel Postulate fails is said to be non-Euclidean. By taking simple measurements of path lengths on a sphere, students discover that arcs of Great Circles are the shortest paths between two points on a sphere and so obtain a model of spherical geometry. Using this model they the discover that both the Line and Parallel Postulates fail in spherical geometry**.**

**Geometry on a Cube**

A triangle in the plane is formed by the line segments joining three points in the plane. Triangles can be formed on any surface on which line segments have been specified. For instance, the arcs of Great Circles joining three points on the sphere form a spherical triangle. The Cable Guy forms a triangle on a cube when he runs wire efficiently between three points in a room or on the walls of the pyramid shaped lobby of the Luxor Hotel in Las Vegas or on the geodesic dome at Epcot Center.

In the plane the sum of the angles of a triangle is always 180°, a consequence of the Parallel Postulate. But will the sum of the angles of a triangle always be 180° in a geometry in which the Parallel Postulate fails? If not, can the sum of the angles be related to the shape of the surface?

The plane is flat, whereas a sphere is not flat - it is curved. A soccer ball and a basketball are both spherical, so they have the same curvature at every point unlike a football which has greater curvature at the two ends than near the middle. The faces of a cube are flat but the corners are sharp - we can even measure how sharp each corner is using the angle deficit idea from the Day 1. In a tiling of the plane by squares, four squares meet each vertex unlike a cube where three squares meet at each corner. Thus each vertex of a cube is 90° from being flat. The Luxor Hotel is the upper half of an octahedron, so the top of the Luxor Hotel is 120° from being flat - it has greater curvature than the corner of a cube. This lesson discovers the remarkable result that the sum of the angles of a triangle on a cube is equal to 180° plus 90° times the number of corners inside the triangle. It is done first by direct measurement and then by a short proof using the sum of angles of triangles and quadrilaterals in the plane. Extensions to other polyhedra are made.

By totalling the curvature at all the corners of a cube we obtain another remarkable result relating the total curvature of a cube to its Euler Characteristic. Extensions to other polyhedra are again made.

## NOTES

# COMPLEX SURFACES

## Institute Notes

| Lesson | Pages | Time | TEKS |
|---|---|---|---|
| Fractal Tetrahedron | | 60 min. | 5b, 8 |

**MATERIALS AND SETUP**

Each group requires: one regular tetrahedron constructed from card stock (with 6” edge), 6 card stock equilateral triangles (6” edge), three squares forming one corner of a cube (edge $3\sqrt{2}$ ), fast drying glue or rubber cement, two-sided tape

**LEARNING STYLES**

Hands on, cooperative groups, technology

**MATH LINKS**

Algebra, sequences and series, topology, similarity

**OTHER LINKS**

Fractals

**LESSON OVERVIEW**

**The Fractal Tetrahedron**

We will look again at tetrahedra and their relation to cubes. On previous days the tetrahedron kite was constructed by attaching many tetrahedra together at their vertices to create a larger tetrahedron. In the study of cross-sections of a cube, a boxed tetrahedron was constructed by slicing off pyramids from corners of a cube.

Now we shall reassemble the boxing cube for a tetrahedron by adding ever smaller tetrahedra to each face of fractal polyhedra, starting with the tetrahedron to be boxed. If we could think of doing this construction infinitely many times, then ultimately the box would be reassembled, but in reality the boxing cube can never be put back together again with this construction.

Participants will follow the instructions and build the fractal beginning with a prefabricated tetrahedron and precut triangles. The trainer will demonstrate that the volume of the fractal polyhedra approaches the volume of the boxing cube using the sum of an infinite geometric series. (The lesson could be used in an algebra class as a concrete illustration of geometric series.) Point out that the volume of the fractal polyhedron at each stage can be determined with a calculator without knowing anything about geometric series.

If time allows, participants will create the Koch snowflake or see its construction demonstrated on *Sketchpad*.

## NOTES

# GEOMETRY ON A CUBE

## Teacher Guide

**GOAL:**

Discover the relationship between the sum of the angles of a triangle on a cube and the number of vertices contained in the cube. Understand curvature.

**MATERIALS:**

Pre-assembled models Platonic solids; protractors; colored straight pins; rubber bands.

**PREREQUISITES:**

Sum of interior angles of a triangle and a quadrilateral in the plane. Solution to the Cable Guy problem for cube.

**PROCEDURE:**

Students work in small groups to create triangles on a cube using pins to mark the vertices and rubber bands stretched between the pins to create line segments forming the sides of the triangle. Encourage different groups to create different triangles, some triangles lying entirely on one face, others containing one vertex of the cube, others containing two vertices of the cube. Students draw the rubber bandtriangle on the cube. The pins have created the vertices of the triangle and the use of rubber bands ensures that the pencil lines are the shortest paths between the pinholes. The choice of pinholes should be sufficiently far from the sides of the cube so that the angles can be measured accurately.

Students tabulate the angle measurements in the table. On comparing data they should conjecture that the sum of the angles of a triangle on a cube is always 180° plus 90° times the number of vertices contained within the triangle. For a triangle on a tetrahedron the sum of the angles is always 180° plus 180° times the number of vertices of the tetrahedron contained inside the triangle; for an octahedron it would 180° plus 120° times the number of vertices of the octahedron inside the triangle. Notice that number of degrees a polyhedron loses at a vertex by not being flat has been added to the sum of the angles of any triangle surrounding that vertex.

This conjecture is substantiated for triangles containing one vertex of the cube. It uses the fact that the shortest path between points on adjacent faces of a cube consists of the unbroken path formed by straight lines on each face that meet the common edge between adjacent faces in supplementary angles. This property was established earlier in the Cable Guy problem.

# GEOMETRY ON A CUBE

**GOAL:**

Investigate the relationship between the sum of the angles of a triangle on a cube and the number of vertices of the cube contained in the triangle.

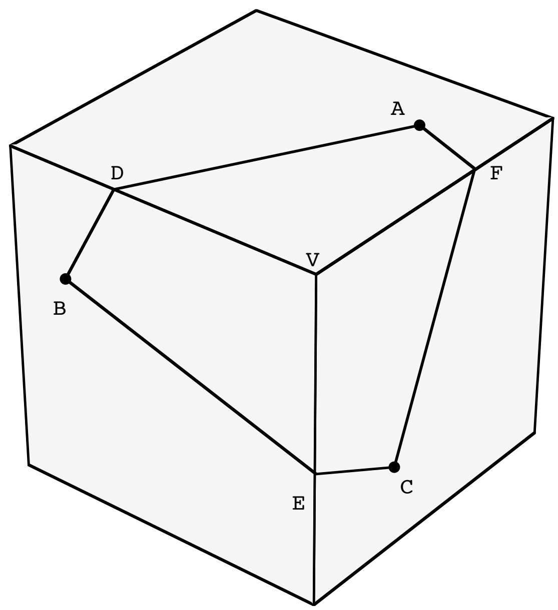
**PROCEDURE:**

1. Working in small groups insert the colored pins in the faces of a cube at any three points other than vertices and label them A, B, C. Use a rubber band to form a triangle on the cube having A, B, C as vertices.
   Is there more than one way to place the rubber band?
   If so, which is the most natural choice?
2. Trace the path of the rubberband:
   Mark with a pencil where the rubber band crosses the edges of the cube.
   Remove the rubber band and pins.
   Draw straight lines on the cube joining a pin-hole (a vertex A, B, or C) on a face of the cube to the points marked on the edges of that face.
   If two or more pin-holes are on the same face, draw straight lines joining the pin-holes on that face. Repeat this for every face for which there is a pin-hole.
3. Measure the angle between the straight lines at each vertex A, B, C and record the angles in the data table and find their total.
4. Compare totals with other groups.
   What formula do these totals suggest for the sum of the angles of a triangle on a cube?

| Vertex | Angle |
|---|---|
| A | |
| B | |
| C | |
| TOTAL | |

5. Confirm the conjecture in special cases:
   a) A, B, C all lie on the same face of the cube,
   b) A, B, C are the centers of three adjacent faces of the cube,
   c) A is the centers of a face and B, C are the midpoints of adjacent edges of the opposite face.

6. To confirm your conjecture for the sum of the angles of a general triangle ABC containing one vertex, use the following figure to show that ∠A + ∠B+ ∠C = 270°.




The figure contains 3 quadrilaterals ADVF, BDVE, and CEVF which meet at the vertex V of the cube.

The sum of the angles of ADVF = ∠A + ∠ADV + ∠DVF + ∠AFV.

The sum of the angles of BDVE = ∠B + ∠BDV + ∠DVE + ∠BEV.

The sum of the angles of CFVE = ∠C + ∠CFV + ∠FVE + ∠CEV.

What is the relationship between angles $\angle ADV$ and $\angle BDV$ formed by the line segment AB and the edge DV?

What is the sum of the angles of any quadrilateral?

Now compute $\angle A + \angle B + \angle C$ using the above information.

7. How would your calculations change for a triangle ABC on a tetrahedron when the triangle contains one vertex of the tetrahedron?

8. What results would you expect for an octahedron?

9. How would you modify your calculations when ABC is a triangle on a cube containing two vertices of the cube?

# OPTIMAL PATHS

## Institute Notes

| Lesson | Pages | Time | TEKS |
|---|---|---|---|
| The Texas Bullet Train | | 30 min. | 5b, 9a |
| Connect the Dots | | 15 min. | 1a, 3e |

**MATERIALS AND SETUP**

*Geometer's Sketchpad* Sketches: "bullet train"

Large map of Texas with Dallas, Houston, and San Antonio highlighted

**LEARNING STYLES**

Technology, hands on, cooperative groups, deductive reasoning

**MATH LINKS**

Optimization, network theory (minimum spanning trees), isometries

**OTHER LINKS**

Physics, engineering, management science

**LESSON OVERVIEW**

Optimal paths are a recurring theme in many fields including mathematics, physics, network theory management science, navigation, and the design of work spaces. Even ants are required to solve such problems as they wear out a path from their nest to the garbage can. Navigators of ships and airplanes plan their routes using the geodesics (great circles) of our spherical planet.

Each optimal path problem includes restrictions and requirements that arise from the context of the problem. Mathematicians model the problems, together with their specific contexts. In creating models, it is sometimes observed that problems arising from very different context are fundamentally the same. Frequently, geometry provides the best model for a problem.

**The Texas Bullet Train**

This problem models the problem of connecting three cites with the least amount of train track.

Participants model the problem with technology and calculate the sums of the distances, moving the tracks until the sum is a minimum. The solution involves connecting the three cities to a point known as the Steiner point. Any junction point in a network that is formed by three branches coming together at 120° angles is called a Steiner Point of the network.

**Connect the Dots**

This proof of the general solution to the problem of characterizing the minimum path connecting three points requires only that students are familiar with rotations, equilateral triangles, and the shortest distance between two points. It should be presented in the classroom after posing the Texas Bullet Train problem. Both can be done with or without technology.

Demonstrate the proof with participant input using *Geometer's Sketchpad*.

**NOTES**

# THE TEXAS BULLET TRAIN

## Teacher Guide

**GOAL:**

Find the path of minimum length that connects three points, and investigate its properties.

**PREREQUISITES:**

Knowledge of the Measure menu on *Geometer's Sketchpad*.

**MATERIALS:**

*Sketchpad* sketch "bullet train", large map of Texas with Dallas, Houston, and San Antonio highlighted.

**NOTES TO TEACHER:**

This is a lesson in optimization. The goal is to minimize a path length and thereby minimize the cost of the track. See C. V. Boys for a soap solution method to find minimal paths connecting many points.

A proof is found in CONNECT THE DOTS

**TI-92 Adaptation:**

The lesson may be done on the TI-92 calculator. Have students open a geometry session and place three points on the screen. Then they should draw segments and make measurements to find the minimum path that connects the three points. They can use the student lab worksheet. The only difference is that the *Sketchpad* lesson represents the actual distances between Dallas, Houston and San Antonio.

**PROCEDURE:**

- Put students in pairs at the computer to investigate solutions to the problem.
- Record results on the lab worksheet.
- It is unlikely that students will consider adding points. Point out the HINT at the bottom of the screen after they have tried the shortest path. Mention that other points may be added to the picture. Such points are call Steiner Points or Fermat Points.

**EXTENSION:**

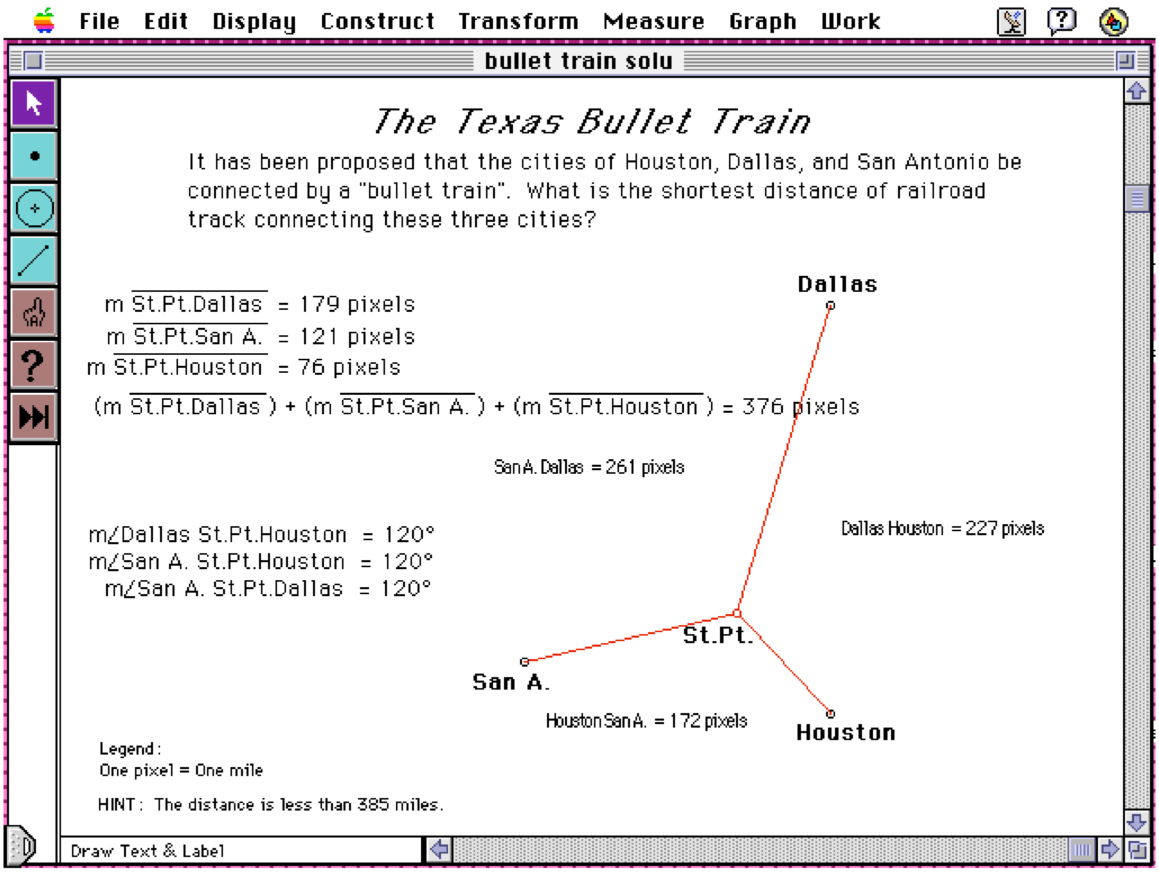
- Prove that the segments forming a minimum path meet at 120°angles.

  The CONNECT THE DOTS lesson provides one proof.

- Investigate Network Theory (Graph theory) and minimum spanning trees.

**SOLUTIONS:**

1. The student logs will vary. The optimal solution has a total distance of 376 miles.
2. Add another point. The path should be 376 miles long and the segments from the cities to the new point meet at 120° angles.




3. The segments connecting the three cities to the Steiner Point always create 120° angles.
4. To connect four points with a minimal path, two Steiner Points are required. The angles around those Steiner Points are all 120° angles.

# THE TEXAS BULLET TRAIN

**<u>Problem</u>:**

It has been proposed that the cities of Houston, Dallas and San Antonio be connected by a "bullet train". Building a rail line is very expensive, hence it is important to design the layout of the tracks so that the amount of track laid is minimal. The focus of this problem is to minimize the total cost of the track used, not to accommodate the passenger's travel time. The only restriction is that a passenger must be able to travel between any two cities.

Where should the tracks be laid? Assume the land is flat and that there are no impassable obstacles (such as cities or rivers) to any possible track.


ck
Wichita Falls
Denison
Sherman
Texarkana
Dallas
Fort Worth
Marshall
Shreveport
Abilene
Tyler
Longview
TEXAS
LOUISIANA
Waco
San Angelo
Ale
Killeen
Temple
College Station
Baton Ro
Austin
Lake Charles
Houston
San Antonio
Galveston
UNITED STATES
MEXICO
Victoria
Corpus Christi

**PROCEDURE:**

- *Open the sketch "bullet train" on Sketchpad.*

1. Write a log of the process you go through to minimize the distance of the track laid. Include measurements and drawings of each significant stage in your solution.
2. Describe your final track design.
   Include a drawing with length and angle measurements.

3. Open a new sketch and draw three random points. Find the minimal path connecting the three points. Drag the points around and make generalizations about the minimal path connecting any three points.

4. Open a new sketch and draw four random points. Find the minimal path connecting the four points. Drag the points around and make generalizations about the minimal path connecting any four points.

# CONNECT THE DOTS

## Teacher Guide

**GOAL:**

Find the path of minimum length that connects three points and its properties..

**PREREQUISITES:**

Equilateral triangle, rotations, congruent triangles, straight angles.

**PROCEDURE:**

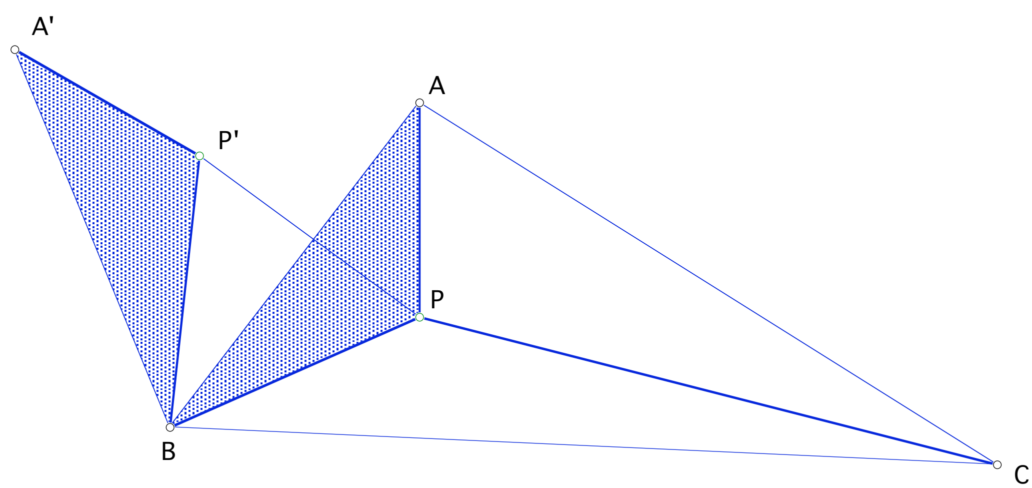
Assume the three points A,B, C are the vertices of an acute triangle.

Restatement of the goal: Find point P such that AP + BP + CP is minimal.

Assume P is any point and consider AP + BP + CP.

Rotate ΔAPB 60°about point B to get ΔC’P’B.

ΔAPB is congruent to ΔC’P’B so AP = C’P’




Draw PP’ creating ΔPBP’.

Angle PBP’ is 60°and BP = BP’ by the rotation.

Thus ΔPBP’ is an isosceles Δ with vertex angle 60°.

Therefore, ΔPBP’ must be an equilateral Δ and BP=P’P.

By substitution, AP + BP + CP = C’P’ + P’P + PC, a broken line joining C’ and C, with bends in the line at P’ and P.

A line joining C’ and C will have minimal length if it is a straight line,

that is, if the angles at P’ and P are 180°.

One of the angles at each of P and P’ is 60°, since the angles are in equilateral ΔPBP’.

Therefore, for the distance to be minimal, angle BPC = 120° and ∠C’P’B = 120°.

Notice that ∠C’P’B = ∠APB by the rotation.

Hence, in order for AP + BP + CP to be minimal, point P must be located so that the segments AP, BP, and CP meet at 120°angles.

How can one locate such a point P?

Returning to the original figure, draw C’A.

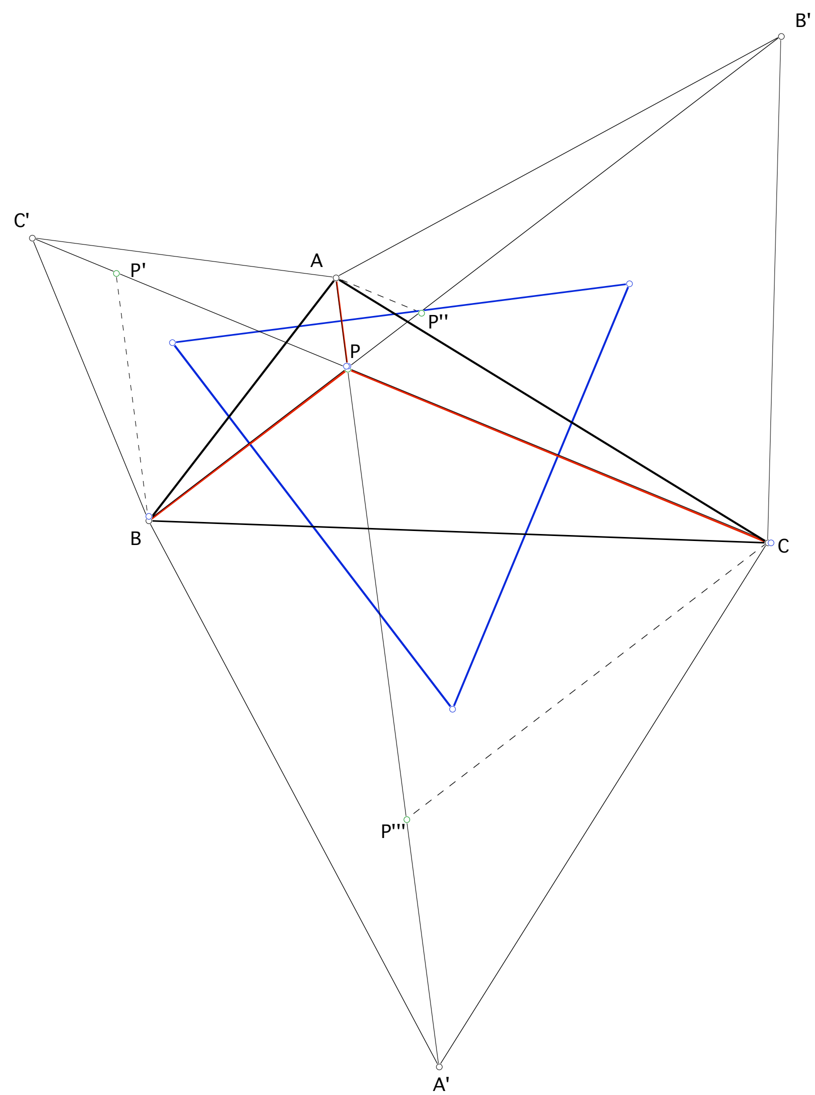
By the rotation, ΔABC’ is an equilateral Δ. P lies on C’C.

The proof could also have been done by rotating ΔAPC 60° about A with P lying on B’B, or by rotating ΔBPC 60° about C with P lying on A’A..

Thus, to find point P, construct equilateral Δs on any two faces of ΔABC.

Point P is the intersection of CC’ and BB’ and AA’.

This also proves that CC’, BB’ and AA’ are concurrent.



**EXTENSIONS USING *SKETCHPAD*:**

- Demonstrate the proof if the triangle is obtuse, but with all angles less than 120°.
- Where is P if one angle of ΔABC is ≥ 120°?
- Construct the circumcircles of the three equilateral Δs built on the sides of ΔABC.

  How is P related to those circles? *P lies on the three circles.*

  Join the centers of the three circles. What kind of triangle results? *Equilateral*
- Find the minimal path and its properties when connecting 4 or more points.

  *Add points to the diagram so that the angles formed at those points are 120°.*

**NOTES:**

The added points are called Steiner points (Jakob Steiner, 1796-1863) or Fermat points.

Soap solution can be used to locate the Fermat points. See C.V.Boys.

**REFERENCES:**

Boys, C. V. *Soap Bubbles.*

Hildebrandt & Tromba, *Mathematics and Optimal Form.*

# FRACTAL TETRAHEDRON

## Teacher Guide

**GOAL:**

Understand volumes of simple polyhedra and the relation of volume to dimension and dilation.

**MATERIALS:**

Each group requires: one regular tetrahedron constructed from card stock (with 6" edge), 6 card stock equilateral triangles (6" edge), three squares forming one corner of a cube (edge $3\sqrt{2}$ ), fast drying glue or rubber cement, two-sided tape, ruler, stella octangula.

**PREREQUISITES:**

Volume and surface area of a pyramid, Pythagorean theorem, stella octangula, volume and surface area of similar figures.

**PROCEDURE:**

The basic construction begins with a tetrahedron and at each step consists of placing a tetrahedron of edge length $\frac{e}{2}$ on every equilateral triangle of side length $e$ so that the vertices of one face of the smaller tetrahedron lie on the midpoints of the sides of the larger triangle. This creates six new equilateral triangles each having side length $\frac{e}{2}$ where there had been just one - three on the uncovered part of the larger triangle and the remaining three are the uncovered portions of the smaller tetrahedron. Repeat the processes by adding ever smaller tetrahedra to all equilateral triangles.

Students record the volume and surface area of the figure at each generation of the construction.

Lead students to see a pattern in the volumes and surface areas as more and more generations are constructed.

The classroom construction can be terminated at any generation.

**NOTES TO TEACHER:**

**Volumes:**

Students should compute the volume of a tetrahedron and compare it to the volume of the tetrahedron's circumscribing cube. $V_{Tet} = \frac{1}{3} V_{Cube}$.

This is worked out in detail below.

**Similar Figures:** Successive generations involve similar figures with each new generation's tetrahedron having edge length $\frac{1}{2}$ that of the previous generation.

Hence, the volume of the new tetrahedra are $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ the volume of the tetrahedrons of the previous generation.

Also, the area of each new face is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ the area of the face of the previous generation.

**Patterns:**

It may be easier to see patterns emerge in the charts if the volumes and surface areas are presented as sums and products and not simplified. Both simplified and non-simplified representations appear in the completed charts.

**Geometric Series:**

The sum of an infinite geometric series

$a + ar + ar^2 + ar^3 + ... = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if $-1 < r < 1$.

So, $\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k = \frac{1}{1-\frac{3}{4}} = 4$ since a = 1 and $r = \frac{3}{4}$.

This sum can be found for larger and larger k using a calculator.

Charting the partial sums will show them approaching 4.

**Sequences:**

The limit of a sequence in the form t, $t^2$, $t^3$, $t^4$, ..., $t^n$ as $n \to \infty$ depends on the value of t.

If $-1 < t < 1$, then $t^n \to o$ as $n \to \infty$.

For example, if $t = \frac{1}{2}$ then $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \to 0$. If $t = -\frac{1}{2}$ , then $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$ also approaches 0.

If $t > 1$, then $t^n \to \infty$ as $n \to \infty$.

For example, if $t = \frac{3}{2}$, then $\frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{3^4}{2^4}$, = 1.5, 2.25, 3.375, 5.0625.

These numbers are getting arbitrarily large.

These patterns can be observed on a graphing calculator by graphing

$y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{3}{2}\right)^x$ for $x \geq 0$.

## Volume of a Tetrahedron

Tetrahedron with edge e, volume = $v = \frac{\sqrt{2}}{12}e^3$

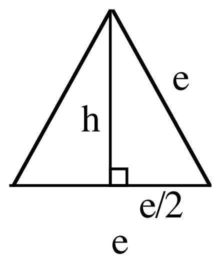
Derivation:

Find B = Area of base of tetrhedron = $\frac{1}{2}eh = \frac{1}{2}e\left(\frac{e\sqrt{3}}{2}\right) = \frac{e^2\sqrt{3}}{4}$

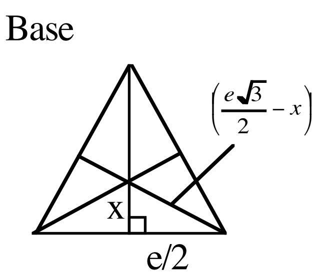
$$e^2 = h^2 + \frac{e^2}{4}$$

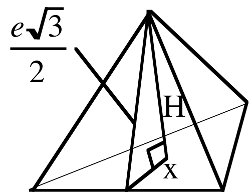
$$h^2 = \frac{3e^2}{4}$$

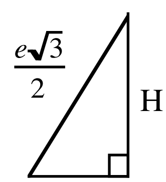
$$h = \frac{e\sqrt{3}}{2}$$




Find H = Height of tetrahedron.




$x = \frac{e}{2\sqrt{3}}$

There are two possible methods for computing x:

$$x = \frac{1}{3}h$$

$$= \frac{1}{3}\cdot\frac{\sqrt{3}}{2}e$$

$$x = \frac{1}{2\sqrt{3}}e$$

or

$$\left(\frac{e}{2}\right)^2 + x^2 = \left(\frac{e\sqrt{3}}{2} - x\right)^2$$

$$\frac{e^2}{4} + x^2 = \frac{3e^2}{4} - ex\sqrt{3} + x^2$$

$$ex\sqrt{3} = \frac{1}{2}e^2$$

$$x = \frac{\frac{e^2}{2}}{e\sqrt{3}} = \frac{e}{2\sqrt{3}}$$

Compute the height of the tetrahedron:

$$H^2 + e^2 = \left(\frac{e\sqrt{3}}{2}\right)^2$$

$$H^2 = \frac{3e^2}{4} - \frac{e^2}{12} = \frac{8}{12}e^2$$

$$H = \sqrt{\frac{2}{3}}e$$

$$H^2 = \frac{2}{3}e^2$$

$$\text{Volume.of.Pyramid} = \frac{1}{3} \cdot \text{base} \cdot \text{height}$$

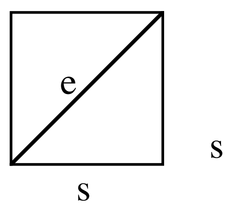
$$v = \frac{1}{3}BH = \frac{1}{3} \cdot \frac{\sqrt{3}}{4}e^2 \cdot \frac{\sqrt{2}}{\sqrt{3}}e = \frac{\sqrt{2}}{12}e^3$$

Compare the volume of the tetrahedron to the volume of its boxing cube:

Volume of the boxing cube with edge s, $v_c = s^3$

The edge, *e*, of the tetrahedron is the diagonal of the face of the boxing cube.

$$e^2 = s^2 + s^2 \therefore e = s\sqrt{2}$$




Since $e = s\sqrt{2}$, $\text{v} = \frac{\sqrt{2}}{12}\left(\text{s}\sqrt{2}\right)^3 = \frac{4}{12}\text{s}^3 = \frac{1}{3}\text{s}^3$

So $v = \frac{1}{3}V_c$ or $\text{V}_\text{c} = 3\text{v}$

**SOLUTIONS:**

See charts and notes to the teacher.

There is no generation at which the fractal tetrahedron fills the cube completely.

**EXTENSIONS:**

- Repeat with the dual octahedron and its boxing cube, building tetrahedra on each triangular face as above.

- Create a fractal from an equilateral triangle in the plane by attaching smaller equilateral triangles to each edges of the previous generation. Analyze area and perimeter.

# FRACTAL TETRAHEDRON

**GOAL:**

Understand volumes of polyhedra and the relation of volume to dimension and dilation.

**PROCEDURE:**

- The basic construction begins with a tetrahedron and at each step consists of placing a tetrahedron of edge length $\frac{e}{2}$ on every equilateral triangle of side length $e$ so that the vertices of one face of the smaller tetrahedron lie on the midpoints of the sides of the larger triangle. This creates six new equilateral triangles each having side length $\frac{e}{2}$ where there had been just one - three being the uncovered part of the larger triangle and the remaining three being the uncovered portions of the smaller tetrahedron. This is repeated as ever smaller tetrahedra are applied to all equilateral triangles.
- Record the volume and surface area of the figure at each generation of the construction.
- Compare the fractal tetrahedron to the corner of the cube at each generation.
- Look for a pattern in the volumes and a pattern in the surface areas as more and more generations are constructed.

**Constructing the Fractal Tetrahedron**

**Generation 0:**

- Begin with the pre-assembled tetrahedron of edge length $e$.
- Record its volume and surface area.

**Generation 1:**

- Assemble 4 tetrahedra using the equilateral triangles as nets and attach one to each face of the first tetrahedron so that the vertices of one face of the small tetrahedron are at the midpoints of the edges of a face of the large tetrahedron. The construction produces a polyhedron which is a Stella Octangula.

- Determine and record the number of faces of the Stella Octangula and its volume and surface area.

**Generation 2:**

- Cut two of the equilateral triangles along midsegments into 8 congruent equilateral triangles (four from each triangle).
- Assemble 6 of these into tetrahedra having half the edge length of generation 1 tetrahedra and attach them to 6 adjacent faces of the previous polyhedron. (It would be too time consuming to create 24 tetrahedra and attach them to all the faces.)
- Determine and record the number of faces, volume and surface area of the polyhedron that would be obtained if this construction were applied to each face of the previous polyhedron.

**Generation 3:**

- Cut one of the remaining small equilateral triangles into 4 equilateral triangles.
- Assemble them into tetrahedra having edge length half those in generation 2 and attach them to four of the faces of generation 2 tetrahedra in the previous polyhedron.
- Determine the number of faces, the volume and surface area of the polyhedron that would be obtained if this construction were applied to each face of the previous polyhedron.

**Patterns:**

- Conjecture what the polyhedron would look like if this construction were applied numerous times to each face at each stage.
- Look for patterns in the entries in the tables as the generations increase.
- Complete the tables for the $n^{th}$ generation.
- Conjecture what the volume and surface area of the polyhedron would be after an infinite number of generations. Test your conjecture (by calculator or by summing a geometric series).
- Does the polyhedron totally fill the boxing cube after infinitely many generations?
- If the edge of a tetrahedron is $e$, what is the edge, $S$, of its boxing cube?

# Fractal Tetrahedron

## Surface Area through the Generations

| Generation | # of Faces | Area of each face | Total Surface area = $S_n$ |
|---|---|---|---|
| **0** | 4 | $\frac{e^2\sqrt{3}}{4}$ | $e^2\sqrt{3}$ |
| **1** | $4 \cdot 6$ | | |
| **2** | | | |
| **3** | | | |
| **4** | | | |
| $n$ | | | |

**Pattern:**

# Fractal Tetrahedron

# Volumes through the Generations

Let $v$ = volume of original tetrahedron

| **Generation** | **# of Faces** | **# of New Tetrahedra** | **Volume of 1 New tetrahedron** | **Total New Volume** | **Total cummulativeVolume =** $V_n$ | $V_n$ **Simplified** |
|---|---|---|---|---|---|---|
| **0** | 4 | 1 | $v$ | $v$ | $v$ | $v = 1v$ |
| **1** | $4 \cdot 6$ | 4 | $\frac{1}{8}v$ | $4\left(\frac{1}{8}\right)v$ | $v + 4\left(\frac{1}{8}\right)v = v\left[1 + \frac{4}{8}\right] = v\left[1 + \frac{1}{2}\right]$ | $\frac{3}{2}v = 1.5v$ |
| **2** | | | | | | |
| **3** | | | | | | |
| **4** | | | | | | |
| **n** | | | | | | |

**Pattern:**

# FRACTAL TETRAHEDRON

## Surface Area through the Generations

| Generation | # of Faces | Area of each face | Total Surface area = $S_n$ |
|---|---|---|---|
| **0** | 4 | $\frac{e^2\sqrt{3}}{4}$ | $e^2\sqrt{3}$ |
| **1** | $4\cdot 6$ | $\frac{1}{4}\left(\frac{e^2\sqrt{3}}{4}\right)$ | $\left(\frac{3}{2}\right)\cdot e^2\sqrt{3}$ |
| **2** | $4\cdot 6^2$ | $\left(\frac{1}{4}\right)^2\left(\frac{e^2\sqrt{3}}{4}\right)$ | $\left(\frac{3}{2}\right)^2\cdot e^2\sqrt{3}$ |
| **3** | $4\cdot 6^3$ | $\left(\frac{1}{4}\right)^3\left(\frac{e^2\sqrt{3}}{4}\right)$ | $\left(\frac{3}{2}\right)^3\cdot e^2\sqrt{3}$ |
| **4** | $4\cdot 6^4$ | $\left(\frac{1}{4}\right)^4\left(\frac{e^2\sqrt{3}}{4}\right)$ | $\left(\frac{3}{2}\right)^4\cdot e^2\sqrt{3}$ |
| $n$ | $4\cdot 6^n$ | $\left(\frac{1}{4}\right)^n\left(\frac{e^2\sqrt{3}}{4}\right)$ | $\left(\frac{3}{2}\right)^n\cdot e^2\sqrt{3}$ |

What happens to the surface area, $S_n$ as $n$ gets large?

As $n \to \infty$, $\left(\frac{3}{2}\right)^n \to \infty$

So, $S_n \to \infty$.

The surface area of the fractal tetrahedron grows without bound.

Thus, this is an example of a solid with finite volume and infinite surface area.

FRACTAL TETRAHEDRON

# Volume through the Generations

Let $v$ = volume of original tetrahedron

| Generation | **# of Faces** | **# of New Tetra-hedra** | **Volume of 1 New Tetra-hedron** | **Total New Volume** | **Total cummulative Volume = $V_n$** | **$V_n$ Simplified** |
|---|---|---|---|---|---|---|
| **0** | 4 | 1 | $v$ | $v$ | $v$ | $v = 1v$ |
| **1** | $4\cdot 6$ | 4 | $\frac{1}{8}v$ | $4\left(\frac{1}{8}\right)v$ | $v + 4\left(\frac{1}{8}\right)v = v\left[1+\frac{4}{8}\right] = v\left[1+\frac{1}{2}\right]$ | $\frac{3}{2}v = 1.5v$ |
| **2** | $4\cdot 6^2$ | $4\cdot 6$ | $\left(\frac{1}{8}\right)^2 v$ | $4\cdot 6\left(\frac{1}{8}\right)^2 v$ | $v + 4\left(\frac{1}{8}\right)v + 4\cdot 6\left(\frac{1}{8}\right)^2 v = v\left[1+\frac{4}{8}\left(1+\frac{6}{8}\right)\right]$ | $\frac{15}{8}v = 1.875v$ |
| **3** | $4\cdot 6^3$ | $4\cdot 6^2$ | $\left(\frac{1}{8}\right)^3 v$ | $4\cdot 6^2\left(\frac{1}{8}\right)^3 v$ | $v + 4\left(\frac{1}{8}\right)v + 4\cdot 6\left(\frac{1}{8}\right)^2 v + 4\cdot 6^2\left(\frac{1}{8}\right)^3$ $= v\left[1+\frac{4}{8}\left(1+\frac{6}{8}+\frac{6^2}{8}\right)\right] = v\left[1+\frac{1}{2}\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2\right)\right]$ | $\frac{69}{32}v = 2.15625v$ |
| **4** | $4\cdot 6^4$ | $4\cdot 6^3$ | $\left(\frac{1}{8}\right)^4 v$ | $4\cdot 6^3\left(\frac{1}{8}\right)^4 v$ | $v\left[1+\frac{1}{2}\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3\right)\right]$ | $\frac{303}{128}v = 2.3671875v$ |
| **n** | $4\cdot 6^n$ | $4\cdot 6^{n-1}$ | $\left(\frac{1}{8}\right)^n v$ | $4\cdot 6^{n-1}\left(\frac{1}{8}\right)^n v$ $= \frac{4}{8}\cdot\left(\frac{6}{8}\right)^{n-1} v$ $= \frac{1}{2}\cdot\left(\frac{3}{4}\right)^{n-1} v$ | $v\left[1+\frac{1}{2}\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\ldots+\left(\frac{3}{4}\right)^{n-1}\right)\right]$ | $v\left[1+\frac{1}{2}\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k\right]$ |

What happens to the volume, $V_n$ ,as n gets large?

As $n$ gets large, $\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k$ approaches 4.

So, $V_n$ approaches $v\left[1+\frac{1}{2}(4)\right] = 3v$

**LAST DAY CLOSURE:**

- ♦ After experiencing the institute, what changes would participants make in their teaching.
- ♦ Write down 2 goals for themselves as geometry trainers/teachers and put in a self-addressed envelope to be sent to them at the end of the year. For the trainers too.
- ♦ Suggest the trainers work with a partner in their area or region for support.
- ♦ Look back at ideal vision - has it changed?
- ♦ What further training would they be interested in?
- ♦ Evaluation form
  - * do feel that a 3-day institue has adequately prepared to present a five day
  - * what kind of support will you need
  - *

**WHERE DO WE GO FROM HERE:** the Institue will show how this vision can be realized in the high school classroom. One of the goals is to provide participants with experiences in a variety of teaching and learnng styles, while allowing time to reflect on personal classroom goals. The Institute can be a starting point for participants to make changes at various levels, to take the ideas developed here and adapt them to their own classrooms.