Geometry

Gerard Venema, Calvin College (Chair)
William Barker, Bowdoin College
Frank Farris, Santa Clara University
Sarah Greenwald, Appalachian State University

1 Basic Recommendations

The Geometry Study Group (GSG) was charged by the CUPM Steering Committee with making recommendations about geometry in the undergraduate mathematics curriculum. We have three basic recommendations: 1) certain geometric concepts and methods belong in the education of every mathematics major, whether that content is delivered in a geometry course or not, 2) every undergraduate mathematics program should include at least one course devoted primarily to geometry, and 3) the curricular needs in geometry of future high school mathematics teachers are best satisfied within geometry courses for mathematics majors, not by a separate *Geometry for Teachers* course.

1.1 Recommendation 1

Every mathematics major program should include substantial geometric content. As pointed out in many documents over the past twenty-five years, instruction in geometric mathematics has suffered from neglect in the recent past. With the "push to calculus," many students never recover from a high school education that skips too quickly over geometry. This must be remedied. Geometry, in a centuries-long interplay with algebra, is one of the vital halves of mathematics, and this is just as true today as in ancient times.

The value of geometry to students cannot be contested. In no other field can students make such a strong connection between intuition, discovery, proof, and applications. Geometry is built on experiences in the physical and artistic worlds, and in turn is an essential skill in many areas of applied mathematics. Without geometry, students will suffer when it comes time to apply theory to physical situations. And the geometric viewpoint is central to many current areas of inquiry, from climate science to the mathematics of film-making.

To help departments respond to our call, we offer a list in Section 3 of the concepts and methods from geometry that should be part of the education of every undergraduate mathematics major. Some topics can be covered naturally in courses not exclusively devoted to geometry. These ideas are offered to help institutions determine for themselves how to ensure that all mathematics majors are most appropriately engaged with geometry.

1.2 Recommendation 2

Every mathematics department should offer at least one undergraduate course devoted primarily to geometry. There will probably never be a consensus on what that course should cover. However, in Section 6 we offer sample syllabi for some of the many valid course choices that institutions might make. We also outline in Section 4 some of the important issues institutions should consider as they decide which geometry course, or courses, to offer.

1.3 Recommendation 3

The GSG recognizes that geometry plays a special role in the education of future high school mathematics teachers. Teaching teachers is vital, but also vital is our call **not** to reduce geometry to something viewed as necessary only for teacher preparation. In particular, the GSG believes that future high school mathematics teachers are best served by appropriately designed geometry courses for the general mathematics major that also take into consideration the needs of secondary mathematics teachers, not by courses specifically designed only for future teachers. In Section 5 we discuss requirements for the preparation of teachers and give suggestions of how they can be met within courses for the general major.

2 Process

The GSG studied geometry in the undergraduate curriculum in several phases. In addition to internal discussions and a survey of the literature, the group invited geometry instructors to respond to an online survey. We also conducted extensive interviews with a number of respected geometry educators. Neither the survey nor the interviews in any way represents a random sample of opinion about geometry and our report is not meant merely to summarize the opinions of others. Rather, we drew from the ideas and opinions expressed in the survey and interviews to inform our discussion. The report represents our considered opinions, based on consultation with the larger community.

The following geometers were personally interviewed by the GSG: Jerry Alexanderson, Thomas Banchoff, Thomas Cecil, Herb Clemens, Scott Crass, Carolyn Gordon, Robin Hartshorne, David Henderson, Roger Howe, Joseph Malkevitch, John McCleary, Thomas Sibley, Nathalie Sinclair, and Walter Whiteley.

3 Concepts and methods from geometry that every mathematics major should learn

Most mathematical concepts involve both number and space, embodying the deep interplay between algebra and geometry. The GSG believes that the geometric/spatial side of mathematics does not receive enough emphasis in the current undergraduate curriculum. In an effort to help departments achieve a more balanced treatment of both sides of mathematics, we offer the following two lists of geometric concepts and methods. The first list consists of items that we believe should be learned by all undergraduate mathematics majors. The second list consists of additional topics that should be available to all undergraduate mathematics majors.

3.1 Topics required for any mathematics major

We think it is critical that every undergraduate mathematics major program include the following geometric topics and we recommend that each department review its course offerings to ensure that their mathematics majors do not miss them. While these topics naturally fit in a geometry course, some could just as easily be included in other mathematics courses.

Visualization, diagrams, and spatial reasoning. A major mathematical skill is to transfer understanding between abstract concepts and visual representations. Diagrams, both as communication methodology and guide to intuition and reasoning, must be part of an student's working vocabulary. Spatial reasoning is also essential. Some students enter college with weak visualization ability, but the good news is that these skills are malleable. Undergraduate students should strengthen and extend their visualization proficiency in calculus, multivariable calculus and linear algebra, for example. Ideally, students should also be challenged to apply visual and geometric reasoning to higher-dimensional objects such as four-dimensional space-time.

Euclidean geometry. Euclidean geometry, especially in the plane, cannot be ignored in the undergraduate curriculum. Every student must be fluidly conversant with the basics of Cartesian analytic geometry, which is all to often passed over in favor of calculus. In addition, the basics of synthetic Euclidean geometry, whether in a fully rigorous or more general way, should be part of every student's repertoire. Although not all mathematics majors will proceed this far, non-trivial and surprising objects such as the Nine Point Circle, the Euler line, and the orthic triangle can show students the beauty and depth of a subject they studied only briefly in high school.

Axiomatic systems. An understanding of the axiomatic method should be part of every mathematics major's education. For historical reasons axiomatic systems have traditionally been part of a geometry course, but some mathematics instructors feel they would be better studied in other areas such as abstract algebra (e.g., the axiom system for a group) or in courses on the history and philosophy of mathematics. While its location is a matter of heated debate, its importance as part of the mathematics curriculum is unquestioned.

Non-Euclidean geometries. The discovery of non-Euclidean geometries was a foundation-shaking event in the history of mathematics. All mathematics majors, with the possible exception of those specializing in certain applied subdisciplines, should know about these developments and how they changed the human understanding of the relationship between mathematics and the real world. This could be part of a geometry course or it could be studied in some other course (such as a history of mathematics course). Ideally, students should know the examples of hyperbolic, elliptic, and spherical geometries.

Transformations and invariants. Transformations have a hallowed place in geometry via Felix Klein's *Erlanger Programm*, where geometry is defined to be the study of properties of a space that are invariant under a specified group of transformations. For instance, angle measure is invariant in Euclidean geometry but not in affine geometry. Students should understand this formulation of geometry and work with specific examples such as the groups of isometries or similarities for Euclidean geometry. Every geometry of relevance to an undergraduate curriculum can be beneficially developed via transformations. When possible, exposure to the analytic representations of these transformations via matrices is recommended, either in a geometry course proper or linear algebra.

Symmetry and symmetry groups. Studying the symmetries of a figure in a geometry—the transformations of the geometry under which the figure is invariant—is a staple of geometric analysis. Students should understand how the various rosette groups, frieze groups, and wallpaper groups are all examples of symmetry groups for figures in Euclidean plane geometry, subgroups of the full group of isometries of the Euclidean plane. Symmetry groups provided some of the first instances of the concept of an abstract group in the nineteenth century. Indeed, the transformation groups associated to geometric systems via the *Erlanger Programm* can be viewed as extensions of the symmetry groups of figures.

3.2 Topics available for any mathematics major

Departments should ensure that the following geometric topics and observations are available in their course offerings.

Geometric intuition connected with other courses. The geometric aspects of multivariable calculus as captured by the dot and cross products, the gradient, curvature, etc., should be emphasized. Similarly the vast geometric content of linear algebra as expressed via matrix rank, inner products, dimension, the rank-nullity theorem, etc., also should be emphasized. Many of the principles of inversive geometry will be found in courses on complex variables. Much geometry can be taught in these and related courses.

The impossibility of certain compass and straightedge constructions. It is important that students be familiar with the famous impossible constructions of “trisecting an angle” and “squaring the circle.” However, they need to understand these facts *in an accurate fashion* and not to continue propagating incorrect mathematical myths. These constructions are impossible under rather restrictive conditions but we can indeed employ a compass and unmarked straight edge to build a tool that will trisect angles, and we can also approximate the squaring of a circle with a finite number of points to as accurate a level as desired. To explain exactly what it is that’s impossible requires a careful and sophisticated discussion, but needs to be understood by our students.

Historical perspectives and modern applications. Geometry has a rich and interesting history that students should know about but it is also fundamental as a tool in the sciences and in applied mathematics. Applications of geometry arise in many surprising and newfound contexts and these can be highlighted in a variety of undergraduate courses, including geometry courses. Undergraduate majors should be exposed to both perspectives.

Graph theory. Graph theory has a great deal of geometric content. Though more often than not it is combined in a course with combinatorics, graph theory can in fact serve as an organizing scaffolding for an effective survey of geometry course.

Projective geometry. Arthur Cayley’s famous quote “Projective geometry is all geometry” may be an overstatement but it has enough truth to justify the inclusion of projective geometry in the undergraduate curriculum. Indeed, all the best known (and many lesser known) plane geometries are subgeometries of $\mathbb{RP}^2$, the real projective plane. The elegance of $\mathbb{RP}^2$, along with the duality that exists between points and lines, makes this a delight to investigate with undergraduates, and can form the basis of a comprehensive survey of other geometries. The study of conics is nicely begun in projective geometry: all real, non-degenerate conics are projectively equivalent to a circle. However, finer and finer divisions of the conics can be made by progressing to affine geometry, Euclidean similarity geometry, and finally Euclidean congruence geometry.

4 Guiding principles that will inform the design of a geometry course

Every mathematics major program should include at least one geometry course. Before it decides on the geometry course (or courses) it will offer, a department should consider several questions to determine the kind of course that will best suit its desired goals. In this section of the report we discuss a number of these questions. We do not attempt to give one-size-fits-all answers to the questions, but simply raise and discuss the issues that a department should consider in answering the questions for itself.

4.1 What is the role of rigorous proof in a college geometry course? Must geometric proofs be done in the context of an axiomatic treatment of geometry?

Rigorous proof is one of the hallmarks of mathematics and geometry is the branch of mathematics in which our concept of proof evolved. The strict axiomatic organization of Euclid's *Elements* has for centuries been revered as the way in which every branch of knowledge should be systematized. A course in high school geometry is the place where most of us were first introduced to rigorous proof and current recommendations regarding the high school curriculum suggest that this will continue to be the case. For all these reasons, proof has traditionally had a special place in the teaching of geometry.

> "There is no geometry without proof." John McCleary.[1]

The GSG believes that rigorous proof continues to be an essential element of a college-level geometry course. It does not believe, however, that proofs must necessarily be taught in the context of axiomatics. An axiomatic approach remains a standard and legitimate way in which to organize a rigorous study of geometry, but it is not the only way. Some geometry educators caution against an overemphasis on formal proof and axiomatics. They point to the needs of future applied mathematicians, statisticians and more, who need geometric intuition and spatial reasoning.

We believe that the role of proof in a geometry course need not be substantially different from the role it plays in other upper-level undergraduate mathematics courses. A geometry course should be taught in such a way that students appreciate the need for proof, come to understand proofs of the major theorems of the subject, and learn to write their own proofs of other results. Proofs should support the student's intuition and not be seen as distinct from (or an impediment to) intuitive understanding. The GSG recommends against the use of the two-column proof format and believes that proofs should be written in plain English in paragraph form. We also support proofs that make use of extensive use of diagrams and diagrammatic reasoning.

An option that some departments will choose is to use the geometry course as the context in which students are introduced to proofs. The proofs of the propositions in the first few books of Euclid's Elements are at just the right level for those who are learning the discipline of writing proofs: the objects of study are familiar, it is possible to list all relevant assumptions regarding them, we can make careful definitions regarding them, and the proofs of the propositions have a relatively simple logical structure. At the same time that the proofs are accessible to students, the

[1] This and further quotes attributed to individuals are from the presonal interviews conducted by the GSG.

propositions are still surprising enough that students can appreciate the need for proof. (For more on this subject, see the article by James McClure in the Resources section of this report.)

4.2 What is the role of technology in an undergraduate geometry course?

We strongly endorse the use of technology in college classrooms. However, we caution that technology should not replace deep abstract reasoning and proof, but should support it as a venue for other deep and essential reasoning.

Many educators who are not already using technology in their geometry classrooms still recognize the benefits of visualization software. We doubt that many geometry courses in the future will use no technology at all. Reported obstacles to in-class use include time constraints in the classroom and concerns about time spent learning a new technology. While these are clearly valid concerns and need to be handled carefully, we believe the benefits of the thoughtful and appropriate use of geometric software outweigh the drawbacks. Since the impact of technology and manipulatives is heavily influenced by the prior professional development of the teacher in when (and how) to use these and when not to use these, it is essential that the geometry course encourage reflective practices for technology and manipulatives.

A wide variety of choices are available for appropriate technologies. Since as long ago as the 1980s, dynamic geometry software programs, computer algebra systems, and graphing calculators have allowed mathematicians to create videos, applets, and other geometric visualizations for their geometry classrooms. Examples of Interactive Geometry Software applications (IGSs) are Cabri and Cabri 3D, Cinderella, GeoGebra, Sketchpad, and Spherical Easel. Examples of computer algebra systems are Maple, Mathematica, Sage, MATLAB, and Wolfram Alpha. Graphing calculators, computer algebra programs, videos, and web applets can all be used to visualize sophisticated curves and surfaces and their interconnections.

We particularly endorse the use of Interactive Geometry Software. Effective use of IGSs encourages geometric discovery and helps students connect to material, in a cycle of experiment, conjecture, and proof. Students manipulate and investigate geometric constructions while preserving the underlying mathematical relationships that define a geometric object in a given metric geometry. In this way students can discover for themselves geometric invariants like the existence of the nine-point circle for any triangle in Euclidean geometry.

Knowledge about incorporating geometry in teaching is also arising in fields outside mathematics, and sometimes this brings a caveat about the role of technology. Biologists and psychologists, among others, have advanced geometric teaching and learning; research in these areas is active and ongoing. Of special interest to college educators is research on the balance between two-dimensional perspectives and three-dimensional perspectives. Research is also available on geometric learning models and the role of technology and other active learning methods in classrooms. These are all interconnected research topics. For example, the van Hiele learning model, developed in 1957 by Dutch educators Dina van Hiele-Geldof and Pierre van Hiele, defined five levels of geometric thinking as visualization, analysis, informal deduction, deduction, and rigor. IGSs are immediately connected to the first three levels. However, as in other settings, it is important to extend these explorations and intuition—teachers should complement IGS activities with proofs and abstract reasoning. The National Mathematics Advisory Panel (2008) cautioned that:

> Despite the widespread use of mathematical manipulatives such as geoboards and dynamic software, evidence regarding their usefulness in helping children learn geometry

> is tenuous at best. Students must eventually transition from concrete (hands-on) or visual representations to internalized abstract representations. The crucial steps in making such transitions are not clearly understood at present and need to be a focus of learning and curriculum research.

Other researchers and teachers counter the claims of insufficient evidence by citing a vast amount of educational literature (see for instance, [29], [34], [35], [37], [45], [48]). The GSG recommends the use of IGSs and other hands-on activities. While we agree that there is more to be learned about how students transfer knowledge from experiential to abstract, we are persuaded by the existing literature. A huge array of print and online resources can inform teachers who want to incorporate technology or manipulatives into their geometry classes. Examples can be found in the resource section of this report.

Finally, we note that new technologies have historically affected the teaching of geometry, and this is likely to continue in the future. For example, spherical trigonometry, once a staple of a geometry education, was virtually eliminated from the curriculum in the 20th century because navigational computations were now relegated to machines. On the other hand, the rise of video gaming and computer graphics in film makes knowledge of topics like projective geometry and computational geometry a marketable job skill. Twenty years from now, we would like to know: What new skills and understandings might students need to take advantage of new technologies?

4.3 How much emphasis should be placed on transformations?

Felix Klein, in the written version of his 1872 Inaugural Lecture as Professor at Erlangen University, defined geometry as the study of properties of a space that are invariant under a designated group of transformations, the "symmetries" of the geometry. This definition, which provided a unified intellectual structure for all the geometric systems that had appeared by the mid-19th century, had profound effects on the directions of subsequent geometric research. Known as the *Erlanger Programm*, this framework remains today as the definitive definition of a geometric system, both for its clean intellectual elegance and its practical applicability in mathematics and the physical sciences.

This leads us to recommend that any first course in undergraduate geometry have at least some discussion of transformations and the *Erlanger Programm*. Different departments will choose how central a role this should play, and there is no one simple answer; it will vary depending on the geometric topics under consideration, and also on the beliefs and preferences of the faculty. As will be seen in our collection of sample course syllabi in Section 6, the theory of transformations can be developed utilizing either a synthetic or an analytic approach.

Most geometric notions (such as congruence in Euclidean geometry) are elegantly defined by using the designated symmetries of the geometry. Additionally, thinking transformationally—- "seeing" the movement of one figure onto another via a symmetry motion—can enhance the intuitive understanding of a problem as well as provide rigorous techniques of proof. Indeed, as will be discussed in Section 5, the Common Core State Standards for Mathematics [CCSS] are increasing the focus on the transformational approach to Euclidean geometry, another reason to mirror this emphasis at the undergraduate level. We believe that any first course in undergraduate geometry should develop enough transformational geometry to support the K–12 curriculum recommendations in the CCSS.

The transformational approach is particularly important when moving beyond Euclidean geometry to study other geometric systems. While other geometries can be defined and developed axiomatically, it is often difficult to discern the analogies and differences between the geometries by comparing the corresponding axiomatic systems. However, given the natural ways in which symmetry groups for differing geometries can be realized via subgroup identifications, it is then obvious when one geometry is a "subgeometry" of another, making many relationships between the geometries readily apparent.

The study of concrete transformation groups within a geometry course is also excellent preparation for subsequent courses in abstract algebra. Geometry courses stressing a transformational point of view can also point the way to more advanced topics of current importance such as Lie theory and representation theory.

5 Geometry for future high school mathematics teachers

A major portion of the clientele for an undergraduate geometry course has traditionally consisted of future high school mathematics teachers. Their needs should be given particular consideration in designing a geometry course. Our recommendations regarding a geometry course for teachers closely align with those in *The Mathematical Education of Teachers II* [MET II].

The first and most basic recommendation of MET II is that "Prospective teachers need mathematics courses that develop a solid understanding of the mathematics they will teach."[2] A course in geometry is essential to meeting that goal for high school teachers and the GSG firmly believes that one or more courses in geometry should be a key element of every secondary education mathematics program. An operating principle that applies to all areas of mathematics is that teachers need substantial content at the level one higher than what they will teach, so the college-level geometry course for teachers should go well beyond what is included in the high school syllabus.

There are certain distinct emphases that should be part of a geometry course for future teachers, but that does not necessarily mean there must be a separate *Geometry for Teachers* course. In fact, the GSG believes it would be preferable to have a course that prepares teachers but does not separate them from other mathematics majors. The needs of these future teachers—to learn in depth the high school mathematics they will teach—can certainly be met in many of the courses we describe in Section 6. It would be particularly unfortunate if departments whose limited resources allow them to offer only one course in geometry chose to restrict that course to students in education programs.

The college geometry course that future teachers take should prepare them to teach in conformity with the Common Core State Standards for Mathematics [CCSS]. The high school geometry course recommended by CCSS is mostly devoted to two-dimensional Euclidean geometry. It builds on the more experiential geometry of earlier grades and formalizes those experiences using precise definitions and careful proofs. Despite the emphasis on proof, the CCSS high school course does not strictly follow the traditional axiomatic approach. Instead it makes certain assumptions about transformations and builds from there. The *basic rigid motions* are rotations, reflections, and translations. In middle school the existence and essential properties of these basic rigid motions are taken for granted. The high school course defines the basic rigid motions more rigorously; the fact that they preserve lines, distance, and angle measure is explicitly assumed (see page 110 in the article by H. Wu in the Resources section of this report). A *congruence* is defined to be a

[2] MET II, Recommendation 1, page 17.

transformation that can be written as a composition of basic rigid motions and two figures are *congruent* if there is a congruence that takes one to the other. Once this groundwork is laid, the basic theorems about triangle congruence and similar triangles are proved using transformations. A college geometry course for future teachers need not follow this outline exactly. In particular, a college course is likely to define a rigid motion to be a transformation that preserves distance (an isometry). But the college course should still be taught in a way that supports the CCSS approach.

Another aspect of the CCSS recommendations that might affect the design of a geometry course for teachers is the fact that CCSS geometry is built on the real numbers. The existence of the real number system and the properties of that system are assumed and used throughout, so there is minimal attention paid to issues of betweenness or continuity.

To be prepared to teach a geometry course based on CCSS, future teachers should take a college geometry course in which definitions and proof are emphasized. In addition, the course they take should include coverage of the following topics:

Proof. Since CCSS calls for rigorous proof in the high school geometry course, it is essential that future high school teachers have a positive experience with proofs in their college-level geometry course.

> "It's necessary that [proof] should be part of the course, for many reasons including the fact that in the K-12 curriculum there is going to be an increasing emphasis on proof." Nathalie Sinclair.

Transformations. The course should study reflections, rotations, translations, and glide reflections. The treatment should include the fact that these transformations constitute **all** of the "rigid motions," i.e., the mappings that preserve distance and angle measure. Transformational proofs of the triangle congruence conditions should also be covered. A future teacher should understand why every distance-preserving transformation of the plane can be written as the composition of reflections and that at most three reflections are required. They should also understand why every transformation of the Euclidean plane that is the composition of two reflections is either a translation or a rotation and know that every composition of three reflections is another reflection or a glide reflection.

Parallel Postulate. Future teachers should understand the parallel postulate and the role it plays in Euclidean geometry. They should be sensitive to which results of plane Euclidean geometry depend on the parallel postulate and they should be aware of the fact that there are other two-dimensional geometries in which the parallel postulate is not satisfied.

Pythagorean Theorem. The Pythagorean Theorem is a fundamental topic in high school geometry. Future teachers should have a solid understanding of the theorem and be confident that they understand at least one proof of the theorem and its converse. They should be familiar with proofs based on area and proofs based on similar triangles.

Dynamic geometry software. Future high school teachers should be familiar with at least one dynamic geometry software package and should see the appropriate use of this technology modeled in their college course.

Historical perspectives. History of mathematics is one topic that has become increasingly important in the preparation of prospective teachers. For example, the Program Standards for Initial Preparation of Mathematics Teachers (NCATE/NCTM, 2003) requires knowledge of the historical development in the number and number systems, algebra, Euclidean and non-Euclidean geometries, calculus, discrete mathematics, probability and statistics, and measurement and measurement system for secondary mathematics teachers, and also includes the recommendation for knowledge of the 'contributions from diverse cultures.' Some departments require their majors to take a separate history of math course or a capstone course/project that includes history. Another possibility is to incorporate historical content in the context of geometry courses.

> "When we think about mathematical concepts in our undergraduate courses we often fail to motivate them, to show how concepts emerge from the need to solve problems. This can be shown very easily using historical examples. I'm not suggesting that studying history per se has to be part of the geometry curriculum or even that assessing that has to be part of it, but when we think about what concepts we want to focus on I would opt for ones that show students well their pedigree, where they've come from and why they were needed." Nathalie Sinclair

Real-life applications. Future teachers should understand that geometry research is ongoing and is found in numerous applications. Some faculty incorporate projects so that students research geometry in their daily life. Others highlight the connections of topics to real-life. For instance

> "Mathematics is isolated enough from the real world—we should look for ways to connect it." David Henderson

6 Sample syllabi for a variety of undergraduate geometry courses

There has long been debate about what to teach and how to teach in geometry classrooms, including a tension between practical applications and theoretical considerations that remains today. In some cultures and times the focus of the geometry curriculum was on geometric techniques for practical applications like those in architecture, surveying, and navigation, while in others, it was the axioms of Euclidean geometry that were a fundamental part of a liberal arts education. Geometry teaching continues to evolve with the needs of society as well as to new geometric discoveries in mathematics and mathematics education research.

The first seven courses described below are standard, mainstream geometry courses for which a variety of textbooks exist. We have chosen to include descriptions of two additional courses because they contain interesting ideas that may be of use to those developing new courses. Textbooks do not currently exist for the last two courses described.

We offer our list of syllabi to illustrate the wondrous variety of effective and exciting ways to deliver geometry to our students. The GSG cannot, in good conscience, recommend any one of these courses over the others (though many individual mathematicians will have their own strongly held preferences). We hope our list will stimulate your thinking and will offer useful ideas and guidance to help improve your department's instruction in geometry.

6.1 A survey of geometries

> "Students have to know that geometry is not just one thing. They should understand the distinction between analytic and synthetic approaches, know basics about spherical, hyperbolic, and Euclidean geometries, and, ideally, understand that they are all part of projective geometry." Thomas Banchoff

> "The power of geometry comes from being able to think new thoughts and see connections as a result of wide exposure." Tom Sibley

This course aims for breadth, while sacrificing some depth. It assumes that students do remember some of the Euclidean geometry they learned in high school. The level of rigor is purposefully sacrificed in order to develop intuition and to cover some of the breadth of geometry.

In addition to the seven topics listed, the course asks students to carry out an individual discovery project. There is no shortage of possibilities for these: finite geometries, 4-dimensional Euclidean geometry, taxicab geometry, ...

Learning Goals:

- Students should demonstrate general understanding of the three major plane geometries: Euclidean, hyperbolic, and spherical.
- Students should show expertise in one area through deeper study or a research project.
- Students should be able to explain the interplay of synthetic and analytic approaches and to prove theorems at the elementary level using each system.
- Students should know enough about transformation geometry to be able to apply their knowledge to classifying patterns by their symmetries.
- Students should glimpse the broad picture of how the three major plane geometries are subcases of projective geometry.

Topics:

- Euclidean geometry. Building on what students presumably remember from high school, the course should develop strength in three areas of Euclidean geometry: The ability to prove basic theorems from axioms; knowledge of representative constructions, such as the various triangle center constructions; and a broad understanding of the historical narrative of Euclidean plane geometry, from Euclid to Hilbert. The role of Euclids Fifth Postulate (and its equivalents) is highlighted.
- Analytic geometry. Students should understand the analytic method as a powerful alternative to synthetic reasoning. For instance, students should be able to use the concept of slope to prove that the midpoints of a quadrilateral form a parallelogram.
- Hyperbolic geometry. Using an axiomatic approach, possibly aided by one of the standard models, students should learn how the negation of Euclids Fifth Postulate leads to a rich body of theorems. These should include the non-existence of similar triangles and the fact that the angles in a triangle sum to less than 180 degrees.

- Spherical geometry. With only as much detail as time permits, students should understand the sphere (and its quotient by the antipodal map), both as a physically useful space and a model to illustrate a third alternative system of geometry. (If time permits, the sphere provides an interesting ground for debating the fine points of axiom systems.)
- Transformations. Whether through connection to linear algebra, complex numbers, or a synthetic approach, students should learn the vocabulary of Euclidean translations, rotations, reflections, and glides. The large goal is to understand the structure of isometries as being direct or indirect, and as always being the product of at most three reflections.
- Symmetries. The knowledge of transformations is applied to an overview of symmetry groups in the plane, including rosettes, friezes, and wallpaper patterns.
- Projective geometry. The unity of Euclidean, hyperbolic, and spherical geometry is glimpsed through a projective approach.

6.2 Axiomatic geometry

> "…the study of axiomatic systems in important in geometry since students don't get exposed to them in any other course." Robin Hartshorne.

Learning goals:

- Students should learn the historical roots of the axiomatic method in geometry, understand how views of what an axiom is have evolved over time, and be aware of some of the philosophical implications of those changing views.
- Students should see a complete set of axioms for 2-dimensional geometry, learn to formulate precise definition and theorem statements in that axiomatic system, and learn to justify all the steps in a proof within that system.
- Students should become sensitive to which results of plane geometry can be proved without the use of the parallel postulate and which require a parallel postulate.
- Students should have experience proving theorems in more than one geometry.

Topics:

- The axiomatic method and axiomatic systems. The axiomatic method originated with Euclid and the ancient Greeks. Over time the understanding of "axiom" was refined to the point that gaps were seen in Euclid's proofs. A modern axiomatic system includes undefined terms and a complete list of *all* assumptions.
- A system of axioms for geometry. An objective of this course is to produce a system of axioms for plane geometry that is rich enough to include all of Euclid's assumptions, both those that he explicitly stated in his postulates as well as those he left unstated (but felt free to use in his proofs). One fundamental choice that must be made in designing an axiomatic geometry course is whether to use a system of axioms that builds on the real numbers (following Birkhoff) or to use axioms for geometry that are independent of other branches of mathematics

(following Hilbert). It is much easier to make connections with high school geometry when the real numbers and measurement are allowed, so axioms based on the real numbers are usually preferable for a course populated with future high school teachers. Hilbert's axioms are probably preferable for students who will go on to graduate work in mathematics.

- Neutral geometry. This is the part of 2-dimensional geometry in which no parallel postulate is assumed. Many standard results, such as the triangle congruence conditions, can be proved in this setting. Students should be able to give axiomatic proofs that various familiar statements are equivalent to Euclid's Fifth Postulate and they should have a clear understanding of what this "equivalence" means. (It does not mean that the statements are logically equivalent to the parallel postulate in isolation, but only that they are equivalent in a context in which other assumptions have been made.)
- Euclidean geometry. Students should learn how the basic results of Euclidean geometry (such as theorems about similar triangles and the Pythagorean Theorem) relate to the parallel postulate.
- Hyperbolic geometry. Students should learn to give axiomatic proofs of results in hyperbolic geometry and understand that there are two essentially different kinds of parallel lines in the hyperbolic plane.
- Models. In order to demonstrate the independence of the parallel postulate, students should be familiar with models for Euclidean, hyperbolic, and spherical geometries and they should have experience working in all three of those geometries.
- Transformations. The ways in which transformations interact with the formulation of the axioms can be explored, especially in a course for future high school teachers.
- Other axiom systems. As time allows, instructors may include other axiom systems, selecting from among those for finite geometry, projective geometry, spherical geometry, or origami.

6.3 Euclidean geometry

Euclidean geometry has continued to expand and develop since the time of Euclid. The subject contains many of the most surprising and beautiful results of elementary mathematics. Euler initiated a revival of the subject in the eighteenth century when he discovered that the three classical triangle centers are collinear. Since that time many mathematicians have contributed to its development. Euclidean geometry is currently experiencing another revival and is the subject of intense research even today, probably because dynamic geometry software makes it simple to construct and explore intricate diagrams.

Learning Goals:

- Students should learn that Euclidean geometry continued to develop after the time of Euclid and that the subject is still expanding today.
- Students should learn the basic results and techniques of post-Euclid Euclidean geometry. They should come to appreciate the great beauty of the results as well as how surprising some of them are.

- Students should learn to prove theorems in Euclidean geometry. They can assume the basic results of high school geometry and build from there. The Euclidean geometry course is a natural setting in which to learn to appreciate the power of logical deduction since the results in the subject are non-obvious but still susceptible to proofs that are relatively simple in their logical structure and easy to understand.
- Students should become familiar with dynamic geometry software. They should learn to use the software to discover, explore, and illustrate the results of Euclidean geometry. Students should be able to make tools of their own that perform standard constructions. (For example, they should be able to make a tool that constructs the circumscribed circle of a triangle.)

Topics:

- A review of elementary Euclidean geometry. Given the great differences in high school geometry courses, it is necessary to review the basic facts and techniques of high school Euclidean geometry.
- Triangle centers. The three classical triangle centers (centroid, orthocenter, and circumcenter) as well as the Euler line should definitely be included. In addition, a sampling of other triangle centers can be discussed—see the *Encyclopedia of Triangle Centers* listed in the Resources section of this report.
- Circumscribed, inscribed, and exscribed (or escribed) circles. The course should cover both the construction of these circles and proofs that they exist.
- The theorems of Ceva and Menalaus. A good place to discuss duality and to make connections with projective geometry.
- Transformations. A proof of the classification of rigid motions of the plane could be included.
- Constructions. See the discussion of constructions in §3 of this report.
- Other topics. There are many additional topics that could be included: The nine point circle and Feuerbach's theorem, the theorems Miquel, Morley, Desargue, Brianchon, Pappus, Simson, and Ptolemy, as well as Pascal's Mystic Hexagram.

6.4 Transformational Euclidean geometry

A transformational approach to establishing geometric results, even when restricted solely to Euclidean geometry, is a powerful tool for developing proofs, solving geometric problems, and developing deep geometric understanding. While transformational geometry is closely associated with the use of analytic (most commonly linear algebraic) techniques, it is indeed possible to develop transformational geometry using a synthetic approach. A synthetic development is particularly effective when studying Euclidean geometry.

Learning goals:

- Students should learn how to approach geometric problems from a transformational viewpoint and to visualize the movement inherent in a transformational approach.

- Students should learn the basic history of the development of geometry, at least enough to appreciate the reasons leading to Felix Klein's formulation of the *Erlanger Programm* in 1872. They should understand the unification achieved by this point-of-view.
- Students should understand and master the use of the group structure of the collection of symmetries of a geometry.
- Students should understand the procedures used to classify patterns via symmetries.
- Students should deeply understand the structure of the group of isometries and the group of similarities, and be able to apply this structure in developing important and interesting results in Euclidean geometry.

Topics:

- Basic Euclidean plane geometry. The goal is to develop enough geometric tools upon which to build the geometric transformations that yield the isometries and similarities of Euclidean plane geometry. This is efficiently accomplished by using a metric axiomatic approach.
- Congruence and Similarity Geometry. These two flavors of Euclidean plane geometry should be developed, governed respectively by the group of isometries and the larger group of similarities. This is often not a well understood distinction.
- The Groups of Isometries and Similarities. Students need to master the structure of these groups of transformations. For the group of isometries this includes the Three Reflection Theorem; the classification of isometries into reflections, rotations, translations, and glide reflections; the concept of a direct or indirect isometry; and other factorization theorems for isometries. For similarities the need is to understand them as compositions of dilations with isometries, and to classify the non-isometric similarities via use of their unique fixed points.
- Geometric Equivalence via Congruence, Similarity, and Conjugation. Central to the transformational viewpoint is understanding the concept of geometric equivalence as embodied in the symmetries of the geometry, in this case the isometries or the similarities. Two sets in the Euclidean plane are geometrically equivalent if they are congruent or similar, that is, if one can be mapped onto the other by an isometry or a similarity. Two transformations of the plane are geometrically equivalent if they are conjugate to each other via an isometry or a similarity.
- Plane Euclidean Geometric Theorems. There should be a rich collection of compelling Euclidean theorems established using the transformational tools developed in the course. These could include the various triangle coincidence points, the Nine Point Circle and the Euler Line, the properties of the orthic triangle, symmetries groups such as the Rosette, Frieze, and Wallpaper groups, etc.

6.5 Transformational geometry: beyond Euclidean

A transformational approach to establishing geometric results is a powerful tool for developing proofs, solving geometric problems, and developing deep geometric understanding. This is particularly true when studying geometries beyond plane Euclidean geometry.

The transformational approach allows a clean and elegant method for comparing different geometries via containment relationships between the different groups of symmetry transformations. For example, similarity geometry is a subgeometry of affine geometry, which in turn is a subgeometry of projective geometry. This allows for a logical approach to the classification of invariants in each geometric system, as well as for a clearer understanding of how to assign particular theorems to their "proper" geometries.

When focusing on geometries beyond Euclidean we feel that an analytic approach via linear algebra is probably more effective than a synthetic approach.

Learning goals:

- Students should learn how to approach geometric problems from a transformational viewpoint and to visualize the movement inherent in a transformational approach.
- Students should learn the basic history of the development of geometry, at least enough to appreciate the reasons leading to Felix Klein's formulation of the *Erlanger Programm* in 1872. They should understand the unification achieved by this point-of-view.
- Students should understand and master the use of the group structure of the collection of symmetries of a geometry.
- Students should develop facility with the basics of the real projective plane and its symmetries.
- Students must develop an intuitive understanding of projective duality and its embodiment in the linear algebra model of the real projective plane.
- Students should come to appreciate Arthur Cayley's (slightly exaggerated) claim that *"projective geometry is all geometry"* by seeing the other classical plane geometries as subgeometries of the real projective plane.

Topics:

- The Real Projective Plane $\mathbb{RP}^2$. Emphasis is recommended on the analytic model as equivalence classes of non-zero vectors in $\mathbb{R}^3$. Development of the linear algebraic tools for handling incidence relations and the cross-ratio. The group of projective transformations, developed as collineations. Perspectivities. Four-fold transitivity of projective transformations. Theorems of Desargues and Pappus. Projective conics.
- The Affine Plane. Motivated and modeled via projectivization, that is, embedding as the plane $z = 1$ in $\mathbb{R}^3$ when $\mathbb{R}^3$ is converted to $\mathbb{RP}^2$. The group of affine transformations seen as projective transformations fixing the line at infinity. The invariance of the affine ratio. The centroid of a triangle. Barycentric coordinates. Three-fold transitivity of affine transformations. Theorems of Ceva and Menelaus. Classification of affine conics.
- The Euclidean Plane: Isometries and Similarities. Developed as subgeometries of affine geometry and projective geometry. The invariance of angle measure and distance ratios in similarity geometry. The invariance of distance in congruence geometry. Examples of Euclidean theorems that are not affine theorems. Classification of conics under similarities and isometries. Laguerre's Formula for angle measure between lines.

- Hyperbolic and Elliptic Geometries. Developed as subgeometries of the real projective plane via the choice of the "absolute" projective conic. Distance and angle measure obtained from the cross-ratio and the absolute conic via the Cayley-Klein method. The subgroups of hyperbolic and elliptic symmetries on the hyperboloid and spherical models. The Beltrami-Klein disk model for hyperbolic geometry. A selection of theorems in hyperbolic and elliptic geometry.
- If time permits: The Complex Projective Line $\mathbb{CP}^1$ and Inversive Geometry. Inversions, circular transformations, and Möbius transformations. The Riemann sphere and stereographic projection. The cross-ratio in $\mathbb{CP}^1$. Feuerbach's Theorem. Hyperbolic and Elliptic geometry as subgeometries of $\mathbb{CP}^1$. The Poincaré disk and upper half-plane models for hyperbolic geometry.
- If time permits: The Geometry of Space-time. The Poincaré and Lorentz groups as the symmetry groups for 4-dimensional space-time in special relativity.

6.6 Computational geometry

Although geometry is as old as mathematics itself, discrete geometry only fully emerged in the 20th century, and computational geometry was only christened in the late 1970s. The terms "discrete" and "computational" fit well together as the geometry must be discretized in preparation for computations. "Discrete" here means concentration on finite sets of points, lines, triangles, and other geometric objects, and is used to contrast with "continuous" geometry, for example, smooth surfaces. Although the two endeavors were growing naturally on their own, it has been the interaction between discrete and computational geometry that has generated the most excitement, with each advance in one field spurring an advance in the other. The interaction also draws upon two traditions: theoretical pursuits in pure mathematics and applications-driven directions, often arising in computer science. The confluence has made the topic an ideal bridge between mathematics and computer science.

Learning Goals: The field has expanded greatly since its origins and now the new connections to areas of mathematics (such as algebraic topology) and new application areas (such as data mining) seems only to be accelerating.

- Students must show understanding of the core pillars of this subject: polygons, convex hulls, triangulations, and Voronoi diagrams.
- Students must grasp questions from algorithmic standpoints, not just addressing whether something can be done, but how it can be constructed, and how efficient such constructions can be.
- Students should become comfortable experimenting with (freely available) applets and programs that emphasize the applicability of the subject.
- Interplay between algorithms should not be ignored. There are numerous areas related to this topic, as it bridges mathematics and computer science.
- To experience the richness of this growing field, students should pursue in depth one or two extra topics not classically covered among the topics below.

Topics:

- Polygons: We introduce the worlds of the "discrete" and the "computational" to a mathematical audience. The key tool will be the study of polygons and polyhedra, the building blocks of 2D and 3D discrete geometry. Topics will include triangulations, enumerations, dissections, and art gallery theorems.

- Convex Hulls: Although a convex hull of a set of points in the plane is easy enough to define, how does one go about computing it? What does it mean to construct a geometric algorithm, and how can one measure better algorithms? We look at several powerful algorithms for 2D hulls, and glimpse into the difficulties with 3D hulls, along with framing the big-Oh notation. In particular, consider Incremental, Divide-and-Conquer, Gift Wrapping, and Graham Scan algorithms.

- Triangulations: This focuses on the partitioning a set of points in the plane into triangles, forming the basis for numerous real-world applications such as terrain meshing and face recognition. We start with basic algorithms and combinatorics and then consider the discrete space of all triangulations (the flip graph). We then concentrate on what is arguably the most important triangulation, the Delaunay triangulation, having striking properties and playing a central role in many applications.

- Voronoi diagrams: Our interest is now on which point of a point set is closest to an arbitrary point in the plane. This focus on "nearest neighbors" leads to the rich geometry of the Voronoi diagram. Moreover, there is an intimate connection via duality between Voronoi diagrams and the Delaunay triangulations from above. And there is a beautiful and deep connection between both these structures and convex hulls in 3D.

- Curves: We extend the Voronoi diagram to apply to curves rather than to just sites, leading to two generalizations: the medial axis (useful in biology) and the straight skeleton (useful in origami). We consider issues in curve reconstruction, an important practical task whose algorithms employ Voronoi diagrams, Delaunay triangulations, and the medial axis.

- Polyhedra: Although we encounter polyhedra in earlier lectures, we study them more systematically with the dual goal of strengthening 3D intuition and presenting several theorem gems. Considering shortest paths on convex polyhedra, a topic which brings us back to the ubiquitous Voronoi diagram, presents one of the most beautiful open problems on edge-unfolding polyhedra. We follow that with two beautiful and useful theorems: The Gauss-Bonnet Theorem and Cauchy's Rigidity Theorem.

- Configuration Spaces: This explores configuration spaces for the simplest articulated objects, the open polygonal chains, which brings us to several famous problems on locked chains. This in turn leads to an investigation of closed polygonal chains, concentrating on the topology of the space of polygons, along with issues in motion planning.

6.7 Differential geometry

The undergraduate differential geometry course should include theoretical and computational components, intrinsic and extrinsic viewpoints, and numerous applications:

- Geometry of curves in space, including the Frenet frame
- Theory of surfaces, including parameterizations, first and second fundamental forms, curvature and geodesics
- The concluding part of the course could be a focus that depends on the interest of the instructor and students, such as the Gauss-Bonnet Theorem, the theory of minimal surfaces, or the geometry of space-time with applications to general relativity.

Ideally a course in differential geometry allows students to see the connections between such topics as calculus, geometry, spatial visualization, linear algebra, differential equations, and complex variables, as well as various topics from the sciences, including physics. The course may serve as an introduction to these topics or a review of them. The course is not only for mathematics majors—it encompasses techniques and ideas relevant to many students in the sciences, such as physics and computer science.

One inescapable prerequisite for this course is multivariable calculus. Some schools have successfully taught differential geometry with nothing more than multivariable calculus as a prerequisite, and so this is a feasible single requirement, especially if computer algebra software will be utilized for computations and visualization. The immediate benefit is that more students in other majors could take the course. However, students coming out of a multivariable calculus course may not have the mathematical maturity needed for the course, depending on the focus and level. Some schools require linear algebra, a proof-writing course, and/or differential equations as additional prerequisites for the class.

6.8 Geometric structures: Axiomatics, graphs, polygons, polyhedra, and surfaces

Joseph Malkevitch teaches a course at York College (CUNY) whose goal is to cover as broad a range of topics with geometrical flavor as possible in one course. The desire is to have students explore geometry and hence become more broadly aware of geometric phenomena and applications of geometry. This course clearly and purposely emphasizes breath over depth. Malkevitch believes that it's much easier for a student to tackle a mathematical topic if he or she has been exposed earlier to some of the basic concepts and results.

Learning Goals:

- Students will come to understand the rudiments of axiomatic systems via studying finite affine, finite projective, and finite hyperbolic planes, as well as by comparing Desarguesian and non-Desarguesian planes.
- Students should be able to explain how the real projective plane $\mathbb{RP}^2$ is constructed from the Euclidean plane and reasons for the central importance of homogeneous coordinates for $\mathbb{RP}^2$.
- Students will gain exposure to planar graphs and Euler's polyhedral formula, which further requires the Jordan Curve Theorem. This leads into a proof of the existence of the five platonic solids and a discussion of frieze and wallpaper patterns.
- Students will gain facility with basic ideas concerning surfaces via explorations of Möebius strips, spheres with handles, and nets of polyhedra.

- Students will consider the concept of distance via extensive work with Taxicab geometry.

Topics:

- What is Geometry? Geometry as the study of space, shapes, and visual phenomena. the role of careful looking. Geometry as a branch of mathematics and as a branch of physics.
- Definitions, Axioms, and Models. Different kinds of geometry, Euclidean geometry, Bolyai-Lobachevsky geometry, projective geometry, affine geometry, and taxicab geometry. Axioms systems and rule systems in sports.
- Graph Theory. The unifier of course topics.
- Geometrical transformations. Translations, rotations, reflections, shears, homothetic mappings, projective transformations, applications to computer vision and robotics, Felix Klein's *Erlanger Programm*. The role of distance functions in geometry.
- Symmetry. Transformations that preserve symmetry, symmetry groups of strips, polyhedra and tilings.
- Polygons. Convex and non-convex polygons, simple polygons, space polygons, orthogonal polygons, visibility theory, art gallery theorems, Bolyai-Gerwien Theorem.
- Polyhedra. Convex polyhedra, regular and semi-regular polyhedra, symmetry issues, deltahedra, origami models, rod models, membrane models, nets, Steinitz's Theorem, Euler's Polyhedral formula.
- Tilings. Regular polygon tilings, polyomino tilings, reptiles, symmetry properties of tilings.
- Lattice point geometry. Pick's theorem, Sylvester's theorem.
- Convexity geometry. Helly's theorem, curves of constant breadth, polyhedra, tilings, packing and covering problems.
- Geometry of surfaces. Basic topology of surfaces, folding and unfolding, Möebius strip, spheres with handles.

6.9 Exploring geometries with hand and eye

Walter Whiteley teaches a course at York University whose goal is to develop a wide sense that "geometry can be found everywhere", to re-present familiar plane geometric results in the context of spherical geometry and 3-D geometry and develop visual reasoning and communication and spatial reasoning as key contributors to geometric reasoning and applications. The course is supported by dynamic geometry software (Geometer's Sketchpad, Spherical Easel) as well as an array of manipulatives and hands-on activities such as using spheres (with elastics), cylinders, mirrors, Mira, sand pouring and origami. The course is also supported by active group work and investigations and developing embodied cognition, cognitive blends of multiple representations and the impact of spatial visual reasoning on the practices of mathematics and on applications of mathematics.

Learning Goals

- Students will explore several networks of definitions and propositions on the plane, the sphere and other surfaces, using multiple approaches drawing on transformations symmetry and isometries, supported by dynamic geometry programs, physical manipulatives and objects, and methods of visual and spatial reasoning;
- Students will gain facility with basic ideas of groups of isometries generated by products of reflections, and the application of these ideas in geometry, including Klein's Hierarchy of geometries, in algebra and in various applied settings;
- Students will reflect on presentations, experiences and course readings connected to (i) the significance and development of mathematical reasoning through multiple representations and switching strategies; (ii) spatial reasoning and visual reasoning, as well as illustrations of embodied cognition (iii) "folding back" as a learning strategy (Pirie-Kieran), and (iv) the key role of rich and flexible spaces of examples and non-examples in learning concepts, illustrated in geometry in their individual learning of geometry (John Mason).
- Develop their individual skills at asking geometric questions drawing on the student's experiences and observations, as well as explore ways these questions might be answered. Working in groups or individually, they will investigate one of these questions in some depth as a project, and present portions of this project to the rest of the class.

Topics

- Explore a network of geometric definitions and relationships (theorems), such as 'straight line', 'isosceles triangle' on plane and sphere and angle measurements using symmetries and isometries, manipulatives such as the ribbon test, other symmetry based reasoning observed in physical methods of construction and test, and (visual) diagrammatic reasoning and communication on plane and sphere;
- Gain facility with a net work of congruence relationships including triangle congruence theorems in plane and sphere (including identifying singular cases where they fail), using transformations, and investigating multiple forms;
- Classify quadrilaterals based on symmetries, and the lattice of subgroups of symmetries of a square, using paper folding, dynamic geometry programs, hand gestures and movement. Explore the use these symmetries for related geometric reasoning, in plane, sphere, 3D, including the fundamental role of inclusive definitions in mathematical reasoning. For example, recognize that parallelograms are best defined as quadrilaterals with half-turn symmetry, and derive all related theorems from this symmetry,
- Derive properties of angle bisectors and edge bisectors (plane and sphere), and apply the properties to constructions with compass and straightedge, in origami and to Voronoi diagrams via sand pouring with applications to computational geometry;
- Develop all isometries of the plane and sphere as products of reflections, using mirrors, Mira, dynamic geometry, and an algebra of reflections, give a proof that all isometries are a product of at most three reflections and apply these results to other geometric patterns;

- Develop the ability to represent all plane isometries with 3x3 matrices using barycentric coordinates, and all spherical isometries with 3x3 orthonormal matrices, and connect key features of the isometry with the eigenvectors of these matrices, both numerically and visually, and solve additional geometric problems with these reasoning tools. Connect these matrix representations to trigonometry identities as well as current methods in computer graphics;
- Distinguish chiral and achiral shapes, including molecules, and their impact on patterns in 2 and 3 dimensions, including biochemistry such as Vitamin E, drug design;
- Explore the properties of the sum of the internal and external angles of polygons, by motions, scaling (in the plane), dynamic geometry programs, and by holonomy and areas of polygons on the sphere (discrete Gauss-Bonnet), using dynamic geometry programs and proofs with parallel transport;
- Explore 'Parallel Postulates' in the presence of the first four Euclidean postulates, through the equivalence of various 'parallel properties' in the plane (and how each breaks or is modified in spherical geometry);
- Klein's Hierarchy of Geometries, including the fundamental role of groups of transformations, and corresponding invariants;
- Spherical Polyhehdra, supported by proof(s) of Euler's formula and extensions to Descartes' formula for polyhedral angle deficits (another form of discrete Gauss-Bonnet), with illustrations through Platonic solids (built with Polydron) and when the formula fails or needs modification. Possible extension to 4-D (through projects).
- Classify frieze patterns, and wallpaper tessellations by their isometries, and apply reasoning from products of isometries to complete patterns and prove the impossibility of other patterns, using results on products of reflections as well as geometry programs such as Kali, i-Ornament and Kaleidotile for tiling and symmetries of polyhedral..
- Projections from sphere to plane (gnomic, stereographic, cylindrical), and liftings from plane to sphere, with applications to map making, conic sections and symmetries.
- Explore: (i) Pierre Cure's Principle: that the symmetry of inputs to an event should appear as symmetry in the output; and (ii) Noether's Theorem: the equivalence of symmetries in the laws of physics and conservation principles in the physical situation; and be able to describe how these principles apply in new situations, including the choice of frame of reference for solving physics problems.
- Project topics chosen by individuals or groups in the class, including possible topics on geometry education at multiple age levels.

7 Resources

Remark: *The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support the various types of geometry courses described in this report. Please note that some of the books listed below were written by the authors of this report.*

Suggested textbooks for *A survey of geometries*

1. Henderson, David W. and Daina Taimina, *Experiencing Geometry, Euclidean and Non-Euclidean with History*, 3rd edition, Pearson, Upper Saddle River, NJ, 2005.

2. Sibley, Thomas Q. *Thinking Geometrically: A survey of Geometries.* Mathematical Association of America, Washington DC, 2015

Suggested textbooks for *Axiomatic Geometry*:

3. Greenberg, Marvin Jay, *Euclidean and Non-Euclidean Geometries: Development and History* (4th edition), Freeman, 2008 (based on Hilbert's axioms).

4. Hartshorne, Robin, *Geometry: Euclid and Beyond*, Springer, 2000 (based on Hilbert's axioms).

5. Lee, John M., *Axiomatic Geometry*, American Mathematical Society, 2013 (based on metric axioms).

6. Venema, Gerard A., *Foundations of Geometry* (2nd edition), Pearson, 2001 (based on metric axioms).

Suggested textbooks for *Euclidean geometry:*

7. Coxeter, H. S. M. and S. L. Grietzer , *Geometry Revisited.* MAA, 1967. (A classic, but not written in the style of modern textbooks.)

8. Isaacs, Martin, *Geometry for College Students.* Brooks Cole 2000.

Suggested textbooks for *Transformational Euclidean geometry*:

9. Barker, William and Roger Howe, *Continuous Symmetry.* Providence, RI: American Mathematical Society, 2007.

10. Martin, George E., *Transformation Geometry: An Introduction to Symmetry*, New York: Springer-Verlag, 1982.

Suggested textbooks for *Transformational geometry: beyond Euclidean*:

11. Brannan, David A., Matthew F. Esplen, and Jeremy J. Gray, *Geometry*, 2^{nd} Edition. Cambridge University Press, 2012.

12. Cederberg, Judith N., *A Course in Modern Geometries*, 2^{nd} Edition. New York: Springer-Verlag, 2001.

13. Coxeter, H. S. M., *The Real Projective Plane*, 3^{rd} Edition. New York: Springer-Verlag, 1993.

14. Ryan, Patrick J., *Euclidean and Non-Euclidean Geometry: An Analytic Approach*, Cambridge University Press, 1986.

Suggested textbooks for *Computational geometry*:

15. de Berg, M. , O. Cheong, M. van Kreveld, and M. Overmars, *Computational Geometry: Algorithms and Applications*, Spring-Verlag, 3rd edition, 2008.

16. Devadoss, S. and J. O'Rourke, *Discrete and Computational Geometry*, Princeton University Press, 2012.

17. O'Rourke, J., *Computational Geometry in C*, Oxford University Press, 2nd edition, 1998.

Suggested textbooks for *Differential geometry*:

18. Banchoff, Thomas F., and Stephen T. Lovett, *Differential Geometry of Curves and Surfaces.* A K Peters, 2010 (note the java applets).

19. Do Carmo, Manfredo P., *Differential Geometry Curves and Surfaces.* Prentice-Hall, 1976 (standard reference).

20. Henderson, David W., *Differential Geometry: A Geometric Introduction.* Prentice Hall, 1997. Self Study Edition, Project Euclid e-book, 2014 available for free online. (Note the geometric intuition).

21. McCleary, John, *Geometry from a Differentiable Viewpoint*, Second Edition, Cambridge University Press, 2013.

22. Oprea, John, *Differential Geometry and its Applications.* MAA Classroom Resource Materials, 2007.

Suggested resources for *Geometric structures: Axiomatics, graphs, polygons, polyhedra, and surfaces*: There currently are no textbooks that cover all the topics suggested for this course. (Malkevitch hopes to write such a book.) The best source for materials is the extensive set of notes Malkevitch has produced for this course, all available on his website:

23. Malkevitch materials.

Additionally, an extensive bibliography for the course (as of 2001) can be found here.

Suggested resources for *Exploring geometries with hand and eye:* There currently are no textbooks that cover all the topics suggested for this course. The course draws materials from the following texts:

24. Henderson, David W. and Daina Taimina, *Experiencing Geometry, Euclidean and Non-Euclidean with History*, 3rd edition, Pearson, Upper Saddle River, NJ, 2005.

25. Tall, David et al, *Cognitive Development of Proof* in New ICMI Study Series, Vol. 15 Proof and Proving in Mathematics Education, Michael de Villiers and Gila Hanna (eds) 2012.

26. Whiteley, Walter , *Learning to see Like a Mathematician. In Multidisciplinary Approaches to Visual Representation and Interpretation* (G. Malcom Ed), Elsevier 2005, 279-292.

Other materials are available directly from Walter Whiteley (whiteley@mathstat.yorku.ca) and from wiki pages linked to from this website

Other relevant sources

27. Apostol, Tom M. and Mamikon A. Mnatsakanian, *New Horizons in Geometry*, MAA Dolciani Mathematical Exposions #47, 2012.

28. Berger, Marcel , *Geometry Revealed: A Jacob's Ladder to Modern Higher Geometry*, Springer, 2010.

29. Carbonneau, Kira J., Scott C. Marley, and James P. Selig, *A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives.* Journal of Educational Psychology, Vol 105(2), May 2013, 380-400.

30. Freudenthal, Hans, *Geometry between the Devil and the Deep Sea.* Educational Studies in Mathematics, vol. 3 (1971), 413–435.

31. Gorini, Catherine A. (editor), *Geometry at Work: Papers in Applied Geometry*, MAA Notes 53, Washington, DC: Mathematical Association of America, 2000.

32. Gray, Jeremy, *Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century.* New York: Springer, 2007.

33. Greenberg, Marvin, *Euclidean and Non-Euclidean Geometry: Development and History* 4th ed. New York: W. H. Freeman, 2007.

34. Grover, Barbara W. and Jeffery Connor, *Characteristics of the College Geometry Course for Preservice Secondary Teachers*, Journal of Mathematics Teacher Education, January 2000, Volume 3, Issue 1, pp 47-67

35. Hanna, Gila and Michael de Villiers (editors), *Proof and Proving in Mathematics Education: The 19th ICMI Study.* Springer, 2012.

36. Holme, Audun, *Geometry: Our Cultural Heritage.* Berlin: Springer, 2002.

37. Hoyles, Celia and Lagrange, Jean-Baptiste (Eds.) Mathematics Education and Technology-Rethinking the Terrain. The 17th ICMI Study Series: New ICMI Study Series, Vol. 13.

38. Johnston-Wilder, Sue and John Mason, *Developing Thinking in Geometry.* SAGE Publications Ltd, 2005.

39. Kimberling, Clark has produced an online resource: *The Encyclopedia of Triangle Centers*

40. Lumpkin, Beatrice, *Geometry Activities From Many Cultures.* Portland, ME: J. Weston Walch, 1997.

41. Malkevitch, Joseph (editor), *Geometry's Future: Conference Proceedings*, 2^{nd} Edition, COMAP, 1991.

42. McClure, James E., *Start Where They Are: Geometry as an Introduction to Proof.* American Mathematical Monthly, 107 (2000), 44 – 52.

43. Pritchard, Chris (editor), *The Changing Shape of Geometry: Celebrating a Century of Geometry and Geometry Teaching.* Cambridge: Cambridge University Press and Washington, DC: Mathematical Association of America, 2002.

44. Rozenfeld, Boris, *A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space.* Translated by Abe Shenitzer. New York: Springer-Verlag, 1988.

45. Schattschneider, Doris and James King (editors), *Geometry Turned on: Dynamic Software in Learning, Teaching, and Research.* MAA Notes 41, Washington, DC: Mathematical Association of America, 1997.

46. Sinclair, Nathalie, *The History of the Geometry Curriculum in the United States.* Charlotte, NC: Information Age Publishing, 2008.

47. Whiteley, Walter, *The Decline and Rise of Geometry in 20th Century North America.* Canadian Mathematics Study Group Conference Proceedings. Edited by J.G. McLoughlin. St John's: Memorial University of Newfoundland, 1999. This is also available online .

48. Whiteley, Walter, *Teaching to see like a mathematician.* Studies in Multidisciplinarity: Multidisciplinary Approaches to Visual Representations and Interpretations, Volume 2, 2005, Pages 279292

49. Wu, H., *Teaching Geometry in Grade 8 and High School According to the Common Core Standards* can be found online.

50. Yaglom, I. M., *Geometric Transformations* (4 volumes), MAA New Mathematical Library, No. 8 (1962), 21 (1968), 24 (1973), 44 (2009).

51. *Common Core State Standards Initiative: Mathematics* can be found online.

52. *The Mathematical Education of Teachers II.* CBMS Issues in Mathematics Education, vol. 17.