Giving more realistic definitions of trigonometric ratios II

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This paper being an extension of Bhattacharjee (2012) is very much relevant to Year 9 to Year 10A in the *Australian Curriculum: Mathematics*. It also falls within the purview of class IX to class XII curriculum of Mathematics in India (Revised NCERT curriculum) for students aged 14–17 years. In Bhattacharjee (2012), the discovery of flaw in the traditional definitions of trigonometric ratios, which make use of the most unrealistic concept of negative length or distance has been reported. With a view to getting rid of such unrealistic concept of negative length or distance, which has been in regular use in the sign convention of geometrical optics, in solving typical problems of elementary mechanics, efforts have already been made by the author earlier in Bhattacharjee (2002, 2011, 2012).

To uproot the misleading concept of negative length or distance from the basic level of trigonometry, realistic definitions of trigonometric ratios have been offered in Bhattacharjee (2012) with the help of vector algebra and they have been subsequently employed to derive the basic formulae of trigonometry in an unambiguous manner. Such a vectorial portrayal of realistic definitions of trigonometric ratios offered in Bhattacharjee (2012) had been much clearer leaving no room for confusion.

Now, with the development of the realistic definitions of trigonometric ratios in Bhattacharjee (2012), there is an urgent need how they lead to other useful formulae of trigonometry. With that point in mind, the realistic definitions of trigonometric ratios offered in Bhattacharjee (2012) have been applied in this paper to derive some of the other useful formulae of trigonometry. The study reveals that the application of the realistic definitions of trigonometric ratios offered in Bhattacharjee (2012) leads directly to those useful formulae of trigonometry. The approach, in all cases considered, is novel, analytical and straight forward unlike the well known geometrical or vectorial approaches found in the traditional literature, Hall and Knight (1906), Spiegel (1959). As a result, the present work will not only enrich the relevant branch of mathematics but it will also enhance the same as well.

Realistic definitions of trigonometric ratios

In this section the realistic definitions of trigonometric ratios offered in Bhattacharjee (2012) are being presented from the point of view of the interest of the readership of the journal.

If θ be the angle which a ray *OA* makes with the positive direction of *x*-axis (measured anti-clockwise with respect to *OX*) as shown in Figure 1, then the realistic definitions of the six trigonometric ratios offered in Bhattacharjee (2012) are as follows:

1. $\sin\theta = \dfrac{(\mathbf{i}\times\mathbf{r}).\mathbf{k}}{r}$

2. $\cos\theta = \dfrac{\mathbf{i}.\mathbf{r}}{r}$

3. $\tan\theta = \dfrac{(\mathbf{i}\times\mathbf{r}).\mathbf{k}}{\mathbf{i}.\mathbf{r}}$

4. $\cot\theta = \dfrac{\mathbf{i}.\mathbf{r}}{(\mathbf{i}\times\mathbf{r}).\mathbf{k}}$

5. $\sec\theta = \dfrac{r}{\mathbf{i}.\mathbf{r}}$

6. $\mathrm{cosec}\theta = \dfrac{r}{(\mathbf{i}\times\mathbf{r}).\mathbf{k}}$

where r is the scalar value of **r**.

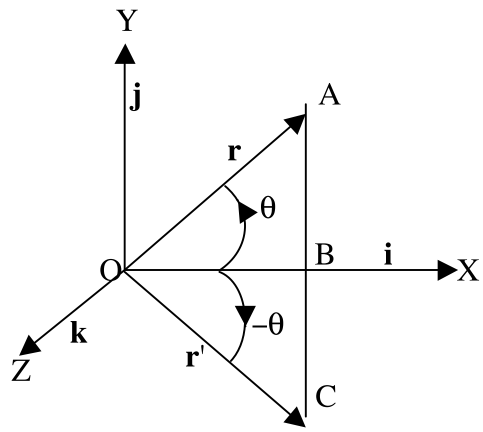

Figure 1. Diagram used for giving realistic definitions of trigonometric ratios.

While dealing with negative angles, it follows readily from Figure 1 that the position vector to be considered for the angle –θ would be **r**' as shown in that Figure.

Thus replacing θ by –θ, **r** by **r**' and r by r' in the definitions of the novel trigonometric ratios defined earlier we have the following results:

1. $\sin(-\theta) = \dfrac{(\mathbf{i}\times\mathbf{r}').\mathbf{k}}{r'}$

2. $\cos(-\theta) = \dfrac{\mathbf{i}.\mathbf{r}'}{r'}$

3. $\tan(-\theta) = \dfrac{(\mathbf{i} \times \mathbf{r}').\mathbf{k}}{\mathbf{i}.\mathbf{r}'}$

4. $\cot(-\theta) = \dfrac{\mathbf{i}.\mathbf{r}'}{(\mathbf{i} \times \mathbf{r}').\mathbf{k}}$

5. $\sec(-\theta) = \dfrac{r'}{\mathbf{i}.\mathbf{r}'}$

6. $\operatorname{cosec}(-\theta) = \dfrac{r'}{(\mathbf{i} \times \mathbf{r}').\mathbf{k}}$

where r' is the scalar value of $\mathbf{r}'$.

Novel derivation based on realistic definitions

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Proof

Considering Figure 2, we have from the definition of novel trigonometric ratio,

$$\sin(A+B) = \frac{(\mathbf{i} \times \mathbf{r}_2).\mathbf{k}}{r_2}$$

$$= \frac{\{\mathbf{i} \times (r_2 \cos B\,\mathbf{I} + r_2 \sin B\,\mathbf{J})\}.\mathbf{k}}{r_2}$$

$$= \frac{\{r_2 \sin A \cos B + r_2 \sin B \sin(90° + A)\}(\mathbf{k}.\mathbf{k})}{r_2}$$

$$= \sin A \cos B + \cos A \sin B \quad \text{(proved)}$$

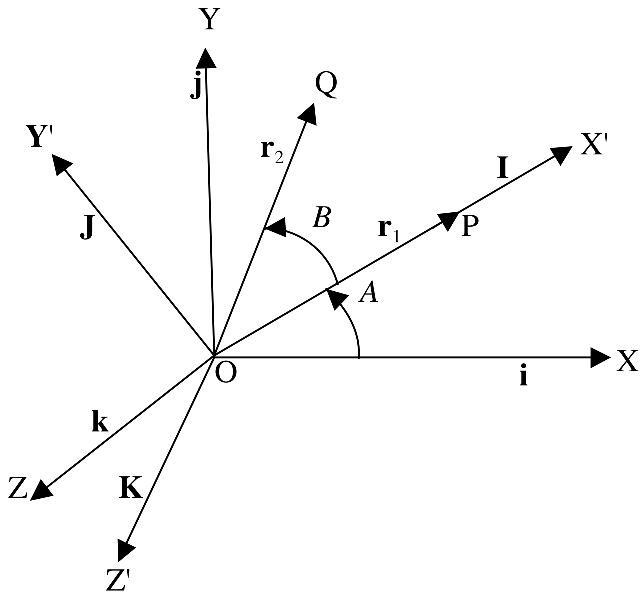

Figure 2. Diagram for the derivation of the formulae for $\sin(A + B)$ *and* $\cos(A + B)$.

2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Proof

Considering Figure 2, we have from the definition of novel trigonometric ratio,

$$\cos(A+B) = \frac{\mathbf{i}.\mathbf{r}_2}{r_2}$$
$$= \frac{\mathbf{i}.\left(r_2 \cos B\,\mathbf{I} + r_2 \sin B\,\mathbf{J}\right)}{r_2}$$
$$= \cos A \cos B + \cos(90° + A) \sin B$$
$$= \cos A \cos B - \sin A \sin B \quad \text{(proved)}$$

3. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Proof

Considering Figure 3, we have from the definition of novel trigonometric ratio,

$$\sin(A-B) = \frac{\left(\mathbf{i} \times \mathbf{r}_2\right).\mathbf{k}}{r_2}$$
$$= \frac{\left\{\mathbf{i} \times \left(r_2 \cos B\,\mathbf{I} - r_2 \sin B\,\mathbf{J}\right)\right\}.\mathbf{k}}{r_2}$$
$$= \frac{\left\{r_2 \sin A \cos B - r_2 \sin B \sin(90° + A)\right\}(\mathbf{k}.\mathbf{k})}{r_2}$$
$$= \sin A \cos B - \cos A \sin B \quad \text{(proved)}$$

4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Proof

Considering Figure 3, we have from the definition of novel trigonometric ratio,

$$\cos(A-B) = \frac{\mathbf{i}.\mathbf{r}_2}{r_2}$$
$$= \frac{\mathbf{i}.\left(r_2 \cos B\,\mathbf{I} - r_2 \sin B\,\mathbf{J}\right)}{r_2}$$
$$= \cos A \cos B - \cos(90° + A) \sin B$$
$$= \cos A \cos B + \sin A \sin B \quad \text{(proved)}$$

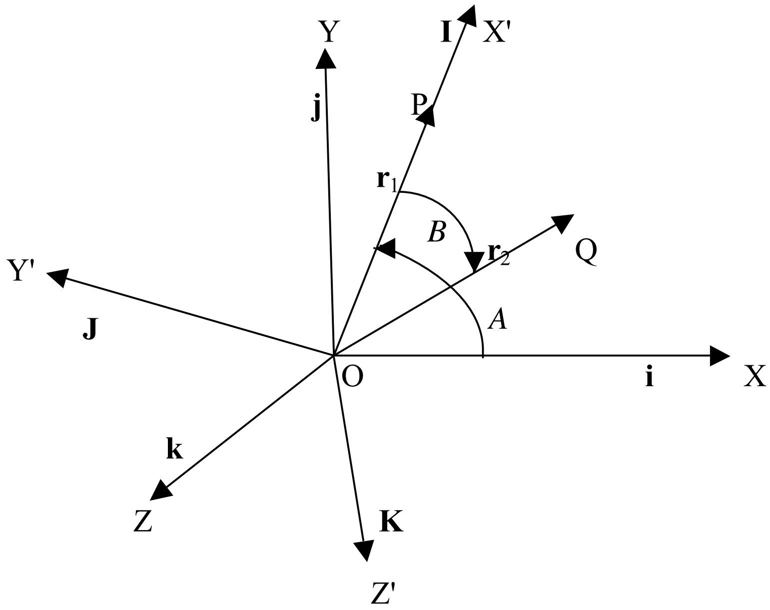

Figure 3. Diagram for the derivation of the formulae for $\sin(A-B)$ *and* $\cos(A-B)$.

5. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Proof

Considering Figure 4, we have from the definition of novel trigonometric ratio,

$$\sin B = \frac{(\mathbf{i}\times\mathbf{c}).\mathbf{k}}{c}$$

$$= \frac{\{\mathbf{i}\times(\mathbf{a}+\mathbf{b})\}.\mathbf{k}}{c}$$

$$= \frac{\left[\mathbf{i}\times\left\{a\,\mathbf{i}+(-b\cos C\,\mathbf{i}+b\sin C\,\mathbf{j})\right\}\right].\mathbf{k}}{c}$$

$$= \frac{b\sin C}{c}$$

or $$\frac{b}{\sin B} = \frac{c}{\sin C} \qquad (1)$$

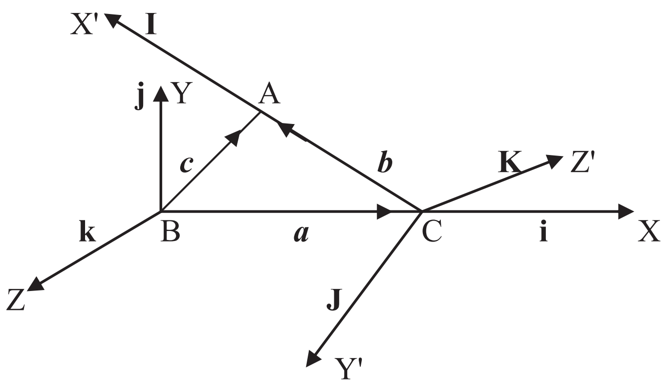

Figure 4. Diagram for the derivation of the law of sines for plane triangles.

Again from Figure 4, we have from the definition of novel trigonometric ratio,

$$\sin(180° - A) = \frac{\{\mathbf{I} \times (-\mathbf{c})\}.\mathbf{k}}{c}$$

$$= \frac{\{\mathbf{I} \times (-b\mathbf{I} - \mathbf{a}\}.\mathbf{k}}{c}$$

$$\frac{\{\mathbf{I} \times (a\cos C\,\mathbf{I} + a\sin C\,\mathbf{J})\}.\mathbf{k}}{c}$$

$$= \frac{a\sin C}{c}$$

or $$\sin A = \frac{a\sin C}{c}$$

or $$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad (2)$$

From equations (1) and (2) it then readily follows that,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{(proved)}$$

6. $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Proof

Considering Figure 5, we have from the definition of novel trigonometric ratio,

$$\cos A = \frac{\mathbf{i}.\mathbf{r}}{r}$$

$$= -\frac{\mathbf{i}.\mathbf{b}}{b}$$

$$= \frac{\{\mathbf{i}.(\mathbf{c} + \mathbf{a})\}}{b}$$

$$= \frac{\{\mathbf{i}.(c\,\mathbf{i} + \mathbf{a})\}}{b}$$

$$= \frac{c + \mathbf{i}.\mathbf{a}}{b}$$

$$= \frac{\mathbf{c}.\mathbf{c} + c\,\mathbf{i}.\mathbf{a}}{bc}$$

$$= \frac{2\mathbf{c}.\mathbf{c} + 2c\,\mathbf{i}.\mathbf{a}}{2bc}$$

$$= \frac{c^2 + \mathbf{c}.\mathbf{c} + 2\mathbf{c}.\mathbf{a}}{2bc}$$

$$= \frac{\{c^2 + (-\mathbf{b} - \mathbf{a}).(-\mathbf{b} - \mathbf{a}) + 2(-\mathbf{b} - \mathbf{a}).\mathbf{a}\}}{2bc}$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \quad \text{(proved)}$$

Again by considering a second right handed coordinate system with the point B as origin and proceeding similarly, the aforesaid formula for cos$B$ can be derived. Also, by choosing a third right handed coordinate system with the point C as origin and following the same procedure, the formula for cos$C$ mentioned above can be easily arrived at.

7. $a = b \cos C + c \cos B$
   $b = c \cos A + a \cos C$
   $c = a \cos B + b \cos A$

Proof

Considering Figure 5, we have from the definition of novel trigonometric ratio,

$$\cos A = \frac{\{\mathbf{i}.(-\mathbf{b})\}}{b}$$

$$= \frac{\{\mathbf{i}.(\mathbf{c} + \mathbf{a})\}}{b}$$

$$= \frac{\{c + a\cos(180° - B)\}}{b}$$

or $$\cos A = \frac{c - a\cos B}{b}$$

or $$c = a \cos B + b \cos A$$

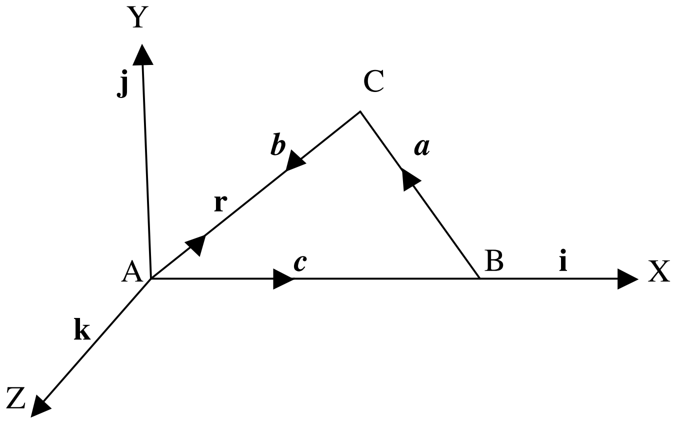

Figure 5. Diagram for the derivation of the law of cosines for plane triangles.

By considering two other right handed coordinate systems, one with origin at the point B, and the other with origin at the point C, and proceeding in a similar manner, the other two formulae, viz. $a = b \cos C + c \cos B$ and $b = c \cos A + a \cos C$ can be readily obtained.

Conclusion

In this paper, the fundamental formulae of compound angle, viz. the formulae for $\sin(A + B)$, $\sin(A - B)$, $\cos(A + B)$ and $\cos(A - B)$, have been derived by direct application of the definitions of realistic trigonometric ratios offered by the author in Bhattacharjee (2012). Those definitions are also applied for the derivation of some other formulae in relation to properties of triangle. The approach for the derivation of formulae in all cases considered is novel, analytical and straight forward unlike the traditional geometrical approaches, (Hall & Knight, 1906) and the vectorial approaches (Spiegel, 1959). As a result, the present contribution will enrich the relevant branch of mathematics there by enhancing it as well.

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