MIGSAA Extended Project

Global Dynamics Above the Ground State Energy for the Three-Dimensional Cubic Nonlinear Klein-Gordon Equation

Author:
William J. Trenberth
Supervisors:
Tadahiro Oh and Oana Pocovnicu
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Abstract

This report presents established results on the long term behavior of solutions to the cubic nonlinear Klein-Gordon equation in three dimensions. In particular the Payne-Sattinger dichotomy for solutions with energy less than that of the ground state, the Bates and Jones approach to the construction of Invariant Manifolds for semilinear partial differential equations and the recent work of Nakanishi and Schlag giving rise to a nonchotomy classifying the behavior of solutions to the nonlinear Klein-Gordon equation with energy perhaps slightly greater than that of the ground state are presented.

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Chapter 1

Introduction

1.1 Introduction

This report concerns the three dimensional cubic nonlinear Klein-Gordon equation (NLKG)

$$\begin{cases} \partial_t^2 u - \Delta u + u - u^3 = 0 \\ u(0) = u_0, \quad \partial_t u(0) = u_1 \end{cases}, \quad u : \mathbb{R}^3 \times I \to \mathbb{R}. \tag{1.1.1}$$

This report closely follows the Book *Invariant Manifolds and Dispersive Hamiltonian Evolution Equations* by Kenji Nakanishi and Wilhelm Schlag [13].The aim of this report is to reiterate recent work on understanding the long term behavior of solutions to NLKG. As far as the author knows, NLKG is not used in the modeling of any physical phenomenon, but, the linear Klein-Gordon equation

$$\begin{cases} \partial_t^2 u - \Delta u + u = 0 \\ u(0) = u_0, \quad \partial_t u(0) = u_1 \end{cases}, \quad u : \mathbb{R}^3 \times I \to \mathbb{R}. \tag{1.1.2}$$

is of great importance in quantum physics in describing relativistic electrons. From a mathematical point of view, NLKG is interesting because it is one of the simplest equations whose solutions collectively exhibit the phenomenon Nakanishi and Schlag wanted to describe in [12]. It is hoped that techniques that are used to study NLKG can be applied to other, perhaps more physically relevant, equations such as the nonlinear Schrodinger Equation. Therefore in the work of Nakanishi and Schlag, and hence in this report, there is an emphasis on robustness of results. That is results that can be easily generalized to other equations.

1.2 Outline of Chapters

The author closely studied this project with Justin Forlano who is also writing a report on this topic. In order to obtain an equitable splitting of content, the authors of their respective reports decided on a nonlinear splitting of the content. Hence if an important result is proved but not stated in this report it is likely proved in the report of Justin Forlano. To see omitted proofs, the reader is encouraged to obtain the report of Justin Forlano or consult the book *Invariant Manifolds and Dispersive Hamiltonian Evolution Equations* [13].

In Chapter Two of this report we will state results in the basic theory of NLKG that will be needed later in this report. In particular the local well possessedness of NLKG and a scattering result: If u is a global solution to NLKG with maximal forward time of existence $T^* = \infty$ and

$\|u\|_{L^3([0,\infty),L^6(\mathbb{R}^3))} < \infty$ then u scatters. The results in this first section are very standard in the field of dispersive PDE and so the proofs are omitted. The existence of ground states for NLKG is then discussed in section two of Chapter Two. In short, there exist a distinguished time independent positive radial solution of NLKG which plays a important role in classifying the behavior of other solutions of NLKG. This section of the report uses results from the theory of elliptic PDE. In the final section of Chapter Two a very important finite time blowup/Global existence result is obtained for solutions with energy less than that of Q. The sign of a certain nonlinear functional, K_0, plays an important role in determining the behavior of solutions to NLKG. This was first noticed by Payne and Sattinger in [15]. In fact this is not the complete picture. Many years later in [9] it was proved that solutions with energy less than that of the ground state that exist globally in fact scatter to 0 both forwards and backwards in time. Due to the complexity of the proof of this result no attempt is made at pursuing its proof.

In Chapter Three of this report we turn to looking at the dynamics of NLKG near $\pm Q$. By rewriting NLKG in a certain way, one can view NLKG as an infinite dimensional dynamical system with equilibrium points $\pm Q$. The aim of this chapter is then to try and apply techniques from dynamical system theory to study the behavior of solutions to NLKG near $\pm Q$. An immediate corollary of the Payne-Sattinger theory in Chapter Two shows that the ground states are unstable. However, more information is needed. One important result in *finite* dimensional dynamical system theory is the *center manifold theorem* which in short asserts the existence of certain invariant manifolds which can simplify the study of a perhaps complicated dynamical system by reducing its behavior to behavior on these invariant manifolds. Of course we cannot directly apply the Center Manifold Theorem as NLKG is an infinite dimensional ODE. Chapter Three is mostly devoted to describing an abstract construction of invariant manifolds for an infinite dimensional ODE as done by Bates and Jones in [2]. There is another method of constructing invariant manifolds for NLKG. This is the method of Lyapunov-Perron method which is stated but not proved in this report.

In Chapter Four, we look at the new ideas of Nakanishi and Schlag in describing global dynamics of NLKG. Chapter Three gives a good picture about how solutions to NLKG behave very close to the ground states. To obtain a complete dynamical picture we need information about how the solutions behave away from the ground states. The most important result in this section, is the *One Pass Theorem* which says that almost (hetro)homo-clinic orbits do not exist.

In Chapter Five the results of the previous sections are pieced together to give a complete description of the behavior of solutions to NLKG with energy perhaps slightly greater than that of the ground state. Basically using the One Pass Theorem and other results in Chapter Four, one can reduce the analysis in the slightly above the ground state regime to a situation similar to the below the ground state regime. In short there are three possible behaviors in each time direction. This gives rise to a nonchotomy giving a complete description of possible dynamical behavior of solutions to NLKG. The geometric information obtained in Chapter Three is combined with this classification to give geometric flavor to this nonchotomy.

Chapter 2

The Nonlinear Klein-Gordon Equation Below the Ground State Energy

2.1 Basic Theory

In this section we briefly state a few basic results for the nonlinear Klein-Gordon equation:

$$\begin{cases} \partial_t^2 u - \Delta u + u - u^3 = 0 \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \tag{2.1.1}$$

The results in this section are fairly standard and we do not pursue the proof of any. For proofs of these results we direct the reader to [13]. To progress far in the study of NLKG, as is usual in PDE one needs to abandon the classical definition of a solution and seek a broader definition. As is usual in dispersive PDE this is done using Duhamel's formula.

Definition We say that u is a *strong solution* to (2.1.1) on the time interval $[0, T)$ with data (u_0, u_1) if

$$u \in C([0,T); H^1) \cap C^1([0,T); L^2) \tag{2.1.2}$$

and u satisfies the Duhamel formula,

$$u(t) = \cos(t\langle\nabla\rangle)u_0 + \frac{\sin(t\langle\nabla\rangle)u_1}{\langle\nabla\rangle} + \int_0^t \frac{\sin((t-s)\langle\nabla\rangle}{\langle\nabla\rangle} u^3(s)\, ds. \tag{2.1.3}$$

We look for solutions to NLKG with initial data from a specific space. A natural space to seek solutions to NLKG is the space $\mathcal{H} := H^1 \times L^2$ endowed with the norm $\|(u_0, u_1)\|_{\mathcal{H}}^2 = \|\langle\nabla\rangle u_0\|_2^2 + \|u_1\|_2^2$. In Chapter Three and beyond we will insist on having radial initial data. That is in Chapter Three and beyond we will study NLKG with initial data from the space

$$\mathcal{H}_{\mathrm{rad}} := \{\vec{u} \in \mathcal{H} : u \text{ is radial}\}. \tag{2.1.4}$$

If u is a solution to (2.1.1) we use the notation

$$\vec{u}(t) := (u(t), \partial_t u(t)). \tag{2.1.5}$$

Then we see that $\vec{u}(t) := (u(t), \partial_t u(t)) \in H^1 \times L^2 = \mathcal{H}$. An immediate consequence of (2.1.3) and the Sobolev embedding $H^1 \hookrightarrow L^6$ is the following energy estimate:

$$\|\vec{u}(t)\|_{\mathcal{H}} \lesssim \|u(0)\|_{\mathcal{H}} + \int_0^t \|u(s)\|_{L^6}^3 \, ds. \tag{2.1.6}$$

This energy estimate hints at the importance of the space $L_t^3 L_x^6$ to NLKG. In the study of the NLKG it is essential to consider the linear Klein-Gordon equation

$$\begin{cases} \partial_t^2 u - \Delta u + u = 0 \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \tag{2.1.7}$$

Associated to the linear Klein-Gordon equation is the operator

$$S_0(t)(u_0, u_1) = \begin{pmatrix} \cos(t\langle\nabla\rangle) & \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} \\ -\sin(t\langle\nabla\rangle)\langle\nabla\rangle & \cos(t\langle\nabla\rangle) \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{2.1.8}$$

which is the linear evolution of the linear Klein-Gordon equation. It is easy to see that $S_0(t)$ is unitary on $\mathcal{H}$ for all t and has adjoint $(S_0(t))^* = S_0(-t)$. We now state a result on local existence of solutions for NLKG.

Theorem 2.1. *(Existence) For any $(u_0, u_1) \in \mathcal{H}$ there exists a unique strong solution $u \in C([0,T); H^1) \cap C^1([0,T); L^2)$ for some $T \geq T_0 > 0$ where T_0 depends only on $\|(u_0, u_1)\|_{\mathcal{H}}$. The solution depends continuously on the initial data.*

The proof of this theorem involves showing a certain nonlinear map coming from the Duhamel formula is a contraction mapping and then applying the Contraction Mapping Theorem to show the existence of a fixed point. This result allows us to define the nonlinear evolution of NLKG. For $(u_0, u_1) \in \mathcal{H}$ let $u(t)$ be the solution to (2.1.1) on some interval I containing 0 with initial data (u_0, u_1). For a given $t \in I$ we define the map $S(t) : \mathcal{H} \to \mathcal{H}$ by $S(t)(u_0, u_1) := \vec{u}(t)$. $S(t)$ is the nonlinear evolution of NLKG (2.1.1). By the previous Theorem $S(\cdot)\vec{u}$ is well defined for at least a small time interval containing 0. If u is a solution to (2.1.1) we define its energy (a priori at a time t) to be

$$E(\vec{u}(t)) = \int_{R^3} \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4 \right) dx. \tag{2.1.9}$$

NLKG is a Hamiltonian PDE with Hamiltonian (2.1.9). Hence the energy, being in fact the Hamiltonian, is a conserved quantity. Before we state the next theorem we need another definition. We say that u scatters if there exists $(\tilde{u_0}, \tilde{u_1}) \in \mathcal{H}$ such that with $\vec{v}(t) = S_0(t)(\tilde{u_0}, \tilde{u_1})$ one has

$$\lim_{t\to\infty} \|(u(t), \partial_t u(t)) - (v(t), \partial_t v(t))\|_{\mathcal{H}} = 0. \tag{2.1.10}$$

We write (2.1.10) as

$$(u(t), \partial_t u(t)) = (v(t), \partial_t v(t)) + o_{\mathcal{H}}(1) \quad t \to \infty. \tag{2.1.11}$$

Intuitively, a solution scatters if it asymptotically behaves like a linear solution.

Theorem 2.2. *Let $u \in C([0,T); H^1) \cap C^1([0,T); L^2)$ be the unique solution to (2.1.1) with initial data (u_0, u_1) guaranteed by the previous theorem. Then u has the following properties.*

1. *If $(u_0, u_1) \in H^2 \times H^1$, then $\vec{u}(t) \in H^2 \times H^1$ for all $0 \leq t \leq T$ where T is any time up to which a strong solution exists.*

2. *The energy of u, $E(\vec{u}(t))$, does not depend on time.*

3. *If $\|(u_0, u_1)\|_{\mathcal{H}} \ll 1$ then the solution exists globally in time and*
$$\|u\|_{L^3([0,\infty),L^6(\mathbb{R}^3)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}} \tag{2.1.12}$$

4. *If $T^* > 0$ is the maximal forward time of existence, then $T^* < \infty$ implies that $\|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3)} = \infty$.*

5. *If $T^* = \infty$ and $\|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3)} < \infty$, then u scatters. On the other hand if u scatters then $\|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3)} < \infty$.*

We define the forward scattering set to be
$$\mathcal{S}_+ = \{(u_0, u_1) \in \mathcal{H} : S(t)(u_0, u_1) \text{ exists for all times and scatters to zero}\}. \tag{2.1.13}$$

Theorem 2.3. *The forward scattering set satisfies the following properties*

1. *There exists $\delta > 0$ such that $B_\delta(0) \subset \mathcal{S}_+$ in $\mathcal{H}$.*

2. *$\mathcal{S}_+ \neq \mathcal{H}$.*

3. *$\mathcal{S}_+$ is unbounded in $\mathcal{H}$.*

4. *$\mathcal{S}_+$ is an open set in $\mathcal{H}$.*

5. *$\mathcal{S}_+$ is path-connected in $\mathcal{H}$.*

2.2 Stationary Solutions and the Ground State

This section is concerned with the definition and properties of the ground state solution to NLKG. The ground state is a distinguished solution of NLKG which has an important role in characterizing behavior of other solutions to NLKG. A time independent solution $u(x,t) = \varphi(x)$ of NLKG satisfies the following elliptic problem.
$$-\Delta\varphi + \varphi = \varphi^3. \tag{2.2.1}$$

By a standard elliptic regularity argument, see [8], any weak solution of (2.2.1) is a classical solution. It was shown in [5] that there is a unique positive radial solution of (2.2.1). We call the unique positive radial solution of (2.2.1) the *ground state* and we denote it by Q. Moreover, it was shown in [18] that the minimization problem
$$\inf\{\|\varphi\|_{H^1}^2 : \varphi \in H^1, \|\varphi\|_{L^4} = 1\} \tag{2.2.2}$$

has a unique positive radial solution φ_∞ which decays exponentially and $\varphi = \lambda\varphi_\infty$ satisfies (2.2.1) for some $\lambda > 0$. This gives a characterization of the ground state as a solution of the minimization problem (2.2.2). The rest of this section concerns results which are connected to the ground state and are of importance latter in this report.

We define functionals K_0, G_0 and J on H^1 by,
$$K_0(\varphi) := \int_{\mathbb{R}^3} \left(|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4\right)\, dx \tag{2.2.3}$$
$$G_0(\varphi) := J(\varphi) - \frac{1}{4}K_0(\varphi) = \frac{1}{4}\|\varphi\|_{H^1}^2 \tag{2.2.4}$$
$$J(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4\right)\, dx. \tag{2.2.5}$$

We call the functional J the stationary energy. This comes from the fact that it is the energy without the $|\partial_t u|^2$ term. These functionals are important in classifying the behavior of solutions of NLKG with energy less than that of the ground state. In fact the sign of $K_0(u)$ plays an important role in determining the behavior of solutions in the below the ground state regime. This fact was first noticed by Payne and Sattinger in [15]. As we shall see in Chapter Five, the sign of K_0 is also of importance in classifying behavior of solutions with energy slightly above that of the ground state.

Lemma 2.4. *Every weak solution of 2.2.1 satisfies $J'(\varphi) = 0$ and $K_0(\varphi) = 0$. Moreover, for any $\varphi \in H^1 \backslash \{0\}$ the function $j_\varphi(\lambda) := Je^\lambda\varphi)$ is continuously differentiable for all $\lambda \in \mathbb{R}$, is strictly convex near $\lambda = -\infty$ and satisfies $j'_\varphi(\lambda_*) = 0$ for a unique λ_*. Further, $j_\varphi(\lambda)$ decreases strictly for $\lambda > \lambda_*$ with $j_\varphi(\lambda) \to -\infty$.*

Proof. The first part of the lemma follows easily from the definition of a weak solution of an elliptic PDE and the definition of the Frechet derivative. For the second part of the theorem, note that

$$j'_\varphi(\lambda) = ae^{2\lambda} - be^{4\lambda} = 0 \tag{2.2.6}$$

for a unique λ. The rest of the lemma is basic real analysis. □

Before we get to the main result of this section we need two technical lemmas. The first lemma improves the usual Sobolev embedding to a compact embedding. The second is the well known result of Polya-Szego on symmetric decreasing arrangements, see [16].

Lemma 2.5. *For all $\varphi \in H^1_{\text{rad}}(\mathbb{R}^3)$ we have the estimate,*

$$|\varphi(x)| \leq C|x|^{-1}\|\varphi\|_{H^1}. \tag{2.2.7}$$

and the embedding $H^1_{\text{rad}}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $2 < p < 6$.

Theorem 2.6. *(Polya-Szego) If $u \in W^{1,p}(\mathbb{R}^n)$ then the symmetric decreasing rearrangement, u^*, of u is also in $W^{1,p}(\mathbb{R}^n)$ and further*

$$\int_{\mathbb{R}^n} |\nabla u^*|^p \, dx \leq \int_{\mathbb{R}^n} |\nabla u^*|^p \, dx. \tag{2.2.8}$$

Lemma 2.7. *We have*

$$\begin{aligned} J(Q) = h_0 &:= \inf\{J(Q) : \varphi \in H^1\backslash\{0\}, \;\; K_0(\varphi) = 0\} \\ &= \inf\{G_0(\varphi) : \varphi \in H^1\backslash\{0\}, \;\; K_0(\varphi) \leq 0\} \end{aligned} \tag{2.2.9}$$

and these infima are achieved uniquely by the ground states $\pm Q$, up to translations.

Proof. We first show the infima are equal. If $K_0(\varphi) < 0$ then $K_0(\lambda_*\varphi) = 0$ for some $0 < \lambda_* < 1$ while $G_0(\lambda_*\varphi) < G_0(\varphi)$. On the other hand, if $K_0(\varphi) = 0$ then $J(\varphi) = G_0(\varphi)$. This argument also shows that a minimizing sequence for the second infima can be made into a minimizing sequence for both, that is with $K_0 = 0$. We now show that the infima are in fact $J(Q)$. For this note that from Lemma 2.4 $J'(Q) = 0$ and $K_0(Q) = 0$. This shows the infima are not larger than $J(Q)$. Let $\{\varphi_n\}_{n\geq 1} \subset H^1\{0\}$ be a minimizing sequence for both infima. By Polya-Szego, $\{\varphi_n^*\}_{n\geq 1}$ is also a minimizing sequence. Hence we assume φ_n is radial. As stated above we can modify this resulting sequence so that $\{\varphi_n\}_{n\geq 1}$ is a minimizing sequence for both infima. That is

$$K_0(\varphi_n) = 0, \quad J(\varphi_n) \to h_0. \tag{2.2.10}$$

Since $G_0(\varphi_n)$ is bounded, φ_n is bounded in H^1. By basic facts about weak convergence and (2.5), $\{\varphi_n\}_{n\geq 1}$ converges weakly to some φ_∞ in H^1 and strongly to φ_∞ in L^4. If $\varphi_\infty = 0$ then $K_0(\varphi) = 0$ and strong convergence to 0 in L^4 implies $\varphi_n \to 0$ strongly in H^1. But using the Sobolev embedding $H^1 \hookrightarrow L^4$,

$$K_0(\varphi_n) \geq \|\varphi_n\|_{H^1}^2(1 - \|\varphi_n\|_{H^1}^2) > 0 \tag{2.2.11}$$

if n is large enough. This is a contradiction. It follows $\varphi_\infty \neq 0$. By weak lower semi-continuity of the H^1 norm we have

$$\|\varphi_\infty\|_{H^1} \leq \liminf_{n\to\infty} \|\varphi_n\|_{H^1} \leq \|\varphi_\infty\|_{L^4} \tag{2.2.12}$$

hence $K_0(\varphi_\infty) \leq 0$. If $K_0(\varphi_\infty) < 0$ then there exists $0 < \lambda_* < 1$ such that $K_0(\lambda_*\varphi_\infty) = 0$. Then

$$h_0 \leq G_0(\lambda_*\varphi_\infty) = \lambda_*^2 G_0(\varphi_\infty) < h_0 \tag{2.2.13}$$

which is a contradiction. It follows $K_0(\varphi_\infty) = 0$. Hence $\|\varphi_n\|_{H^1} \to \|\varphi_\infty\|_{H^1}$ and combining this with the weak convergence of $\{\varphi_n\}_{n\geq 1}$ shows $\varphi_n \to \varphi_\infty$ strongly in H^1. Hence $J(\varphi_\infty) = h_0$ and so φ_∞ is a minimizer. We have shown at least one minimizer exists. We will now show that any minimizer is a translate of $\pm Q$. To this end suppose φ is a minimizer of (2.2.9). The theory of Lagrange multipliers then implies that

$$J'(\varphi) = \mu K_0'(\varphi) \quad \text{and} \quad K_0(\varphi) = 0 \tag{2.2.14}$$

for some $\mu \in \mathbb{R}$. From this one can show that $J'(\varphi) = 0$ and so φ is a solution of the elliptic problem (2.2.1). A basic elliptic regularity argument, see [8], shows φ is C^∞. We can apply the same elliptic regularity argument to $\pm|\varphi|$. The maximum principle shows that φ does not change sign. But by [7] any positive (negative) solution of (2.2.1) is radial around some origin and hence is a translation of the ground state by [5]. This shows φ is a translation of the ground state which is what we wanted to show. Since φ_∞ is a radial minimizer we must have $\varphi_\infty = Q$ and so we get (2.2.9). □

2.3 Payne-Sattinger

The sign of the functional K_0 turns out to be very important in classifying behavior in the below the ground state regime. This was first noticed in [15].

Definition We define the *Payne-Sattinger* regions by

$$\mathcal{PS}_+ := \{(u_0, u_1) \in \mathcal{H} : E(u_0, u_1) < J(Q), \;\; K_0(u_0) \geq 0\} \tag{2.3.1}$$
$$\mathcal{PS}_- := \{(u_0, u_1) \in \mathcal{H} : E(u_0, u_1) < J(Q), \;\; K_0(u_0) < 0\} \tag{2.3.2}$$

$\mathcal{PS}_+$ and $\mathcal{PS}_-$ give rise to a global existence/finite time blow up dichotomy which we shall see in this section.

Lemma 2.8. *If $J(\varphi) < J(Q)$ and $K_0(\varphi) < 0$ then*

$$-K_0(\varphi) \geq 2(J(Q) - J(\varphi)). \tag{2.3.3}$$

Proof. Let $j_\varphi(\lambda)$ and λ_* be as in Lemma 2.4. Then,

$$j_\varphi''(\lambda) = 2K_0(e^\lambda\varphi) - 2e^{4\lambda}\|\varphi\|_{L^4}^4 \leq 2j_\varphi'(\lambda). \tag{2.3.4}$$

Integrating both sides of this inequality between λ_* and 0 gives

$$j'_\varphi(0) - j'_\varphi(\lambda_*) \le 2\left(j_\varphi(0) - j_\varphi(\lambda_*)\right). \tag{2.3.5}$$

From Lemma 2.4 this is the same as

$$K_0(\varphi) \le 2(J(\varphi) - J(e^{\lambda_*}\varphi). \tag{2.3.6}$$

From Lemma 2.7 $J(e^{\lambda_*}\varphi) \ge J(Q)$. The result follows. $\square$

Using Lemma 2.7 we can obtain an important invariant result.

Theorem 2.9. *The regions $\mathcal{PS}_+$ and $\mathcal{PS}_-$ are invariant under the flow of 2.1.1 in the following sense: if $(u(0), \partial_t u(0)) \in \mathcal{PS}_+$ then $(u(t), \partial_t u(t)) \in \mathcal{PS}_+$ for as long as the solution exists, and the same result holds for $\mathcal{PS}_-$.*

Proof. Suppose by contradiction $\mathcal{PS}_+$ (or $\mathcal{PS}_-$) is not invariant. Let u be defined on a time interval I around $t = 0$. As $t \mapsto K_0(u(t))$ is continuous if u moves from $\mathcal{PS}_+$ to $\mathcal{PS}_-$ (or vice versa) by elementary analysis $K_0(u(t^*)) = 0$ for some $t^* \in I$ and $K_0 u(\cdot))$ changes sign on a small neighborhood of t^*. Here we define sign0 $= +1$. Since,

$$J(u(t^*)) \le E\left(u(t^*), \partial_t u(t^*)\right) = E(u_0, u_1) < J(Q) \tag{2.3.7}$$

Lemma 2.7 implies that $u(t^*) = 0$. From Sobolev embedding we have

$$\|u\|_{L^4}^4 < \frac{1}{2}\|u\|_{H^1}^4. \tag{2.3.8}$$

Hence for t close enough to t^* so that $\|u(t)\|_{H^1}^4 \ll \|u(t)|_{H^1}^2$ we have

$$K_0(u(t)) \simeq \|u(t)\|_{H^1}^2 \ge 0 \tag{2.3.9}$$

which contradicts that $K_0 u(\cdot))$ changes sign on a small neighborhood of t^*. $\square$

Based on the above Theorem we will occasionally abuse notation and write $u(t) \in \mathcal{PS}_\pm$. The main result in [15] was the following theorem which gives an important finite time blowup/global existence dichotomy.

Theorem 2.10. *Solutions of 2.1.1 which lie in $\mathcal{PS}_+$ exist for all times, whereas those in $\mathcal{PS}_-$ blow up in finite time (in both temporal directions). In particular, data of negative energy blow up in finite time, and $\mathcal{S} \ne \mathcal{H}$.*

Proof. If $u(t) \in \mathcal{PS}_+$ then,

$$\frac{1}{2}\|u(t)\|_{H^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 \simeq G_0(u(t)) + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 \le E(u, \partial_t u) \tag{2.3.10}$$

Hence $\|u(t)\|_{\mathcal{H}}$ is bounded and so by iterating the local existence result in Theorem 2.1 we see that u exists globally. For finite time blow up in $\mathcal{PS}_-$ we use the convexity argument of Payne and Sattinger. Suppose for a contradiction that $u(t) \in \mathcal{PS}_-$ is a global solution to NLKG and let $y(t) := \|u(t)\|_{L^2}^2$. Then,

$$\begin{aligned} \ddot{y} &= 2\|\partial_t u\|_{L_x^2}^2 + \int_{\mathbb{R}^3} u\partial_t^2 u\,dx \\ &= 2\|\partial_t u\|_{L_x^2}^2 - K_0(u(t)) \\ &= 6\|\partial_t u\|_{L_x^2}^2 - 8E(u, \partial_t u) + 2\|u\|_{H^1}^2. \end{aligned} \tag{2.3.11}$$

Since $J(u(t)) \leq E(u, \partial_t u) = E(u(0), \partial_t u(0)) < J(Q)$ Lemma 2.8 implies that

$$-K_0(u(t)) \geq \delta > 0, \quad \forall t \geq 0 \tag{2.3.12}$$

for fixed

$$\delta = 2(J(Q) - E(u, \partial_t u)). \tag{2.3.13}$$

Hence $y(t) \to \infty$ as $t \to \infty$. But then $\|u(t)\|_{H^1}^2 \to \infty$ as $t \to \infty$ so from (2.3.11) we have for large t,

$$\ddot{y}(t) \geq 6\|\partial_t u\|_{L_x^2}^2 \geq \frac{3}{2}\frac{\dot{y}(t)^2}{y(t)}. \tag{2.3.14}$$

This means

$$\partial_t^2(y^{-\frac{1}{2}}) = -\frac{1}{2}y^{-\frac{5}{2}}\left(y\ddot{y} - \frac{3}{2}\dot{y}^2\right) \leq 0. \tag{2.3.15}$$

However since $y^{-\frac{1}{2}} \to 0$ as $t \to \infty$ one has that $\partial_t(y^{-\frac{1}{2}})(t_0) < 0$ for some t_0. Hence from (2.3.15) we have

$$\partial_t(y^{-\frac{1}{2}})(t) \leq \partial_t(y^{-\frac{1}{2}})(t_0) < 0, \quad \forall t \geq t_0 \tag{2.3.16}$$

but this would mean $y^{-\frac{1}{2}}$ would vanish at some time which is impossible. □

This result of Payne and Sattinger does not quite give us a complete picture of the dynamics of solutions with energy less than that of the ground state. We know solutions in $\mathcal{PS}_+$ exist globally but do not know exactly how they behave. In 2011 Ibrahim, Masmoudi and Nakanishi [9] proved that solutions in $\mathcal{PS}_+$ in fact scatter to zero. They proved the following result.

Theorem 2.11. *All solutions $u(t)$ of 2.1.1 which are associated with $\mathcal{PS}_+$ scatter as $t \to \pm\infty$ and $\|u\|_{L_t^3 L_x^6} < \infty$. Moreover, there exists a function $N : (0, J(Q)) \to (0, \infty)$ so that*

$$\|u\|_{L_t^3 L_x^6} < N\left(E(u, \partial_t u)\right) \tag{2.3.17}$$

for all solutions belonging to $\mathcal{PS}_+$.

The proof of this result is quite complicated and we omit it. The complicatedness of the proof was perhaps the main reason for the 36 year gap between [15] and [9]. To prove this result, Ibrahim, Masmoudi and Nakanishi used the technique of profile decomposition which was developed in [1] and [11] more than twenty years after the result of Payne and Sattinger.

Chapter 3

Dynamics of NLKG Close to the Ground State

3.1 Instability of the Ground State

In this section we consider solutions of NLKG written in the form $u = Q + v$ where v is small in a suitable sense. We can then write NLKG around the point $(Q, 0)$,

$$\partial_t^2 v + L_+ v = N(v) \tag{3.1.1}$$

where

$$L_+ = -\Delta + 1 - 3Q^2 \quad \text{and} \quad N(v) = 3Qv^2 + v^3. \tag{3.1.2}$$

The reason for writing NLKG this way is that we can view NLKG as an infinite dimensional dynamical system with equilibrium points $\pm Q$. There are many powerful results finite dimensional dynamical systems theory. One such result is the center manifold theorem. For a reference on center manifold theory see [3]. In some sense the aim of this chapter is to find an appropriate generalization of the Center Manifold Theorem that can be applied to NLKG. Doing this would enable us to understand the behavior of NLKG locally around $\pm Q$. First we prove the ground state is unstable and so the dynamics near the ground state are in fact sufficient complex to warrant this approach.

We can write the energy and K_0 as,

$$\begin{aligned} E(Q+v, \partial_t v) &= J(Q) + \frac{1}{2}\langle L_+ v, v\rangle + \frac{1}{2}\|\partial_t v\|_{L^2}^2 - \frac{1}{4}\int_{\mathbb{R}^3} (4Qv^3 + v^4)\, dx \\ &= J(Q) + \frac{1}{2}\langle L_+ v, v\rangle + \frac{1}{2}\|\partial_t v\|_{L^2}^2 + O\left(\|v\|_{H^1}^3\right) \\ K_0(Q+v) &= -2\langle Q^3, v\rangle + \big\langle (L_+ - 3Q^2)v, v\big\rangle + O(\|v\|_{H^1}^3). \end{aligned} \tag{3.1.3}$$

Using these expansions we can prove the ground state is unstable.

Corollary 3.1. *For every $\varepsilon > 0$ the ball $B_\varepsilon(Q, 0)$ in $\mathcal{H}$ satisfies*

$$B_\varepsilon(Q,0) \cap \mathcal{PS}_+ \neq \emptyset, \quad B_\varepsilon(Q,0) \cap \mathcal{PS}_- \neq \emptyset. \tag{3.1.4}$$

Hence for every $\varepsilon > 0$ there exists two nonempty open subsets of $B_\varepsilon(Q,0)$ that lead to blowup in finite positive time and global forward existence along with scattering to 0 respectively.

Proof. Considering $v = \varepsilon Q$ and $\partial_t v = 0$ in the above formula one has

$$E(Q + \varepsilon Q, 0) = J(Q) - \varepsilon^2 \|Q\|_{L^4}^4 + O(\varepsilon^3) \tag{3.1.5}$$

and

$$K_0(Q + \varepsilon Q) = -2\varepsilon \|Q\|_{L^4}^4 + O(\varepsilon^2). \tag{3.1.6}$$

Hence for $\varepsilon > 0$ small enough $Q + \varepsilon Q \in \mathcal{PS}_+$ and $Q - \varepsilon Q \in \mathcal{PS}_-$. Hence from Theorem 2.10 we reach the desired conclusion. □

3.2 Construction of Invariant Manifolds via Bates and Jones

In the rest of this chapter we prove an appropriate generalization of the Center Manifold Theorem and apply it to NLKG. This was first done by Bates and Jones [2]. In short they constructed stable, unstable, center, center stable and center unstable manifolds for abstract ODEs of the form

$$\partial_t u = Au + f(u) \tag{3.2.1}$$

where X is a Banach space and A is a linear operator on X satisfying certain conditions. The approach of Bates and Jones requires notations and definitions which we will now state. Given a linear operator A on some Banach space we define

$$\begin{aligned} \sigma^s(A) &= \{\lambda \in \sigma(A) : \mathrm{Re}\lambda < 0\}, \\ \sigma^c(A) &= \{\lambda \in \sigma(A) : \mathrm{Re}\lambda = 0\}, \\ \sigma^u(A) &= \{\lambda \in \sigma(A) : \mathrm{Re}\lambda > 0\}, \end{aligned} \tag{3.2.2}$$

and define $X^s(A), X^c(A)$ and $X^u(A)$ to be the associated spectral subspaces. We will write $\sigma^s(A)$ as σ^s, $\sigma^c(A)$ as σ^c and $\sigma^u(A)$ as σ^u when no confusion arises and similarly for $X^s(A), X^c(A)$ and $X^u(A)$. We define $X^{cs} = X^c + X^s$ and $X^{cu} = X^c + X^u$ and we let $\pi^c, \pi^s, \pi^u, \pi^{cs}$ and π^{cu} denote the projection maps of X onto X^c, X^s, X^u, X^{cs} and X^{cu} respectively.

Definition We call the following collection of hypothesis on (3.2.1) *hypothesis H.*

(H1) $u \in X$ and X is a Banach space.

(H2) $A : X \to X$ is a densely defined linear operator on X generating a strongly continuous group $S(t)$.

(H3) X^s and X^u are both finite dimensional.

(H4) The evolution of $S(t)$ restricted to X^c which we denote by $S^c(t)$, satisfies the following bound for all $\rho > 0$,

$$\|S^c(t)u\| \leq M(\rho)e^{\rho|t|}\|u\|, \qquad \text{for all } t > 0. \tag{3.2.3}$$

(H5) The nonlinearity $f(u)$ is defined on all of X and is locally Lipschitz with $f(0) = 0$. For all $\varepsilon > 0$, there exists a neighborhood $U = U(\varepsilon)$ of 0 such that f has Lipschitz constant ε on $U(\varepsilon)$.

Let Φ_t denote the flow of (3.2.1). If $V \subset U$ we say that V is *positively invariant relative to U* if for any $v \in V$

$$\bigcup_{s\in[0,t]} \{\Phi_s(v)\} \subset U \implies \bigcup_{s\in[0,t]} \{\Phi_s(v)\} \subset V, \quad \text{for all } t > 0. \tag{3.2.4}$$

By replacing $[0,t]$ with $[-t,0]$ in the previous definition one can define *negatively invariant relative to U*. *Invariant relative to U* means both positively and negatively invariant. If $X = U$ in the previous definitions then we drop the 'relative to U' part.

Definition Given a neighborhood U of 0 we define:

- The stable manifold to be

$$W^s(U) := \{u \in U : \phi_t(u) \in U \text{ for all } t \geq 0, \phi_t(u) \to 0 \text{ exponentially as } t \to \infty\}.$$

  That is, $u \in W^s(U)$ if and only if there exists $\alpha > 0$ such that $\|\Phi_t(u)\| \lesssim_u e^{-\alpha t}$ for all $t > 0$.

- The unstable manifold to be

$$W^s(U) := \{u \in U : \phi_t(u) \in U \text{ for all } t \leq 0, \phi_t(u) \to 0 \text{ exponentially as } t \to -\infty\}.$$

- A Lipschitz manifold $Y \subset U$ to be center stable if
  (i) Y is invariant relative to U,
  (ii) $\pi^{cs}(U)$ contains a neighborhood of 0 in X^{cs},
  (iii) $Y \cap W^u(U) = \{0\}$.

- A Lipschitz manifold $Y \subset U$ to be center unstable if
  (i) Y is invariant relative to U,
  (ii) $\pi^{cu}(U)$ contains a neighborhood of 0 in X^{cu},
  (iii) $Y \cap W^s(U) = \{0\}$.

- A Lipschitz manifold $Y \subset U$ to be a center manifold if
  (i) Y is invariant relative to U,
  (ii) $\pi^{c}(U)$ contains a neighborhood of 0 in X^{c},
  (iii) $Y \cap W^u(U) = Y \cap W^s(U) = \{0\}$.

Even though the center stable, center unstable and center manifolds are not necessarily unique, when the meaning is clear we will use the notation W^{cs}, W^{cu} and W^c to denote the center stable, center unstable and center manifolds respectively. The main result in this chapter is the existence of these invariant manifolds.

Theorem 3.2. *Under hypothesis H on (3.2.1) there exists a neighborhood U of X such that $W^s(U)$ and $W^u(U)$ are Lipschitz manifolds in U and are tangent to X^s and X^u respectively. Furthermore there exist $W^{cs}(U), W^{cu}(U), W^c(U) \subset U$ such that $W^{cs}(U), W^{cu}(U)$ and $W^c(U)$ are center stable, center unstable and center manifolds respectively. Each of these manifolds is invariant relative to U; in fact, U can be chosen such that X^s is positively invariant and X^u is negatively invariant. Furthermore, the following properties hold:*

(P1) $W^c = W^{cs} \cap W^{cu}$. *If* $W^u = \{0\}$, *then* $W^{cu} = W^c$ *etc. Moreover,* W^{cu} *is* W^{cs} *backwards in time.*

(P2) W^{cs} *has the following repulsion property: there exists a neighborhood* $V \subset U$ *of* 0 *such that if* $u_0 \notin W^{cs}$ *but* $u_0 \in V$, *then* $\Phi_t(u_0)$ *leaves* V *in positive time.*

As a consequence of (P2), we see that if $W^u \neq \{0\}$ then the equilibrium 0 is unstable in the following sense: for any small neighborhood of 0 there exists a solution that leaves the neighborhood in a positive time.

3.3 Proof of Theorem

This section is devoted to the proof of Theorem 3.2. Consider the abstract ODE

$$\partial_t u = Au + g(u). \tag{3.3.1}$$

One should think of $g(u)$ as a truncated $f(u)$. First we will prove results on (3.3.1) under hypothesis J, which are stated below. Hypothesis J are related to hypothesis H so it will not be hard to prove our modification of f indeed satisfies hypothesis J. The idea is to construct invariant manifolds as the the image of certain Lipschitz functions.

Definition We call the following collection of hypothesis on (3.3.1) *hypothesis J*

(J1) $u \in X$ and X is a Banach space.

(J2) A is a linear operator on X such that X admits an A-invariant splitting $X = X^- \oplus X^+$.

(J3) X^+ is finite dimensional and X^- is closed.

(J4) $A^+ := A|_{X^+}$ and generate $A^- := A|_{X^-}$ C_0-groups $S^+(t)$ and $S^-(t)$ respectively.

(J5) The groups $S^+(t)$ and $S^-(t)$ satisfy the bounds

$$\|S^-(t)\| \leq Me^{\alpha t} \text{ for all } t \geq 0 \text{ and } \|S^+(t)\| \leq Me^{\beta t} \text{ for all } t \leq 0. \tag{3.3.2}$$

respectively where $\beta > \alpha$.

(J6) $g(0) = 0$ and g is globally Lipschitz with sufficiently small Lipschitz constant $\varepsilon > 0$, depending on $M, \alpha, \beta, |\pi^{\pm}\|$ where $\pi\pm$ are the projections associated with the splitting of X, satisfying

$$-\beta + 2\varepsilon < -\alpha - 2\varepsilon. \tag{3.3.3}$$

The next Lemma involves re-norming $X^{\pm}$ in a way that makes future calculations easier.

Lemma 3.3. *Under hypothesis J one can re-norm X^+ and X^- separately so that each inequality in (3.3.2) holds with $M = 1$.*

Proof. On X^- define

$$\|x\|_- = \sup_{t>0} e^{-\alpha t}\|S^-(t)x\| \tag{3.3.4}$$

and on X^+ define

$$\|x\|_+ = \sup_{t\leq 0} e^{-\beta t}\|S^+(t)x\|. \tag{3.3.5}$$

One can show these are norms which have the desired properties. $\square$

Under hypothesis J we denote the norms on X^+ and X^- given by Lemma 3.3 to be $\|\cdot\|_+$ and $\|\cdot\|_-$ respectively. We define a norm on X by $|\cdot| = \|\pi^+(\cdot)\|_+ + \|\pi^-(\cdot)\|_-$. One of the benefits of working with (3.3.1) under hypothesis J is that (3.3.1) is globally well posed.

Theorem 3.4. *Under hypothesis J, the nonlinear flow $\Phi_t(\cdot)$ associated with (3.3.1) is globally defined in $\mathbb{R} \times X$.*

Proof. The proof of this theorem is just sketched at it is standard. The group $S(t)$ generated by A satisfies the bound

$$\|S(t)\| \leq c_0 e^{c_1|t|}. \tag{3.3.6}$$

We define

$$\Gamma u(t) = S(t)u(0) + \int_0^t S(t-s)g(u(s))\,ds \tag{3.3.7}$$

for u in a suitable Banach space. For T small enough one can show that Γ is a contraction mapping on a suitable subset of the Banach space and Γ has a unique fixed point. This gives local existence. One can also show T does not depend on the initial data and so the local existence result can be iterated to show the flow $\Phi_t(\cdot)$ is globally defined. $\square$

For (3.3.1) under hypothesis J let γ be such that

$$-\beta + 2\varepsilon < \gamma < -\alpha - 2\varepsilon \tag{3.3.8}$$

and then define

$$\begin{aligned} W^+ &:= \{u \in X : e^{\gamma t}\Phi_t(u) \to 0 \text{ as } t \to \infty\}, \\ W^- &:= \{u \in X : e^{\gamma t}\Phi_t(u) \to 0 \text{ as } t \to -\infty\}. \end{aligned} \tag{3.3.9}$$

For $\lambda > 0$ we define

$$\mathcal{K}_\lambda := \{(v,w) \in X^- \oplus X^+ : \lambda|v| \leq |w|\}. \tag{3.3.10}$$

It is easy to see from the definition of $W^\pm$ that $W^\pm$ are invariant. Further, one can pick μ and ν so that $0 < \mu < 1 < \nu$, $\mu\nu^{-1} < 1$ and

$$\varepsilon < \frac{\beta - \alpha}{2 + \nu + \mu^{-1}}, \tag{3.3.11}$$

$$\varepsilon(1+\mu^{-1}) - \beta < \gamma < -\varepsilon(1+\nu) - \alpha. \tag{3.3.12}$$

Remark Setting $u(t) = (v(t), w(t)) \in X^- \oplus X^+$ one can write (3.3.1) in the form

$$\begin{aligned} \partial_t v &= A^- v + g^-(v,w) \\ \partial_t w &= A^+ + g^+(v,w) \end{aligned} \tag{3.3.13}$$

where $g^\pm := \pi^\pm g$. The solutions to this system are given by

$$\begin{aligned} v(t) &= S^-(t)v(0) + \int_0^t S^-(t-s)g^-\left(v(s), w(s)\right)\,ds, \quad t \geq 0 \\ w(t+\tau) &= S^-(\tau)v(0) + \int_0^\tau S^-(\tau - s)g^-\left(v(t+s), w(t+s)\right)\,ds, \quad 0 \leq -\tau \leq t. \end{aligned} \tag{3.3.14}$$

Lemma 3.5. *If $|w(s)| \leq k_1|v(s)|$ for all $0 \leq s \leq t$ then*

$$|v(t)| \leq |v(0)|e^{(\varepsilon(1+k_1)+\alpha)t}. \tag{3.3.15}$$

If $|v(s)| \leq k_2|w(s)|$ for all $0 \leq s \leq t$ then

$$|w(t-\tau)| \leq |w(t)|e^{((\varepsilon(1+k_2)-\beta)\tau}. \tag{3.3.16}$$

Proof. The Lemma is a straightforward application of Gronwall's inequality. We just prove the first claim. If $|w(s)| \leq k_1|v(s)|$ for all $0 \leq s \leq t$ for some $k_1, t > 0$ then taking the $|\cdot|$ norm of the splitting in 3.3.14 and using the Lipschitz condition on g we have

$$|v(t)| \leq |v(0)|e^{\alpha t} + \varepsilon \int_0^t e^{\alpha(t-s)}(|v(s)| + |w(s)|)\, ds. \tag{3.3.17}$$

From Gronwall's inequality we get,

$$|v(t)| \leq |v(0)|e^{(\varepsilon(1+k_1)+\alpha)t}. \tag{3.3.18}$$

□

Lemma 3.6. *With $\mathcal{K}_\lambda$ as defined above the following is true:*

(a) For $\lambda \in [\mu, \nu]$, $\mathcal{K}_\lambda$ is positively invariant. Furthermore, if $(v_0, u_0) \in \mathcal{K}_\lambda$, then

$$|w(t)| \geq |w(0)| \exp\left((\beta - \varepsilon(1 + \lambda^{-1}))t\right), \quad \textit{for all } t \geq 0. \tag{3.3.19}$$

(b) More generally, if $u_2(0) \in u_1(0) + \mathcal{K}_\lambda$, then the corresponding solutions satisfy $u_2(t) \in u_1(t) + \mathcal{K}_\lambda$. Furthermore,

$$|w_1(t) - w_2(t)| \geq |w_1(0) - w_2(0)| \exp\left((\beta - \varepsilon(1 + \lambda^{-1}))t\right), \textit{ for all } t \geq 0. \tag{3.3.20}$$

Proof. We just prove (a) as the proof of (b) is similar. To prove (a) we prove $\mathcal{K}_\lambda$ is locally invariant (in a sense to be defined below) and then by a continuity argument extend this to invariance. We show $\mathcal{K}_\lambda$ is locally positively invariant (in time): that is if $(v_0, w_0) \in \mathcal{K}_\lambda$ then there exists $t > 0$ such that $(v(s), w(s)) \in \mathcal{K}_\lambda$ for all $0 \leq s \leq t$. This is true by continuity if $\lambda|v_0| < |w_0|$. If $\lambda|v_0| = |w_0|$, then by continuity for any $\delta > 0$ one has

$$|w(s)| \leq (\lambda + \delta)|v(s)|, \quad |v(s)| \leq (\lambda - \delta)^{-1}|w(s)| \quad \forall\ 0 \leq s \leq t \tag{3.3.21}$$

where $t > 0$ is small. Then applying (3.3.18) and (3.3.16) with $k_1 = (\lambda - \delta)^{-1}$ and $k_2 = \lambda + \delta$ we have,

$$\frac{|v(t)|}{|w(t)|} \leq \frac{|v_0|}{|w_0|} \exp\left((\alpha - \beta + \varepsilon(2 + \lambda + \delta + (\lambda - \delta)^{-1}))t\right) \leq \lambda^{-1}. \tag{3.3.22}$$

Now we want to show that $\mathcal{K}_\lambda$ is in invariant. We do so by using the local positive invariance of $\mathcal{K}_\lambda$ and a simple continuity argument. Set

$$B := \{t \geq 0 : \lambda|v(t)| \leq |w(t)|\}. \tag{3.3.23}$$

As v and w are continuous, B is a closed set in $[0, \infty)$. Define

$$T = \inf\{t \geq 0 : \lambda|v(t)| > |w(t)|\}. \tag{3.3.24}$$

Suppose in order to obtain a contradiction that $T < \infty$. As B is closed, $T \in B$. By the local invariance of $\mathcal{K}_\lambda$ we have that $[T, T + t] \subset B$ for small t which contradicts the infimum. It follows that $\mathcal{K}_\lambda$ is positively invariant. The inequality (3.3.19) follows from the invariance of $\mathcal{K}_\lambda$ and (3.3.16) with $k_2 = \lambda^{-1}$. □

Lemma 3.7. *There exist Lipschitz functions $h^\pm$ with the following properties:*

(a) $h^- : X^- \to X^+$ so that $W^- = graph(h^-)$ and $h^-(0) = 0$.

(b) $h^+ : X^+ \to X^-$ so that $W^+ = graph(h^+)$ and $h^+(0) = 0$.

Moreover, $W^+ \cap W^- = \{0\}$.

Proof. We prove (a) first. The construction of W^- hinges on the assumption that $\dim X^+ < \infty$. For $v_0 \in X^- \backslash \{0\}$ and for $t \geq 0$ set

$$\begin{aligned} B &:= \{w_0 \in X^+ : |w_0| \leq \mu |v_0|\} \\ G^t_{v_0} &:= \{w_0 \in B : |w(t)| \leq \mu |v(t)|\}. \end{aligned} \tag{3.3.25}$$

By Lemma 3.6, $G^t_{v_0} \subset G^s_{v_0}$ for $0 \leq s \leq t$. Note that for each t, G_t is closed and hence compact as each G_t is contained in the compact B. Hence if we can show $G_t \neq \emptyset$ for each t it would follow that

$$G^\infty_{v_0} := \bigcap_{t>0} G^t_{v_0} \neq \emptyset. \tag{3.3.26}$$

To this end consider the map $\varphi : w_0 \mapsto w(t)$ which is continuous for every t and as φ_t is locally Lipschitz in time,

$$|w(t) - w(0)| \leq C|t| \tag{3.3.27}$$

hence,

$$|w_0| - Ct \leq |\varphi_t(w_0)| \tag{3.3.28}$$

so for a fixed t we see that φ_t is proper. Using the Brouwer degree (which is admissible as X^+ is finite dimensional). We get,

$$\deg(\varphi_t, B, 0) = \deg(\varphi_0, B, 0) = 1 \neq 0 \tag{3.3.29}$$

so there exists $w_0 \in B$ such that $\varphi_t(w_0) = 0$ hence $w_0 \in G^t_{v_0}$. As reasoned above, this means that $G^\infty_{v_0} \neq \emptyset$. Now we will show G_∞ is a singleton and hence $G^\infty_{v_0}$ defines a unique function $h^- : X^- \to X^+$ by

$$\{h^-(v_0)\} = G^\infty_{v_0}. \tag{3.3.30}$$

In fact we will show h^- is a function and is Lipschitz simultaneously. We claim:

$$|w_1 - w_2| \leq \mu |v_1 - v_2| \tag{3.3.31}$$

when $w_1 \in G^\infty_{v_1}$ and $w_2 \in G^\infty_{v_2}$. Note that with $v_1 = v_2$ this proves h^- is a function. To prove this claim assume that

$$|w_1 - w_2| > \mu |v_1 - v_2| \tag{3.3.32}$$

where $w_1 \in G^\infty_{v_1}$ and $w_2 \in G^\infty_{v_2}$. Then $(v_2, w_2) \in (v_1, w_2) + \mathcal{K}_\mu$ and so by Lemma 3.6 we have

$$|w_1(t) - w_2(t)| \geq |w_1 - w_2| \exp\left((\beta - \varepsilon(1 + \mu^{-1})t\right). \tag{3.3.33}$$

As $w_j \in G^t_{v_j}$ for every t, for $j = 1, 2$ we have that

$$|v_j(t) \leq |v_j(0)| \exp\left((\alpha + \varepsilon(1 + \mu))t\right), \quad j = 1, 2. \tag{3.3.34}$$

From Lemma 3.5 we have that $|w_j| \leq \mu |v_j|$ for $j = 1, 2$. Thus, for $t \geq 0$ we have,

$$|w_2(t) - w_1(t)| \leq (|v_1| + |v_2|) \exp\left((\alpha t + \varepsilon(1 + \mu))t\right). \tag{3.3.35}$$

But 3.3.33 corresponds to exponential growth while 3.3.35 corresponds to exponential decay. This is a contradiction so the claim follows. It remains to show $W^- = \mathrm{graph}(h^-)$. To do this first we show $\mathrm{graph}(h^-) \subset W^-$. If $(v_0, h^-(v_0) \in \mathrm{graph}(h^-)$, from the construction of h^-, in particular from (3.3.34) and $|w(t)| \leq \mu |v(t)|$ we have

$$e^{\gamma t} \Phi_t(v_0, h(v_0) \to 0 \text{ as } t \to \infty \tag{3.3.36}$$

which implies $(w,v) \in W^-$. Now we want to show the opposite inclusion. This follows from the following claim: W^- is a graph, that is,

$$|\{w : (v_0, w) \in W^-\}| \leq 1 \quad \forall v_0 \in X^-. \tag{3.3.37}$$

To prove this claim suppose $(v_0, w_1) \in W^-$ and $(v_0, w_2) \in W^-$. Then $(v_0, w_2) \in (v_0, w_1) + \mathcal{K}_\lambda$ so by Lemma 3.6 we have

$$|w_2(t) - w_1(t)| \geq |w_1(0) - w_2(0)| \exp\big((\beta - \varepsilon(1 + \mu^{-1})t\big) \tag{3.3.38}$$

which implies exponential growth. But as as $(v_0, w_j) \in W^-$ for $j = 1, 2$ we also have

$$|w_2(t) - w_1(t)|e^{\gamma t} \to 0 \text{ as } t \to \infty \tag{3.3.39}$$

which corresponds to exponential decay. This is a contradiction unless $w_1 = w_2$. This finishes the proof of (a). The proof of (b) proceeds differently than that of (a). We will construct h^+ as the fixed point of a contraction map. Define

$$\mathcal{L} = \{h \in C^0(X^+, X^-) : h(0) = 0, h \in \mathrm{Lip}_{\nu^{-1}}\} \tag{3.3.40}$$

and for any $h \in \mathcal{L}$ set $\mathcal{G}_h = \mathrm{graph}(h)$. Endow $\mathcal{L}$ with the norm

$$\|h\|_{\mathcal{L}} := \sup_{w \neq 0} \frac{|h(w)|}{|w|}. \tag{3.3.41}$$

For any $h \in \mathcal{L}$ and $t \geq 0$ we make the following two claims:

$$\pi^+(\Phi_t(\mathcal{G}_h)) = X^+, \tag{3.3.42}$$

$$\Phi_t(\mathcal{G}_h) = G_{h_t}, \text{ for some unique } h_t \in \mathcal{L}. \tag{3.3.43}$$

The first claim follows from the Brouwer degree. That is, the map $w \mapsto \pi^+(\Phi_t((h(w), w)))$ is proper so by the homotopy invariance of the Brouwer degree,

$$\deg(\pi^+(\Phi_t((h(\cdot), \cdot))), X^+, w_0) = \deg(\mathrm{id}, X^+, w_0) = 1 \quad \forall w_0 \in X^+ \tag{3.3.44}$$

By properties of the Brouwer degree this shows the first claim. The second claim follows from Lemma 3.6. Indeed if $(v_j(t), w_j(t)) := \Phi_t(h(w_j), w_j)$ for $j = 1, 2$ where $w_j \in X^+$ we have

$$|h(w_1) - h(w_2)| \leq \nu^{-1}|w_1 - w_2| \tag{3.3.45}$$

and so by Lemma 3.6

$$|v_1(t) - v_2(t)| \leq \nu^{-1}|w_1(t) - w_2(t)|. \tag{3.3.46}$$

Hence $w_1(t) = w_2(t) \implies w_1 = w_2$. This shows

$$\pi^+(\Phi_t(h(w), w)) \mapsto \pi^-(\Phi_t(h(w), w)) \tag{3.3.47}$$

is a well defined ν^{-1}-Lipschitz map and is the desired map. Denote the map $t \mapsto h_t$ by $T_t : \mathcal{L} \to \mathcal{L}$. We claim:

$$T_t \text{ is a contraction for large } t. \tag{3.3.48}$$

Fix t and take $h_1, h_2 \in \mathcal{L}, w \in X^+$. To prove (3.3.48) we want to estimate

$$\frac{|T_t(h_1)(w) - T_t(h_2)(w)|}{|w|}. \tag{3.3.49}$$

Choose $w_j \in X^+$ with

$$(T_t(h_j)(w), w) = \Phi_t(h_j(w_j), w_j), j = 1, 2. \tag{3.3.50}$$

First we estimate the denominator of 3.3.49. As $(h(w_j), w_j) \in \mathcal{K}_\nu$ for $j = 1, 2$ by Lemma 3.6 we have

$$|w| \geq |w_j| \exp\left((\beta - \varepsilon(1+\nu^{-1}))t\right) \quad , j = 1, 2. \tag{3.3.51}$$

Now we estimate the numerator of 3.3.49. Assume that $T_t(h_1) \neq T_t(h_2)$ (as if not then 3.3.49 is just 0). Then

$$(T_t(h_1)(w), w) \notin (T_t(h_2)(w), w) + \mathcal{K}_\mu \tag{3.3.52}$$

and so by Lemma 3.6

$$\Phi_s(h_2(w_2), w_2) \notin \Phi_s(h_1(w_1), w_1) + \mathcal{K}_\mu, \quad \forall 0 \leq s \leq t. \tag{3.3.53}$$

Hence by (3.3.18) we have

$$|T_t(h_2)(w) - T_t(h_1)(w)| \leq |h_2(w_2) - h_1(w_2)| \exp\left((\alpha + \varepsilon(1+\mu))t\right). \tag{3.3.54}$$

But also using that $h_2 \in \mathrm{Lip}_{\nu^{-1}}$ and then (3.3.53) with $s = 0$,

$$\begin{aligned} |h_2(w_2) - h_1(w_1)| &\leq |h_2(w_2) - h_2(w_1)| + |h_2(w_1) - h_1(w_1)| \\ &\leq \nu^{-1}|w_2 - w_1| + |h_2(w_1) - h_1(w_1)| \\ &\leq \nu^{-1}\mu|h_2(w_2) - h_2(w_1)| + |h_2(w_1) - h_1(w_1)|. \end{aligned} \tag{3.3.55}$$

As $\nu^{-1}\mu < 1$ we can rearrange this,

$$|h_2(w_2) - h_1(w_1)| \leq (1 - \mu\nu^{-1})^{-1}|h_2(w_1) - h_1(w_1)|. \tag{3.3.56}$$

With 3.3.54 in mind this gives a numerator estimate for 3.3.49,

$$|T_t(h_2)(w) - T_t(h_1)(w)| \leq (1 - \mu\nu^{-1})^{-1}|h_2(w_1) - h_1(w_1)| \exp\left((\alpha + \varepsilon(1+\mu))t\right). \tag{3.3.57}$$

Combing the denominator estimate, (3.3.51) and the numerator estimate, (3.3.57) gives

$$\frac{|T_t(h_1)(w) - T_t(h_2)(w)|}{|w|} \leq (1 - \mu\nu^{-1})^{-1}\frac{|h_2(w_1) - h_1(w_1)|}{|w_1|} \exp\left((\alpha - \beta + \varepsilon(2 + \mu + \nu^{-1}))t\right). \tag{3.3.58}$$

Noting that

$$\frac{|h_2(w_1) - h_1(w_1)|}{|w_1|} \leq \|h_2 - h_1\|_{\mathcal{L}} \tag{3.3.59}$$

and taking a supremum over $w \in X^+\backslash\{0\}$ we get

$$\|T_t(h_1)(w) - T_t(h_2)(w)\|_{\mathcal{L}} \leq (1 - \mu\nu^{-1})^{-1}\|h_2 - h_1\|_{\mathcal{L}} \exp\left((\alpha - \beta + \varepsilon(2 + \mu + \nu^{-1}))t\right). \tag{3.3.60}$$

This shows that T_t is a contraction for large enough t and so by the contraction mapping theorem has a unique fixed point. For large enough t let h_t^* be the fixed point of T_t. Note that for t, τ large

$$T_t(T_\tau(h_t^*) = T_\tau(T_t(h_t^*) = T_\tau(h_t^*) \tag{3.3.61}$$

and so by the uniqueness of h_t^*, $h_t^* = h_\tau^*$. Let h^+ denote the common unique fixed point of T_t for all t large enough. To complete the proof we need to show that graph$(h^*) = W^+$. First we show graph$(h^*) \subset W^+$. Note that as $h^+ \in \mathcal{L}$, graph$(h^+) \subset \mathcal{K}_\nu$. If $u = (v, w) \in$ graph(h^+) then we set $(v(t), w(t)) := \Phi_t(u)$. By graph$(h^+) \subset \mathcal{K}_\nu$ and Lemma (3.6) we have

$$|w(t)|e^{\gamma t} \leq |w| \exp\left((\beta - \varepsilon(1+\nu^{-1}) + \gamma)t\right) \to 0 \text{ as } t \to -\infty. \tag{3.3.62}$$

As graph$(h_+) = \Phi_t($graph$(h_+))$ by (3.3.43) and graph$(h^+) \subset \mathcal{K}_\nu$ we also have $|v(t)| \leq \nu^{-1}|w(t)|$. This with (3.3.62) gives $e^{\gamma t}\Phi_t(u) \to 0$ as $t \to 0$. This shows graph$(h^*) \subset W^+$. It remains to show $W^+ \subset$ graph(h^*). This follows from the fact that $W^+$is a graph. The proof of this fact is similar to the proof that W^- is a graph in (a). Finally we need to show $W^+ \cap W^- = \{0\}$. This follows from the following two inclusions from the construction of h^- and h^+:

$$W^+ \subset \mathcal{K}_\nu \quad \text{and} \quad W^- \subset X^- \times X^+\backslash\mathcal{K}_\nu.$$

□

The previous lemma will be used to construct the invariant Lipschitz manifolds as in Theorem 3.2. To show tangency we will need the following basic facts about Frechet derivatives.

Lemma 3.8. *Suppose $h : X \to Y$ where X and Y are Banach spaces.*

(i) If $\lim_{x \to 0} \frac{\|h(x)\|}{\|x\|} = 0$ then h is Frechet differentiable at 0 and $dh(0) = 0$.

(ii) If h is Lipschitz and $dh(0) = 0$ then graph$(h) \subset X \times Y$ is a Lipschitz submanifold of $X \times Y$ and $T_{(0,h(0))}$graph$(h) = X$.

The importance of the previous lemma is that to prove $W^{\pm}$ is tangent to $X^{\pm}$ it suffices to show $dh^{\pm}(0) = 0$.

Lemma 3.9. *We have the following:*

$$\mathcal{L}_0 := \{h \in \mathcal{L} : \ h \text{ is differentiable at 0 and } Dh(0) = 0\} \tag{3.3.63}$$

is a closed set in $\mathcal{L}$.

The proof of the previous lemma is straightforward and so is omitted.

Lemma 3.10. *Assume that g is differentiable at 0 and that $Dg(0) = 0$. Then h^+ and h^- as constructed in the previous proof satisfy $dh^{\pm}(0) = 0$.*

Proof. The proof of this Lemma is just sketched for h^+. The idea is to show that T_t as defined in the proof of Lemma 3.7 satisfies

$$T_t : \mathcal{L}_0 \to \mathcal{L}_0 \text{ for all } t > 0. \tag{3.3.64}$$

One can show that T_t has a unique fixed point on $\mathcal{L}_0$ and by the uniqueness of fixed point this must in fact be h^+, the fixed point of T_t on $\mathcal{L}$. □

We are now in a position to prove Theorem 3.2. As mentioned previously, the general idea in this proof is to reduce 3.2.1 under hypothesis H to 3.3.1 under hypothesis J. We can then apply Lemma 3.7 to construct the various invariant manifolds.

Proof. (Proof of Theorem 3.2). Consider (3.2.1) under hypothesis H. Then for $\varepsilon > 0$ small enough there exists a set U_ε such that Lip$(f|_{U_\varepsilon}) = \varepsilon$. Define a function $g : X \to X$ such that $g = f$ on U_ε and Lip$(g) = \varepsilon$. We now construct center unstable and stable manifolds. One can show that hypothesis J are satisfied with the invariant A-splitting of X being $X^+ = X^c \oplus X^u$ and $X^- = X^s$ and the parameters α, β being chosen such that

$$0 < \alpha < \beta < \inf\{\text{Re}z : z \in \sigma^u(A)\}. \tag{3.3.65}$$

By Lemma 3.7 and Lemma 3.10, W^+ and W^- are Lipschitz manifolds and are tangent to X^+ and X^- at 0. We claim that $W^s(U) := W^- \cap U$ where U is a sufficiently small neighborhood of 0. By construction $W^- \subset X \backslash \mathcal{K}_\nu \cup \{0\}$ which shows $W^- \subset W^u$. We also claim that $W^{cu} = W^+ \cap U$ is a center unstable manifold. The invariance of $W^+ \cap U$ relative to U follows from the invariance of W^+. From the construction of h^+ and (3.3.42) we have $\pi^+(W^+) = X^-$ and so $\pi^{cs}(W^- \cap U)$ is a neighborhood of 0. Further from Lemma 3.7 $W^+ \cap W^s = W^+ \cap W^- = \{0\}$. The previous 3 facts show that $W^+ \cap U$ is a center stable manifold.

One can also show that hypothesis J are satisfied with the invariant A-splitting of X being $X^+ = X^u$ and $X^- = X^s \oplus X^c$. By Lemma 3.7 and Lemma 3.10 W^+ and W^- are Lipschitz manifolds and are tangent to X^- and X^+ at 0. Using a similar argument to the above we can show $W^s(U) := W^- \cap U$ where U is a sufficiently small neighborhood of 0 and that $W^{cu} := W^- \cap U$ is a center unstable manifold.

We now construct a center manifold. Denote by $h^+ : X^c \oplus X^s \to X^u$ and $h^- : X^{\oplus} X^u \to X^s$ the functions whose graphs are W^u and W^s respectively. These functions are obtained by applying Lemma 3.7 to the different A-invariant splitting of X as done above. For a fixed $x \in X^c$ define $F(y,z) = (h^-(x+z), h^+(x+y))$ By construction h^- is $\mu-$Lipschitz, h^+ is ν^{-1}-Lipschitz and $\mu, \nu^{-1} < 1$. Hence, F is a contraction and so has a unique fixed point. This defines a function $h^c : X^c \to X^s \oplus X^u$. From the construction of h^c is is easy to see that h^s is Lipschitz. We let $W^c := \text{graph}(h^c) \cap U$. By construction $W^c = W^{cu} \cap W^{cs}$ and (P1) holds. One can verify that W^c satisfies the other desired properties. □

3.4 Spectral Properties of L_+

Before we apply Theorem 3.2 to 2.1.1 we need to know information about the spectrum of L_+.

Lemma 3.11. *As an operator in L^2_{rad}, L_+ has only one negative eiqenvalue, which is non-degenerate, and no eigenvalue at 0 or in the continuous spectrum $[1, \infty)$.*

Lemma 3.12. *We have*

$$\sigma(L_+) = \{-k^2\} \cup [1, \infty) \tag{3.4.1}$$

where $-k^2$ is the unique negative eigenvalue of L_+.

3.5 Application to the NLKG Equation

To apply the Bates and Jones construction to NLKG, first we need to write NLKG in the form of 3.2.1. We can write 3.1.1 as a first order system

$$\partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -L_+ & 0 \end{pmatrix} \begin{pmatrix} v \\ \partial_t v \end{pmatrix} + \begin{pmatrix} 0 \\ 3Qv^2 + v^3 \end{pmatrix}, \tag{3.5.1}$$

We set

$$A = \begin{pmatrix} 0 & 1 \\ -L_+ & 0 \end{pmatrix} \tag{3.5.2}$$

and

$$f \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3Qu_1^2 + u_1^3 \end{pmatrix} \tag{3.5.3}$$

and attempt to verify the hypothesis of Theorem 3.2. Hypothesis (H1) is obvious with $X = \mathcal{H}_{\text{rad}}$. Hypothesis (H5) can be verified using the Sobolev embedding $H^1 \hookrightarrow L^6$.

Lemma 3.13. *With f as defined above f is defined on all of $\mathcal{H}_{rad}$ and is locally Lipschitz with $f(0) = 0$. For all $\varepsilon > 0$ there exists a neighborhood $U = U(\varepsilon)$ of 0 such that f has Lipschitz constant at most ε on U.*

Proof. For $u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{H}_{\text{rad}}$ we have

$$\begin{aligned} \|f(u) - f(v)\|_{\mathcal{H}} &= \|3Q(v_1^2 - w_1) + (v_1^3 - w_1^3)\|_{L^2} \\ &\lesssim \left(\|v_1\|_{L^6} + \|w_1\|_{L^6} + \|v_1\|_{L^6}^2 + \|w_1\|_{L^6}^2\right) \|v_1 - w_1\|_{L^6} \\ &\lesssim \left(\|v\|_{\mathcal{H}} + \|w\|_{\mathcal{H}} + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2\right) \|v - w\|_{\mathcal{H}} \end{aligned} \tag{3.5.4}$$

hence $\|f(u) - f(v)\|_{\mathcal{H}} \leq C(R + R^2)\|v - w\|_{\mathcal{H}}$ if $u, v \in B_R(0) \subset \mathcal{H}$. □

Now we determine the spectrum of A.

Lemma 3.14. *For* Re $z > 0$ *the resolvent of A on $L^2_{\text{rad}} \times L^2_{\text{rad}}$ is*

$$(A - z)^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -(L_+ + z^2)^{-1}(zf + g) \\ f - z(L_+ + z^2)^{-1}(zf + g) \end{pmatrix} \tag{3.5.5}$$

for all $(f, g) \in L^2_{\text{rad}} \times L^2_{\text{rad}}$. Hence

$$\sigma(A) = \{z \in \mathbb{C} : z^2 \in -\sigma(L_+)\} = \{\pm k\} \cup i[1, \infty) \cup i(-\infty, -1] \tag{3.5.6}$$

where $-k^2$ is the negative eigenvalue of L_+.

Proof. Simply using algebra to rearrange

$$\begin{pmatrix} 0 & 1 \\ -L_+ & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - z \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{3.5.7}$$

gives the result. □

The previous theorem verifies (H3). It remains to verify (H2) and (H4). Spectral theory is required to do this. We refer the reader to the appendix for relevant spectral theory results.

Lemma 3.15. *There exists $m > 0$ such that*

(i) $A - mId$ is dissipative. That is,

$$\left\langle (A - mId)\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right), \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\right\rangle_{\mathcal{H}} \leq 0 \quad \forall \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathcal{H}. \tag{3.5.8}$$

(ii) The range of $(\lambda + m)Id - A$ equals $\mathcal{H}$ for all $\lambda > 0$,

(iii) A generates a semigroup $S(t)$ on $t \geq 0$ with $\|S(t)\| \leq e^{mt}$, and $S(t)$ extends to a group on $\mathbb{R}$ with $\|S(t)\| \leq e^{m|t|}$ for all $t \in \mathbb{R}$.

Proof. We first prove (i). This follows from integration by parts,

$$\begin{aligned} \langle A\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right), \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\rangle_{\mathcal{H}} &= \langle g, f\rangle_{H^1} + \langle -L_+ f, g\rangle_{L^2}\rangle \\ &= \langle \nabla g, \nabla f\rangle_{L^2} + \langle g, f\rangle_{L^2} + \langle (\Delta - 1)f, g\rangle_{L^2} + \langle 3Q^2 f, g\rangle_{L^2} \\ &= \langle 3Q^2 f, g\rangle_{L^2} \\ &\leq 3\|Q\|_{L^\infty}^2 \langle f, g\rangle_{L^2} \\ &\leq 3\|Q\|_{L^\infty}^2 \left(\|f\|_{H^1}^2 + \|g\|_{L^2}\right) \\ &= 3\|Q\|_{L^\infty}^2 \langle \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right), \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\rangle_{\mathcal{H}}. \end{aligned} \tag{3.5.9}$$

So (i) holds if $m \geq 3\|Q\|^2_{L^\infty}$. For (ii) note that

$$\sigma(A) = \{\pm k\} \cup i[1,\infty) \cup i(-\infty,-1] \tag{3.5.10}$$

and so if m is large enough $\lambda + m \notin \sigma(A)$ for all $\lambda > 0$. This means $((\lambda+m)\mathrm{Id} - A)^{-1}$ exists and so the range of $((\lambda+m)\mathrm{Id} - A)$ is $\mathcal{H}$ for all $\lambda > 0$. For (iii), by Theorem 7.1 and the previous two facts $A - m\mathrm{Id}$ generates a contraction semigroup $\{T(t)\}_{t\geq 0}$. As $\{e^{mt\mathrm{Id}}\}_{t\geq 0}$ is the semigroup generated by $m\mathrm{Id}$, A generates the semigroup $\{S(t)\}_{t\geq 0}$ with $S(t) = T(t)e^{mt\mathrm{Id}}$. Hence

$$\|S(t)\| \leq \|T(t)\| \cdot \|e^{mt\mathrm{Id}}\| \leq e^{mt}, \quad \forall t \geq 0. \tag{3.5.11}$$

By reversing time $\{S(t)\}_{t\geq 0}$ extends to a group on $\mathbb{R}$. □

As $\mathrm{Dom}(A) = H^2 \times L^2$, A is densely defined on $\mathcal{H}$ the above result proves (H2). We denote

$$S^c(t) = S(t)|_{X^c}. \tag{3.5.12}$$

To verify (H4) we need the following Lemma.

Lemma 3.16. *Let $S_0(t)$ be the group generated by*

$$A_0 = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix} \tag{3.5.13}$$

on $\mathcal{H}$. Then for every $t \in \mathbb{R}$ $S(t) - S_0(t)$ is compact. In particular, $r(S^c(t)) = 1$ for every $t \in \mathbb{R}$ where r is the spectral radius.

Proof. This proof involves using spectral theory to analyze the spectrum of $S^c(t)$. In particular Weyl's Theorem, Theorem 7.2, and Theorem 7.3 which can be found in the appendix of this report. We will briefly sketch the proof and refer the reader to [13] for details. From Duhamel's formula

$$S(t)\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) - S_0(t)\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) = \int_0^t S_0(t-s)KS(s)\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) ds \quad \text{where} \quad K = -3\begin{pmatrix} 0 & 0 \\ Q^2 & 0 \end{pmatrix}. \tag{3.5.14}$$

From Lemma 2.5 multiplication by Q^2 is a compact operator from $H^1_{\mathrm{rad}} \to L^2$. Hence K is a compact operator. Thus $S(t) - S_0(t)$ is compact by the above equation. By the conservation of energy for the linear Klein-Gordon equation, $S_0(t)$ is a unitary operator every t. It follows that

$$\sigma(S_0(t)) = \sigma_{\mathrm{ess}}(S_0(t)) = \mathbb{S}^1. \tag{3.5.15}$$

One can verify that the conditions in Weyl's Theorem are true. Hence

$$\sigma_{\mathrm{ess}}(S(t)) = \sigma_{\mathrm{ess}}(S_0(t)) = \mathbb{S}^1. \tag{3.5.16}$$

Suppose for a contradiction that $S(t)$ has spectrum outside $\mathbb{S}^1$. One can show spectrum of $S(t)$ outside of $\mathbb{S}$ must be discrete. Hence by Lemma 7.3 in the appendix this implies that the generator of $S^c(t)$, A^c, needs to have discrete spectrum in $\{z \in \mathbb{C} : \mathrm{Re} z \geq 0\}$. This is a contradiction and so $r(S^c(t) = 1$.

□

From the above Lemma and the spectral radius formula we have

$$\lim_{n\to\infty} \|S^c(t)^n\|^{\frac{1}{n}} = r(S^c(t)) = 1. \tag{3.5.17}$$

Hence given any $t > 0$ and $\varepsilon > 0$ one can find a positive integer N such that for $n \geq N$,

$$\|S^c(t)^n\|^{\frac{1}{n}} \leq 1 + t\varepsilon \implies \|S^c(tn)\| \leq (1 + t\varepsilon)^n \leq e^{\varepsilon t n}. \tag{3.5.18}$$

This shows that (H4) holds. All of the hypothesis of Theorem 3.2 have been verified. We now have the following information dynamical behavior close to the ground state.

Theorem 3.17. *In a small neighborhood of $(Q, 0)$ in $H^1_{rad} \times L^2_{rad}$ there exists one-dimensional stable and unstable Lipschitz manifolds, as well as a center manifold of co-dimension two for NLKG. These manifolds have X^s, X^u and X^c respectively as tangent spaces at $(Q, 0)$. Moreover there exist center-stable and center-unstable manifolds for with X^{cs} and X^{cu} as tangent spaces at $(Q.0)$ respectively. Properties (P1) and (P2) hold as in the statement of Theorem 3.2*

In fact with more effort one can prove the following result which shows the importance of the center manifold in understanding the dynamics of NLKG around $(Q, 0)$.

Theorem 3.18. *Restricted to the local center manifold W^c around $(Q, 0)$ the flow of NLKG is orbitally stable as $t \to \pm\infty$ in the following sense: There exists a neighborhood U of $(Q, 0)$ with the property that a solution staring in U remains in U for all times (both positive and negative) if and only if the solution intersects $W^c \cap U$ (in which case it will lie entirely in $W^c \cap U$.*

3.6 Alternative Construction of Invariant Manifolds

This section states a result that comes from an alternative method of constructing invariant manifolds. The upside of this new method is that it provides more information. The downside is that its proof requires more information, such as Strichartz estimates for NLKG. One important property of L_+ is the following:

Theorem 3.19. *L_+ has no eigenvalue in $(0, 1]$ and no resonance at the threshold 1.*

We call the previous property of L_+ the gap property. Giving a precise definition of the terms in the statement of the previous Theorem and attempting to prove the Theorem goes beyond the scope of this report and neither are pursued here. It is important to note however that the gap property of L_+ is crucial in proving the scattering statement in the coming Theorem.

We decompose any solution u to (2.1.1) by putting

$$u(t) = Q + v(t), \quad v(t) = \lambda(t)\rho + \gamma(t), \quad \gamma \perp \rho. \tag{3.6.1}$$

Recall that $-k^2$ is the unique negative eigenvalue of L_+ with associated L^2 normalized eigenvalue ρ. From (2.1.1) we obtain the following system of equations for $\lambda, \gamma \in \mathbb{R} \times P^{\perp}_{\rho}(H^1)$.

$$\begin{cases} \ddot{\lambda} - k^2\lambda = N_\rho(v), \\ \ddot{\gamma} + \omega^2\gamma = P^{\perp}_{\rho} N(v) \end{cases} \tag{3.6.2}$$

where $N(v) = Qv^2 + v^3$, and $N_\rho(v) = \langle N(v), \rho\rangle_{L^2}$.

Theorem 3.20. *Assume that the gap property for L_+ holds. Then there exists $\nu > 0$ small and a smooth graph $\mathcal{M}$ in $B_\nu(Q,0) \subset \mathcal{H}_{\rm rad}$ so that $(Q,0) \in \mathcal{M}$, with tangent plane*

$$T_Q\mathcal{M} = \{(v_0, v_1) \in \mathcal{H} : \langle kv_0 + v_1, \rho\rangle = 0\} \tag{3.6.3}$$

at $(Q,0)$ in the sense that

$$\sup_{(Q+v_0,v_1)\in\partial B_\delta(Q,0)\cap\mathcal{M}} |\langle kv_0 + v_1, \rho\rangle| \lesssim \delta^2, \quad \forall 0 < \delta < \nu. \tag{3.6.4}$$

Any data $(u_0, u_1) \in \mathcal{M}$ lead to global solutions of (2.1.1) of the form $u = Q + v$ where v satisfies

$$\|(v, \dot{v})\|_{L_t^\infty((0,\infty);\mathcal{H})} + \|v\|_{L^3((0,\infty);L^6)} \lesssim \nu \tag{3.6.5}$$

and scatter to a free Klein-Gordon solution. That is there exists a unique free Klein-Gordon solution γ_∞ such that

$$|\lambda(t)| + |\dot{\lambda}(t)| + \|\vec{\gamma}(t) - \vec{\gamma}_\infty(t)\|_{\mathcal{H}} \to 0 \text{ as } t \to \infty. \tag{3.6.6}$$

In particular we have

$$E(\vec{u}) = J(Q) + \frac{1}{2}\|\vec{\gamma}_\infty\|_{\mathcal{H}}^2. \tag{3.6.7}$$

Finally, any solution that remains inside $B_\nu(Q,0)$ for all $t \geq 0$ necessarily lies entirely on $\mathcal{M}$, and $\mathcal{M}$ is invariant under the flow of (2.1.1) for all $t \geq 0$.

The above theorem gives an alternative construction of the stable manifold around $(Q,0)$. The unstable manifold can be constructed by time reversal.

Corollary 3.21. *Let ν be as in Theorem 3.20. Then any $(u_0, u_1) \in B_\nu(Q,0)$ with the property that the associated solution $u(t)$ of (2.1.1) satisfies*

$$(u(t), \partial_t u(t)) \to (Q,0) \text{ as } t \to \infty \tag{3.6.8}$$

belongs to a one dimensional smooth manifold W^s in B_ν which is tangent to the line

$$\{(Q,0) + \lambda(\rho, -k\rho) : \lambda \in \mathbb{R}\} \tag{3.6.9}$$

at $(Q,0)$. Moreover, $W^s\backslash\{(Q,0)\}$ consists of two distinct trajectories of solutions to (2.1.1) which approach $(Q,0)$ exponentially fast. Any two solutions belonging to one of these two halves differ by only time translation.

Chapter 4

The Nonlinear Distance Function and Moving Away From the Ground State

4.1 The Nonlinear Distance Function

In this chapter we state but do not prove some important results on the nonlinear distance function which are used in Chapter Five of this report. The nonlinear distance function is important in understanding the global dynamics of NLKG. Let

$$u = \sigma[Q + v], \quad v = \lambda\rho + \gamma, \quad \gamma \perp \rho \tag{4.1.1}$$

for $\sigma = \pm 1$. The choice of σ determines which ground state we are decomposing u about. We define the linearized energy as

$$\|\vec{v}\|_E^2 = \frac{1}{2}\left[k^2\langle v, \rho\rangle + \|\omega P_\rho^\perp v\|_{L^2}^2 + \|\dot{v}\|_{L^2}^2\right] \tag{4.1.2}$$

where $-k^2$ is the sole negative eigenvalue of L_+. Then

$$E(\vec{u}) - J(Q) + k^2\lambda^2 = \|\vec{v}\|_E^2 - C(v), \quad C(v) = \langle Q, v^3\rangle_{L^2} + \frac{1}{4}\|v\|_{L^4}^4. \tag{4.1.3}$$

On can show that

$$\|\vec{v}\|_E^2 \simeq \|v\|_{\mathcal{H}}^2. \tag{4.1.4}$$

There exists $\delta_E \ll 1$ such that

$$\|v\|_E \leq 4\delta_E \implies |C(v)| \leq \frac{1}{2}\|v\|_E^2. \tag{4.1.5}$$

Let χ be a smooth cut off function on $\mathbb{R}$ such that $\chi(r) = 1$ for $|r| \leq 1$ and $\chi(r) = 0$ for $|r| \geq 2$. We define

$$d_\sigma(\vec{u}) = \sqrt{|v|_E^2 - \chi(\|\vec{v}\|_E^2/(2\delta_E)C(v)}. \tag{4.1.6}$$

Some important facts about d_σ are as follows.

$$\|\vec{v}\|_E/2 \leq d_\sigma(\vec{u}) \leq 2\|\vec{v}\|_E, \quad d_\sigma(\vec{u}) = \|\vec{v}\|_E + O(\|\vec{v}\|_E^2, \tag{4.1.7}$$

$$d_\sigma(\vec{u}) \leq \delta_E \implies d_\sigma^2(\vec{u}) = E(\vec{u}) - J(Q) + k^2\lambda^2. \tag{4.1.8}$$

We set

$$d_Q(\vec{u}) = \inf_{\sigma=\pm1} d_\sigma(\vec{u}). \tag{4.1.9}$$

It is sometimes helpful to set

$$\lambda_+ = \frac{1}{2}(\lambda + \frac{1}{k}\dot{\lambda}), \quad \lambda_- = \frac{1}{2}(\lambda - \frac{1}{k}\dot{\lambda}) \tag{4.1.10}$$

so we have the following decomposition of λ and $\dot{\lambda}$.

$$\lambda = \lambda_+ + \lambda_-, \quad \dot{\lambda} = k(\lambda_+ - \lambda_-). \tag{4.1.11}$$

4.2 Moving Away from the Ground State: Important Results

In this section we list important results related to the nonlinear distance function. For proofs of the following results see [13].

Lemma 4.1. *(Eigenmode dominance) For any $\vec{u} \in \mathcal{H}$ satisfying*

$$E(\vec{u})) < J(Q) + d_Q(\vec{u})/2, \quad d_Q(\vec{u}) \leq \delta_E, \tag{4.2.1}$$

one has $d_Q(\vec{u}) \simeq |\lambda|$. In particular, λ has a fixed sign on each connected component of the region (4.2.1).

Lemma 4.2. *(Ejection lemma) There exists a constant $0 < \delta_X \leq \delta_E$ with the following property. Let $u(t)$ be a local solution on maximal interval of existence $[0, T)$ satisfying*

$$R := d_Q(\vec{u}(0)) \leq \delta_X, \quad E(\vec{u}) < J(Q) + R^2/2 \tag{4.2.2}$$

and for some $t_0 \in (0, T)$,

$$d_Q(\vec{u}(t)) \geq R \quad (0 < \forall t < t_0). \tag{4.2.3}$$

Alternatively, assume that $\frac{d}{dt} d_Q(\vec{u}(t))|_{t=0} \geq 0$. Then $d_Q(\vec{u}(t))$ increases monotonically until reaching δ_X and meanwhile,

$$d_Q(\vec{u}(t)) \simeq -\mathfrak{s}\lambda(t) \simeq -\mathfrak{s}\lambda_+(t) \simeq e^{kt}R, \tag{4.2.4}$$

$$|\lambda_-(t) + \|\vec{\gamma}(t)\|_E \lesssim R + d_Q^2(\vec{u}(t)), \tag{4.2.5}$$

$$\min_{s=0,2} \mathfrak{s}K_s(u(t)) \gtrsim d_Q(\vec{u}(t)) - C_* d_Q(\vec{u}(0)), \tag{4.2.6}$$

with a fixed sign $\mathfrak{s} = +1$ or $\mathfrak{s} = -1$, where $C_* \geq 1$ is an absolute constant.

Lemma 4.3. *For any $\delta > 0$, there exists $\varepsilon_0(\delta), \kappa_0, \kappa_1(\delta) > 0$ such that for any $\vec{u} \in \mathcal{H}$ satisfying*

$$E(\vec{u}) < J(Q) + \varepsilon_0^2(\delta), \quad d_Q(\vec{u}) \geq \delta, \tag{4.2.7}$$

one has either

$$K_0(u) \leq -\kappa_1(\delta) \quad \textit{and} \quad K_2(u) \leq -\kappa_1(\delta), \tag{4.2.8}$$

or

$$K_0(u) \geq \min(\kappa_1(\delta), \kappa_0\|u\|_{H^1}^2) \quad \textit{and} \quad K_2(u) \geq \min(\kappa_1(\delta), \kappa_0\|\nabla u\|_{L^2}^2). \tag{4.2.9}$$

Lemma 4.4. *Let $\delta_S := \delta_X/(2C_*) > 0$ where δ_X and $C_* \geq 1$ are constant from Lemma 4.2. Let $0 \leq \delta \leq \delta_S$ and*

$$\mathcal{H}_{(\delta)} = \{\vec{u} \in \mathcal{H} : E(\vec{u}) < J(Q) + \min(d_Q^2(\vec{u}), \varepsilon_0^2(\delta))\}, \tag{4.2.10}$$

where $\varepsilon_0(\delta)$ is given by Lemma 4.3. Then there exists a unique continuous function $\mathfrak{S} : \mathcal{H}_{(\delta)} \to \{\pm 1\}$ satisfying

$$\begin{cases} \vec{u} \in \mathcal{H}_{(\delta)}, \;\; d_Q(\vec{u}) \leq \delta_E & \Longrightarrow \;\; \mathfrak{S}(\vec{u}) = -\mathrm{sign}\lambda, \\ \vec{u} \in \mathcal{H}_{(\delta)}, \;\; d_Q(\vec{u}) \geq \delta & \Longrightarrow \;\; \mathfrak{S}(\vec{u}) = -\mathrm{sign}K_0(u) = -\mathrm{sign}K_2(u), \end{cases} \tag{4.2.11}$$

where we set $\mathrm{sign}0 = +1$.

Lemma 4.5. *There exists a constant $M_* \simeq J(Q)^{1/2}$ such that for any $\vec{u} \in \mathcal{H}_{(\delta_S)}$ satisfying $\mathfrak{S}(\vec{u}) = +1$ we have $\|\vec{u}\|_{\mathcal{H}} \leq M_*$.*

Lemma 4.6. *For any $M > 0$, there exists $\mu_0(M) > 0$ with the following property. Let $u(t)$ be a finite energy solution of NLKG on $[0, 2]$ satisfying*

$$\|\vec{u}\|_{L_t^\infty((0,2);\mathcal{H})} \leq M, \qquad \int_0^2 \|\nabla u(t)\|_{L^2}^2 \, dt \leq \mu^2 \tag{4.2.12}$$

for some $\mu \in (0, \mu_0]$. Then u extends to a global solution and scatters to 0 as $t \to \pm\infty$, and moreover $\|u\|_{L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^3)} \ll \mu^2$.

Using the constants in the previous lemmas we choose $\varepsilon_*, \delta_*, R_*, \mu > 0$ such that

$$\delta_* \leq \delta_S, \quad \delta_* \ll \delta_X, \quad \varepsilon_* \leq \varepsilon_0(R_*), \quad \mu < \mu_0(M_*) \tag{4.2.13}$$

and

$$\varepsilon_* \ll R_* \ll \min(\delta_*, \kappa_1(\delta_*)^{\frac{1}{2}}, \kappa_0^{\frac{1}{2}}\mu, J(Q)^{\frac{1}{2}}). \tag{4.2.14}$$

Theorem 4.7. *(One pass theorem) Let $\varepsilon_*, R_* > 0$ be small constants as specified above. If a solution to NLKG on an interval I satisfies for some $\varepsilon \in (0, \varepsilon_*]$, $R \in (2\varepsilon, R_*]$, and $\tau_1 < \tau_2 \in I$,*

$$E(\vec{u}) < J(Q) + \varepsilon^2, \quad d_Q(\vec{u}(\tau_1)) < R = d_Q(\vec{u}(\tau_2)), \tag{4.2.15}$$

then for all $t \in (\tau_2, \infty) \cap I =: I'$ we have $d_Q(\vec{u}) \geq R$.

Chapter 5

Classification of Global NLKG Dynamics

5.1 Main Result

We now prove a classification result concerning the global behavior of solutions with energy at most slightly above that of the ground state. That is solutions in the set

$$\mathcal{H}^{\varepsilon} := \{\vec{u} \in \mathcal{H} : E(\vec{u} < E(Q,0) + \varepsilon^2\}. \tag{5.1.1}$$

where ε is sufficiently small. The proof of this classification brings together results and ideas proved or stated in previous chapters of this report. Few new ideas are needed. We now state the main classification result.

Theorem 5.1. *Consider all solutions of NLKG (2.1.1) with radial initial data $\vec{u}(0) \in \mathcal{H}^{\varepsilon}$ for some small $\varepsilon > 0$. The set of all these solutions splits into nine nonempty sets characterized as*

(1) Scattering to 0 for both $t \to \pm\infty$,

(2) Finite time blow up on both sides $\pm t > 0$,

(3) Scattering to 0 as $t \to \infty$ and finite time blow up in $t < 0$,

(4) Finite time blow up in $t > 0$ and scattering to 0 as $t \to -\infty$,

(5) Trapped by $\pm Q$ for $t \to \infty$ and scattering to 0 as $t \to -\infty$,

(6) Scattering to 0 as $t \to \infty$ and trapped by $\pm Q$ as $t \to \infty$,

(7) Trapped by $\pm Q$ for $t \to \infty$ and finite time blow up in $t < 0$,

(8) Finite time blow up in $t > 0$ and trapped by $\pm Q$ as $t \to -\infty$,

(9) Trapped by $\pm Q$ as $t \to \infty$.

where 'trapped by $\pm Q$" means that the solution stays in a $O(\varepsilon)$ neighborhood of $\pm Q$ forever after some time (or before some time). The initial data for the sets (1)-(4), respectively, are open in $\mathcal{H}$.

To prove this nonchotomy we will show that in each time direction there are 3 only 3 possible behaviors:

1. scattering to 0,
2. finite time blowup,
3. trapped by $\pm Q$.

Here, *trapped by* $\pm Q$ *for* $\sigma > 0$ where $\sigma = \pm 1$ means there exists $T > 0$ such that for all $\sigma t \geq T$ one has

$$d_Q(\vec{u}(t)) \leq 2\varepsilon. \tag{5.1.2}$$

The three distinct behaviors in each time direction give the 9 sets as in the above Theorem. Before we prove this result we first state a Lemma that summarizes much of the previous chapter.

Lemma 5.2. *Let $\vec{u}$ be a solution to NLKG with $E(\vec{u}) < J(Q) + \varepsilon(R_*)$ that is not trapped by the ground state. Then eventually $d_Q(\vec{u}(t)) \geq R_*$ and eventually $\mathrm{sign}K_0(u(t)) = \mathrm{sign}K_2(u(t))$ is constant.*

Proof. If $\vec{u}(t)$ is not trapped by the ground state then by the the One Pass Theorem (Theorem 4.7) eventually $d_Q(\vec{u}(t)) \geq R_*$. Then by Lemma 4.3 $\mathrm{sign}K_0(u(t)) = \mathrm{sign}K_2(u(t))$ is constant as it cannot change unless it enters the region $d_Q(\vec{u}(t)) < R_*$. □

The previous Lemma shows that we are now in a situation similar to the below the ground state regime. Recall that in Chapter Two a convexity argument was used to show finite time blowup/global existence and scattering depending on the sign of K_0. The aim is to try and adapt the arguments in Chapter Two to our current setting. We know the sign of K_0 stabilities if a solution is not trapped by the ground state. We will show that if the sign stabilities to -1 then the solution experiences blowup in finite time and if the sign stabilizes to $+1$ then the solution exists globally forward in time. This is the following Lemma

Lemma 5.3. *Suppose $\vec{u}$ is a solution to NLKG with $E(\vec{u}) < J(Q) + \varepsilon^2(R_*)$ that is not trapped by the ground state such that. If $\mathrm{sign}K_0(u(t)) = \mathrm{sign}K_2(u(t)) = -1$ eventually then then $\vec{u}(t)$ blows up in finite time. On the other hand if $\mathrm{sign}K_0(u(t)) = \mathrm{sign}K_2(u(t)) = +1$ eventually then then $\vec{u}(t)$ exists globally.*

Proof. The proof of the first part of this result is similar to the proof of Theorem 2.10 and so we will just explain how the proof of Theorem 2.10 can be adapted to our current situation. Examining the proof of Theorem 2.10 one can see that if we can get a lower bound

$$-K_0(u(t)) \geq \delta > 0 \quad \text{eventually} \tag{5.1.3}$$

for a fixed δ then one could proceed in exactly the same way and conclude finite time blowup by a convexity argument. In the proof of Theorem 2.10 one got this lower bound by Lemma 2.8 which required the solution to have energy less than that of the ground state. For the current situation the required lower bound comes from Lemma 4.3 which can be applied as eventually $d_Q(\vec{u}(t)) \geq R_*$ by Lemma 5.2. For the proof of the second part, if $\mathrm{sign}K_0(u(t)) = \mathrm{sign}K_2(u(t)) = +1$ eventually then by Lemma 4.5, $\|\vec{u}(t)\|_{\mathcal{H}}$ is bounded and so by iterating local existence, $u(t)$ is a global solution forward in time. □

Similar to the below the ground state regime global existence if $\mathrm{sign}K_0(u(t)) = \mathrm{sign}K_2(u(t)) = +1$ eventually is not the complete picture. Similarly, in this situation we also have the following scattering result.

Lemma 5.4. *For each $\varepsilon \in (0, \varepsilon^*]$, there exists $0 < M(J(Q) + \varepsilon^2) < \infty$ such that if a solution u of NLKG on $[0, \infty)$ satisfies $E(\vec{u}) \leq J(Q) + \varepsilon^2$, $d_Q(\vec{u}(t)) \geq R_*$ and $\mathfrak{S}(\vec{u}(t)) = \mathrm{sign} K_0(u) + 1$ for all $t \geq 0$, then u scatters to 0 as $t \to \infty$ and $\|u\|_{L_t^3 L_x^6(0,\infty)} \leq M$.*

The proof of this Lemma is similar to the proof of Theorem 2.11 and we omit the proof for similar reasons. Note that we now have a complete classification of behavior for solutions with energy perhaps slightly above that of the ground state. We have proved Theorem 5.1 expect for showing sets (1) through (9) are non empty and sets (1)-(4) are open. We do this now.

Proof. (Proof of Theorem 5.1) Fix $0 < \varepsilon < \varepsilon^*$ and define

$$\mathcal{H}^\varepsilon = \{\vec{u} : E(\vec{u}) < J(Q) + \varepsilon^2\}. \tag{5.1.4}$$

We define the following sets which correspond to to the global behavior of the solution $u(t)$ forward in time, $\sigma = +1$, and backwards in time, $\sigma = -1$

$$\begin{aligned}
\mathcal{S}_\sigma^\varepsilon &= \{\vec{u}(0) \in \mathcal{H}^\varepsilon : u(t) \text{ scatters as } \sigma t \to \infty\} \\
\mathcal{T}_\sigma^\varepsilon &= \{\vec{u}(0) \in \mathcal{H}^\varepsilon : u(t) \text{ is trapped by } \{\pm Q\} \text{ as } \sigma t \to \infty\} \\
\mathcal{B}_\sigma^\varepsilon &= \{\vec{u}(0) \in \mathcal{H}^\varepsilon : u(t) \text{ blows up in } \sigma t > 0\}.
\end{aligned} \tag{5.1.5}$$

Note that (1)$= \mathcal{S}_+^\varepsilon \cap \mathcal{S}_-^\varepsilon$, (2)$= \mathcal{B}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon$, (3)$=\mathcal{S}_+^\varepsilon \cap \mathcal{B}_-^\varepsilon$ and etcetera. Hence to show that sets (1)-(4) are open it suffices to show that $\mathcal{S}_-^\varepsilon$, $\mathcal{S}_+^\varepsilon$, $\mathcal{B}_-^\varepsilon$ and $\mathcal{B}_+^\varepsilon$ are respectively open. $\mathcal{S}_+^\varepsilon$ being opens from the basic theory in Chapter Two. In particular as $\mathcal{S}_+$ is open from Lemma 2.3 and

$$\mathcal{S}_+^\varepsilon = \mathcal{S}_+ \cap E^{-1}\left(-\infty, J(Q) + \varepsilon^2)\right) \tag{5.1.6}$$

it follows $\mathcal{S}_+^\varepsilon$ is also open. The same argument applies to $\mathcal{S}_-^\varepsilon$. Proving that $\mathcal{B}_\sigma^\varepsilon$ is open is more complicated. We will just prove it for $\mathcal{B}_+^\varepsilon$ for ease of notation, the proof for $\mathcal{B}_-^\varepsilon$ is similar. Select a solution $\vec{u}(t)$ with initial data in $\mathcal{B}_+^\varepsilon$. Then there exists $T^* > 0$ such that $\|\vec{u}(t)\| \to \infty$ as $t \to T^*-$. We will construct a $\mathcal{H}$-open set containing $\vec{u}(0)$ such that any solution $\vec{w}$ with initial data in this open set eventually satisfies $-K_0(w) \geq \delta > 0$ and hence a Payne-Sattinger argument similar to Chapter Two and Lemma 5.3 will show that $\vec{w} \in \mathcal{B}_+^\varepsilon$. To this end note that $\partial_t \|u(t)\|_{L_x^2}^2 = 2\langle u, \partial_t u\rangle$ and

$$\partial_t^2 \|u(t)\|_{L_x^2}^2 = 2\left(\|\partial_t u\|_{L_x^2}^2 - K_0(u(t))\right) \geq 6\|\partial_t u\|_{L_x^2}^2 + 2\|u\|_{H_x^1}^2 - 8E(\vec{u}). \tag{5.1.7}$$

The inequality above shows that $K_0(u(t)) \to -\infty$ as $t \to T^*-$ and $\langle u, \partial_t u\rangle \to \infty$ as $t \to T^*-$. Let $T^{**} < T^*$ be sufficiently close to T^*. By the continuity of K_0 there exists $r > 0$ such that $-K_0(w) \geq \kappa_0(\varepsilon^*)$ for all $\vec{w} \in B_r(u(T^{**}))$. Here $\kappa_0(\varepsilon^*)$ is the constant in Lemma 4.3. Let $\vec{w} \in B_r(u(T^{**}))$, then by Lemma 4.3, $-K_0(w(t)) \geq \kappa_0(\varepsilon^*)$ can only be violated if $\vec{w}(t)$ returns to the δ_S ball (with respect to d_Q). If this did happen then

$$|\langle w(t), \partial w(t)\rangle_{L_x^2}| \lesssim \|\vec{w}(t)\|_{L_x^2}^2 \leq \|\vec{w}(t) - (Q, 0)\|_{\mathcal{H}}^2 + \|Q\|_{H^1} \lesssim \delta_S^2 + \|Q\|_{H^1} \lesssim 1. \tag{5.1.8}$$

However this leads to a contradiction as, if r is perhaps chosen to be smaller, $|\langle w(t), \partial w(t)\rangle_{L_x^2}$ starts of large and increases as long as $K_0(w(t)) < 0$. Hence $-K_0(w(t)) \geq \kappa_0(\varepsilon^*)$ for all $t \geq 0$ and so $\vec{w}(t)$ blows up in finite time by a Payne-Sattinger argument. The required open set is then

$$\Phi(-t)\left(B_r(u(T^{**}))\right). \tag{5.1.9}$$

This completes showing sets (1) through (4) are open. Now we show the sets are non-empty. Sets (1) and (2) are respectively non-empty from the theory in Chapter Two, in particular Theorem 2.10. Obviously (9) contains $(\pm Q, 0)$. For the other sets, note that by time reversal is suffices to show (3), (4) and (5) are non-empty. Proving that these sets are non-empty requires methods used to prove the results in Chapter Four. The proofs of the results in Chapter Four are beyond the scope of this report and hence we will just sketch the proof that (4) is non-empty. To this end, choose initial data, $\vec{u}$, with

$$\lambda(0) = 0, \quad \dot{\lambda}(0) = k\theta\varepsilon, \quad \vec{\gamma}(0) = 0 \tag{5.1.10}$$

for some $0 < \theta \ll 1$. Here we are considering the decomposition

$$u = \lambda\rho + \gamma, \quad \gamma \perp \rho \tag{5.1.11}$$

as in the previous chapter. Then $E(\vec{u}) < J(Q) + \varepsilon^2$. One can show that

$$\|\lambda(t) - \lambda_0(t)\| \lesssim e^{2k|t|}\theta^2\varepsilon^2, \quad \|\vec{\gamma}(t)\|_E \lesssim \theta\varepsilon + e^{2k|t|}\theta^2\varepsilon^2 \tag{5.1.12}$$

where

$$\lambda_0(t) = \sinh(kt)\theta\varepsilon \tag{5.1.13}$$

as long as $e^{k|t|}\theta\varepsilon \ll \delta_X$. This shows that $\vec{u}(t)$ is not trapped in $\pm t > 0$. Since

$$\lambda(t)t > 0 \quad \text{if} \quad \theta\varepsilon \ll e^{k|t|}\theta\varepsilon \ll \delta_X \tag{5.1.14}$$

one can show that $\text{sign}K_0(u(t)) = \text{sign}K_2(u(t)) = -1$ after exiting in positive time and so $\vec{u}(t)$ blows up in $t > 0$ and that $\text{sign}K_0(u(t)) = \text{sign}K_2(u(t)) = +1$ after exiting in negative time and so $\vec{u}(t)$ scatters to 0 as $t \to -\infty$. □

Although (1) and (2) are immediately non-empty from Chapter Two, we can use a construction similar to the one in the proof of Theorem 5.1 to construct solutions in (1) and (2) with energy greater than that of the ground state.

Note that the geometric information provided by Chapter 3 was not used in the proof of Theorem 5.1. The role of Chapter 3 in the classification of global dynamics is to provide geometric information on the sets (1)-(9). This is the following Theorem.

Theorem 5.5. *Let the sets (1)-(9) be the same as in the statement of Theorem 5.1. The sets* $(5) \cup (7) \cup (9)$ *and* $(6) \cup (8) \cup (9)$ *are co-dimension one Lipschitz manifolds in* $\mathcal{H}_{rad}$ *and are the center-stable and center-unstable manifolds around* $\pm(Q, 0)$ *respectively. Moreover one can show that solutions on the center-stable manifold scatter to* $\pm(Q, 0)$ *forward in time and similarly for the center-unstable manifolds backwards in time.*

Proof. The proof of the first part of this Theorem is a direct application of Theorems 3.17 and 3.18. The proof of the second part of this Theorem comes from the scattering result in Theorem 3.20. □

Chapter 6

Concluding Remarks

6.1 Summary of Results

This paragraphs explains how the results in this report fit together to give the final classification results in Chapter Five.The classification of Chapter Five required techniques similar to those used to classify solutions with energy less than that of the ground state. Chapter Two presented some of these techniques in a simpler context. Chapter Three provided information on the geometric structure of certain sets in the classification of Chapter 5. That is certain sets in the classification are Lipschitz invariant manifolds. Chapter Four stated results on the behavior of solutions away from the ground states. This is the new work of Nakanishi and Schlag [12]. This was necessary as Chapter Three only gave local information about the dynamics of NLKG. Chapter Five combined these techniques and results to give Theorems 5.1 and 5.5, first obtained by Nakanishi and Schlag [12].

6.2 Future Research Directions

There are many ways one could attempt to build on the result of Nakanishi and Schlag. One possible direction could be attempting to remove the radial assumption in Theorem 5.1. This could be difficult as one would have to deal with extra symmetries but there is no immediate reason why this could no be done. Another possible direction could be investigating exactly how big ε^2 can be made. This would be interesting for the following reason. The elliptic problem (2.2.1) which the ground state satisfies in fact has infinitely many smooth nodal solutions. That is solutions which change their sign a finite number of time. For any $n \in \mathbb{N}$, there is a smooth nodal solutions whose sign changes n times. This gives a sequence of stationary solutions $\{Q_j\}$ with $Q_0 = Q$. One can show that $\{J(Q_j)\}$ is an increasing sequence. The point here is that there is no reason why the classification result Theorem **??** should change as long as $J(Q) + \varepsilon^2 < J(Q_1)$. Another research direction could be, using similar techniques as in [12], attempting to classify solutions with energy slightly greater than that of $(Q_1, 0)$ and more generally $(Q_j, 0)$. Preceding in this direction one could perhaps give a complete classification of radial solutions of NLKG. It would also be interesting to apply the methods of [12] to other equations. This has already been done for the cubic radial NLS in $\mathbb{R}^3$ [14]. In [10] the energy critical wave was studied in dimensions three and five. A suitable one pass theorem was obtained but the authors were unable to describe the behavior of solutions that did not move away from the ground states.

Chapter 7

Appendix

7.1 Spectral Theory

In this section we briefly state spectral theory results that were needed in Chapter 3. For proofs of the following results or for more information see [17], [4] or [6].

If T is an operator on a Banach space X then we define the *point spectrum* of T to be

$$P_\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\} \tag{7.1.1}$$

and the *residual spectrum* of T to be

$$R_\sigma(T) = \{\lambda \in \mathbb{C} : \text{range}(\lambda \text{Id} - A) \text{ is not dense in } X\} \tag{7.1.2}$$

Theorem 7.1. *(Lumer-Phillips) Let A be a linear operator defined on a linear subspace $D(A)$ of the Banach space X. Assume*

(i) A is dissipative

(ii) $\lambda I - A$ is surjective for some $\lambda > 0$

then A generates a contraction semigroup in $\overline{D(A)}$.

Theorem 7.2. *Let A and B be bounded operator on a Hilbert space such that $A - B$ is compact, $\text{int}_{\mathbb{C}}(\sigma(A)) = \emptyset$ and each component of $\mathbb{C} \backslash \sigma(A)$ contains a point of $\rho(B)$, the resolvent set of B. Then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$.*

Theorem 7.3. *Let A be the generator of a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ on a Banach space X. Then*

$$\begin{aligned} P_\sigma(S(t)) \backslash \{0\} &= e^{tP_\sigma(A)} \\ R_\sigma(S(t)) \backslash \{0\} &= e^{tR_\sigma(A)} \end{aligned} \tag{7.1.3}$$

for all $t \geq 0$.

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