Golomb Rulers

Roger C. Alperin
San Jose State University
San Jose, CA 95192
alperin@math.sjsu.edu

Vladimir Drobot
Department of Computer Science
San Jose State University
San Jose, CA 95192
drobot@pacbell.net

The Math Factor podcast posed the problem of finding the smallest number of inch marks on a 12 inch ruler so that one could still measure any integer length from 1 to 12. One needs only four additional marks besides 0 and 12; for example 1, 4, 7, 10 works. This entertaining problem lead to others during the next few minutes (you can listen at mathfactor.uark.edu/2005/10) and inspired us to look for generalizations. After several false starts and numerous literature searches we uncovered the fascinating theory of Golomb and minimal spanning rulers, a generalization to the natural numbers and relations to an unsolved conjecture of Erdös and Turan.

We begin the discussion with our first problem—– which led us to Golomb rulers. A property of the ruler of length 6 with marks at 0, 1, 4, 6 is that each of the lengths 1, 2, 3, 4, 5, and 6 can be measured and it can be done in only one way. *Can one choose marks on a ruler of length 12 so that each length from 1 to 12 can measured in only one way?*

Golomb rulers are sets of integers (marks) with the property that if a distance can be measured using these marks then it can be done in a unique way.

Definition 1. A set $\mathcal{G}$ of integers

$$a_1 < a_2 < \cdots < a_{p-1} < a_p$$

is called a *Golomb ruler* if for every two distinct pairs of these integers, say $a_i < a_j$ and $a_m < a_n$, we have $a_j - a_i \neq a_n - a_m$.

The *size* of $\mathcal{G}$ is defined to be p (the number of marks in $\mathcal{G}$) and is denoted $\#\mathcal{G}$. The *length* of $\mathcal{G}$ is defined to be $a_p - a_1$ (the largest distance that can be measured using the marks from $\mathcal{G}$).

It is clear that we can translate these sets: if $\mathcal{G} = \{a_1, a_2, \cdots, a_p\}$ is a Golomb ruler then so is $\{\mathrm{a}_1 + b, a_2 + b, \cdots, a_p + b\}$. This makes the choice of a_1 immaterial, so it will usually be taken to be 0. It is also clear that if $\mathcal{G} = \{a_1, a_2, \cdots, a_p\}$ is a Golomb ruler, then so is the reflection of $\mathcal{G}$ around the midpoint $(a_1 + a_p)/2$. For example, $\{0, 1, 4, 6\}$ is a Golomb ruler, as is $\{0, 2, 5, 6\}$ obtained by reflecting the first ruler around the point 3. To simplify the statements of some of the theorems, a set $\{a_1\}$ consisting of a single point is considered to be a Golomb ruler.

Golomb rulers have numerous applications. The best known is an application to radio astronomy. Radio telescopes (antennas) are placed in a linear array. For each pair of these antennas, the received signals are subtracted from each other and an inference can be then made as to the location of the source. These inferences can be made much more accurate if all the distances between the antennas are multiples of the same common length, and many such pairs with distinct distances between them are available and can be utilized. The problem maximizing the number of distinct distances between the pairs, while minimizing the number of the antennas and the length of the array, was first considered by Solomon W. Golomb [8, 1, 2, 10].

Other applications include assignments of channels in radio communications, X-ray crystallography, and self-orthogonal codes. Rankin [12] gives more information about these applications. There is also a wealth of information in various writings by Martin Gardner [5, 6, 7].

The Golomb ruler $\{0, 1, 4, 6\}$ has the additional property that every integral distance between 1 and 6 can be measured. We call such a ruler *perfect*.

Definition 2. A Golomb ruler $\mathcal{G}$ of length N is called *perfect* if every integer d, $1 \leq d \leq N$, can be expressed as $d = a - a'$, for some $a, a' \in \mathcal{G}$.

Since $\mathcal{G}$ is a Golomb ruler, the representation of each d is unique. Unfortunately, there are very few perfect Golomb rulers.

Theorem 1. (Golomb) *Together with their translations and reflections around the midpoint the only perfect Golomb rulers are* $\{0\}, \{0, 1\}$, $\{0, 1, 3\}$, *and* $\{0, 1, 4, 6\}$.

This theorem was proved by Golomb, but apparently he never published it. There are several places where the proof appears (A. Dimitromanolakis [4] or W. Rankin [12]), but they are not very easily accessible, so we present here a slight modification of the original argument.

Proof. Suppose $\mathcal{G}$ is a perfect Golomb ruler of size p and length N then we must have $N = \binom{p}{2} = \frac{1}{2}p(p-1)$, so N is a triangular number. This is easy to see since there are N distances to be measured and and the number of distinct pairs of these points is $\binom{p}{2}$. The triangular numbers below 10 are $0, 1, 3, 6$ corresponding to the rulers listed in the theorem

Assume then that $\mathcal{G}$ is a perfect Golomb ruler of length $N > 9$ and we seek a contradiction. Without loss of generality we may assume that a_1, the smallest number in $\mathcal{G}$, is equal to 0 and so the largest number is $a_p = N$. By hypothesis, every number $1 \leq d \leq N$ is uniquely realizable as a difference of two marks in $\mathcal{G}$. Since N is realizable, 0 and N must belong to $\mathcal{G}$. Since $N-1$ is realizable, either 1 or $N-1$ belongs to $\mathcal{G}$. By reflecting $\mathcal{G}$ around $N/2$, we may assume that $1 \in \mathcal{G}$. Next, since $N > 3$ then $N-2$ must be realized. Since $N-2 > 1$ then $\mathcal{G}$ must contain another point.

The possible pairs realizing $N-2$ are $\{2, N\}$, $\{1, N-1\}$, $\{0, N-2\}$. The first two produce duplications: $1-0 = 2-1$ and $1-0 = N-(N-1)$. The third is the only possibility so $\mathcal{G}$ contains $N-2$ as well as $0, 1, N$. The realized distances are $1, 2, N-3, N-2, N-1$, and N.

Since $N-4 \notin \{1, 2\}$ we need one of the pairs $\{0, N-4\}$, $\{1, N-3\}$, $\{2, N-2\}$, $\{3,\ N-1\}$, $\{4, N\}$ to realize $N-4$.. All but the last case yield duplications: $(N-2)-(N-4) = N-(N-2)$, $1-0 = (N-2)-(N-3)$, $2-0 = N-(N-2)$, $1-0 = N-(N-1)$.

The last case is okay, so $\mathcal{G}$ contains 0, 1, 4, $N-2$, and N. The distances which can be realized by $\mathcal{G}$ are $1, 2, 3, 4, N-6, N-4, N-3, N-2, N-1$ and N.

Finally, consider the distance $N-5$. Since $N-5 \notin \{1, 2, 3, 4\}$ and $N > 9$ this distance has not been realized. The possible pairs for realizing the distance $N-5$ are $\{0, N-5\}$, $\{1, N-4\}$, $\{2, N-3\}$, $\{3, N-2\}$, $\{4, N-1\}$, $\{5, N\}$. The reader may easily check that each of these case leads to a duplication. This contradiction shows that $N < 9$ and the constructions above give the perfect rulers asserted by the theorem. ∎

Since perfect Golomb rulers essentially do not exist, we seek "almost perfect" rulers. Roughly speaking, given a length N, we try to place as many points as possible in the interval $[0, N]$ so that the resulting set forms a Golomb ruler. Alternatively, given the size p of the ruler (the number of marks), we try to construct a Golomb ruler of shortest possible length N with p points. Such rulers are called optimal.

Definition 3. For every positive integer p, let $G(p)$ be the shortest possible length of a Golomb ruler with p marks.

A Golomb ruler with p marks is called *optimal* if its length is $G(p)$. Dimitromanolakis [4] has a detailed discussion of optimal Golomb rulers. For example, $G(6) = 17$, and there are 4 optimal rulers of size 6 and length 17: $\{0, 1, 4, 10, 12, 17\}, \{0, 1, 4, 10, 15, 17\}, \{0, 1, 8, 11, 13, 17\}, \{0, 1, 8, 12, 14, 17\}$.

Computer searches give the largest known value of $G(p)$. The current record is $G(26) = 492$ and the corresponding optimal Golomb ruler has marks

$$0\ 1\ 33\ 83\ 104\ 110\ 124\ 163\ 185\ 200\ 203\ 249\ 251\ 258$$

$$314\ 318\ 343\ 356\ 386\ 430\ 440\ 456\ 464\ 475\ 487\ 492.$$

The search took several years but it is not known if it is unique [13, 14]. *Wikipedia* is also a good source of information on the latest status of the values of $G(p)$.

Given a Golomb ruler with p marks, there are $\binom{p}{2} \sim \frac{1}{2}p^2$ distinct distances one can measure with this ruler. Thus, one expects $G(p)$ to be roughly at least $\frac{1}{2}p^2$. It is a conjecture, with strong empirical evidence, that $G(p) < p^2$; but it is only a conjecture.

Golomb rulers also have a close connection with additive number theory. It is completely outside the scope of this paper to discuss this connection in any depth, and we only state some facts and invite the reader to investigate further.

Definition 4. A subset $\mathcal{B}$ of integers contained in $[1, N]$ is called a B_2 *basis*, if for any two distinct pairs of integers from $\mathcal{B}$, say a, a' and b, b' we have

$$a + a' \neq b + b'.$$

There is an old conjecture of Erdös and Turan which states that a B_2 basis with $\lfloor\sqrt{N}\rfloor$ elements can be constructed in $[1, N]$ for any N. This is very closely related to the conjecture that $G(p) < p^2$. Halberstam and Roth [9] give a comprehensive discussion of additive number theory and the connection with Golomb rulers.

Perfect rulers on $\mathbb{N}$

In this section we study *infinite rulers*. These are sets $\mathcal{G}$ of nonnegative integers, such any positive integer d is realized as a distance between some two elements of $\mathcal{G}$. In addition, if we require that this representation is unique, we may speak of infinite perfect Golomb rulers.

Definition 5. A subset $\mathcal{G}$ of the set $\mathbb{N}$ of natural numbers is called an *infinite perfect Golomb ruler* if

1) for every positive integer d, there are elements $a, a' \in \mathcal{G}$ so that $d = a - a'$, and
2) for every such d this representation is unique.

It is not entirely clear that such things exist, but in fact they do and also they can be made arbitrarily *thin* (sparse) depending on the choice of a function φ.

Theorem 2. *Let $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be strictly increasing with $\varphi(x) \to \infty$ as $x \to \infty$. Then there is a subset $\mathcal{G}$ of $\mathbb{N}$ which is an infinite perfect Golomb* *ruler and such that for $x > x_0 = x_0(\mathcal{G}, \varphi)$*

$$\#\{k \mid k \in \mathcal{G},\ k \le x\} \le \varphi(x). \tag{1}$$

Proof. The basic idea to construct the set $\mathcal{G}$ is as follows: first we choose a rapidly increasing sequence $\gamma_k, k = 1, 2, \cdots$, and then construct $\mathcal{G}$ by successively adding the points $\{\gamma_k, \gamma_k + k\}$. If a duplication should occur as a result of this addition then we do not add the pair. Various things have to be proved, for example, that skipping a pair does not result in some integer d not being realized as the difference of two element of $\mathcal{G}$, etc. The details follow.

Choose a strictly increasing function $\psi(x)$such that

$$x < \frac{1}{2}\varphi(\psi(x)) \tag{2}$$

and define a sequence $\{\gamma_k\}$ by

$$\begin{aligned} &\gamma_1 = 0 \\ &\gamma_{k+1} > \psi(k+1) + 2(\gamma_k + k) + 1, \text{ for } k \ge 1. \end{aligned} \tag{3}$$

Define $\mathcal{A}_1 = \{\gamma_1, \gamma_1 + 1\}$ which of course equals $\{0, 1\}$ and

$$\mathcal{A}_{k+1} = \begin{cases} \mathcal{A}_k \cup \{\gamma_{k+1}, \gamma_{k+1} + (k+1)\} \\ \qquad \text{if this set } \mathcal{A}_k \cup \{\gamma_{k+1}, \gamma_{k+1} + (k+1)\} \text{ has no duplicate distances} \\ \mathcal{A}_k \text{ otherwise.} \end{cases}$$

This just says that we start with $\{\gamma_1, \gamma_1 + 1\}$ and for each $k > 1$ add two new points $\gamma_k, \gamma_k + k$ provided that this does not introduce a duplication. Finally, we set

$$\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{A}_k.$$

First of all we show that the set $\mathcal{G}$ satisfies the density condition (1) in the statement of the Theorem. Let $x > 1$ be given and let k_0 be the largest integer such that $\gamma_{k_0} \le x$. Then

$$\#\{k \mid k \in \mathcal{G}, k \le x\} \le 2k_0$$

because the elements of $\mathcal{G}$ come in pairs: γ_p and $\gamma_p + p$. Now

$$k_0 < \frac{1}{2}\varphi(\psi(k_0)) < \frac{1}{2}\varphi(\gamma_{k_0}) \leq \frac{1}{2}\varphi(x).$$

The first inequality follows from (2) and the second and third follow from (3) and the fact that φ is monotonically increasing. Thus the density claim (1) of the Theorem is true.

By construction, there are is no duplication of distances in $\mathcal{G}$. This is quite clear, since we made sure that there is no duplication of distances in any of the sets $\mathcal{A}_k$.

It remains to show that every distance d is realized as a difference of two elements of $\mathcal{G}$. It suffices to analyze the pairs which are not included by our process: When a duplication occurs by inclusion of $\{\gamma_p, \gamma_p + p\}$, then we claim that

$$p = a - a' \text{ where } a, a' \in \mathcal{A}_{p-1} \tag{4}$$

i.e., p is already realized as a distance in the set $\mathcal{A}_{p-1}$. This would occur, for example, in the following situation: Let a and a' be two points in a set $\mathcal{A}_q$, for some q such that $a - a' > q$. Then, further along in the process, adding the pair $\{\gamma_{a-a'}, \gamma_{a-a'} + (a - a')\}$ would surely create a duplication. The claim is that this is essentially the only way it could happen. Now, if this claim is true, then either every distance d occurs in $\mathcal{G}$ through the addition of the pair $\{\gamma_d, \gamma_d + d\}$ or d occurs already as a distance in the set $\mathcal{A}_{d-1}$.

We now prove the assertion (4). Suppose that the addition of the pair $\{\gamma_p, \gamma_p + p\}$ to the set $\mathcal{A}_{p-1}$ results in duplications. Because there are no duplications in the set $\mathcal{A}_{p-1}$ these duplications must involve the points from the pair under discussion. It follows, because of (3), that both points of the pair are larger than any of the points in $\mathcal{A}_{p-1}$, and so the possibilities are:

$i)$ $(\gamma_p + p) - \gamma_p = a - a'$

$ii)$ $(\gamma_p + p) - a = \gamma_p - a'$

$iii)$ $(\gamma_p + p) - a = a' - a''$

$iv)$ $\gamma_p - a = a' - a''$

where the numbers a, a', a'' are elements of the set $\mathcal{A}_{p-1}$. In cases i) and ii) then p is a difference of some elements in $\mathcal{A}_{p-1}$, hence (4) holds. The possibilities iii) and iv) cannot occur because the largest element of $\mathcal{A}_{p-1}$ is at most $\gamma_{p-1} + (p-1)$ and from (3) then $\gamma_p > 2(\gamma_{p-1} + p - 1)$. But, if either iii) or iv) were true, then either γ_p or $\gamma_p + p$ would be at most twice the largest element of $\mathcal{A}_{p-1}$. Thus our claim (4) is shown and the theorem is proved. ∎

Thus, thin infinite perfect Golomb rulers do exist. The construction in Theorem 2 does not give a "formula" for the nth mark— it just constructs these marks one by one.

It would be interesting to know how *thick* an infinite perfect Golomb can be. In particular is it possible to have

$$\delta_{\mathcal{G}}(x) = \#\{\ k \mid k \in \mathcal{G}, k \leq x\} \sim \sqrt{x}? \tag{5}$$

By arguments similar to the discussion of finite perfect Golomb rulers, it is easy to see that $\delta_{\mathcal{G}}(x)$ should roughly be at least $\frac{1}{2}x^2$, and (5) is motivated by the Erdös-Turan conjecture about B_2 bases (it does not follow from nor does it imply the conjecture).

Minimal spanning rulers

Next we return to rulers of finite length and discuss those that can be used to measure every distance. They differ from Golomb rulers in that there might be a distance that can be measured in two different ways, but we require that every eligible distance can be measured. We call such rulers spanning.

Definition 6. Let $\mathcal{S} = \{0 = a_1 < a_2 < \cdots < a_p = N\}$ be a set of integers. We say that $\mathcal{S}$ is a *spanning ruler on* $[0, N]$ if every integer $1 \leq d \leq N$ can be expressed as $d = a - a'$, with a and $a' \in \mathcal{S}$.

We say that a spanning ruler $\mathcal{M}$ is *minimal* on $[0, N]$, if whenever $\mathcal{M}'$ is a proper subset of $\mathcal{M}$ then the set $\mathcal{M}'$ is not a spanning ruler on $[0, N]$.

Minimal spanning rulers obviously exist. Just start with $\{0, 1, \cdots, N\}$ and remove one point at a time until you can't do it anymore.

However, minimal rulers cannot be very "thin". If $\mathcal{M}$ is a minimal ruler of length N and $p = \#\mathcal{M}$, then $\binom{p}{2} = \frac{1}{2}p(p-1) \geq N$; so p is roughly at least $\sqrt{2N}$. We now show that we can come fairly close to this lower bound.

Theorem 3. *For every integer $N \geq 4$ there is a minimal spanning ruler $\mathcal{M}_N \subset [0, N]$ such that*

$$2\sqrt{N} - 1 \leq \#\mathcal{M}_N < 2\sqrt{N} \tag{6}$$

and the equality on the left side holds only when N is a perfect square.

Proof. The basic idea of the proof can best be seen by an example of a thin minimal ruler for $N = 100$. The ruler $\mathcal{M}_{100}$ is in this case taken to be

$$\mathcal{M}_{100} = \{0, 1, 2, 3, \cdots, 9, 20, 30, 40, \cdots, 90, 100\}.$$

Notice that the number 10 is *not* included. The number of elements in $\mathcal{M}_{100}$ is 19 which is equal to $2\sqrt{100} - 1$. Every distance $1 \leq d \leq 100$ is realizable: for $d = 10 = 30 - 20$ say whereas any other multiple d of 10 is $d = d - 0$. If $d = q \cdot 10 + j$, $1 \leq q, j \leq 9$ then $d = (q+1) \cdot 10 - (10 - j)$.

Finally, if $1 \leq d \leq 9$ then $d = d - 0$. None of the numbers can be removed. For example, $d = 7$ cannot be removed because then $13 = 20 - 7$ would not be realizable. The number 30 cannot be removed because then $21 = 30 - 9$ would not be realizable. If 10 is included the ruler is not minimal. The actual proof is based on this example although some care must be taken when N is not a perfect square. Here are the details.

By inspection, when $N \in \{5, 6, 7, 8\}$ then the minimal spanning rulers satisfying (6) are, respectively:

$$\{0, 1, 3, 5\}, \{0, 1, 4, 6\}, \{0, 1, 4, 5, 7\}, \{0, 1, 4, 6, 8\}.$$

Incidentally, there are no minimal spanning rulers satisfying the condition (6) for $N \in \{1, 2, 4\}$ and there is one for $N = 3$, namely $\{0, 1, 3\}$.

Let $\xi = \left\lfloor \sqrt{N} \right\rfloor$ so that $\xi^2 \le N < (\xi+1)^2 = (\xi+2)\xi+1$. We assume that $\xi \ge 3$. There are two possibilities:

$$\alpha)\ N = 0 \bmod \xi \text{ so that } N = K\xi,\ K = \xi,\ \xi+1, \text{ or } \xi+2; \tag{7}$$

$\beta)$ $N \ne 0 \bmod \xi$ so that $N = K\xi + \eta$, $K = \xi$ or $\xi + 1$, and $1 \le \eta < \xi$.

In case $\alpha)$ take

$$\mathcal{M}_N = \{0, 1, \cdots, \xi - 1, 2\xi, 3\xi, \cdots, K\xi\} \quad (\xi \text{ is not included})$$

with K as in (7).

Every distance $1 \le d \le N$ is realized as the following analysis shows: If $1 \le d \le \xi - 1$ then $d = d - 0$; When $d = \xi$, then $d = 3\xi - 2\xi$ since both $2\xi, 3\xi \in \mathcal{M}_N$ for $\xi \ge 3$; For $d = q\xi, q > 1$ then $d = q\xi - 0$; Finally if $d = q\xi + \eta, 1 \le q < K, 1 \le \eta < \xi$, then $d = (q+1)\xi - (\xi - \eta)$.

Also, we see that none of the marks can be removed: The endpoints 0 and N cannot be deleted because $N = N - 0$; The points $1 \le d < \xi$ cannot be deleted because of the distance $\xi + (\xi - d) = 2\xi - d$; Finally, the points $q\xi, 2 \le q \le K$, cannot be deleted because of the distance $(q-1)\xi + 1 = q\xi - (\xi - 1)$.

In addition we have that $\#\mathcal{M}_N = \xi + K - 1$. Thus, to show (6) we must prove that for $t = 0, 1, 2$ then

$$2\sqrt{\xi(\xi+t)} - 1 \le 2\xi + t - 1 < 2\sqrt{\xi(\xi+t)}$$

with equality holding on the left side only when $t = 0$. This is done in a straightforward mannner by squaring each side of the inequality to eliminate radical expressions.

In case $\beta)$ take

$$\mathcal{M}_N = \{0, 1, \cdots, \xi - 1, 2\xi, 3\xi, \cdots, K\xi, K\xi + \eta\} \quad (\xi \text{ not included})$$

where K, η are as specified in (7). Again, all the distances $1 \le d \le N = K\xi + \eta$ can be realized: If $1 \le d \le K\xi$ then the argument is the same as in case $\alpha)$; If $d = K\xi + \delta, 1 \le \delta \le \eta$, $d = (K\xi + \eta) - (\eta - \delta)$.

None of the marks can be removed: The endpoints cannot be removed because of the distance $N = N - 0$; The points $\xi - c$, $1 \leq c < \xi$, $c \neq \eta$ cannot be removed because of the distance $\xi + c = 2\xi - (\xi - c)$.

The point $\xi - \eta$ cannot be removed because of the distance

$$(K-1)\xi + \eta = K\xi - (\xi - \eta). \tag{7}$$

However also $(K-1)\xi + \eta = (K\xi + \eta) - \xi$ and we see that (7) is the only way to realize the distance $(K-1)\xi + \eta$ since $\xi \notin \mathcal{M}_N$

The points $2\xi, 3\xi, \cdots, K\xi$ cannot be removed for the following reason: Let τ be such that $\tau \neq \eta$, $1 \leq \tau < \xi$. If $k\xi$ is removed then the distance $(k-1)\xi + \tau = k\xi - (\xi - \tau)$ is not realizable. (It can't be realized using the mark $K\xi + \eta$.)

Finally, $\#\mathcal{M}_N = \xi + K$ and to show (6) we must prove that for $t = 0, 1$ and $1 \leq \eta < \xi$ then

$$2\sqrt{\xi(\xi + t) + \eta} - 1 < 2\xi + t < 2\sqrt{\xi(\xi + t) + \eta}.$$

Again, squaring both sides of each inequality to eliminate radicals and some algebra does the job. ∎

The minimal spanning rulers can also be quite *thickly* marked.

Theorem 4. *For any $N > 0$ there is a minimal spanning ruler $\mathcal{M}_N$ with*

$$\#\mathcal{M}_N > \frac{1}{2}N.$$

Proof. A moment's reflection shows that if $N = 2n$ or $N = 2n + 1$ then $\mathcal{M}_N = \{0, 1, \cdots, n, N\}$ works. ∎

Acknowledgment. We thank the referees for many valuable suggestions, including a very nice reformulation of the statement of Theorem 3.

References

[1] E. J. Blum, F. Biraud, and J. C. Ribes, On optimal synthetic linear arrays with applications to radio astronomy, *IEEE Transactions on Antennas and Propagation*, AP-22 (1974), 108–112.

[2] Gary S. Bloom and Solomon W. Golomb, Applications of numbered undirected graphs, *Proceedings of the IEEE*, 65 (April 1977), 562–570.

[3] A. K. Dewdney, *The Armchair Universe*, W. H. Freeman, New York, 1988.

[4] A. Dimitromanolakis, Analysis of the Golomb ruler and the Sidon set problem, and determination of large near-optinmal Golomb rulers, Diploma thesis, Technical University of Crete (Greece) 2002. (The English version can be downloaded from www.cs.toronto.edu/∼apostol/golomb.)

[5] Martin Gardner, *The Magic Numbers of Dr. Matrix*, Prometheus Books, Amherst N. Y., 1985.

[6] Martin Gardner, Mathematical Games, *Sci. Amer.*, March 1972, 108–112.

[7] Martin Gardner, *Wheels of Life*, W. H. Freeman, New York, 1988.

[8] Solomon W. Golomb, The use of combinatorial structures in communication signal analysis, *Applications of Combinatorial Mathematics*, Oxford, 1994, 59–78.

[9] H. Halberstam and F. K. Roth, *Sequences*, Springer, New York. 1983.

[10] Alan T. Moffet, Minimum redundancy linear arrays, *IEEE Transactions on Antennas and Propagation*, AP-16, March 1968, 172–175.

[11] Ed Pegg, Math Games, Rulers, and Arrays, www.maa.org/editorial/mathgames/mathgames_11_15_04.html.

[12] William T. Rankin, Optimal Golomb Rulers: An exhaustive parallel search implementation, M.S. thesis, Duke University, 1993. (Available at people.ee.duke.edu/∼wrankin/golomb/golomb.html).

[13] James B. Shearer, Golomb rulers webpage, www.research.ibm.com/people/s/shearer/grule.html.

[14] Eric W. Weisstein, Golomb Ruler, MathWorld-A Wolfram Web Resource, mathworld.wolfram.com/GolombRuler.html.