Government of Karnataka

MATHEMATICS

10

TENTH STANDARD

PART-I

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

KARNATAKA TEXT BOOK SOCIETY (R)
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Foreword

The National Curriculum Framework, 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

The National Council of Educational Research and Training (NCERT) appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisors for this book, Professor P. Sinclair of IGNOU, New Delhi and Professor G.P. Dikshit (Retd.) of Lucknow University, Lucknow for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. We are especially grateful to the members of the National Monitoring Committee, appointed by the Department of Secondary and Higher Education, Ministry of Human Resource Development under the Chairpersonship of Professor Mrinal Miri and Professor G.P. Deshpande, for their valuable time and contribution. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

New Delhi
15 November 2006

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ಮುನ್ನುಡಿ

2005ನೇ ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯಕ್ರಮ ಆಧರಿಸಿ ರಚಿತವಾದ ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯವಸ್ತುವಿನಂತೆ ರಚಿಸಲಾಗಿರುವ 10ನೇ ತರಗತಿ ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ ಗಣಿತ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ರಾಜ್ಯದಲ್ಲಿ 2018–19 ನೇ ಸಾಲಿನಿಂದ ಜಾರಿಗೊಳಿಸಲಾಗುತ್ತಿದೆ.

ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ ಪ್ರಕಟಿಸಿರುವ ಆಂಗ್ಲ ಮಾಧ್ಯಮದ ಗಣಿತ ಪುಸ್ತಕವನ್ನು ರಾಜ್ಯದಲ್ಲಿ ಕನ್ನಡ, ಮರಾಠಿ, ತೆಲುಗು ಮತ್ತು ತಮಿಳು ಮಾಧ್ಯಮಗಳಿಗೆ ಭಾಷಾಂತರಿಸಲಾಗಿದೆ. ಇಂಗ್ಲೀಷ್, ಹಿಂದಿ ಮತ್ತು ಉರ್ದು ಮಾಧ್ಯಮದ ಪುಸ್ತಕಗಳನ್ನು ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ ಯಿಂದ ನೇರವಾಗಿ ಪಡೆದು ಭಾಗ–1 ಮತ್ತು ಭಾಗ–2 ಎಂಬುದಾಗಿ ವಿಭಾಗಿಸಿ ಮುದ್ರಿಸಿ ಜಾರಿಗೊಳಿಸಲಾಗಿದೆ.

ಈ ಪುಸ್ತಕವು 2005 ರ ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯಕ್ರಮದ ಎಲ್ಲಾ ವೈಶಿಷ್ಟ್ಯಗಳನ್ನು ಹೊಂದಿದೆ. 10ನೇ ತರಗತಿಯ ಗಣಿತ ಪುಸ್ತಕದಲ್ಲಿ ಅಂತರ್ಗತ ವಿಧಾನ (Integrated Approach), ರಚನಾತ್ಮಕ ವಿಧಾನ (Constructive Approach) ಹಾಗೂ ಸುರುಳಿಯಾಕಾರದ ವಿಧಾನ (Spiral Approach) ಗಳನ್ನು ಅಳವಡಿಸಲಾಗಿದೆ.

ಪಠ್ಯಪುಸ್ತಕದ ವಿಷಯ ಹಾಗೂ ಅಭ್ಯಾಸಗಳು ವಿದ್ಯಾರ್ಥಿಗಳನ್ನು ಚಿಂತನಾಶೀಲರನ್ನಾಗಿ ಮಾಡಿ, ಚಟುವಟಿಕೆಗಳ ಮೂಲಕ ಜ್ಞಾನ ಹಾಗೂ ಸಾಮರ್ಥ್ಯಗಳನ್ನು ಪಡೆಯುವಂತೆ ಮಾಡುವ ಪ್ರಯತ್ನ ಮಾಡಲಾಗಿದೆ. ಪಠ್ಯ ವಸ್ತುಗಳೊಂದಿಗೆ ಅತ್ಯಂತ ಅವಶ್ಯಕ ಜೀವನ ಮೌಲ್ಯಗಳನ್ನು ಅಂತರ್ಗತವಾಗಿ ಬಳಸಲಾಗಿದೆ. ಈ ಪುಸ್ತಕಗಳು ಪರೀಕ್ಷಾ ದೃಷ್ಟಿಯಿಂದ ರಚಿತವಾಗಿಲ್ಲ ಬದಲಾಗಿ ವಿದ್ಯಾರ್ಥಿಗಳ ಸರ್ವಾಂಗೀಣ ವ್ಯಕ್ತಿತ್ವ ವಿಕಸನಕ್ಕೆ ಪೂರಕವಾಗಿದೆ.

ನಿತ್ಯಜೀವನದಲ್ಲಿ ಗಣಿತವು ಎಲ್ಲಾ ಹಂತಗಳಲ್ಲೂ ಅತ್ಯಾವಶ್ಯಕವಾಗಿದೆ. ರಾಷ್ಟ್ರೀಯ ಪಠ್ಯಕ್ರಮ–2005 ರಂತೆ ಗಣಿತದ ಪರಿಕಲ್ಪನೆಗಳನ್ನು ಅರ್ಥಮಾಡಿಕೊಂಡು ಸಿದ್ಧಾಂತಗಳನ್ನು ನಿರೂಪಿಸಿ, ಪ್ರಮೇಯಗಳನ್ನು ಸಾಧಿಸಿ, ಲೆಕ್ಕಗಳನ್ನು ಬಿಡಿಸುವ ಜೊತೆಗೆ ಗಣಿತವನ್ನು ಜೀವನದ ಸಕಲ ಕ್ಷೇತ್ರಗಳಲ್ಲೂ ಬಳಸುವ ಸಾಮರ್ಥ್ಯವನ್ನು ಬೆಳೆಸಿಕೊಂಡು ಜೀವನದಲ್ಲಿ ಯಶಸ್ಸುಗಳಿಸುವಂತೆ ಮಾಡಲು ಸಹಕಾರಿ ಕಲಿಕೆಗೆ ಪೂರಕವಾಗಿದೆ.

ಪ್ರಾಚೀನ ಭಾರತದ ಶ್ರೇಷ್ಠ ಗಣಿತ ಶಾಸ್ತ್ರಜ್ಞರ ಕೊಡುಗೆಗಳನ್ನು ಸೂಕ್ತ ಸಂದರ್ಭಗಳಲ್ಲಿ ಕೊಡಲಾಗಿದೆ. ಈ ಪಠ್ಯಪುಸ್ತಕಗಳು ಶೈಕ್ಷಣಿಕವಾಗಿ ವೈಶಿಷ್ಟ್ಯಪೂರ್ಣವಾಗಿವೆ. ಇತರೆ ವಿಷಯಗಳ ಪಠ್ಯಪುಸ್ತಕಗಳಂತೆ ಈ ಪಠ್ಯಪುಸ್ತಕಗಳು ವಿದ್ಯಾರ್ಥಿಗಳಿಗೆ ಸಾಮರ್ಥ್ಯ ಹಾಗೂ ಕೌಶಲ್ಯಗಳನ್ನು ಬೆಳೆಸಿಕೊಳ್ಳಲು ಸಹಾಯ ಮಾಡುತ್ತವೆ.

ರಾಷ್ಟ್ರೀಯ ಮಟ್ಟದ ಶೈಕ್ಷಣಿಕ ಸುಧಾರಣೆಗಳನ್ನು ಗಮನದಲ್ಲಿರಿಸಿಕೊಂಡು ಕರ್ನಾಟಕ ರಾಜ್ಯದ ವಿದ್ಯಾರ್ಥಿಗಳೂ ಸಹ ರಾಷ್ಟ್ರೀಯ ಮಟ್ಟದ ಸ್ಪರ್ಧಾತ್ಮಕ ಪರೀಕ್ಷೆಗಳಲ್ಲಿ ಯಶಸ್ಸನ್ನು ಸಾಧಿಸಲು ಈ ಪುಸ್ತಕ ಸಹಕಾರಿಯಾಗಲಿ ಎಂಬುದೇ ಸಾರ್ವಜನಿಕ ಶಿಕ್ಷಣ ಇಲಾಖೆಯ ಪ್ರಮುಖ ಆಶಯವಾಗಿದೆ.

ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ ಪ್ರಕಟಿಸಿರುವ ಗಣಿತ ಪುಸ್ತಕಗಳನ್ನು ನಮ್ಮ ರಾಜ್ಯದಲ್ಲಿ ಪ್ರಥಮವಾಗಿ ಜಾರಿಗೆ ತರಲು ದಿಟ್ಟ ನಿರ್ಧಾರವನ್ನು ತೆಗೆದುಕೊಂಡು ಸಹಕರಿಸಿದ ಸರ್ಕಾರ ಮತ್ತು ಇಲಾಖೆಯ ಉನ್ನತ ಅಧಿಕಾರಿಗಳಿಗೆ ಆಭಾರಿಯಾಗಿದ್ದೇನೆ. ಈ ಪಠ್ಯಪುಸ್ತಕವನ್ನು ನಮ್ಮ ರಾಜ್ಯದಲ್ಲಿ ಜಾರಿಗೆ ತರಲು ಸಕಾಲದಲ್ಲಿ ಅನುಮತಿ ನೀಡಿ ಸಹಕರಿಸಿದ ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ ಯ ಪ್ರಕಾಶನ ವಿಭಾಗ ಹಾಗೂ ಎನ್.ಸಿ.ಇ.ಆರ್.ಟಿ ಸಂಸ್ಥೆ ಎಲ್ಲ ಅಧಿಕಾರಿ, ಸಿಬ್ಬಂದಿಗಳಿಗೆ ಇಲಾಖೆ ತನ್ನ ಹೃತ್ಪೂರ್ವಕ ಕೃತಜ್ಞತೆಗಳನ್ನು ಅರ್ಪಿಸುತ್ತದೆ.

ರಾಜ್ಯದಲ್ಲಿ, ಈ ಪುಸ್ತಕವನ್ನು ಕನ್ನಡ, ಮರಾಠಿ, ತೆಲುಗು ಮತ್ತು ತಮಿಳು ಮಾಧ್ಯಮಗಳಿಗೆ ಭಾಷಾಂತರಿಸಿದ ಎಲ್ಲ ಸಂಪನ್ಮೂಲ ವ್ಯಕ್ತಿಗಳಿಗೆ, ಕಾರ್ಯಕ್ರಮ ಸಂಯೋಜನೆ ಮಾಡಿದ ಕಾರ್ಯಕ್ರಮಾಧಿಕಾರಿಗೆ, ಸುಂದರವಾಗಿ ಡಿಟಿಪಿ ಕಾರ್ಯವನ್ನು ನಿರ್ವಹಿಸಿರುವ ಡಿಟಿಪಿ ಆಪರೇಟರ್ ಗಳು ಹಾಗೂ ಸಂಸ್ಥೆಗೆ, ಪುಸ್ತಕವನ್ನು ಅಚ್ಚುಕಟ್ಟಾಗಿ ಮುದ್ರಿಸಿ ವಿತರಿಸಿರುವ ಮುದ್ರಕರುಗಳಿಗೆ ಈ ಸಂದರ್ಭದಲ್ಲಿ ಕರ್ನಾಟಕ ಪಠ್ಯಪುಸ್ತಕ ಸಂಘವು ಹೃತ್ಪೂರ್ವಕ ಕೃತಜ್ಞತೆಗಳನ್ನು ಅರ್ಪಿಸುತ್ತದೆ.

ನರಸಿಂಹಯ್ಯ
ವ್ಯವಸ್ಥಾಪಕ ನಿರ್ದೇಶಕರು
ಕರ್ನಾಟಕ ಪಠ್ಯಪುಸ್ತಕ ಸಂಘ (ರಿ)
ಬೆಂಗಳೂರು – 85

PREFACE

Through the years, from the time of the Kothari Commission, there have been several committees looking at ways of making the school curriculum meaningful and enjoyable for the learners. Based on the understanding developed over the years, a National Curriculum Framework (NCF) was finalised in 2005. As part of this exercise, a National Focus Group on Teaching of Mathematics was formed. Its report, which came in 2005, highlighted a constructivist approach to the teaching and learning of mathematics.

The essence of this approach is that children already know, and do some mathematics very naturally in their surroundings, before they even join school. The syllabus, teaching approach, textbooks etc., should build on this knowledge in a way that allows children to enjoy mathematics, and to realise that mathematics is more about a way of reasoning than about mechanically applying formulae and algorithms. The students and teachers need to perceive mathematics as something natural and linked to the world around us. While teaching mathematics, the focus should be on helping children to develop the ability to particularise and generalise, to solve and pose meaningful problems, to look for patterns and relationships, and to apply the logical thinking behind mathematical proof. And, all this in an environment that the children relate to, without overloading them.

This is the philosophy with which the mathematics syllabus from Class I to Class XII was developed, and which the textbook development committee has tried to realise in the present textbook. More specifically, while creating the textbook, the following broad guidelines have been kept in mind.

- The matter needs to be linked to what the child has studied before, and to her experiences.
- The language used in the book, including that for 'word problems', must be clear, simple and unambiguous.
- Concepts/processes should be introduced through situations from the children's environment.
- For each concept/process give several examples and exercises, but not of the same kind. This ensures that the children use the concept/process again and again, but in varying contexts. Here 'several' should be within reason, not overloading the child.
- Encourage the children to see, and come out with, diverse solutions to problems.
- As far as possible, give the children motivation for results used.

- All proofs need to be given in a non-didactic manner, allowing the learner to see the flow of reason. The focus should be on proofs where a short and clear argument reinforces mathematical thinking and reasoning.
- Whenever possible, more than one proof is to be given.
- Proofs and solutions need to be used as vehicles for helping the learner develop a clear and logical way of expressing her arguments.
- All geometric constructions should be accompanied by an analysis of the construction and a proof for the steps taken to do the required construction. Accordingly, the children would be trained to do the same while doing constructions.
- Add such small anecdotes, pictures, cartoons and historical remarks at several places which the children would find interesting.
- Include optional exercises for the more interested learners. These would not be tested in the examinations.
- Give answers to all exercises, and solutions/hints for those that the children may require.
- Whenever possible, propagate constitutional values.

As you will see while studying this textbook, these points have been kept in mind by the Textbook Development Committee. The book has particularly been created with the view to giving children space to explore mathematics and develop the abilities to reason mathematically. Further, two special appendices have been given — Proofs in Mathematics, and Mathematical Modelling. These are placed in the book for interested students to study, and are only optional reading at present. These topics may be considered for inclusion in the main syllabi in due course of time.

As in the past, this textbook is also a team effort. However, what is unusual about the team this time is that teachers from different kinds of schools have been an integral part at each stage of the development. We are also assuming that teachers will contribute continuously to the process in the classroom by formulating examples and exercises contextually suited to the children in their particular classrooms. Finally, we hope that teachers and learners would send comments for improving the textbook to the NCERT.

Parvin Sinclair
G.P. Dikshit
Chief Advisors
Textbook Development Committee

CONTENTS

PART - I

ARITHMETIC PROGRESSIONS 1

1.1 Introduction

You must have observed that in nature, many things follow a certain pattern, such as the petals of a sunflower, the holes of a honeycomb, the grains on a maize cob, the spirals on a pineapple and on a pine cone etc.

We now look for some patterns which occur in our day-to-day life. Some such examples are :

(i) Reena applied for a job and got selected. She has been offered a job with a starting monthly salary of ₹ 8000, with an annual increment of ₹ 500 in her salary. Her salary (in ₹) for the 1st, 2nd, 3rd, . . . years will be, respectively

8000, 8500, 9000, ...

(ii) The lengths of the rungs of a ladder decrease uniformly by 2 cm from bottom to top (see Fig. 5.1). The bottom rung is 45 cm in length. The lengths (in cm) of the 1st, 2nd, 3rd, . . ., 8th rung from the bottom to the top are, respectively

45, 43, 41, 39, 37, 35, 33, 31

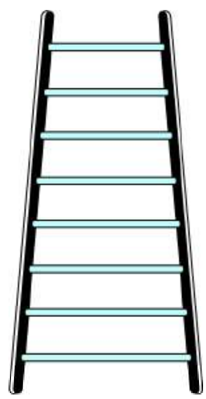


Fig. 1.1

(iii) In a savings scheme, the amount becomes $\frac{5}{4}$ times of itself after every 3 years. The maturity amount (in ₹) of an investment of ₹ 8000 after 3, 6, 9 and 12 years will be, respectively :

10000, 12500, 15625, 19531.25

(iv) The number of unit squares in squares with side 1, 2, 3, . . . units (see Fig. 1.2) are, respectively

$1^2, 2^2, 3^2, \ldots$

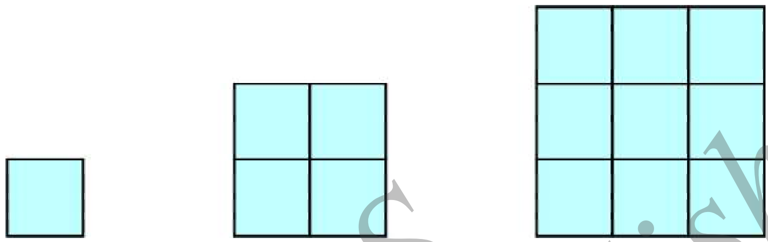


Fig. 1.2

(v) Shakila puts ₹ 100 into her daughter's money box when she was one year old and increased the amount by ₹ 50 every year. The amounts of money (in ₹) in the box on the 1st, 2nd, 3rd, 4th, . . . birthday were

100, 150, 200, 250, . . ., respectively.

(vi) A pair of rabbits are too young to produce in their first month. In the second, and every subsequent month, they produce a new pair. Each new pair of rabbits produce a new pair in their second month and in every subsequent month (see Fig. 1.3). Assuming no rabbit dies, the number of pairs of rabbits at the start of the 1st, 2nd, 3rd, . . ., 6th month, respectively are :

1, 1, 2, 3, 5, 8

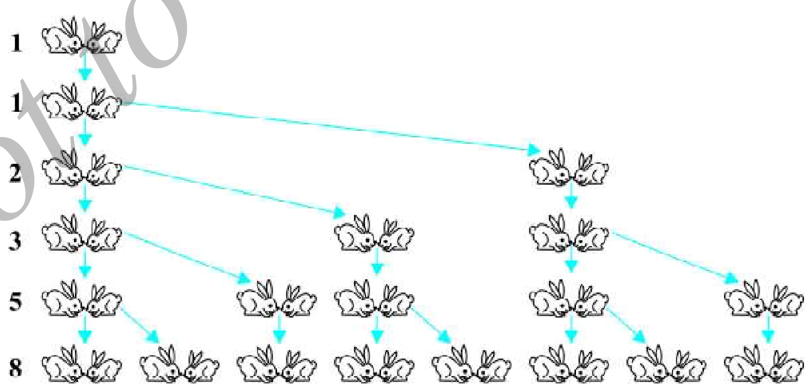


Fig. 1.3

In the examples above, we observe some patterns. In some, we find that the succeeding terms are obtained by adding a fixed number, in other by multiplying with a fixed number, in another we find that they are squares of consecutive numbers, and so on.

In this chapter, we shall discuss one of these patterns in which succeeding terms are obtained by adding a fixed number to the preceding terms. We shall also see how to find their nth terms and the sum of n consecutive terms, and use this knowledge in solving some daily life problems.

1.2 Arithmetic Progressions

Consider the following lists of numbers :

(i) 1, 2, 3, 4, . . .

(ii) 100, 70, 40, 10, . . .

(iii) –3, –2, –1, 0, . . .

(iv) 3, 3, 3, 3, . . .

(v) –1.0, –1.5, –2.0, –2.5, . . .

Each of the numbers in the list is called a **term**.

Given a term, can you write the next term in each of the lists above? If so, how will you write it? Perhaps by following a pattern or rule. Let us observe and write the rule.

In (i), each term is 1 more than the term preceding it.

In (ii), each term is 30 less than the term preceding it.

In (iii), each term is obtained by adding 1 to the term preceding it.

In (iv), all the terms in the list are 3 , i.e., each term is obtained by adding (or subtracting) 0 to the term preceding it.

In (v), each term is obtained by adding – 0.5 to (i.e., subtracting 0.5 from) the term preceding it.

In all the lists above, we see that successive terms are obtained by adding a fixed number to the preceding terms. Such list of numbers is said to form an **Arithmetic Progression (AP)**.

So, **an arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.**

This fixed number is called the **common difference** of the AP. Remember that it **can be positive, negative or zero**.

Let us denote the first term of an AP by a_1, second term by a_2, . . ., nth term by a_n and the common difference by d. Then the AP becomes $a_1, a_2, a_3, \ldots, a_n$.

So, $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$.

Some more examples of AP are:

(a) The heights (in cm) of some students of a school standing in a queue in the morning assembly are 147 , 148, 149, . . ., 157.

(b) The minimum temperatures (in degree celsius) recorded for a week in the month of January in a city, arranged in ascending order are

$$-3.1, -3.0, -2.9, -2.8, -2.7, -2.6, -2.5$$

(c) The balance money (in ₹) after paying 5 % of the total loan of ₹ 1000 every month is 950, 900, 850, 800, . . ., 50.

(d) The cash prizes (in ₹) given by a school to the toppers of Classes I to XII are, respectively, 200, 250, 300, 350, . . ., 750.

(e) The total savings (in ₹) after every month for 10 months when ₹ 50 are saved each month are 50, 100, 150, 200, 250, 300, 350, 400, 450, 500.

It is left as an exercise for you to explain why each of the lists above is an AP.

You can see that

$$a, a + d, a + 2d, a + 3d, \ldots$$

represents an arithmetic progression where a is the first term and d the common difference. This is called the **general form of an AP.**

Note that in examples (a) to (e) above, there are only a finite number of terms. Such an AP is called a **finite AP.** Also note that each of these Arithmetic Progressions (APs) has a last term. The APs in examples (i) to (v) in this section, are not finite APs and so they are called **infinite Arithmetic Progressions**. Such APs do not have a last term.

Now, to know about an AP, what is the minimum information that you need? Is it enough to know the first term? Or, is it enough to know only the common difference? You will find that you will need to know both – the first term a and the common difference d.

For instance if the first term a is 6 and the common difference d is 3, then the AP is

$$6, 9,12, 15, \ldots$$

and if a is 6 and d is – 3, then the AP is

$$6, 3, 0, -3, \ldots$$

Similarly, when

$a = -7, \quad d = -2, \quad$ the AP is $-7, -9, -11, -13, \ldots$

$a = 1.0, \quad d = 0.1, \quad$ the AP is $1.0, 1.1, 1.2, 1.3, \ldots$

$a = 0, \quad d = 1\frac{1}{2}, \quad$ the AP is $0, 1\frac{1}{2}, 3, 4\frac{1}{2}, 6, \ldots$

$a = 2, \quad d = 0, \quad$ the AP is $2, 2, 2, 2, \ldots$

So, if you know what a and d are, you can list the AP. What about the other way round? That is, if you are given a list of numbers can you say that it is an AP and then find a and d? Since a is the first term, it can easily be written. We know that in an AP, every succeeding term is obtained by adding d to the preceding term. So, d found by subtracting any term from its succeeding term, i.e., the term which immediately follows it should be same for an AP.

For example, for the list of numbers :

$$6, 9, 12, 15, \ldots,$$

We have

$$a_2 - a_1 = 9 - 6 = 3,$$
$$a_3 - a_2 = 12 - 9 = 3,$$
$$a_4 - a_3 = 15 - 12 = 3$$

Here the difference of any two consecutive terms in each case is 3. So, the given list is an AP whose first term a is 6 and common difference d is 3.

For the list of numbers : $6, 3, 0, -3, \ldots$,

$$a_2 - a_1 = 3 - 6 = -3$$
$$a_3 - a_2 = 0 - 3 = -3$$
$$a_4 - a_3 = -3 - 0 = -3$$

Similarly this is also an AP whose first term is 6 and the common difference is -3.

In general, for an AP $a_1, a_2, \ldots, a_n$, we have

$$d = a_{k+1} - a_k$$

where a_{k+1} and a_k are the $(k + 1)$th and the kth terms respectively.

To obtain d in a given AP, we need not find all of $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$. It is enough to find only one of them.

Consider the list of numbers $1, 1, 2, 3, 5, \ldots$. By looking at it, you can tell that the difference between any two consecutive terms is not the same. So, this is not an AP.

Note that to find d in the AP : 6, 3, 0, – 3, . . ., we have subtracted 6 from 3 and not 3 from 6, i.e., we should subtract the kth term from the $(k + 1)$ th term even if the $(k + 1)$ th term is smaller.

Let us make the concept more clear through some examples.

Example 1 : For the AP : $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, . . ., write the first term a and the common difference d.

Solution : Here, $a = \frac{3}{2}, d = \frac{1}{2} - \frac{3}{2} = -1$.

Remember that we can find d using any two consecutive terms, once we know that the numbers are in AP.

Example 2 : Which of the following list of numbers form an AP? If they form an AP, write the next two terms :

(i) 4, 10, 16, 22, . . .

(ii) 1, – 1, – 3, – 5, . . .

(iii) – 2, 2, – 2, 2, – 2, . . .

(iv) 1, 1, 1, 2, 2, 2, 3, 3, 3, . . .

Solution : (i) We have

$$a_2 - a_1 = 10 - 4 = 6$$
$$a_3 - a_2 = 16 - 10 = 6$$
$$a_4 - a_3 = 22 - 16 = 6$$

i.e., $a_{k+1} - a_k$ is the same every time.

So, the given list of numbers forms an AP with the common difference $d = 6$.

The next two terms are: 22 + 6 = 28 and 28 + 6 = 34.

(ii) $a_2 - a_1 = -1 - 1 = -2$

$a_3 - a_2 = -3 - (-1) = -3 + 1 = -2$

$a_4 - a_3 = -5 - (-3) = -5 + 3 = -2$

i.e., $a_{k+1} - a_k$ is the same every time.

So, the given list of numbers forms an AP with the common difference $d = -2$.

The next two terms are:

$$-5 + (-2) = -7 \quad \text{and} \quad -7 + (-2) = -9$$

(iii) $a_2 - a_1 = 2 - (-2) = 2 + 2 = 4$

$a_3 - a_2 = -2 - 2 = -4$

As $a_2 - a_1 \neq a_3 - a_2$, the given list of numbers does not form an AP.

(iv) $a_2 - a_1 = 1 - 1 = 0$
$a_3 - a_2 = 1 - 1 = 0$
$a_4 - a_3 = 2 - 1 = 1$

Here, $a_2 - a_1 = a_3 - a_2 \neq a_4 - a_3$.

So, the given list of numbers does not form an AP.

EXERCISE 1.1

1. In which of the following situations, does the list of numbers involved make an arithmetic progression, and why?
   (i) The taxi fare after each km when the fare is ₹ 15 for the first km and ₹ 8 for each additional km.
   (ii) The amount of air present in a cylinder when a vacuum pump removes $\frac{1}{4}$ of the air remaining in the cylinder at a time.
   (iii) The cost of digging a well after every metre of digging, when it costs ₹ 150 for the first metre and rises by ₹ 50 for each subsequent metre.
   (iv) The amount of money in the account every year, when ₹ 10000 is deposited at compound interest at 8 % per annum.

2. Write first four terms of the AP, when the first term a and the common difference d are given as follows:
   (i) $a = 10$, $d = 10$ (ii) $a = -2$, $d = 0$
   (iii) $a = 4$, $d = -3$ (iv) $a = -1$, $d = \frac{1}{2}$
   (v) $a = -1.25$, $d = -0.25$

3. For the following APs, write the first term and the common difference:
   (i) $3, 1, -1, -3, \ldots$ (ii) $-5, -1, 3, 7, \ldots$
   (iii) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}, \ldots$ (iv) $0.6, 1.7, 2.8, 3.9, \ldots$

4. Which of the following are APs ? If they form an AP, find the common difference d and write three more terms.
   (i) $2, 4, 8, 16, \ldots$ (ii) $2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$
   (iii) $-1.2, -3.2, -5.2, -7.2, \ldots$ (iv) $-10, -6, -2, 2, \ldots$
   (v) $3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2}, \ldots$ (vi) $0.2, 0.22, 0.222, 0.2222, \ldots$
   (vii) $0, -4, -8, -12, \ldots$ (viii) $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots$

(ix) $1, 3, 9, 27, \ldots$

(x) $a, 2a, 3a, 4a, \ldots$

(xi) $a, a^2, a^3, a^4, \ldots$

(xii) $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, \ldots$

(xiii) $\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \ldots$

(xiv) $1^2, 3^2, 5^2, 7^2, \ldots$

(xv) $1^2, 5^2, 7^2, 73, \ldots$

1.3 *n*th Term of an AP

Let us consider the situation again, given in Section 1.1 in which Reena applied for a job and got selected. She has been offered the job with a starting monthly salary of ₹ 8000, with an annual increment of ₹ 500. What would be her monthly salary for the fifth year?

To answer this, let us first see what her monthly salary for the second year would be.

It would be ₹ (8000 + 500) = ₹ 8500. In the same way, we can find the monthly salary for the 3rd, 4th and 5th year by adding ₹ 500 to the salary of the previous year. So, the salary for the 3rd year = ₹ (8500 + 500)

= ₹ (8000 + 500 + 500)

= ₹ (8000 + 2 × 500)

= ₹ [8000 + **(3 – 1)** × 500] (for the **3rd year**)

= ₹ 9000

Salary for the 4th year = ₹ (9000 + 500)

= ₹ (8000 + 500 + 500 + 500)

= ₹ (8000 + 3 × 500)

= ₹ [8000 + **(4 – 1)** × 500] (for the **4th year**)

= ₹ 9500

Salary for the 5th year = ₹ (9500 + 500)

= ₹ (8000+500+500+500 + 500)

= ₹ (8000 + 4 × 500)

= ₹ [8000 + **(5 – 1)** × 500] (for the **5th year**)

= ₹ 10000

Observe that we are getting a list of numbers

$$8000, 8500, 9000, 9500, 10000, \ldots$$

These numbers are in AP. (Why?)

Now, looking at the pattern formed above, can you find her monthly salary for the 6th year? The 15th year? And, assuming that she will still be working in the job, what about the monthly salary for the 25th year? You would calculate this by adding ₹ 500 each time to the salary of the previous year to give the answer. Can we make this process shorter? Let us see. You may have already got some idea from the way we have obtained the salaries above.

Salary for the 15th year

$$= \text{Salary for the 14th year} + ₹\ 500$$

$$= ₹\left[8000 + \underbrace{500 + 500 + 500 + \ldots + 500}_{13 \text{ times}}\right] + ₹\ 500$$

$$= ₹\ [8000 + 14 \times 500]$$

$$= ₹\ [8000 + \mathbf{(15 - 1)} \times 500] = ₹\ 15000$$

i.e., First salary + (15 – 1) × Annual increment.

In the same way, her monthly salary for the 25th year would be

$$₹\ [8000 + \mathbf{(25 - 1)} \times 500] = ₹\ 20000$$

$$= \text{First salary} + \mathbf{(25 - 1)} \times \text{Annual increment}$$

This example would have given you some idea about how to write the 15th term, or the 25th term, and more generally, the nth term of the AP.

Let $a_1, a_2, a_3, \ldots$ be an AP whose first term a_1 is a and the common difference is d.

Then,

the **second term** $a_2 = a + d = a + \mathbf{(2 - 1)}\ d$

the **third** term $a_3 = a_2 + d = (a + d) + d = a + 2d = a + \mathbf{(3 - 1)}\ d$

the **fourth** term $a_4 = a_3 + d = (a + 2d) + d = a + 3d = a + \mathbf{(4 - 1)}\ d$

. . . .

. . . .

Looking at the pattern, we can say that the ***n*th** term $a_n = a + (n - 1)\ d$.

So, **the *n*th term a_n of the AP with first term a and common difference d is given by $a_n = a + (n - 1)\ d$.**

a_n is also called the **general term of the AP**. If there are m terms in the AP, then a_m represents the **last term which is sometimes also denoted by l**.

Let us consider some examples.

Example 3 : Find the 10th term of the AP : 2, 7, 12, . . .

Solution : Here, $a = 2$, $d = 7 - 2 = 5$ and $n = 10$.

We have $a_n = a + (n - 1)\, d$

So, $a_{10} = 2 + (10 - 1) \times 5 = 2 + 45 = 47$

Therefore, the 10th term of the given AP is 47.

Example 4 : Which term of the AP : 21, 18, 15, . . . is -81? Also, is any term 0? Give reason for your answer.

Solution : Here, $a = 21$, $d = 18 - 21 = -3$ and $a_n = -81$, and we have to find n.

As $a_n = a + (n - 1)\, d$,

we have $-81 = 21 + (n - 1)(-3)$

$-81 = 24 - 3n$

$-105 = -3n$

So, $n = 35$

Therefore, the 35th term of the given AP is -81.

Next, we want to know if there is any n for which $a_n = 0$. If such an n is there, then

$$21 + (n - 1)(-3) = 0,$$

i.e., $3(n - 1) = 21$

i.e., $n = 8$

So, the eighth term is 0.

Example 5 : Determine the AP whose 3rd term is 5 and the 7th term is 9.

Solution : We have

$$a_3 = a + (3 - 1)\, d = a + 2d = 5 \quad (1)$$

and $$a_7 = a + (7 - 1)\, d = a + 6d = 9 \quad (2)$$

Solving the pair of linear equations (1) and (2), we get

$$a = 3, \quad d = 1$$

Hence, the required AP is 3, 4, 5, 6, 7, . . .

Example 6 : Check whether 301 is a term of the list of numbers 5, 11, 17, 23, . . .

Solution : We have :

$$a_2 - a_1 = 11 - 5 = 6, \quad a_3 - a_2 = 17 - 11 = 6, \quad a_4 - a_3 = 23 - 17 = 6$$

As $a_{k+1} - a_k$ is the same for $k = 1, 2, 3$, etc., the given list of numbers is an AP.

Now, $a = 5$ and $d = 6$.

Let 301 be a term, say, the nth term of this AP.

We know that

$$a_n = a + (n - 1)\, d$$

So, $$301 = 5 + (n - 1) \times 6$$

i.e., $$301 = 6n - 1$$

So, $$n = \frac{302}{6} = \frac{151}{3}$$

But n should be a positive integer (Why?). So, 301 is not a term of the given list of numbers.

Example 7 : How many two-digit numbers are divisible by 3?

Solution : The list of two-digit numbers divisible by 3 is :

$$12, 15, 18, \ldots, 99$$

Is this an AP? Yes it is. Here, $a = 12$, $d = 3$, $a_n = 99$.

As $$a_n = a + (n - 1)\, d,$$

we have $$99 = 12 + (n - 1) \times 3$$

i.e., $$87 = (n - 1) \times 3$$

i.e., $$n - 1 = \frac{87}{3} = 29$$

i.e., $$n = 29 + 1 = 30$$

So, there are 30 two-digit numbers divisible by 3.

Example 8 : Find the 11th term from the last term (towards the first term) of the AP : 10, 7, 4, . . ., – 62.

Solution : Here, $a = 10$, $d = 7 - 10 = -3$, $l = -62$,

where $$l = a + (n - 1)\, d$$

To find the 11th term from the last term, we will find the total number of terms in the AP.

So, $-62 = 10 + (n-1)(-3)$

i.e., $-72 = (n-1)(-3)$

i.e., $n - 1 = 24$

or $n = 25$

So, there are 25 terms in the given AP.

The 11th term from the last term will be the 15th term. (Note that it will not be the 14th term. Why?)

So, $a_{15} = 10 + (15-1)(-3) = 10 - 42 = -32$

i.e., the 11th term from the last term is -32.

Alternative Solution :

If we write the given AP in the reverse order, then $a = -62$ and $d = 3$ (Why?)

So, the question now becomes finding the 11th term with these a and d.

So, $a_{11} = -62 + (11-1) \times 3 = -62 + 30 = -32$

So, the 11th term, which is now the required term, is -32.

Example 9 : A sum of ₹ 1000 is invested at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an AP? If so, find the interest at the end of 30 years making use of this fact.

Solution : We know that the formula to calculate simple interest is given by

$$\text{Simple Interest} = \frac{\text{P} \times \text{R} \times \text{T}}{100}$$

So, the interest at the end of the 1st year = ₹ $\frac{1000 \times 8 \times 1}{100}$ = ₹ 80

The interest at the end of the 2nd year = ₹ $\frac{1000 \times 8 \times 2}{100}$ = ₹ 160

The interest at the end of the 3rd year = ₹ $\frac{1000 \times 8 \times 3}{100}$ = ₹ 240

Similarly, we can obtain the interest at the end of the 4th year, 5th year, and so on.

So, the interest (in ₹) at the end of the 1st, 2nd, 3rd, . . . years, respectively are

80, 160, 240, . . .

It is an AP as the difference between the consecutive terms in the list is 80, i.e., $d = 80$. Also, $a = 80$.

So, to find the interest at the end of 30 years, we shall find a_{30}.

Now, $a_{30} = a + (30 - 1)\,d = 80 + 29 \times 80 = 2400$

So, the interest at the end of 30 years will be ₹ 2400.

Example 10 : In a flower bed, there are 23 rose plants in the first row, 21 in the second, 19 in the third, and so on. There are 5 rose plants in the last row. How many rows are there in the flower bed?

Solution : The number of rose plants in the 1st, 2nd, 3rd, . . ., rows are :

$$23, 21, 19, \ldots, 5$$

It forms an AP (Why?). Let the number of rows in the flower bed be n.

Then $a = 23$, $d = 21 - 23 = -2$, $a_n = 5$

As, $a_n = a + (n - 1)\,d$

We have, $5 = 23 + (n - 1)(-2)$

i.e., $-18 = (n - 1)(-2)$

i.e., $n = 10$

So, there are 10 rows in the flower bed.

EXERCISE 1.2

1. Fill in the blanks in the following table, given that a is the first term, d the common difference and a_n the nth term of the AP:

	a	d	n	a_n
(i)	7	3	8	. . .
(ii)	– 18	. . .	10	0
(iii)	. . .	– 3	18	– 5
(iv)	– 18.9	2.5	. . .	3.6
(v)	3.5	0	105	. . .

2. Choose the correct choice in the following and justify :

(i) 30th term of the AP: 10, 7, 4, . . . , is

(A) 97 (B) 77 (C) –77 (D) – 87

(ii) 11th term of the AP: $-3, -\frac{1}{2}, 2, \ldots$, is

(A) 28 (B) 22 (C) –38 (D) $-48\frac{1}{2}$

3. In the following APs, find the missing terms in the boxes :

(i) 2, $\square$, 26

(ii) $\square$, 13, $\square$, 3

(iii) 5, $\square$, $\square$, $9\frac{1}{2}$

(iv) – 4, $\square$, $\square$, $\square$, $\square$, 6

(v) $\square$, 38, $\square$, $\square$, $\square$, – 22

4. Which term of the AP : 3, 8, 13, 18, . . . ,is 78?

5. Find the number of terms in each of the following APs :

(i) 7, 13, 19, . . . , 205 (ii) 18, $15\frac{1}{2}$, 13, . . . , – 47

6. Check whether – 150 is a term of the AP : 11, 8, 5, 2 . . .

7. Find the 31st term of an AP whose 11th term is 38 and the 16th term is 73.

8. An AP consists of 50 terms of which 3rd term is 12 and the last term is 106. Find the 29th term.

9. If the 3rd and the 9th terms of an AP are 4 and – 8 respectively, which term of this AP is zero?

10. The 17th term of an AP exceeds its 10th term by 7. Find the common difference.

11. Which term of the AP : 3, 15, 27, 39, . . . will be 132 more than its 54th term?

12. Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?

13. How many three-digit numbers are divisible by 7?

14. How many multiples of 4 lie between 10 and 250?

15. For what value of n, are the nth terms of two APs: 63, 65, 67, . . . and 3, 10, 17, . . . equal?

16. Determine the AP whose third term is 16 and the 7th term exceeds the 5th term by 12.

17. Find the 20th term from the last term of the AP : 3, 8, 13, . . ., 253.

18. The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

19. Subba Rao started work in 1995 at an annual salary of ₹ 5000 and received an increment of ₹ 200 each year. In which year did his income reach ₹ 7000?

20. Ramkali saved ₹ 5 in the first week of a year and then increased her weekly savings by ₹ 1.75. If in the nth week, her weekly savings become ₹ 20.75, find n.

1.4 Sum of First *n* Terms of an AP

Let us consider the situation again given in Section 5.1 in which Shakila put ₹ 100 into her daughter's money box when she was one year old, ₹ 150 on her second birthday, ₹ 200 on her third birthday and will continue in the same way. How much money will be collected in the money box by the time her daughter is 21 years old?

Here, the amount of money (in ₹) put in the money box on her first, second, third, fourth . . . birthday were respectively 100, 150, 200, 250, . . . till her 21st birthday. To find the total amount in the money box on her 21st birthday, we will have to write each of the 21 numbers in the list above and then add them up. Don't you think it would be a tedious and time consuming process? Can we make the process shorter? This would be possible if we can find a method for getting this sum. Let us see.

We consider the problem given to Gauss (about whom you will read in Chapter 8), to solve when he was just 10 years old. He was asked to find the sum of the positive integers from 1 to 100. He immediately replied that the sum is 5050. Can you guess how did he do? He wrote :

$$S = 1 + 2 + 3 + . . . + 99 + 100$$

And then, reversed the numbers to write

$$S = 100 + 99 + . . . + 3 + 2 + 1$$

Adding these two, he got

$$2S = (100 + 1) + (99 + 2) + . . . + (3 + 98) + (2 + 99) + (1 + 100)$$
$$= 101 + 101 + . . . + 101 + 101 \quad \text{(100 times)}$$

So, $$S = \frac{100 \times 101}{2} = 5050 \text{, i.e., the sum} = 5050.$$

We will now use the same technique to find the sum of the first n terms of an AP :

$$a, a + d, a + 2d, \ldots$$

The nth term of this AP is $a + (n - 1) d$. Let S denote the sum of the first n terms of the AP. We have

$$S = a + (a + d) + (a + 2d) + \ldots + [a + (n - 1) d] \qquad (1)$$

Rewriting the terms in reverse order, we have

$$S = [a + (n - 1) d] + [a + (n - 2) d] + \ldots + (a + d) + a \qquad (2)$$

On adding (1) and (2), term-wise. we get

$$2S = \underbrace{[2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d] + [2a + (n-1)d]}_{n \text{ times}}$$

or, $2S = n\,[2a + (n - 1)\,d]$ (Since, there are n terms)

or, $S = \dfrac{n}{2}\,[2a + (n - 1)\,d]$

So, **the sum of the first *n* terms of an AP is given by**

$$\mathbf{S = \frac{n}{2}\,[2a + (n - 1)\,d]}$$

We can also write this as $S = \dfrac{n}{2}\,[a + a + (n - 1)\,d]$

i.e., $$\mathbf{S = \frac{n}{2}\,(a + a_n)} \qquad (3)$$

Now, if there are only n terms in an AP, then $a_n = l$, the last term. From (3), we see that

$$\mathbf{S = \frac{n}{2}\,(a + l)} \qquad (4)$$

This form of the result is useful when the first and the last terms of an AP are given and the common difference is not given.

Now we return to the question that was posed to us in the beginning. The amount of money (in ₹) in the money box of Shakila's daughter on 1st, 2nd, 3rd, 4th birthday, . . ., were 100, 150, 200, 250, . . ., respectively.

This is an AP. We have to find the total money collected on her 21st birthday, i.e., the sum of the first 21 terms of this AP.

Here, $a = 100$, $d = 50$ and $n = 21$. Using the formula :

$$S = \frac{n}{2}[2a + (n-1)\,d],$$

we have $$S = \frac{21}{2}[2 \times 100 + (21-1) \times 50] = \frac{21}{2}[200 + 1000]$$

$$= \frac{21}{2} \times 1200 = 12600$$

So, the amount of money collected on her 21st birthday is ₹ 12600.

Hasn't the use of the formula made it much easier to solve the problem?

We also use S_n in place of S to denote the sum of first n terms of the AP. We write S_{20} to denote the sum of the first 20 terms of an AP. The formula for the sum of the first n terms involves four quantities S, a, d and n. If we know any three of them, we can find the fourth.

Remark : The nth term of an AP is the difference of the sum to first n terms and the sum to first $(n-1)$ terms of it, i.e., $a_n = S_n - S_{n-1}$.

Let us consider some examples.

Example 11 : Find the sum of the first 22 terms of the AP : 8, 3, –2, . . .

Solution : Here, $a = 8$, $d = 3 - 8 = -5$, $n = 22$.

We know that

$$S = \frac{n}{2}[2a + (n-1)\,d]$$

Therefore, $$S = \frac{22}{2}[16 + 21(-5)] = 11(16 - 105) = 11(-89) = -979$$

So, the sum of the first 22 terms of the AP is – 979.

Example 12 : If the sum of the first 14 terms of an AP is 1050 and its first term is 10, find the 20th term.

Solution : Here, $S_{14} = 1050$, $n = 14$, $a = 10$.

As $$S_n = \frac{n}{2}[2a + (n-1)d],$$

so, $$1050 = \frac{14}{2}[20 + 13d] = 140 + 91d$$

i.e., $910 = 91d$

or, $d = 10$

Therefore, $a_{20} = 10 + (20 - 1) \times 10 = 200$, i.e. 20th term is 200.

Example 13 : How many terms of the AP : 24, 21, 18, . . . must be taken so that their sum is 78?

Solution : Here, $a = 24$, $d = 21 - 24 = -3$, $S_n = 78$. We need to find n.

We know that $S_n = \frac{n}{2}[2a + (n-1)d]$

So, $78 = \frac{n}{2}[48 + (n-1)(-3)] = \frac{n}{2}[51 - 3n]$

or $3n^2 - 51n + 156 = 0$

or $n^2 - 17n + 52 = 0$

or $(n - 4)(n - 13) = 0$

or $n = 4$ or 13

Both values of n are admissible. So, the number of terms is either 4 or 13.

Remarks:

1. In this case, the sum of the first 4 terms = the sum of the first 13 terms = 78.
2. Two answers are possible because the sum of the terms from 5th to 13th will be zero. This is because a is positive and d is negative, so that some terms will be positive and some others negative, and will cancel out each other.

Example 14 : Find the sum of :

(i) the first 1000 positive integers (ii) the first n positive integers

Solution :

(i) Let $S = 1 + 2 + 3 + . . . + 1000$

Using the formula $S_n = \frac{n}{2}(a + l)$ for the sum of the first n terms of an AP, we have

$$S_{1000} = \frac{1000}{2}(1 + 1000) = 500 \times 1001 = 500500$$

So, the sum of the first 1000 positive integers is 500500.

(ii) Let $S_n = 1 + 2 + 3 + . . . + n$

Here $a = 1$ and the last term l is n.

Therefore, $S_n = \frac{n(1+n)}{2}$ or $S_n = \frac{n(n+1)}{2}$

So, **the sum of first *n* positive integers is given by**

$$\mathbf{S_n = \frac{n(n+1)}{2}}$$

Example 15 : Find the sum of first 24 terms of the list of numbers whose nth term is given by

$$a_n = 3 + 2n$$

Solution :

As $a_n = 3 + 2n$,

so, $a_1 = 3 + 2 = 5$

$a_2 = 3 + 2 \times 2 = 7$

$a_3 = 3 + 2 \times 3 = 9$

$\vdots$

List of numbers becomes 5, 7, 9, 11, . . .

Here, $7 - 5 = 9 - 7 = 11 - 9 = 2$ and so on.

So, it forms an AP with common difference $d = 2$.

To find S_{24}, we have $n = 24$, $a = 5$, $d = 2$.

Therefore, $S_{24} = \frac{24}{2}\left[2 \times 5 + (24-1) \times 2\right] = 12\left[10 + 46\right] = 672$

So, sum of first 24 terms of the list of numbers is 672.

Example 16 : A manufacturer of TV sets produced 600 sets in the third year and 700 sets in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find :

(i) the production in the 1st year

(ii) the production in the 10th year

(iii) the total production in first 7 years

Solution : (i) Since the production increases uniformly by a fixed number every year, the number of TV sets manufactured in 1st, 2nd, 3rd, . . ., years will form an AP.

Let us denote the number of TV sets manufactured in the nth year by a_n.

Then, $a_3 = 600$ and $a_7 = 700$

or, $a + 2d = 600$

and $a + 6d = 700$

Solving these equations, we get $d = 25$ and $a = 550$.

Therefore, production of TV sets in the first year is 550.

(ii) Now $a_{10} = a + 9d = 550 + 9 \times 25 = 775$

So, production of TV sets in the 10th year is 775.

(iii) Also, $$S_7 = \frac{7}{2}[2 \times 550 + (7-1) \times 25]$$

$$= \frac{7}{2}[1100 + 150] = 4375$$

Thus, the total production of TV sets in first 7 years is 4375.

EXERCISE 1.3

1. Find the sum of the following APs:

(i) 2, 7, 12, . . ., to 10 terms.

(ii) –37, –33, –29, . . ., to 12 terms.

(iii) 0.6, 1.7, 2.8, . . ., to 100 terms.

(iv) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$, . . ., to 11 terms.

2. Find the sums given below :

(i) $7 + 10\frac{1}{2} + 14 + \ldots + 84$

(ii) $34 + 32 + 30 + \ldots + 10$

(iii) $-5 + (-8) + (-11) + \ldots + (-230)$

3. In an AP:

(i) given $a = 5$, $d = 3$, $a_n = 50$, find n and S_n.

(ii) given $a = 7$, $a_{13} = 35$, find d and S_{13}.

(iii) given $a_{12} = 37$, $d = 3$, find a and S_{12}.

(iv) given $a_3 = 15$, $S_{10} = 125$, find d and a_{10}.

(v) given $d = 5$, $S_9 = 75$, find a and a_9.

(vi) given $a = 2$, $d = 8$, $S_n = 90$, find n and a_n.

(vii) given $a = 8$, $a_n = 62$, $S_n = 210$, find n and d.

(viii) given $a_n = 4$, $d = 2$, $S_n = -14$, find n and a.

(ix) given $a = 3$, $n = 8$, $S = 192$, find d.

(x) given $l = 28$, $S = 144$, and there are total 9 terms. Find a.

4. How many terms of the AP : 9, 17, 25, . . . must be taken to give a sum of 636?

5. The first term of an AP is 5, the last term is 45 and the sum is 400. Find the number of terms and the common difference.

6. The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

7. Find the sum of first 22 terms of an AP in which $d = 7$ and 22nd term is 149.

8. Find the sum of first 51 terms of an AP whose second and third terms are 14 and 18 respectively.

9. If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

10. Show that $a_1, a_2, \ldots, a_n, \ldots$ form an AP where a_n is defined as below :

(i) $a_n = 3 + 4n$ (ii) $a_n = 9 - 5n$

Also find the sum of the first 15 terms in each case.

11. If the sum of the first n terms of an AP is $4n - n^2$, what is the first term (that is S_1)? What is the sum of first two terms? What is the second term? Similarly, find the 3rd, the 10th and the nth terms.

12. Find the sum of the first 40 positive integers divisible by 6.

13. Find the sum of the first 15 multiples of 8.

14. Find the sum of the odd numbers between 0 and 50.

15. A contract on construction job specifies a penalty for delay of completion beyond a certain date as follows: ₹ 200 for the first day, ₹ 250 for the second day, ₹ 300 for the third day, etc., the penalty for each succeeding day being ₹ 50 more than for the preceding day. How much money the contractor has to pay as penalty, if he has delayed the work by 30 days?

16. A sum of ₹ 700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is ₹ 20 less than its preceding prize, find the value of each of the prizes.

17. In a school, students thought of planting trees in and around the school to reduce air pollution. It was decided that the number of trees, that each section of each class will plant, will be the same as the class, in which they are studying, e.g., a section of Class I will plant 1 tree, a section of Class II will plant 2 trees and so on till Class XII. There are three sections of each class. How many trees will be planted by the students?

18. A spiral is made up of successive semicircles, with centres alternately at A and B, starting with centre at A, of radii 0.5 cm, 1.0 cm, 1.5 cm, 2.0 cm, . . . as shown in Fig. 1.4. What is the total length of such a spiral made up of thirteen consecutive semicircles? (Take $\pi = \frac{22}{7}$)

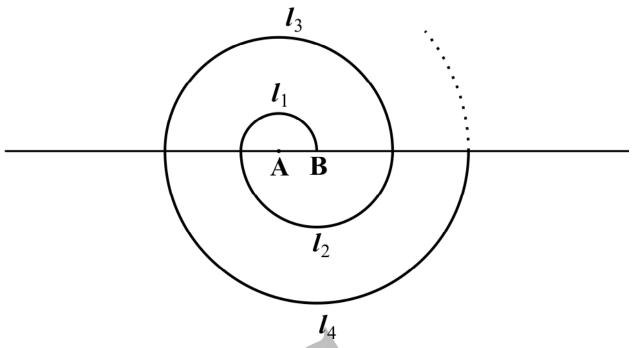


Fig. 1.4

[**Hint :** Length of successive semicircles is $l_1, l_2, l_3, l_4, \ldots$ with centres at A, B, A, B, . . ., respectively.]

19. 200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on (see Fig. 1.5). In how many rows are the 200 logs placed and how many logs are in the top row?

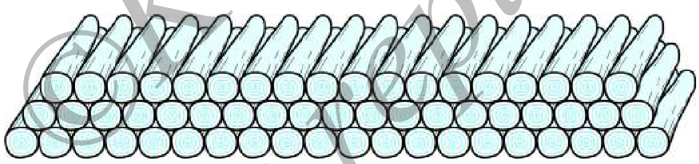


Fig. 1.5

20. In a potato race, a bucket is placed at the starting point, which is 5 m from the first potato, and the other potatoes are placed 3 m apart in a straight line. There are ten potatoes in the line (see Fig. 1.6).

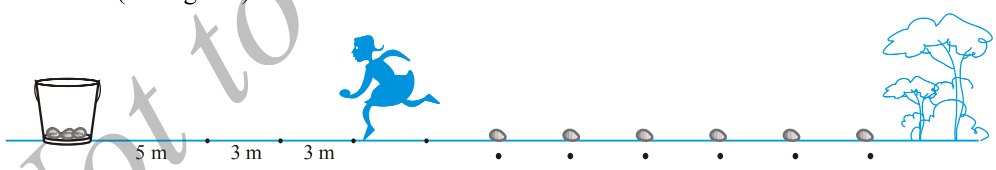


Fig. 1.6

A competitor starts from the bucket, picks up the nearest potato, runs back with it, drops it in the bucket, runs back to pick up the next potato, runs to the bucket to drop it in, and she continues in the same way until all the potatoes are in the bucket. What is the total distance the competitor has to run?

[**Hint :** To pick up the first potato and the second potato, the total distance (in metres) run by a competitor is $2 \times 5 + 2 \times (5 + 3)$]

EXERCISE 1.4 (Optional)*

1. Which term of the AP : 121, 117, 113, . . ., is its first negative term?

   [**Hint :** Find n for $a_n < 0$]

2. The sum of the third and the seventh terms of an AP is 6 and their product is 8. Find the sum of first sixteen terms of the AP.

3. A ladder has rungs 25 cm apart. (see Fig. 1.7). The rungs decrease uniformly in length from 45 cm at the bottom to 25 cm at the top. If the top and the bottom rungs are $2\frac{1}{2}$ m apart, what is the length of the wood required for the rungs?

   [**Hint :** Number of rungs $= \frac{250}{25} + 1$]

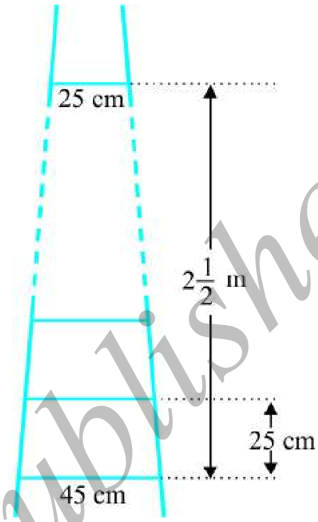


Fig. 1.7

4. The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. Find this value of x.

   [**Hint :** $S_{x-1} = S_{49} - S_x$]

5. A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete.

   Each step has a rise of $\frac{1}{4}$ m and a tread of $\frac{1}{2}$ m. (see Fig. 1.8). Calculate the total volume of concrete required to build the terrace.

   [**Hint :** Volume of concrete required to build the first step $= \frac{1}{4} \times \frac{1}{2} \times 50$ m^3]

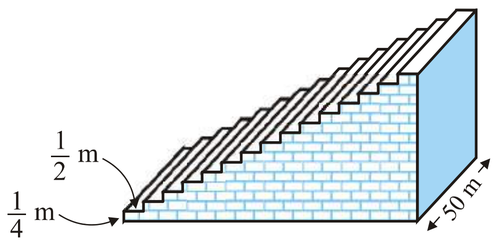


Fig. 1.8

* These exercises are not from the examination point of view.

1.5 Summary

In this chapter, you have studied the following points :

1. An **arithmetic progression** (AP) is a list of numbers in which each term is obtained by adding a fixed number d to the preceding term, except the first term. The fixed number d is called the **common difference**.

The general form of an AP is $a,\ a + d,\ a + 2d,\ a + 3d, \ldots$

2. A given list of numbers $a_1, a_2, a_3, \ldots$ is an AP, if the differences $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$, give the same value, i.e., if $a_{k+1} - a_k$ is the same for different values of k.

3. In an AP with first term a and common difference d, the nth term (or the general term) is given by $a_n = a + (n - 1)\, d$.

4. The sum of the first n terms of an AP is given by :

$$S = \frac{n}{2}\left[2a + (n-1)d\right]$$

5. If l is the last term of the finite AP, say the nth term, then the sum of all terms of the AP is given by :

$$S = \frac{n}{2}(a + l)$$

A NOTE TO THE READER

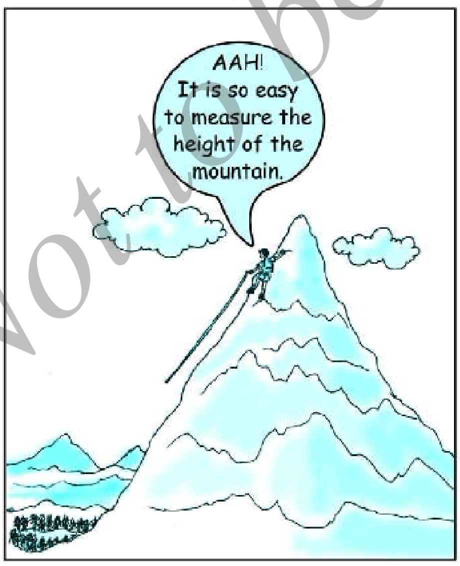
If a, b, c are in AP, then $b = \dfrac{a+c}{2}$ and b is called the arithmetic mean of a and c.

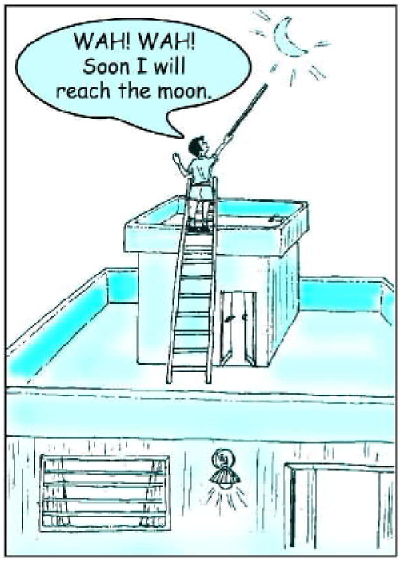
TRIANGLES 2

2.1 Introduction

You are familiar with triangles and many of their properties from your earlier classes. In Class IX, you have studied congruence of triangles in detail. Recall that two figures are said to be *congruent*, if they have the same shape and the same size. In this chapter, we shall study about those figures which have the same shape but not necessarily the same size. Two figures having the same shape (and not necessarily the same size) are called *similar figures*. In particular, we shall discuss the similarity of triangles and apply this knowledge in giving a simple proof of Pythagoras Theorem learnt earlier.

Can you guess how heights of mountains (say Mount Everest) or distances of some long distant objects (say moon) have been found out? Do you think these have





been measured directly with the help of a measuring tape? In fact, all these heights and distances have been found out using the idea of indirect measurements, which is based on the principle of similarity of figures (see Example 7, Q.15 of Exercise 2.3 and also Chapters 11 and 12 of this book).

2.2 Similar Figures

In Class IX, you have seen that all circles with the same radii are congruent, all squares with the same side lengths are congruent and all equilateral triangles with the same side lengths are congruent.

Now consider any two (or more) circles [see Fig. 2.1 (i)]. Are they congruent? Since all of them do not have the same radius, they are not congruent to each other. Note that some are congruent and some are not, but all of them have the same shape. So they all are, what we call, *similar*. Two similar figures have the same shape but not necessarily the same size. Therefore, all circles are similar. What about two (or more) squares or two (or more) equilateral triangles [see Fig.2.1 (ii) and (iii)]? As observed in the case of circles, here also all squares are similar and all equilateral triangles are similar.

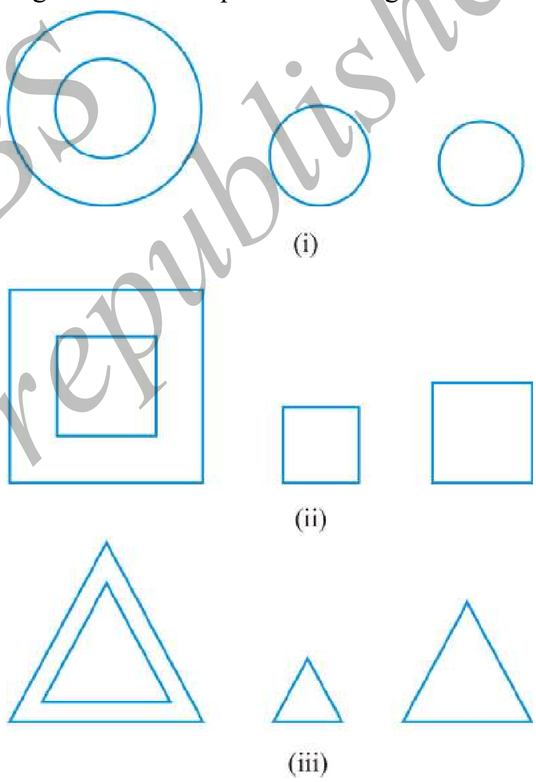


Fig. 2.1

From the above, we can say *that all congruent figures are similar but the similar figures need not be congruent.*

Can a circle and a square be similar? Can a triangle and a square be similar? These questions can be answered by just looking at the figures (see Fig. 2.1). Evidently these figures are not similar. (Why?)

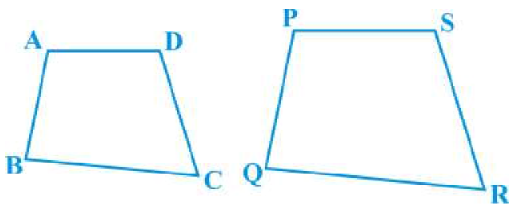


Fig. 2.2

What can you say about the two quadrilaterals ABCD and PQRS (see Fig 2.2)?Are they similar? These figures appear to be similar but we cannot be certain about it.Therefore, we must have some definition of similarity of figures and based on this definition some rules to decide whether the two given figures are similar or not. For this, let us look at the photographs given in Fig. 2.3:

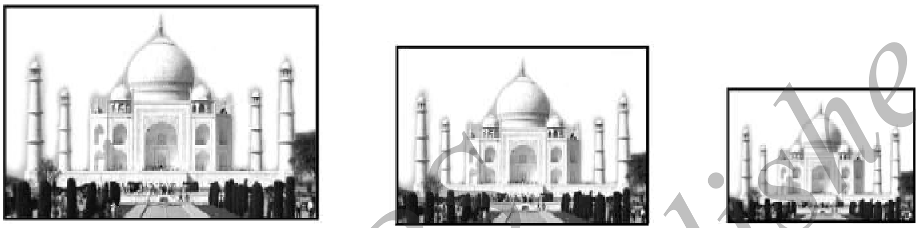


Fig. 2.3

You will at once say that they are the photographs of the same monument (Taj Mahal) but are in different sizes. Would you say that the three photographs are similar? Yes,they are.

What can you say about the two photographs of the same size of the same person one at the age of 10 years and the other at the age of 40 years? Are these photographs similar? These photographs are of the same size but certainly they are not of the same shape. So, they are not similar.

What does the photographer do when she prints photographs of different sizes from the same negative? You must have heard about the stamp size, passport size and postcard size photographs. She generally takes a photograph on a small size film, say of 35mm size and then enlarges it into a bigger size, say 45mm (or 55mm). Thus, if we consider any line segment in the smaller photograph (figure), its corresponding line segment in the bigger photograph (figure) will be $\frac{45}{35}\left(\text{or } \frac{55}{35}\right)$ of that of the line segment. This really means that every line segment of the smaller photograph is enlarged (increased) *in the ratio* 35:45 (or 35:55). It can also be said that every line segment of the bigger photograph is reduced (decreased) in the ratio 45:35 (or 55:35). Further, if you consider inclinations (or angles) between any pair of corresponding line segments in the two photographs of different sizes, you shall see that these inclinations(or angles) *are always equal.* This is the essence of the similarity of two figures and in particular of two polygons. We say that:

Two polygons of the same number of sides are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (or proportion).

Note that the same ratio of the corresponding sides is referred to as *the scale factor* (or the *Representative Fraction*) for the polygons. You must have heard that world maps (i.e., global maps) and blue prints for the construction of a building are prepared using a suitable scale factor and observing certain conventions.

In order to understand similarity of figures more clearly, let us perform the following activity:

Activity 1 : Place a lighted bulb at a point O on the ceiling and directly below it a table in your classroom. Let us cut a polygon, say a quadrilateral ABCD, from a plane cardboard and place this cardboard parallel to the ground between the lighted bulb and the table. Then a shadow of ABCD is cast on the table. Mark the outline of this shadow as A′B′C′D′ (see Fig.2.4).

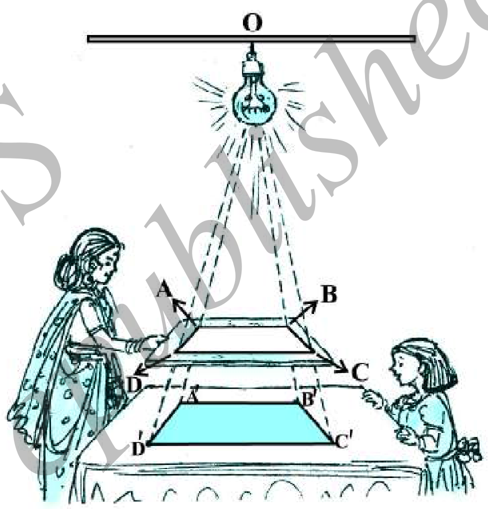


Fig.2.4

Note that the quadrilateral A′B′C′D′ is an enlargement (or magnification) of the quadrilateral ABCD. This is because of the property of light that light propogates in a straight line. You may also note that A′ lies on ray OA, B′ lies on ray OB, C′ lies on OC and D′ lies on OD. Thus, quadrilaterals A′B′C′D′ and ABCD are of the same shape but of different sizes.

So, quadrilateral A′B′C′D′ is similiar to quadrilateral ABCD. We can also say that quadrilateral ABCD is similar to the quadrilateral A′B′C′D′.

Here, you can also note that vertex A′ corresponds to vertex A, vertex B′ corresponds to vertex B, vertex C′ corresponds to vertex C and vertex D′ corresponds to vertex D. Symbolically, these correspondences are represented as A′ $\leftrightarrow$ A, B′ $\leftrightarrow$ B, C′ $\leftrightarrow$ C and D′ $\leftrightarrow$ D. By actually measuring the angles and the sides of the two quadrilaterals, you may verify that

(i) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$ and

(ii) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{DA}{D'A'}$.

This again emphasises that *two polygons of the same number of sides are similar, if* (*i*) *all the corresponding angles are equal and* (*ii*) *all the corresponding sides are in the same ratio* (*or proportion*).

From the above, you can easily say that quadrilaterals ABCD and PQRS of Fig. 2.5 are similar.

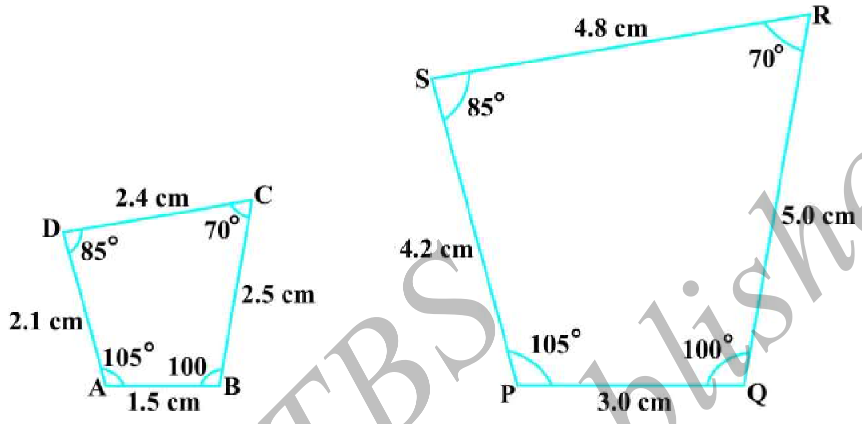


Fig. 2.5

Remark : You can verify that if one polygon is similar to another polygon and this second polygon is similar to a third polygon, then the first polygon is similar to the third polygon.

You may note that in the two quadrilaterals (a square and a rectangle) of Fig. 2.6, corresponding angles are equal, but their corresponding sides are not in the same ratio.

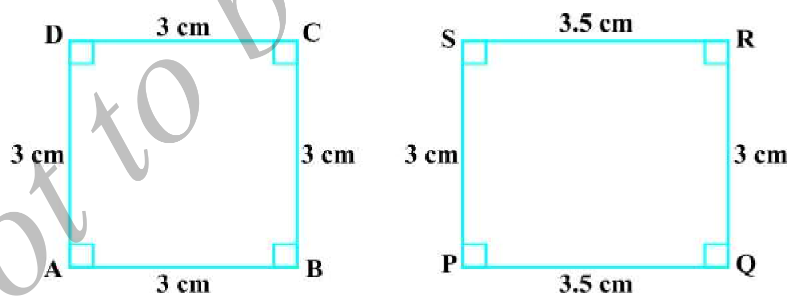


Fig. 2.6

So, the two quadrilaterals are not similar. Similarly, you may note that in the two quadrilaterals (a square and a rhombus) of Fig. 2.7, corresponding sides are in the same ratio, but their corresponding angles are not equal. Again, the two polygons (quadrilaterals) are not similar.

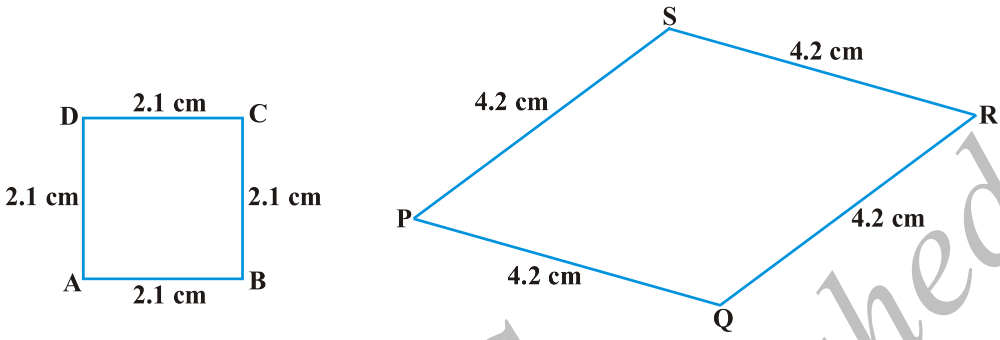


Fig. 2.7

Thus, *either of the above two conditions* (*i*) *and* (*ii*) *of similarity of two polygons is not sufficient for them to be similar*.

EXERCISE 2.1

1. Fill in the blanks using the correct word given in brackets :

(i) All circles are ________. (congruent, similar)

(ii) All squares are ________. (similar, congruent)

(iii) All ________ triangles are similar. (isosceles, equilateral)

(iv) Two polygons of the same number of sides are similar, if (a) their corresponding angles are ________ and (b) their corresponding sides are ________. (equal, proportional)

2. Give two different examples of pair of

(i) similar figures. (ii) non-similar figures.

3. State whether the following quadrilaterals are similar or not:

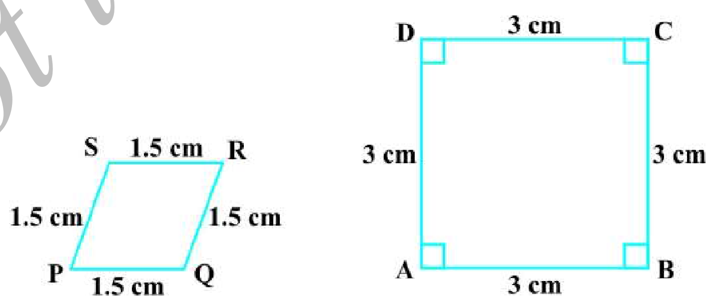


Fig. 2.8

2.3 Similarity of Triangles

What can you say about the similarity of two triangles?

You may recall that triangle is also a polygon. So, we can state the same conditions for the similarity of two triangles. That is:

Two triangles are similiar, if

(*i*) *their corresponding angles are equal and*

(*ii*) *their corresponding sides are in the same ratio* (*or proportion*).

Note that if corresponding angles of two triangles are equal, then they are known as *equiangular triangles*. A famous Greek mathematician Thales gave an important truth relating to two equiangular triangles which is as follows:

Thales
(640 – 546 B.C.)

The ratio of any two corresponding sides in two equiangular triangles is always the same.

It is believed that he had used a result called the *Basic Proportionality Theorem* (now known as the *Thales Theorem*) for the same.

To understand the Basic Proportionality Theorem, let us perform the following activity:

Activity 2 : Draw any angle XAY and on its one arm AX, mark points (say five points) P, Q, D, R and B such that AP = PQ = QD = DR = RB.

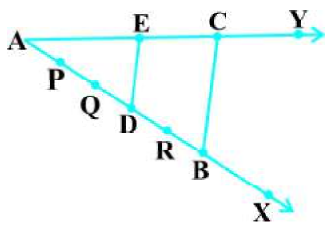


Fig. 2.9

Now, through B, draw any line intersecting arm AY at C (see Fig. 2.9).

Also, through the point D, draw a line parallel to BC to intersect AC at E. Do you observe from your constructions that $\frac{AD}{DB} = \frac{3}{2}$? Measure AE and EC. What about $\frac{AE}{EC}$? Observe that $\frac{AE}{EC}$ is also equal to $\frac{3}{2}$. Thus, you can see that in Δ ABC, DE || BC and $\frac{AD}{DB} = \frac{AE}{EC}$. Is it a coincidence? No, it is due to the following theorem (known as the Basic Proportionality Theorem):

Theorem 2.1 : *If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio.*

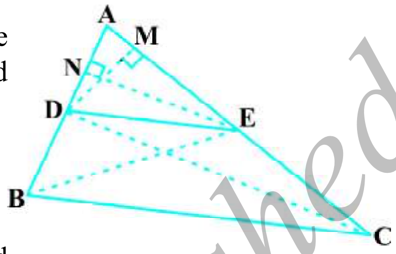


Fig. 2.10

Proof : We are given a triangle ABC in which a line parallel to side BC intersects other two sides AB and AC at D and E respectively (see Fig. 2.10).

We need to prove that $\frac{AD}{DB} = \frac{AE}{EC}$.

Let us join BE and CD and then draw DM $\perp$ AC and EN $\perp$ AB.

Now, area of Δ ADE (= $\frac{1}{2}$ base $\times$ height) = $\frac{1}{2}$ AD $\times$ EN.

Recall from Class IX, that area of Δ ADE is denoted as ar(ADE).

So, $$\text{ar(ADE)} = \frac{1}{2}\ \text{AD} \times \text{EN}$$

Similarly, $$\text{ar(BDE)} = \frac{1}{2}\ \text{DB} \times \text{EN},$$

$$\text{ar(ADE)} = \frac{1}{2}\ \text{AE} \times \text{DM} \quad \text{and} \quad \text{ar(DEC)} = \frac{1}{2}\ \text{EC} \times \text{DM}.$$

Therefore, $$\frac{\text{ar(ADE)}}{\text{ar(BDE)}} = \frac{\frac{1}{2}\ \text{AD} \times \text{EN}}{\frac{1}{2}\ \text{DB} \times \text{EN}} = \frac{\text{AD}}{\text{DB}} \qquad (1)$$

and $$\frac{\text{ar(ADE)}}{\text{ar(DEC)}} = \frac{\frac{1}{2}\ \text{AE} \times \text{DM}}{\frac{1}{2}\ \text{EC} \times \text{DM}} = \frac{\text{AE}}{\text{EC}} \qquad (2)$$

Note that Δ BDE and DEC are on the same base DE and between the same parallels BC and DE.

So, $$\text{ar(BDE)} = \text{ar(DEC)} \qquad (3)$$

Therefore, from (1), (2) and (3), we have :

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Is the converse of this theorem also true (For the meaning of converse, see Appendix 1)? To examine this, let us perform the following activity:

Activity 3 : Draw an angle XAY on your notebook and on ray AX, mark points B_1, B_2, B_3, B_4 and B such that $AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B$.

Similarly, on ray AY, mark points C_1, C_2, C_3, C_4 and C such that $AC_1 = C_1C_2 = C_2C_3 = C_3C_4 = C_4C$. Then join B_1C_1 and BC (see Fig. 2.11).

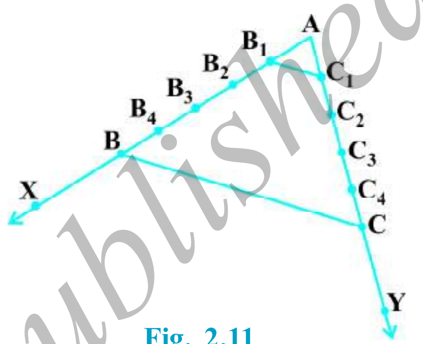


Fig. 2.11

Note that $\frac{AB_1}{B_1B} = \frac{AC_1}{C_1C}$ (Each equal to $\frac{1}{4}$)

You can also see that lines B_1C_1 and BC are parallel to each other, i.e.,

$$B_1C_1 \parallel BC \quad (1)$$

Similarly, by joining B_2C_2, B_3C_3 and B_4C_4, you can see that:

$$\frac{AB_2}{B_2B} = \frac{AC_2}{C_2C} \left(= \frac{2}{3}\right) \text{ and } B_2C_2 \parallel BC \quad (2)$$

$$\frac{AB_3}{B_3B} = \frac{AC_3}{C_3C} \left(= \frac{3}{2}\right) \text{ and } B_3C_3 \parallel BC \quad (3)$$

$$\frac{AB_4}{B_4B} = \frac{AC_4}{C_4C} \left(= \frac{4}{1}\right) \text{ and } B_4C_4 \parallel BC \quad (4)$$

From (1), (2), (3) and (4), it can be observed that if a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.

You can repeat this activity by drawing any angle XAY of different measure and taking any number of equal parts on arms AX and AY . Each time, you will arrive at the same result. Thus, we obtain the following theorem, which is the converse of Theorem 2.1:

Theorem 2.2 : *If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.*

This theorem can be proved by taking a line DE such that $\frac{\text{AD}}{\text{DB}} = \frac{\text{AE}}{\text{EC}}$ and assuming that DE is not parallel to BC (see Fig. 2.12).

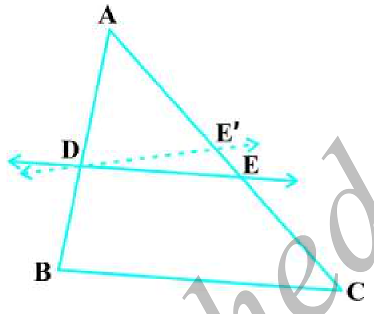


Fig. 2.12

If DE is not parallel to BC, draw a line DE′ parallel to BC.

So, $$\frac{\text{AD}}{\text{DB}} = \frac{\text{AE}'}{\text{E}'\text{C}} \quad \text{(Why ?)}$$

Therefore, $$\frac{\text{AE}}{\text{EC}} = \frac{\text{AE}'}{\text{E}'\text{C}} \quad \text{(Why ?)}$$

Adding 1 to both sides of above, you can see that E and E′ must coincide. (Why ?)

Let us take some examples to illustrate the use of the above theorems.

Example 1 : If a line intersects sides AB and AC of a Δ ABC at D and E respectively and is parallel to BC, prove that $\frac{\text{AD}}{\text{AB}} = \frac{\text{AE}}{\text{AC}}$ (see Fig. 2.13).

Solution : DE || BC (Given)

So, $$\frac{\text{AD}}{\text{DB}} = \frac{\text{AE}}{\text{EC}} \quad \text{(Theorem 2.1)}$$

or, $$\frac{\text{DB}}{\text{AD}} = \frac{\text{EC}}{\text{AE}}$$

or, $$\frac{\text{DB}}{\text{AD}} + 1 = \frac{\text{EC}}{\text{AE}} + 1$$

or, $$\frac{\text{AB}}{\text{AD}} = \frac{\text{AC}}{\text{AE}}$$

So, $$\frac{\text{AD}}{\text{AB}} = \frac{\text{AE}}{\text{AC}} 2$$

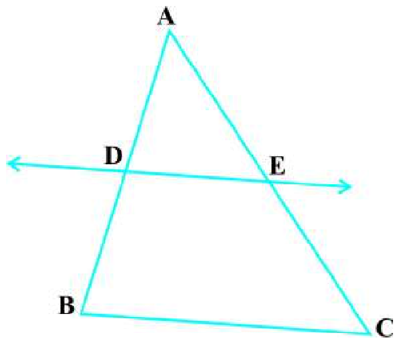


Fig. 2.13

Example 2 : ABCD is a trapezium with AB || DC. E ar sides AD and BC respectively such that (see Fig. 2.14). Show that $\frac{AE}{ED} = \frac{BF}{FC}$.

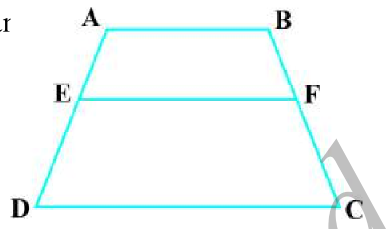


Fig. 2.14

Solution : Let us join AC to intersect EF at G (see Fig. 2.15).

AB || DC and EF || AB (Given)

So, EF || DC (Lines parallel to the same line are parallel to each other)

Now, in Δ ADC,

EG || DC (As EF || DC)

So, $\frac{AE}{ED} = \frac{AG}{GC}$ (Theorem 2.1) (1)

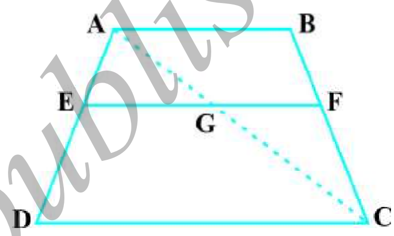


Fig. 2.15

Similarly, from Δ CAB,

$$\frac{CG}{AG} = \frac{CF}{BF}$$

i.e., $\frac{AG}{GC} = \frac{BF}{FC}$ (2)

Therefore, from (1) and (2),

$$\frac{AE}{ED} = \frac{BF}{FC}$$

Example 3 : In Fig. 2.16, $\frac{PS}{SQ} = \frac{PT}{TR}$ and ∠ PST = ∠ PRQ. Prove that PQR is an isosceles triangle.

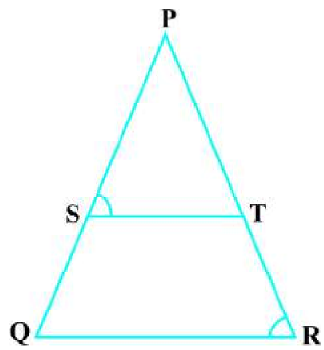


Fig. 2.16

Solution : It is given that $\frac{PS}{SQ} = \frac{PT}{TR}$.

So, ST || QR (Theorem2.2)

Therefore, ∠ PST = ∠ PQR (Corresponding angles) (1)

Also, it is given that

$$\angle PST = \angle PRQ \qquad (2)$$

So, $\angle PRQ = \angle PQR$ [From (1) and (2)]

Therefore, $PQ = PR$ (Sides opposite the equal angles)

i.e., PQR is an isosceles triangle.

EXERCISE 2.2

1. In Fig. 2.17, (i) and (ii), DE || BC. Find EC in (i) and AD in (ii).

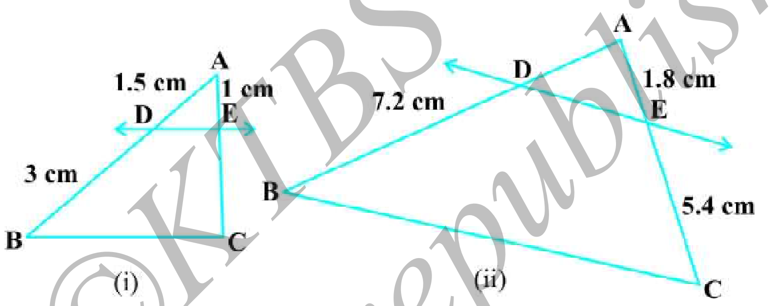


Fig. 2.17

2. E and F are points on the sides PQ and PR respectively of a Δ PQR. For each of the following cases, state whether EF || QR :

(i) PE = 3.9 cm, EQ = 3 cm, PF = 3.6 cm and FR = 2.4 cm

(ii) PE = 4 cm, QE = 4.5 cm, PF = 8 cm and RF = 9 cm

(iii) PQ = 1.28 cm, PR = 2.56 cm, PE = 0.18 cm and PF = 0.36 cm

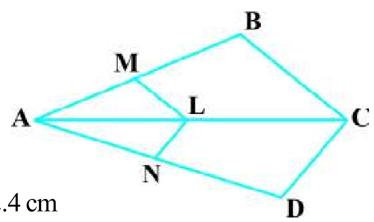


Fig. 2.18

3. In Fig. 2.18, if LM || CB and LN || CD, prove that

$$\frac{AM}{AB} = \frac{AN}{AD}.$$

4. In Fig. 2.19, DE || AC and DF || AE. Prove that

$$\frac{BF}{FE} = \frac{BE}{EC}.$$

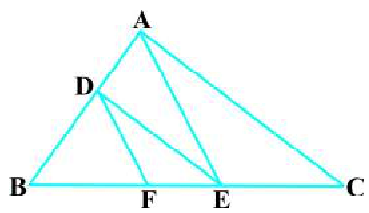


Fig. 2.19

5. In Fig. 2.20, DE || OQ and DF || OR. Show that EF || QR.

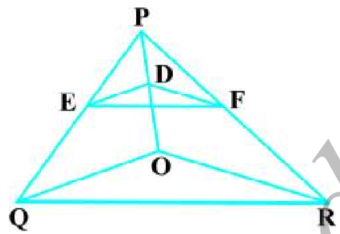


Fig. 2.20

6. In Fig. 2.21, A, B and C are points on OP, OQ and OR respectively such that AB || PQ and AC || PR. Show that BC || QR.

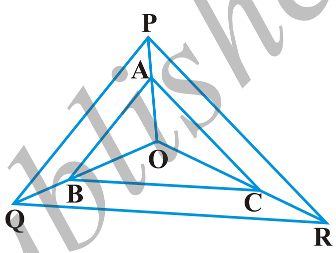


Fig. 2.21

7. Using Theorem 2.1, prove that a line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side. (Recall that you have proved it in Class IX).

8. Using Theorem 2.2, prove that the line joining the mid-points of any two sides of a triangle is parallel to the third side. (Recall that you have done it in Class IX).

9. ABCD is a trapezium in which AB || DC and its diagonals intersect each other at the point O. Show that $\frac{AO}{BO} = \frac{CO}{DO}$.

10. The diagonals of a quadrilateral ABCD intersect each other at the point O such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that ABCD is a trapezium.

2.4 Criteria for Similarity of Triangles

In the previous section, we stated that two triangles are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (or proportion).

That is, in Δ ABC and Δ DEF, if

(i) ∠ A = ∠ D, ∠ B = ∠ E, ∠ C = ∠ F and

(ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$, then the two triangles are similar (see Fig.2.22).

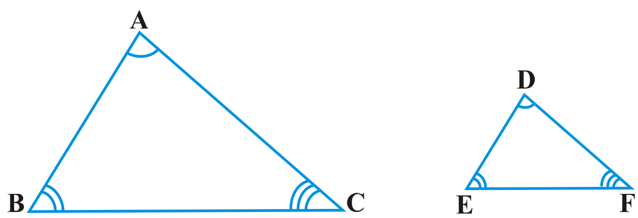


Fig.2.22

Here, you can see that A corresponds to D, B corresponds to E and C corresponds to F. Symbolically, we write the similarity of these two triangles as 'Δ ABC ~ Δ DEF' and read it as 'triangle ABC is similar to triangle DEF'. The symbol '~' stands for 'is similar to'. Recall that you have used the symbol '≅' for 'is congruent to' in Class IX.

It must be noted that as done in the case of congruency of two triangles, the similarity of two triangles should also be expressed symbolically, using correct correspondence of their vertices. For example, for the triangles ABC and DEF of Fig. 2.22, we cannot write Δ ABC ~ Δ EDF or Δ ABC ~ Δ FED. However, we can write Δ BAC ~ Δ EDF.

Now a natural question arises : For checking the similarity of two triangles, say ABC and DEF, should we always look for all the equality relations of their corresponding angles ($\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E, $\angle$ C = $\angle$ F) and all the equality relations of the ratios of their corresponding sides $\left(\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}\right)$? Let us examine. You may recall that in Class IX, you have obtained some criteria for congruency of two triangles involving only three pairs of corresponding parts (or elements) of the two triangles. Here also, let us make an attempt to arrive at certain criteria for similarity of two triangles involving relationship between less number of pairs of corresponding parts of the two triangles, instead of all the six pairs of corresponding parts. For this, let us perform the following activity:

Activity 4 : Draw two line segments BC and EF of two different lengths, say 3 cm and 5 cm respectively. Then, at the points B and C respectively, construct angles PBC and QCB of some measures, say, 60° and 40°. Also, at the points E and F, construct angles REF and SFE of 60° and 40° respectively (see Fig. 2.23).

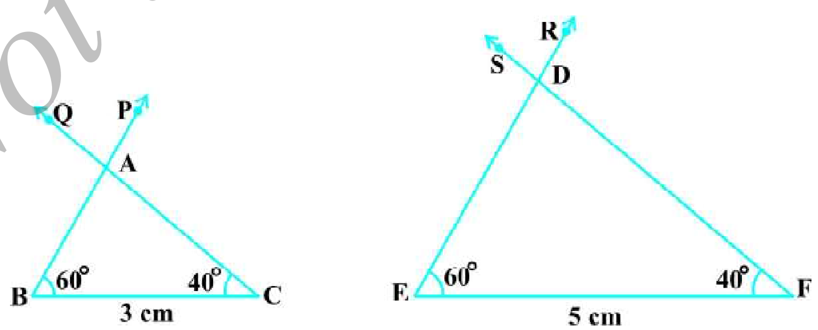


Fig. 2.23

Let rays BP and CQ intersect each other at A and rays ER and FS intersect each other at D. In the two triangles ABC and DEF, you can see that $\angle$ B = $\angle$ E, $\angle$ C = $\angle$ F and $\angle$ A = $\angle$ D. That is, corresponding angles of these two triangles are equal. What can you say about their corresponding sides ? Note that $\frac{BC}{EF} = \frac{3}{5} = 0.6$. What about $\frac{AB}{DE}$ and $\frac{CA}{FD}$? On measuring AB, DE, CA and FD, you will find that $\frac{AB}{DE}$ and $\frac{CA}{FD}$ are also equal to 0.6 (or nearly equal to 0.6, if there is some error in the measurement). Thus, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$. You can repeat this activity by constructing several pairs of triangles having their corresponding angles equal. Every time, you will find that their corresponding sides are in the same ratio (or proportion). This activity leads us to the following criterion for similarity of two triangles.

Theorem 2.3 : *If in two triangles, corresponding angles are equal, then their corresponding sides are in the same ratio (or proportion) and hence the two triangles are similar.*

This criterion is referred to as the AAA (Angle–Angle–Angle) criterion of similarity of two triangles.

This theorem can be proved by taking two triangles ABC and DEF such that $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F (see Fig. 2.24)

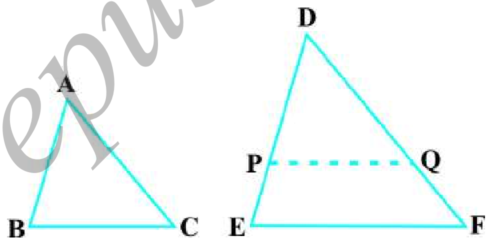


Fig. 2.24

Cut DP = AB and DQ = AC and join PQ.

So, Δ ABC $\cong$ Δ DPQ (Why ?)

This gives $\angle$ B = $\angle$ P = $\angle$ E and PQ || EF (How?)

Therefore, $\frac{DP}{PE} = \frac{DQ}{QF}$ (Why?)

i.e., $\frac{AB}{DE} = \frac{AC}{DF}$ (Why?)

Similarly, $\frac{AB}{DE} = \frac{BC}{EF}$ and so $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

Remark : If two angles of a triangle are respectively equal to two angles of another triangle, then by the angle sum property of a triangle their third angles will also be equal. Therefore, AAA similarity criterion can also be stated as follows:

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

This may be referred to as the AA *similarity criterion* for two triangles.

You have seen above that if the three angles of one triangle are respectively equal to the three angles of another triangle, then their corresponding sides are proportional (i.e., in the same ratio). What about the converse of this statement? Is the converse true? In other words, if the sides of a triangle are respectively proportional to the sides of another triangle, is it true that their corresponding angles are equal? Let us examine it through an activity :

Activity 5 : Draw two triangles ABC and DEF such that AB = 3 cm, BC = 6 cm, CA = 8 cm, DE = 4.5 cm, EF = 9 cm and FD = 12 cm (see Fig. 2.25).

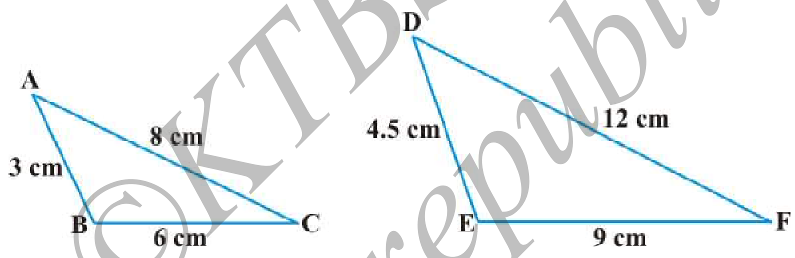


Fig. 2.25

So, you have : $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (each equal to $\frac{2}{3}$)

Now measure $\angle$ A, $\angle$ B, $\angle$ C, $\angle$ D, $\angle$ E and $\angle$ F. You will observe that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$, i.e., the corresponding angles of the two triangles are equal.

You can repeat this activity by drawing several such triangles (having their sides in the same ratio). Everytime you shall see that their corresponding angles are equal. It is due to the following criterion of similarity of two triangles:

Theorem 2.4 : *If in two triangles, sides of one triangle are proportional to (i.e., in the same ratio of) the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similiar.*

This criterion is referred to as the SSS (Side–Side–Side) *similarity criterion for two triangles*.

This theorem can be proved by taking two triangles ABC and DEF such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (< 1) (see Fig. 2.26):

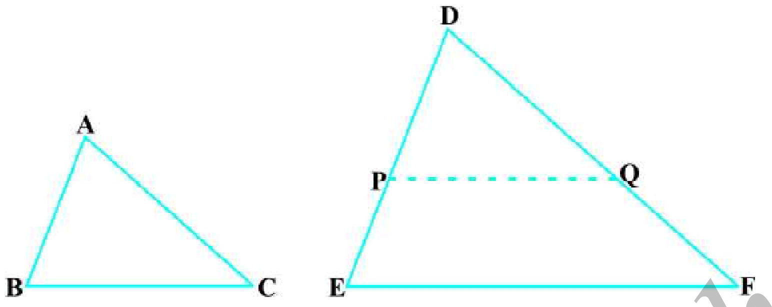


Fig. 2.26

Cut DP = AB and DQ = AC and join PQ.

It can be seen that $\frac{DP}{PE} = \frac{DQ}{QF}$ and PQ || EF (How?)

So, $\angle P = \angle E$ and $\angle Q = \angle F$.

Therefore, $$\frac{DP}{DE} = \frac{DQ}{DF} = \frac{PQ}{EF}$$

So, $$\frac{DP}{DE} = \frac{DQ}{DF} = \frac{BC}{EF} \quad \text{(Why?)}$$

So, BC = PQ (Why?)

Thus, $\Delta\, ABC \cong \Delta\, DPQ$ (Why ?)

So, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ (How ?)

Remark : You may recall that either of the two conditions namely, (i) corresponding angles are equal and (ii) corresponding sides are in the same ratio is not sufficient for two polygons to be similar. However, on the basis of Theorems 2.3 and 2.4, you can now say that in case of similarity of the two triangles, it is not necessary to check both the conditions as one condition implies the other.

Let us now recall the various criteria for congruency of two triangles learnt in Class IX. You may observe that SSS similarity criterion can be compared with the SSS congruency criterion.This suggests us to look for a similarity criterion comparable to SAS congruency criterion of triangles. For this, let us perform an activity.

Activity 6 : Draw two triangles ABC and DEF such that AB = 2 cm, $\angle A = 50°$, AC = 4 cm, DE = 3 cm, $\angle D = 50°$ and DF = 6 cm (see Fig.2.27).

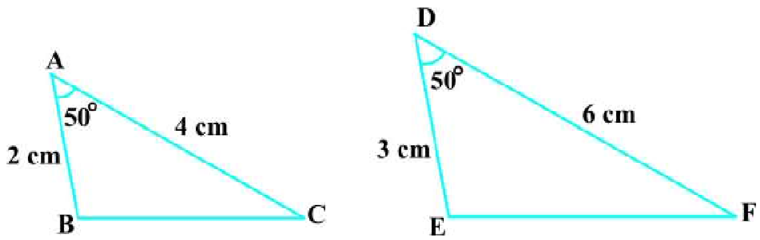


Fig. 2.27

Here, you may observe that $\frac{AB}{DE} = \frac{AC}{DF}$ (each equal to $\frac{2}{3}$) and $\angle$ A (included between the sides AB and AC) = $\angle$ D (included between the sides DE and DF). That is, one angle of a triangle is equal to one angle of another triangle and sides including these angles are in the same ratio (i.e., proportion). Now let us measure $\angle$ B, $\angle$ C, $\angle$ E and $\angle$ F.

You will find that $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F. That is, $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F. So, by AAA similarity criterion, Δ ABC ~ Δ DEF. You may repeat this activity by drawing several pairs of such triangles with one angle of a triangle equal to one angle of another triangle and the sides including these angles are proportional. Everytime, you will find that the triangles are similar. It is due to the following criterion of similarity of triangles:

Theorem 2.5 : *If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar.*

This criterion is referred to as the SAS (Side–Angle–Side) similarity criterion for two triangles.

As before, this theorem can be proved by taking two triangles ABC and DEF such that $\frac{AB}{DE} = \frac{AC}{DF}$ (< 1) and $\angle$ A = $\angle$ D (see Fig. 2.28). Cut DP = AB, DQ = AC and join PQ.

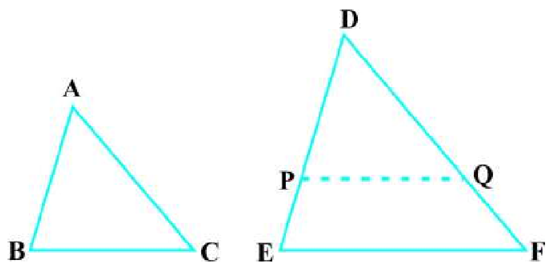


Fig. 2.28

Now, PQ || EF and Δ ABC $\cong$ Δ DPQ (How ?)

So, $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ P and $\angle$ C = $\angle$ Q

Therefore, Δ ABC ~ Δ DEF (Why?)

We now take some examples to illustrate the use of these criteria.

Example 4 : In Fig. 2.29, if PQ || RS, prove that Δ POQ ~ Δ SOR.

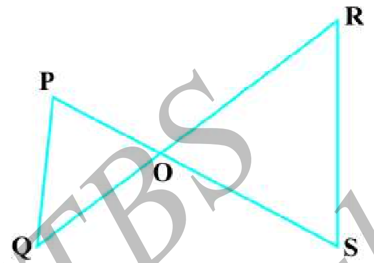


Fig. 2.29

Solution : PQ || RS (Given)

So, $\angle$ P = $\angle$ S (Alternate angles)

and $\angle$ Q = $\angle$ R

Also, $\angle$ POQ = $\angle$ SOR (Vertically opposite angles)

Therefore, Δ POQ ~ Δ SOR (AAA similarity criterion)

Example 5 : Observe Fig. 2.30 and then find $\angle$ P.

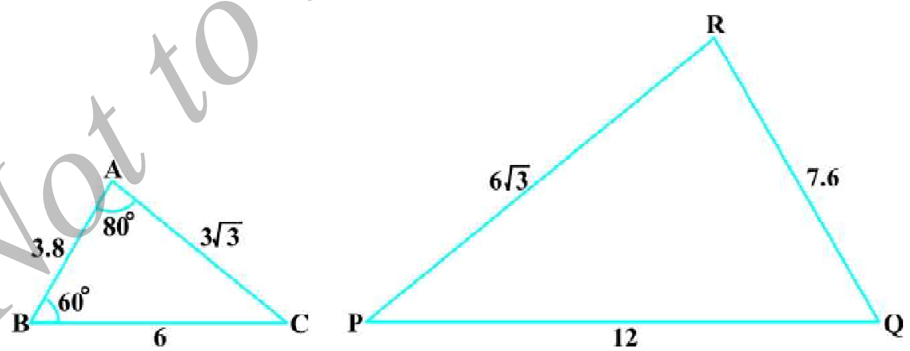


Fig. 2.30

Solution : In Δ ABC and Δ PQR,

$$\frac{AB}{RQ} = \frac{3.8}{7.6} = \frac{1}{2},\ \frac{BC}{QP} = \frac{6}{12} = \frac{1}{2} \text{ and } \frac{CA}{PR} = \frac{3\sqrt{3}}{6\sqrt{3}} = \frac{1}{2}$$

That is, $\dfrac{AB}{RQ} = \dfrac{BC}{QP} = \dfrac{CA}{PR}$

So, $\Delta\, ABC \sim \Delta\, RQP$ (SSS similarity)

Therefore, $\angle C = \angle P$ (Corresponding angles of similar triangles)

But $\angle C = 180° - \angle A - \angle B$ (Angle sum property)

$= 180° - 80° - 60° = 40°$

So, $\angle P = 40°$

Example 6 : In Fig. 2.31,

$OA \,.\, OB = OC \,.\, OD.$

Show that $\angle A = \angle C$ and $\angle B = \angle D$.

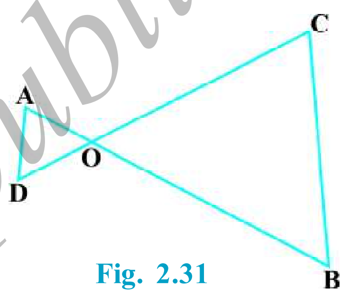


Fig. 2.31

Solution : $OA \,.\, OB = OC \,.\, OD$ (Given)

So, $\dfrac{OA}{OC} = \dfrac{OD}{OB}$ (1)

Also, we have $\angle AOD = \angle COB$ (Vertically opposite angles) (2)

Therefore, from (1) and (2), $\Delta\, AOD \sim \Delta\, COB$ (SAS similarity criterion)

So, $\angle A = \angle C$ and $\angle D = \angle B$

(Corresponding angles of similar triangles)

Example 7 : A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

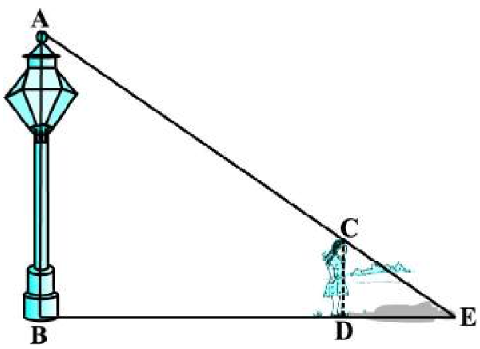


Fig. 2.32

Solution : Let AB denote the lamp-post and CD the girl after walking for 4 seconds away from the lamp-post (see Fig. 2.32).

From the figure, you can see that DE is the shadow of the girl. Let DE be x metres.

Now, BD = 1.2 m × 4 = 4.8 m.

Note that in Δ ABE and Δ CDE,

$\angle B = \angle D$ (Each is of 90° because lamp-post as well as the girl are standing vertical to the ground)

and $\angle E = \angle E$ (Same angle)

So, Δ ABE ~ Δ CDE (AA similarity criterion)

Therefore, $$\frac{BE}{DE} = \frac{AB}{CD}$$

i.e., $$\frac{4.8 + x}{x} = \frac{3.6}{0.9} \quad \left(90 \text{ cm} = \frac{90}{100} \text{ m} = 0.9 \text{ m}\right)$$

i.e., $4.8 + x = 4x$

i.e., $3x = 4.8$

i.e., $x = 1.6$

So, the shadow of the girl after walking for 4 seconds is 1.6 m long.

Example 8 : In Fig. 2.33, CM and RN are respectively the medians of Δ ABC and Δ PQR. If Δ ABC ~ Δ PQR, prove that :

(i) Δ AMC ~ Δ PNR

(ii) $\frac{CM}{RN} = \frac{AB}{PQ}$

(iii) Δ CMB ~ Δ RNQ

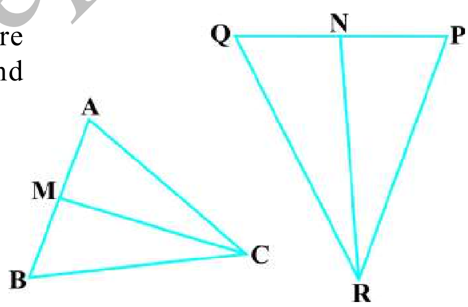


Fig. 2.33

Solution : (i) Δ ABC ~ Δ PQR (Given)

So, $$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} \quad (1)$$

and $\angle A = \angle P$, $\angle B = \angle Q$ and $\angle C = \angle R$ (2)

But AB = 2 AM and PQ = 2 PN

(As CM and RN are medians)

So, from (1), $$\frac{2\,AM}{2\,PN} = \frac{CA}{RP}$$

i.e., $$\frac{AM}{PN} = \frac{CA}{RP} \quad (3)$$

Also, $$\angle MAC = \angle NPR \quad \text{[From (2)]} \quad (4)$$

So, from (3) and (4),

$$\Delta AMC \sim \Delta PNR \quad \text{(SAS similarity)} \quad (5)$$

(ii) From (5), $$\frac{CM}{RN} = \frac{CA}{RP} \quad (6)$$

But $$\frac{CA}{RP} = \frac{AB}{PQ} \quad \text{[From (1)]} \quad (7)$$

Therefore, $$\frac{CM}{RN} = \frac{AB}{PQ} \quad \text{[From (6) and (7)]} \quad (8)$$

(iii) Again, $$\frac{AB}{PQ} = \frac{BC}{QR} \quad \text{[From (1)]}$$

Therefore, $$\frac{CM}{RN} = \frac{BC}{QR} \quad \text{[From (8)]} \quad (9)$$

Also, $$\frac{CM}{RN} = \frac{AB}{PQ} = \frac{2\ BM}{2\ QN}$$

i.e., $$\frac{CM}{RN} = \frac{BM}{QN} \quad (10)$$

i.e., $$\frac{CM}{RN} = \frac{BC}{QR} = \frac{BM}{QN} \quad \text{[From (9) and (10)]}$$

Therefore, $$\Delta CMB \sim \Delta RNQ \quad \text{(SSS similarity)}$$

[**Note :** You can also prove part (iii) by following the same method as used for proving part (i).]

EXERCISE 2.3

1. State which pairs of triangles in Fig. 6.34 are similar. Write the similarity criterion used by you for answering the question and also write the pairs of similar triangles in the symbolic form :

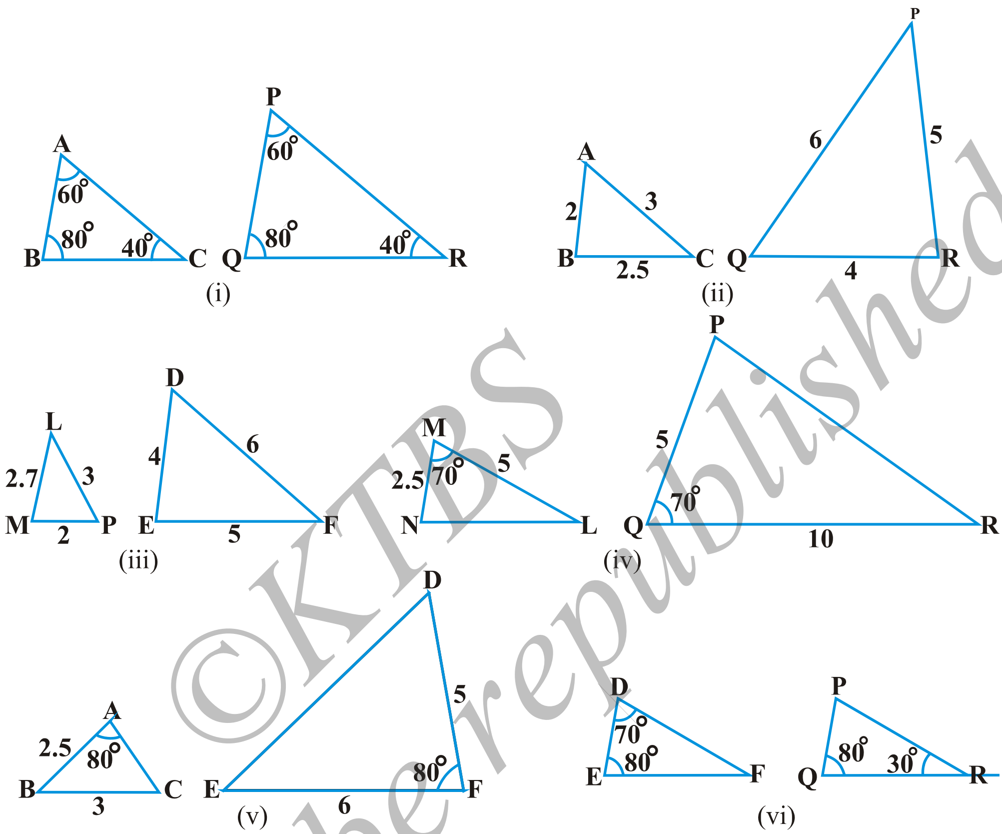


Fig. 2.34

2. In Fig. 2.35, Δ ODC ~ Δ OBA, $\angle$BOC = 125° and $\angle$CDO = 70°. Find $\angle$DOC, $\angle$DCO and $\angle$OAB.

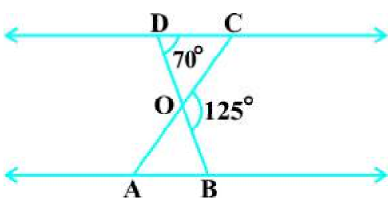


Fig. 2.35

3. Diagonals AC and BD of a trapezium ABCD with AB || DC intersect each other at the point O. Using a similarity criterion for two triangles, show that $\frac{OA}{OC} = \frac{OB}{OD}$.

4. In Fig. 2.36, $\frac{QR}{QS} = \frac{QT}{PR}$ and $\angle 1 = \angle 2$. Show that Δ PQS ~ Δ TQR.

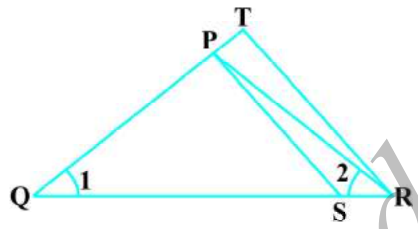


Fig. 2.36

5. S and T are points on sides PR and QR of Δ PQR such that $\angle$ P = $\angle$ RTS. Show that Δ RPQ ~ Δ RTS.

6. In Fig. 2.37, if Δ ABE $\cong$ Δ ACD, show that Δ ADE ~ Δ ABC.

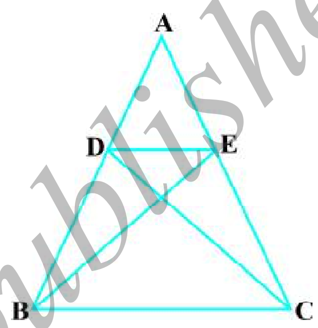


Fig. 2.37

7. In Fig. 2.38, altitudes AD and CE of Δ ABC intersect each other at the point P. Show that:

(i) Δ AEP ~ Δ CDP

(ii) Δ ABD ~ Δ CBE

(iii) Δ AEP ~ Δ ADB

(iv) Δ PDC ~ Δ BEC

8. E is a point on the side AD produced of a parallelogram ABCD and BE intersects CD at F. Show that Δ ABE ~ Δ CFB.

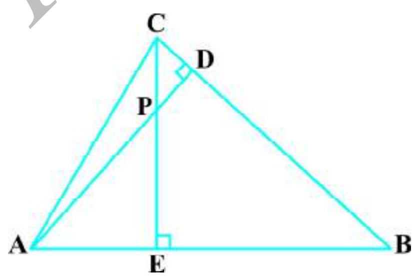


Fig. 2.38

9. In Fig. 2.39, ABC and AMP are two right triangles, right angled at B and M respectively. Prove that:

(i) Δ ABC ~ Δ AMP

(ii) $\frac{CA}{PA} = \frac{BC}{MP}$

10. CD and GH are respectively the bisectors of $\angle$ ACB and $\angle$ EGF such that D and H lie on sides AB and FE of Δ ABC and Δ EFG respectively. If Δ ABC ~ Δ FEG, show that:

(i) $\frac{CD}{GH} = \frac{AC}{FG}$

(ii) Δ DCB ~ Δ HGE

(iii) Δ DCA ~ Δ HGF

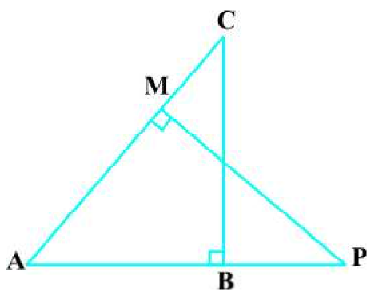


Fig. 2.39

11. In Fig. 2.40, E is a point on side CB produced of an isosceles triangle ABC with AB = AC. If AD ⊥ BC and EF ⊥ AC, prove that Δ ABD ~ Δ ECF.

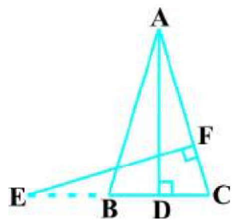


Fig. 2.40

12. Sides AB and BC and median AD of a triangle ABC are respectively proportional to sides PQ and QR and median PM of Δ PQR (see Fig.2.41). Show that Δ ABC ~ Δ PQR.

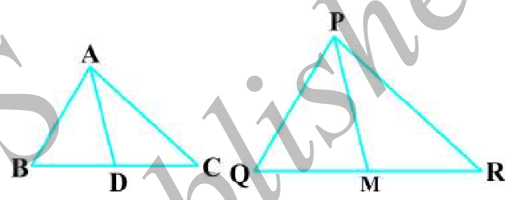


Fig. 2.41

13. D is a point on the side BC of a triangle ABC such that ∠ ADC = ∠ BAC. Show that $CA^2 = CB.CD$.

14. Sides AB and AC and median AD of a triangle ABC are respectively proportional to sides PQ and PR and median PM of another triangle PQR. Show that Δ ABC ~ Δ PQR.

15. A vertical pole of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower.

16. If AD and PM are medians of triangles ABC and PQR, respectively where Δ ABC ~ Δ PQR, prove that $\frac{AB}{PQ} = \frac{AD}{PM}$.

2.5 Areas of Similar Triangles

You have learnt that in two similar triangles, the ratio of their corresponding sides is the same. Do you think there is any relationship between the ratio of their areas and the ratio of the corresponding sides? You know that area is measured in square units. So, you may expect that this ratio is the square of the ratio of their corresponding sides. This is indeed true and we shall prove it in the next theorem.

Theorem 2.6 : *The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.*

Proof : We are given two triangles ABC and PQR such that Δ ABC ~ Δ PQR (see Fig. 2.42).

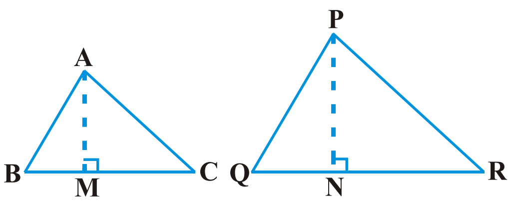


Fig. 2.42

We need to prove that $\dfrac{\text{ar (ABC)}}{\text{ar (PQR)}} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{CA}{RP}\right)^2$.

For finding the areas of the two triangles, we draw altitudes AM and PN of the triangles.

Now, $\text{ar (ABC)} = \dfrac{1}{2}\, BC \times AM$

and $\text{ar (PQR)} = \dfrac{1}{2}\, QR \times PN$

So, $$\frac{\text{ar (ABC)}}{\text{ar (PQR)}} = \frac{\frac{1}{2} \times BC \times AM}{\frac{1}{2} \times QR \times PN} = \frac{BC \times AM}{QR \times PN} \qquad (1)$$

Now, in Δ ABM and Δ PQN,

$\angle B = \angle Q$ (As $\Delta ABC \sim \Delta PQR$)

and $\angle M = \angle N$ (Each is of 90°)

So, $\Delta ABM \sim \Delta PQN$ (AA similarity criterion)

Therefore, $$\frac{AM}{PN} = \frac{AB}{PQ} \qquad (2)$$

Also, $\Delta ABC \sim \Delta PQR$ (Given)

So, $$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} \qquad (3)$$

Therefore, $$\frac{\text{ar (ABC)}}{\text{ar (PQR)}} = \frac{AB}{PQ} \times \frac{AM}{PN} \qquad \text{[From (1) and (3)]}$$

$$= \frac{AB}{PQ} \times \frac{AB}{PQ} \qquad \text{[From (2)]}$$

$$= \left(\frac{AB}{PQ}\right)^2$$

Now using (3), we get

$$\frac{\text{ar (ABC)}}{\text{ar (PQR)}} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$$

Let us take an example to illustrate the use of this theorem.

Example 9 : In Fig. 2.43, the line segment XY is parallel to side AC of Δ ABC and it divides the triangle into two parts of equal areas. Find the ratio $\frac{AX}{AB}$.

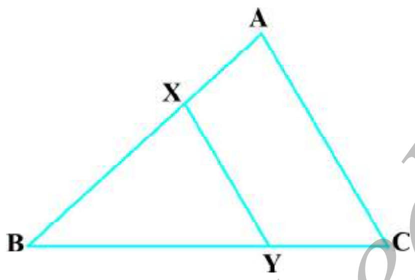


Fig. 2.43

Solution : We have $XY \parallel AC$ (Given)

So, $\angle BXY = \angle A$ and $\angle BYX = \angle C$ (Corresponding angles)

Therefore, $\Delta ABC \sim \Delta XBY$ (AA similarity criterion)

So, $$\frac{ar(ABC)}{ar(XBY)} = \left(\frac{AB}{XB}\right)^2 \quad \text{(Theorem 2.6)} \quad (1)$$

Also, $ar(ABC) = 2\, ar(XBY)$ (Given)

So, $$\frac{ar(ABC)}{ar(XBY)} = \frac{2}{1} \quad (2)$$

Therefore, from (1) and (2),

$$\left(\frac{AB}{XB}\right)^2 = \frac{2}{1}, \text{ i.e., } \frac{AB}{XB} = \frac{\sqrt{2}}{1}$$

or, $$\frac{XB}{AB} = \frac{1}{\sqrt{2}}$$

or, $$1 - \frac{XB}{AB} = 1 - \frac{1}{\sqrt{2}}$$

or, $$\frac{AB - XB}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}}, \text{ i.e., } \frac{AX}{AB} = \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}.$$

EXERCISE 2.4

1. Let Δ ABC ~ Δ DEF and their areas be, resp2ctively, 64 cm^2 and 121 cm^2. If EF = 15.4 cm, find BC.
2. Diagonals of a trapezium ABCD with AB || DC intersect each other at the point O. If AB = 2 CD, find the ratio of the areas of triangles AOB and COD.

3. In Fig. 2.44, ABC and DBC are two triangles on the same base BC. If AD intersects BC at O, show that

$$\frac{\text{ar (ABC)}}{\text{ar (DBC)}} = \frac{\text{AO}}{\text{DO}}.$$

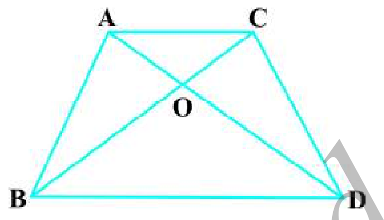


Fig. 2.44

4. If the areas of two similar triangles are equal, prove that they are congruent.

5. D, E and F are respectively the mid-points of sides AB, BC and CA of Δ ABC. Find the ratio of the areas of Δ DEF and Δ ABC.

6. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians.

7. Prove that the area of an equilateral triangle described on one side of a square is equal to half the area of the equilateral triangle described on one of its diagonals.

Tick the correct answer and justify :

8. ABC and BDE are two equilateral triangles such that D is the mid-point of BC. Ratio of the areas of triangles ABC and BDE is

   (A) 2 : 1 (B) 1 : 2 (C) 4 : 1 (D) 1 : 4

9. Sides of two similar triangles are in the ratio 4 : 9. Areas of these triangles are in the ratio

   (A) 2 : 3 (B) 4 : 9 (C) 81 : 16 (D) 16 : 81

2.6 Pythagoras Theorem

You are already familiar with the Pythagoras Theorem from your earlier classes. You had verified this theorem through some activities and made use of it in solving certain problems. You have also seen a proof of this theorem in Class IX. Now, we shall prove this theorem using the concept of similarity of triangles. In proving this, we shall make use of a result related to similarity of two triangles formed by the perpendicular to the hypotenuse from the opposite vertex of the right triangle.

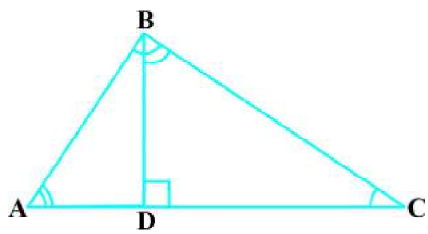


Fig. 2.45

Now, let us take a right triangle ABC, right angled at B. Let BD be the perpendicular to the hypotenuse AC (see Fig. 2.45).

You may note that in Δ ADB and Δ ABC

$$\angle \text{A} = \angle \text{A}$$

and $\angle \text{ADB} = \angle \text{ABC}$ (Why?)

So, $\Delta \text{ ADB} \sim \Delta \text{ ABC}$ (How?) (1)

Similarly, $\Delta \text{ BDC} \sim \Delta \text{ ABC}$ (How?) (2)

So, from (1) and (2), triangles on both sides of the perpendicular BD are similar to the whole triangle ABC.

Also, since $\quad \Delta\, ADB \sim \Delta\, ABC$

and $\quad \Delta\, BDC \sim \Delta\, ABC$

So, $\quad \Delta\, ADB \sim \Delta\, BDC \quad$ (From Remark in Section 2.2)

The above discussion leads to the following theorem :

Theorem 2.7 : *If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse then triangles on both sides of the perpendicular are similar to the whole triangle and to each other.*

Let us now apply this theorem in proving the Pythagoras Theorem:

Pythagoras
(569 – 479 B.C.E.)

Theorem 2.8 : *In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*

Proof : We are given a right triangle ABC right angled at B.

We need to prove that $AC^2 = AB^2 + BC^2$

Let us draw $\quad BD \perp AC \quad$ (see Fig. 2.46).

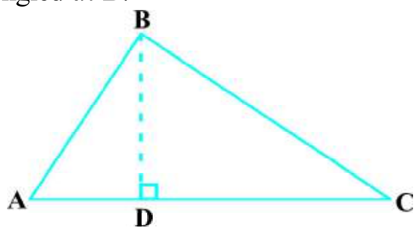


Fig. 2.46

Now, $\quad \Delta\, ADB \sim \Delta\, ABC \quad$ (Theorem 2.7)

So, $$\frac{AD}{AB} = \frac{AB}{AC} \quad \text{(Sides are proportional)}$$

or, $\quad AD \,.\, AC = AB^2 \qquad (1)$

Also, $\quad \Delta\, BDC \sim \Delta\, ABC \quad$ (Theorem 2.7)

So, $$\frac{CD}{BC} = \frac{BC}{AC}$$

or, $\quad CD \,.\, AC = BC^2 \qquad (2)$

Adding (1) and (2),

$$AD \,.\, AC + CD \,.\, AC = AB^2 + BC^2$$

or, $$AC\,(AD + CD) = AB^2 + BC^2$$

or, $$AC \,.\, AC = AB^2 + BC^2$$

or, $$AC^2 = AB^2 + BC^2$$

The above theorem was earlier given by an ancient Indian mathematician Baudhayan (about 800 B.C.E.) in the following form :

The diagonal of a rectangle produces by itself the same area as produced by its both sides (*i.e., length and breadth*).

For this reason, this theorem is sometimes also referred to as the *Baudhayan Theorem*.

What about the converse of the Pythagoras Theorem? You have already verified, in the earlier classes, that this is also true. We now prove it in the form of a theorem.

Theorem 2.9 : *In a triangle, if square of one side is equal to the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.*

Proof : Here, we are given a triangle ABC in which $AC^2 = AB^2 + BC^2$.

We need to prove that $\angle B = 90°$.

To start with, we construct a Δ PQR right angled at Q such that PQ = AB and QR = BC (see Fig. 2.47).

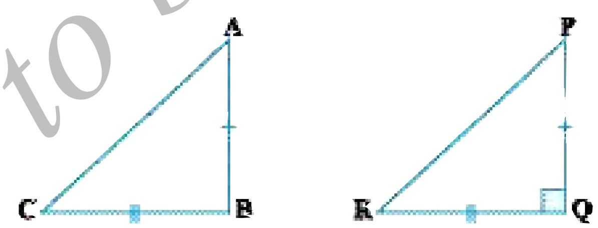


Fig. 2.47

Now, from Δ PQR, we have :

$$PR^2 = PQ^2 + QR^2$$ (Pythagoras Theorem, as $\angle Q = 90°$)

or, $$PR^2 = AB^2 + BC^2$$ (By construction) (1)

But $AC^2 = AB^2 + BC^2$ (Given) (2)

So, $AC = PR$ [From (1) and (2)] (3)

Now, in Δ ABC and Δ PQR,

$AB = PQ$ (By construction)

$BC = QR$ (By construction)

$AC = PR$ [Proved in (3) above]

So, $\Delta ABC \cong \Delta PQR$ (SSS congruence)

Therefore, $\angle B = \angle Q$ (CPCT)

But $\angle Q = 90°$ (By construction)

So, $\angle B = 90°$

Note : Also see Appendix 1 for another proof of this theorem.

Let us now take some examples to illustrate the use of these theorems.

Example 10 : In Fig. 2.48, $\angle ACB = 90°$ and $CD \perp AB$. Prove that $\frac{BC^2}{AC^2} = \frac{BD}{AD}$.

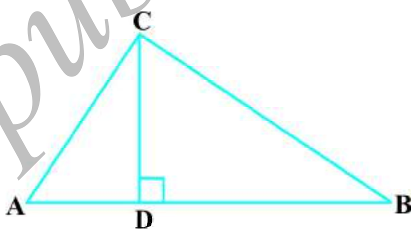


Fig. 2.48

Solution : $\Delta ACD \sim \Delta ABC$ (Theorem 2.7)

So, $\frac{AC}{AB} = \frac{AD}{AC}$

or, $AC^2 = AB \cdot AD$ (1)

Similarly, $\Delta BCD \sim \Delta BAC$ (Theorem 2.7)

So, $\frac{BC}{BA} = \frac{BD}{BC}$

or, $BC^2 = BA \cdot BD$ (2)

Therefore, from (1) and (2),

$$\frac{BC^2}{AC^2} = \frac{BA \cdot BD}{AB \cdot AD} = \frac{BD}{AD}$$

Example 11 : A ladder is placed against a wall such that its foot is at a distance of 2.5 m from the wall and its top reaches a window 6 m above the ground. Find the length of the ladder.

Solution : Let AB be the ladder and CA be the wall with the window at A (see Fig. 2.49).

Also, BC = 2.5 m and CA = 6 m

From Pythagoras Theorem, we have:

$$AB^2 = BC^2 + CA^2$$
$$= (2.5)^2 + (6)^2$$
$$= 42.25$$

So, $AB = 6.5$

Thus, length of the ladder is 6.5 m.

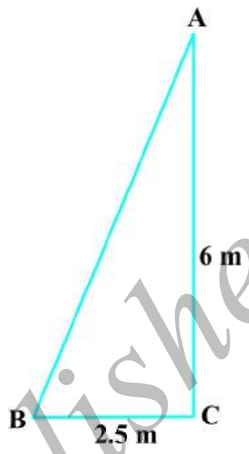


Fig. 2.49

Example 12 : In Fig. 2.50, if AD ⊥ BC, prove that $AB^2 + CD^2 = BD^2 + AC^2$.

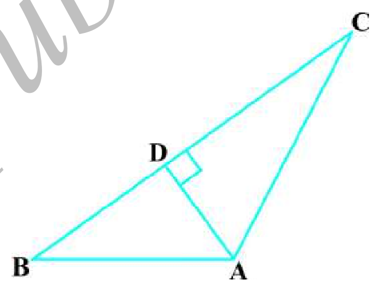


Fig. 2.50

Solution : From Δ ADC, we have

$$AC^2 = AD^2 + CD^2 \quad \text{(Pythagoras Theorem)} \quad (1)$$

From Δ ADB, we have

$$AB^2 = AD^2 + BD^2 \quad \text{(Pythagoras Theorem)} \quad (2)$$

Subtracting (1) from (2), we have

$$AB^2 - AC^2 = BD^2 - CD^2$$

or, $AB^2 + CD^2 = BD^2 + AC^2$

Example 13 : BL and CM are medians of a triangle ABC right angled at A. Prove that $4 (BL^2 + CM^2) = 5\ BC^2$.

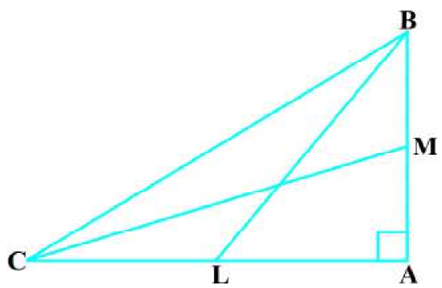


Fig. 2.51

Solution : BL and CM are medians of the Δ ABC in which ∠ A = 90° (see Fig. 2.51).

From Δ ABC,

$$BC^2 = AB^2 + AC^2 \quad \text{(Pythagoras Theorem)} \quad (1)$$

From Δ ABL,

$$BL^2 = AL^2 + AB^2$$

or, $$BL^2 = \left(\frac{AC}{2}\right)^2 + AB^2 \quad \text{(L is the mid-point of AC)}$$

or, $$BL^2 = \frac{AC^2}{4} + AB^2$$

or, $$4\,BL^2 = AC^2 + 4\,AB^2 \qquad (2)$$

From Δ CMA,

$$CM^2 = AC^2 + AM^2$$

or, $$CM^2 = AC^2 + \left(\frac{AB}{2}\right)^2 \quad \text{(M is the mid-point of AB)}$$

or, $$CM^2 = AC^2 + \frac{AB^2}{4}$$

or $$4\,CM^2 = 4\,AC^2 + AB^2 \qquad (3)$$

Adding (2) and (3), we have

$$4\,(BL^2 + CM^2) = 5\,(AC^2 + AB^2)$$

i.e., $$4\,(BL^2 + CM^2) = 5\,BC^2 \qquad \text{[From (1)]}$$

Example 14 : O is any point inside a rectangle ABCD (see Fig. 2.52). Prove that $OB^2 + OD^2 = OA^2 + OC^2$.

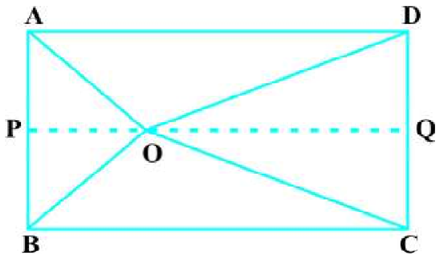


Fig. 2.52

Solution :

Through O, draw PQ || BC so that P lies on AB and Q lies on DC.

Now, $$PQ \parallel BC$$

Therefore, $$PQ \perp AB \text{ and } PQ \perp DC \; (\angle B = 90° \text{ and } \angle C = 90°)$$

So, $$\angle BPQ = 90° \text{ and } \angle CQP = 90°$$

Therefore, BPQC and APQD are both rectangles.

Now, from Δ OPB,

$$OB^2 = BP^2 + OP^2 \qquad (1)$$

Similarly, from Δ OQD,

$$OD^2 = OQ^2 + DQ^2 \qquad (2)$$

From Δ OQC, we have

$$OC^2 = OQ^2 + CQ^2 \qquad (3)$$

and from Δ OAP, we have

$$OA^2 = AP^2 + OP^2 \qquad (4)$$

Adding (1) and (2),

$$OB^2 + OD^2 = BP^2 + OP^2 + OQ^2 + DQ^2$$
$$= CQ^2 + OP^2 + OQ^2 + AP^2$$
(As BP = CQ and DQ = AP)
$$= CQ^2 + OQ^2 + OP^2 + AP^2$$
$$= OC^2 + OA^2 \qquad \text{[From (3) and (4)]}$$

EXERCISE 2.5

1. Sides of triangles are given below. Determine which of them are right triangles. In case of a right triangle, write the length of its hypotenuse.
   (i) 7 cm, 24 cm, 25 cm
   (ii) 3 cm, 8 cm, 6 cm
   (iii) 50 cm, 80 cm, 100 cm
   (iv) 13 cm, 12 cm, 5 cm
2. PQR is a triangle right angled at P and M is a point on QR such that PM $\perp$ QR. Show that $PM^2 = QM \cdot MR$.
3. In Fig. 2.53, ABD is a triangle right angled at A and AC $\perp$ BD. Show that
   (i) $AB^2 = BC \cdot BD$
   (ii) $AC^2 = BC \cdot DC$
   (iii) $AD^2 = BD \cdot CD$

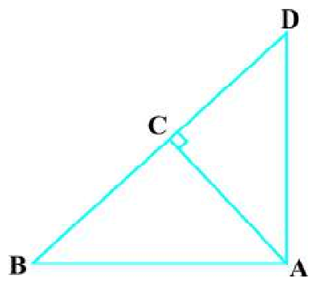


Fig. 2.53

4. ABC is an isosceles triangle right angled at C. Prove that $AB^2 = 2AC^2$.
5. ABC is an isosceles triangle with AC = BC. If $AB^2 = 2\,AC^2$, prove that ABC is a right triangle.
6. ABC is an equilateral triangle of side $2a$. Find each of its altitudes.
7. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.

8. In Fig. 2.54, O is a point in the interior of a triangle ABC, OD $\perp$ BC, OE $\perp$ AC and OF $\perp$ AB. Show that

(i) $OA^2 + OB^2 + OC^2 - OD^2 - OE^2 - OF^2 = AF^2 + BD^2 + CE^2$,

(ii) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$.

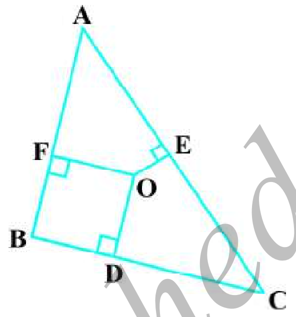


Fig. 2.54

9. A ladder 10 m long reaches a window 8 m above the ground. Find the distance of the foot of the ladder from base of the wall.

10. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?

11. An aeroplane leaves an airport and flies due north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?

12. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between the feet of the poles is 12 m, find the distance between their tops.

13. D and E are points on the sides CA and CB respectively of a triangle ABC right angled at C. Prove that $AE^2 + BD^2 = AB^2 + DE^2$.

14. The perpendicular from A on side BC of a Δ ABC intersects BC at D such that DB = 3 CD (see Fig. 2.55). Prove that $2\,AB^2 = 2\,AC^2 + BC^2$.

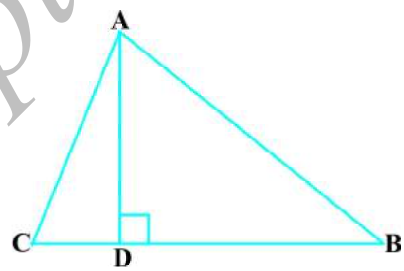


Fig. 2.55

15. In an equilateral triangle ABC, D is a point on side BC such that BD = $\frac{1}{3}$ BC. Prove that $9\,AD^2 = 7\,AB^2$.

16. In an equilateral triangle, prove that three times the square of one side is equal to four times the square of one of its altitudes.

17. Tick the correct answer and justify : In Δ ABC, AB = $6\sqrt{3}$ cm, AC = 12 cm and BC = 6 cm. The angle B is :

(A) 120° (B) 60°

(C) 90° (D) 45°

EXERCISE 2.6 (Optional)*

1. In Fig. 2.56, PS is the bisector of $\angle$ QPR of Δ PQR. Prove that $\dfrac{QS}{SR} = \dfrac{PQ}{PR}$.

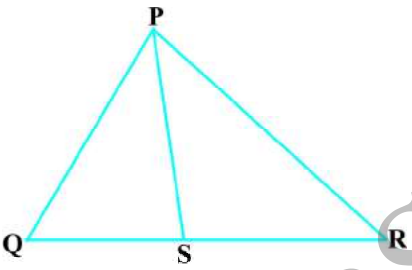


Fig. 2.56

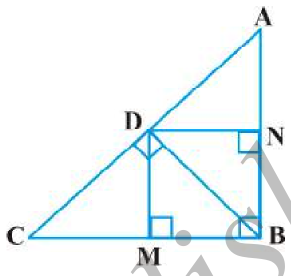


Fig. 2.57

2. In Fig. 2.57, D is a point on hypotenuse AC of Δ ABC, such that BD $\perp$ AC, DM $\perp$ BC and DN $\perp$ AB. Prove that :

   (i) $DM^2 = DN \,.\, MC$ (ii) $DN^2 = DM \,.\, AN$

3. In Fig. 2.58, ABC is a triangle in which $\angle ABC > 90°$ and AD $\perp$ CB produced. Prove that $AC^2 = AB^2 + BC^2 + 2\, BC \,.\, BD$.

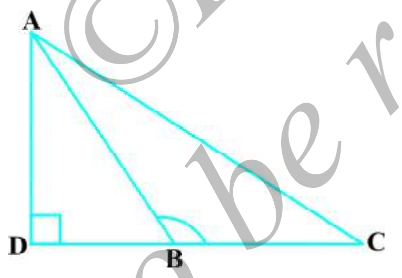


Fig. 2.58

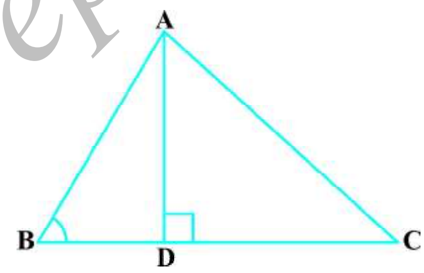


Fig. 2.59

4. In Fig. 2.59, ABC is a triangle in which $\angle ABC < 90°$ and AD $\perp$ BC. Prove that $AC^2 = AB^2 + BC^2 - 2\, BC \,.\, BD$.

5. In Fig. 2.60, AD is a median of a triangle ABC and AM $\perp$ BC. Prove that :

   (i) $AC^2 = AD^2 + BC \,.\, DM + \left(\dfrac{BC}{2}\right)^2$

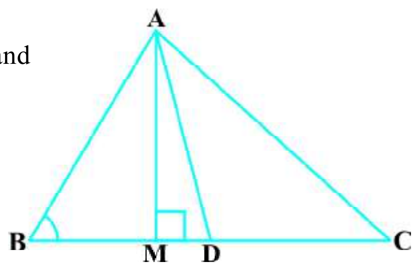


Fig. 2.60

* These exercises are not from examination point of view.

(ii) $AB^2 = AD^2 - BC \cdot DM + \left(\frac{BC}{2}\right)^2$ (iii) $AC^2 + AB^2 = 2\,AD^2 + \frac{1}{2}\,BC^2$

6. Prove that the sum of the squares of the diagonals of parallelogram is equal to the sum of the squares of its sides.

7. In Fig. 2.61, two chords AB and CD intersect each other at the point P. Prove that :

(i) $\Delta APC \sim \Delta DPB$ (ii) $AP \cdot PB = CP \cdot DP$

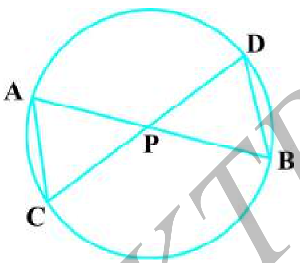


Fig. 2.61

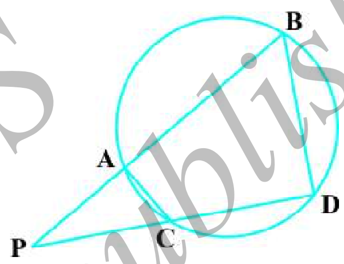


Fig. 2.62

8. In Fig. 2.62, two chords AB and CD of a circle intersect each other at the point P (when produced) outside the circle. Prove that

(i) $\Delta PAC \sim \Delta PDB$ (ii) $PA \cdot PB = PC \cdot PD$

9. In Fig. 2.63, D is a point on side BC of ΔABC such that $\frac{BD}{CD} = \frac{AB}{AC}$. Prove that AD is the bisector of $\angle BAC$.

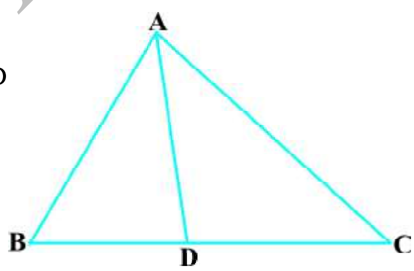


Fig. 2.63

10. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out (see Fig. 2.64)? If she pulls in the string at the rate of 5 cm per second, what will be the horizontal distance of the fly from her after 12 seconds?

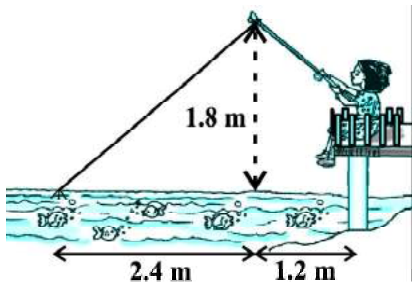


Fig. 2.64

2.7 Summary

In this chapter you have studied the following points :

1. Two figures having the same shape but not necessarily the same size are called similar figures.
2. All the congruent figures are similar but the converse is not true.
3. Two polygons of the same number of sides are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (i.e., proportion).
4. If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.
5. If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.
6. If in two triangles, corresponding angles are equal, then their corresponding sides are in the same ratio and hence the two triangles are similar (AAA similarity criterion).
7. If in two triangles, two angles of one triangle are respectively equal to the two angles of the other triangle, then the two triangles are similar (AA similarity criterion).
8. If in two triangles, corresponding sides are in the same ratio, then their corresponding angles are equal and hence the triangles are similar (SSS similarity criterion).
9. If one angle of a triangle is equal to one angle of another triangle and the sides including these angles are in the same ratio (proportional), then the triangles are similar (SAS similarity criterion).
10. The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.
11. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the whole triangle and also to each other.
12. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides (Pythagoras Theorem).
13. If in a triangle, square of one side is equal to the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.

A Note to the Reader

If in two right triangles, hypotenuse and one side of one triangle are proportional to the hypotenuse and one side of the other triangle, then the two triangles are similar. This may be referred to as the RHS Similarity Criterion.

If you use this criterion in Example 2, Chapter 11, the proof will become simpler.

PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

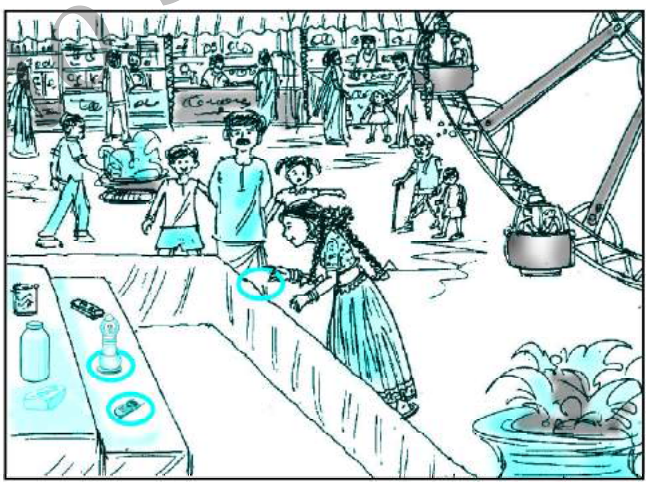
3

3.1 Introduction

You must have come across situations like the one given below :

Akhila went to a fair in her village. She wanted to enjoy rides on the Giant Wheel and play Hoopla (a game in which you throw a ring on the items kept in a stall, and if the ring covers any object completely, you get it). The number of times she played Hoopla is half the number of rides she had on the Giant Wheel. If each ride costs ₹ 3, and a game of Hoopla costs ₹ 4, how would you find out the number of rides she had and how many times she played Hoopla, provided she spent ₹ 20.

May be you will try it by considering different cases. If she has one ride, is it possible? Is it possible to have two rides? And so on. Or you may use the knowledge of Class IX, to represent such situations as linear equations in two variables.



Let us try this approach.

Denote the number of rides that Akhila had by x, and the number of times she played Hoopla by y. Now the situation can be represented by the two equations:

$$y = \frac{1}{2}x \quad (1)$$

$$3x + 4y = 20 \quad (2)$$

Can we find the solutions of this pair of equations? There are several ways of finding these, which we will study in this chapter.

3.2 Pair of Linear Equations in Two Variables

Recall, from Class IX, that the following are examples of linear equations in two variables:

$$2x + 3y = 5$$

$$x - 2y - 3 = 0$$

and $$x - 0y = 2, \text{ i.e., } x = 2$$

You also know that an equation which can be put in the form $ax + by + c = 0$, where a, b and c are real numbers, and ***a* and *b* are not both zero**, is called a linear equation in two variables x and y. (We often denote the condition a and b are not both zero by $a^2 + b^2 \neq 0$). You have also studied that a solution of such an equation is a pair of values, one for x and the other for y, which makes the two sides of the equation equal.

For example, let us substitute $x = 1$ and $y = 1$ in the left hand side (LHS) of the equation $2x + 3y = 5$. Then

$$\text{LHS} = 2(1) + 3(1) = 2 + 3 = 5,$$

which is equal to the right hand side (RHS) of the equation.

Therefore, $x = 1$ and $y = 1$ is a solution of the equation $2x + 3y = 5$.

Now let us substitute $x = 1$ and $y = 7$ in the equation $2x + 3y = 5$. Then,

$$\text{LHS} = 2(1) + 3(7) = 2 + 21 = 23$$

which is not equal to the RHS.

Therefore, $x = 1$ and $y = 7$ is **not** a solution of the equation.

Geometrically, what does this mean? It means that the point (1, 1) lies on the line representing the equation $2x + 3y = 5$, and the point (1, 7) does not lie on it. So, **every solution of the equation is a point on the line representing it**.

In fact, this is true for any linear equation, that is, **each solution (x, y) of a linear equation in two variables, $ax + by + c = 0$, corresponds to a point on the line representing the equation, and vice versa**.

Now, consider Equations (1) and (2) given above. These equations, **taken together**, represent the information we have about Akhila at the fair.

These two linear equations are **in the same two variables x and y**. Equations like these are called a *pair of linear equations in two variables*.

Let us see what such pairs look like algebraically.

The general form for a pair of linear equations in two variables x and y is

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0,$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are all real numbers and $a_1^2 + b_1^2 \neq 0$, $a_2^2 + b_2^2 \neq 0$.

Some examples of pair of linear equations in two variables are:

$2x + 3y - 7 = 0$ and $9x - 2y + 8 = 0$

$5x = y$ and $-7x + 2y + 3 = 0$

$x + y = 7$ and $17 = y$

Do you know, what do they look like geometrically?

Recall, that you have studied in Class IX that the geometrical (i.e., graphical) representation of a linear equation in two variables is a straight line. Can you now suggest what a pair of linear equations in two variables will look like, geometrically? There will be two straight lines, both to be considered together.

You have also studied in Class IX that given two lines in a plane, only one of the following three possibilities can happen:

(i) The two lines will intersect at one point.

(ii) The two lines will not intersect, i.e., they are parallel.

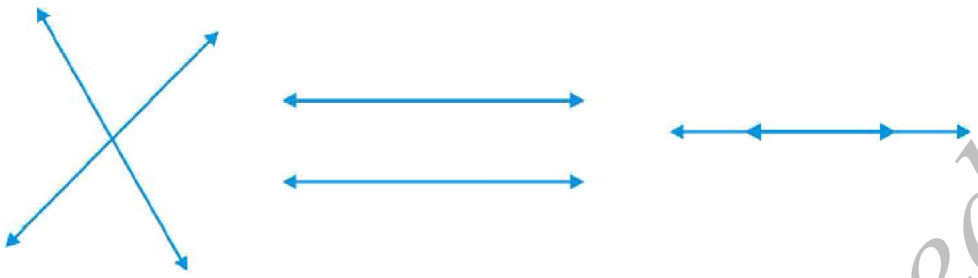
(iii) The two lines will be coincident.

We show all these possibilities in Fig. 3.1:

In Fig. 3.1 (a), they intersect.

In Fig. 3.1 (b), they are parallel.

In Fig. 3.1 (c), they are coincident.



(a) (b) (c)

Fig. 3.1

Both ways of representing a pair of linear equations go hand-in-hand — the algebraic and the geometric ways. Let us consider some examples.

Example 1 : Let us take the example given in Section 3.1. Akhila goes to a fair with ₹ 20 and wants to have rides on the Giant Wheel and play Hoopla. Represent this situation algebraically and graphically (geometrically).

Solution : The pair of equations formed is :

$$y = \frac{1}{2}x$$

i.e., $$x - 2y = 0 \qquad (1)$$

$$3x + 4y = 20 \qquad (2)$$

Let us represent these equations graphically. For this, we need at least two solutions for each equation. We give these solutions in Table 3.1.

Table 3.1

x	0	2
$y = \frac{x}{2}$	0	1

(i)

x	0	$\frac{20}{3}$	4
$y = \frac{20-3x}{4}$	5	0	2

(ii)

Recall from Class IX that there are infinitely many solutions of each linear equation. So each of you can choose any two values, which may not be the ones we have chosen. Can you guess why we have chosen $x = 0$ in the first equation and in the second equation? When one of the variables is zero, the equation reduces to a linear equation in one variable, which can be solved easily. For instance, putting $x = 0$ in Equation (2), we get $4y = 20$, i.e., $y = 5$. Similarly, putting $y = 0$ in Equation (2), we get $3x = 20$, i.e., $x = \frac{20}{3}$. But as $\frac{20}{3}$ is not an integer, it will not be easy to plot exactly on the graph paper. So, we choose $y = 2$ which gives $x = 4$, an integral value.

Plot the points A(0, 0), B(2, 1) and P(0, 5), Q(4, 2), corresponding to the solutions in Table 3.1. Now draw the lines AB and PQ, representing the equations $x - 2y = 0$ and $3x + 4y = 20$, as shown in Fig. 3.2.

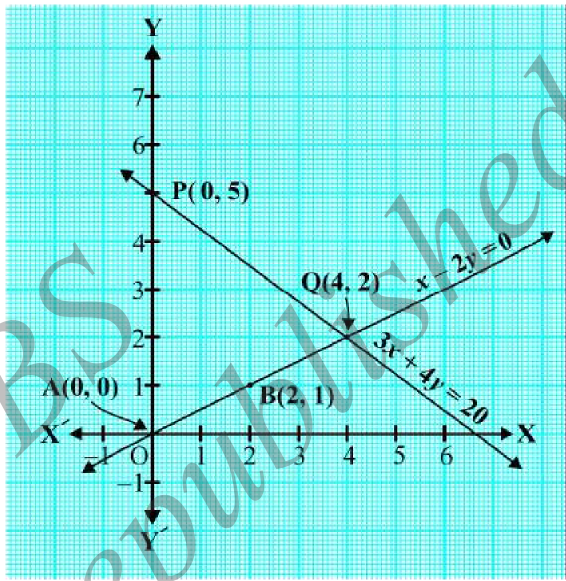


Fig. 3.2

In Fig. 3.2, observe that the two lines representing the two equations are intersecting at the point (4, 2). We shall discuss what this means in the next section.

Example 2 : Romila went to a stationery shop and purchased 2 pencils and 3 erasers for ₹ 9. Her friend Sonali saw the new variety of pencils and erasers with Romila, and she also bought 4 pencils and 6 erasers of the same kind for ₹ 18. Represent this situation algebraically and graphically.

Solution : Let us denote the cost of 1 pencil by ₹ x and one eraser by ₹ y. Then the algebraic representation is given by the following equations:

$$2x + 3y = 9 \qquad (1)$$

$$4x + 6y = 18 \qquad (2)$$

To obtain the equivalent geometric representation, we find two points on the line representing each equation. That is, we find two solutions of each equation.

These solutions are given below in Table 3.2.

Table 3.2

x	0	4.5
$y = \dfrac{9-2x}{3}$	3	0

(i)

x	0	3
$y = \dfrac{18-4x}{6}$	3	1

(ii)

We plot these points in a graph paper and draw the lines. We find that both the lines coincide (see Fig. 3.3). This is so, because, both the equations are equivalent, i.e., one can be derived from the other.

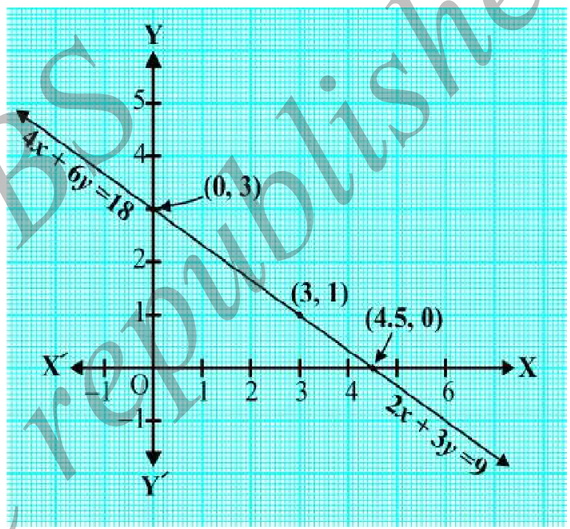


Fig. 3.3

Example 3 : Two rails are represented by the equations $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Represent this situation geometrically.

Solution : Two solutions of each of the equations :

$$x + 2y - 4 = 0 \qquad (1)$$

$$2x + 4y - 12 = 0 \qquad (2)$$

are given in Table 3.3

Table 3.3

x	0	4
$y = \dfrac{4-x}{2}$	2	0

(i)

x	0	6
$y = \dfrac{12-2x}{4}$	3	0

(ii)

To represent the equations graphically, we plot the points R(0, 2) and S(4, 0), to get the line RS and the points P(0, 3) and Q(6, 0) to get the line PQ.

We observe in Fig. 3.4, that the lines do not intersect anywhere, i.e., they are parallel.

So, we have seen several situations which can be represented by a pair of linear equations. We have seen their algebraic and geometric representations. In the next few sections, we will discuss how these representations can be used to look for solutions of the pair of linear equations.

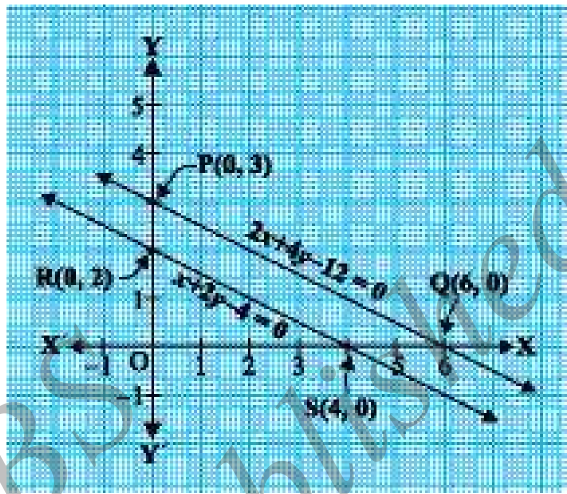


Fig. 3.4

EXERCISE 3.1

1. Aftab tells his daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." (Isn't this interesting?) Represent this situation algebraically and graphically.
2. The coach of a cricket team buys 3 bats and 6 balls for ₹ 3900. Later, she buys another bat and 3 more balls of the same kind for ₹ 1300. Represent this situation algebraically and geometrically.
3. The cost of 2 kg of apples and 1kg of grapes on a day was found to be ₹ 160. After a month, the cost of 4 kg of apples and 2 kg of grapes is ₹ 300. Represent the situation algebraically and geometrically.

3.3 Graphical Method of Solution of a Pair of Linear Equations

In the previous section, you have seen how we can graphically represent a pair of linear equations as two lines. You have also seen that the lines may intersect, or may be parallel, or may coincide. Can we solve them in each case? And if so, how? We shall try and answer these questions from the geometrical point of view in this section.

Let us look at the earlier examples one by one.

- In the situation of Example 1, find out how many rides on the Giant Wheel Akhila had, and how many times she played Hoopla.

In Fig. 3.2, you noted that the equations representing the situation are geometrically shown by two lines intersecting at the point (4, 2). Therefore, the point (4, 2) lies on the lines represented by both the equations $x - 2y = 0$ and $3x + 4y = 20$. And this is the only common point.

Let us verify algebraically that $x = 4$, $y = 2$ is a solution of the given pair of equations. Substituting the values of x and y in each equation, we get $4 - 2 \times 2 = 0$ and $3(4) + 4(2) = 20$. So, we have verified that $x = 4$, $y = 2$ is a solution of both the equations. **Since (4, 2) is the only common point on both the lines, there is one and only one solution for this pair of linear equations in two variables.**

Thus, the number of rides Akhila had on Giant Wheel is 4 and the number of times she played Hoopla is 2.

- In the situation of Example 2, can you find the cost of each pencil and each eraser?

In Fig. 3.3, the situation is geometrically shown by a pair of coincident lines. The solutions of the equations are given by the common points.

Are there any common points on these lines? From the graph, we observe that every point on the line is a common solution to both the equations. So, the equations $2x + 3y = 9$ and $4x + 6y = 18$ have **infinitely many solutions.** This should not surprise us, because if we divide the equation $4x + 6y = 18$ by 2 , we get $2x + 3y = 9$, which is the same as Equation (1). That is, both the equations are equivalent. From the graph, we see that any point on the line gives us a possible cost of each pencil and eraser. For instance, each pencil and eraser can cost ₹ 3 and ₹ 1 respectively. Or, each pencil can cost ₹ 3.75 and eraser can cost ₹ 0.50, and so on.

- In the situation of Example 3, can the two rails cross each other?

In Fig. 3.4, the situation is represented geometrically by two parallel lines. Since the lines do not intersect at all, the rails do not cross. This also means that the equations have no common solution.

A pair of linear equations which has no solution, is called an *inconsistent* pair of linear equations. A pair of linear equations in two variables, which has a solution, is called a *consistent* pair of linear equations. A pair of linear equations which are equivalent has infinitely many distinct common solutions. Such a pair is called a *dependent* pair of linear equations in two variables. Note that a dependent pair of linear equations is always consistent.

We can now summarise the behaviour of lines representing a pair of linear equations in two variables and the existence of solutions as follows:

(i) the lines may intersect in a single point. In this case, the pair of equations has a unique solution (consistent pair of equations).

(ii) the lines may be parallel. In this case, the equations have no solution (inconsistent pair of equations).

(iii) the lines may be coincident. In this case, the equations have infinitely many solutions [dependent (consistent) pair of equations].

Let us now go back to the pairs of linear equations formed in Examples 1, 2, and 3, and note down what kind of pair they are geometrically.

(i) $x - 2y = 0$ and $3x + 4y - 20 = 0$ (The lines intersect)

(ii) $2x + 3y - 9 = 0$ and $4x + 6y - 18 = 0$ (The lines coincide)

(iii) $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$ (The lines are parallel)

Let us now write down, and compare, the values of $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$ in all the three examples. Here, a_1, b_1, c_1 and a_2, b_2, c_2 denote the coefficents of equations given in the general form in Section 3.2.

Table 3.4

Sl No.	Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Compare the ratios	Graphical representation	Algebraic interpretation
1.	$x - 2y = 0$ $3x + 4y - 20 = 0$	$\frac{1}{3}$	$\frac{-2}{4}$	$\frac{0}{-20}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Exactly one solution (unique)
2.	$2x + 3y - 9 = 0$ $4x + 6y - 18 = 0$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{-9}{-18}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines	Infinitely many solutions
3.	$x + 2y - 4 = 0$ $2x + 4y - 12 = 0$	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{-4}{-12}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution

From the table above, you can observe that if the lines represented by the equation

$$a_1x + b_1y + c_1 = 0$$

and

$$a_2x + b_2y + c_2 = 0$$

are (i) intersecting, then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

(ii) coincident, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(iii) parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

In fact, the converse is also true for any pair of lines. You can verify them by considering some more examples by yourself.

Let us now consider some more examples to illustrate it.

Example 4 : Check graphically whether the pair of equations

$$x + 3y = 6 \quad (1)$$

and

$$2x - 3y = 12 \quad (2)$$

is consistent. If so, solve them graphically.

Solution : Let us draw the graphs of the Equations (1) and (2). For this, we find two solutions of each of the equations, which are given in Table 3.5

Table 3.5

x	0	6
$y = \frac{6-x}{3}$	2	0

x	0	3
$y = \frac{2x-12}{3}$	– 4	–2

Plot the points A(0, 2), B(6, 0), P(0, – 4) and Q(3, – 2) on graph paper, and join the points to form the lines AB and PQ as shown in Fig. 3.5.

We observe that there is a point B (6, 0) common to both the lines AB and PQ. So, the solution of the pair of linear equations is $x = 6$ and $y = 0$, i.e., the given pair of equations is consistent.

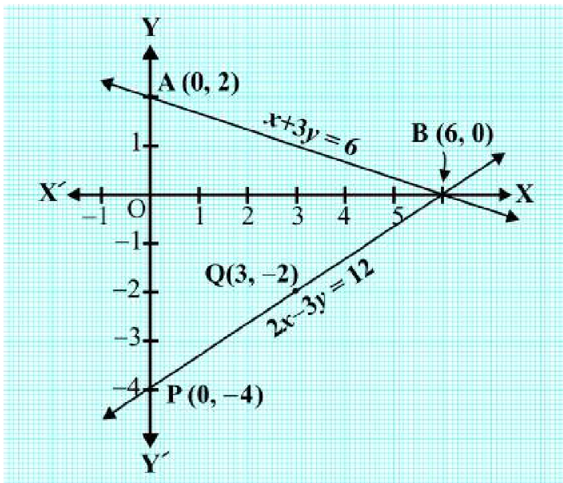


Fig. 3.5

Example 5 : Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$5x - 8y + 1 = 0 \qquad (1)$$

$$3x - \frac{24}{5}y + \frac{3}{5} = 0 \qquad (2)$$

Solution : Multiplying Equation (2) by $\frac{5}{3}$, we get

$$5x - 8y + 1 = 0$$

But, this is the same as Equation (1). Hence the lines represented by Equations (1) and (2) are coincident. Therefore, Equations (1) and (2) have infinitely many solutions.

Plot few points on the graph and verify it yourself.

Example 6 : Champa went to a 'Sale' to purchase some pants and skirts. When her friends asked her how many of each she had bought, she answered, "The number of skirts is two less than twice the number of pants purchased. Also, the number of skirts is four less than four times the number of pants purchased". Help her friends to find how many pants and skirts Champa bought.

Solution : Let us denote the number of pants by x and the number of skirts by y. Then the equations formed are :

$$y = 2x - 2 \qquad (1)$$

and

$$y = 4x - 4 \qquad (2)$$

Let us draw the graphs of Equations (1) and (2) by finding two solutions for each of the equations. They are given in Table 3.6.

Table 3.6

x	2	0
$y = 2x - 2$	2	– 2

x	0	1
$y = 4x - 4$	– 4	0

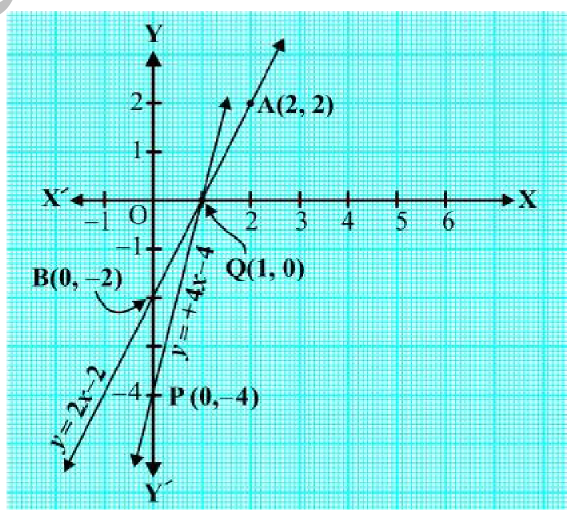


Fig. 3.6

Plot the points and draw the lines passing through them to represent the equations, as shown in Fig. 3.6.

The two lines intersect at the point (1, 0). So, $x = 1$, $y = 0$ is the required solution of the pair of linear equations, i.e., the number of pants she purchased is 1 and she did not buy any skirt.

Verify the answer by checking whether it satisfies the conditions of the given problem.

EXERCISE 3.2

1. Form the pair of linear equations in the following problems, and find their solutions graphically.

(i) 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.

(ii) 5 pencils and 7 pens together cost ₹ 50, whereas 7 pencils and 5 pens together cost ₹ 46. Find the cost of one pencil and that of one pen.

2. On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident:

(i) $5x - 4y + 8 = 0$
$7x + 6y - 9 = 0$

(ii) $9x + 3y + 12 = 0$
$18x + 6y + 24 = 0$

(iii) $6x - 3y + 10 = 0$
$2x - y + 9 = 0$

3. On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following pair of linear equations are consistent, or inconsistent.

(i) $3x + 2y = 5$; $2x - 3y = 7$

(ii) $2x - 3y = 8$; $4x - 6y = 9$

(iii) $\frac{3}{2}x + \frac{5}{3}y = 7$; $9x - 10y = 14$

(iv) $5x - 3y = 11$; $-10x + 6y = -22$

(v) $\frac{4}{3}x + 2y = 8$; $2x + 3y = 12$

4. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution graphically:

(i) $x + y = 5$, $2x + 2y = 10$
(ii) $x - y = 8$, $3x - 3y = 16$
(iii) $2x + y - 6 = 0$, $4x - 2y - 4 = 0$
(iv) $2x - 2y - 2 = 0$, $4x - 4y - 5 = 0$

5. Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.

6. Given the linear equation $2x + 3y - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

(i) intersecting lines (ii) parallel lines
(iii) coincident lines

7. Draw the graphs of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

3.4 Algebraic Methods of Solving a Pair of Linear Equations

In the previous section, we discussed how to solve a pair of linear equations graphically. The graphical method is not convenient in cases when the point representing the solution of the linear equations has non-integral coordinates like $\left(\sqrt{3}, 2\sqrt{7}\right)$, $(-1.75, 3.3)$, $\left(\frac{4}{13}, \frac{1}{19}\right)$, etc. There is every possibility of making mistakes while reading such coordinates. Is there any alternative method of finding the solution? There are several algebraic methods, which we shall now discuss.

3.4.1 Substitution Method : We shall explain the method of substitution by taking some examples.

Example 7 : Solve the following pair of equations by substitution method:

$$7x - 15y = 2 \quad (1)$$

$$x + 2y = 3 \quad (2)$$

Solution :

Step 1 : We pick either of the equations and write one variable in terms of the other. Let us consider the Equation (2) :

$$x + 2y = 3$$

and write it as

$$x = 3 - 2y \quad (3)$$

Step 2 : Substitute the value of x in Equation (1). We get

$$7(3 - 2y) - 15y = 2$$

i.e., $$21 - 14y - 15y = 2$$

i.e., $$-29y = -19$$

Therefore, $$y = \frac{19}{29}$$

Step 3 : Substituting this value of y in Equation (3), we get

$$x = 3 - 2\left(\frac{19}{29}\right) = \frac{49}{29}$$

Therefore, the solution is $x = \frac{49}{29}, y = \frac{19}{29}$.

Verification : Substituting $x = \frac{49}{29}$ and $y = \frac{19}{29}$, you can verify that both the Equations (1) and (2) are satisfied.

To understand the substitution method more clearly, let us consider it stepwise:

Step 1 : Find the value of one variable, say y in terms of the other variable, i.e., x from either equation, whichever is convenient.

Step 2 : Substitute this value of y in the other equation, and reduce it to an equation in one variable, i.e., in terms of x, which can be solved. Sometimes, as in Examples 9 and 10 below, you can get statements with no variable. If this statement is true, you can conclude that the pair of linear equations has infinitely many solutions. If the statement is false, then the pair of linear equations is inconsistent.

Step 3 : Substitute the value of x (or y) obtained in Step 2 in the equation used in Step 1 to obtain the value of the other variable.

Remark : We have **substituted** the value of one variable by expressing it in terms of the other variable to solve the pair of linear equations. That is why the method is known as the *substitution method*.

Example 8 : Solve Q.1 of Exercise 3.1 by the method of substitution.

Solution : Let s and t be the ages (in years) of Aftab and his daughter, respectively. Then, the pair of linear equations that represent the situation is

$$s - 7 = 7\,(t - 7), \text{ i.e., } s - 7t + 42 = 0 \qquad (1)$$

and $$s + 3 = 3\,(t + 3), \text{ i.e., } s - 3t = 6 \qquad (2)$$

Using Equation (2), we get $s = 3t + 6$.

Putting this value of s in Equation (1), we get

$$(3t + 6) - 7t + 42 = 0,$$

i.e., $4t = 48$, which gives $t = 12$.

Putting this value of t in Equation (2), we get

$$s = 3\,(12) + 6 = 42$$

So, Aftab and his daughter are 42 and 12 years old, respectively.

Verify this answer by checking if it satisfies the conditions of the given problems.

Example 9 : Let us consider Example 2 in Section 3.3, i.e., the cost of 2 pencils and 3 erasers is ₹ 9 and the cost of 4 pencils and 6 erasers is ₹ 18. Find the cost of each pencil and each eraser.

Solution : The pair of linear equations formed were:

$$2x + 3y = 9 \quad (1)$$

$$4x + 6y = 18 \quad (2)$$

We first express the value of x in terms of y from the equation $2x + 3y = 9$, to get

$$x = \frac{9 - 3y}{2} \quad (3)$$

Now we substitute this value of x in Equation (2), to get

$$\frac{4(9 - 3y)}{2} + 6y = 18$$

i.e., $18 - 6y + 6y = 18$

i.e., $18 = 18$

This statement is true for all values of y. However, we do not get a specific value of y as a solution. Therefore, we cannot obtain a specific value of x. This situation has arisen because both the given equations are the same. Therefore, Equations (1) and (2) have *infinitely many solutions*. Observe that we have obtained the same solution graphically also. (Refer to Fig. 3.3, Section 3.2.) We cannot find a unique cost of a pencil and an eraser, because there are many common solutions, to the given situation.

Example 10 : Let us consider the Example 3 of Section 3.2. Will the rails cross each other?

Solution : The pair of linear equations formed were:

$$x + 2y - 4 = 0 \quad (1)$$

$$2x + 4y - 12 = 0 \quad (2)$$

We express x in terms of y from Equation (1) to get

$$x = 4 - 2y$$

Now, we substitute this value of x in Equation (2) to get

$$2(4 - 2y) + 4y - 12 = 0$$

i.e., $$8 - 12 = 0$$

i.e., $$-4 = 0$$

which is a false statement.

Therefore, the equations do not have a common solution. So, the two rails will not cross each other.

EXERCISE 3.3

1. Solve the following pair of linear equations by the substitution method.
   (i) $x + y = 14$, $x - y = 4$
   (ii) $s - t = 3$, $\frac{s}{3} + \frac{t}{2} = 6$
   (iii) $3x - y = 3$, $9x - 3y = 9$
   (iv) $0.2x + 0.3y = 1.3$, $0.4x + 0.5y = 2.3$
   (v) $\sqrt{2}\,x + \sqrt{3}\,y = 0$, $\sqrt{3}\,x - \sqrt{8}\,y = 0$
   (vi) $\frac{3x}{2} - \frac{5y}{3} = -2$, $\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$
2. Solve $2x + 3y = 11$ and $2x - 4y = -24$ and hence find the value of 'm' for which $y = mx + 3$.
3. Form the pair of linear equations for the following problems and find their solution by substitution method.
   (i) The difference between two numbers is 26 and one number is three times the other. Find them.
   (ii) The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.
   (iii) The coach of a cricket team buys 7 bats and 6 balls for ₹ 3800. Later, she buys 3 bats and 5 balls for ₹ 1750. Find the cost of each bat and each ball.

(iv) The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹ 105 and for a journey of 15 km, the charge paid is ₹ 155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km?

(v) A fraction becomes $\frac{9}{11}$, if 2 is added to both the numerator and the denominator. If, 3 is added to both the numerator and the denominator it becomes $\frac{5}{6}$. Find the fraction.

(vi) Five years hence, the age of Jacob will be three times that of his son. Five years ago, Jacob's age was seven times that of his son. What are their present ages?

3.4.2 Elimination Method

Now let us consider another method of eliminating (i.e., removing) one variable. This is sometimes more convenient than the substitution method. Let us see how this method works.

Example 11 : The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly incomes.

Solution : Let us denote the incomes of the two person by ₹ $9x$ and ₹ $7x$ and their expenditures by ₹ $4y$ and ₹ $3y$ respectively. Then the equations formed in the situation is given by :

$$9x - 4y = 2000 \qquad (1)$$

and

$$7x - 3y = 2000 \qquad (2)$$

Step 1 : Multiply Equation (1) by 3 and Equation (2) by 4 to make the coefficients of y equal. Then we get the equations:

$$27x - 12y = 6000 \qquad (3)$$

$$28x - 12y = 8000 \qquad (4)$$

Step 2 : Subtract Equation (3) from Equation (4) to *eliminate* y, because the coefficients of y are the same. So, we get

$$(28x - 27x) - (12y - 12y) = 8000 - 6000$$

i.e.,

$$x = 2000$$

Step 3 : Substituting this value of x in (1), we get

$$9(2000) - 4y = 2000$$

i.e.,

$$y = 4000$$

So, the solution of the equations is $x = 2000$, $y = 4000$. Therefore, the monthly incomes of the persons are ₹ 18,000 and ₹ 14,000, respectively.

Verification : 18000 : 14000 = 9 : 7. Also, the ratio of their expenditures = 18000 – 2000 : 14000 – 2000 = 16000 : 12000 = 4 : 3

Remarks :

1. The method used in solving the example above is called the *elimination* method, because we eliminate one variable first, to get a linear equation in one variable. In the example above, we eliminated y. We could also have eliminated x. Try doing it that way.
2. You could also have used the substitution, or graphical method, to solve this problem. Try doing so, and see which method is more convenient.

Let us now note down these steps in the elimination method :

Step 1 : First multiply both the equations by some suitable non-zero constants to make the coefficients of one variable (either x or y) numerically equal.

Step 2 : Then add or subtract one equation from the other so that one variable gets eliminated. If you get an equation in one variable, go to Step 3.

If in Step 2, we obtain a true statement involving no variable, then the original pair of equations has infinitely many solutions.

If in Step 2, we obtain a false statement involving no variable, then the original pair of equations has no solution, i.e., it is inconsistent.

Step 3 : Solve the equation in one variable (x or y) so obtained to get its value.

Step 4 : Substitute this value of x (or y) in either of the original equations to get the value of the other variable.

Now to illustrate it, we shall solve few more examples.

Example 12 : Use elimination method to find all possible solutions of the following pair of linear equations :

$$2x + 3y = 8 \qquad (1)$$

$$4x + 6y = 7 \qquad (2)$$

Solution :

Step 1 : Multiply Equation (1) by 2 and Equation (2) by 1 to make the coefficients of x equal. Then we get the equations as :

$$4x + 6y = 16 \qquad (3)$$

$$4x + 6y = 7 \qquad (4)$$

Step 2 : Subtracting Equation (4) from Equation (3),

$$(4x - 4x) + (6y - 6y) = 16 - 7$$

i.e., $$0 = 9, \text{ which is a false statement.}$$

Therefore, the pair of equations has no solution.

Example 13 : The sum of a two-digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

Solution : Let the ten's and the unit's digits in the first number be x and y, respectively. So, the first number may be written as $10x + y$ in the expanded form (for example, $56 = 10(5) + 6$).

When the digits are reversed, x becomes the unit's digit and y becomes the ten's digit. This number, in the expanded notation is $10y + x$ (for example, when 56 is reversed, we get $65 = 10(6) + 5$).

According to the given condition.

$$(10x + y) + (10y + x) = 66$$

i.e., $$11(x + y) = 66$$

i.e., $$x + y = 6 \qquad (1)$$

We are also given that the digits differ by 2, therefore,

either $$x - y = 2 \qquad (2)$$

or $$y - x = 2 \qquad (3)$$

If $x - y = 2$, then solving (1) and (2) by elimination, we get $x = 4$ and $y = 2$. In this case, we get the number 42.

If $y - x = 2$, then solving (1) and (3) by elimination, we get $x = 2$ and $y = 4$. In this case, we get the number 24.

Thus, there are two such numbers 42 and 24.

Verification : Here $42 + 24 = 66$ and $4 - 2 = 2$. Also $24 + 42 = 66$ and $4 - 2 = 2$.

EXERCISE 3.4

1. Solve the following pair of linear equations by the elimination method and the substitution method :

   (i) $x + y = 5$ and $2x - 3y = 4$ (ii) $3x + 4y = 10$ and $2x - 2y = 2$

   (iii) $3x - 5y - 4 = 0$ and $9x = 2y + 7$ (iv) $\frac{x}{2} + \frac{2y}{3} = -1$ and $x - \frac{y}{3} = 3$

2. Form the pair of linear equations in the following problems, and find their solutions (if they exist) by the elimination method :

(i) If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$ if we only add 1 to the denominator. What is the fraction?

(ii) Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?

(iii) The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.

(iv) Meena went to a bank to withdraw ₹ 2000. She asked the cashier to give her ₹ 50 and ₹ 100 notes only. Meena got 25 notes in all. Find how many notes of ₹ 50 and ₹ 100 she received.

(v) A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹ 27 for a book kept for seven days, while Susy paid ₹ 21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.

3.4.3 Cross - Multiplication Method

So far, you have learnt how to solve a pair of linear equations in two variables by graphical, substitution and elimination methods. Here, we introduce one more algebraic method to solve a pair of linear equations which for many reasons is a very useful method of solving these equations. Before we proceed further, let us consider the following situation.

The cost of 5 oranges and 3 apples is ₹ 35 and the cost of 2 oranges and 4 apples is ₹ 28. Let us find the cost of an orange and an apple.

Let us denote the cost of an orange by ₹ x and the cost of an apple by ₹ y. Then, the equations formed are :

$$5x + 3y = 35, \text{ i.e., } 5x + 3y - 35 = 0 \tag{1}$$

$$2x + 4y = 28, \text{ i.e., } 2x + 4y - 28 = 0 \tag{2}$$

Let us use the elimination method to solve these equations.

Multiply Equation (1) by 4 and Equation (2) by 3. We get

$$(4)(5)x + (4)(3)y + (4)(-35) = 0 \tag{3}$$

$$(3)(2)x + (3)(4)y + (3)(-28) = 0 \tag{4}$$

Subtracting Equation (4) from Equation (3), we get

$$[(5)(4) - (3)(2)]x + [(4)(3) - (3)(4)]y + [4(-35) - (3)(-28)] = 0$$

Therefore, $$x = \frac{-[(4)(-35) - (3)(-28)]}{(5)(4) - (3)(2)}$$

i.e., $$x = \frac{(3)(-28) - (4)(-35)}{(5)(4) - (2)(3)} \qquad (5)$$

If Equations (1) and (2) are written as $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then we have

$a_1 = 5$, $b_1 = 3$, $c_1 = -35$, $a_2 = 2$, $b_2 = 4$, $c_2 = -28$.

Then Equation (5) can be written as $x = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$,

Similarly, you can get $$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

By simplyfing Equation (5), we get

$$x = \frac{-84 + 140}{20 - 6} = 4$$

Similarly, $$y = \frac{(-35)(2) - (5)(-28)}{20 - 6} = \frac{-70 + 140}{14} = 5$$

Therefore, $x = 4$, $y = 5$ is the solution of the given pair of equations.

Then, the cost of an orange is ₹ 4 and that of an apple is ₹ 5.

Verification : Cost of 5 oranges + Cost of 3 apples = ₹ 20 + ₹ 15 = ₹ 35. Cost of 2 oranges + Cost of 4 apples = ₹ 8 + ₹ 20 = ₹ 28.

Let us now see how this method works for any pair of linear equations in two variables of the form

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$

and $$a_2x + b_2y + c_2 = 0 \qquad (2)$$

To obtain the values of x and y as shown above, we follow the following steps:

Step 1 : Multiply Equation (1) by b_2 and Equation (2) by b_1, to get

$$b_2a_1x + b_2b_1y + b_2c_1 = 0 \qquad (3)$$

$$b_1a_2x + b_1b_2y + b_1c_2 = 0 \qquad (4)$$

Step 2 : Subtracting Equation (4) from (3), we get:

$$(b_2a_1 - b_1a_2)\,x + (b_2b_1 - b_1b_2)\,y + (b_2c_1 - b_1c_2) = 0$$

i.e., $(b_2a_1 - b_1a_2)\, x = b_1c_2 - b_2c_1$

So, $$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0 \qquad (5)$$

Step 3 : Substituting this value of x in (1) or (2), we get

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \qquad (6)$$

Now, two cases arise :

Case 1 : $a_1b_2 - a_2b_1 \neq 0$. In this case $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. Then the pair of linear equations has a unique solution.

Case 2 : $a_1b_2 - a_2b_1 = 0$. If we write $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$, then $a_1 = k\,a_2$, $b_1 = k\,b_2$.

Substituting the values of a_1 and b_1 in the Equation (1), we get

$$k\,(a_2x + b_2y) + c_1 = 0. \qquad (7)$$

It can be observed that the Equations (7) and (2) can both be satisfied only if $c_1 = k\,c_2$, i.e., $\frac{c_1}{c_2} = k$.

If $c_1 = k\,c_2$, any solution of Equation (2) will satisfy the Equation (1), and vice versa. So, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$, then there are infinitely many solutions to the pair of linear equations given by (1) and (2).

If $c_1 \neq k\,c_2$, then any solution of Equation (1) will not satisfy Equation (2) and vice versa. Therefore the pair has no solution.

We can summarise the discussion above for the pair of linear equations given by (1) and (2) as follows:

(i) When $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, we get a unique solution.

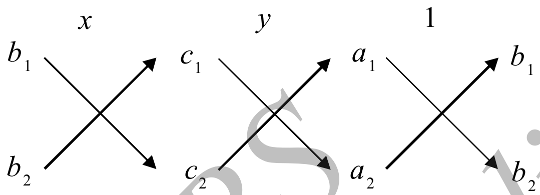
(ii) When $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, there are infinitely many solutions.

(iii) When $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, there is no solution.

Note that you can write the solution given by Equations (5) and (6) in the following form :

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \qquad (8)$$

In remembering the above result, the following diagram may be helpful to you :



The arrows between the two numbers indicate that they are to be multiplied and the second product is to be subtracted from the first.

For solving a pair of linear equations by this method, we will follow the following steps :

Step 1 : Write the given equations in the form (1) and (2).

Step 2 : Taking the help of the diagram above, write Equations as given in (8).

Step 3 : Find x and y, provided $a_1b_2 - a_2b_1 \neq 0$

Step 2 above gives you an indication of why this method is called the **cross-multiplication method**.

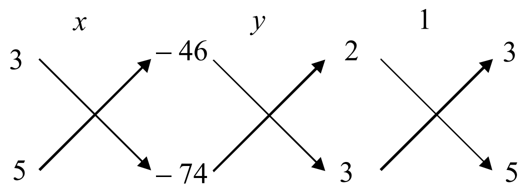
Example 14 : From a bus stand in Bangalore , if we buy 2 tickets to Malleswaram and 3 tickets to Yeshwanthpur, the total cost is ₹ 46; but if we buy 3 tickets to Malleswaram and 5 tickets to Yeshwanthpur the total cost is ₹ 74. Find the fares from the bus stand to Malleswaram, and to Yeshwanthpur.

Solution : Let ₹ x be the fare from the bus stand in Bangalore to Malleswaram, and ₹ y to Yeshwanthpur. From the given information, we have

$$2x + 3y = 46, \text{ i.e., } 2x + 3y - 46 = 0 \qquad (1)$$

$$3x + 5y = 74, \text{ i.e., } 3x + 5y - 74 = 0 \qquad (2)$$

To solve the equations by the cross-multiplication method, we draw the diagram as given below.



Then $$\frac{x}{(3)(-74)-(5)(-46)} = \frac{y}{(-46)(3)-(-74)(2)} = \frac{1}{(2)(5)-(3)(3)}$$

i.e., $$\frac{x}{-222+230} = \frac{y}{-138+148} = \frac{1}{10-9}$$

i.e., $$\frac{x}{8} = \frac{y}{10} = \frac{1}{1}$$

i.e., $$\frac{x}{8} = \frac{1}{1} \text{ and } \frac{y}{10} = \frac{1}{1}$$

i.e., $$x = 8 \text{ and } y = 10$$

Hence, the fare from the bus stand in Bangalore to Malleswaram is ₹ 8 and the fare to Yeshwanthpur is ₹ 10.

Verification : You can check from the problem that the solution we have got is correct.

Example 15 : For which values of p does the pair of equations given below has unique solution?

$$4x + py + 8 = 0$$
$$2x + 2y + 2 = 0$$

Solution : Here $a_1 = 4$, $a_2 = 2$, $b_1 = p$, $b_2 = 2$.

Now for the given pair to have a unique solution : $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

i.e., $$\frac{4}{2} \neq \frac{p}{2}$$

i.e., $$p \neq 4$$

Therefore, for all values of p, except 4, the given pair of equations will have a unique solution.

Example 16 : For what values of k will the following pair of linear equations have infinitely many solutions?

$$kx + 3y - (k-3) = 0$$
$$12x + ky - k = 0$$

Solution : Here, $\frac{a_1}{a_2} = \frac{k}{12}$, $\frac{b_1}{b_2} = \frac{3}{k}$, $\frac{c_1}{c_2} = \frac{k-3}{k}$

For a pair of linear equations to have infinitely many solutions : $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

So, we need $$\frac{k}{12} = \frac{3}{k} = \frac{k-3}{k}$$

or, $$\frac{k}{12} = \frac{3}{k}$$

which gives $k^2 = 36$, i.e., $k = \pm 6$.

Also, $$\frac{3}{k} = \frac{k-3}{k}$$

gives $3k = k^2 - 3k$, i.e., $6k = k^2$, which means $k = 0$ or $k = 6$.

Therefore, the value of k, that satisfies both the conditions, is $k = 6$. For this value, the pair of linear equations has infinitely many solutions.

EXERCISE 3.5

1. Which of the following pairs of linear equations has unique solution, no solution, or infinitely many solutions. In case there is a unique solution, find it by using cross multiplication method.

(i) $x - 3y - 3 = 0$
$3x - 9y - 2 = 0$

(ii) $2x + y = 5$
$3x + 2y = 8$

(iii) $3x - 5y = 20$
$6x - 10y = 40$

(iv) $x - 3y - 7 = 0$
$3x - 3y - 15 = 0$

2. (i) For which values of a and b does the following pair of linear equations have an infinite number of solutions?

$2x + 3y = 7$

$(a - b)x + (a + b)y = 3a + b - 2$

(ii) For which value of k will the following pair of linear equations have no solution?

$3x + y = 1$

$(2k - 1)x + (k - 1)y = 2k + 1$

3. Solve the following pair of linear equations by the substitution and cross-multiplication methods :

$8x + 5y = 9$

$3x + 2y = 4$

4. Form the pair of linear equations in the following problems and find their solutions (if they exist) by any algebraic method :

(i) A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. When a student A takes food for 20 days she has to pay ₹ 1000 as hostel charges whereas a student B, who takes food for 26 days, pays ₹ 1180 as hostel charges. Find the fixed charges and the cost of food per day.

(ii) A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes $\frac{1}{4}$ when 8 is added to its denominator. Find the fraction.

(iii) Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?

(iv) Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars?

(v) The area of a rectangle gets reduced by 9 square units, if its length is reduced by 5 units and breadth is increased by 3 units. If we increase the length by 3 units and the breadth by 2 units, the area increases by 67 square units. Find the dimensions of the rectangle.

3.5 Equations Reducible to a Pair of Linear Equations in Two Variables

In this section, we shall discuss the solution of such pairs of equations which are not linear but can be reduced to linear form by making some suitable substitutions. We now explain this process through some examples.

Example 17 : Solve the pair of equations:

$$\frac{2}{x} + \frac{3}{y} = 13$$

$$\frac{5}{x} - \frac{4}{y} = -2$$

Solution : Let us write the given pair of equations as

$$2\left(\frac{1}{x}\right) + 3\left(\frac{1}{y}\right) = 13 \qquad (1)$$

$$5\left(\frac{1}{x}\right) - 4\left(\frac{1}{y}\right) = -2 \qquad (2)$$

These equations are not in the form $ax + by + c = 0$. However, if we substitute

$\frac{1}{x} = p$ and $\frac{1}{y} = q$ in Equations (1) and (2), we get

$$2p + 3q = 13 \quad (3)$$

$$5p - 4q = -2 \quad (4)$$

So, we have expressed the equations as a pair of linear equations. Now, you can use any method to solve these equations, and get $p = 2$, $q = 3$.

You know that $p = \frac{1}{x}$ and $q = \frac{1}{y}$.

Substitute the values of p and q to get

$$\frac{1}{x} = 2, \text{ i.e., } x = \frac{1}{2} \text{ and } \frac{1}{y} = 3, \text{ i.e., } y = \frac{1}{3}.$$

Verification : By substituting $x = \frac{1}{2}$ and $y = \frac{1}{3}$ in the given equations, we find that both the equations are satisfied.

Example 18 : Solve the following pair of equations by reducing them to a pair of linear equations :

$$\frac{5}{x-1} + \frac{1}{y-2} = 2$$

$$\frac{6}{x-1} - \frac{3}{y-2} = 1$$

Solution : Let us put $\frac{1}{x-1} = p$ and $\frac{1}{y-2} = q$. Then the given equations

$$5\left(\frac{1}{x-1}\right) + \frac{1}{y-2} = 2 \quad (1)$$

$$6\left(\frac{1}{x-1}\right) - 3\left(\frac{1}{y-2}\right) = 1 \quad (2)$$

can be written as :

$$5p + q = 2 \quad (3)$$

$$6p - 3q = 1 \quad (4)$$

Equations (3) and (4) form a pair of linear equations in the general form. Now, you can use any method to solve these equations. We get $p = \frac{1}{3}$ and $q = \frac{1}{3}$.

Now, substituting $\frac{1}{x-1}$ for p, we have

$$\frac{1}{x-1} = \frac{1}{3},$$

i.e., $x - 1 = 3$, i.e., $x = 4$.

Similarly, substituting $\frac{1}{y-2}$ for q, we get

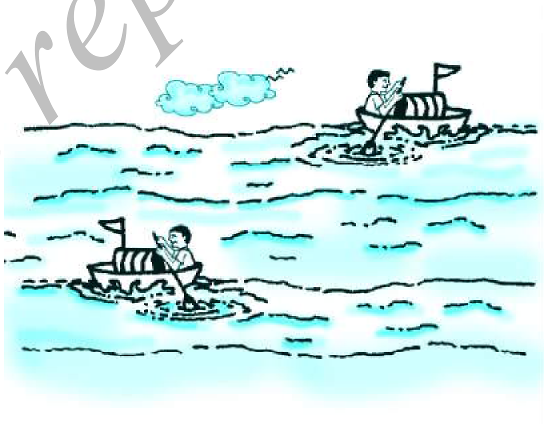
$$\frac{1}{y-2} = \frac{1}{3}$$

i.e., $3 = y - 2$, i.e., $y = 5$

Hence, $x = 4$, $y = 5$ is the required solution of the given pair of equations.

Verification : Substitute $x = 4$ and $y = 5$ in (1) and (2) to check whether they are satisfied.

Example 19 : A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km down-stream. Determine the speed of the stream and that of the boat in still water.



Solution : Let the speed of the boat in still water be x km/h and speed of the stream be y km/h. Then the speed of the boat downstream $= (x + y)$ km/h,

and the speed of the boat upstream $= (x - y)$ km/h

Also, $$\text{time} = \frac{\text{distance}}{\text{speed}}$$

In the first case, when the boat goes 30 km upstream, let the time taken, in hour, be t_1. Then

$$t_1 = \frac{30}{x-y}$$

Let t_2 be the time, in hours, taken by the boat to go 44 km downstream. Then $t_2 = \frac{44}{x+y}$. The total time taken, $t_1 + t_2$, is 10 hours. Therefore, we get the equation

$$\frac{30}{x-y} + \frac{44}{x+y} = 10 \qquad (1)$$

In the second case, in 13 hours it can go 40 km upstream and 55 km downstream. We get the equation

$$\frac{40}{x-y} + \frac{55}{x+y} = 13 \qquad (2)$$

Put $$\frac{1}{x-y} = u \text{ and } \frac{1}{x+y} = v \qquad (3)$$

On substituting these values in Equations (1) and (2), we get the pair of linear equations:

$$30u + 44v = 10 \quad \text{or} \quad 30u + 44v - 10 = 0 \qquad (4)$$

$$40u + 55v = 13 \quad \text{or} \quad 40u + 55v - 13 = 0 \qquad (5)$$

Using Cross-multiplication method, we get

$$\frac{u}{44(-13) - 55(-10)} = \frac{v}{40(-10) - 30(-13)} = \frac{1}{30(55) - 44(40)}$$

i.e., $$\frac{u}{-22} = \frac{v}{-10} = \frac{1}{-110}$$

i.e., $$u = \frac{1}{5}, \quad v = \frac{1}{11}$$

Now put these values of u and v in Equations (3), we get

$$\frac{1}{x-y} = \frac{1}{5} \text{ and } \frac{1}{x+y} = \frac{1}{11}$$

i.e., $$x - y = 5 \text{ and } x + y = 11 \qquad (6)$$

Adding these equations, we get

$$2x = 16$$

i.e., $$x = 8$$

Subtracting the equations in (6), we get

$$2y = 6$$

i.e., $$y = 3$$

Hence, the speed of the boat in still water is 8 km/h and the speed of the stream is 3 km/h.

Verification : Verify that the solution satisfies the conditions of the problem.

EXERCISE 3.6

1. Solve the following pairs of equations by reducing them to a pair of linear equations:

(i) $\frac{1}{2x} + \frac{1}{3y} = 2$

$\frac{1}{3x} + \frac{1}{2y} = \frac{13}{6}$

(ii) $\frac{2}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 2$

$\frac{4}{\sqrt{x}} - \frac{9}{\sqrt{y}} = -1$

(iii) $\frac{4}{x} + 3y = 14$

$\frac{3}{x} - 4y = 23$

(iv) $\frac{5}{x-1} + \frac{1}{y-2} = 2$

$\frac{6}{x-1} - \frac{3}{y-2} = 1$

(v) $\frac{7x-2y}{xy} = 5$

$\frac{8x+7y}{xy} = 15$

(vi) $6x + 3y = 6xy$

$2x + 4y = 5xy$

(vii) $\frac{10}{x+y} + \frac{2}{x-y} = 4$

$\frac{15}{x+y} - \frac{5}{x-y} = -2$

(viii) $\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}$

$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8}$

2. Formulate the following problems as a pair of equations, and hence find their solutions:

(i) Ritu can row downstream 20 km in 2 hours, and upstream 4 km in 2 hours. Find her speed of rowing in still water and the speed of the current.

(ii) 2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the work, and also that taken by 1 man alone.

(iii) Roohi travels 300 km to her home partly by train and partly by bus. She takes 4 hours if she travels 60 km by train and the remaining by bus. If she travels 100 km by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus separately.

EXERCISE 3.7 (Optional)*

1. The ages of two friends Ani and Biju differ by 3 years. Ani's father Dharam is twice as old as Ani and Biju is twice as old as his sister Cathy. The ages of Cathy and Dharam differ by 30 years. Find the ages of Ani and Biju.
2. One says, "Give me a hundred, friend! I shall then become twice as rich as you". The other replies, "If you give me ten, I shall be six times as rich as you". Tell me what is the amount of their (respective) capital? [From the Bijaganita of Bhaskara II]
[**Hint :** $x + 100 = 2(y - 100)$, $y + 10 = 6(x - 10)$].
3. A train covered a certain distance at a uniform speed. If the train would have been 10 km/h faster, it would have taken 2 hours less than the scheduled time. And, if the train were slower by 10 km/h; it would have taken 3 hours more than the scheduled time. Find the distance covered by the train.
4. The students of a class are made to stand in rows. If 3 students are extra in a row, there would be 1 row less. If 3 students are less in a row, there would be 2 rows more. Find the number of students in the class.
5. In a Δ ABC, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.
6. Draw the graphs of the equations $5x - y = 5$ and $3x - y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.
7. Solve the following pair of linear equations:

(i) $px + qy = p - q$
$qx - py = p + q$

(ii) $ax + by = c$
$bx + ay = 1 + c$

(iii) $\dfrac{x}{a} - \dfrac{y}{b} = 0$
$ax + by = a^2 + b^2$.

(iv) $(a - b)x + (a + b)y = a^2 - 2ab - b^2$
$(a + b)(x + y) = a^2 + b^2$

(v) $152x - 378y = -74$
$-378x + 152y = -604$

8. ABCD is a cyclic quadrilateral (see Fig. 3.7). Find the angles of the cyclic quadrilateral.

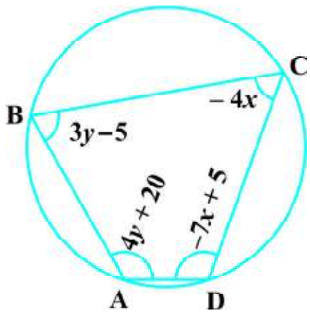


Fig. 3.7

* These exercises are not from the examination point of view.

3.6 Summary

In this chapter, you have studied the following points:

1. Two linear equations in the same two variables are called a pair of linear equations in two variables. The most general form of a pair of linear equations is

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers, such that $a_1^2 + b_1^2 \neq 0,\ a_2^2 + b_2^2 \neq 0$.

2. A pair of linear equations in two variables can be represented, and solved, by the:
   (i) graphical method
   (ii) algebraic method

3. Graphical Method :

   The graph of a pair of linear equations in two variables is represented by two lines.
   (i) If the lines intersect at a point, then that point gives the unique solution of the two equations. In this case, the pair of equations is **consistent**.
   (ii) If the lines coincide, then there are infinitely many solutions — each point on the line being a solution. In this case, the pair of equations is **dependent (consistent)**.
   (iii) If the lines are parallel, then the pair of equations has no solution. In this case, the pair of equations is **inconsistent**.

4. Algebraic Methods : We have discussed the following methods for finding the solution(s) of a pair of linear equations :
   (i) Substitution Method
   (ii) Elimination Method
   (iii) Cross-multiplication Method

5. If a pair of linear equations is given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the following situations can arise :
   (i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_1}$: In this case, the pair of linear equations is consistent.
   (ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$: In this case, the pair of linear equations is inconsistent.
   (iii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$: In this case, the pair of linear equations is dependent and consistent.

6. There are several situations which can be mathematically represented by two equations that are not linear to start with. But we alter them so that they are reduced to a pair of linear equations.

CIRCLES 4

4.1 Introduction

You have studied in Class IX that a circle is a collection of all points in a plane which are at a constant distance (radius) from a fixed point (centre). You have also studied various terms related to a circle like chord, segment, sector, arc etc. Let us now examine the different situations that can arise when a circle and a line are given in a plane.

So, let us consider a circle and a line PQ. There can be three possibilities given in Fig. 4.1 below:

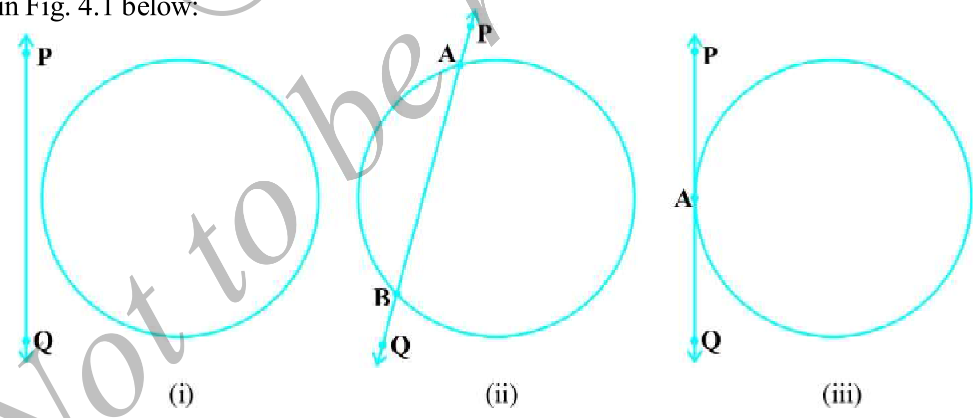


Fig. 4.1

In Fig. 4.1 (i), the line PQ and the circle have no common point. In this case, PQ is called a **non-intersecting** line with respect to the circle. In Fig. 10.1 (ii), there are two common points A and B that the line PQ and the circle have. In this case, we call the line PQ a **secant** of the circle. In Fig. 10.1 (iii), there is only one point A which is common to the line PQ and the circle. In this case, the line is called a **tangent** to the circle.

You might have seen a pulley fitted over a well which is used in taking out water from the well. Look at Fig. 4.2. Here the rope on both sides of the pulley, if considered as a ray, is like a tangent to the circle representing the pulley.

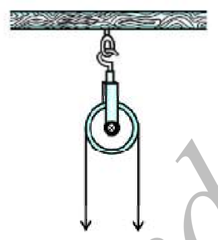


Fig. 4.2

Is there any position of the line with respect to the circle other than the types given above? You can see that there cannot be any other type of position of the line with respect to the circle. In this chapter, we will study about the existence of the tangents to a circle and also study some of their properties.

4.2 Tangent to a Circle

In the previous section, you have seen that **a tangent* to a circle is a line that intersects the circle at only one point**.

To understand the existence of the tangent to a circle at a point, let us perform the following activities:

Activity 1 : Take a circular wire and attach a straight wire AB at a point P of the circular wire so that it can rotate about the point P in a plane. Put the system on a table and gently rotate the wire AB about the point P to get different positions of the straight wire [see Fig. 4.3(i)].

In various positions, the wire intersects the circular wire at P and at another point Q_1 or Q_2 or Q_3, etc. In one position, you will see that it will intersect the circle at the point P only (see position A′B′ of AB). This shows that a tangent exists at the point P of the circle. On rotating further, you can observe that in all other positions of AB, it will intersect the circle at P and at another point, say R_1 or R_2 or R_3, etc. So, you can observe that **there is only one tangent at a point of the circle**.

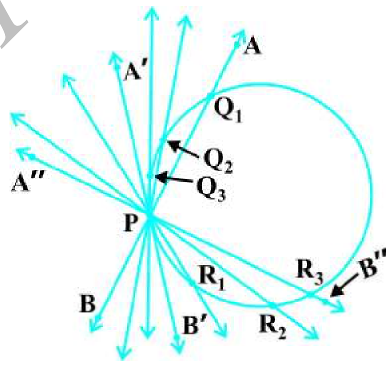


Fig. 4.3 (i)

While doing activity above, you must have observed that as the position AB moves towards the position A′ B′, the common point, say Q_1, of the line AB and the circle gradually comes nearer and nearer to the common point P. Ultimately, it coincides with the point P in the position A′B′ of A″B″. Again note, what happens if 'AB' is rotated rightwards about P? The common point R_3 gradually comes nearer and nearer to P and ultimately coincides with P. So, what we see is:

The tangent to a circle is a special case of the secant, when the two end points of its corresponding chord coincide.

*The word 'tangent' comes from the Latin word 'tangere', which means to touch and was introduced by the Danish mathematician Thomas Fineke in 1583.

Activity 2 : On a paper, draw a circle and a secant PQ of the circle. Draw various lines parallel to the secant on both sides of it. You will find that after some steps, the length of the chord cut by the lines will gradually decrease, i.e., the two points of intersection of the line and the circle are coming closer and closer [see Fig. 4.3(ii)]. In one case, it becomes zero on one side of the secant and in another case, it becomes zero on the other side of the secant. See the positions P′Q′ and P″Q″ of the secant in Fig. 4.3 (ii). These are the tangents to the circle parallel to the given secant PQ. This also helps you to see that there cannot be more than two tangents parallel to a given secant.

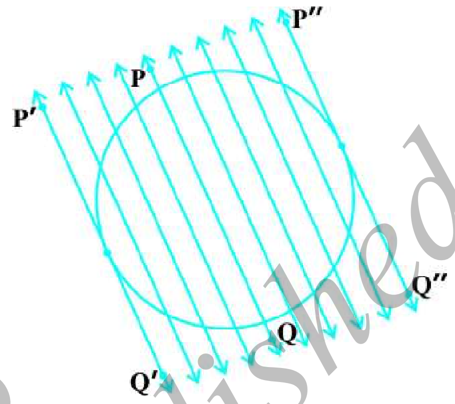


Fig. 4.3 (ii)

This activity also establishes, what you must have observed, while doing Activity 1, namely, a tangent is the secant when both of the end points of the corresponding chord coincide.

The common point of the tangent and the circle is called the **point of contact** [the point A in Fig. 4.1 (iii)]and the tangent is said to **touch** the circle at the common point.

Now look around you. Have you seen a bicycle or a cart moving? Look at its wheels. All the spokes of a wheel are along its radii. Now note the position of the wheel with respect to its movement on the ground. Do you see any tangent anywhere? (See Fig. 4.4). In fact, the wheel moves along a line which is a tangent to the circle representing the wheel. Also, notice that in all positions, the radius through the point of contact with the ground appears to be at right angles to the tangent (see Fig. 4.4). We shall now prove this property of the tangent.

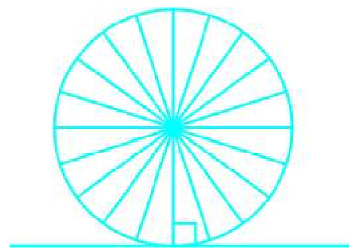


Fig. 4.4

Theorem 4.1 : *The tangent at any point of a circle is perpendicular to the radius through the point of contact.*

Proof : We are given a circle with centre O and a tangent XY to the circle at a point P. We need to prove that OP is perpendicular to XY.

Take a point Q on XY other than P and join OQ (see Fig. 10.5).

The point Q must lie outside the circle. (Why? Note that if Q lies inside the circle, XY will become a secant and not a tangent to the circle). Therefore, OQ is longer than the radius OP of the circle. That is,

$$OQ > OP.$$

Since this happens for every point on the line XY except the point P, OP is the shortest of all the distances of the point O to the points of XY. So OP is perpendicular to XY. (as shown in Theorem A1.7.)

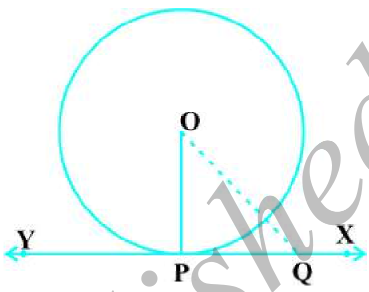


Fig. 4.5

Remarks :

1. By theorem above, we can also conclude that at any point on a circle there can be one and only one tangent.

2. The line containing the radius through the point of contact is also sometimes called the 'normal' to the circle at the point.

EXERCISE 4.1

1. How many tangents can a circle have?
2. Fill in the blanks :
   (i) A tangent to a circle intersects it in ________ point (s).
   (ii) A line intersecting a circle in two points is called a ________ .
   (iii) A circle can have ________ parallel tangents at the most.
   (iv) The common point of a tangent to a circle and the circle is called ________ .
3. A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q so that OQ = 12 cm. Length PQ is :
   (A) 12 cm (B) 13 cm (C) 8.5 cm (D) $\sqrt{119}$ cm.
4. Draw a circle and two lines parallel to a given line such that one is a tangent and the other, a secant to the circle.

4.3 Number of Tangents from a Point on a Circle

To get an idea of the number of tangents from a point on a circle, let us perform the following activity:

Activity 3 : Draw a circle on a paper. Take a point P inside it. Can you draw a tangent to the circle through this point? You will find that all the lines through this point intersect the circle in two points. So, it is not possible to draw any tangent to a circle through a point inside it [see Fig. 4.6 (i)].

Next take a point P on the circle and draw tangents through this point. You have already observed that there is only one tangent to the circle at such a point [see Fig. 4.6 (ii)].

Finally, take a point P outside the circle and try to draw tangents to the circle from this point. What do you observe? You will find that you can draw exactly two tangents to the circle through this point [see Fig. 4.6 (iii)].

We can summarise these facts as follows:

Case 1 : There is no tangent to a circle passing through a point lying inside the circle.

Case 2 : There is one and only one tangent to a circle passing through a point lying on the circle.

Case 3 : There are exactly two tangents to a circle through a point lying outside the circle.

In Fig. 4.6 (iii), T_1 and T_2 are the points of contact of the tangents PT_1 and PT_2 respectively.

The length of the segment of the tangent from the external point P and the point of contact with the circle is called the **length of the tangent** from the point P to the circle.

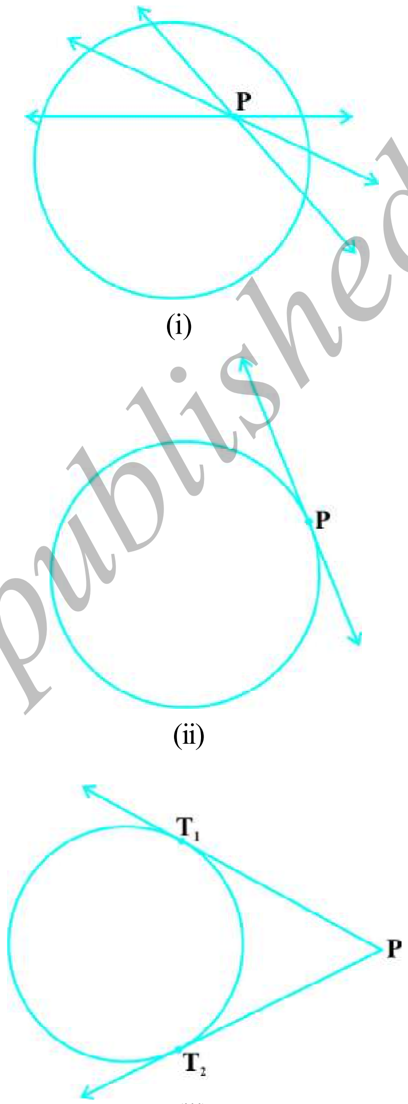


Fig. 4.6

Note that in Fig. 4.6 (iii), PT_1 and PT_2 are the lengths of the tangents from P to the circle. The lengths PT_1 and PT_2 have a common property. Can you find this? Measure PT_1 and PT_2. Are these equal? In fact, this is always so. Let us give a proof of this fact in the following theorem.

Theorem 4.2 : *The lengths of tangents drawn from an external point to a circle are equal.*

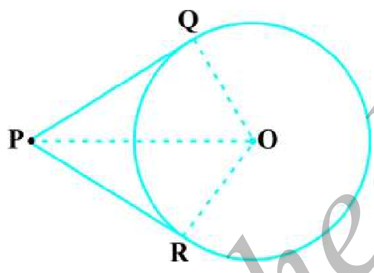


Fig. 4.7

Proof : We are given a circle with centre O, a point P lying outside the circle and two tangents PQ, PR on the circle from P (see Fig. 4.7). We are required to prove that PQ = PR.

For this, we join OP, OQ and OR. Then $\angle$ OQP and $\angle$ ORP are right angles, because these are angles between the radii and tangents, and according to Theorem 4.1 they are right angles. Now in right triangles OQP and ORP,

	OQ = OR	(Radii of the same circle)
	OP = OP	(Common)
Therefore,	Δ OQP $\cong$ Δ ORP	(RHS)
This gives	PQ = PR	(CPCT)

Remarks :

1. The theorem can also be proved by using the Pythagoras Theorem as follows:

$$PQ^2 = OP^2 - OQ^2 = OP^2 - OR^2 = PR^2 \text{ (As OQ = OR)}$$

which gives PQ = PR.

2. Note also that $\angle$ OPQ = $\angle$ OPR. Therefore, OP is the angle bisector of $\angle$ QPR, i.e., the centre lies on the bisector of the angle between the two tangents.

Let us take some examples.

Example 1 : Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.

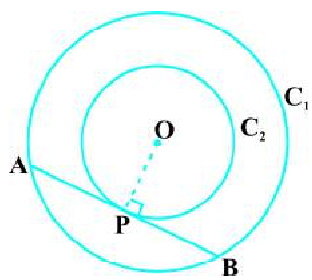


Fig. 4.8

Solution : We are given two concentric circles C_1 and C_2 with centre O and a chord AB of the larger circle C_1 which touches the smaller circle C_2 at the point P (see Fig. 4.8). We need to prove that AP = BP.

Let us join OP. Then, AB is a tangent to C_2 at P and OP is its radius. Therefore, by Theorem 4.1,

$$OP \perp AB$$

Now AB is a chord of the circle C_1 and $OP \perp AB$. Therefore, OP is the bisector of the chord AB, as the perpendicular from the centre bisects the chord,

i.e., $AP = BP$

Example 2 : Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that $\angle PTQ = 2 \angle OPQ$.

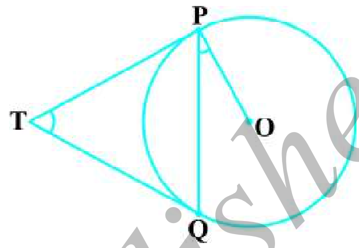


Fig. 4.9

Solution : We are given a circle with centre O, an external point T and two tangents TP and TQ to the circle, where P, Q are the points of contact (see Fig. 4.9). We need to prove that

$$\angle PTQ = 2 \angle OPQ$$

Let $\angle PTQ = \theta$

Now, by Theorem 4.2, TP = TQ. So, TPQ is an isosceles triangle.

Therefore, $\angle TPQ = \angle TQP = \frac{1}{2}(180° - \theta) = 90° - \frac{1}{2}\theta$

Also, by Theorem 4.1, $\angle OPT = 90°$

So, $\angle OPQ = \angle OPT - \angle TPQ = 90° - \left(90° - \frac{1}{2}\theta\right)$

$$= \frac{1}{2}\theta = \frac{1}{2}\angle PTQ$$

This gives $\angle PTQ = 2 \angle OPQ$

Example 3 : PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T (see Fig. 4.10). Find the length TP.

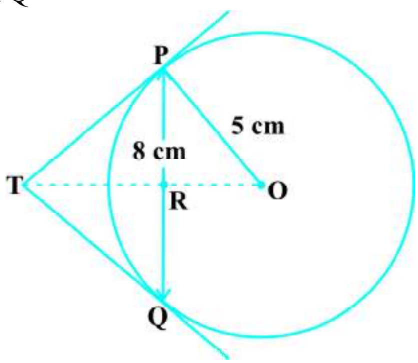


Fig. 4.10

Solution : Join OT. Let it intersect PQ at the point R. Then Δ TPQ is isosceles and TO is the angle bisector of $\angle$ PTQ. So, $OT \perp PQ$ and therefore, OT bisects PQ which gives PR = RQ = 4 cm.

Also, $OR = \sqrt{OP^2 - PR^2} = \sqrt{5^2 - 4^2}$ cm = 3 cm.

Now, $\angle$ TPR + $\angle$ RPO = 90° = $\angle$ TPR + $\angle$ PTR (Why?)

So, $\angle$ RPO = $\angle$ PTR

Therefore, right triangle TRP is similar to the right triangle PRO by AA similarity.

This gives $\dfrac{TP}{PO} = \dfrac{RP}{RO}$, i.e., $\dfrac{TP}{5} = \dfrac{4}{3}$ or TP = $\dfrac{20}{3}$ cm.

Note : TP can also be found by using the Pythagoras Theorem, as follows:

Let TP = x and TR = y. Then

$$x^2 = y^2 + 16 \quad \text{(Taking right } \Delta \text{ PRT)} \qquad (1)$$

$$x^2 + 5^2 = (y + 3)^2 \quad \text{(Taking right } \Delta \text{ OPT)} \qquad (2)$$

Subtracting (1) from (2), we get

$$25 = 6y - 7 \quad \text{or} \quad y = \frac{32}{6} = \frac{16}{3}$$

Therefore, $$x^2 = \left(\frac{16}{3}\right)^2 + 16 = \frac{16}{9}(16 + 9) = \frac{16 \times 25}{9} \qquad \text{[From (1)]}$$

or $$x = \frac{20}{3} \text{ cm}$$

EXERCISE 4.2

In Q.1 to 3, choose the correct option and give justification.

1. From a point Q, the length of the tangent to a circle is 24 cm and the distance of Q from the centre is 25 cm. The radius of the circle is

   (A) 7 cm (B) 12 cm

   (C) 15 cm (D) 24.5 cm

2. In Fig. 4.11, if TP and TQ are the two tangents to a circle with centre O so that $\angle$ POQ = 110°, then $\angle$ PTQ is equal to

   (A) 60° (B) 70°

   (C) 80° (D) 90°

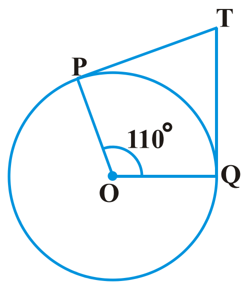


Fig. 4.11

3. If tangents PA and PB from a point P to a circle with centre O are inclined to each other at angle of 80°, then $\angle$ POA is equal to

   (A) 50° (B) 60°

   (C) 70° (D) 80°

4. Prove that the tangents drawn at the ends of a diameter of a circle are parallel.

5. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.

6. The length of a tangent from a point A at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

7. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.

8. A quadrilateral ABCD is drawn to circumscribe a circle (see Fig. 4.12). Prove that

$$AB + CD = AD + BC$$

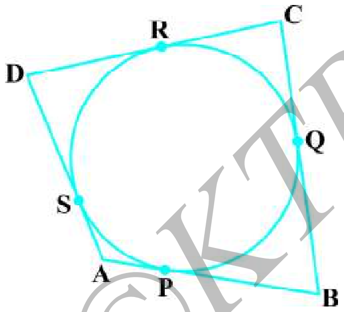


Fig. 4.12

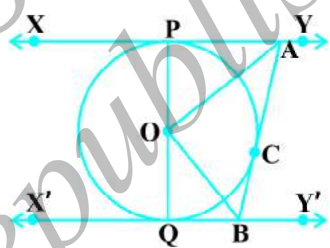


Fig. 4.13

9. In Fig. 4.13, XY and X′Y′ are two parallel tangents to a circle with centre O and another tangent AB with point of contact C intersecting XY at A and X′Y′ at B. Prove that $\angle$ AOB = 90°.

10. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.

11. Prove that the parallelogram circumscribing a circle is a rhombus.

12. A triangle ABC is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC into which BC is divided by the point of contact D are of lengths 8 cm and 6 cm respectively (see Fig. 4.14). Find the sides AB and AC.

13. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.

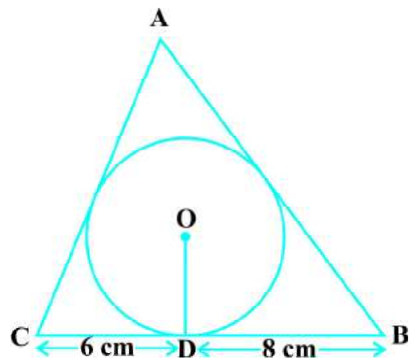


Fig. 4.14

4.4 Summary

In this chapter, you have studied the following points :

1. The meaning of a tangent to a circle.

2. The tangent to a circle is perpendicular to the radius through the point of contact.

3. The lengths of the two tangents from an external point to a circle are equal.

AREAS RELATED TO CIRCLES 5

5.1 Introduction

You are already familiar with some methods of finding perimeters and areas of simple plane figures such as rectangles, squares, parallelograms, triangles and circles from your earlier classes. Many objects that we come across in our daily life are related to the circular shape in some form or the other. Cycle wheels, wheel barrow (*thela*), dartboard, round cake, *papad*, drain cover, various designs, bangles, brooches, circular paths, washers, flower beds, etc. are some examples of such objects (see Fig. 5.1). So, the problem of finding perimeters and areas related to circular figures is of great practical importance. In this chapter, we shall begin our discussion with a review of the concepts of perimeter (circumference) and area of a circle and apply this knowledge in finding the areas of two special 'parts' of a circular region (or briefly of a circle) known as *sector* and *segment*. We shall also see how to find the areas of some combinations of plane figures involving circles or their parts.

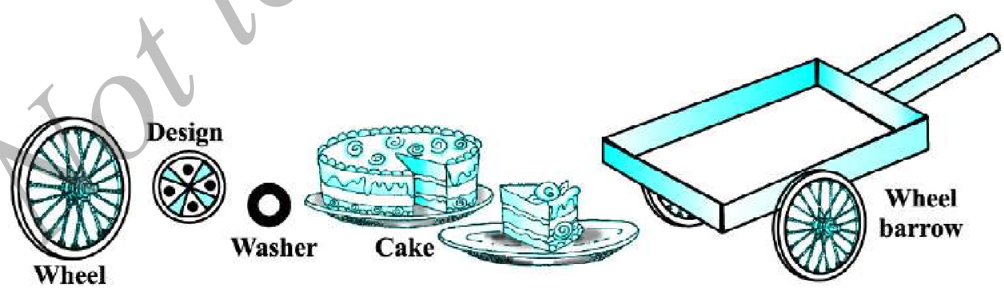


Fig. 5.1

5.2 Perimeter and Area of a Circle — A Review

Recall that the distance covered by travelling once around a circle is its *perimeter,* usually called its *circumference*. You also know from your earlier classes, that circumference of a circle bears a constant ratio with its diameter. This constant ratio is denoted by the Greek letter π (read as 'pi'). In other words,

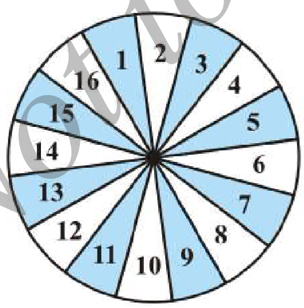
$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

or,

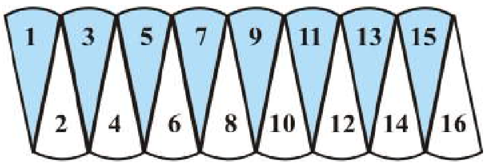
$$\begin{aligned}\text{circumference} &= \pi \times \text{diameter}\\ &= \pi \times 2r \quad (\text{where } r \text{ is the radius of the circle})\\ &= 2\pi r\end{aligned}$$

The great Indian mathematician Aryabhata (C.E. 476 – 550) gave an approximate value of π. He stated that $\pi = \frac{62832}{20000}$, which is nearly equal to 3.1416. It is also interesting to note that using an identity of the great mathematical genius Srinivas Ramanujan (1887–1920) of India, mathematicians have been able to calculate the value of π correct to million places of decimals. As you know from Chapter 1 of Class IX, π is an irrational number and its decimal expansion is non-terminating and non-recurring (non-repeating). However, for practical purposes, we generally take the value of π as $\frac{22}{7}$ or 3.14, approximately.

You may also recall that area of a circle is πr^2, where r is the radius of the circle. Recall that you have verified it in Class VII, by cutting a circle into a number of sectors and rearranging them as shown in Fig. 5.2.



(i)



(ii)

Fig 5.2

You can see that the shape in Fig. 5.2 (ii) is nearly a rectangle of length $\frac{1}{2} \times 2\pi r$ and breadth r. This suggests that the area of the circle $= \frac{1}{2} \times 2\pi r \times r = \pi r^2$. Let us recall the concepts learnt in earlier classes, through an example.

Example 1 : The cost of fencing a circular field at the rate of ₹ 24 per metre is ₹ 5280. The field is to be ploughed at the rate of ₹ 0.50 per m^2. Find the cost of ploughing the field (Take $\pi = \frac{22}{7}$).

Solution : Length of the fence (in metres) $= \frac{\text{Total cost}}{\text{Rate}} = \frac{5280}{24} = 220$

So, circumference of the field = 220 m

Therefore, if r metres is the radius of the field, then

$$2\pi r = 220$$

or, $$2 \times \frac{22}{7} \times r = 220$$

or, $$r = \frac{220 \times 7}{2 \times 22} = 35$$

i.e., radius of the field is 35 m.

Therefore, area of the field $= \pi r^2 = \frac{22}{7} \times 35 \times 35 \text{ m}^2 = 22 \times 5 \times 35 \text{ m}^2$

Now, cost of ploughing 1 m^2 of the field = ₹ 0.50

So, total cost of ploughing the field = ₹ $22 \times 5 \times 35 \times 0.50$ = ₹ 1925

EXERCISE 5.1

Unless stated otherwise, use $\pi = \frac{22}{7}$.

1. The radii of two circles are 19 cm and 9 cm respectively. Find the radius of the circle which has circumference equal to the sum of the circumferences of the two circles.
2. The radii of two circles are 8 cm and 6 cm respectively. Find the radius of the circle having area equal to the sum of the areas of the two circles.
3. Fig. 5.3 depicts an archery target marked with its five scoring regions from the centre outwards as Gold, Red, Blue, Black and White. The diameter of the region representing Gold score is 21 cm and each of the other bands is 10.5 cm wide. Find the area of each of the five scoring regions.

Fig. 5.3

4. The wheels of a car are of diameter 80 cm each. How many complete revolutions does each wheel make in 10 minutes when the car is travelling at a speed of 66 km per hour?

5. Tick the correct answer in the following and justify your choice : If the perimeter and the area of a circle are numerically equal, then the radius of the circle is

(A) 2 units (B) π units (C) 4 units (D) 7 units

5.3 Areas of Sector and Segment of a Circle

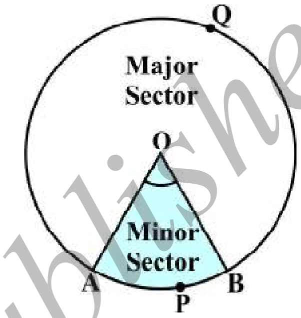


Fig. 5.4

You have already come across the terms *sector* and *segment* of a circle in your earlier classes. Recall that the portion (or part) of the circular region enclosed by two radii and the corresponding arc is called a *sector* of the circle and the portion (or part) of the circular region enclosed between a chord and the corresponding arc is called a *segment* of the circle. Thus, in Fig. 5.4, shaded region OAPB is a *sector* of the circle with centre O. $\angle$ AOB is called the *angle* of the sector. Note that in this figure, unshaded region OAQB is also a sector of the circle. For obvious reasons, OAPB is called the *minor sector* and OAQB is called the *major sector*. You can also see that angle of the major sector is $360° - \angle$ AOB.

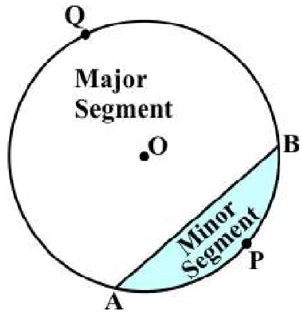


Fig. 5.5

Now, look at Fig. 5.5 in which AB is a chord of the circle with centre O. So, shaded region APB is a segment of the circle. You can also note that unshaded region AQB is another segment of the circle formed by the chord AB. For obvious reasons, APB is called the *minor segment* and AQB is called the *major segment*.

Remark : When we write 'segment' and 'sector' we will mean the 'minor segment' and the 'minor sector' respectively, unless stated otherwise.

Now with this knowledge, let us try to find some relations (or formulae) to calculate their areas.

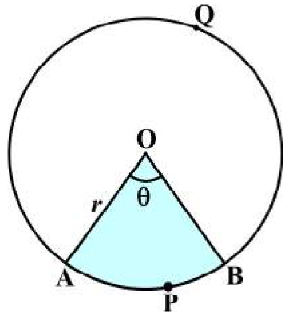


Fig. 5.6

Let OAPB be a sector of a circle with centre O and radius r (see Fig. 5.6). Let the degree measure of $\angle$ AOB be θ.

You know that area of a circle (in fact of a circular region or disc) is πr^2.

In a way, we can consider this circular region to be a sector forming an angle of 360° (i.e., of degree measure 360) at the centre O. Now by applying the Unitary Method, we can arrive at the area of the sector OAPB as follows:

When degree measure of the angle at the centre is 360, area of the sector = πr^2

So, when the degree measure of the angle at the centre is 1, area of the sector = $\frac{\pi r^2}{360}$.

Therefore, when the degree measure of the angle at the centre is θ, area of the sector = $\frac{\pi r^2}{360} \times \theta = \frac{\theta}{360} \times \pi r^2$.

Thus, we obtain the following relation (or formula) for area of a sector of a circle:

Area of the sector of angle $\theta = \boldsymbol{\frac{\theta}{360} \times \pi r^2}$,

where r is the radius of the circle and θ the angle of the sector in degrees.

Now, a natural question arises : Can we find the length of the arc APB corresponding to this sector? Yes. Again, by applying the Unitary Method and taking the whole length of the circle (of angle 360°) as $2\pi r$, we can obtain the required length of the arc APB as $\frac{\theta}{360} \times 2\pi r$.

So, **length of an arc of a sector of angle** $\theta = \boldsymbol{\frac{\theta}{360} \times 2\pi r}$.

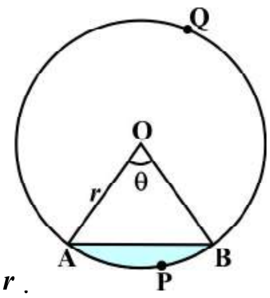


Fig. 5.7

Now let us take the case of the area of the segment APB of a circle with centre O and radius r (see Fig. 12.7). You can see that :

Area of the segment APB = Area of the sector OAPB – Area of Δ OAB

$$= \frac{\theta}{360} \times \pi r^2 - \text{area of } \Delta\text{OAB}$$

Note : From Fig. 5.6 and Fig. 12.7 respectively, you can observe that :

Area of the major sector OAQB = πr^2 – Area of the minor sector OAPB

and Area of major segment AQB = πr^2 – Area of the minor segment APB

Let us now take some examples to understand these concepts (or results).

Example 2 : Find the area of the sector of a circle with radius 4 cm and of angle 30°. Also, find the area of the corresponding major sector (Use $\pi = 3.14$).

Solution : Given sector is OAPB (see Fig. 5.8).

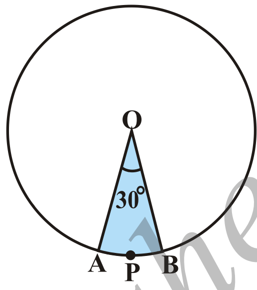


Fig. 5.8

$$\text{Area of the sector} = \frac{\theta}{360} \times \pi r^2$$

$$= \frac{30}{360} \times 3.14 \times 4 \times 4 \text{ cm}^2$$

$$= \frac{12.56}{3} \text{cm}^2 = 4.19 \text{cm}^2 \text{ (approx.)}$$

Area of the corresponding major sector

$$= \pi r^2 - \text{area of sector OAPB}$$

$$= (3.14 \times 16 - 4.19) \text{ cm}^2$$

$$= 46.05 \text{ cm}^2 = 46.1 \text{ cm}^2 \text{(approx.)}$$

Alternatively, area of the major sector $= \frac{(360 - \theta)}{360} \times \pi r^2$

$$= \left(\frac{360 - 30}{360}\right) \times 3.14 \times 16 \text{ cm}^2$$

$$= \frac{330}{360} \times 3.14 \times 16 \text{cm}^2 = 46.05 \text{ cm}^2$$

$$= 46.1 \text{ cm}^2 \text{ (approx.)}$$

Example 3 : Find the area of the segment AYB shown in Fig. 5.9, if radius of the circle is 21 cm and $\angle AOB = 120°$. (Use $\pi = \frac{22}{7}$)

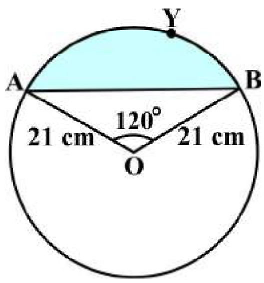


Fig. 5.9

Solution : Area of the segment AYB

$$= \text{Area of sector OAYB} - \text{Area of } \Delta \text{ OAB} \quad (1)$$

Now, area of the sector OAYB $= \frac{120}{360} \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2 = 462 \text{ cm}^2$ (2)

For finding the area of Δ OAB, draw OM ⊥ AB as shown in Fig. 5.10.

Note that OA = OB. Therefore, by RHS congruence, Δ AMO ≅ Δ BMO.

So, M is the mid-point of AB and ∠ AOM = ∠ BOM $= \frac{1}{2} \times 120° = 60°$.

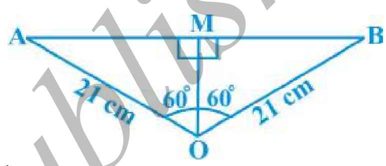


Fig. 5.10

Let $\quad \text{OM} = x \text{ cm}$

So, from Δ OMA, $\quad \frac{\text{OM}}{\text{OA}} = \cos 60°$

or, $\quad \frac{x}{21} = \frac{1}{2} \quad \left(\cos 60° = \frac{1}{2}\right)$

or, $\quad x = \frac{21}{2}$

So, $\quad \text{OM} = \frac{21}{2} \text{ cm}$

Also, $\quad \frac{\text{AM}}{\text{OA}} = \sin 60° = \frac{\sqrt{3}}{2}$

So, $\quad \text{AM} = \frac{21\sqrt{3}}{2} \text{ cm}$

Therefore, $\quad \text{AB} = 2 \text{ AM} = \frac{2 \times 21\sqrt{3}}{2} \text{ cm} = 21\sqrt{3} \text{ cm}$

So, area of Δ OAB $= \frac{1}{2} \text{AB} \times \text{OM} = \frac{1}{2} \times 21\sqrt{3} \times \frac{21}{2} \text{ cm}^2$

$$= \frac{441}{4}\sqrt{3} \text{ cm}^2 \quad (3)$$

Therefore, area of the segment AYB $= \left(462 - \frac{441}{4}\sqrt{3}\right) \text{cm}^2$ [From (1), (2) and (3)]

$$= \frac{21}{4}(88 - 21\sqrt{3}) \text{cm}^2$$

EXERCISE 5.2

Unless stated otherwise, use $\pi = \frac{22}{7}$.

1. Find the area of a sector of a circle with radius 6 cm if angle of the sector is 60°.
2. Find the area of a quadrant of a circle whose circumference is 22 cm.
3. The length of the minute hand of a clock is 14 cm. Find the area swept by the minute hand in 5 minutes.
4. A chord of a circle of radius 10 cm subtends a right angle at the centre. Find the area of the corresponding : (i) minor segment (ii) major sector. (Use $\pi = 3.14$)
5. In a circle of radius 21 cm, an arc subtends an angle of 60° at the centre. Find:
   (i) the length of the arc (ii) area of the sector formed by the arc
   (iii) area of the segment formed by the corresponding chord
6. A chord of a circle of radius 15 cm subtends an angle of 60° at the centre. Find the areas of the corresponding minor and major segments of the circle.
   (Use $\pi = 3.14$ and $\sqrt{3} = 1.73$)
7. A chord of a circle of radius 12 cm subtends an angle of 120° at the centre. Find the area of the corresponding segment of the circle.
   (Use $\pi = 3.14$ and $\sqrt{3} = 1.73$)
8. A horse is tied to a peg at one corner of a square shaped grass field of side 15 m by means of a 5 m long rope (see Fig. 5.11). Find
   (i) the area of that part of the field in which the horse can graze.
   (ii) the increase in the grazing area if the rope were 10 m long instead of 5 m. (Use $\pi = 3.14$)

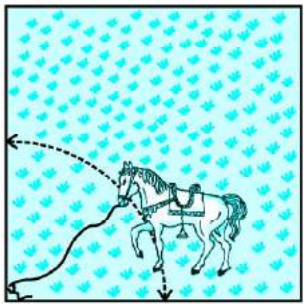


Fig. 5.11

9. A brooch is made with silver wire in the form of a circle with diameter 35 mm. The wire is also used in making 5 diameters which divide the circle into 10 equal sectors as shown in Fig. 5.12. Find :
   (i) the total length of the silver wire required.
   (ii) the area of each sector of the brooch.

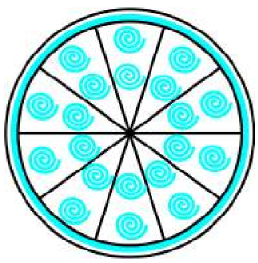


Fig. 5.12

10. An umbrella has 8 ribs which are equally spaced (see Fig. 5.13). Assuming umbrella to be a flat circle of radius 45 cm, find the area between the two consecutive ribs of the umbrella.

Fig. 5.13

11. A car has two wipers which do not overlap. Each wiper has a blade of length 25 cm sweeping through an angle of 115°. Find the total area cleaned at each sweep of the blades.

12. To warn ships for underwater rocks, a lighthouse spreads a red coloured light over a sector of angle 80° to a distance of 16.5 km. Find the area of the sea over which the ships are warned. (Use $\pi = 3.14$)

13. A round table cover has six equal designs as shown in Fig. 12.14. If the radius of the cover is 28 cm, find the cost of making the designs at the rate of ₹ 0.35 per cm². (Use $\sqrt{3} = 1.7$)

Fig. 5.14

14. Tick the correct answer in the following :

Area of a sector of angle p (in degrees) of a circle with radius R is

(A) $\frac{p}{180} \times 2\pi R$ (B) $\frac{p}{180} \times \pi R^2$ (C) $\frac{p}{360} \times 2\pi R$ (D) $\frac{p}{720} \times 2\pi R^2$

5.4 Areas of Combinations of Plane Figures

So far, we have calculated the areas of different figures separately. Let us now try to calculate the areas of some combinations of plane figures. We come across these types of figures in our daily life and also in the form of various interesting designs. Flower beds, drain covers, window designs, designs on table covers, are some of such examples. We illustrate the process of calculating areas of these figures through some examples.

Example 4 : In Fig. 5.15, two circular flower beds have been shown on two sides of a square lawn ABCD of side 56 m. If the centre of each circular flower bed is the point of intersection O of the diagonals of the square lawn, find the sum of the areas of the lawn and the flower beds.

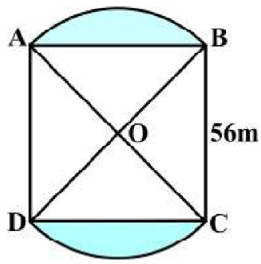


Fig. 5.15

Solution : Area of the square lawn ABCD = $56 \times 56\ m^2$ (1)

Let OA = OB = x metres

So, $x^2 + x^2 = 56^2$

or, $2x^2 = 56 \times 56$

or, $x^2 = 28 \times 56$ (2)

Now, area of sector OAB $= \frac{90}{360} \times \pi x^2 = \frac{1}{4} \times \pi x^2$

$= \frac{1}{4} \times \frac{22}{7} \times 28 \times 56\ m^2$ [From (2)] (3)

Also, area of Δ OAB $= \frac{1}{4} \times 56 \times 56\ m^2$ ($\angle$ AOB = 90°) (4)

So, area of flower bed AB $= \left(\frac{1}{4} \times \frac{22}{7} \times 28 \times 56 - \frac{1}{4} \times 56 \times 56\right) m^2$

[From (3) and (4)]

$= \frac{1}{4} \times 28 \times 56 \left(\frac{22}{7} - 2\right) m^2$

$= \frac{1}{4} \times 28 \times 56 \times \frac{8}{7}\ m^2$ (5)

Similarly, area of the other flower bed

$= \frac{1}{4} \times 28 \times 56 \times \frac{8}{7}\ m^2$ (6)

Therefore, total area $= \left(56 \times 56 + \frac{1}{4} \times 28 \times 56 \times \frac{8}{7}\right.$

$\left. + \frac{1}{4} \times 28 \times 56 \times \frac{8}{7}\right) m^2$ [From (1), (5) and (6)]

$= 28 \times 56 \left(2 + \frac{2}{7} + \frac{2}{7}\right) m^2$

$= 28 \times 56 \times \frac{18}{7}\ m^2 = 4032\ m^2$

Alternative Solution :

Total area = Area of sector OAB + Area of sector ODC + Area of Δ OAD + Area of Δ OBC

$$= \left(\frac{90}{360} \times \frac{22}{7} \times 28 \times 56 + \frac{90}{360} \times \frac{22}{7} \times 28 \times 56 + \frac{1}{4} \times 56 \times 56 + \frac{1}{4} \times 56 \times 56\right) \text{m}^2$$

$$= \frac{1}{4} \times 28 \times 56 \left(\frac{22}{7} + \frac{22}{7} + 2 + 2\right) \text{m}^2$$

$$= \frac{7 \times 56}{7} (22 + 22 + 14 + 14) \text{m}^2$$

$$= 56 \times 72 \text{ m}^2 = 4032 \text{ m}^2$$

Example 5 : Find the area of the shaded region in Fig. 5.16, where ABCD is a square of side 14 cm.

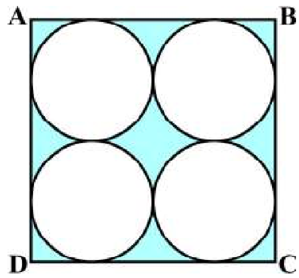


Fig. 5.16

Solution : Area of square ABCD

$$= 14 \times 14 \text{ cm}^2 = 196 \text{ cm}^2$$

Diameter of each circle $= \frac{14}{2}$ cm $= 7$ cm

So, radius of each circle $= \frac{7}{2}$ cm

So, area of one circle $= \pi r^2 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \text{ cm}^2$

$$= \frac{77}{2} \text{ cm}^2$$

Therefore, area of the four circles $= 4 \times \frac{77}{2} \text{cm}^2 = 154 \text{ cm}^2$

Hence, area of the shaded region $= (196 - 154) \text{ cm}^2 = 42 \text{ cm}^2$.

Example 6 : Find the area of the shaded design in Fig. 5.17, where ABCD is a square of side 10 cm and semicircles are drawn with each side of the square as diameter. (Use $\pi = 3.14$)

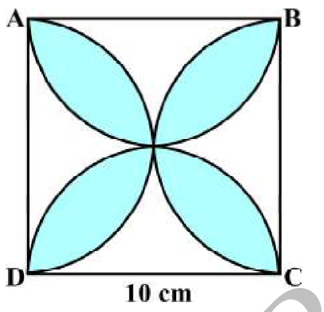


Fig. 5.17

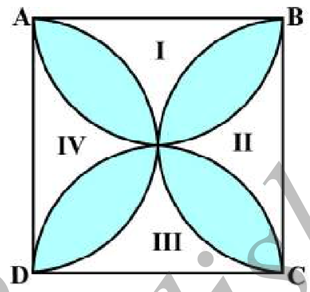


Fig. 5.18

Solution : Let us mark the four unshaded regions as I, II, III and IV (see Fig. 5.18).

Area of I + Area of III

= Area of ABCD – Areas of two semicircles of each of radius 5 cm

$$= \left(10 \times 10 - 2 \times \frac{1}{2} \times \pi \times 5^2\right) \text{cm}^2 = (100 - 3.14 \times 25) \text{ cm}^2$$

$= (100 - 78.5) \text{ cm}^2 = 21.5 \text{ cm}^2$

Similarly, Area of II + Area of IV = 21.5 cm^2

So, area of the shaded design = Area of ABCD – Area of (I + II + III + IV)

$= (100 - 2 \times 21.5) \text{ cm}^2 = (100 - 43) \text{ cm}^2 = 57 \text{ cm}^2$

EXERCISE 5.3

Unless stated otherwise, use $\pi = \frac{22}{7}$.

1. Find the area of the shaded region in Fig. 5.19, if PQ = 24 cm, PR = 7 cm and O is the centre of the circle.

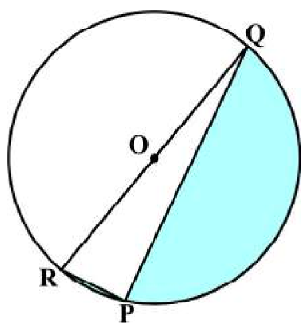


Fig. 5.19

2. Find the area of the shaded region in Fig. 5.20, if radii of the two concentric circles with centre O are 7 cm and 14 cm respectively and $\angle AOC = 40°$.

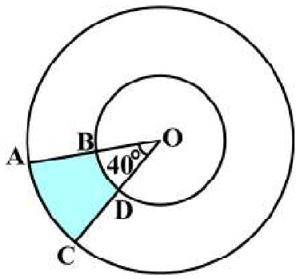


Fig. 5.20

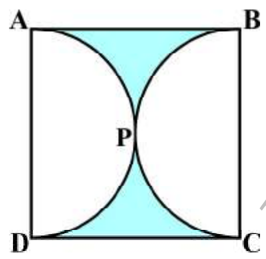


Fig. 5.21

3. Find the area of the shaded region in Fig. 5.21, if ABCD is a square of side 14 cm and APD and BPC are semicircles.

4. Find the area of the shaded region in Fig. 5.22, where a circular arc of radius 6 cm has been drawn with vertex O of an equilateral triangle OAB of side 12 cm as centre.

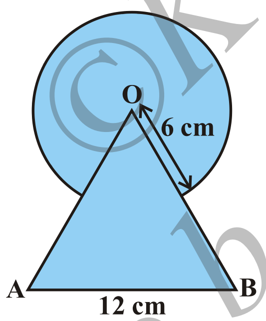


Fig. 5.22

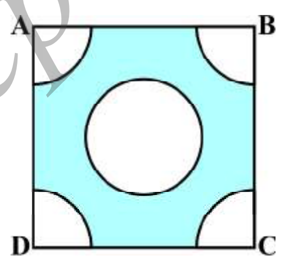


Fig. 5.23

5. From each corner of a square of side 4 cm a quadrant of a circle of radius 1 cm is cut and also a circle of diameter 2 cm is cut as shown in Fig. 5.23. Find the area of the remaining portion of the square.

6. In a circular table cover of radius 32 cm, a design is formed leaving an equilateral triangle ABC in the middle as shown in Fig. 5.24. Find the area of the design.

Fig. 5.24

7. In Fig. 5.25, ABCD is a square of side 14 cm. With centres A, B, C and D, four circles are drawn such that each circle touch externally two of the remaining three circles. Find the area of the shaded region.

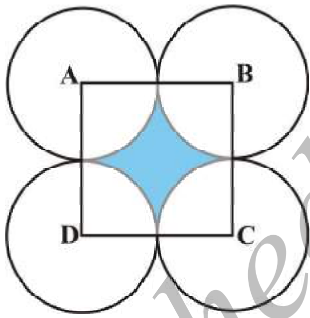


Fig. 5.25

8. Fig. 5.26 depicts a racing track whose left and right ends are semicircular.

Fig. 5.26

The distance between the two inner parallel line segments is 60 m and they are each 106 m long. If the track is 10 m wide, find :

(i) the distance around the track along its inner edge

(ii) the area of the track.

9. In Fig. 5.27, AB and CD are two diameters of a circle (with centre O) perpendicular to each other and OD is the diameter of the smaller circle. If OA = 7 cm, find the area of the shaded region.

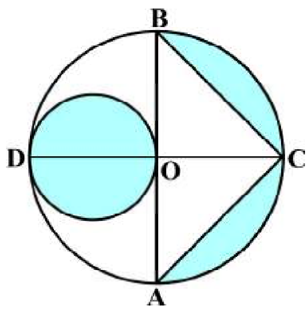


Fig. 5.27

10. The area of an equilateral triangle ABC is 17320.5 cm^2. With each vertex of the triangle as centre, a circle is drawn with radius equal to half the length of the side of the triangle (see Fig. 5.28). Find the area of the shaded region. (Use π = 3.14 and $\sqrt{3} = 1.73205$)

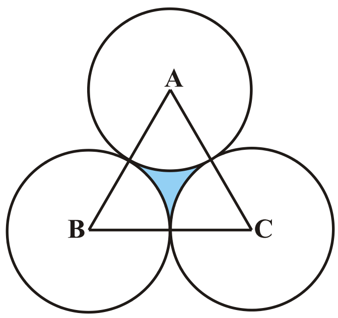


Fig. 5.28

11. On a square handkerchief, nine circular designs each of radius 7 cm are made (see Fig. 5.29). Find the area of the remaining portion of the handkerchief.

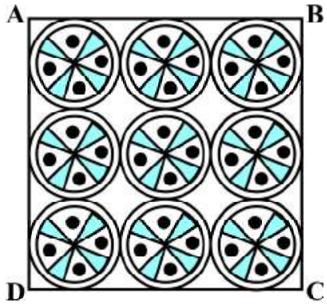


Fig. 5.29

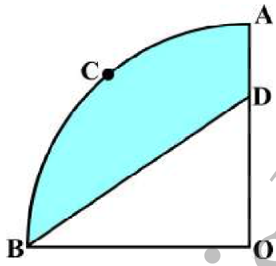


Fig. 5.30

12. In Fig. 5.30, OACB is a quadrant of a circle with centre O and radius 3.5 cm. If OD = 2 cm, find the area of the

(i) quadrant OACB, (ii) shaded region.

13. In Fig. 5.31, a square OABC is inscribed in a quadrant OPBQ. If OA = 20 cm, find the area of the shaded region. (Use $\pi = 3.14$)

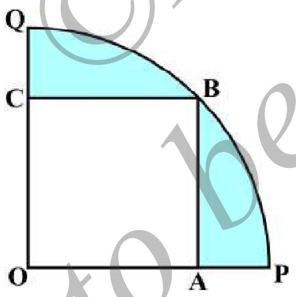


Fig. 5.31

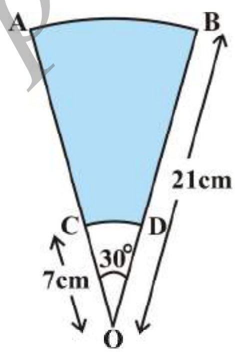


Fig. 5.32

14. AB and CD are respectively arcs of two concentric circles of radii 21 cm and 7 cm and centre O (see Fig. 5.32). If $\angle AOB = 30°$, find the area of the shaded region.

15. In Fig. 5.33, ABC is a quadrant of a circle of radius 14 cm and a semicircle is drawn with BC as diameter. Find the area of the shaded region.

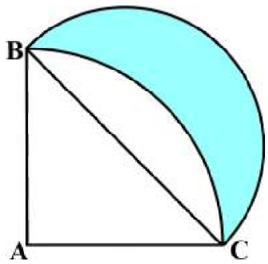


Fig. 5.33

16. Calculate the area of the designed region in Fig. 5.34 common between the two quadrants of circles of radius 8 cm each.

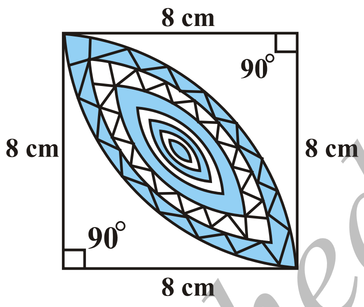


Fig. 5.34

5.5 Summary

In this chapter, you have studied the following points :

1. Circumference of a circle = $2\pi r$.

2. Area of a circle = πr^2.

3. Length of an arc of a sector of a circle with radius r and angle with degree measure θ is $\frac{\theta}{360} \times 2\pi r$.

4. Area of a sector of a circle with radius r and angle with degree measure θ is $\frac{\theta}{360} \times \pi r^2$.

5. Area of segment of a circle

= Area of the corresponding sector – Area of the corresponding triangle.

CONSTRUCTIONS 6

6.1 Introduction

In Class IX, you have done certain constructions using a straight edge (ruler) and a compass, e.g., bisecting an angle, drawing the perpendicular bisector of a line segment, some constructions of triangles etc. and also gave their justifications. In this chapter, we shall study some more constructions by using the knowledge of the earlier constructions. You would also be expected to give the mathematical reasoning behind why such constructions work.

6.2 Division of a Line Segment

Suppose a line segment is given and you have to divide it in a given ratio, say 3 : 2. You may do it by measuring the length and then marking a point on it that divides it in the given ratio. But suppose you do not have any way of measuring it precisely, how would you find the point? We give below two ways for finding such a point.

Construction 6.1 : *To divide a line segment in a given ratio.*

Given a line segment AB, we want to divide it in the ratio $m : n$, where both m and n are positive integers. To help you to understand it, we shall take $m = 3$ and $n = 2$.

Steps of Construction :

1. Draw any ray AX, making an acute angle with AB.
2. Locate 5 ($= m + n$) points A_1, A_2, A_3, A_4 and A_5 on AX so that $AA_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5$.
3. Join BA_5.
4. Through the point A_3 ($m = 3$), draw a line parallel to A_5B (by making an angle equal to $\angle AA_5B$) at A_3 intersecting AB at the point C (see Fig. 6.1). Then, AC : CB = 3 : 2.

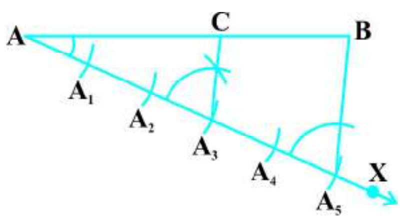


Fig. 6.1

Let us see how this method gives us the required division.

Since A_3C is parallel to A_5B, therefore,

$$\frac{AA_3}{A_3A_5} = \frac{AC}{CB} \qquad \text{(By the Basic Proportionality Theorem)}$$

By construction, $\frac{AA_3}{A_3A_5} = \frac{3}{2}\cdot$ Therefore, $\frac{AC}{CB} = \frac{3}{2}\cdot$

This shows that C divides AB in the ratio 3 : 2.

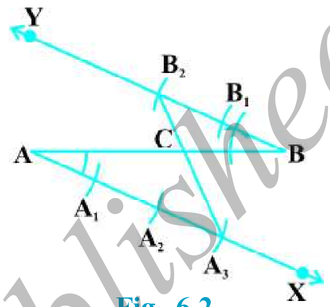


Fig. 6.2

Alternative Method

Steps of Construction :

1. Draw any ray AX making an acute angle with AB.
2. Draw a ray BY parallel to AX by making $\angle$ ABY equal to $\angle$ BAX.
3. Locate the points A_1, A_2, A_3 (m = 3) on AX and B_1, B_2 (n = 2) on BY such that $AA_1 = A_1A_2 = A_2A_3 = BB_1 = B_1B_2$.
4. Join A_3B_2. Let it intersect AB at a point C (see Fig. 6.2).

Then AC : CB = 3 : 2.

Why does this method work? Let us see.

Here $\Delta\ AA_3C$ is similar to $\Delta\ BB_2C$. (Why ?)

Then $$\frac{AA_3}{BB_2} = \frac{AC}{BC}.$$

Since by construction, $\frac{AA_3}{BB_2} = \frac{3}{2}$, therefore, $\frac{AC}{BC} = \frac{3}{2}\cdot$

In fact, the methods given above work for dividing the line segment in any ratio.

We now use the idea of the construction above for constructing a triangle similar to a given triangle whose sides are in a given ratio with the corresponding sides of the given triangle.

Construction 6.2 : *To construct a triangle similar to a given triangle as per given scale factor.*

This construction involves two different situations. In one, the triangle to be constructed is smaller and in the other it is larger than the given triangle. Here, the **scale factor** means the ratio of the sides of the triangle to be constructed with the corresponding sides of the given triangle (see also Chapter 2). Let us take the following examples for understanding the constructions involved. **The same methods would apply for the general case also.**

Example 1 : Construct a triangle similar to a given triangle ABC with its sides equal to $\frac{3}{4}$ of the corresponding sides of the triangle ABC (i.e., of scale factor $\frac{3}{4}$).

Solution : Given a triangle ABC, we are required to construct another triangle whose sides are $\frac{3}{4}$ of the corresponding sides of the triangle ABC.

Steps of Construction :

1. Draw any ray BX making an acute angle with BC on the side opposite to the vertex A.
2. Locate 4 (the greater of 3 and 4 in $\frac{3}{4}$) points B_1, B_2, B_3 and B_4 on BX so that $BB_1 = B_1B_2 = B_2B_3 = B_3B_4$.
3. Join B_4C and draw a line through B_3 (the 3rd point, 3 being smaller of 3 and 4 in $\frac{3}{4}$) parallel to B_4C to intersect BC at C′.
4. Draw a line through C′ parallel to the line CA to intersect BA at A′ (see Fig. 6.3).

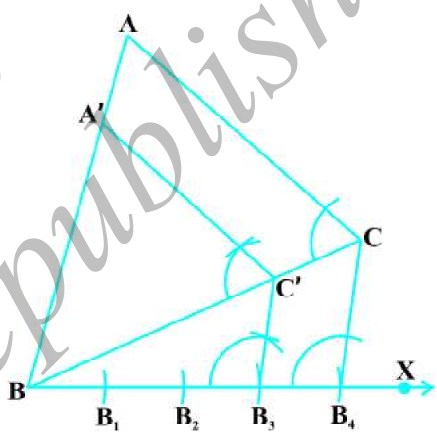


Fig. 6.3

Then, Δ A′BC′ is the required triangle.

Let us now see how this construction gives the required triangle.

By Construction 6.1, $\frac{BC'}{C'C} = \frac{3}{1}$.

Therefore, $\frac{BC}{BC'} = \frac{BC' + C'C}{BC'} = 1 + \frac{C'C}{BC'} = 1 + \frac{1}{3} = \frac{4}{3}$, i.e., $\frac{BC'}{BC} = \frac{3}{4}$.

Also C′A′ is parallel to CA. Therefore, Δ A′BC′ ~ Δ ABC. (Why ?)

So, $\frac{A'B}{AB} = \frac{A'C'}{AC} = \frac{BC'}{BC} = \frac{3}{4}$.

Example 2 : Construct a triangle similar to a given triangle ABC with its sides equal to $\frac{5}{3}$ of the corresponding sides of the triangle ABC (i.e., of scale factor $\frac{5}{3}$).

Solution : Given a triangle ABC, we are required to construct a triangle whose sides are $\frac{5}{3}$ of the corresponding sides of Δ ABC.

Steps of Construction :

1. Draw any ray BX making an acute angle with BC on the side opposite to the vertex A.
2. Locate 5 points (the greater of 5 and 3 in $\frac{5}{3}$) B_1, B_2, B_3, B_4 and B_5 on BX so that $BB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5$.
3. Join B_3(the 3rd point, 3 being smaller of 3 and 5 in $\frac{5}{3}$) to C and draw a line through B_5 parallel to B_3C, intersecting the extended line segment BC at C′.
4. Draw a line through C′ parallel to CA intersecting the extended line segment BA at A′ (see Fig. 6.4).

   Then A′BC′ is the required triangle.

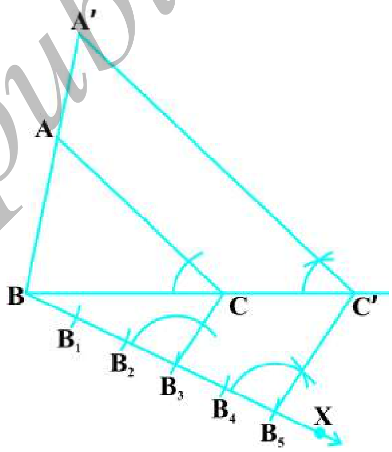


Fig. 6.4

For justification of the construction, note that Δ ABC ~ Δ A′BC′. (Why ?)

Therefore, $\frac{AB}{A'B} = \frac{AC}{A'C'} = \frac{BC}{BC'}$.

But, $\frac{BC}{BC'} = \frac{BB_3}{BB_5} = \frac{3}{5}$,

So, $\frac{BC'}{BC} = \frac{5}{3}$, and, therefore, $\frac{A'B}{AB} = \frac{A'C'}{AC} = \frac{BC'}{BC} = \frac{5}{3}$.

Remark : In Examples 1 and 2, you could take a ray making an acute angle with AB or AC and proceed similarly.

EXERCISE 6.1

In each of the following, give the justification of the construction also:

1. Draw a line segment of length 7.6 cm and divide it in the ratio 5 : 8. Measure the two parts.

2. Construct a triangle of sides 4 cm, 5 cm and 6 cm and then a triangle similar to it whose sides are $\frac{2}{3}$ of the corresponding sides of the first triangle.

3. Construct a triangle with sides 5 cm, 6 cm and 7 cm and then another triangle whose sides are $\frac{7}{5}$ of the corresponding sides of the first triangle.

4. Construct an isosceles triangle whose base is 8 cm and altitude 4 cm and then another triangle whose sides are $1\frac{1}{2}$ times the corresponding sides of the isosceles triangle.

5. Draw a triangle ABC with side BC = 6 cm, AB = 5 cm and $\angle$ ABC = 60°. Then construct a triangle whose sides are $\frac{3}{4}$ of the corresponding sides of the triangle ABC.

6. Draw a triangle ABC with side BC = 7 cm, $\angle$ B = 45°, $\angle$ A = 105°. Then, construct a triangle whose sides are $\frac{4}{3}$ times the corresponding sides of Δ ABC.

7. Draw a right triangle in which the sides (other than hypotenuse) are of lengths 4 cm and 3 cm. Then construct another triangle whose sides are $\frac{5}{3}$ times the corresponding sides of the given triangle.

6.3 Construction of Tangents to a Circle

You have already studied in chapter 4 that if a point lies inside a circle, there cannot be a tangent to the circle through this point. However, if a point lies on the circle, then there is only one tangent to the circle at this point and it is perpendicular to the radius through this point. Therefore, if you want to draw a tangent at a point of a circle, simply draw the radius through this point and draw a line perpendicular to this radius through this point and this will be the required tangent at the point.

You have also seen that if the point lies outside the circle, there will be two tangents to the circle from this point.

We shall now see how to draw these tangents.

Construction 6.3 : *To construct the tangents to a circle from a point outside it.*

We are given a circle with centre O and a point P outside it. We have to construct the two tangents from P to the circle.

Steps of Construction:

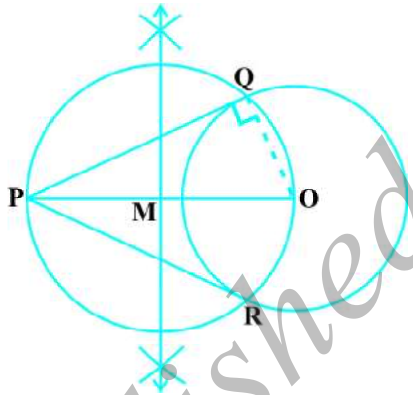


Fig. 6.5

1. Join PO and bisect it. Let M be the mid-point of PO.
2. Taking M as centre and MO as radius, draw a circle. Let it intersect the given circle at the points Q and R.
3. Join PQ and PR.

   Then PQ and PR are the required two tangents (see Fig. 6.5).

Now let us see how this construction works. Join OQ. Then $\angle$ PQO is an angle in the semicircle and, therefore,

$$\angle \text{PQO} = 90^\circ$$

Can we say that PQ $\perp$ OQ?

Since, OQ is a radius of the given circle, PQ has to be a tangent to the circle. Similarly, PR is also a tangent to the circle.

Note : If centre of the circle is not given, you may locate its centre first by taking any two non-parallel chords and then finding the point of intersection of their perpendicular bisectors. Then you could proceed as above.

EXERCISE 6.2

In each of the following, give also the justification of the construction:

1. Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

2. Construct a tangent to a circle of radius 4 cm from a point on the concentric circle of radius 6 cm and measure its length. Also verify the measurement by actual calculation.

3. Draw a circle of radius 3 cm. Take two points P and Q on one of its extended diameter each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points P and Q.

4. Draw a pair of tangents to a circle of radius 5 cm which are inclined to each other at an angle of 60°.

5. Draw a line segment AB of length 8 cm. Taking A as centre, draw a circle of radius 4 cm and taking B as centre, draw another circle of radius 3 cm. Construct tangents to each circle from the centre of the other circle.

6. Let ABC be a right triangle in which AB = 6 cm, BC = 8 cm and $\angle$ B = 90°. BD is the perpendicular from B on AC. The circle through B, C, D is drawn. Construct the tangents from A to this circle.

7. Draw a circle with the help of a bangle. Take a point outside the circle. Construct the pair of tangents from this point to the circle.

6.4 Summary

In this chapter, you have learnt how to do the following constructions:

1. To divide a line segment in a given ratio.
2. To construct a triangle similar to a given triangle as per a given scale factor which may be less than 1 or greater than 1.
3. To construct the pair of tangents from an external point to a circle.

A NOTE TO THE READER

Construction of a quadrilateral (or a polygon) similar to a given quadrilateral (or a polygon) with a given scale factor can also be done following the similar steps as used in Examples 1 and 2 of Construction 6.2.

7 COORDINATE GEOMETRY

7.1 Introduction

In Class IX, you have studied that to locate the position of a point on a plane, we require a pair of coordinate axes. The distance of a point from the y-axis is called its ***x*-coordinate**, or **abscissa**. The distance of a point from the x-axis is called its ***y*-coordinate**, or **ordinate**. The coordinates of a point on the x-axis are of the form $(x, 0)$, and of a point on the y-axis are of the form $(0, y)$.

Here is a play for you. Draw a set of a pair of perpendicular axes on a graph paper. Now plot the following points and join them as directed: Join the point A(4, 8) to B(3, 9) to C(3, 8) to D(1, 6) to E(1, 5) to F(3, 3) to G(6, 3) to H(8, 5) to I(8, 6) to J(6, 8) to K(6, 9) to L(5, 8) to A. Then join the points P(3.5, 7), Q (3, 6) and R(4, 6) to form a triangle. Also join the points X(5.5, 7), Y(5, 6) and Z(6, 6) to form a triangle. Now join S(4, 5), T(4.5, 4) and U(5, 5) to form a triangle. Lastly join S to the points (0, 5) and (0, 6) and join U to the points (9, 5) and (9, 6). What picture have you got?

Also, you have seen that a linear equation in two variables of the form $ax + by + c = 0$, (a, b are not simultaneously zero), when represented graphically, gives a straight line. Further, in Chapter 9, you will see that the graph of $y = ax^2 + bx + c$ ($a \neq 0$), is a parabola. In fact, coordinate geometry has been developed as an algebraic tool for studying geometry of figures. It helps us to study geometry using algebra, and understand algebra with the help of geometry. Because of this, coordinate geometry is widely applied in various fields such as physics, engineering, navigation, seismology and art!

In this chapter, you will learn how to find the distance between the two points whose coordinates are given, and to find the area of the triangle formed by three given points. You will also study how to find the coordinates of the point which divides a line segment joining two given points in a given ratio.

7.2 Distance Formula

Let us consider the following situation:

A town B is located 36 km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it. Let us see. This situation can be represented graphically as shown in Fig. 7.1. You may use the Pythagoras Theorem to calculate this distance.

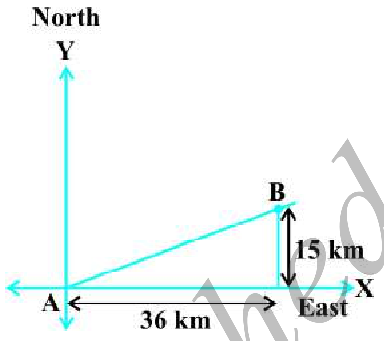


Fig. 7.1

Now, suppose two points lie on the x-axis. Can we find the distance between them? For instance, consider two points A(4, 0) and B(6, 0) in Fig. 7.2. The points A and B lie on the x-axis.

From the figure you can see that OA = 4 units and OB = 6 units.

Therefore, the distance of B from A, i.e., AB = OB – OA = 6 – 4 = 2 units.

So, if two points lie on the x-axis, we can easily find the distance between them.

Now, suppose we take two points lying on the y-axis. Can you find the distance between them. If the points C(0, 3) and D(0, 8) lie on the y-axis, similarly we find that CD = 8 – 3 = 5 units (see Fig. 7.2).

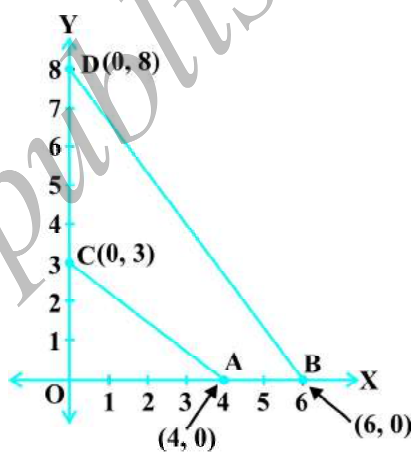


Fig. 7.2

Next, can you find the distance of A from C (in Fig. 7.2)? Since OA = 4 units and OC = 3 units, the distance of A from C, i.e., AC = $\sqrt{3^2 + 4^2}$ = 5 units. Similarly, you can find the distance of B from D = BD = 10 units.

Now, if we consider two points not lying on coordinate axis, can we find the distance between them? Yes! We shall use Pythagoras theorem to do so. Let us see an example.

In Fig. 7.3, the points P(4, 6) and Q(6, 8) lie in the first quadrant. How do we use Pythagoras theorem to find the distance between them? Let us draw PR and QS perpendicular to the x-axis from P and Q respectively. Also, draw a perpendicular from P on QS to meet QS at T. Then the coordinates of R and S are (4, 0) and (6, 0), respectively. So, RS = 2 units. Also, QS = 8 units and TS = PR = 6 units.

Therefore, QT = 2 units and PT = RS = 2 units.

Now, using the Pythagoras theorem, we have

$$PQ^2 = PT^2 + QT^2$$
$$= 2^2 + 2^2 = 8$$

So, $PQ = 2\sqrt{2}$ units

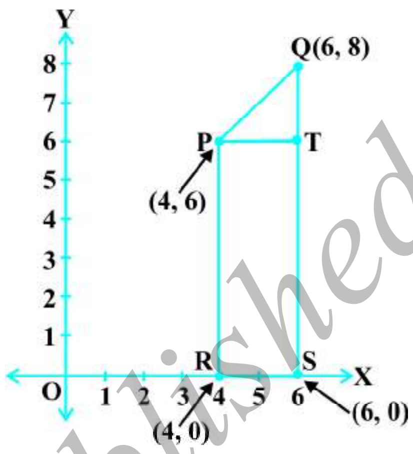


Fig. 7.3

How will we find the distance between two points in two different quadrants?

Consider the points P(6, 4) and Q(–5, –3) (see Fig. 7.4). Draw QS perpendicular to the *x*-axis. Also draw a perpendicular PT from the point P on QS (extended) to meet *y*-axis at the point R.

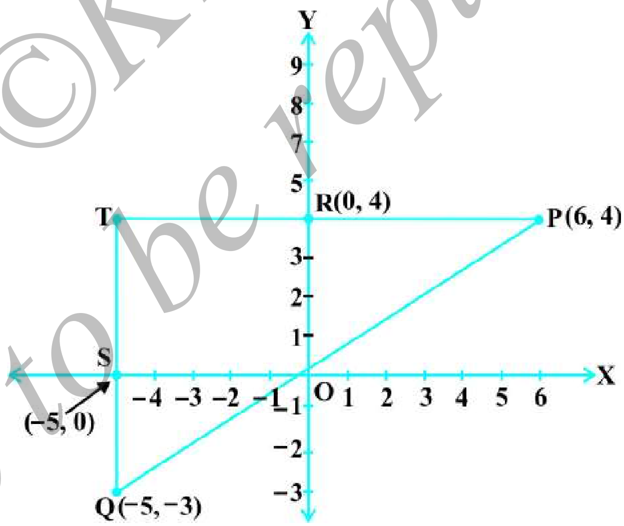


Fig. 7.4

Then PT = 11 units and QT = 7 units. (Why?)

Using the Pythagoras Theorem to the right triangle PTQ, we get $PQ = \sqrt{11^2 + 7^2} = \sqrt{170}$ units.

Let us now find the distance between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Draw PR and QS perpendicular to the x-axis. A perpendicular from the point P on QS is drawn to meet it at the point T (see Fig. 7.5).

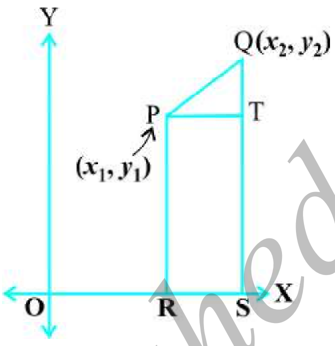


Fig. 7.5

Then, $OR = x_1$, $OS = x_2$. So, $RS = x_2 - x_1 = PT$.

Also, $SQ = y_2$, $ST = PR = y_1$. So, $QT = y_2 - y_1$.

Now, applying the Pythagoras theorem in Δ PTQ, we get

$$PQ^2 = PT^2 + QT^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Therefore, $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note that since distance is always non-negative, we take only the positive square root. So, the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\mathbf{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}},$$

which is called the **distance formula**.

Remarks :

1. In particular, the distance of a point $P(x, y)$ from the origin $O(0, 0)$ is given by

$$OP = \sqrt{x^2 + y^2}.$$

2. We can also write, $PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. (Why?)

Example 1 : Do the points (3, 2), (–2, –3) and (2, 3) form a triangle? If so, name the type of triangle formed.

Solution : Let us apply the distance formula to find the distances PQ, QR and PR, where P(3, 2), Q(–2, –3) and R(2, 3) are the given points. We have

$$PQ = \sqrt{(3+2)^2 + (2+3)^2} = \sqrt{5^2 + 5^2} = \sqrt{50} = 7.07 \text{ (approx.)}$$

$$QR = \sqrt{(-2-2)^2 + (-3-3)^2} = \sqrt{(-4)^2 + (-6)^2} = \sqrt{52} = 7.21 \text{ (approx.)}$$

$$PR = \sqrt{(3-2)^2 + (2-3)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2} = 1.41 \text{ (approx.)}$$

Since the sum of any two of these distances is greater than the third distance, therefore, the points P, Q and R form a triangle.

Also, $PQ^2 + PR^2 = QR^2$, by the converse of Pythagoras theorem, we have $\angle P = 90°$. Therefore, PQR is a right triangle.

Example 2 : Show that the points (1, 7), (4, 2), (–1, –1) and (– 4, 4) are the vertices of a square.

Solution : Let A(1, 7), B(4, 2), C(–1, –1) and D(– 4, 4) be the given points. One way of showing that ABCD is a square is to use the property that all its sides should be equal and both its digonals should also be equal. Now,

$$AB = \sqrt{(1-4)^2 + (7-2)^2} = \sqrt{9+25} = \sqrt{34}$$

$$BC = \sqrt{(4+1)^2 + (2+1)^2} = \sqrt{25+9} = \sqrt{34}$$

$$CD = \sqrt{(-1+4)^2 + (-1-4)^2} = \sqrt{9+25} = \sqrt{34}$$

$$DA = \sqrt{(1+4)^2 + (7-4)^2} = \sqrt{25+9} = \sqrt{34}$$

$$AC = \sqrt{(1+1)^2 + (7+1)^2} = \sqrt{4+64} = \sqrt{68}$$

$$BD = \sqrt{(4+4)^2 + (2-4)^2} = \sqrt{64+4} = \sqrt{68}$$

Since, AB = BC = CD = DA and AC = BD, all the four sides of the quadrilateral ABCD are equal and its diagonals AC and BD are also equal. Thereore, ABCD is a square.

Alternative Solution : We find the four sides and one diagonal, say, AC as above. Here $AD^2 + DC^2 = 34 + 34 = 68 = AC^2$. Therefore, by the converse of Pythagoras theorem, $\angle D = 90°$. A quadrilateral with all four sides equal and one angle 90° is a square. So, ABCD is a square.

Example 3 : Fig. 7.6 shows the arrangement of desks in a classroom. Ashima, Bharti and Camella are seated at A(3, 1), B(6, 4) and C(8, 6) respectively. Do you think they are seated in a line? Give reasons for your answer.

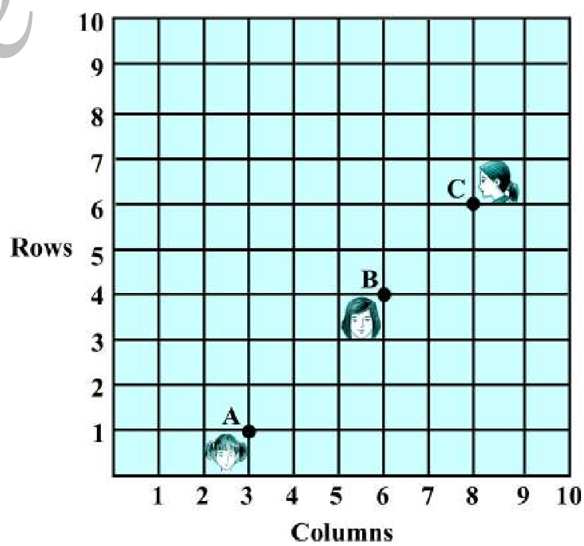


Fig. 7.6

Solution : Using the distance formula, we have

$$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(8-6)^2 + (6-4)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$AC = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

Since, $AB + BC = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = AC$, we can say that the points A, B and C are collinear. Therefore, they are seated in a line.

Example 4 : Find a relation between x and y such that the point (x, y) is equidistant from the points (7, 1) and (3, 5).

Solution : Let $P(x, y)$ be equidistant from the points A(7, 1) and B(3, 5).

We are given that AP = BP. So, $AP^2 = BP^2$

i.e., $$(x-7)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$$

i.e., $$x^2 - 14x + 49 + y^2 - 2y + 1 = x^2 - 6x + 9 + y^2 - 10y + 25$$

i.e., $$x - y = 2$$

which is the required relation.

Remark : Note that the graph of the equation $x - y = 2$ is a line. From your earlier studies, you know that a point which is equidistant from A and B lies on the perpendicular bisector of AB. Therefore, the graph of $x - y = 2$ is the perpendicular bisector of AB (see Fig. 7.7).

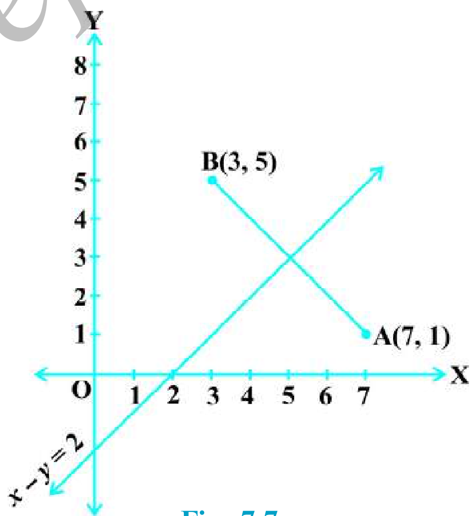


Fig. 7.7

Example 5 : Find a point on the y-axis which is equidistant from the points A(6, 5) and B(– 4, 3).

Solution : We know that a point on the y-axis is of the form $(0, y)$. So, let the point $P(0, y)$ be equidistant from A and B. Then

$$(6-0)^2 + (5-y)^2 = (-4-0)^2 + (3-y)^2$$

i.e., $$36 + 25 + y^2 - 10y = 16 + 9 + y^2 - 6y$$

i.e., $$4y = 36$$

i.e., $$y = 9$$

So, the required point is (0, 9).

Let us check our solution : $AP = \sqrt{(6-0)^2 + (5-9)^2} = \sqrt{36+16} = \sqrt{52}$

$BP = \sqrt{(-4-0)^2 + (3-9)^2} = \sqrt{16+36} = \sqrt{52}$

Note : Using the remark above, we see that (0, 9) is the intersection of the y-axis and the perpendicular bisector of AB.

EXERCISE 7.1

1. Find the distance between the following pairs of points :
   (i) (2, 3), (4, 1) (ii) (– 5, 7), (– 1, 3) (iii) $(a, b), (-a, -b)$
2. Find the distance between the points (0, 0) and (36, 15). Can you now find the distance between the two towns A and B discussed in Section 7.2.
3. Determine if the points (1, 5), (2, 3) and (– 2, – 11) are collinear.
4. Check whether (5, – 2), (6, 4) and (7, – 2) are the vertices of an isosceles triangle.
5. In a classroom, 4 friends are seated at the points A, B, C and D as shown in Fig. 7.8. Champa and Chameli walk into the class and after observing for a few minutes Champa asks Chameli, "Don't you think ABCD is a square?" Chameli disagrees. Using distance formula, find which of them is correct.

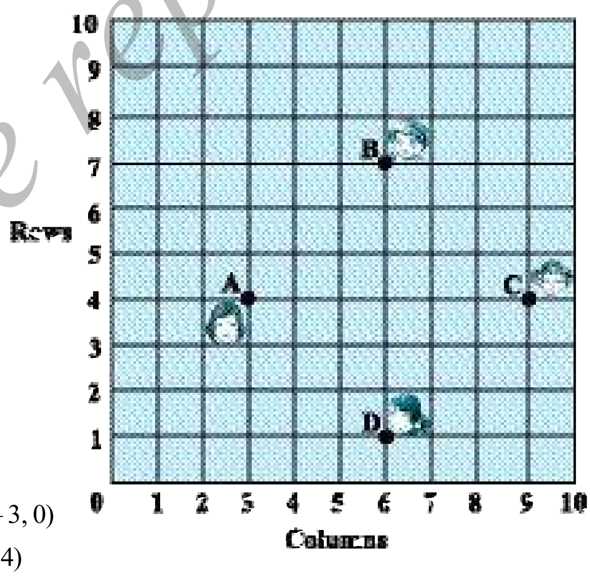


Fig. 7.8

6. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer:
   (i) (– 1, – 2), (1, 0), (– 1, 2), (– 3, 0)
   (ii) (–3, 5), (3, 1), (0, 3), (–1, – 4)
   (iii) (4, 5), (7, 6), (4, 3), (1, 2)
7. Find the point on the x-axis which is equidistant from (2, –5) and (–2, 9).
8. Find the values of y for which the distance between the points P(2, – 3) and Q(10, y) is 10 units.

9. If Q(0, 1) is equidistant from P(5, –3) and R(x, 6), find the values of x. Also find the distances QR and PR.

10. Find a relation between x and y such that the point (x, y) is equidistant from the point (3, 6) and (– 3, 4).

7.3 Section Formula

Let us recall the situation in Section 7.2. Suppose a telephone company wants to position a relay tower at P between A and B is such a way that the distance of the tower from B is twice its distance from A. If P lies on AB, it will divide AB in the ratio 1 : 2 (see Fig. 7.9). If we take A as the origin O, and 1 km as one unit on both the axis, the coordinates of B will be (36, 15). In order to know the position of the tower, we must know the coordinates of P. How do we find these coordinates?

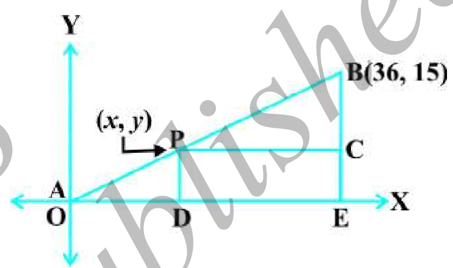


Fig. 7.9

Let the coordinates of P be (x, y). Draw perpendiculars from P and B to the x-axis, meeting it in D and E, respectively. Draw PC perpendicular to BE. Then, by the AA similarity criterion, studied in Chapter 6, Δ POD and Δ BPC are similar.

Therefore , $\frac{\text{OD}}{\text{PC}} = \frac{\text{OP}}{\text{PB}} = \frac{1}{2}$, and $\frac{\text{PD}}{\text{BC}} = \frac{\text{OP}}{\text{PB}} = \frac{1}{2}$

So, $\frac{x}{36 - x} = \frac{1}{2}$ and $\frac{y}{15 - y} = \frac{1}{2}$.

These equations give $x = 12$ and $y = 5$.

You can check that P(12, 5) meets the condition that OP : PB = 1 : 2.

Now let us use the understanding that you may have developed through this example to obtain the general formula.

Consider any two points A(x_1, y_1) and B(x_2, y_2) and assume that P (x, y) divides AB internally in the ratio $m_1 : m_2$, i.e., $\frac{\text{PA}}{\text{PB}} = \frac{m_1}{m_2}$ (see Fig. 7.10).

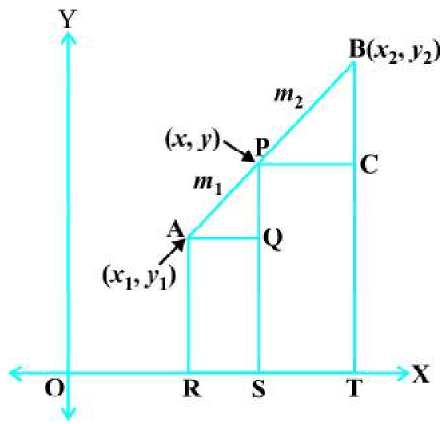


Fig. 7.10

Draw AR, PS and BT perpendicular to the x-axis. Draw AQ and PC parallel to the x-axis. Then, by the AA similarity criterion,

$$\Delta \text{ PAQ} \sim \Delta \text{ BPC}$$

Therefore, $$\frac{\text{PA}}{\text{BP}} = \frac{\text{AQ}}{\text{PC}} = \frac{\text{PQ}}{\text{BC}} \qquad (1)$$

Now,
$$\text{AQ} = \text{RS} = \text{OS} - \text{OR} = x - x_1$$
$$\text{PC} = \text{ST} = \text{OT} - \text{OS} = x_2 - x$$
$$\text{PQ} = \text{PS} - \text{QS} = \text{PS} - \text{AR} = y - y_1$$
$$\text{BC} = \text{BT} - \text{CT} = \text{BT} - \text{PS} = y_2 - y$$

Substituting these values in (1), we get

$$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

Taking $\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}$, we get $x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$

Similarly, taking $\frac{m_1}{m_2} = \frac{y - y_1}{y_2 - y}$, we get $y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$

So, the coordinates of the point P(x, y) which divides the line segment joining the points A(x_1, y_1) and B(x_2, y_2), internally, in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right) \qquad (2)$$

This is known as the **section formula**.

This can also be derived by drawing perpendiculars from A, P and B on the y-axis and proceeding as above.

If the ratio in which P divides AB is $k : 1$, then the coordinates of the point P will be

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}\right).$$

Special Case : The mid-point of a line segment divides the line segment in the ratio 1 : 1. Therefore, the coordinates of the mid-point P of the join of the points A(x_1, y_1) and B(x_2, y_2) is

$$\left(\frac{1 \cdot x_1 + 1 \cdot x_2}{1+1}, \frac{1 \cdot y_1 + 1 \cdot y_2}{1+1}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Let us solve a few examples based on the section formula.

Example 6 : Find the coordinates of the point which divides the line segment joining the points (4, – 3) and (8, 5) in the ratio 3 : 1 internally.

Solution : Let P(x, y) be the required point. Using the section formula, we get

$$x = \frac{3(8) + 1(4)}{3 + 1} = 7, \quad y = \frac{3(5) + 1(-3)}{3 + 1} = 3$$

Therefore, (7, 3) is the required point.

Example 7 : In what ratio does the point (– 4, 6) divide the line segment joining the points A(– 6, 10) and B(3, – 8)?

Solution : Let (– 4, 6) divide AB internally in the ratio $m_1 : m_2$. Using the section formula, we get

$$(-4, 6) = \left(\frac{3m_1 - 6m_2}{m_1 + m_2}, \frac{-8m_1 + 10m_2}{m_1 + m_2}\right) \qquad (1)$$

Recall that if (x, y) = (a, b) then $x = a$ and $y = b$.

So, $$-4 = \frac{3m_1 - 6m_2}{m_1 + m_2} \text{ and } 6 = \frac{-8m_1 + 10m_2}{m_1 + m_2}$$

Now, $$-4 = \frac{3m_1 - 6m_2}{m_1 + m_2}$$ gives us

$$-4m_1 - 4m_2 = 3m_1 - 6m_2$$

i.e., $$7m_1 = 2m_2$$

i.e., $$m_1 : m_2 = 2 : 7$$

You should verify that the ratio satisfies the y-coordinate also.

Now, $$\frac{-8m_1 + 10m_2}{m_1 + m_2} = \frac{-8\dfrac{m_1}{m_2} + 10}{\dfrac{m_1}{m_2} + 1} \quad \text{(Dividing throughout by } m_2\text{)}$$

$$= \frac{-8 \times \dfrac{2}{7} + 10}{\dfrac{2}{7} + 1} = 6$$

Therefore, the point (– 4, 6) divides the line segment joining the points A(– 6, 10) and B(3, – 8) in the ratio 2 : 7.

Alternatively : The ratio $m_1 : m_2$ can also be written as $\frac{m_1}{m_2} : 1$, or $k : 1$. Let (– 4, 6) divide AB internally in the ratio $k : 1$. Using the section formula, we get

$$(-4,\ 6) = \left(\frac{3k-6}{k+1},\ \frac{-8k+10}{k+1}\right) \qquad (2)$$

So,
$$-4 = \frac{3k-6}{k+1}$$

i.e.,
$$-4k - 4 = 3k - 6$$

i.e.,
$$7k = 2$$

i.e.,
$$k : 1 = 2 : 7$$

You can check for the y-coordinate also.

So, the point (– 4, 6) divides the line segment joining the points A(– 6, 10) and B(3, – 8) in the ratio 2 : 7.

Note : You can also find this ratio by calculating the distances PA and PB and taking their ratios provided you know that A, P and B are collinear.

Example 8 : Find the coordinates of the points of trisection (i.e., points dividing in three equal parts) of the line segment joining the points A(2, – 2) and B(– 7, 4).

Solution : Let P and Q be the points of trisection of AB i.e., AP = PQ = QB (see Fig. 7.11).

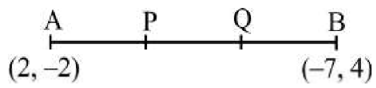


Fig. 7.11

Therefore, P divides AB internally in the ratio 1 : 2. Therefore, the coordinates of P, by applying the section formula, are

$$\left(\frac{1(-7)+2(2)}{1+2},\ \frac{1(4)+2(-2)}{1+2}\right),\ \text{i.e., } (-1,\ 0)$$

Now, Q also divides AB internally in the ratio 2 : 1. So, the coordinates of Q are

$$\left(\frac{2(-7)+1(2)}{2+1},\ \frac{2(4)+1(-2)}{2+1}\right),\ \text{i.e., } (-4,\ 2)$$

Therefore, the coordinates of the points of trisection of the line segment joining A and B are (–1, 0) and (– 4, 2).

Note : We could also have obtained Q by noting that it is the mid-point of PB. So, we could have obtained its coordinates using the mid-point formula.

Example 9 : Find the ratio in which the y-axis divides the line segment joining the points (5, – 6) and (–1, – 4). Also find the point of intersection.

Solution : Let the ratio be k : 1. Then by the section formula, the coordinates of the point which divides AB in the ratio k : 1 are $\left(\frac{-k+5}{k+1}, \frac{-4k-6}{k+1}\right)$.

This point lies on the y-axis, and we know that on the y-axis the abscissa is 0.

Therefore, $$\frac{-k+5}{k+1} = 0$$

So, $$k = 5$$

That is, the ratio is 5 : 1. Putting the value of k = 5, we get the point of intersection as $\left(0, \frac{-13}{3}\right)$.

Example 10 : If the points A(6, 1), B(8, 2), C(9, 4) and D(p, 3) are the vertices of a parallelogram, taken in order, find the value of p.

Solution : We know that diagonals of a parallelogram bisect each other.

So, the coordinates of the mid-point of AC = coordinates of the mid-point of BD

i.e., $$\left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{2+3}{2}\right)$$

i.e., $$\left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

so, $$\frac{15}{2} = \frac{8+p}{2}$$

i.e., $$p = 7$$

EXERCISE 7.2

1. Find the coordinates of the point which divides the join of (–1, 7) and (4, –3) in the ratio 2 : 3.
2. Find the coordinates of the points of trisection of the line segment joining (4, –1) and (–2, –3).
3. To conduct Sports Day activities, in your rectangular shaped school ground ABCD, lines have been drawn with chalk powder at a distance of 1m each. 100 flower pots have been placed at a distance of 1m from each other along AD, as shown in Fig. 7.12. Niharika runs $\frac{1}{4}$ th the distance AD on the 2nd line and posts a green flag. Preet runs $\frac{1}{5}$ th the distance AD on the eighth line and posts a red flag. What is the distance between both the flags? If Rashmi has to post a blue flag exactly halfway between the line segment joining the two flags, where should she post her flag?

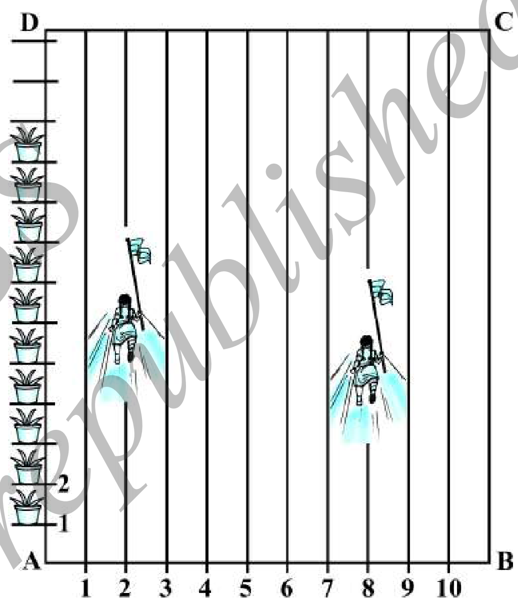


Fig. 7.12

4. Find the ratio in which the line segment joining the points (– 3, 10) and (6, – 8) is divided by (– 1, 6).
5. Find the ratio in which the line segment joining A(1, – 5) and B(– 4, 5) is divided by the x-axis. Also find the coordinates of the point of division.
6. If (1, 2), (4, y), (x, 6) and (3, 5) are the vertices of a parallelogram taken in order, find x and y.
7. Find the coordinates of a point A, where AB is the diameter of a circle whose centre is (2, – 3) and B is (1, 4).
8. If A and B are (– 2, – 2) and (2, – 4), respectively, find the coordinates of P such that $AP = \frac{3}{7} AB$ and P lies on the line segment AB.
9. Find the coordinates of the points which divide the line segment joining A(– 2, 2) and B(2, 8) into four equal parts.
10. Find the area of a rhombus if its vertices are (3, 0), (4, 5), (– 1, 4) and (– 2, – 1) taken in order. [**Hint :** Area of a rhombus = $\frac{1}{2}$ (product of its diagonals)]

7.4 Area of a Triangle

In your earlier classes, you have studied how to calculate the area of a triangle when its base and corresponding height (altitude) are given. You have used the formula :

$$\textbf{Area of a triangle} = \frac{1}{2} \times \textbf{base} \times \textbf{altitude}$$

In Class IX, you have also studied Heron's formula to find the area of a triangle. Now, if the coordinates of the vertices of a triangle are given, can you find its area? Well, you could find the lengths of the three sides using the distance formula and then use Heron's formula. But this could be tedious, particularly if the lengths of the sides are irrational numbers. Let us see if there is an easier way out.

Let ABC be any triangle whose vertices are A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3). Draw AP, BQ and CR perpendiculars from A, B and C, respectively, to the x-axis. Clearly ABQP, APRC and BQRC are all trapezia (see Fig. 7.13).

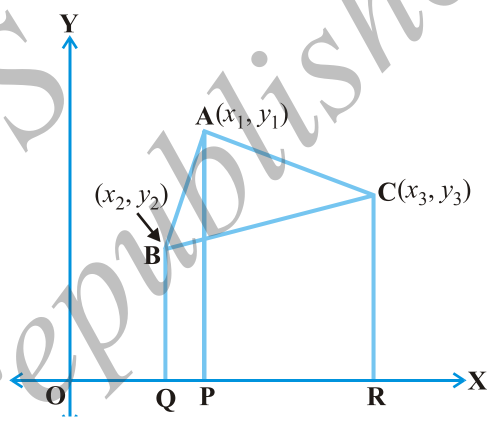


Fig. 7.13

Now, from Fig. 7.13, it is clear that

area of Δ ABC = area of trapezium ABQP + area of trapezium APRC – area of trapezium BQRC.

You also know that the

$$\text{area of a trapezium} = \frac{1}{2}\,\text{(sum of parallel sides)(distance between them)}$$

Therefore,

$$\text{Area of } \Delta\,\text{ABC} = \frac{1}{2}(\text{BQ} + \text{AP})\,\text{QP} + \frac{1}{2}(\text{AP} + \text{CR})\,\text{PR} - \frac{1}{2}(\text{BQ} + \text{CR})\,\text{QR}$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)\right]$$

Thus, the area of Δ ABC is the numerical value of the expression

$$\frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Let us consider a few examples in which we make use of this formula.

Example 11 : Find the area of a triangle whose vertices are (1, –1), (– 4, 6) and (–3, –5).

Solution : The area of the triangle formed by the vertices A(1, –1), B(– 4, 6) and C (–3, –5), by using the formula above, is given by

$$\frac{1}{2}\left[1\,(6+5)+(-4)\,(-5+1)+(-3)\,(-1-6)\right]$$

$$=\frac{1}{2}(11+16+21)=24$$

So, the area of the triangle is 24 square units.

Example 12 : Find the area of a triangle formed by the points A(5, 2), B(4, 7) and C (7, – 4).

Solution : The area of the triangle formed by the vertices A(5, 2), B(4, 7) and C (7, – 4) is given by

$$\frac{1}{2}\left[5\,(7+4)+4\,(-4-2)+7\,(2-7)\right]$$

$$=\frac{1}{2}(55-24-35)=\frac{-4}{2}=-2$$

Since area is a measure, which cannot be negative, we will take the numerical value of – 2, i.e., 2. Therefore, the area of the triangle = 2 square units.

Example 13 : Find the area of the triangle formed by the points P(–1.5, 3), Q(6, –2) and R(–3, 4).

Solution : The area of the triangle formed by the given points is equal to

$$\frac{1}{2}\left[-1.5(-2-4)+6(4-3)+(-3)(3+2)\right]$$

$$=\frac{1}{2}(9+6-15)=0$$

Can we have a triangle of area 0 square units? What does this mean?

If the area of a triangle is 0 square units, then its vertices will be collinear.

Example 14 : Find the value of k if the points A(2, 3), B(4, k) and C(6, –3) are collinear.

Solution : Since the given points are collinear, the area of the triangle formed by them must be 0, i.e.,

$$\frac{1}{2}[2(k+3)+4(-3-3)+6(3-k)]=0$$

i.e., $\frac{1}{2}(-4k)=0$

Therefore, $k=0$

Let us verify our answer.

$$\text{area of } \Delta\,ABC = \frac{1}{2}[2(0+3)+4(-3-3)+6(3-0)]=0$$

Example 15 : If A(–5, 7), B(– 4, –5), C(–1, –6) and D(4, 5) are the vertices of a quadrilateral, find the area of the quadrilateral ABCD.

Solution : By joining B to D, you will get two triangles ABD and BCD.

Now the area of Δ ABD $= \frac{1}{2}[-5(-5-5)+(-4)(5-7)+4(7+5)]$

$$= \frac{1}{2}(50+8+48) = \frac{106}{2} = 53 \text{ square units}$$

Also, the area of Δ BCD $= \frac{1}{2}[-4(-6-5)-1(5+5)+4(-5+6)]$

$$= \frac{1}{2}(44-10+4) = 19 \text{ square units}$$

So, the area of quadrilateral ABCD = 53 + 19 = 72 square units.

Note : To find the area of a polygon, we divide it into triangular regions, which have no common area, and add the areas of these regions.

EXERCISE 7.3

1. Find the area of the triangle whose vertices are :
   (i) (2, 3), (–1, 0), (2, – 4)   (ii) (–5, –1), (3, –5), (5, 2)
2. In each of the following find the value of 'k', for which the points are collinear.
   (i) (7, –2), (5, 1), (3, k)   (ii) (8, 1), (k, – 4), (2, –5)
3. Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are (0, –1), (2, 1) and (0, 3). Find the ratio of this area to the area of the given triangle.
4. Find the area of the quadrilateral whose vertices, taken in order, are (– 4, – 2), (– 3, – 5), (3, – 2) and (2, 3).
5. You have studied in Class IX, (Chapter 9, Example 3), that a median of a triangle divides it into two triangles of equal areas. Verify this result for Δ ABC whose vertices are A(4, – 6), B(3, –2) and C(5, 2).

EXERCISE 7.4 (Optional)*

1. Determine the ratio in which the line $2x + y - 4 = 0$ divides the line segment joining the points A(2, – 2) and B(3, 7).
2. Find a relation between x and y if the points (x, y), (1, 2) and (7, 0) are collinear.
3. Find the centre of a circle passing through the points (6, – 6), (3, – 7) and (3, 3).
4. The two opposite vertices of a square are (–1, 2) and (3, 2). Find the coordinates of the other two vertices.
5. The Class X students of a secondary school in Krishinagar have been allotted a rectangular plot of land for their gardening activity. Sapling of Gulmohar are planted on the boundary at a distance of 1m from each other. There is a triangular grassy lawn in the plot as shown in the Fig. 7.14. The students are to sow seeds of flowering plants on the remaining area of the plot.

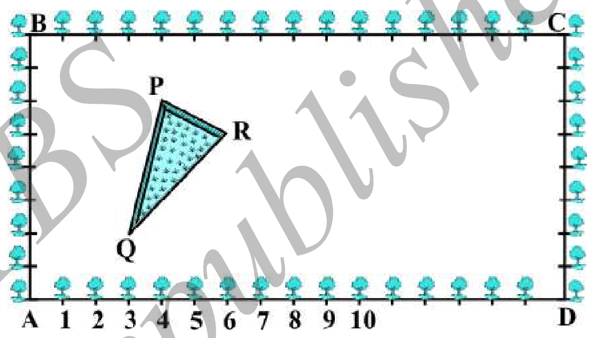


Fig. 7.14

   (i) Taking A as origin, find the coordinates of the vertices of the triangle.
   (ii) What will be the coordinates of the vertices of Δ PQR if C is the origin?
   Also calculate the areas of the triangles in these cases. What do you observe?
6. The vertices of a Δ ABC are A(4, 6), B(1, 5) and C(7, 2). A line is drawn to intersect sides AB and AC at D and E respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}\cdot$ Calculate the area of the Δ ADE and compare it with the area of Δ ABC. (Recall Theorem 6.2 and Theorem 6.6).
7. Let A (4, 2), B(6, 5) and C(1, 4) be the vertices of Δ ABC.
   (i) The median from A meets BC at D. Find the coordinates of the point D.
   (ii) Find the coordinates of the point P on AD such that AP : PD = 2 : 1
   (iii) Find the coordinates of points Q and R on medians BE and CF respectively such that BQ : QE = 2 : 1 and CR : RF = 2 : 1.
   (iv) What do yo observe?
   [**Note :** The point which is common to all the three medians is called the *centroid* and this point divides each median in the ratio 2 : 1.]

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* These exercises are not from the examination point of view.

(v) If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of Δ ABC, find the coordinates of the centroid of the triangle.

8. ABCD is a rectangle formed by the points A(–1, –1), B(– 1, 4), C(5, 4) and D(5, – 1). P, Q, R and S are the mid-points of AB, BC, CD and DA respectively. Is the quadrilateral PQRS a square? a rectangle? or a rhombus? Justify your answer.

7.5 Summary

In this chapter, you have studied the following points :

1. The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. The distance of a point $P(x, y)$ from the origin is $\sqrt{x^2 + y^2}$.

3. The coordinates of the point $P(x, y)$ which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m_1 : m_2$ are $\left(\dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$.

4. The mid-point of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

5. The area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is the numerical value of the expression

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

A NOTE TO THE READER

Section 7.3 discusses the Section Formula for the coordinates (x, y) of a point P which divides internally the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m_1 : m_2$ as follows :

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

Note that, here, PA : PB = $m_1 : m_2$.

However, if P does not lie between A and B but lies on the line AB, outside the line segment AB, and PA : PB = $m_1 : m_2$, we say that P divides externally the line segment joining the points A and B. You will study Section Formula for such case in higher classes.

REAL NUMBERS

8

8.1 Introduction

In Class IX, you began your exploration of the world of real numbers and encountered irrational numbers. We continue our discussion on real numbers in this chapter. We begin with two very important properties of positive integers in Sections 8.2 and 8.3, namely the Euclid's division algorithm and the Fundamental Theorem of Arithmetic.

Euclid's division algorithm, as the name suggests, has to do with divisibility of integers. Stated simply, it says any positive integer a can be divided by another positive integer b in such a way that it leaves a remainder r that is smaller than b. Many of you probably recognise this as the usual long division process. Although this result is quite easy to state and understand, it has many applications related to the divisibility properties of integers. We touch upon a few of them, and use it mainly to compute the HCF of two positive integers.

The Fundamental Theorem of Arithmetic, on the other hand, has to do something with multiplication of positive integers. You already know that every composite number can be expressed as a product of primes in a unique way — this important fact is the Fundamental Theorem of Arithmetic. Again, while it is a result that is easy to state and understand, it has some very deep and significant applications in the field of mathematics. We use the Fundamental Theorem of Arithmetic for two main applications. First, we use it to prove the irrationality of many of the numbers you studied in Class IX, such as $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$. Second, we apply this theorem to explore when exactly the decimal expansion of a rational number, say $\frac{p}{q} (q \neq 0)$, is terminating and when it is non-terminating repeating. We do so by looking at the prime factorisation of the denominator q of $\frac{p}{q}$. You will see that the prime factorisation of q will completely reveal the nature of the decimal expansion of $\frac{p}{q}$.

So let us begin our exploration.

8.2 Euclid's Division Lemma

Consider the following folk puzzle*.

A trader was moving along a road selling eggs. An idler who didn't have much work to do, started to get the trader into a wordy duel. This grew into a fight, he pulled the basket with eggs and dashed it on the floor. The eggs broke. The trader requested the Panchayat to ask the idler to pay for the broken eggs. The Panchayat asked the trader how many eggs were broken. He gave the following response:

If counted in pairs, one will remain;
If counted in threes, two will remain;
If counted in fours, three will remain;
If counted in fives, four will remain;
If counted in sixes, five will remain;
If counted in sevens, nothing will remain;
My basket cannot accomodate more than 150 eggs.

So, how many eggs were there? Let us try and solve the puzzle. Let the number of eggs be a. Then working backwards, we see that a is less than or equal to 150:

If counted in sevens, nothing will remain, which translates to $a = 7p + 0$, for some natural number p. If counted in sixes, $a = 6q + 5$, for some natural number q.

If counted in fives, four will remain. It translates to $a = 5w + 4$, for some natural number w.

If counted in fours, three will remain. It translates to $a = 4s + 3$, for some natural number s.

If counted in threes, two will remain. It translates to $a = 3t + 2$, for some natural number t.

If counted in pairs, one will remain. It translates to $a = 2u + 1$, for some natural number u.

That is, in each case, we have a and a positive integer b (in our example, b takes values 7, 6, 5, 4, 3 and 2, respectively) which divides a and leaves a remainder r (in our case, r is 0, 5, 4, 3, 2 and 1, respectively), that is smaller than b. The

* This is modified form of a puzzle given in 'Numeracy Counts!' by A. Rampal, and others.

moment we write down such equations we are using Euclid's division lemma, which is given in Theorem 8.1.

Getting back to our puzzle, do you have any idea how you will solve it? Yes! You must look for the multiples of 7 which satisfy all the conditions. By trial and error (using the concept of LCM), you will find he had 119 eggs.

In order to get a feel for what Euclid's division lemma is, consider the following pairs of integers:

17, 6; 5, 12; 20, 4

Like we did in the example, we can write the following relations for each such pair:

$17 = 6 \times 2 + 5$ (6 goes into 17 twice and leaves a remainder 5)

$5 = 12 \times 0 + 5$ (This relation holds since 12 is larger than 5)

$20 = 4 \times 5 + 0$ (Here 4 goes into 20 five-times and leaves no remainder)

That is, for each pair of positive integers a and b, we have found whole numbers q and r, satisfying the relation:

$$a = bq + r,\ 0 \leq r < b$$

Note that q or r can also be zero.

Why don't you now try finding integers q and r for the following pairs of positive integers a and b?

(i) 10, 3; (ii) 4, 19; (iii) 81, 3

Did you notice that q and r are unique? These are the only integers satisfying the conditions $a = bq + r$, where $0 \leq r < b$. You may have also realised that this is nothing but a restatement of the long division process you have been doing all these years, and that the integers q and r are called the *quotient* and *remainder,* respectively.

A formal statement of this result is as follows :

Theorem 8.1 (Euclid's Division Lemma) : *Given positive integers a and b, there exist unique integers q and r satisfying* $a = bq + r,\ 0 \leq r < b$.

This result was perhaps known for a long time, but was first recorded in Book VII of Euclid's Elements. Euclid's division algorithm is based on this lemma.

An **algorithm** is a series of well defined steps which gives a procedure for solving a type of problem.

The word *algorithm* comes from the name of the 9th century Persian mathematician al-Khwarizmi. In fact, even the word 'algebra' is derived from a book, he wrote, called *Hisab al-jabr w'al-muqabala.*

A **lemma** is a proven statement used for proving another statement.

Muhammad ibn Musa al-Khwarizmi
(C.E. 780 – 850)

Euclid's division algorithm is a technique to compute the Highest Common Factor (HCF) of two given positive integers. Recall that the HCF of two positive integers a and b is the largest positive integer d that divides both a and b.

Let us see how the algorithm works, through an example first. Suppose we need to find the HCF of the integers 455 and 42. We start with the larger integer, that is, 455. Then we use Euclid's lemma to get

$$455 = 42 \times 10 + 35$$

Now consider the divisor 42 and the remainder 35, and apply the division lemma to get

$$42 = 35 \times 1 + 7$$

Now consider the divisor 35 and the remainder 7, and apply the division lemma to get

$$35 = 7 \times 5 + 0$$

Notice that the remainder has become zero, and we cannot proceed any further. **We claim** that the HCF of 455 and 42 is the divisor at this stage, i.e., 7. You can easily verify this by listing all the factors of 455 and 42. Why does this method work? It works because of the following result.

So, let us state **Euclid's division algorithm** clearly.

To obtain the HCF of two positive integers, say c and d, with c > d, follow the steps below:

Step 1 : Apply Euclid's division lemma, to c and d. So, we find whole numbers, q and r such that $c = dq + r$, $0 \leq r < d$.

Step 2 : If $r = 0$, d is the HCF of c and d. If $r \neq 0$, apply the division lemma to d and r.

Step 3 : Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.

This algorithm works because HCF (c, d) = HCF (d, r) where the symbol HCF (c, d) denotes the HCF of c and d, etc.

Example 1 : Use Euclid's algorithm to find the HCF of 4052 and 12576.

Solution :

Step 1 : Since 12576 > 4052, we apply the division lemma to 12576 and 4052, to get

$$12576 = 4052 \times 3 + 420$$

Step 2 : Since the remainder $420 \neq 0$, we apply the division lemma to 4052 and 420, to get

$$4052 = 420 \times 9 + 272$$

Step 3 : We consider the new divisor 420 and the new remainder 272, and apply the division lemma to get

$$420 = 272 \times 1 + 148$$

We consider the new divisor 272 and the new remainder 148, and apply the division lemma to get

$$272 = 148 \times 1 + 124$$

We consider the new divisor 148 and the new remainder 124, and apply the division lemma to get

$$148 = 124 \times 1 + 24$$

We consider the new divisor 124 and the new remainder 24, and apply the division lemma to get

$$124 = 24 \times 5 + 4$$

We consider the new divisor 24 and the new remainder 4, and apply the division lemma to get

$$24 = 4 \times 6 + 0$$

The remainder has now become zero, so our procedure stops. Since the divisor at this stage is 4, the HCF of 12576 and 4052 is 4.

Notice that 4 = HCF (24, 4) = HCF (124, 24) = HCF (148, 124) = HCF (272, 148) = HCF (420, 272) = HCF (4052, 420) = HCF (12576, 4052).

Euclid's division algorithm is not only useful for calculating the HCF of very large numbers, but also because it is one of the earliest examples of an algorithm that a computer had been programmed to carry out.

Remarks :

1. Euclid's division lemma and algorithm are so closely interlinked that people often call former as the division algorithm also.

2. Although Euclid's Division Algorithm is stated for only positive integers, it can be extended for all integers except zero, i.e., $b \neq 0$. However, we shall not discuss this aspect here.

Euclid's division lemma/algorithm has several applications related to finding properties of numbers. We give some examples of these applications below:

Example 2 : Show that every positive even integer is of the form $2q$, and that every positive odd integer is of the form $2q + 1$, where q is some integer.

Solution : Let a be any positive integer and $b = 2$. Then, by Euclid's algorithm, $a = 2q + r$, for some integer $q \geq 0$, and $r = 0$ or $r = 1$, because $0 \leq r < 2$. So, $a = 2q$ or $2q + 1$.

If a is of the form $2q$, then a is an even integer. Also, a positive integer can be either even or odd. Therefore, any positive odd integer is of the form $2q + 1$.

Example 3 : Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where q is some integer.

Solution : Let us start with taking a, where a is a positive odd integer. We apply the division algorithm with a and $b = 4$.

Since $0 \leq r < 4$, the possible remainders are 0, 1, 2 and 3.

That is, a can be $4q$, or $4q + 1$, or $4q + 2$, or $4q + 3$, where q is the quotient. However, since a is odd, a cannot be $4q$ or $4q + 2$ (since they are both divisible by 2).

Therefore, any odd integer is of the form $4q + 1$ or $4q + 3$.

Example 4 : A sweetseller has 420 *kaju barfis* and 130 *badam barfis*. She wants to stack them in such a way that each stack has the same number, and they take up the least area of the tray. What is the number of that can be placed in each stack for this purpose?

Solution : This can be done by trial and error. But to do it systematically, we find HCF (420, 130). Then this number will give the maximum number of *barfis* in each stack and the number of stacks will then be the least. The area of the tray that is used up will be the least.

Now, let us use Euclid's algorithm to find their HCF. We have :

$$420 = 130 \times 3 + 30$$

$$130 = 30 \times 4 + 10$$

$$30 = 10 \times 3 + 0$$

So, the HCF of 420 and 130 is 10.

Therefore, the sweetseller can make stacks of 10 for both kinds of *barfi*.

EXERCISE 8.1

1. Use Euclid's division algorithm to find the HCF of :
   (i) 135 and 225 (ii) 196 and 38220 (iii) 867 and 255
2. Show that any positive odd integer is of the form $6q + 1$, or $6q + 3$, or $6q + 5$, where q is some integer.
3. An army contingent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march?
4. Use Euclid's division lemma to show that the square of any positive integer is either of the form $3m$ or $3m + 1$ for some integer m.

   [**Hint :** Let x be any positive integer then it is of the form $3q$, $3q + 1$ or $3q + 2$. Now square each of these and show that they can be rewritten in the form $3m$ or $3m + 1$.]
5. Use Euclid's division lemma to show that the cube of any positive integer is of the form $9m$, $9m + 1$ or $9m + 8$.

8.3 The Fundamental Theorem of Arithmetic

In your earlier classes, you have seen that any natural number can be written as a product of its prime factors. For instance, $2 = 2$, $4 = 2 \times 2$, $253 = 11 \times 23$, and so on. Now, let us try and look at natural numbers from the other direction. That is, can any natural number be obtained by multiplying prime numbers? Let us see.

Take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce a large collection of positive integers (In fact, infinitely many). Let us list a few :

$7 \times 11 \times 23 = 1771$ $\qquad$ $3 \times 7 \times 11 \times 23 = 5313$

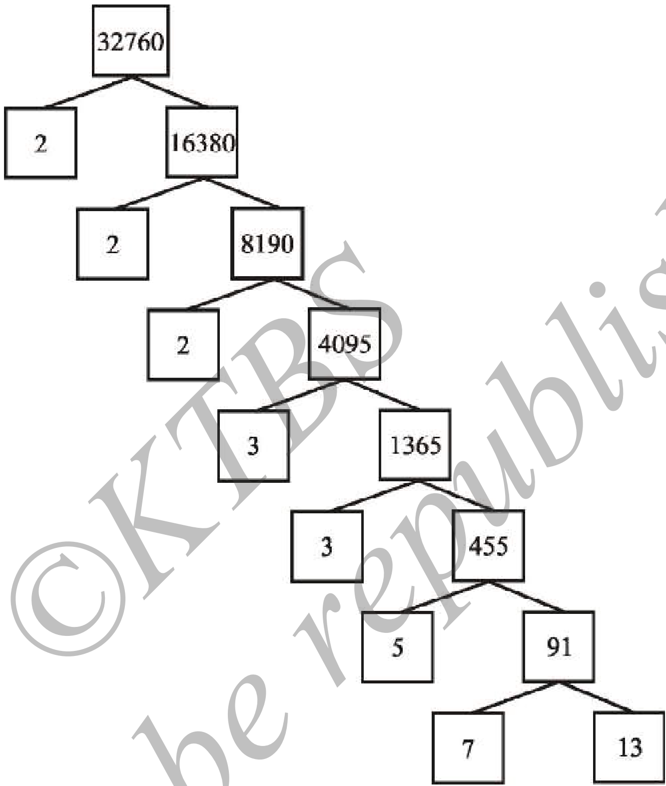
$2 \times 3 \times 7 \times 11 \times 23 = 10626$ $\qquad$ $2^3 \times 3 \times 7^3 = 8232$

$2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

and so on.

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of integers, or infinitely many? Infact, there are infinitely many primes. So, if we combine all these primes in all possible ways, we will get an infinite collection of numbers, all the primes and all possible products of primes. The question is – can we produce all the composite numbers this way? What do you think? Do you think that there may be a composite number which is not the product of powers of primes? Before we answer this, let us factorise positive integers, that is, do the opposite of what we have done so far.

We are going to use the factor tree with which you are all familiar. Let us take some large number, say, 32760, and factorise it as shown :



So we have factorised 32760 as $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$ as a product of primes, i.e., $32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$ as a product of powers of primes. Let us try another number, say, 123456789. This can be written as $3^2 \times 3803 \times 3607$. Of course, you have to check that 3803 and 3607 are primes! (Try it out for several other natural numbers yourself.) This leads us to a conjecture that every composite number can be written as the product of powers of primes. In fact, this statement is true, and is called the **Fundamental Theorem of Arithmetic** because of its basic crucial importance to the study of integers. Let us now formally state this theorem.

Theorem 8.2 (Fundamental Theorem of Arithmetic) : *Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.*

An equivalent version of Theorem 8.2 was probably first recorded as Proposition 14 of Book IX in Euclid's Elements, before it came to be known as the Fundamental Theorem of Arithmetic. However, the first correct proof was given by Carl Friedrich Gauss in his *Disquisitiones Arithmeticae*.

Carl Friedrich Gauss is often referred to as the 'Prince of Mathematicians' and is considered one of the three greatest mathematicians of all time, along with Archimedes and Newton. He has made fundamental contributions to both mathematics and science.

Carl Friedrich Gauss
(1777 – 1855)

The Fundamental Theorem of Arithmetic says that every composite number can be factorised as a product of primes. Actually it says more. It says that given any composite number it can be factorised as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur. So, for example, we regard $2 \times 3 \times 5 \times 7$ as the same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. This fact is also stated in the following form:

The prime factorisation of a natural number is unique, except for the order of its factors.

In general, given a composite number x, we factorise it as $x = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq \ldots \leq p_n$. If we combine the same primes, we will get powers of primes. For example,

$$32760 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5 \times 7 \times 13$$

Once we have decided that the order will be ascending, then the way the number is factorised, is unique.

The Fundamental Theorem of Arithmetic has many applications, both within mathematics and in other fields. Let us look at some examples.

Example 5 : Consider the numbers 4^n, where n is a natural number. Check whether there is any value of n for which 4^n ends with the digit zero.

Solution : If the number 4^n, for any n, were to end with the digit zero, then it would be divisible by 5. That is, the prime factorisation of 4^n would contain the prime 5. This is not possible because $4^n = (2)^{2n}$; so the only prime in the factorisation of 4^n is 2. So, the uniqueness of the Fundamental Theorem of Arithmetic guarantees that there are no other primes in the factorisation of 4^n. So, there is no natural number n for which 4^n ends with the digit zero.

You have already learnt how to find the HCF and LCM of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realising it! This method is also called the *prime factorisation method*. Let us recall this method through an example.

Example 6 : Find the LCM and HCF of 6 and 20 by the prime factorisation method.

Solution : We have : $6 = 2^1 \times 3^1$ and $20 = 2 \times 2 \times 5 = 2^2 \times 5^1$.

You can find HCF(6, 20) = 2 and LCM(6, 20) = $2 \times 2 \times 3 \times 5 = 60$, as done in your earlier classes.

Note that HCF(6, 20) = 2^1 = **Product of the smallest power of each common prime factor in the numbers.**

LCM (6, 20) = $2^2 \times 3^1 \times 5^1$ = **Product of the greatest power of each prime factor, involved in the numbers**.

From the example above, you might have noticed that HCF(6, 20) × LCM(6, 20) = 6 × 20. In fact, we can verify that **for any two positive integers *a* and *b*, HCF (*a*, *b*) × LCM (*a*, *b*) = *a* × *b*.** We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 7 : Find the HCF of 96 and 404 by the prime factorisation method. Hence, find their LCM.

Solution : The prime factorisation of 96 and 404 gives :

$$96 = 2^5 \times 3, \ 404 = 2^2 \times 101$$

Therefore, the HCF of these two integers is $2^2 = 4$.

Also, $$\text{LCM}(96, 404) = \frac{96 \times 404}{\text{HCF}(96, 404)} = \frac{96 \times 404}{4} = 9696$$

Example 8 : Find the HCF and LCM of 6, 72 and 120, using the prime factorisation method.

Solution : We have :

$$6 = 2 \times 3, \ 72 = 2^3 \times 3^2, \ 120 = 2^3 \times 3 \times 5$$

Here, 2^1 and 3^1 are the smallest powers of the common factors 2 and 3, respectively.

So, $\text{HCF}(6, 72, 120) = 2^1 \times 3^1 = 2 \times 3 = 6$

2^3, 3^2 and 5^1 are the greatest powers of the prime factors 2, 3 and 5 respectively involved in the three numbers.

So, $\text{LCM}(6, 72, 120) = 2^3 \times 3^2 \times 5^1 = 360$

Remark : Notice, $6 \times 72 \times 120 \neq \text{HCF}(6, 72, 120) \times \text{LCM}(6, 72, 120)$. So, the product of three numbers is not equal to the product of their HCF and LCM.

EXERCISE 8.2

1. Express each number as a product of its prime factors:
   (i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429
2. Find the LCM and HCF of the following pairs of integers and verify that LCM × HCF = product of the two numbers.
   (i) 26 and 91 (ii) 510 and 92 (iii) 336 and 54
3. Find the LCM and HCF of the following integers by applying the prime factorisation method.
   (i) 12, 15 and 21 (ii) 17, 23 and 29 (iii) 8, 9 and 25
4. Given that HCF (306, 657) = 9, find LCM (306, 657).
5. Check whether 6^n can end with the digit 0 for any natural number n.
6. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.
7. There is a circular path around a sports field. Sonia takes 18 minutes to drive one round of the field, while Ravi takes 12 minutes for the same. Suppose they both start at the same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point?

8.4 Revisiting Irrational Numbers

In Class IX, you were introduced to irrational numbers and many of their properties. You studied about their existence and how the rationals and the irrationals together made up the real numbers. You even studied how to locate irrationals on the number line. However, we did not prove that they were irrationals. In this section, we will prove that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and, in general, $\sqrt{p}$ is irrational, where p is a prime. One of the theorems, we use in our proof, is the Fundamental Theorem of Arithmetic.

Recall, a number 's' is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Some examples of irrational numbers, with which you are already familiar, are :

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, -\frac{\sqrt{2}}{\sqrt{3}}, 0.10110111011110\ldots, \text{ etc.}$$

Before we prove that $\sqrt{2}$ is irrational, we need the following theorem, whose proof is based on the Fundamental Theorem of Arithmetic.

Theorem 8.3 : *Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.*

***Proof :** Let the prime factorisation of a be as follows :

$a = p_1p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Therefore, $a^2 = (p_1p_2 \ldots p_n)(p_1p_2 \ldots p_n) = p_1^2 p_2^2 \ldots p_n^2$.

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2, \ldots, p_n$. So p is one of $p_1, p_2, \ldots, p_n$.

Now, since $a = p_1 p_2 \ldots p_n$, p divides a.

We are now ready to give a proof that $\sqrt{2}$ is irrational.

The proof is based on a technique called 'proof by contradiction'. (This technique is discussed in some detail in Appendix 1).

Theorem 8.4 : $\sqrt{2}$ *is irrational.*

Proof : Let us assume, to the contrary, that $\sqrt{2}$ is rational.

So, we can find integers r and s ($\neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are coprime.

So, $b\sqrt{2} = a$.

Squaring on both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2.

Now, by Theorem 8.3, it follows that 2 divides a.

So, we can write $a = 2c$ for some integer c.

* Not from the examination point of view.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using Theorem 8.3 with $p = 2$).

Therefore, a and b have at least 2 as a common factor.

But this contradicts the fact that a and b have no common factors other than 1.

This contradiction has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.

So, we conclude that $\sqrt{2}$ is irrational.

Example 9 : Prove that $\sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $\sqrt{3}$ is rational.

That is, we can find integers a and b ($\neq 0$) such that $\sqrt{3} = \frac{a}{b}\cdot$

Suppose a and b have a common factor other than 1, then we can divide by the common factor, and assume that a and b are coprime.

So, $b\sqrt{3} = a\cdot$

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, a^2 is divisible by 3, and by Theorem 8.3, it follows that a is also divisible by 3.

So, we can write $a = 3c$ for some integer c.

Substituting for a, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.

This means that b^2 is divisible by 3, and so b is also divisible by 3 (using Theorem 8.3 with $p = 3$).

Therefore, a and b have at least 3 as a common factor.

But this contradicts the fact that a and b are coprime.

This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational.

So, we conclude that $\sqrt{3}$ is irrational.

In Class IX, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product and quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

Example 10 : Show that $5 - \sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $5 - \sqrt{3}$ is rational.

That is, we can find coprime a and b ($b \neq 0$) such that $5 - \sqrt{3} = \frac{a}{b}$.

Therefore, $5 - \frac{a}{b} = \sqrt{3}$.

Rearranging this equation, we get $\sqrt{3} = 5 - \frac{a}{b} = \frac{5b - a}{b}$.

Since a and b are integers, we get $5 - \frac{a}{b}$ is rational, and so $\sqrt{3}$ is rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our incorrect assumption that $5 - \sqrt{3}$ is rational.

So, we conclude that $5 - \sqrt{3}$ is irrational.

Example 11 : Show that $3\sqrt{2}$ is irrational.

Solution : Let us assume, to the contrary, that $3\sqrt{2}$ is rational.

That is, we can find coprime a and b ($b \neq 0$) such that $3\sqrt{2} = \frac{a}{b}$.

Rearranging, we get $\sqrt{2} = \frac{a}{3b}$.

Since 3, a and b are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that $3\sqrt{2}$ is irrational.

EXERCISE 8.3

1. Prove that $\sqrt{5}$ is irrational.

2. Prove that $3 + 2\sqrt{5}$ is irrational.

3. Prove that the following are irrationals :

(i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6 + \sqrt{2}$

8.5 Revisiting Rational Numbers and Their Decimal Expansions

In Class IX, you studied that rational numbers have either a terminating decimal expansion or a non-terminating repeating decimal expansion. In this section, we are going to consider a rational number, say $\frac{p}{q} (q \neq 0)$, and explore exactly when the decimal expansion of $\frac{p}{q}$ is terminating and when it is non-terminating repeating (or recurring). We do so by considering several examples.

Let us consider the following rational numbers :

(i) 0.375 (ii) 0.104 (iii) 0.0875 (iv) 23.3408.

Now (i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $0.104 = \frac{104}{1000} = \frac{104}{10^3}$

(iii) $0.0875 = \frac{875}{10000} = \frac{875}{10^4}$ (iv) $23.3408 = \frac{233408}{10000} = \frac{233408}{10^4}$

As one would expect, they can all be expressed as rational numbers whose denominators are powers of 10. Let us try and cancel the common factors between the numerator and denominator and see what we get :

(i) $0.375 = \frac{375}{10^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{3}{2^3}$ (ii) $0.104 = \frac{104}{10^3} = \frac{13 \times 2^3}{2^3 \times 5^3} = \frac{13}{5^3}$

(iii) $0.0875 = \frac{875}{10^4} = \frac{7}{2^4 \times 5}$ (iv) $23.3408 = \frac{233408}{10^4} = \frac{2^2 \times 7 \times 521}{5^4}$

Do you see any pattern? It appears that, we have converted a real number whose decimal expansion terminates into a rational number of the form $\frac{p}{q}$, where p and q are coprime, and the prime factorisation of the denominator (that is, q) has only powers of 2, or powers of 5, or both. We should expect the denominator to look like this, since powers of 10 can only have powers of 2 and 5 as factors.

Even though, we have worked only with a few examples, you can see that any real number which has a decimal expansion that terminates can be expressed as a rational number whose denominator is a power of 10. Also the only prime factors of 10 are 2 and 5. So, cancelling out the common factors between the numerator and the denominator, we find that this real number is a rational number of the form $\frac{p}{q}$, where the prime factorisation of q is of the form $2^n 5^m$, and n, m are some non-negative integers.

Let us write our result formally:

Theorem 8.5 : *Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers.*

You are probably wondering what happens the other way round in Theorem 8.5. That is, if we have a rational number of the form $\frac{p}{q}$, and the prime factorisation of q is of the form 2^n5^m, where n, m are non negative integers, then does $\frac{p}{q}$ have a terminating decimal expansion?

Let us see if there is some obvious reason why this is true. You will surely agree that any rational number of the form $\frac{a}{b}$, where b is a power of 10, will have a terminating decimal expansion. So it seems to make sense to convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Let us go back to our examples above and work backwards.

(i) $\frac{3}{8} = \frac{3}{2^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{375}{10^3} = 0.375$

(ii) $\frac{13}{125} = \frac{13}{5^3} = \frac{13 \times 2^3}{2^3 \times 5^3} = \frac{104}{10^3} = 0.104$

(iii) $\frac{7}{80} = \frac{7}{2^4 \times 5} = \frac{7 \times 5^3}{2^4 \times 5^4} = \frac{875}{10^4} = 0.0875$

(iv) $\frac{14588}{625} = \frac{2^2 \times 7 \times 521}{5^4} = \frac{2^6 \times 7 \times 521}{2^4 \times 5^4} = \frac{233408}{10^4} = 23.3408$

So, these examples show us how we can convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Therefore, the decimal expansion of such a rational number terminates. Let us write down our result formally.

Theorem 8.6 : *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.*

We are now ready to move on to the rational numbers whose decimal expansions are non-terminating and recurring. Once again, let us look at an example to see what is going on. We refer to Example 5, Chapter 1, from your Class IX textbook, namely, $\frac{1}{7}$. Here, remainders are 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1, . . . and divisor is 7.

$$\begin{array}{r|l} & 0.1428571 \\ \hline 7 & 10 \\ & \underline{7} \\ & 30 \\ & \underline{28} \\ & 20 \\ & \underline{14} \\ & 60 \\ & \underline{56} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 10 \\ & \underline{7} \\ & 30 \end{array}$$

Notice that the denominator here, i.e., 7 is clearly not of the form 2^n5^m. Therefore, from Theorems 1.5 and 1.6, we know that $\frac{1}{7}$ will not have a terminating decimal expansion. Hence, 0 will not show up as a remainder (Why?), and the remainders will start repeating after a certain stage. So, we will have a block of digits, namely, 142857, repeating in the quotient of $\frac{1}{7}$.

What we have seen, in the case of $\frac{1}{7}$, is true for any rational number not covered by Theorems 8.5 and 8.6. For such numbers we have :

Theorem 8.7 : *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is not of the form 2^n5^m, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).*

From the discussion above, we can conclude that *the decimal expansion of every rational number is either terminating or non-terminating repeating*.

EXERCISE 8.4

1. Without actually performing the long division, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating repeating decimal expansion:

(i) $\frac{13}{3125}$ (ii) $\frac{17}{8}$ (iii) $\frac{64}{455}$ (iv) $\frac{15}{1600}$

(v) $\frac{29}{343}$ (vi) $\frac{23}{2^3 5^2}$ (vii) $\frac{129}{2^2 5^7 7^5}$ (viii) $\frac{6}{15}$

(ix) $\frac{35}{50}$ (x) $\frac{77}{210}$

2. Write down the decimal expansions of those rational numbers in Question 1 above which have terminating decimal expansions.

3. The following real numbers have decimal expansions as given below. In each case, decide whether they are rational or not. If they are rational, and of the form $\frac{p}{q}$, what can you say about the prime factors of q?

   (i) 43.123456789   (ii) 0.120120012000120000. . .   (iii) $43.\overline{123456789}$

8.6 Summary

In this chapter, you have studied the following points:

1. Euclid's division lemma :

   Given positive integers a and b, there exist whole numbers q and r satisfying $a = bq + r$, $0 \le r < b$.

2. Euclid's division algorithm : This is based on Euclid's division lemma. According to this, the HCF of any two positive integers a and b, with $a > b$, is obtained as follows:

   Step 1 : Apply the division lemma to find q and r where $a = bq + r$, $0 \le r < b$.

   Step 2 : If $r = 0$, the HCF is b. If $r \neq 0$, apply Euclid's lemma to b and r.

   Step 3 : Continue the process till the remainder is zero. The divisor at this stage will be HCF (a, b). Also, HCF(a, b) = HCF(b, r).

3. The Fundamental Theorem of Arithmetic :

   Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.

4. If p is a prime and p divides a^2, then p divides a, where a is a positive integer.

5. To prove that $\sqrt{2}, \sqrt{3}$ are irrationals.

6. Let x be a rational number whose decimal expansion terminates. Then we can express x in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorisation of q is of the form $2^n 5^m$, where n, m are non-negative integers.

7. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is of the form $2^n 5^m$, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

8. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is not of the form $2^n 5^m$, where n, m are non-negative integers. Then x has a decimal expansion which is non-terminating repeating (recurring).

A NOTE TO THE READER

You have seen that :

HCF $(p, q, r) \times$ LCM $(p, q, r) \neq p \times q \times r$, where p, q, r are positive integers (see Example 8). However, the following results hold good for three numbers p, q and r :

$$\text{LCM}(p, q, r) = \frac{p \cdot q \cdot r \cdot \text{HCF}(p, q, r)}{\text{HCF}(p, q) \cdot \text{HCF}(q, r) \cdot \text{HCF}(p, r)}$$

$$\text{HCF}(p, q, r) = \frac{p \cdot q \cdot r \cdot \text{LCM}(p, q, r)}{\text{LCM}(p, q) \cdot \text{LCM}(q, r) \cdot \text{LCM}(p, r)}$$

ANSWERS/HINTS

Arithmetic Progressions

EXERCISE 1.1

1. (i) Yes. 15, 23, 31, . . . forms an AP as each succeeding term is obtained by adding 8 in its preceding term.

(ii) No. Volumes are V, $\frac{3V}{4}$, $\left(\frac{3}{4}\right)^2$ V, $\cdots$

(iii) Yes. 150, 200, 250, . . . form an AP.

(iv) No. Amounts are $10000\left(1+\frac{8}{100}\right), 10000\left(1+\frac{8}{100}\right)^2, 10000\left(1+\frac{8}{100}\right)^3, \cdots$

2. (i) 10, 20, 30, 40 (ii) – 2, – 2, – 2, – 2 (iii) 4, 1, – 2, – 5

(iv) $-1, -\frac{1}{2}, 0, \frac{1}{2}$ (v) – 1.25, – 1.50, – 1.75, – 2.0

3. (i) $a = 3,\ d = -2$ (ii) $a = -5,\ d = 4$

(iii) $a = \frac{1}{3}, d = \frac{4}{3}$ (iv) $a = 0.6,\ d = 1.1$

4. (i) No (ii) Yes. $d = \frac{1}{2}; 4, \frac{9}{2}, 5$

(iii) Yes. $d = -2; -9.2, -11.2, -13.2$ (iv) Yes. $d = 4;\ 6, 10, 14$

(v) Yes. $d = \sqrt{2}; 3+4\sqrt{2}, 3+5\sqrt{2}, 3+6\sqrt{2}$ (vi) No

(vii) Yes. $d = -4;\ -16, -20, -24$ (viii) Yes. $d = 0; -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$

(ix) No (x) Yes. $d = a$; $5a, 6a, 7a$

(xi) No (xii) Yes. $d = \sqrt{2}$; $\sqrt{50}, \sqrt{72}, \sqrt{98}$

(xiii) No (xiv) No (xv) Yes. $d = 24$; 97, 121, 145

EXERCISE 1.2

1. (i) $a_n = 28$ (ii) $d = 2$ (iii) $a = 46$ (iv) $n = 10$ (v) $a_n = 3.5$

2. (i) C (ii) B

3. (i) $\boxed{14}$ (ii) $\boxed{18}$, $\boxed{8}$ (iii) $\boxed{6\frac{1}{2}}$, $\boxed{8}$

(iv) $\boxed{-2}$, $\boxed{0}$, $\boxed{2}$, $\boxed{4}$ (v) $\boxed{53}$, $\boxed{23}$, $\boxed{8}$, $\boxed{-7}$

4. 16th term **5.** (i) 34 (ii) 27

6. No **7.** 178 **8.** 64

9. 5th term **10.** 1 **11.** 65th term

12. 100 **13.** 128 **14.** 60

15. 13 **16.** 4, 10, 16, 22, . . .

17. 20th term from the last term is 158.

18. –13, –8, –3 **19.** 11th year **20.** 10

EXERCISE 1.3

1. (i) 245 (ii) –180 (iii) 5505 (iv) $\frac{33}{20}$

2. (i) $1046\frac{1}{2}$ (ii) 286 (iii) –8930

3. (i) $n = 16$, $S_n = 440$ (ii) $d = \frac{7}{3}$, $S_{13} = 273$ (iii) $a = 4$, $S_{12} = 246$

(iv) $d = -1$, $a_{10} = 8$ (v) $a = -\frac{35}{3}$, $a_9 = \frac{85}{3}$ (vi) $n = 5$, $a_n = 34$

(vii) $n = 6$, $d = \frac{54}{5}$ (viii) $n = 7$, $a = -8$ (ix) $d = 6$

(x) $a = 4$

4. 12. By putting $a = 9, d = 8, S = 636$ in the formula $S = \frac{n}{2}[2a + (n-1)d]$, we get a quadratic equation $4n^2 + 5n - 636 = 0$. On solving, we get $n = -\frac{53}{4}, 12$. Out of these two roots only one root 12 is admissible.

5. $n = 16,\ d = \frac{8}{3}$ **6.** $n = 38,\ S = 6973$ **7.** Sum = 1661

8. $S_{51} = 5610$ **9.** n^2 **10.** (i) $S_{15} = 525$ (ii) $S_{15} = -465$

11. $S_1 = 3, S_2 = 4;\ a_2 = S_2 - S_1 = 1;\ S_3 = 3,\ a_3 = S_3 - S_2 = -1,$
$a_{10} = S_{10} - S_9 = -15;\ a_n = S_n - S_{n-1} = 5 - 2n.$

12. 4920 **13.** 960 **14.** 625 **15.** ₹ 27750

16. Values of the prizes (in ₹) are 160, 140, 120, 100, 80, 60, 40.

17. 234 **18.** 143 cm

19. 16 rows, 5 logs are placed in the top row. By putting $S = 200, a = 20, d = -1$ in the formula $S = \frac{n}{2}[2a + (n-1)d]$, we get, $41n - n^2 = 400$. On solving, $n = 16, 25$. Therefore, the number of rows is either 16 or 25. $a_{25} = a + 24\,d = -4$

i.e., number of logs in 25th row is -4 which is not possible. Therefore $n = 25$ is not possible. For $n = 16$, $a_{16} = 5$. Therefore, there are 16 rows and 5 logs placed in the top row.

20. 370 m

EXERCISE 1.4 (Optional)*

1. 32nd term **2.** $S_{16} = 20, 76$ **3.** 385 cm

4. 35 **5.** 750 m^3

Triangles

EXERCISE 2.1

1. (i) Similar (ii) Similar (iii) Equilateral

(iv) Equal, Proportional **3.** No

EXERCISE 2.2

1. (i) 2 cm (ii) 2.4 cm
2. (i) No (ii) Yes (iii) Yes
9. Through O, draw a line parallel to DC, intersecting AD and BC at E and F respectively.

EXERCISE 2.3

1. (i) Yes. AAA, Δ ABC ~ Δ PQR (ii) Yes. SSS, Δ ABC ~ Δ QRP
   (iii) No (iv) Yes. SAS, Δ MNL ~ Δ QPR
   (v) No (vi) Yes. AA, Δ DEF ~ Δ PQR
2. 55°, 55°, 55°
14. Produce AD to a point E such that AD = DE and produce PM to a point N such that PM = MN. Join EC and NR.
15. 42 m

EXERCISE 2.4

1. 11.2 cm 2. 4 : 1 5. 1 : 4 8. C 9. D

EXERCISE 2.5

1. (i) Yes, 25 cm (ii) No (iii) No (iv) Yes, 13 cm
6. $a\sqrt{3}$ 9. 6m 10. $6\sqrt{7}$ m 11. $300\sqrt{61}$ km
12. 13 m 17. C

EXERCISE 2.6 (Optional)*

1. Through R, draw a line parallel to SP to intersect QP produced at T. Show PT = PR.
6. Use result (iii) of Q.5 of this Exercise. 7. 3 m, 2.79 m

Pair Of Linear Equations In Two Variables

EXERCISE 3.1

1. Algebraically the two situations can be represented as follows:
   $x - 7y + 42 = 0$; $x - 3y - 6 = 0$, where x and y are respectively the present ages of Aftab and his daughter. To represent the situations graphically, you can draw the graphs of these two linear equations.

2. Algebraically the two situations can be represented as follows:

$x + 2y = 1300$; $x + 3y = 1300$, where x and y are respectively the costs (in ₹) of a bat and a ball. To represent the situations graphically, you can draw the graphs of these two linear equations.

3. Algebraically the two situations can be represented as follows:

$2x + y = 160$; $4x + 2y = 300$, where x and y are respectively the prices (in ₹ per kg) of apples and grapes. To represent the situations graphically, you can draw the graphs of these two linear equations.

EXERCISE 3.2

1. (i) Required pair of linear equations is

$x + y = 10$; $x - y = 4$, where x is the number of girls and y is the number of boys.

To solve graphically draw the graphs of these equations on the same axes on graph paper.

Girls = 7, Boys = 3.

(ii) Required pair of linear equations is

$5x + 7y = 50$; $7x + 5y = 46$, where x and y represent the cost (in ₹) of a pencil and of a pen respectively.

To solve graphically, draw the graphs of these equations on the same axes on graph paper.

Cost of one pencil = ₹ 3, Cost of one pen = ₹ 5

2. (i) Intersect at a point (ii) Coincident (iii) Parallel

3. (i) Consistent (ii) Inconsistent (iii) Consistent (iv) Consistent (v) Consistent

4. (i) Consistent (ii) Inconsistent (iii) Consistent (iv) Inconsistent

The solution of (i) above, is given by $y = 5 - x$, where x can take any value, i.e., there are infinitely many solutions.

The solution of (iii) above is $x = 2$, $y = 2$, i.e., unique solution.

5. Length = 20 m and breadth = 16 m.

6. One possible answer for the three parts:

(i) $3x + 2y - 7 = 0$ (ii) $2x + 3y - 12 = 0$ (iii) $4x + 6y - 16 = 0$

7. Vertices of the triangle are (–1, 0), (4, 0) and (2, 3).

EXERCISE 3.3

1. (i) $x = 9,\ y = 5$ (ii) $s = 9,\ t = 6$ (iii) $y = 3x - 3$,

where x can take any value, i.e., infinitely many solutions.

(iv) $x = 2,\ y = 3$ (v) $x = 0,\ y = 0$ (vi) $x = 2,\ y = 3$

2. $x = -2,\ y = 5;\ m = -1$

3. (i) $x - y = 26,\ x = 3y$, where x and y are two numbers $(x > y)$; $x = 39, y = 13$.

(ii) $x - y = 18,\ x + y = 180$, where x and y are the measures of the two angles in degrees; $x = 99,\ y = 81$.

(iii) $7x + 6y = 3800, 3x + 5y = 1750$, where x and y are the costs (in ₹) of one bat and one ball respectively; $x = 500, y = 50$.

(iv) $x + 10y = 105,\ x + 15y = 155$, where x is the fixed charge (in ₹) and y is the charge (in ₹ per km); $x = 5,\ y = 10$; ₹ 255.

(v) $11x - 9y + 4 = 0,\ 6x - 5y + 3 = 0$, where x and y are numerator and denominator of the fraction; $\frac{7}{9}$ $(x = 7,\ y = 9)$.

(vi) $x - 3y - 10 = 0, x - 7y + 30 = 0$, where x and y are the ages in years of Jacob and his son; $x = 40, y = 10$.

EXERCISE 3.4

1. (i) $x = \frac{19}{5},\ y = \frac{6}{5}$ (ii) $x = 2,\ y = 1$ (iii) $x = \frac{9}{13}, y = -\frac{5}{13}$

(iv) $x = 2,\ y = -3$

2. (i) $x - y + 2 = 0, 2x - y - 1 = 0$, where x and y are the numerator and denominator of the fraction; $\frac{3}{5}$.

(ii) $x - 3y + 10 = 0,\ x - 2y - 10 = 0$, where x and y are the ages (in years) of Nuri and Sonu respectively. Age of Nuri $(x) = 50$, Age of Sonu $(y) = 20$.

(iii) $x + y = 9,\ 8x - y = 0$, where x and y are respectively the tens and units digits of the number; 18.

(iv) $x + 2y = 40,\ x + y = 25$, where x and y are respectively the number of ₹ 50 and ₹ 100 notes; $x = 10, y = 15$.

(v) $x + 4y = 27,\ x + 2y = 21$, where x is the fixed charge (in ₹) and y is the additional charge (in ₹) per day; $x = 15,\ y = 3$.

EXERCISE 3.5

1. (i) No solution (ii) Unique solution; $x = 2,\ y = 1$
   (iii) Infinitely many solutions (iv) Unique solution; $x = 4,\ y = -1$

2. (i) $a = 5, b = 1$ (ii) $k = 2$ 3. $x = -2,\ y = 5$

4. (i) $x + 20y = 1000,\ x + 26y = 1180$, where x is the fixed charges (in ₹) and y is the charges (in ₹) for food per day; $x = 400,\ y = 30$.
   (ii) $3x - y - 3 = 0,\ 4x - y - 8 = 0$, where x and y are the numerator and denominator of the fraction; $\frac{5}{12}$.
   (iii) $3x - y = 40,\ 2x - y = 25$, where x and y are the number of right answers and wrong answers respectively; 20.
   (iv) $u - v = 20,\ u + v = 100$, where u and v are the speeds (in km/h) of the two cars; $u = 60$, $v = 40$.
   (v) $3x - 5y - 6 = 0,\ 2x + 3y - 61 = 0$, where x and y are respectively the length and breadth (in units) of the rectangle; length $(x) = 17$, breadth $(y) = 9$.

EXERCISE 3.6

1. (i) $x = \frac{1}{2},\ y = \frac{1}{3}$ (ii) $x = 4,\ y = 9$ (iii) $x = \frac{1}{5},\ y = -2$
   (iv) $x = 4,\ y = 5$ (v) $x = 1,\ y = 1$ (vi) $x = 1,\ y = 2$
   (vii) $x = 3,\ y = 2$ (viii) $x = 1,\ y = 1$

2. (i) $u + v = 10,\ u - v = 2$, where u and v are respectively speeds (in km/h) of rowing and current; $u = 6, v = 4$.
   (ii) $\frac{2}{n} + \frac{5}{m} = \frac{1}{4},\ \frac{3}{n} + \frac{6}{m} = \frac{1}{3}$, where n and m are the number of days taken by 1 woman and 1 man to finish the embroidery work; $n = 18, m = 36$.
   (iii) $\frac{60}{u} + \frac{240}{v} = 4,\ \frac{100}{u} + \frac{200}{v} = \frac{25}{6}$, where u and v are respectively the speeds (in km/h) of the train and bus; $u = 60,\ v = 80$.

EXERCISE 3.7 (Optional)*

1. Age of Ani is 19 years and age of Biju is 16 years or age of Ani 21 years and age of Biju 24 years.
2. ₹ 40, ₹ 170. Let the money with the first person (in ₹) be x and the money with the second person (in ₹) be y.

   $x + 100 = 2(y - 100), y + 10 = 6(x - 10)$
3. 600 km
4. 36
5. $\angle A = 20°$, $\angle B = 40°$, $\angle C = 120°$
6. Coordinates of the vertices of the triangle are (1, 0), (0, –3), (0, –5).
7. (i) $x = 1$, $y = -1$ (ii) $x = \dfrac{c(a-b)-b}{a^2-b^2}$, $y = \dfrac{c(a-b)+a}{a^2-b^2}$

   (iii) $x = a$, $y = b$ (iv) $x = a + b$, $y = -\dfrac{2ab}{a+b}$ (v) $x = 2$, $y = 1$
8. $\angle A = 120°$, $\angle B = 70°$, $\angle C = 60°$, $\angle D = 110°$

Circles

EXERCISE 4.1

1. Infinitely many
2. (i) One (ii) Secant (iii) Infinitely many (iv) Point of contact
3. D

EXERCISE 4.2

1. A
2. B
3. A
6. 3 cm
7. 8 cm
12. AB = 15 cm, AC = 13 cm

Areas Related To Circles

EXERCISE 5.1

1. 28 cm
2. 10 cm
3. Gold : 346.5 cm^2; Red : 1039.5 cm^2; Blue : 1732.5 cm^2; Black : 2425.5 cm^2; White : 3118.5 cm^2.
4. 4375
5. A

EXERCISE 5.2

1. $\frac{132}{7}\text{cm}^2$ 2. $\frac{77}{8}\text{cm}^2$ 3. $\frac{154}{3}\text{cm}^2$

4. (i) 28.5 cm^2 (ii) 235.5 cm^2

5. (i) 22 cm (ii) 231 cm^2 (iii) $\left(231-\frac{441\sqrt{3}}{4}\right)\text{cm}^2$

6. 20.4375 cm^2 ; 686.0625 cm^2 7. 88.44 cm^2

8. (i) 19.625 m^2 (ii) 58.875 cm^2 9. (i) 285 mm (ii) $\frac{385}{4}\text{ mm}^2$

10. $\frac{22275}{28}\text{ cm}^2$ 11. $\frac{158125}{126}\text{ cm}^2$ 12. 189.97 km^2

13. ₹ 162.68 14. D

EXERCISE 5.3

1. $\frac{4523}{28}\text{cm}^2$ 2. $\frac{154}{3}\text{cm}^2$ 3. 42 cm^2

4. $\left(\frac{660}{7}+36\sqrt{3}\right)\text{cm}^2$ 5. $\frac{68}{7}\text{ cm}^2$ 6. $\left(\frac{22528}{7}-768\sqrt{3}\right)\text{cm}^2$

7. 42 cm^2 8. (i) $\frac{2804}{7}\text{ m}$ (ii) 4320 m^2

9. 66.5 cm^2 10. 1620.5 cm^2 11. 378 cm^2

12. (i) $\frac{77}{8}\text{ cm}^2$ (ii) $\frac{49}{8}\text{ cm}^2$ 13. 228 cm^2

14. $\frac{308}{3}\text{ cm}^2$ 15. 98 cm^2 16. $\frac{256}{7}\text{ cm}^2$

Coordinate Gometry

EXERCISE 7.1

1. (i) $2\sqrt{2}$ (ii) $4\sqrt{2}$ (iii) $2\sqrt{a^2+b^2}$

2. 39; 39 km 3. No 4. Yes 5. Champa is correct.

6. (i) Square (ii) No quadrilateral (iii) Parallelogram

7. $(-7, 0)$ **8.** $-9, 3$ **9.** ± 4, QR $= \sqrt{41}$, PR $= \sqrt{82}$, $9\sqrt{2}$

10. $3x + y - 5 = 0$

EXERCISE 7.2

1. $(1, 3)$ **2.** $\left(2, -\frac{5}{3}\right); \left(0, -\frac{7}{3}\right)$

3. $\sqrt{61}$ m; 5th line at a distance of 22.5 m **4.** $2:7$

5. $1:1; \left(-\frac{3}{2}, 0\right)$ **6.** $x = 6,\ y = 3$ **7.** $(3, -10)$

8. $\left(-\frac{2}{7}, -\frac{20}{7}\right)$ **9.** $\left(-1, \frac{7}{2}\right), (0, 5), \left(1, \frac{13}{2}\right)$ **10.** 24 sq. units

EXERCISE 7.3

1. (i) $\frac{21}{2}$ sq. units (ii) 32 sq. units **2.** (i) $k = 4$ (ii) $k = 3$

3. 1 sq. unit; 1 : 4 **4.** 28 sq. units

EXERCISE 7.4 (Optional)*

1. $2:9$ **2.** $x + 3y - 7 = 0$ **3.** $(3, -2)$ **4.** $(1, 0), (1, 4)$

5. (i) (4, 6), (3, 2), (6, 5); taking AD and AB as coordinate axes

(ii) (12, 2), (13, 6), (10, 3); taking CB and CD as coordinate axes. $\frac{9}{2}$ sq. units, $\frac{9}{2}$ sq. units; areas are the same in both the cases.

6. $\frac{15}{32}$ sq. units; 1 : 16

7. (i) $D\left(\frac{7}{2}, \frac{9}{2}\right)$ (ii) $P\left(\frac{11}{3}, \frac{11}{3}\right)$

(iii) $Q\left(\frac{11}{3}, \frac{11}{3}\right)$, $R\left(\frac{11}{3}, \frac{11}{3}\right)$ (iv) P, Q, R are the same point.

(v) $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ **8.** Rhombus

Real Numbers

EXERCISE 8.1

1. (i) 45 (ii) 196 (iii) 51
2. An integer can be of the form $6q$, $6q + 1$, $6q + 2$, $6q + 3$, $6q + 4$ or $6q + 5$.
3. 8 columns
4. An integer can be of the form $3q$, $3q + 1$ or $3q + 2$. Square all of these integers.
5. An integer can be of the form $9q$, $9q + 1$, $9q + 2$, $9q + 3$, . . ., or $9q + 8$.

EXERCISE 8.2

1. (i) $2^2 \times 5 \times 7$ (ii) $2^2 \times 3 \times 13$ (iii) $3^2 \times 5^2 \times 17$
   (iv) $5 \times 7 \times 11 \times 13$ (v) $17 \times 19 \times 23$
2. (i) LCM = 182; HCF = 13 (ii) LCM = 23460; HCF = 2 (iii) LCM = 3024; HCF = 6
3. (i) LCM = 420; HCF = 3 (ii) LCM = 11339; HCF = 1 (iii) LCM = 1800; HCF = 1
4. 22338

7. 36 minutes

EXERCISE 8.4

1. (i) Terminating (ii) Terminating
   (iii) Non-terminating repeating (iv) Terminating
   (v) Non-terminating repeating (vi) Terminating

(vii) Non-terminating repeating (viii) Terminating

(ix) Terminating (x) Non-terminating repeating

2. (i) 0.00416 (ii) 2.125 (iv) 0.009375

(vi) 0.115 (viii) 0.4 (ix) 0.7

3. (i) Rational, prime factors of q will be either 2 or 5 or both only.

(ii) Not rational

(iii) Rational, prime factors of q will also have a factor other than 2 or 5.

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