Revision

$6 \; ; \; 17 \; ; \; 34 \; ; \; 57$

First differences: $11 \; ; \; 17 \; ; \; 23$

Second differences: $6 \; ; \; 6$

$a+b+c = 6$

$3a+b = 11$

$2a = 6$

$$T_n = an^2 + bn + c$$

$2a = 6$

$a = 3$

$3a + b = 11$

$3(3) + b = 11$

$b = 2$

$a + b + c = 6$

$3 + 2 + c = 6$

$c = 1$

$$\therefore T_n = 3n^2 + 2n + 1$$

Arithmetic sequence

An arithmetic sequence is a sequence where the difference between the terms is constant.

The general term is $T_n = a + (n-1)d$

a - first term
d - constant difference
n - number of the term
T_n - n^{th} term

Example

The first three terms of an arithmetic sequence is $3p - 4; 4p - 3$ and $7p - 6$.
Determine :

a) The value of p
b) the first three terms of the sequence
c) the 16th term

Solution

a) Whenever we are dealing with consecutive terms in an arithmetic sequence we apply the principle:
$T_{k+1} - T_k = d$
$\therefore T_2 - T_1 = d$
And $T_3 - T_2 = d$
$\therefore T_2 - T_1 = T_3 - T_2$ (both sides equal d)
$\therefore (4p - 3) - (3p - 4) = (7p - 6) - (4p - 3)$
$\therefore 4p - 3 - 3p + 4 = 7p - 6 - 4p + 3$
$\therefore p + 1 = 3p - 3$
$\therefore p = 2$

b) The first three terms of the sequence is 2; 5 and 8
c) $T_{16} = a + 15d$
$= 2 + 15(3)$
$= 47$

Geometric sequence

A geometric sequence is a sequence where the ration between the terms is constant

The general term is $T_n = ar^{(n-1)}$

a - first term
r - constant ration $\left(\frac{T_2}{T_1}\right)$
n - number of the term
T_n - n^{th} term

Example

Determine the first three terms of the geometric sequence with the 2nd term equal to – 4 and 5the term equal to $\frac{4}{125}$.

Solution

$T_2 = ar = -4$ .... (1)

$T_5 = ar^4 = \frac{4}{125}$ .... (2)

$\frac{T_5}{T_2} = \frac{ar^4}{ar} = \frac{\frac{4}{125}}{-4} = -\frac{1}{125}$ .... (2) + (1)

$\therefore r^3 = -\frac{1}{125}$

$\therefore r = -\frac{1}{5}$

Sub in (1): $a\left(-\frac{1}{5}\right) = -4$

$a = 20$

Thus, the first three terms of the geometric sequence is 20 ;- 4; $\frac{4}{5}$.

Example

The geometric sequence 1; $\frac{3}{2}$; $\frac{9}{4}$; .... has a term equal to $\frac{243}{32}$. Which term is this in the sequence?

Solution

a = 1, r = $\frac{3}{2}$ and $T_k = \frac{243}{32}$. We must determine k.

$T_k = ar^{k-1} = \frac{243}{32}$

$1 \times \left(\frac{3}{2}\right)^{k-1} = \left(\frac{3}{2}\right)^5$

$k - 1 = 5$

k = 6

$\frac{243}{32}$ is the 6th term in the sequence.

Arithmetic Series

Sum of arithmetic series: $S_n = \frac{n}{2}[2a + (n-1)d]$

Proof: $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_n = a + (a+d) + (a+2d) + \ ... + [a + (n-2)d] + [a + (n-1)d]$ ……… (1)
$S_n = [a + (n-1)d] \ + [a + (n-2)d] + \ ... + (a + 2d) + (a+d) + \ a$ ……..(2)
(1) + (2): $2S_n = [2a + (n-1)d] \ + [2a + (n-1)d] + \ ... + (2a + (n-1)d) + (2a + (n-1)d)$
$\therefore \quad 2S_n = n[2a + (n-1)d]$
$S_n = \frac{n}{2}[2a + (n-1)d]$

Example

How many terms are there in the following arithmetic series and what is the sum of the series?
2 + 5 + 8 + ... + 62

Solution
a = 2; d= 3 and T_n= 62
$T_n = a + (n-1)d$
$62 = 2 + (n-1)(3)$
$n = 21$

There are 21 terms in the series

$S_n = \frac{n}{2}[a + l]$
$S_{21} = \frac{21}{2}[2 + 62]$
$= \frac{21}{2} \times \frac{61}{1}$
= 672

Geometric Sequence

Sum of geometric sequence: $S_n = \frac{a(r^n - 1)}{r-1}$

Proof: $S_n = \frac{a(r^n - 1)}{r-1}$

$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \ldots \ldots \ldots$ (1)

$r \times S_n = ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \ldots \ldots \ldots$ (2)

(1) − (2): $S_n - rS_n = a + 0 + 0 + 0 + \cdots + 0 - ar^n$

$S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

$S_n = \frac{a(1-r^n)}{1-r}$ for r < 1

Example

The sum of the first n terms of the geometric series $\frac{3}{4} + \frac{3}{2} + 3 + \cdots$ is $23\frac{1}{4}$
Determine n, the number of terms in the series

Solution

$S_n = 23\frac{1}{4}$; a = $\frac{3}{4}$ and r = 2

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\frac{93}{4} = \frac{\frac{3}{4}(2^n - 1)}{2 - 1}$$

$31 = 2^n - 1$

$2^n = 32 = 2^5$

n = 5.

There are 5 terms in the series

Sum to infinity of a geometric series

For a geometric series with first term a and common ratio r, with $-1 < r < 1$,

$$S_\infty = \frac{a}{1-r} \quad \text{of} \quad \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

Note: This does not mean that we have added up an infinite number of terms, which is of course impossible to do.

Example

Convert the repeating decimal $3,\dot{2}\dot{7}$ to a common fraction.

Solution

$3,272\,727\ldots = 3 + \left(\frac{27}{100} + \frac{27}{10000} + \frac{27}{1000000} + \ldots.\right)$

In the brackets we have an infinite geometric series with $a = \frac{27}{100}, r = \frac{1}{100}$ and $-1 < r < 1$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{\frac{27}{100}}{1 - \frac{1}{100}} = \frac{\frac{27}{100}}{\frac{99}{100}} = \frac{3}{11}$$

$$3,\dot{2}\dot{7} = 3 + \frac{33}{11} = 3\frac{33}{11}$$

Example

For which values of x will the infinite geometric series $(1-x) + (1-x)^2 + \ldots$ converge and what is the sum of the series?

Solution

In the series is a = $1 - x$ and r = $1 - x$

The series will converge if $-1 < r < 1$

$-1 < 1 - x < 1$

$-2 < -x < 0$

$2 > x > 0$

For $0 < x < -2$: $S_\infty = \frac{1-x}{1-(1-x)} = \frac{1-x}{x}$

For all other values of x the series will diverges.

Sigma Notation

Example

Calculate the value of:

$$\sum_{j=4}^{8} j(j+2)$$

Solution

$$\sum_{j=4}^{8} j(j+2) = (4\times 6)+(5\times 7)+(6\times 8)+(7\times 9)+(8\times 10)$$

= 24 + 35 + 48 + 63 + 80

= 250

The number of terms in this series is 5 because j started at 4 and not at 1.

In general, $\sum_{k=m}^{n} T_k$ has n – m + 1 terms $(n \geq m)$

E.g. $\sum_{k=3}^{7} T_k = T_3 + T_4 + T_5 + T_6 + T_7$ have 7-3+1 = 5 terms.

Example

Calculate:

$$\sum_{i=1}^{100} (2i-1)$$

Solution

$$\sum_{i=1}^{100} (2i-1) = 1+3+5+\cdots+199$$

This is an arithmetic series with first term a = 1 and l = 199

n = 100 – 1 + 1 = 100 terms

$S_n = \frac{n}{2}[a+l]$

$S_{100} = \frac{100}{2}[1+199]$

= 10 000

Do the following:

QUESTION 2

2.1 Given: **0 ; -1 ; 1 ; 6 ; 14 ; . . .**

2.1.1 Show that this sequence has a constant second difference. (2)

2.1.2 Write down the next term of the sequence. (1)

2.1.3 Determine an expression for the n^{th} term of the sequence. (4)

2.1.4 Calculate the 30^{th} term. (2)

2.2 In the arithmetic series: $\boldsymbol{a + 13 + b + 27 + . . .}$

2.2.1 Prove that $a = 6$ and $b = 20$. (2)

2.2.2 Determine which term of the series will be equal to 230. (3)

2.3 For which value(s) of k will the series:

$\left(\frac{1-k}{5}\right) + \left(\frac{1-k}{5}\right)^2 + \left(\frac{1-k}{5}\right)^3 + . . .$ converge? (3)

2.4 Given: **16 + 3 + 8 + 3 + 4 + 3 + 2 + . . .**

2.4.1 Determine the sum of the first 40 terms of the series, to the nearest integer. (4)

2.4.2 Write the series: **16 + 8 + 4 + 2 + . . .** in the form $\sum_{k=\cdots}^{\cdots} T_k$

where $T_k = ar^{k-1}$ and a and r are rational numbers. (2)

2.4.3 Determine S_∞ of the series in QUESTION 2.4.2. (2)

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QUESTION 3

Chris bought a bonsai (miniature tree) at a nursery. When he bought the tree, its height was 130 mm. Thereafter the height of the tree increased, as shown below.

INCREASE IN HEIGHT OF THE TREE PER YEAR		
During the first year	**During the second year**	**During the third year**
100 mm	70 mm	49 mm

3.1 Chris noted that the sequence of height increases, namely 100 ; 70 ; 49 ..., was geometric. During which year will the height of the tree increase by approximately 11,76 mm? (4)

3.2 Chris plots a graph to represent the height $h(n)$ of the tree (in mm) n years after he bought it. Determine a formula for $h(n)$. (3)

3.3 What height will the tree eventually reach? (3)

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QUESTION 4

4.1 The first 4 terms of an arithmetic sequence are: 3; *p*; *q*; 21. (3)
Determine the values of *p* and *q*

4.2 The sum of *n* terms of an arithmetic sequence is given by $S_n = 4n + 3n^2$, determine the first three terms of the sequence (3)

4.3 Prove that the sum of *n* terms of an arithmetic series is given by the following formula: (4)

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

QUESTION 5

5.1 Determine the value of *n* if $\sum_{t=1}^{n} 3(2)^{t-1} = 381$ (3)

5.2 Given the series: $1 + 3x + 9x^2 + ...$

5.2.1 Determine the possible values of *x* so that the series is convergent. (2)

5.2.2 Determine the value of *x* if $1 + 3x + 9x^2 + ... = \frac{2}{3}$ (3)