

A stylized map of the province of Manitoba is shown in the background. The land area is white, and the surrounding water areas are a light gray. The map is oriented with the top of the province at the top of the page.

Grade 12 Pre-Calculus Mathematics (40S)

A Course for
Independent Study

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

*A Course for
Independent Study*

2007

Manitoba Education, Citizenship and Youth

Manitoba Education, Citizenship and Youth Cataloguing in Publication Data

510 Grade 12 pre-calculus mathematics (40S) : a course for
 independent study

 Previously published as : Senior 4 pre-calculus
 mathematics (40S) : a course for distance learning.
 ISBN-13: 978-0-7711-3797-6

 1. Mathematics—Programmed instruction.
 2. Calculus—Programmed instruction. 3. Mathematics
 —Study and teaching (Secondary). 4. Calculus—Study
 and teaching (Secondary)—Manitoba. 5. Mathematics—Study
 and teaching (Secondary)—Manitoba. 6. Calculus—Study and
 teaching (Secondary)—Manitoba. I. Manitoba. Manitoba
 Education, Citizenship and Youth. II. Title: Senior 4
 pre-calculus mathematics (40S) : a course for distance
 learning.

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Manitoba Education, Citizenship and Youth
School Programs Division
1970 Ness Avenue
Winnipeg, Manitoba R3J 0Y9.

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thanks to the authors and publishers who allowed their original material to be adapted or
reproduced.

Acknowledgements

Manitoba Education, Citizenship and Youth gratefully acknowledges the contributions of the following individuals in the development of *Grade 12 Pre-Calculus Mathematics: A Course for Independent Study (40S)*.

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GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Introduction

Grade 12 Pre-Calculus Mathematics

Introduction

Welcome to the *Grade 12 Pre-Calculus Mathematics* course for distance learning offered through the School Programs Division, Manitoba Education, Citizenship and Youth. **It is expected that you have successfully completed *Grade 11 Pre-Calculus Mathematics*.**

As a student in a course for distance learning, you have taken on a dual role—that of a student and a teacher. As a student, you are responsible for mastering the lessons and completing the exercises assigned at the end of each lesson. As a teacher, you are responsible for checking your work carefully and noting the nature of your errors. Finally, you must work diligently to overcome your difficulties.

You should seek out a study partner for this course. Most students find that a study partner helps them get through the course with greater success. This study partner can help you correct your assignments and module self-tests, as well as help you prepare for the examinations.

Grade 12 Pre-Calculus Mathematics is one of four possible Grade 12 mathematics courses in Manitoba (the other three courses are *Grade 12 Applied Mathematics*, *Grade 12 Consumer Mathematics*, and *Grade 12 Accounting Systems*). This course contains many new and interesting topics in mathematics. You will need to use many of the skills and procedures that you have already learned to solve some of the problems you will find in the exercises. On completion of this course, successful students will be well prepared to study post-secondary mathematics.

This course is divided into eight modules. Each module contains lessons that are followed by assignments. **It is recommended that you complete all of the assigned exercises.** Answer keys are provided to the exercises and are found at the end of each module.

The eight modules are as follows:

Module 1: Transformations

Module 2: Circular Functions I

Module 3: Circular Functions II

Module 4: Exponential and Logarithmic Functions I

Module 5: Exponential and Logarithmic Functions II

Module 6: Permutations and Combinations

Module 7: Probability

Module 8: Conic Sections

The table of contents outlines the topics found in this course. Every student enrolled in *Grade 12 Pre-Calculus Mathematics* is required to complete all eight modules. Each module ends with a test.

These tests should be written without the aid of any books. Your performance on these tests will give you an indication of how well you understand the material. Your study partner can help you by marking some of these tests. Answer keys for Module Tests 2, 4, 6, and 8 are provided in the Module Tests Answer Keys section at the end of the course. You should then correct all errors and use these tests to help you prepare for the midterm and final examinations. Note that Module Tests 1, 3, 5, and 7 are to be sent to the Tutor/Marker as soon as each one is completed. Therefore, no answer keys are provided for these tests.

Tutor/Marker

The person who marks your tests and exams is your tutor/marker. This person is also available to help you with your learning. When you register for the course, you will receive a letter that gives you the name and contact information for the tutor/marker for *Grade 12 Pre-Calculus Mathematics*. Please take advantage of all the resources provided by the Independent Study Office.

Note:

Module Tests 1, 3, 5, and 7 are to be sent to the Tutor/Marker as soon as each one is completed.

Address:
ISO Tutor/Marker
555 Main Street
WINKLER MB
R6W 1C4

Your Tutor/Marker will review the results with you.

Calculator Use

You will need a scientific calculator for this course. A graphing calculator may be helpful but it is not necessary. References made to the graphing calculator in the course are in optional sections. Many of the exercises ask for exact answers where calculator use is not required.

You are permitted to use a scientific calculator for all tests and exams. You are not asked questions requiring a graphing calculator.

Formula Sheet

A formula sheet is provided for the exams. You may also use it for your tests. A copy of the formula sheet is included at the end of this introductory section.

Evaluation

Your final mark in this course will be based on the results of four hand-in tests, and two examinations: a midterm and a final.

The value of these tests and examinations are shown below.

First Term:	Hand-in Tests 1 and 3, 10% each	20%
	Midterm Examination after Module 4 (Based on Modules 1 through 4)	20%
Second Term:	Hand-in Tests 5 and 7, 10% each	20%
	Final Examination after Module 8 (Based on Modules 1 through 8)	40%
Total		100%

The module tests after Modules 1, 3, 5, and 7 must be completed, and then sent in to your Tutor/Marker. You should also note that the final examination is cumulative, meaning that it is based on the entire course.

You are required to send a **coloured cover sheet** with each hand-in test. Cover sheets can be found after page 6 of the Introduction.

Guide Graphics

Graphics have been placed inside the margins of the course to identify a specific task. Each graphic has a specific purpose to help guide you.

The significance of each guide graphic is described below.



Assignment: You are required to do the assignment questions that accompany this graphic.



Note: This graphic will appear when there is a direction or explanation that you should note carefully.



Test Time: This graphic alerts students that it is time to write a test or prepare for a test.



Send In: This graphic indicates that you must send in the assignment or self-test for correcting.



Study/Review: This graphic is to remind you that you should review your material for a test or examination.



Check: Check your answers against Answer Key provided for this lesson.



Exam Time: When this graphic appears, it is time to write an examination

Cautionary Note

Some of the activities in this course involve chance and probability. In some families and communities, the connection between probability and gambling may be problematic; for example, parents/guardians may not approve of playing cards, dice, or prize money. As an alternative, students can use numbered index cards, number cubes, or points or credits.

Applying for Exams

- If you are **attending school**, ask your Independent Study Option (ISO) facilitator to add your name to the ISO examination eligibility list. Do this at least three weeks prior to the next scheduled examination week.
- If you are **not attending school**, check the **Examination Request Form** for options available to you. Fill in this form and mail or fax three weeks before you are ready to write the Grade 12 Pre-Calculus Mathematics Midterm or Final Examination. The address is:

ISO Tutor/Marker
555 Main Street
WINKLER MB R6W 1C4
Fax: 204-325-1719

Contact Information

Use the following mailing address for any materials that you are forwarding to your tutor/marker:

Distance Learning and Information
Technologies Unit
555 Main Street
WINKLER MB R6W 1C4

Telephone: (204) 325-1700

Toll-free: (800) 465-9915

Fax: (204) 325-1719

Website: <www.edu.gov.mb.ca/k12/dl/iso/index.html>

This website contains information concerning policies and procedures for the Independent Study Option as well as other useful information. It also includes a forms section where you can download the application form for writing exams.

Notes

Formula Sheet

Senior 4 Pre-Calculus Mathematics (40S)

$$s = \theta r$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$P(n, r) \text{ or } {}_n P_r = \frac{n!}{(n-r)!}$$

$$C(n, r) \text{ or } {}_n C_r = \frac{n!}{r!(n-r)!}$$

$$t_{k+1} = {}_n C_k a^{n-k} b^k$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$

$$A = Pe^{rt}$$

$$e \approx 2.718 \ 28$$

$$\log_a(MN) = \log_a M + \log_a N$$

$$\log_a \left(\frac{M}{N} \right) = \log_a M - \log_a N$$

$$\log_a(M^n) = n \log_a M$$

$$\log_a M = \frac{\log_b M}{\log_b a}$$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, a > b$$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$(y-k) = a(x-h)^2$$

$$(x-h) = a(y-k)^2$$

$$t_n = t_1 r^{n-1}$$

$$S_n = \frac{t_1(1-r^n)}{1-r} = \frac{t_1(r^n-1)}{r-1}$$

$$S_n = \frac{t_1 - t_n r}{1-r}$$

$$S_\infty = \frac{t_1}{1-r}, |r| < 1$$

Lesson 1

Translations

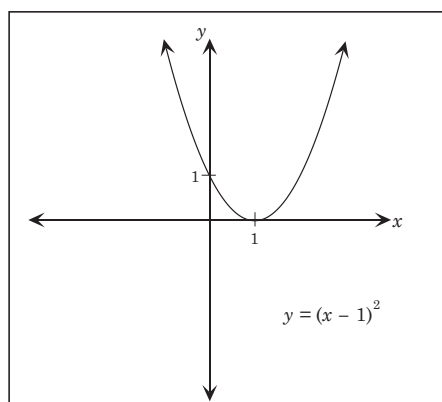
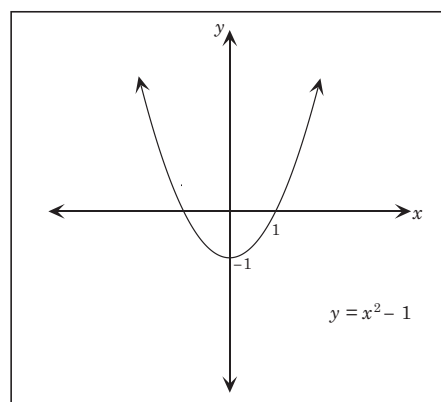
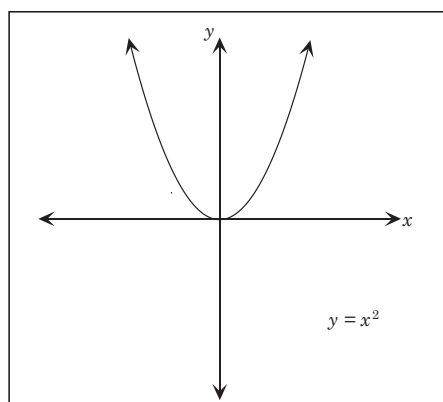
Outcomes

Upon completing this lesson, you will be able to

- describe how a translation affects the graph and properties of a function
- sketch the translation of a function
- state the translation that produced a new sketch from the given sketch

Overview

You have encountered translations when you sketched the following three functions in *Grade 11 Pre-Calculus Mathematics*.



Defining Translations

In $y = x^2 - 1$, each y -value is 1 less than the corresponding y -value in $y = x^2$. Therefore the graph of $y = x^2 - 1$ is 1 unit lower than the graph of $y = x^2$. We say that the graph of $y = x^2$ has been **translated** 1 unit down. The function $y = x^2 - 1$ is a **vertical translation** of the function, $y = x^2$.

Similarly, $y = (x - 1)^2$ is a **horizontal translation** of $y = x^2$. The graph of $y = x^2$ has been shifted one unit to the right.

Definition: A **translation** is a transformation of a geometric figure in which every point is moved the same distance in the same direction.

Example 1

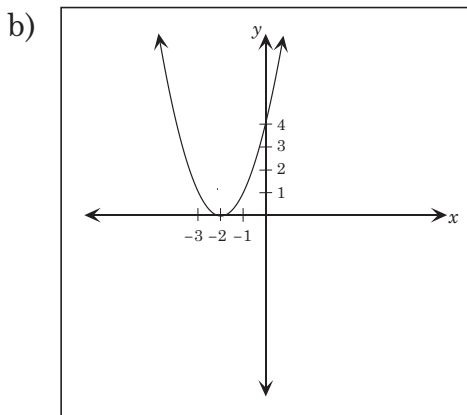
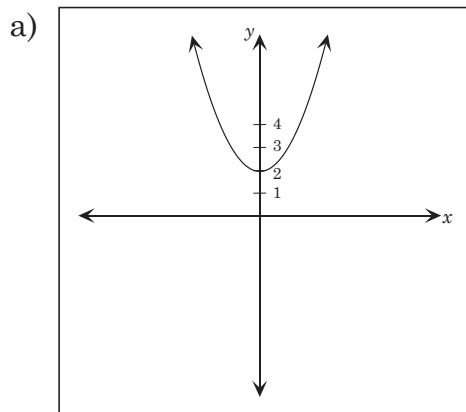
Use the sketch of $y = x^2$ to sketch

a) $y = x^2 + 2$

b) $y = (x + 2)^2$

Note: A graphing calculator is not required for any part of this course.

Solution



As evident from the graphs, the sketch of $y = x^2 + 2$ is a vertical translation of two units up, and the graph of $y = (x + 2)^2$ is a horizontal translation of two units to the left of the graph of $y = x^2$.

Notice that with the vertical translation the “2” is outside the brackets, and affects the y -values by an amount of 2. With the horizontal translation, the “2” is inside the brackets and affects the x -values, not by an amount of 2 but rather by -2 .



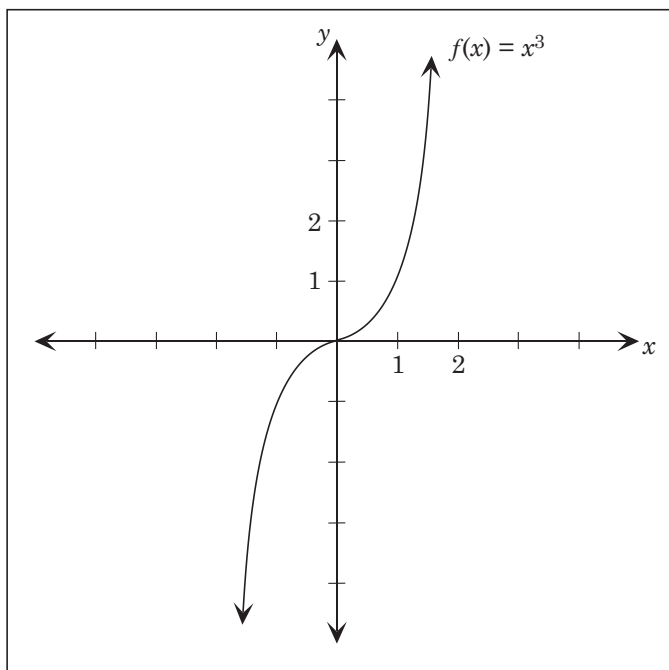


Vertical translations are often easier and more natural to understand. Is there some way to remember in which direction to shift a horizontal translation? One suggestion is to follow the vertex or the intercepts, whichever is more convenient. In the above examples, the smallest value of a squared quantity is zero, thus producing the vertex. Let's follow the vertex as shown in the chart below.

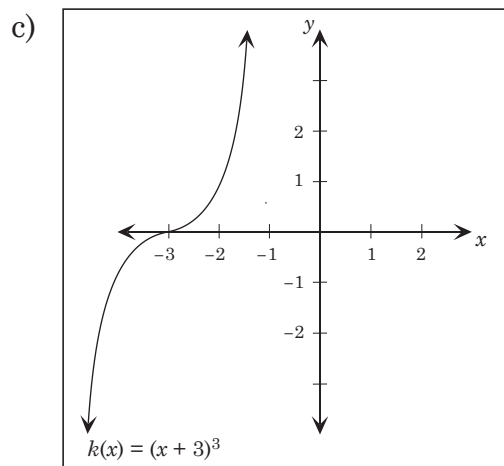
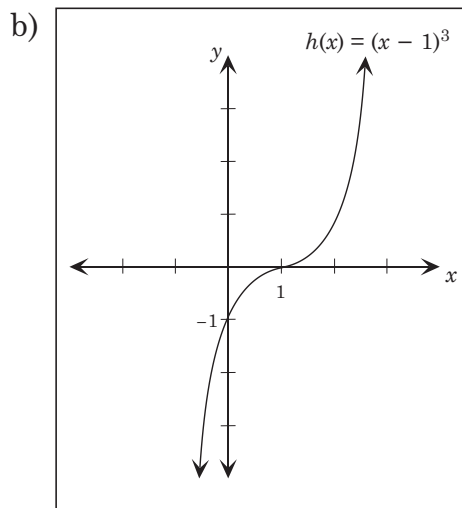
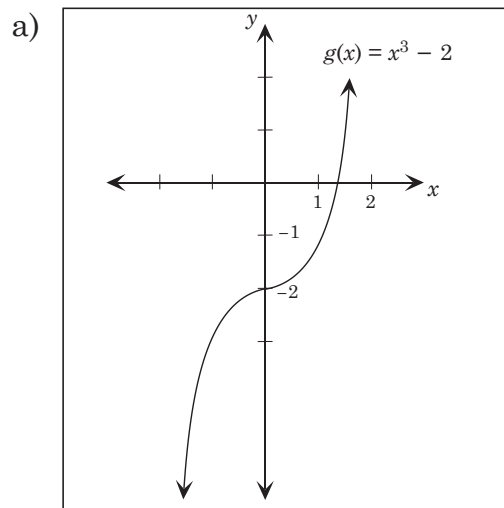
Quadratic Equation	The Value of x Which Makes $y = 0$	Effects on Graph
$y = x^2$	0	basic graph
$y = (x + 2)^2$	-2	2 units left
$y = (x - 2)^2$	2	2 units right

Example 2

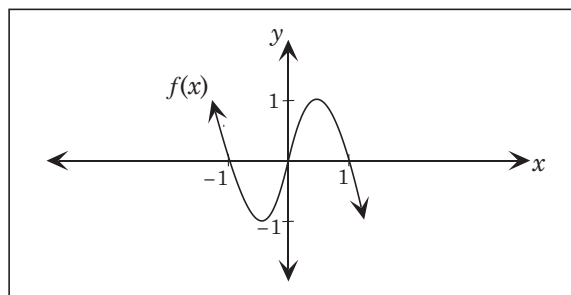
Use the given sketch of $f(x) = x^3$ to sketch the following functions. In this example, there is no vertex. Try following the x -intercept instead. Remember that the x -intercepts are also called the zeros of the function.



- a) $g(x) = x^3 - 2$
- b) $h(x) = (x - 1)^3$
- c) $k(x) = (x + 3)^3$

Solution**Example 3**

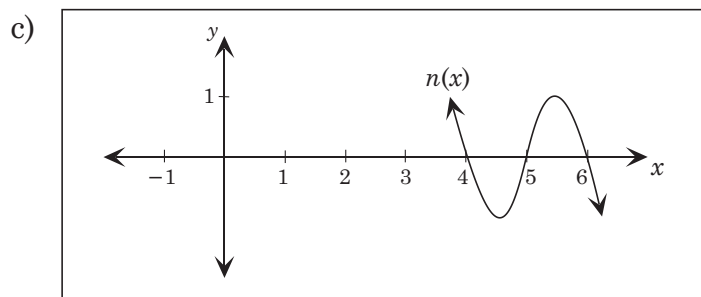
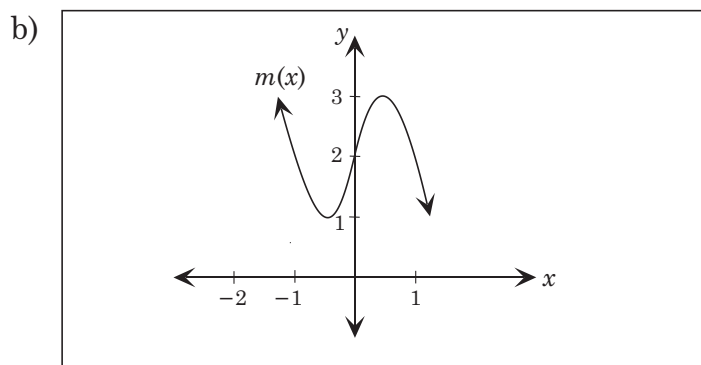
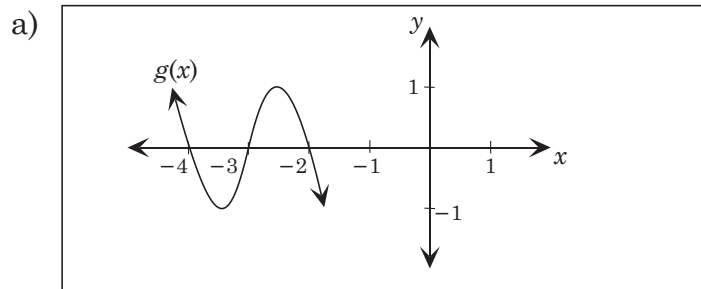
Use the given sketch of $f(x)$ to sketch the following functions. Again, try translating the x - and y -intercepts as a tool for making the graphing easier.



a) $g(x) = f(x + 3)$

b) $m(x) = f(x) + 2$

c) $n(x) = f(x - 5)$

Solution**Summary**

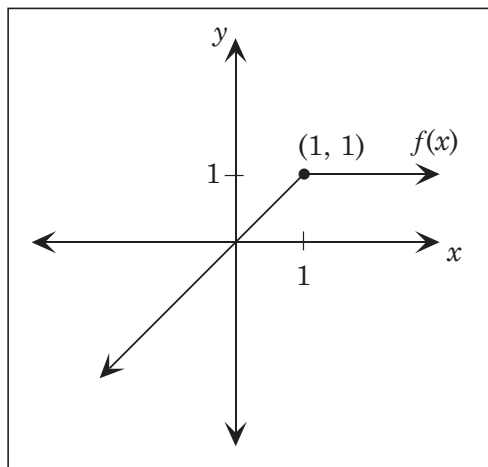
Given a function, $f(x)$, the effect of a translation on $f(x)$ is summarized in the chart below.

Translation	Effect on Graph	Effect on (x, y)
$f(x) + k$	Vertical translation of k units: <ul style="list-style-type: none">• up if $k > 0$• down if $k < 0$	$(x, y + k)$
$f(x - h)$	Horizontal translation of h units: <ul style="list-style-type: none">• to the left if $h < 0$• to the right if $h > 0$	$(x + h, y)$

Assignment

1. Given the sketch of $f(x)$ drawn below, sketch each of the following functions.

- | | |
|-------------------|-------------------|
| a) $f(x - 5)$ | b) $f(x) - 5$ |
| c) $f(x) + 5$ | d) $f(x + 5)$ |
| e) $f(x - 5) - 5$ | f) $f(x + 5) + 5$ |

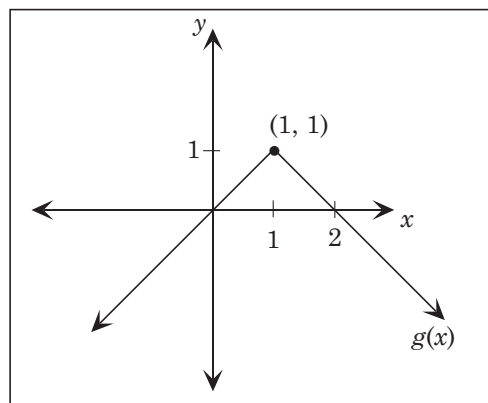


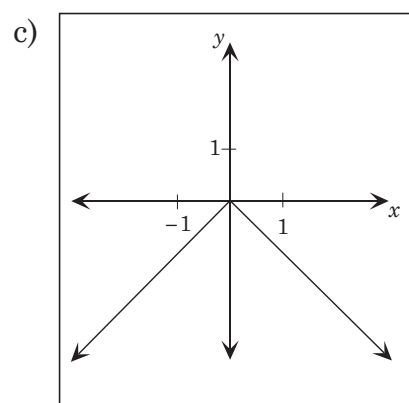
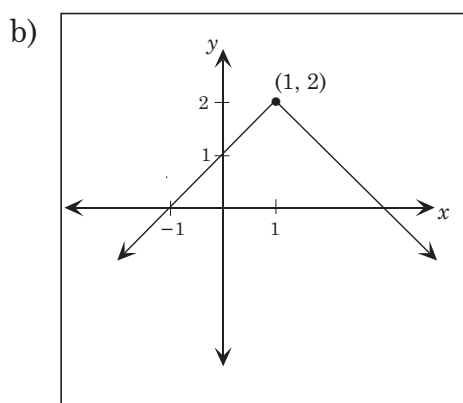
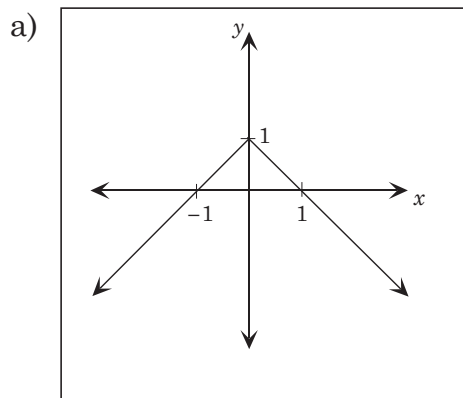
2. Let $f(x) = x^2 + 2$. Sketch each of the following functions.

- | | |
|---------------|-------------------|
| a) $f(x)$ | b) $f(x) - 6$ |
| c) $f(x + 1)$ | d) $f(x - 2) - 3$ |

3. For each of the functions in Question 2 state the properties of the function: the domain, the range, and the values of the intercepts. (**Recall:** To find the y -intercept, let $x = 0$ and solve for y . To find the x -intercept, let $y = 0$ and solve for x . For more practice, see Lesson 3.)

4. Each of graphs (a), (b), and (c) represents a translation of the given function, $g(x)$, drawn below. Write an expression for each new function in terms of $g(x)$.

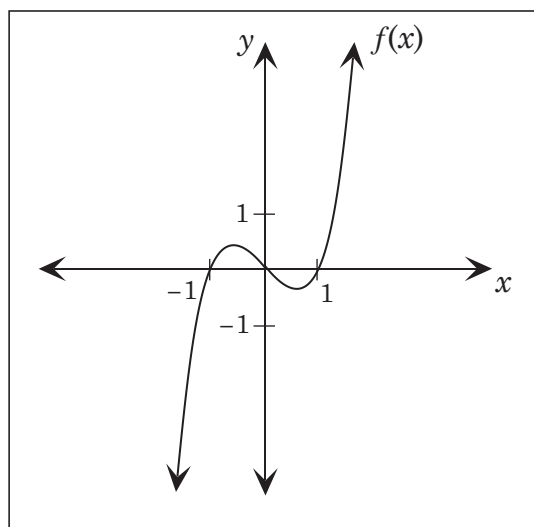




5. If $f(x) = x^3 + 3x^2 - x + 6$, write an unsimplified equation for
- a) $g(x)$ which has the same graph as $f(x)$ moved two units to the left.
 - b) $h(x)$ which has the same graph as $f(x)$ moved three units down.
 - c) $m(x)$ which has the same graph as $f(x)$ moved two units to the right and one unit up.
6. How are the graphs of $g(x) = \frac{|x|}{x}$ and $n(x) = \frac{|x+3|}{x+3}$ related?

(For more practice, see Lesson 3, Assignment, questions 3, 4, and 5.)

7. Below is the graph of $f(x) = x^3 - x$. Sketch the graph of $g(x) = (x + 2)^3 - (x + 2)$.



8. Is the translation of a function still a function?

Check your answers in the Module 1 Answer Key.

