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Grade 6, Math Circles

6/7 March, $[(3 \times 4 + 10^3) - 3] \times 2$

Algebra

Introduction

Algebra is an extremely important topic in mathematics that plays a key part becoming a fantastic mathematician. Today we are going to dive into some of the most important topics in algebra, leaving you with an entirely new set of math skills.

Order of Operations

Did you notice that on today's note there is an algebraic equation beside the date? Were you able to figure out what it evaluates to? Since the expression is where the year usually is, you may have guessed that $[(3 \times 4 + 10^3) - 3] \times 2 = 2018$. Here's why:

$= [(3 \times 4 + 10^3) - 3] \times 2$
$= [(12 + 1000) - 3] \times 2$
$= [1012 - 3] \times 2$
$= 1009 \times 2$
$= 2018$

Calculate the following equation: $3 \times 4 + 10^3 - 3 \times 2$

Did you get the same answer?

Why did we get two different answers for the same question? It has to do with the order in which we calculated each element of the equation. There is an acronym that can help us follow the steps to ensure we get the right answer, it is called BEDMAS.

Brackets	First Priority
Exponents	Second Priority
Division	Third Priority
Multiplication	Third Priority
Addition	Fourth Priority
Subtraction	Fourth Priority

BEDMAS tells us that brackets are the highest priority, then exponents, then both division and multiplication, and finally addition and subtraction. This means that we evaluate exponents before we multiply, divide before we subtract, etc.

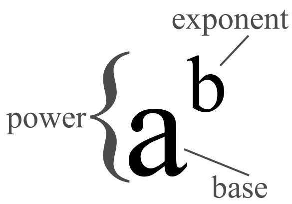
Note: If an expression has two or more operations of the same priority, do those operations from left to right.

Examples

- $10 + 2 \div 2 - 3 \times 3 = ?$
- $(3 + 27 - 8 \times 2) \div 7 = ?$
- $3 \times (5 + 2 \times 6 \div 6 + 8) + 10 = ?$
- $42 \div 7 \times 3 + 9 \times 2 - 4 = ?$

Exponents

Exponentiation (or using exponents) is simply repeated multiplication of a certain number.



The **base** is the number that you are multiplying over and over again, and the **exponent** is the number of times that you are multiplying the base. The entire expression is called a **power**.

There is also an important rule to remember when dealing with exponents:

Any number to the exponent 1 is itself
Any number to the exponent 0 is equal to 1

Examples

- $3^4 = 3 \times 3 \times 3 \times 3 = 81$
- $1^{362} = 1$
- $0^8 = 0$
- $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

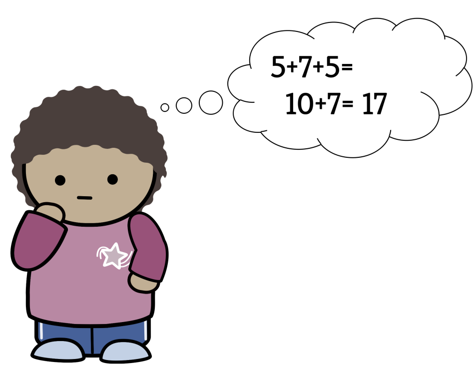
Try it Out

(a) $4^3 =$

(b) $10^4 =$

(c) $627^1 =$

(d) $2^6 =$



Nested Brackets

We know that BEDMAS tells us to evaluate anything inside brackets first, but what if there is a bracket inside a bracket? For example how would we solve the following equation?

$[22 + (4 - 2) \times 3]$

Well we simply preform BEDMAS twice, going to the inner most bracket first, then working our way outwards.

$= [22 + 2 \times 3]$
$= [22 + 6]$
$= 28$

A bracket inside a bracket is called a **nested bracket**. There is no limit on how many sets of brackets can be used in an equation so you always start at the inner most bracket and work outwards. If there are two brackets that are equally nested, evaluate them from left to right! There are different types of brackets including the commonly used parentheses (), the square brackets [ ], and the curly brackets {}.

Examples

$(10 - (3 + 2)) \times 5$
$= (10 - 5) \times 5$
$= 5 \times 5$
$= 25$

$[((12 - 4) \div 4) + 5] \times 2$
$= [(8 \div 4) + 5] \times 2$
$= [2 + 5] \times 2$
$= 7 \times 2$
$= 14$

Try it Out

(a) $14 - 3[10 - (1 + 2) \times 3] + 4 =$

(b) $5 + 2 \times \{[3 + (2 \times 4 - 1) + 4] - 2\} =$

Notation

There is some new notation that I will introduce to you today that you will see a lot of as you continue to learn more and more about math.

Variables

As you may already know, mathematicians often use letters in math to represent an unknown quantity. We call these symbols **variables**. Variables can represent all sorts of different things such as the height of a person, the price of an item, the distance you travelled on your bike, etc. An equation with variables is called an **algebraic equation**.

Examples

- A rectangle has a length of $5cm$ and a width of $x\ cm$
- $16 + x = 22$

Multiplication

Math equations can get very long and complicated so a lot of the time mathematicians will use "hidden multiplication signs" to simplify their work. It is called a hidden multiplication sign because instead of using x between two numbers or expressions they simply use brackets.

Examples

$4(5-2) \Longrightarrow 4 \times (5-2)$

$7(4 \times 5) \Longrightarrow 7 \times (4 \times 5)$

$12(4) \Longrightarrow 12 \times 4$

$15(x+3)(4) \Longrightarrow 15 \times (x+3) \times 4$

Division

Division can also be re-written as a fraction with the dividend as the numerator (top) and the divisor as the denominator (bottom).

Examples

$\frac{18}{3} \Longrightarrow 18 \div 3$

$\frac{x}{10} \Longrightarrow x \div 10$

$\frac{21}{x} \Longrightarrow 21 \div x$

$\frac{1}{2}(6) \Longrightarrow 1 \div 2 \times 6$

Distributive Property

We just learned that $4(1+3)$ is the same as $4 \times (1+3)$ but what happens when we have a variable inside the bracket? For example, can we simplify $4(x+6)$? Well it turns out we can! Notice that $4(1+3) = (4 \times 1) + (4 \times 3) = 4 + 12 = 16$. When you have a number or variable outside a bracket you can simply "distribute" it to each item inside the bracket.

Examples

$3(x+2) \rightarrow (3 \times x) + (3 \times 2) = 3x + 6$

$5(2x+3) \rightarrow (5 \times 2x) + (3 \times 5) = 10x + 15$

$x(3+x) \rightarrow (3 \times x) + (x \times x) = 3x + x^2$

$7(x - 4 + 2y) \rightarrow 7x - 28 + 14y$

Try it Yourself

$14(2+x)$ $\quad$ $8(3x+2-4)$ $\quad$ $x(x-2)$

Solving Equations

A lot of times in mathematics we are required to solve for an unknown variable, most commonly x. What we as mathematicians must do is figure out what number x must be in order for the equation to be true.

Example

$$x + 6 = 14$$

In order to figure out what x is, we can guess what number added to 6 equals 14, or we can solve for x using algebra.

When using algebra to solve algebraic expressions it is important to remember one rule:

Whatever you do to one side of the equation, you must do to the other side

The key to solving for a variable is to isolate, or make the variable all alone, on one side of the equation. Taking $x + 6 = 14$ we see that 6 is being added to x so in order to get x alone we must subtract 6 from it but also from 14 because of our rule.

$$x + 6 - 6 = 14 - 6$$
$$x = 8$$

Try it Yourself

$x - 12 = 5$ $\quad$ $120 + x + 15 = 142$ $\quad$ $x + 8 = 13$

Most of the time there is more than just addition and subtraction involved when solving for x, often we will have to multiply or divide to solve for x.

Examples

$\frac{x}{4} = 6$ $\quad$ $3x = 9$

In the first example we see x is being divided by 4, so to undo it we must multiply by 4. In the second example we see x is being multiplied by 3, so to undo it we must divide by 3.

$\frac{x}{4}(4) = 6(4)$ $\quad$ $\frac{3x}{3} = \frac{9}{3}$

$x = 24$ $\quad$ $x = 3$

Try it Yourself

$8x = 64$ $\quad$ $\frac{x}{3} = 7$ $\quad$ $\frac{2x}{4} = 8$

In general, when solving for a variable, we want to undo all of the operations being applied in a reverse BEDMAS order. Fill out the following chart to show how to undo each operation.

Original Operation		**Opposite Operation**
Addition	$+$	
Subtraction	$-$	
Division	$\div$	
Multiplication	$\times$	

Rarely in mathematics will you only have to undo one operation to solve for a variable, so there are a few simple steps to follow when you have multiple operations being preformed on a variable.

1. If possible, simplify each side of the equation. (Do all addition, subtraction, multiplication and division that can be done.)
2. Look at the equation and determine what number(s) must be removed to get x by itself.
3. Undo any addition or subtraction first
   - Remember **whatever you do to one side you must do to the other**
4. Undo any multiplication or division
   - Remember **whatever you do to one side you must do to the other**

Examples

$$(6+4\times 2)+x=22$$
$$(6+8)+x=22$$
$$14+x=22$$
$$14+x-14=22-14$$
$$x=8$$

$$\frac{x}{4}-11=5$$
$$\frac{x}{4}-11+11=5+11$$
$$\frac{x}{4}=16$$
$$\frac{x}{4}(4)=16(4)$$
$$x=64$$

$$4x-12(\frac{8}{4})=0$$
$$4x-12(2)=0$$
$$4x-24=0$$
$$4x-24+24=0+24$$
$$4x=24$$
$$\frac{4x}{4}=\frac{24}{4}$$
$$x=6$$

Try it Yourself

$2x-3=5$ $\qquad$ $3x=8+4$ $\qquad$ $45+1=2x+28$

Collecting Like Terms

When we say Collecting Like Terms, we mean adding or subtracting the same variables together, and doing the same for the numerical values. We then want to put all our variables on one side of the equation and all of our numerical values on the other. This makes our equations look nicer and will help us in solving them. Here is an example to illustrate:

$$x + x + y + y = 1 + 1 + 1 + 2 + 2$$
$$2(x) + 2(y) = 3(1) + 2(2)$$

We treat variables the same way we treat regular numbers. Looking at the right side of the equation, we can say there are three "1s" and two "2s," (instead of adding the numbers right away) just like the left side for x and y. We know that $1 + 1 + 1$ is the same as 3×1, and $2 + 2$ is the same as 2×2. Using this same logic, $x + x$ is the same $2 \times x$, $(2x)$, and $y + y$ is the same as $2 \times y$, $(2y)$:

$$2x + 2y = 3 + 4$$
$$2x + 2y = 7$$

Try it Yourself

$x + x - x = 5 - 3$

$3x = 2x + 4$

$x + y + 2z + y + z = z - 4z + x + 8$

$2x - 103 + (3 + 2 \times 2) = x(18 \div 9 - 2)$

$\dfrac{x + 2 - 1 + x}{3} = 3$

Word Problems

Algebra is very useful in solving word problems and it can be helpful when the answer is not obvious.

To solve a word problem with algebra, follow these simple steps:

1. Represent the unknown quantity with a variable.
2. Use the information given in the problem to set up an equation with the variable.
3. Solve the equation.
4. Write a conclusion.

Example

Hannah went to the grocery store and bought 18 pieces of fruit. She bought 6 apples, 4 pears, 3 bananas, 1 kiwi and the rest were peaches. How many peaches did Hannah buy?

1. Let x be the amount of peaches purchased.
2. $6 + 4 + 3 + 1 + x = 18$
3. $14 + x = 18$
   $14 + x - 14 = 18 - 14$
   $x = 4$
4. Therefore Hannah bought 4 peaches

Try it Yourself

If you spent $30 at the store and spent $5 on pizza, $7 on cereal, and $4 on vegetables, and the rest on two chicken breasts, how much did you spend on each chicken breast?

Problem Set

1. Evaluate

$$46 + 20 \div 4 - 8 \times (20 + 6 - 7 \times 3) + \frac{4^3}{8}$$

2. Evaluate

$$(7 \times 3 - 2)^2 - 4 \times 2$$

3. Sandy is given the following equation without any brackets. Help Sandy figure out where brackets need to be placed to make the equation true.

$$27 + 6 - 3 \times 4 \div 7 + 8 = 11$$

4. Sam and Rhea are arguing over who answered a question on their test correctly. When they get their tests back they realize they both have gotten the question wrong. Help explain to Sam and Rhea what they did wrong and provide the correct solution.

The Question: $24(6 - 10 \div 2) + 8 \div [6 - 2(2)]$

Sam's Answer
$= 24(-4 \div 2) + 8 \div [6 - 2(2)]$
$= 24(-2) + 8 \div [6 - 2(2)]$
$= 24(-2) + 8 \div [6 - 4]$
$= 24(-2) + 8 \div 4$
$= -48 + 8 \div 4$
$= -48 + 2$
$= -46$

Rhea's Answer
$= 24(6 - 5) + 8 \div [6 - 2(2)]$
$= 24(6 - 5) + 8 \div [4(2)]$
$= 24(6 - 5) + 8 \div [8]$
$= 24(1) + 8 \div 8$
$= 25 + 8 \div 8$
$= 25 + 1$
$= 26$

5. Evaluate the following expression:

$$(4)(2+1) - 7 + ((8)(9))^0 - (2)\left(\frac{97+5}{51}\right) = ?$$

6. Solve the following for x:
   a) $x + 4 - 2 = 12$
   b) $19(2) = 3x - 11$
   c) $-27 = x - 15 - 34$
   d) $71 - 36 = x + 15 + 8$
   e) $14 + x = 6 - 20 - 3x$
   f) $417 - 31 = 16 + x - 58 + x$
   g) $\frac{27}{3} = \frac{x}{9}$
   h) $-x + \frac{(10)(6)}{5} - 3^2 = \frac{x}{4}$
   i) $(6)(3) - \frac{(7)(12)}{4} = \frac{x}{8} - (5)(2)$

7. Simplify the following:

   a) $4[x + 3(2x + 1)]$ b) $9 + 3[x + (3x + 2)] + 4$

   c) $4 + 10[x + (2x + 3)] + 12x$ d) $5 + 2[3 + (2x + 1) + x] - 2$

8. Jessica has 8 nickels, 3 dimes, and 7 quarters in her pocket. What is the total amount of money that she has in her pocket?

9. Gary bought an armadillo for \$126.95. If he had three 50 dollar bills how much change did Gary receive.

10. A classroom of 67 students is divided into two groups, with one of the groups having 9 students more than the other. What is the size of each group?

11. The area of a dog park is $1000m^2$, with the width being $20m$.

    - What is the length of the dog park in meters?
    - If a dog is 2 meters in length and a total of 500 dogs can fit in the pen exactly, what is the width of each dog (assuming they are all the same size)?

12. A train consists of an engine and a number of freight cars. The engine is 18 meters in length and each freight car is 8 meters in length. If the length of the train is 378 meters, then what is the number of cars in the train (including the engine)?

13. Julia went to the candy store with \$32 to get jawbreakers. Each jawbreakers costs \$2. If Julia left the store with \$16 how many jawbreakers did she buy?

14. \$100 is to be split among three friends. Friend B gets double the amount of friend A, and friend C gets \$10 more than friend B. How much money does each friend get?

15. Try different values for the below sequence of steps to find a pattern. Explain the number trick by showing the steps using algebra.

    1) Pick a number.
    2) Multiply the number by itself.
    3) Add the result to the original number.
    4) Divide by the original number.
    5) Add 24.
    6) Subtract the original number.
    7) Divide by 5.

16. Ryan starts running at his top speed of $18km/hr$. He slows down $3km/hr$ after every 17 minutes until he reaches $6km/hr$ which he can run at for 5 hours. *Note:* Ryan runs at a constant rate during each interval. Today Ryan ran $41km$ training for a marathon. How long did it take Ryan to finish his training?