UNIVERSITY OF WATERLOO
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Waterloo, Ontario N2L 3G1

Enriching Mathematics and Computer Science for 50 years

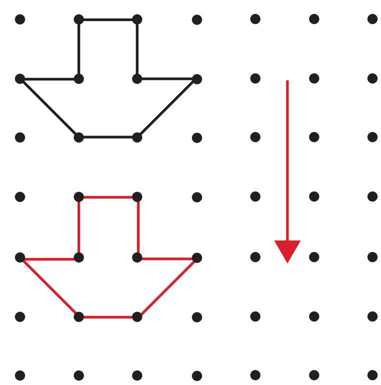
Grade 6 Math Circles
October 29/30, 2013
Shapeshifting

Transformations of Shapes

Translation

A translation is a shift by a certain distance in a particular direction. The orientation and size of the shape remain unchanged.

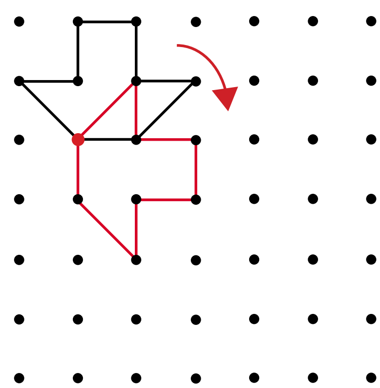
Example: Shift down 3 units.



Rotation

A rotation is a turn by some angle about a given point. The shape is otherwise unchanged.

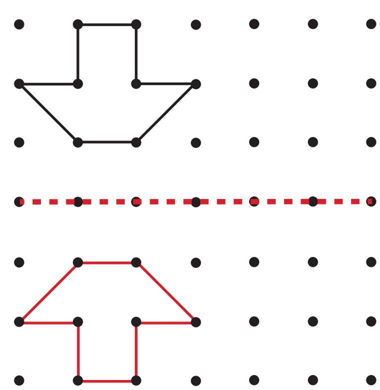
Example: Rotate 90° CW about the red point.



Reflection

A reflection is a flip about some arbitrary axis. The shape is otherwise unchanged.

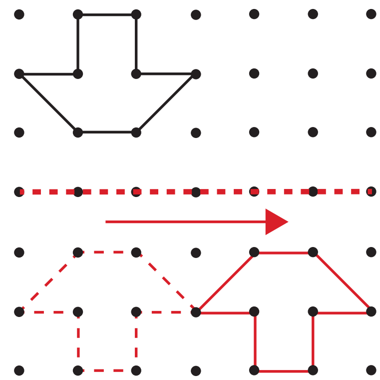
Example: Reflect about the red-axis.



Glide Reflection

A glide reflection is a reflection and then a translation along the axis of reflection. The shape is otherwise unchanged.

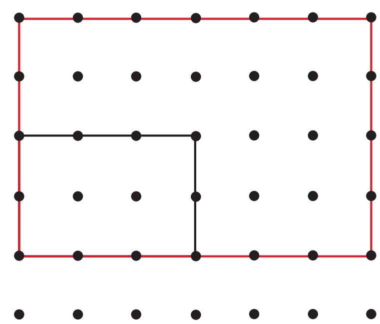
Example: Reflect about the red-axis and shift 3 units to the right.



Scaling

Scaling a shape means to change the size of the shape while maintaining its proportions. The shape is otherwise unchanged. We will only consider scaling edges (not scaling areas or volumes).

Example: Scale by a factor of two.

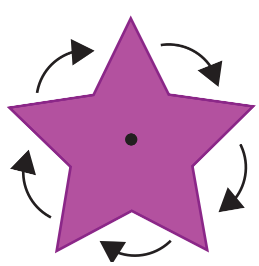


Symmetries of Shapes

A shape (or finite arrangements of shapes) has rotational (or reflectional) symmetry if a rotation (or reflection) can be applied in a non-trivial way to the shape (or arrangement) so that the result of the transformation is the original shape (or arrangement).

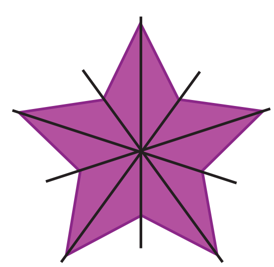
Note: By *trivial* I mean something obvious, like rotating any shape by 360° about its centre and then saying that the original shape is recovered. This isn't really meaningful symmetry.

Rotational



If the star is rotated any integer multiple of 72° (1/5 of a full turn) about its centre, then the original shape is recovered.

Reflectional



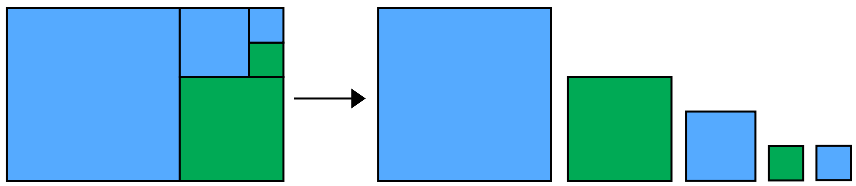
If the star is reflected about any of the axes shown (black lines), then the original shape is recovered. The black lines are called **axes of symmetry**.

Arrangements of shapes can also have scale symmetry, meaning that some (or all) of the shapes in the arrangement are can be scaled so that they are congruent.

Note: Congruent means that the shapes are exactly the same except for that they can be in different positions and be oriented differently.

Scale

The shapes belonging to the arrangement below on the left can be scaled to be congruent to each other.



Symmetries of Patterns of Shapes

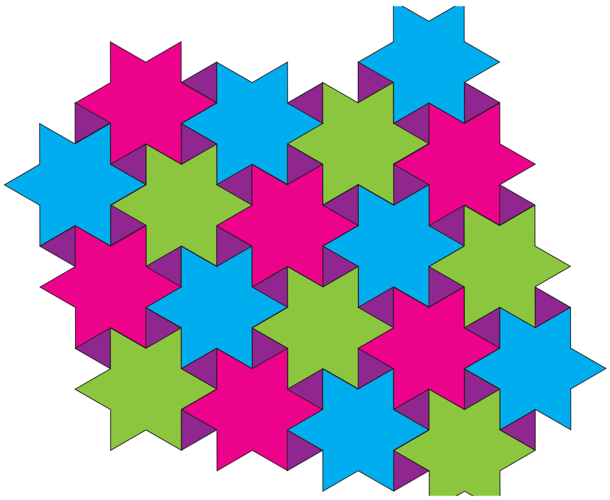
Patterns of shapes are arrangements of shapes that follow a particular sequence of repetition. They have a basic repeating unit, which can be an arrangement of shapes or just a single shape. Patterns can be infinite or finite in size.

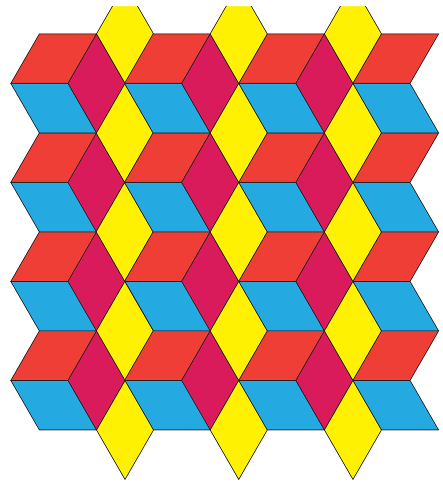
If a unit of the pattern has rotational, reflectional, or scale symmetry, then the entire pattern will have this type of symmetry.

Infinite patterns can have two other types of symmetry: translational, and glide-felectional symmetry. An infinite pattern has translational (or glid-reflectional) symmetry if a translation (or glide-reflection) can be applied to the pattern so that the result of the transformation is the original pattern.

To see examples of these types of symmetry, visit the website "Totally Tessellate" at this link http://library.thinkquest.org/16661/ and view the section on symmetries and transformations.

Tessellations



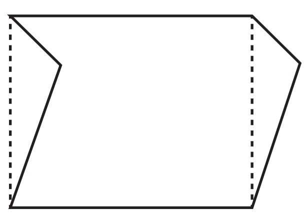


A tessellation is a collection of shapes that fit together with no gaps or overlaps.

Almost all tessellations are patterns and so they can possess many types of symmetry.

Making Tessellations:

- We can make tessellations from more than one shape.
- We can make tessellations from single shapes, or shapes that have been re-formed a bit. For example, we can make a tessellation out of this shape that has been made from a rectangle:



- We can make tessellations out of any triangle or quadrilateral using the rotation method (rotating through midpoints). To see this method, again check out the website “Totally Tessellated”.

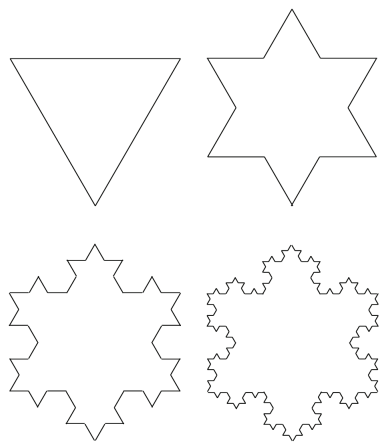
Note: Not all shapes can make tessellations.

Fractals

A fractal is a geometric figure that is formed by a repeated application of a certain process (iteration). Fractals do not change complexity at any level of magnification.

Fractals always have a property of self-similarity (a pattern is found inside itself), so you can often find scale symmetry in fractals!

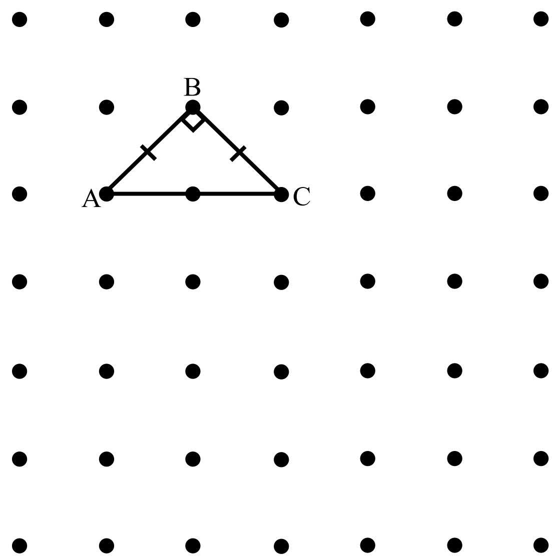
Here are the first few iterations of a famous fractal called the Koch Snowflake. If an infinite number of iterations are completed, the resulting figure will have an infinite perimeter but a finite area!



from galileospendulum.org

Problem Set

1. (a) Perform the following transformations on triangle ABC (for each transformation, start with the **original** triangle):



i. Translation 2 units to the right

ii. 90° rotation CCW about vertex A

iii. Reflection over AC

iv. Translation 4 units down

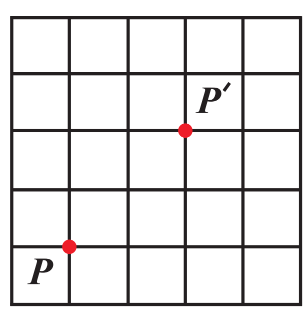
Make sure that you **label** the vertices A, B, and C on the resulting images.

(b) For each transformation in part (a), write down what transformation you would do to get the transformed triangle back to its original position.

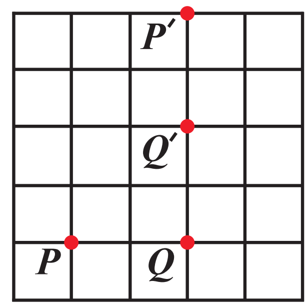
(c) Draw the triangle on grid paper and perform the transformations **in succession**. That means, after you translate the triangle, take the result and rotate it, etc. Try doing the transformations in different orders. Do you get the same result every time?

(d) Come up with a series of transformations to apply to triangle ABC in succession that give a different resulting image if done in reverse order.

2. If point P is reflected to get point P', draw the axis of reflection.

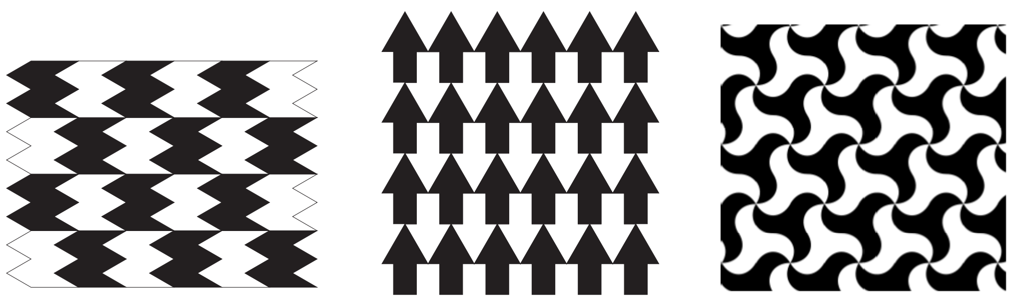


3. If points P and Q are rotated 90° CW to get points P' and Q', draw the point of rotation.



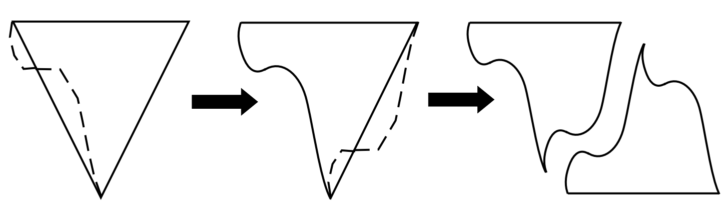
4. Recall that a glide reflection is a combination of a reflection and a translation along the axis of reflection. Draw a shape on grid paper, then reflect it over a horizontal line drawn below it, and then translate it 4 units to the right. Next, start with the same shape, but try doing the translation before the reflection. Why does the order you do the transformations in not matter? Is this fact true for any combination of a reflection and a translation?

5. What types of symmetry are present in the following tessellations? Assume the patterns extend infinitely in all directions.



6. Use the rotation method to create a tessellation with any quadrilateral of your choice.

7. You can create a glide-reflection tessellation by changing one side of a shape, flipping the pattern horizontally and rotating it to the other side of the shape, then fitting the pieces together:



Create your own glide-reflection tessellation.

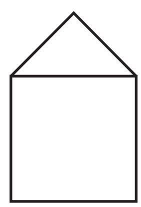
8. Symmetry can be found in the letters of the alphabet as well. Determine which letters have certain types of symmetry. Ignore the trivial case of rotating by any integer multiple of 360°.

	A	B	C	D	E	F	G	H	I	J	K	L	M
Reflectional													
Rotational													
	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Reflectional													
Rotational													

9. Janet is sitting down facing a mirror. Behind her is a clock. In the mirror she sees the image of the clock as seen on the right. What time is it (PM)?

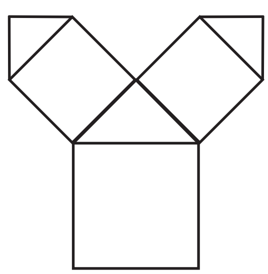
10. One famous fractal is the Pythagoras Tree. To make this fractal, follow the steps below:

(1) Draw a square with a right angle isosceles triangle on top like so:



This is the base shape for this fractal.

(2) Add the base shape to the other two edges of the triangle like so:



(3) Repeat an infinite number of times (or until you get a nice picture)!

11. Try creating an H-fractal! Follow the steps below:

    (1) Draw a horizontal line segment (make it fairly large).

    (2) At both ends of this line, draw a perpendicular line that is half as long as the first line and intersects the first line at its midpoint.

    (3) Repeat step 2 for both the newly added lines.

    (4) Repeat step 3 an infinite number of times!

*12. Another famous fractal is Sierpinski's Triangle. It is created by connecting the middle points of each of the sides of the larger upright triangles to create smaller triangles (ie. make a Tri-Force in each upright triangle).

    (a) Shade in the upside-down triangle. Continue the pattern for one more iteration, shading in the upside-down triangles that you create. (a template is on the back of this page)

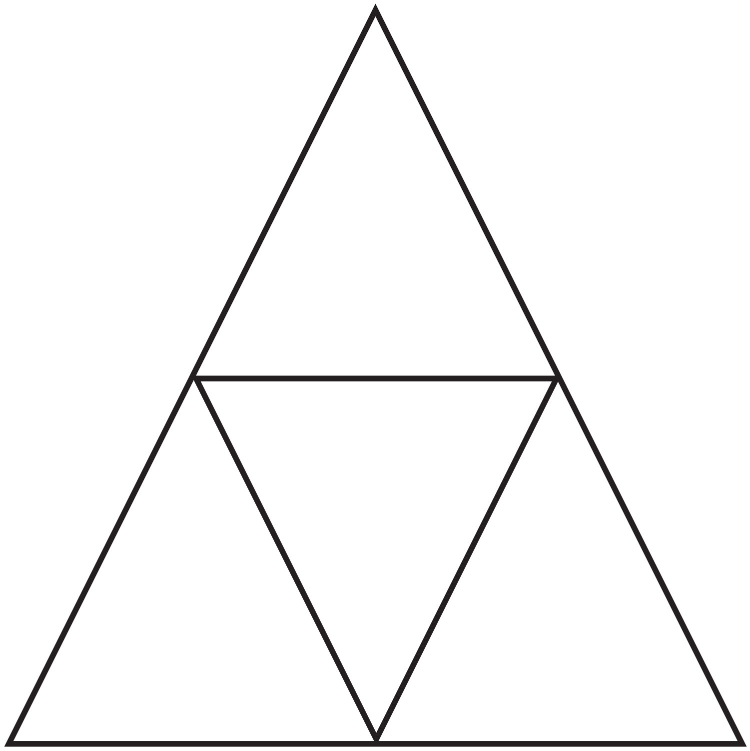
    (b) Record the fraction of the area of the original triangle that is made up of unshaded triangles (FA) in the following chart (for iteration 2):

Iteration	FA
1	3/4
2	
3	
4	

    (c) Continue the drawing the fractal, shading in the upside-down triangles you create as you go. After each iteration, fill in the FA column in the chart.

    **(d) What value goes in the FA column after the nth iteration?

    **(e) What happens to the area of the unshaded triangles after an infinite number of iterations?



*13. If all the side lengths of a shape are equal and all of the interior angles are equal, then the shape is called a regular polygon. For example, a square is a regular polygon.

Which of the following regular polygons can form tessellations?

(a) Equilateral Triangle

(b) Square

(c) Regular Pentagon (5 sides)

(d) Regular Hexagon (6 sides)

Why can you make tessellations out of some shapes and not others?