UNIVERSITY OF
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Waterloo, Ontario N2L 3G1

Enriching Mathematics and
Computer Science for 50 years

Grade 7 & 8 Math Circles

November 7, 2013

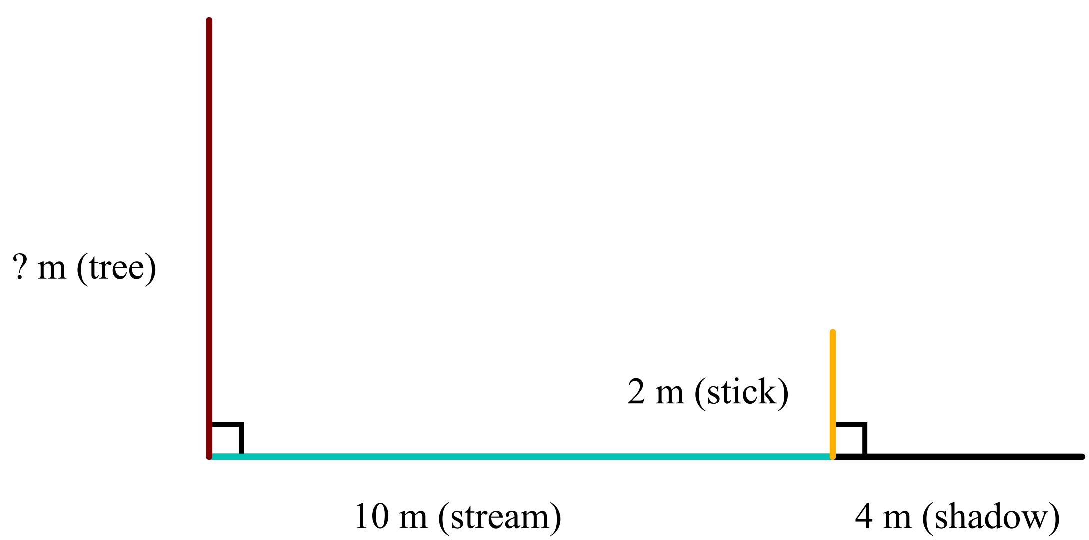
Similarity and Congruences

Intro Problem

Sherlock Holmes has been trying to figure out the height of a tree located on the very edge of a fast moving, 10 m wide stream, which he cannot swim across.

Standing directly across from the tree on the bank on the other side of the stream, Holmes sticks a 2 m long stick at a right angle into the ground. As it is a sunny day, both the stick and the tree cast shadows, and they are perfectly aligned. The shadow cast by the stick is 4 m long from where it stands (away from the stream) and the tip of the shadow cast by the tree perfectly lines up with the tip of the stick's shadow.

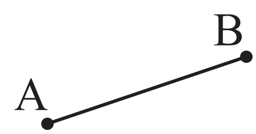
How tall is the tree?



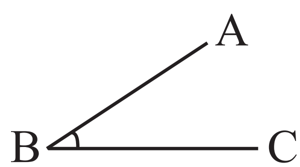
Notation

When dealing with geometry (which often involves diagrams), we need to use good notation so that we don't get confused.

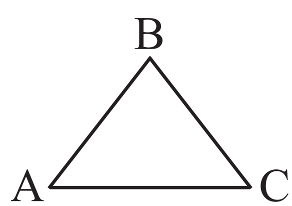
- We represent points with capital letters (such as A)
- We represent line segments with two letters, such as AB, which represent the endpoints of the segment



- The length of a segment is denoted with two vertical bars surrounding the segment's name (e.g. length of AB is written $|AB|$)
- To denote angles, we write the angle created by joining three letters (**order matters**); for example, $\angle ABC$ represents

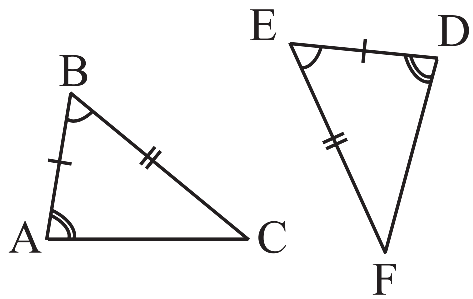


- We write $\triangle ABC$ to mean the triangle created by connecting the points A, B, and C



  - For polygons with more than 3 sides, we would use more letters

- If we have multiple polygons, we use ticks to indicate if certain sides have equal length, and we use arcs to indicate if certain angles have equal measure



In the above diagram,

$$|AB| = |DE| \qquad |BC| = |EF|$$

$$\angle ABC = \angle DEF \qquad \angle BAC = \angle EDF$$

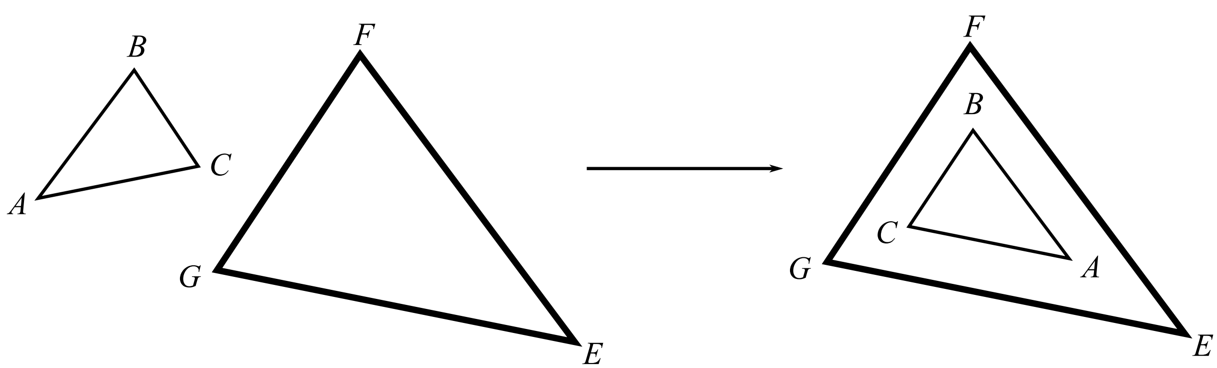
Note: It is important that we always match vertices. For example, it is better to write $\angle ABC = \angle DEF$ then to write $\angle ABC = \angle FED$, since vertex A looks like vertex D, B looks like E, and C looks like F.

Similarity

Two polygons are said to be **similar** if one is a **scaled** copy of the other (either shrunken or enlarged). Informally, they have the same shape and same interior angles, but not necessarily the same size or orientation.

Example

The two triangles $\triangle ABC$ and $\triangle EFG$ are **scaled** copies of one another.



Hence they are similar. We use the symbol "$\sim$" to mean "is similar to".

Thus

$$\triangle ABC \sim \triangle EFG$$

Note: Notice that we matched vertices when comparing these triangles.

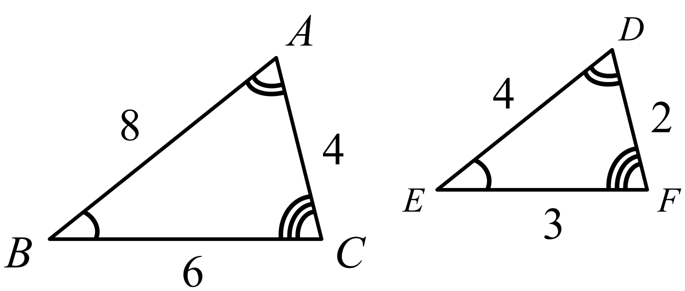
Facts about Similarity

If two polygons are similar, then the following facts are true:

- Corresponding angles in the two polygons are equal
- The ratios of corresponding sides are equal (the corresponding sides are *in proportion*)

We will study similarity of triangles in detail because triangles are the simplest polygons, and because more complicated polygons can be "built" from many triangles. That is, if you understand similar triangles, you can understand similar polygons in general.

Example
If $\triangle ABC \sim \triangle DEF$ then



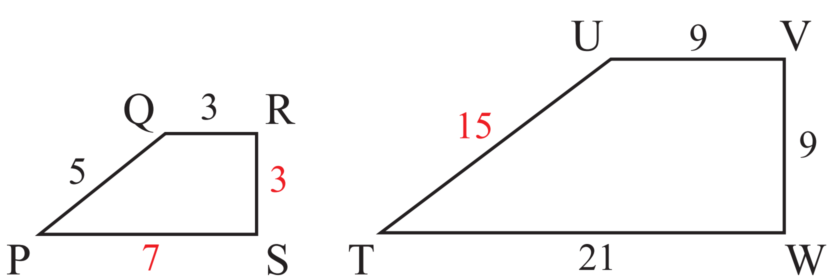
$$\angle ABC = \angle DEF \qquad \frac{|AB|}{|DE|} = \frac{8}{4} = 2$$

$$\angle BAC = \angle EDF \qquad \frac{|BC|}{|EF|} = \frac{6}{3} = 2$$

$$\angle ACB = \angle DFE \qquad \frac{|CA|}{|FD|} = \frac{4}{2} = 2$$

Example

Given that the quadrilaterals below are similar, find the lengths of all the edges of both shapes.



Solution

Since trapezoid $PRQS \sim$ trapezoid $TUVW$, the ratios of the lengths of corresponding sides are equal. That is,

$$\frac{|PQ|}{|TU|} = \frac{|QR|}{|UV|} = \frac{|RS|}{|VW|} = \frac{|PS|}{|TW|}$$

We are given that $\dfrac{|QR|}{|UV|} = \dfrac{3}{9} = \dfrac{1}{3}$. Thus, we get the following equations for the unknown side lengths:

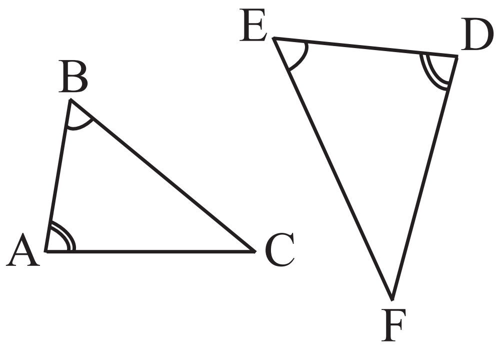
$$\frac{5}{|TU|} = \frac{1}{3} \qquad \frac{|RS|}{9} = \frac{1}{3} \qquad \frac{|PS|}{21} = \frac{1}{3}$$

Solving the equations gives us the values marked in red on the diagram.

Similarity Tests

Angle-Angle Similarity (AA)

If two triangles $\triangle ABC$ and $\triangle DEF$ have **at least two** equal corresponding interior angles, then $\triangle ABC \sim \triangle DEF$.



① $\angle BAC = \angle EDF$ (diagram)

② $\angle ABC = \angle DEF$ (diagram)

Therefore, $\triangle ABC \sim \triangle DEF$ (AA)

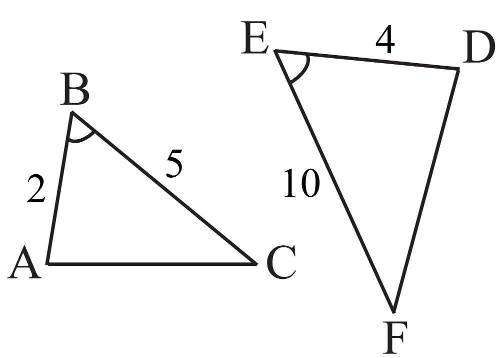
$\square$

Notes:

- Proofs are made of true statements (give your reasoning for the statement) followed by a logical conclusion (a therefore statement). See the structure of the proof above.
- When we complete a proof, we put a little box $\square$ at the end of our solution to show that the proof is finished. You can think of it as a reward for your hard work!!

Side-Angle-Side Similarity (SAS)

If two triangles $\triangle ABC$ and $\triangle DEF$ have an equal angle, and if the ratio of the corresponding sides containing the angles are equal, then $\triangle ABC \sim \triangle DEF$.



① $\angle ABC = \angle DEF$ and are *contained.* (diagram)

② $\dfrac{|AB|}{|DE|} = \dfrac{2}{4} = \dfrac{1}{2}$ (diagram)

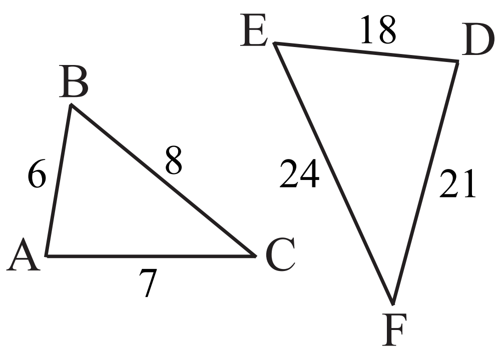
③ $\dfrac{|BC|}{|EF|} = \dfrac{5}{10} = \dfrac{1}{2}$ (diagram)

Therefore, $\triangle ABC \sim \triangle DEF$ (SAS)

$\square$

Side-Side-Side Similarity (SSS)

If two triangles $\triangle ABC$ and $\triangle DEF$ have all their corresponding sides in equal ratio (that is $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|CA|}{|FD|}$) then $\triangle ABC \sim \triangle DEF$.



① $\frac{|AB|}{|DE|} = \frac{6}{18} = \frac{1}{3}$ (diagram)

② $\frac{|BC|}{|EF|} = \frac{8}{24} = \frac{1}{3}$ (diagram)

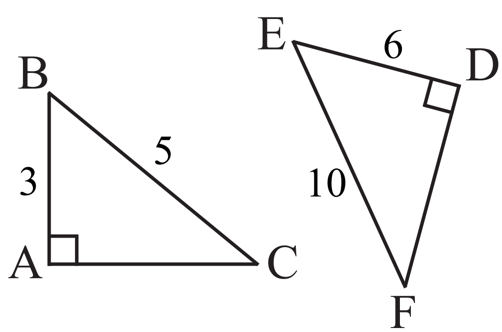
③ $\frac{|CA|}{|FD|} = \frac{7}{21} = \frac{1}{3}$ (diagram)

Therefore, $\triangle ABC \sim \triangle DEF$ (SSS)

$\square$

Side-Side-Right Angle Similarity (SSRA)

For two **right-angled** triangles $\triangle ABC$ and $\triangle DEF$, if they have two pairs of corresponding sides in equal ratio, then $\triangle ABC \sim \triangle DEF$.



① $\angle BAC = \angle EDF = 90^\circ$ are *right angles*. (diagram)

② $\frac{|BA|}{|ED|} = \frac{3}{6} = \frac{1}{2}$ (diagram)

③ $\frac{|BC|}{|EF|} = \frac{5}{10} = \frac{1}{2}$ (diagram)

Therefore, $\triangle ABC \sim \triangle DEF$ (SSRA)

$\square$

Notes:

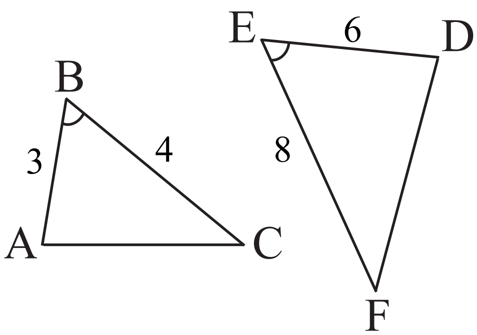
- The SSRA similarity test is not the same as the SAS similarity test! The right angle in SSRA does not have to be contained within the sides in proportion.
- The SSRA test deals with a special case of when we know two pairs of sides in proportion and one angle that is not contained by those sides. In general, this is not enough information to figure out whether the triangles are similar. But if the angle is a right angle, we can use the SSRA test.

Example

Given two triangles $\triangle ABC$ and $\triangle DEF$, if $\angle ABC = \angle DEF = 60°$ and $|AB| = 3$, $|BC| = 4$, $|DE| = 6$ and $|EF| = 8$, **prove** (justify your reasoning) that they are similar.

Solution

First, we can sketch the two triangles based on the information we are given (diagram is not drawn to scale).



This looks like the SAS case!

Formally, we write our proof as follows:

1 $\frac{|AB|}{|DE|} = \frac{3}{6} = \frac{1}{2}$ (given/diagram)

2 $\frac{|BC|}{|EF|} = \frac{4}{8} = \frac{1}{2}$ (given/diagram)

3 $\angle ABC = \angle DEF$ and are *contained.* (given/diagram)

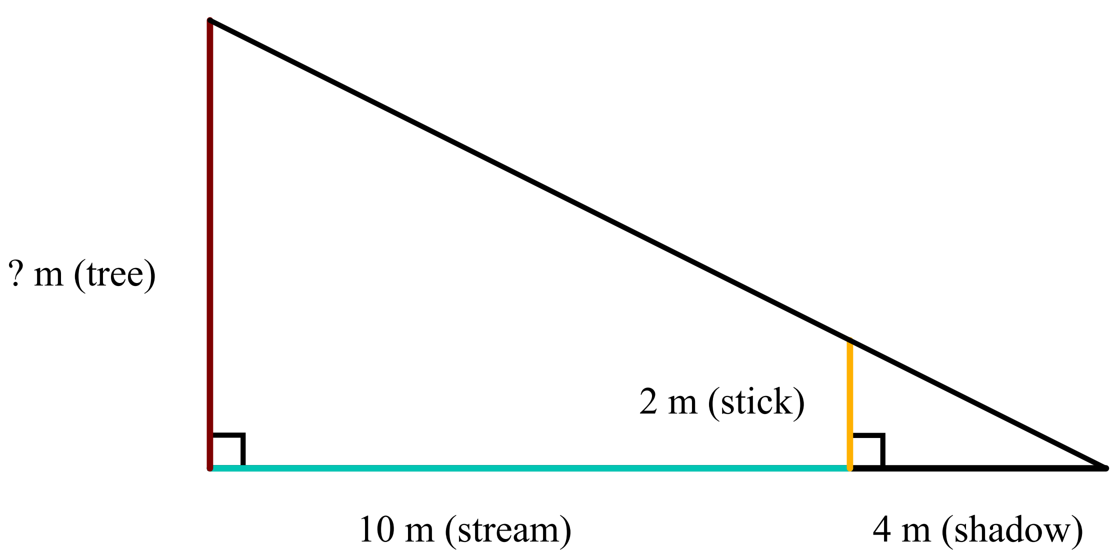
Therefore, $\triangle ABC \sim \triangle DEF$ (SAS).

$\square$

Return to Intro Problem

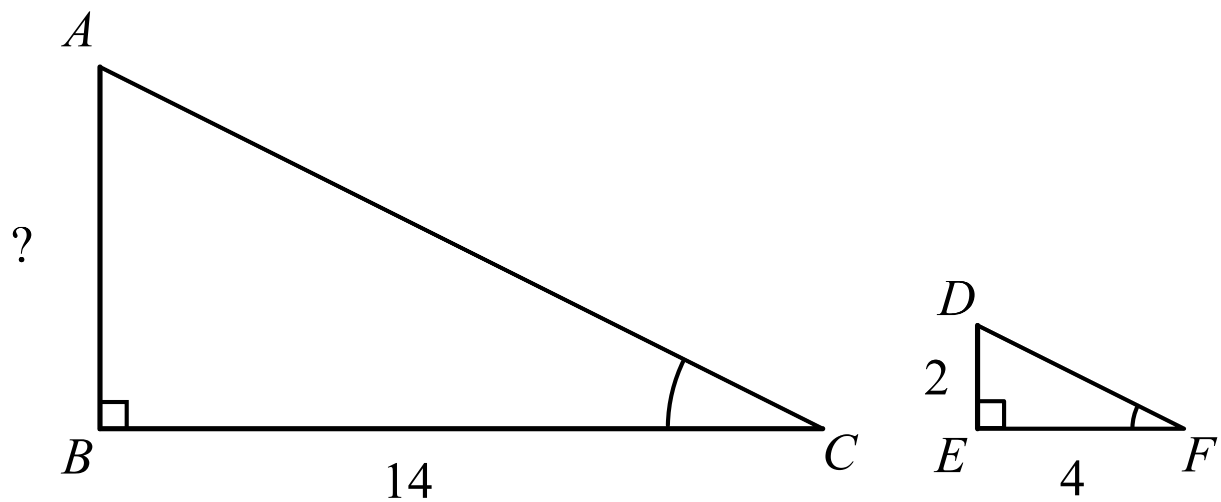
We now have the tools required to solve this problem.

We can connect the top of the tree to the top of the stick and the end of the shadows to make two triangles as seen below.



If only we could show that these triangles are similar, then we could use ratios to solve for the height of the tree.

Fortunately, we can show that these triangles are in fact similar using one of our tests! First, we draw them separately and label their vertices as seen below.



Then we can prove that they are similar as follows:

1 $\angle ABC = \angle DEF$ (diagram)

2 $\angle ACB = \angle DFE$ (diagram)

Therefore, $\triangle ABC \sim \triangle DEF$ (AA)

And since $\triangle ABC \sim \triangle DEF$, by properties of similarity, we have that

$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$$

So using the lengths given in the diagram, we get that

$$\frac{|AB|}{2} = \frac{14}{4}$$
$$\frac{|AB|}{2} = \frac{7}{2}$$
$$\left(\frac{|AB|}{2}\right)(2) = \left(\frac{7}{2}\right)(2)$$
$$|AB| = 7$$

All of our lengths were in units of metres, so our answer will also be in units of metres. Therefore, the tree is 7 metres tall.

Congruence

Remember that if two polygons are similar, then both polygons have the same interior angles, but not necessarily the same side lengths.

If similar polygons also have the same side lengths, they are said to be **congruent**. We use the symbol "≅" to show that two polygons are congruent.

Facts about Similarity

If two polygons are congruent, then the following facts are true:

- Corresponding angles in the two polygons are equal
- Corresponding side lengths in the two polygons are equal
- The polygons are also similar

Once again, we will focus on triangles in this lesson, because if we can show that two triangles are congruent or similar, then we can show that any polygon is congruent of similar by breaking it up into many triangles.

Congruent Triangles

In the case of triangles, if $\triangle ABC$ is congruent to $\triangle DEF$, then we write

$$\triangle ABC \cong \triangle DEF$$

To determine if two triangles are congruent:

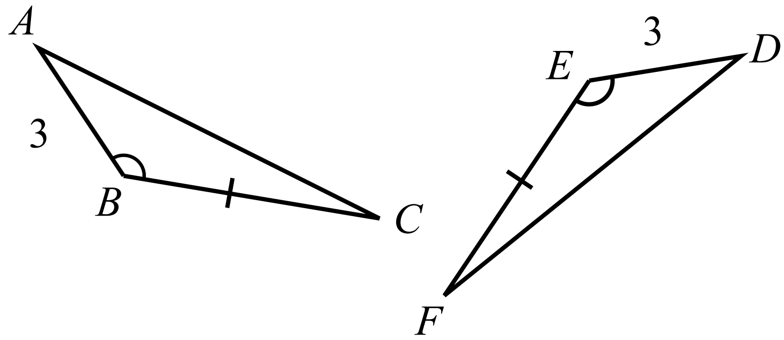
(1) Show they are similar (using the similarity tests)

(2) Show that a pair of corresponding sides are equal in length

This can be thought of as our congruence test.

Example

Are these two triangles similar? Are they congruent?



Solution

The triangles look like they might be congruent, so we will try to prove that they are. But first we must show that they are similar.

① $\dfrac{|DE|}{|AB|} = \dfrac{3}{3} = 1$ (diagram)

Since no length is specified for sides FE and CB, we can give them a variable length, say a. We give them the same length because of the ticks in the diagram.

② $\dfrac{|FE|}{|CB|} = \dfrac{a}{a} = 1$ (diagram)

③ $\angle ABC = \angle DEF$ and is *contained.* (diagram)

Therefore, $\triangle ABC \sim \triangle DEF$ (SAS)

Furthermore, $|AB| = |DE|$ (diagram)

Therefore, it is true that $\triangle ABC \cong \triangle DEF$

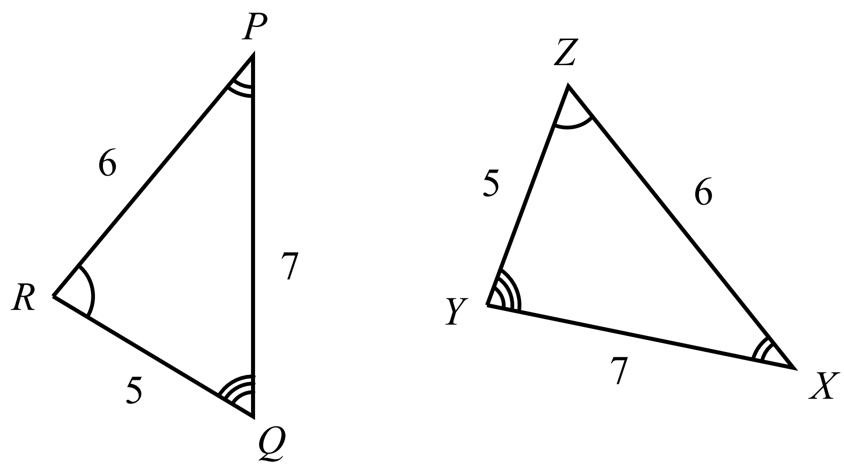
$\square$

Using Congruence

Congruence is a very useful tool. If we know that two triangles are congruent, and we know the side lengths and angles of one triangle, then we immediately know the corresponding side lengths and angles of the other triangle.

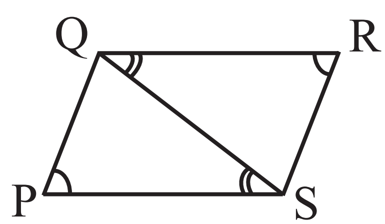
Example

If $\triangle PQR \cong \triangle XYZ$, and $\triangle PQR$ is as shown below, draw and label the side lengths and angles of $\triangle XYZ$ (the example is completed for you).



Example

Given the following diagram, prove that $|PS| = |QR|$



Solution

To prove this result, our strategy will be to first show that $\triangle SPQ \cong \triangle QRS$.

Step 1: Show that the triangles are similar:

1 $\angle SPQ = \angle QRS$ (diagram)

2 $\angle PSQ = \angle RQS$ (diagram)

Therefore, $\triangle SPQ \sim \triangle QRS$ (AA)

Step 2: Show that the triangles are congruent:

$|SQ| = |QS|$ (of course!) which means that one pair of corresponding sides are equal in length.

Therefore, $\triangle SPQ \cong \triangle QRS$ (congruence test)

Step 3: Use properties of congruence:

Since PS and QR are corresponding sides of congruent triangles, by properties of congruence, we must have that $|PS| = |QR|$

$\square$

Example

Given that $\triangle ABC \sim \triangle DEF$, $|BC| = 4$ cm, and $|EF| = 8$ cm, **prove** that $\triangle ABC$ is not congruent to $\triangle DEF$ (ie. $\triangle ABC \ncong \triangle DEF$.

Solution

It seems that the triangles are not similar, because BC and EF are corresponding sides and do not have equal length. How do we prove this formally? We can use a **proof by contradiction**.

1 Assume that $\triangle ABC \cong \triangle DEF$

2 Then, since BC and EF are corresponding sides of congruent triangles, it is true that $|BC| = |EF|$. (by properties of congruence)

3 $|BC| = 4 \text{ cm} \neq 8 \text{ cm} = |EF|$ (given)

Statement 3 contradicts statement 2, so one of the above statements must be incorrect. Since the only statement that was not justified was statement 1 (the assumption), then 1 must be false.

Therefore, the opposite of 1 is true: $\triangle ABC \ncong \triangle DEF$

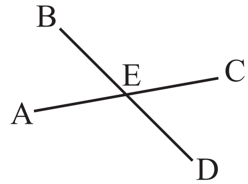
$\square$

Note: The method of proof by contradiction is a powerful tool that can be used to do many proofs. It may come in handy on the problem set (hint hint)! The strategy is always to assume that the opposite statement of what you are trying to prove is true, then use one or more true statements (you must justify these) to show a contradiction somewhere. Because you justified all of the true statements, it is only logical that the assumption was false. This means that the statement you are trying to prove is true!

Problem Set

*difficult **challenge ***Mad Skillz

Here are a couple of results that may be helpful in the completion of this problem set:



Opposite angles

Angles that are opposite each other when two lines cross are always equal in measure.

In the diagram above, $\angle BEC = \angle AED$ and $\angle BEA = \angle CED$

Angles along a straight line

Two or more adjacent angles that lie along a straight line must add up to $180°$.

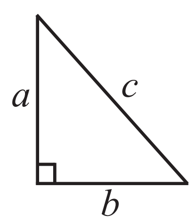
In the diagram above, $\angle BEC$ and $\angle CED$ are adjacent and lie along the straight line segment BD. Therefore, $\angle BEC + \angle CED = 180°$

See if you can spot the three other pairs of angles in the diagram with the same property.

Pythagorean Theorem

Given a **right-angled** triangle with hypotenuse of length c and legs of length a and b, it is true that

$$\boxed{c^2 = a^2 + b^2}$$



Reminders/Notes

- Always draw a picture (and use SCRAP PAPER)
- When a question asks you to *explain* or *prove* something, don't just right down the answer, explain or prove it!
- There may be more than one way to solve a particular question. Don't worry if your method isn't exactly the same as the method found in the solutions. It's a good method as long as it's correct!

1. In each of the following cases, **prove** whether or not $\triangle ABC$ is similar to $\triangle DEF$. Make sure that you show reasons for each step.

   (a) Both triangles are equilateral.

   (b) $\triangle ABC$ is obtuse and $\triangle DEF$ is acute.

   (c) $|AB| = |DE|$, $|AC| = |DF|$, and $\angle BAC = \angle EDF$.

   (d) All the edges of $\triangle DEF$ are two times as long as the corresponding edges of $\triangle ABC$.

2. In each of the following cases, **prove** whether or not $\triangle ABC$ is congruent to $\triangle DEF$. Make sure that you show reasons for each step.

   (a) $|AB| = 5$ cm, $|BC| = 3$ cm, and $|CA| = 4$ cm. $|DE| = 5$ cm, $|EF| = 3$ cm, and $|FD| = 40$ mm.

   (b) $|AB| = |DE|$, $|CA| = |FD|$, and $\angle ABC = \angle DEF$ are right angles.

   (c) $\angle BCA = \angle EFD$ are right angles. $|BC| = 4$ m, $|AC| = 5$ m, $|EF| = 4$ m, and $|DF| = 10$ m.

   (d) $\angle CAB = \angle BCA = 45°$, and $|CA| = 102$ mm. $\angle FDE = \angle EFD = 45°$, and $|FD| = 0.102$ m. Also, what kind of triangles are $\triangle ABC$ and $\triangle DEF$?

3. Prove that any two regular polygons with the same number of edges are similar.

4. **The ambiguous case.** Consider the SSA situation where you know that two triangles, $\triangle ABC$ and $\triangle DEF$, have two pairs of congruent edges and one pair of corresponding non-included congruent angles.

   Suppose that $|AB| = |DE| = 5$ cm, $|BC| = |EF| = 4$ cm, and $\angle CAB = \angle FDE = 45°$. If $\triangle ABC$ is not congruent to $\triangle DEF$, draw the two triangles.

   Hint: Use a compass to help draw the possible orientations for the edges BC and EF.

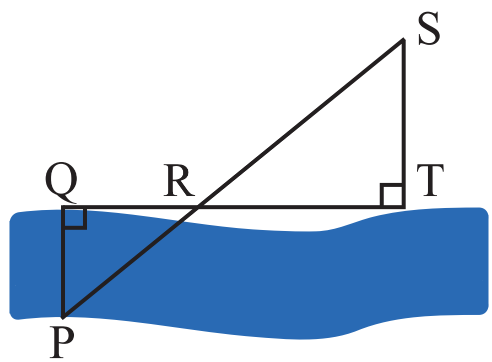
5. (a) A right triangle is scaled by a factor of f (ie. its edges are scaled by a factor of f). What factor is the area of the triangle scaled by? Explain your reasoning.

   (b) What factor is the volume of right triangular prism scaled by when its edges are scaled by a factor of f? Explain your reasoning.

   **(c) Explain why the results from (a) and (b) hold for any triangle or triangular prism. Do they hold for any polygon or any prism with a polygonal base?

6. * You want to determine the height of a flagpole. You have a friend place a small mirror on the ground so that you can see the reflection of the top of the flagpole. Your friend then measures the distance from you to the mirror and finds that it is 2 feet. The distance from the mirror to the flagpole is 7 feet. If your eyes are 5 feet above the ground, how high is the flagpole? **Hint:** Assume you and the flag pole both make a right angle with the ground.

7. Bill wants to find out how wide the Grand River is but he doesn't want to get wet, so he uses rocks and trees as landmarks and makes the following measurements:
   $|QR| = 7$ m, $|RT| = 10.5$ m, and $|TS| = 9$ m.

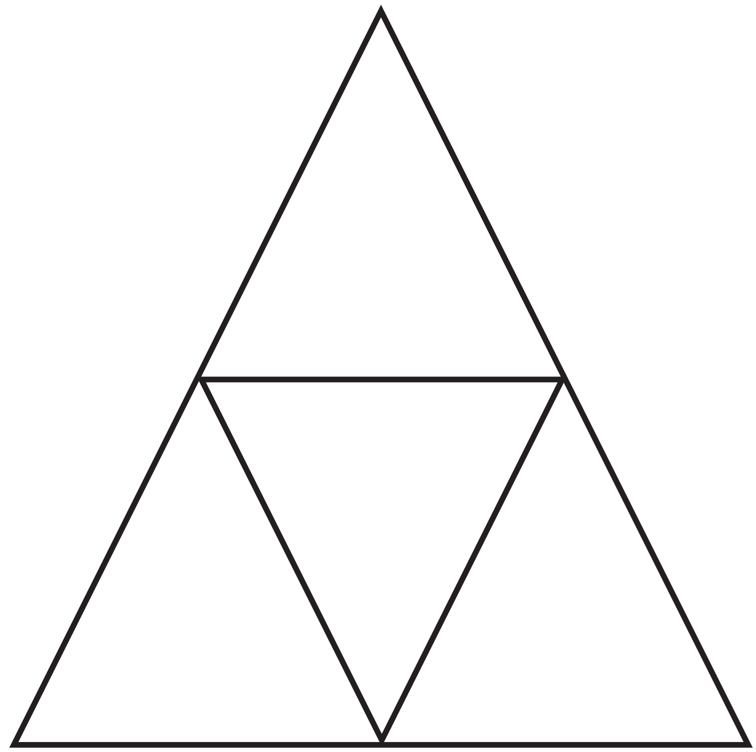
   How wide is the Grand River between points P and Q?

8. An interesting application of similarity is the study of fractals. Fractals are geometric figures that are created by iteration (a repeated process). They have often contain similar shapes within themselves or patterns that repeat in multiple levels of magnification.

   Consider the famous fractal, Sierpinski's Triangle (it's a repeating Tri-Force, for those who enjoy video games). It is created as follows:

   i. Start with an equilateral triangle.
   ii. Connect the mid-points of each of the sides of the triangle to create an inverted equilateral triangle.
   iii. Connect the mid-points of each of the sides of the larger upright triangles to create smaller inverted triangles.
   iv. Repeat step iii over and over again!

   There is a template for you on the next page. Steps i and ii have already been completed.



(a) Shade in the upside-down triangle. Continue to construct the fractal for a few more iterations, shading in the upside-down triangles that you create.

(b) The triangles created in each iteration are similar to the triangles from previous iterations. What is the scale factor between triangles from one iteration to the next?

(c) By what factor does the area of the triangles scale by from one iteration to the next?

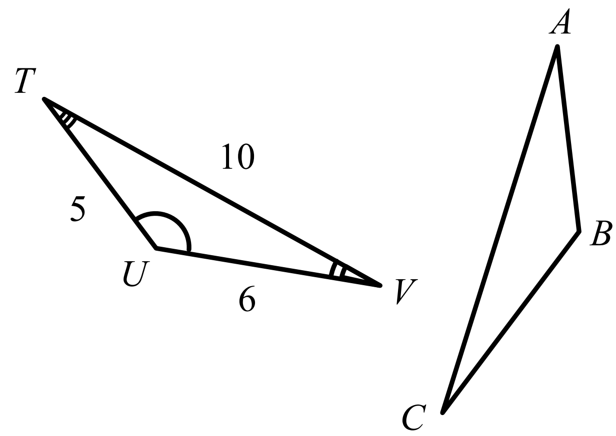
(d) How many unshaded (right-side up) triangles are created out of one unshaded triangle in any given iteration?

*(e) What fraction of the area of the original triangle is made up of unshaded triangles after the nth iteration?

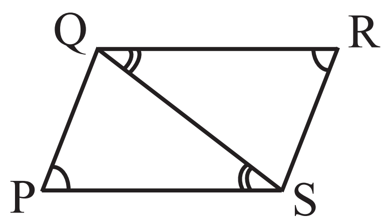
*(f) What happens to the area of the unshaded triangles after an infinite number of iterations?

9. * A research team wishes to determine the altitude of a mountain as follows: They use a laser beam, mounted on a structure of height 2 meters, to shine a beam of light through the top of a pole and then through the top of the mountain. The height of the pole is 20 meters. The distance between the mountain and the pole is 1000 meters. The distance between the pole and the laser is 10 meters (assume that the laser, pole, and the mountain all lie on a straight line). Find the height h, of the mountain.

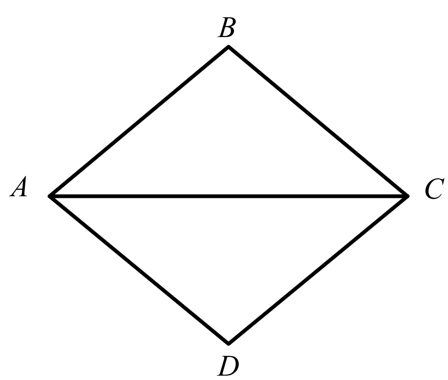
10. If $\triangle ABC \cong \triangle TUV$ in the following diagram, fill in the side lengths and label the corresponding angles (use arcs) of $\triangle ABC$.



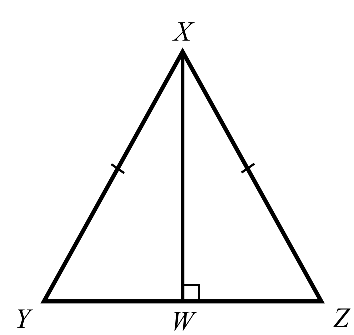
11. Given the following diagram, prove that $PQ = RS$



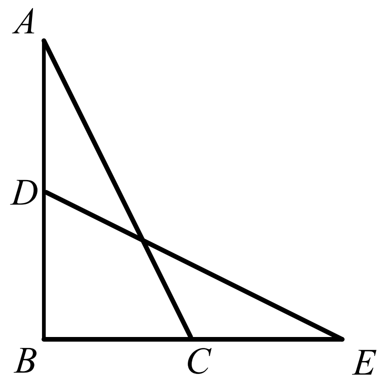
12. In the following diagram, $|BC| = |CD|$ and $\angle BCA = \angle DCA$. Prove that $\triangle ABC \cong \triangle ADC$.



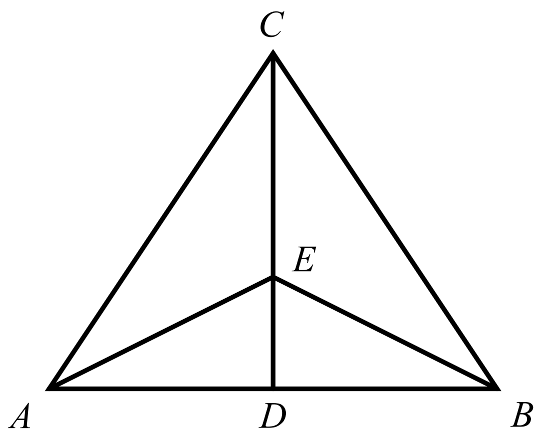
13. In the following diagram, prove that $\angle XZY = \angle XYZ$



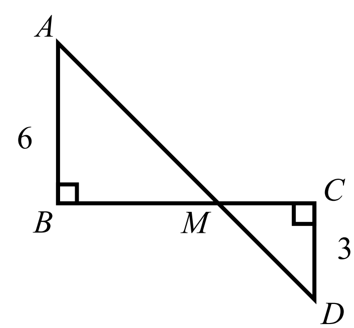
14. If $\angle BAC = \angle DEB$ and $|AB| = |BE|$, prove that $|AD| = |EC|$.



15. ** If $\angle AED = \angle BED$ and $\angle ACD = \angle BCD$, prove that $\angle EAD = \angle EBD$.



16. ** Given the following diagram, if $AD = 15$, find the perimeter of $\triangle ABM$ and $\triangle DCM$.



17. *** In the following diagram, $|AD| \times |AC| = |AE| \times |AB|$. Show that $\angle CDE + \angle CBE = 180°$ and $\angle ADB + \angle BEC = 180°$.