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Mathematics and Computing

Grade 7/8 Math Circles

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Coordinate Systems

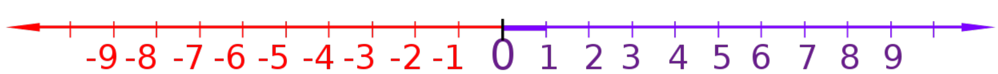
Introduction

Coordinate systems decribe location in *a space*. Specifically, we want them to describe any location in a space that we care about. The common spaces you should be used to thinking about are 1D, 2D, and 3D spaces.

Coordinate systems use variables: the same number of variables as there are dimensions in the space. They allow for graphing equations and looking at them *qualitatively* (as we've done before in Math Circles). Doing math using coordinate systems helps us understand the math we're doing, and gives us a picture of theories and equations. Coordinate systems are incredibly important to physics, which uses math to understand and quantify the real world.

But what makes a good coordinate system? And how can we use them to help us do more and useful math? Let's take a look.

The Number Line



Before we go much further, let's stop for a moment and think about the number line. The number line is a coordinate system. In particular, it's a one-dimensional, 1D, coordinate system. Remeber that a coordinate system describes any location in a space. A 1D space only needs 1 variable to be able to describe any location properly. If we only have one dimension that we care about, then the number line is the best coordinate system you can ask for: it covers every point in 1D space, and it does this by using one variable.

As great as this is, drawing the whole number line every time you just want to talk about one dimension can get pretty annoying. So, before we learn about more coordinate systems, we're going to talk about a better way to draw them.

Quick Aside: Vectors

The regular number you know about in general are called ***scalars***. Scalars have only a size. multiplying by a scalar *scales* the size of whatever you are multiplying.

A ***vector*** has both a size and a direction. As variables they are written with an arrow on top of the variable, ex. $\vec{a}$. Vectors have endless applications in physics and math. They are one of the most important tools for understanding the universe. One of their uses is in coordinate systems.

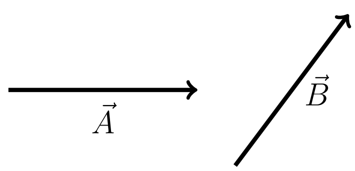
Drawing Vectors

Vectors are drawn using arrows, since they also have a size and direction.

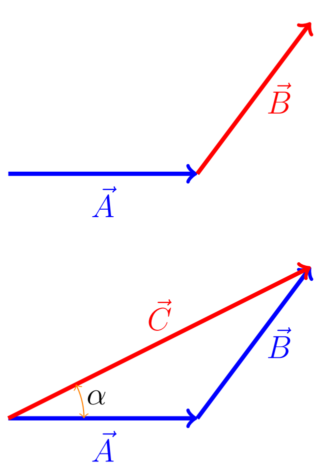
Size is represented by how long the arrow is.

Direction is represented by which way the arrow is pointing.

Vector Operations: addition

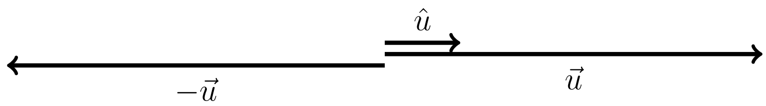


Adding vectors together is as simple as following the arrows. If I want to add these two vectors together, then I need to follow one, and then the other. Try to think about how to subtract vectors. For more details, see the Grade 6 Week 1 Fall 2018 Math Circles lesson.



Unit vectors

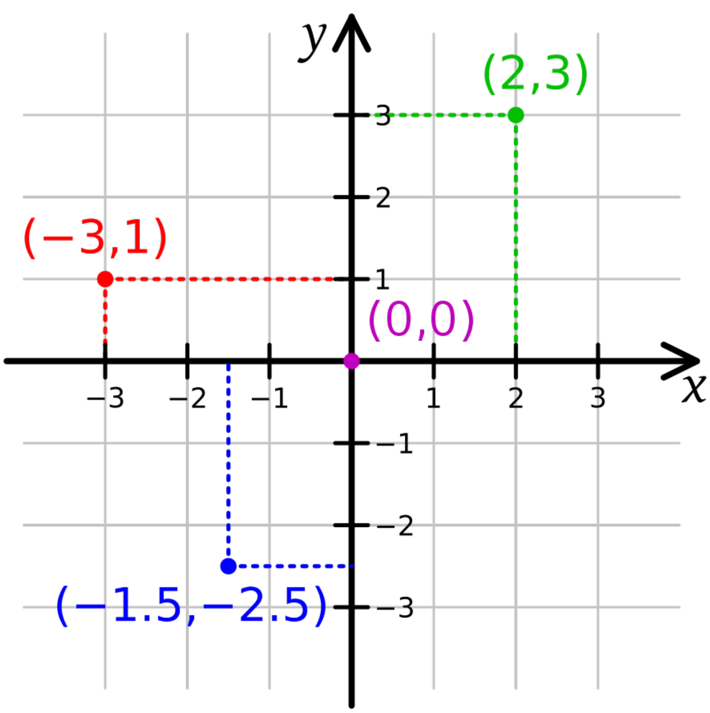
A vector with a size of 1 unit is called a ***unit vector***. As a variable, it is written with a hat on top of the symbol, ex. $\hat{a}$. Any vector at all can be written as (a scalar) $\times$ (a unit vector). For example:



In the above diagram, $\vec{u} = 5\hat{u}$, and $-\vec{u} = -5\hat{u}$.

2D Cartesian Coordinates

Now that we've learned about the most basic coordinate system, for a 1D space, and learned a bit about vectors, let's review another familiar coordinate system: the 2D Cartesian coordinate system.



You can see that the Cartesian coordinate system uses 2 variables to describe any location in the 2D space.

The point (2,3) in the above example tells us 2 things: *how far to go* and *where*. Seeing (2,3), we know to go 2 units along the x axis, and 3 units along the y axis. The information the coordinate gives us is size and direction. This is perfect for using vectors. The 2D Cartesian grid can be defined in terms of unit vectors, $\hat{x}$ and $\hat{y}$. This is a lot more useful, because the axes are not what really matter here. What matters is the direction these axes go, and how long **one unit** is. Both these things are best represented by unit vectors.

Using $\hat{x}$ and $\hat{y}$ as the way we understand the coordinate system, we can also have a better way of *writing coordinates*. Rather than using (2,3) as a coordinate, it can be a lot more helpful to write this position on the grid as a *vector addition*. This way, the point (2,3) can be written as $2\hat{x} + 3\hat{y}$. This tells us the exact same thing (2,3) does, but a lot more clearly: go 2 units along x, and 3 units along y, and we know exactly what directions these are. Thinking of coordinate systems like this will also help understand some of the other coordinate systems we'll look at in this lesson.

Note: For any coordinate system, the point (0,0) is called the ***origin***: it's where all other points are referenced from.

(Aside)

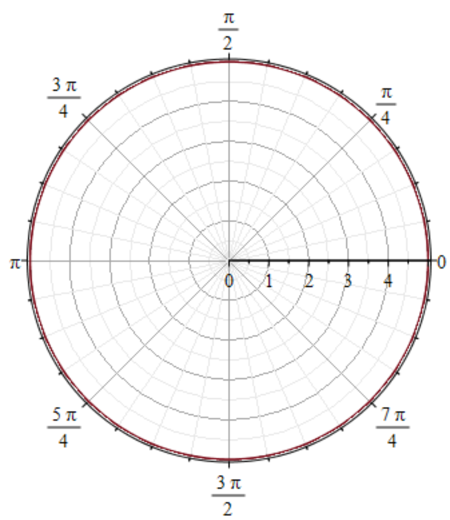
What makes the x and y axes a good set of axes? In fact, any two unit vectors that are not parallel to each other would **span** the 2D space. The important feature of a good coordinate system is that any point in the space you care about can be reached using the axes you decide on, and for this, the axes don't need to be perpendicular. In fact, it can sometimes be more useful to have axes that are not perpendicular. In general though, we use perpendicular axes because they are the easiest to visualize coordinates in, and it makes more of the math we do easier.

Polar Coordinates

How would you describe a circle in 2D Cartesian coordinates? How easy is it? Well, here's the equation:

$$x^2 + y^2 = r^2$$

where r is the radius of the circle you want. This describes a circle with its center on the origin. After choosing a radius for your circle, as long as the coordinates you have satisfy this equation, then the point will be on the circle. Cool. But this equation is annoying. You have to look at two variables, square them, compare them with the square of another number after adding them, there's a lot to do. There's especially a lot to do for something that you find everywhere in nature. Circles are used a lot in applied math, to talk about circular motion, spinning, etc., and so we care about being able to talk about them easily.



Polar coordinates do not use x and y. Instead, we choose a *principle axis*, and look at 2 things: radius and angle. We will use $\hat{r}$ and $\hat{\phi}$ to talk about these (ϕ is the greek letter "*phi*", used often for angles in math and physics). The variables of this coordinate system, then, are r and ϕ. The space that we're looking at is still 2D, so we only need 2 variables. The principle axis is the line where $\phi = 0$.

Values of ϕ are given in ***radians***. Radians are another way to measure angles. To write a point as a vector addition in polar coordinates, we need to use radians for ϕ. The important radian values are in the diagram above. Where on the diagram would the point $4\hat{r} + \frac{\pi}{2}\hat{\phi}$ be?

Looking at this coordinate system, notice that:

- Are the unit vectors perpendicular?

- Does this span 2D space?
- How would you write a coordinate as a vector addition?
- What do the unit vectors look like as you move to different points in the system?

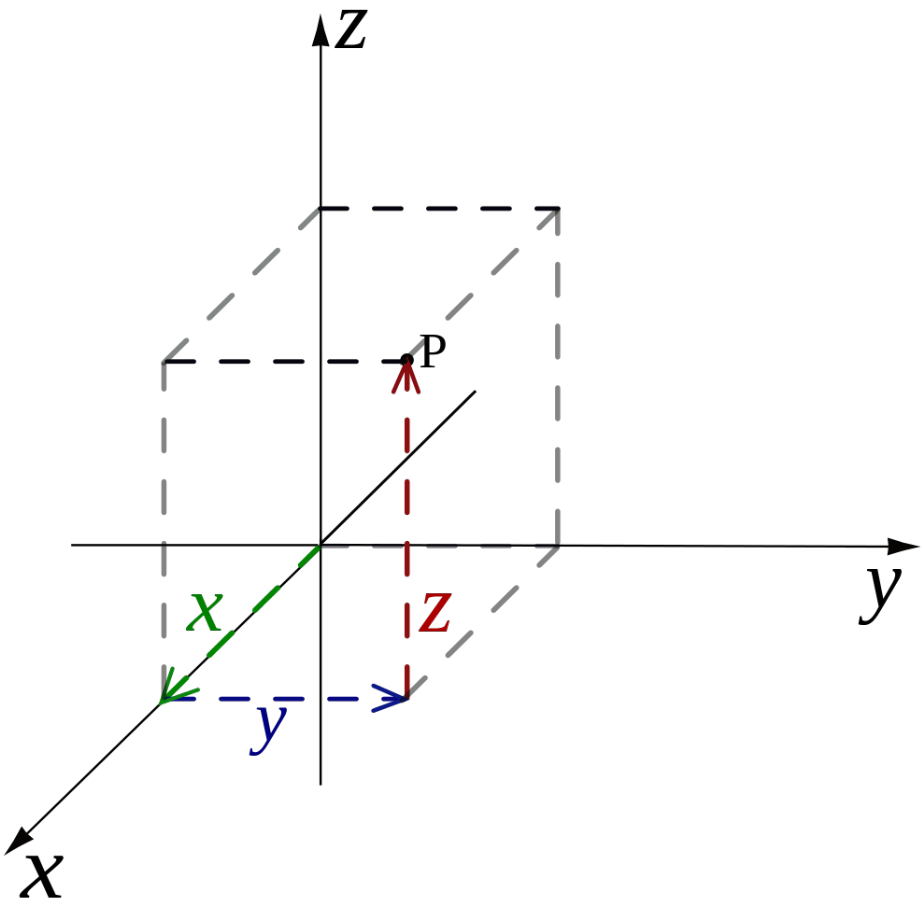
Using polar coordinates makes graphing a circle incredibly easy. In the 2D Cartesian coordinate system the equation of a circle of radius 4 is $x^2 + y^2 = 4^2 = 16$. In the polar coordinate system, the equation for the same circle of radius 4 is just $r = 4$. Try drawing this circle in the polar grid above.

3D Cartesian Coordinates

Let's consider adding dimensions. We'll just do 3 dimensions in this lesson, since that's all we can draw. Now we have x, y, and z, so our coordinates look like (x,y,z). We now know where the point (3,2,4) is like this: 3 units along x, 2 units alond y, and 4 uints along z. Again, this is most best represented with unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. The space that we care about now is 3 dimensional space, like the world we live in.

Looking at this coordinate system, notice that:

- Are the unit vectors perpendicular?
- Does this span 3D space?
- How would you write a coordinate as a vector addition?
- What do the unit vectors look like as you move to different points in the system?



Polar Coordinates in 3D

Just like with 2D Cartesian coordinates, 3D Cartesian coordinates are also very bad at representing circles easiy. So, how can we extend polar coordinates to 3 dimensions? Let's look:

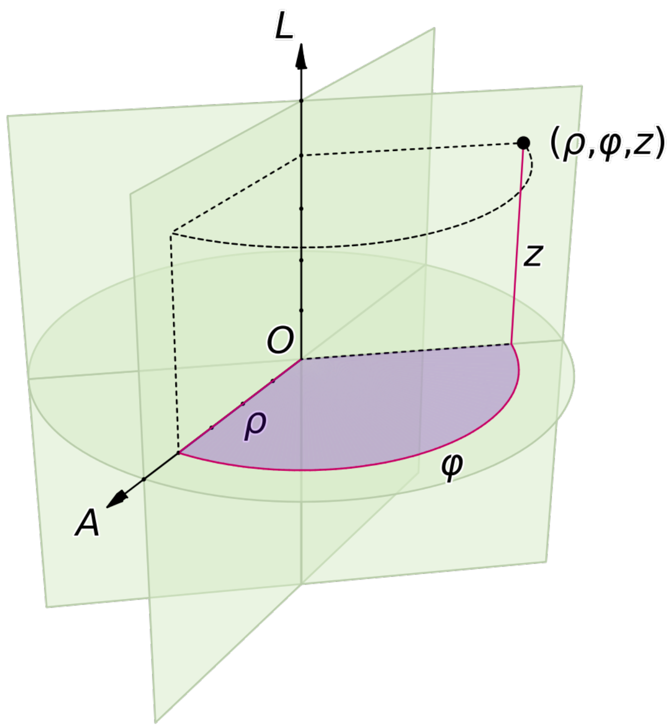
Cylindrical Coordinates

An important 3D circle-based structure we care about is cylinders. Cylindrical coordinates have unit vectors $\hat{r}$, $\hat{\phi}$, and $\hat{z}$. It works in the same way as polar coordinates, except instead of r being the distance from the *origin*, r is now the distance from the z-axis.

Looking at this coordinate system notice that:

- Are the unit vectors perpendicular?
- Does this span 3D space?
- How would you write a coordinate as a vector addition?
- What do the unit vectors look like as you move to different points in the system?

Try plotting some (r, ϕ, z) coordinates.



Note: r is sometimes replaced with ρ, the greek letter "rho", in 3 dimensions.

$r = 4$ here is a cylinder of radius 4.

Spherical Coordinates

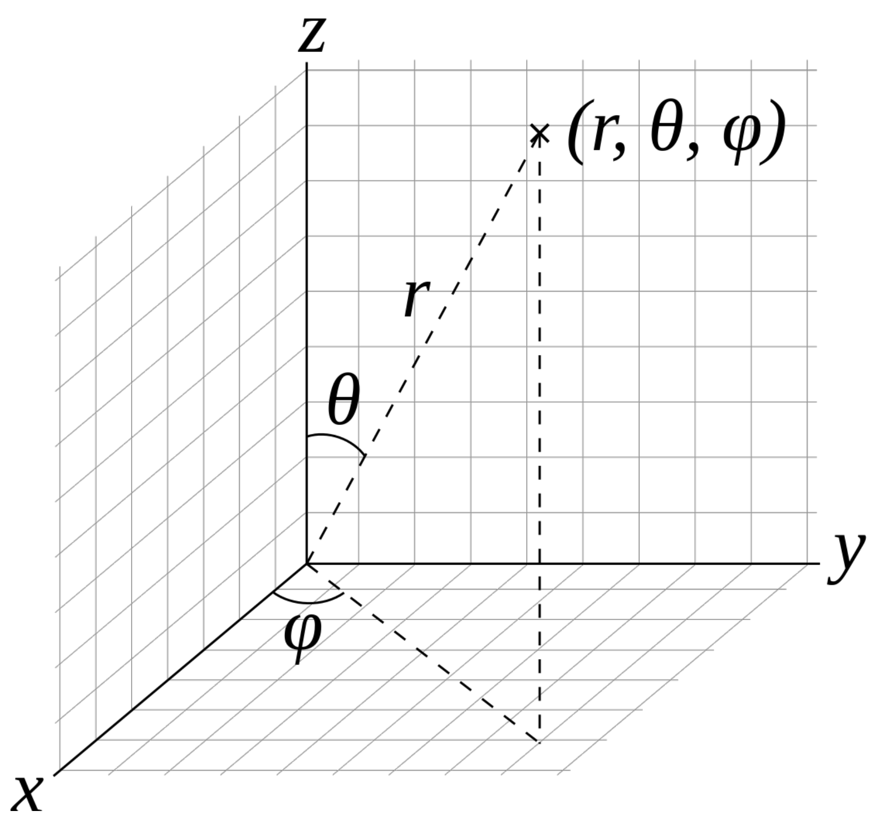
Another one of the 3D structures we really care about are spheres, and so an important coordinate system is spherical coordinates. The unit vectors now are $\hat{r}$, $\hat{\phi}$, and $\hat{\theta}$. r is now once again the distance from the origin, but in 3 dimensions. ϕ and θ are both angluar variables.

Notice that for each of these 3D coordinates systems, the number of variables is always a same, but we change the number of angular vs. distance variables. ϕ is still the angle from the "x-axis", and θ is now the angle from the z-axis.

Looking at this coordinate system, notice that:

- Are the unit vectors perpendicular?
- Does this span 3D space?
- How would you write a coordinate as a vector addition?
- What do the unit vectors look like as you move to different points in the system?

Try plotting some (r, θ, ϕ) coordinates.



Note: $r = 4$ would now be a sphere of radius 4.

Problems

1. What is the difference between a scalar and a vector? What two things will a vector tell you? Give an example of each.

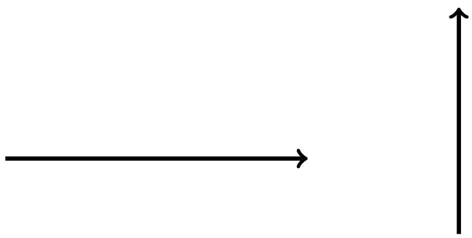
   Scalar: only tells you about a size, or "how much" there is of a unit. Ex. 5 m, 24 km/h, 6 mm are all scalars

   Vector: tells you two things: a size and a direction. 24 km/h [East] is a vector.

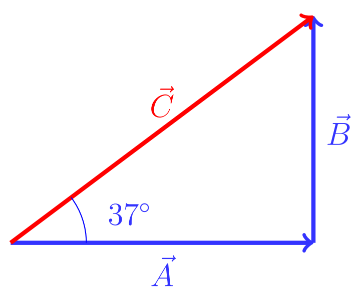
   The difference between them is that vectors have a direction, that you have to be careful of when doing operations like addition or subtraction.

2. Add these vectors together. What is the resulting vector? Measure the size and angle, and write the direction as [degrees above or below the left or right].

   (a)

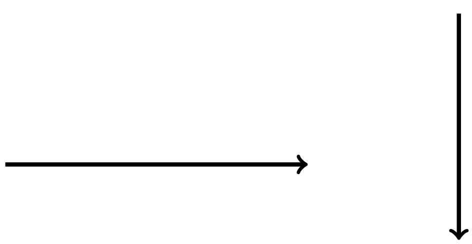
   

   We need to put the vectors together so we can add them together. After following one and then the other, the resultant vector goes from where you started (at the beginning of the first vector) to where you ended (at the end of the second vector).

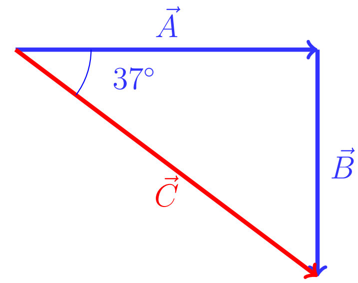
   

   The resultant vector is $\vec{C} = 5$ cm [37° above the right]. It's important that when lining up the vectors, their size and direction is not changed: you're only allowed to move the **whole vector** to line up for addition.

   (b)

Follow the same process as for part (a) above.



The resultant vector is $\vec{C} = 5$ cm [37° below the right].
As an exercise, try adding $\vec{B} + \vec{A}$ and compare your result.

3. What is a unit vector? Write the coordinate (5, 2) as a vector addition if (a) the coordinate (5, 2) is in the 2D Cartesian coordinate system, and if (b) the coordinate (5, 2) is in the polar coordinate system.

   A unit vector is a vector with a size of 1 unit. In 2D Cartesian coordinates, the point (5, 2) would be $5\hat{x} + 2\hat{y}$. In polar coordinates, it would be $5\hat{r} + 2\hat{\phi}$. Notice that the units themselves change depending on the coordiate system: in 2D Cartesian, $2\hat{x}$ is a distance in terms of units, and in polar $2\hat{\phi}$ is an angle in radians.

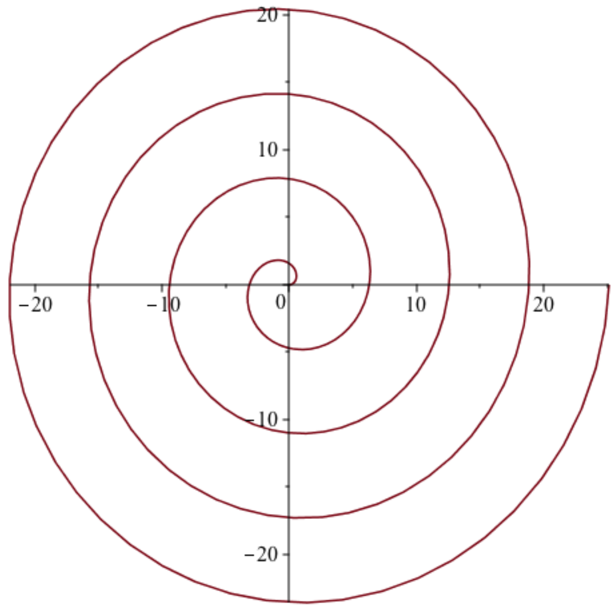
4. Write the coordinate (5, 2, 4) as a vector addition if (a) the coordinate (5, 2, 4) is in the 3D Cartesian coordinate system, (b) the coordinate (5, 2, 4) is in the cylindrical coordinates system, and (c) the coordinate (5, 2, 4) is in the spherical coordinates system.

   In 3D Cartesian: $5\hat{x}+2\hat{y}+4\hat{z}$. In cylindrical: $5\hat{r}+2\hat{\phi}+4\hat{z}$. In spherical, this is actually a bit of a trick, because some people write spherical coordinates as (r, θ, ϕ), and some people as (r, ϕ, θ). We'll use what is shown in the diagram in the lesson and write $5\hat{r} + 2\hat{\theta} + 4\hat{\phi}$

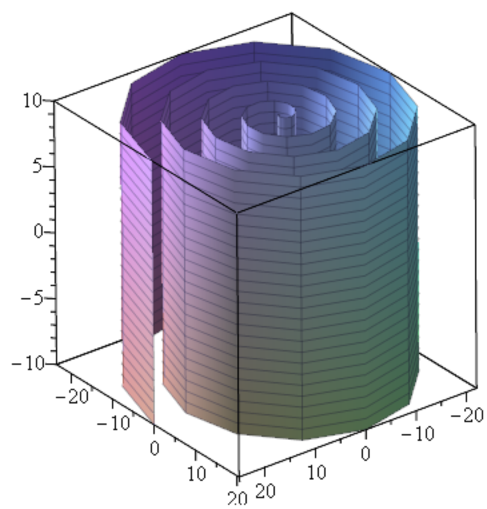
CHALLENGE

5. What does the graph of $r = \phi$ look like in polar coordinates? What does it look like in cylindrical coordinates? What does it look like in spherical coordinates?

   The graph of $r = \phi$ in polar coordinates is a spiral that increases as you go around starting from the origin, like this:



In cylindrical coordinates, you get the same thing as with polar, except now it extends into the 3D space to make a "spiralling cylinder", like this:



In spherical coordinates, this is now similar, but it's a "spiralling sphere", whose radius keeps increasing as you go around. See the animation below (t is the value of ϕ):