

Graduate Electrodynamics

Class Notes

Leon Hostetler

May 4, 2019
Version 0.0

Contents

Preface	iv
1 Introduction	1
1.1 Theories and Unification	1
1.2 A Brief Overview	1
2 Special Relativity and Electrodynamics	3
2.1 Relativity Principle	3
2.2 Lorentz Transformations	6
2.3 Length Contraction and Time Dilation	11
2.4 Minkowski Space	13
2.5 Relativistic Kinematics	19
2.6 Charges in External EM Fields	22
2.7 Groups and Tensors	25
2.8 Maxwell's Equations	26
2.9 Summary: Special Relativity	30
3 Electrostatics	34
3.1 Gauss's Law	34
3.2 Electrostatic Potential	39
3.3 Electric Force and Work	40
3.4 Electric Moments	41
3.5 Multipole Expansion	48
3.6 Multipoles in External Field	50
3.7 Laplace Equation in Spherical Coordinates	52
3.8 Spherical Multipole Moments	60
3.9 Poisson Equation and General Boundary Conditions	63
3.10 Conductors	65
3.11 Image Charges	66
3.12 Green's Theorem	69
3.13 Laplace Equation in Cylindrical Coordinates	73
3.14 Summary: Electrostatics	77
4 Magnetostatics	83
4.1 Laws of Ampere and Biot-Savart	83
4.2 The Vector Potential	84
4.3 Magnetic Dipoles	85
4.4 Field of Moving Point Charge	88
4.5 Larmor Precession	90
4.6 Summary: Magnetostatics	92
5 Electrodynamics	94
5.1 Plane Waves	94

5.2	Energy and Momentum Conservation	100
5.3	Retarded Green's Function	103
5.4	Radiation	108
5.5	Summary: Electrodynamics	116

Preface

About These Notes

My class notes can be found at www.leonhostetler.com/classnotes

These are my class notes for a first graduate course (PHY 841) on classical electrodynamics taken at Michigan State University. The course was taught by Professor Andreas von Manteuffel—one of the most effective instructors I have had the pleasure of learning from. The primary textbooks used were “Classical Electrodynamics” by John David Jackson and “The Classical Theory of Fields” by Landau and Lifshitz. The course material is more aligned with the Jackson textbook although the order and organization differ.

Please bear in mind that these notes will contain errors. Any errors are certainly my own. If you find one, please email me at leonhostetler@gmail.com with the name of the class notes, the page on which the error is found, and the nature of the error.

This work is currently licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License. That means you are free to copy and distribute this document in whole for noncommercial use, but you are not allowed to distribute derivatives of this document or to copy and distribute it for commercial reasons.

Updates

Last Updated: May 4, 2019

Version 0.5 (May 4, 2019): Initial upload

Chapter 1

Introduction

1.1 Theories and Unification

The forces of electricity and magnetism have been truly unified in the $U(1)_{em}$ electromagnetic force in the sense that neither force can be entirely disentangled from the other. Later, the weak force was combined with the electromagnetic force to form the $SU(2)_w \times U(1)_y \longrightarrow U(1)_{em}$ electro-weak theory. However, unlike the electromagnetic force, this is not a true unification. Finally, the strong force was added to the electro-weak force to complete the Standard Model. Again, this is not true unification. It is more like two independent theories have been added together and mixed.

The Standard Model is a true quantum field theory. The Grand Unified Theory is the attempt to combine these theories in true unification.

The interesting thing about electrodynamics is that it is a simple theory which leads to complex phenomena.

1.2 A Brief Overview

In electrodynamics, we have three different representations of the key equations. In the center, and most importantly, we have Maxwell's equations in differential form:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}.$$

Integrating these gives us another representation—Maxwell's equations in integral form. Also known by names like Gauss' Law, Faraday's Law, and Ampere's Laws:

$$\begin{aligned} \int \vec{E} \cdot d\vec{A} &= 4\pi \int_V d^3r \rho \\ &\vdots \end{aligned}$$

Conversely, we can derive Maxwell's equations, from these laws by differentiating.

The third representation is the relativistic formulation. Recall that in special relativity, we have 4-vectors like that of position

$$x^\mu = (ct, \vec{x}),$$

and Lorentz transformations

$$x'^\mu = L^\mu_\nu x^\nu.$$

Mapping from Maxwell's equations in differential form to the relativistic form is just a matter of notation. In the relativistic formulation, all four of Maxwell's equations combine to become the single equation

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\partial^\nu,$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

and the scalar and vector potential are combined to form the 4-potential

$$A^\mu = (\phi, \vec{A}).$$

Recall that the potentials are related to the fields as

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \partial_t \vec{A}.$$

Plugging these into the differential form of Maxwell's equations reduces the number of equations from four to two but at the cost of the first-order differential equations becoming second-order. A useful tool when working with potentials is multipole expansion. For example, for the scalar potential,

$$\phi = \sum_{\ell, m} \frac{Q_m^\ell}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}^*.$$

Related are the Green's functions

$$\Delta G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}').$$

These are closely related to the *propagators* of quantum field theory.

Gauge invariance of the potentials gives us the gauge transformation

$$A'^\mu = A^\mu + \partial^\mu f,$$

and it means the potentials have unphysical degrees of freedom.

We can also think in terms of the Lagrangian formalism

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \dots,$$

where the first term is the Lagrangian density for the photon field.

This leads us to conservation laws

$$\begin{aligned} \partial_\mu J^\mu &= 0 \\ \partial_\mu T^{\mu\nu} &= 0. \end{aligned}$$

Connecting electromagnetism to mechanics is the Lorentz force

$$\vec{F} = q \left(\vec{E} - \frac{1}{c} \vec{v} \times \vec{B} \right).$$

Finally, we have applications and phenomena, such as electrostatics, where the chief equation is

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho,$$

magnetostatics, where

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J},$$

electromagnetic waves

$$\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{x} - \omega t),$$

and radiation

$$\begin{aligned} \frac{dP}{d\Omega} &\propto |\vec{d}|^2 \\ \vec{S} &= \frac{c}{4\pi} \vec{E} \times \vec{B}. \end{aligned}$$

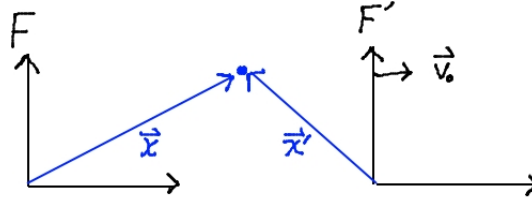
Chapter 2

Special Relativity and Electrodynamics

2.1 Relativity Principle

The relativity principle tells us that the laws of physics take the same form in every inertial (i.e. nonaccelerating) frame.

Consider a stationary frame F and a frame F' that is moving with velocity \vec{v}_0 relative to frame F .



Then **Galilean relativity** tells us that the position vector \vec{x} of a point relative to F is related to the position vector \vec{x}' of the same point relative to F' by

$$\vec{x}'(t) = \vec{x}(t) - \vec{v}_0(t) + \vec{x}_0.$$

Velocities and accelerations in the two frames are related as

$$\begin{aligned}\vec{v}'(t) &= \frac{d}{dt} \vec{x}'(t) = \frac{d\vec{x}}{dt} - \vec{v}_0 = \vec{v}(t) - \vec{v}_0 \\ \vec{a}'(t) &= \frac{d}{dt} \vec{v}'(t) = \frac{d\vec{v}}{dt} = \vec{a}(t).\end{aligned}$$

In frame F , Newton's equations of motion for a system of particles are

$$m_i \vec{a}_i = \sum_{k \neq i} \vec{F}_{ik}(\vec{x}_i - \vec{x}_k).$$

In frame F' ,

$$m_i \vec{a}_i' = \sum_{k \neq i} \vec{F}_{ik}'(\vec{x}_i' - \vec{x}_k').$$

The equations are the same but all vectors now have primes.

Now consider the Lorentz force. In frame F , the Lorentz force is

$$\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right).$$

In frame F' , the Lorentz force is

$$\vec{F}' = q \left(\vec{E}' + \vec{v}' \times \vec{B}' \right).$$

The forces should be equal, so

$$\begin{aligned} \vec{F} &= \vec{F}' \\ \vec{E} + \vec{v} \times \vec{B} &= \vec{E}' + \vec{v}' \times \vec{B}' \\ &= \vec{E}' + (\vec{v} - \vec{v}_0) \times \vec{B}' \\ &= \vec{E}' + \vec{v} \times \vec{B}' - \vec{v}_0 \times \vec{B}'. \end{aligned}$$

The transformation between the two frames should hold for any \vec{v} . In particular, it should hold for $\vec{v} = 0$. In that case, we get

$$\vec{E} = \vec{E}' - \vec{v}_0 \times \vec{B}'.$$

On the other hand,

$$\vec{v} \times \vec{B} = \vec{v} \times \vec{B}'$$

for any \vec{v} , implies that

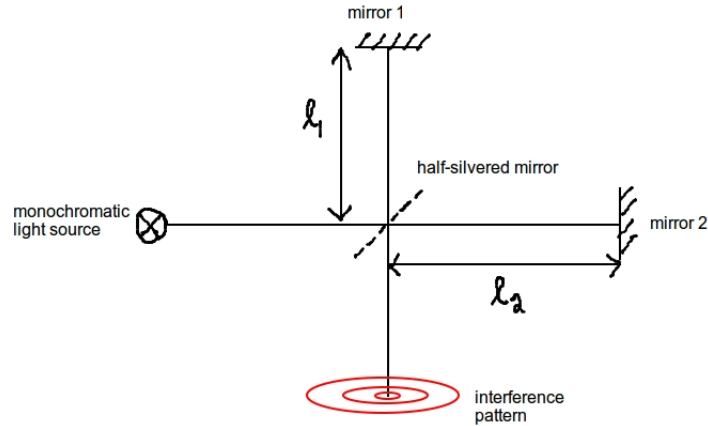
$$\vec{B} = \vec{B}'.$$

But this contradicts experiments. Suppose in frame F , $\vec{B} = 0$, which occurs when a charged particle is not moving. What we found above, this means that $\vec{B}' = 0$, always. However, from the moving frame F' , the particle appears to be moving, and so $\vec{B}' \neq 0$. Logically, there are two possible resolutions of this paradox:

1. The relativity principle holds in mechanics, but maybe it does not hold in electrodynamics. I.e., maybe there are special (“ether”) frames.
2. The relativity principle also holds for electrodynamics. In that case,
 - a) Maxwell’s equations and/or the Lorentz force law are incorrect
 - b) Galilean transformation is incorrect

The Michelson-Morley Experiment

The Michelson-Morley experiments disproved the ether hypothesis, then Einstein showed in 1905 that it is in fact the Galilean transformation that is incorrect.



The idea is to rotate the apparatus and look for differences in the interference pattern due to an optical path difference Δs caused by the Earth traveling through the ether. If the Earth were at rest with respect to the ether (or if there were no ether), then

$$\Delta s = 2(\ell_2 - \ell_1).$$

On the other hand, if the Earth is moving through the ether at a velocity v_0 (assume the direction of motion through the ether is parallel with the line from the light source to mirror 2), then the time for the light ray to travel from the half-silvered mirror to mirror 2 is

$$T = \frac{\ell_2 + x}{c} = \frac{x}{v_0},$$

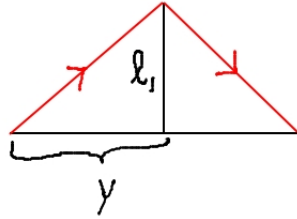
where x is the extra distance the light ray has to travel due to mirror 2 having been carried away from the point at which the light was emitted from the source. On the return trip (i.e. the light ray going from mirror 2 back to the half-silvered mirror), the light ray has to travel a shorter distance due to the half-silvered mirror being carried toward it. This travel time is

$$T' = \frac{\ell_2 - x'}{c} = \frac{x'}{v_0}.$$

We can solve this pair of equations for x and x' , then the total distance traveled by the light ray in going from the half-silvered mirror to mirror 2 and back again can be found after some algebra, to be

$$s_2 = \ell_2 + x + \ell_2 - x' = \frac{2\ell_2}{1 - \left(\frac{v_0}{c}\right)^2}.$$

For the light ray traveling from the half-silvered mirror to mirror 1 and back again, we can draw a diagram.



The distance traveled by this light ray is

$$s_1 = 2\sqrt{\ell_1^2 + y^2} = \frac{2\ell_1}{\sqrt{1 - \left(\frac{v_0}{c}\right)^2}}.$$

Then the optical path difference is

$$\Delta s = s_2 - s_1.$$

If we conduct the experiment multiple times while rotating the apparatus, we expect a change in the interference pattern only if the optical path difference changes as the apparatus is rotated (due to v_0 changing). Such a change would indicate the presence of an absolute background—the ether.

However, the Michelson-Morley experiment found no such change, leading to the conclusion that $v_0 = 0$. So, whereas Galilean relativity claims that

$$\vec{c}' = \vec{c} - \vec{v}_0,$$

the Michelson-Morley experiment found that

$$\vec{c}' = \vec{c}.$$

That is, the speed of light in a vacuum is the same no matter what speed you are moving through the vacuum. I.e., c is the same in all inertial frames.

Suppose you flash a light at the origin in frame F . The flash occurs at $t = 0$ when the origins of frames F and F' coincide. The surface of the sphere corresponding to how far the light wave has traveled, moves at a speed c . So in frame F , a point on this surface satisfies

$$c^2 t^2 = x^2 + y^2 + z^2 \implies c^2 t^2 - (x^2 + y^2 + z^2) = 0.$$

In frame F' , a point on this surface satisfies

$$c^2 t'^2 = x'^2 + y'^2 + z'^2 \implies c^2 t'^2 - (x'^2 + y'^2 + z'^2) = 0,$$

since c is the same in both frames. Thus, the Michelson-Morley condition is that

$$c^2 t^2 - (x^2 + y^2 + z^2) = c^2 t'^2 - (x'^2 + y'^2 + z'^2).$$

2.2 Lorentz Transformations

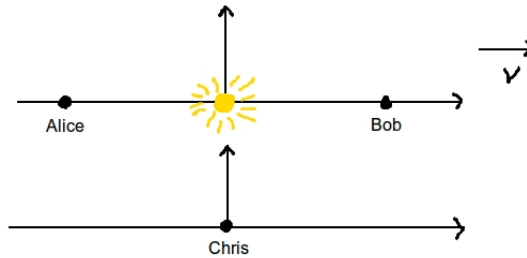
Transformation of Position and Time

Following Einstein's 1905 work, we will start with the axioms:

1. The laws of physics are form invariant, i.e., the relativity principle
2. The speed of light in vacuum is c in every inertial frame, i.e., the Michelson-Morley condition

What are the consequences? What conclusions follow from these axioms?

Consider simultaneous events. Suppose in frame F' , there is a flashing light at the origin. There are two observers—Alice sitting at $-x$ and Bob sitting at $+x$. As soon as they see the light flash, they synchronize their individual clocks. Now, consider observer Chris in frame F such that frame F' is moving to the right (with respect to F) at speed v .



According to Chris, Alice starts her clock first because she is moving toward the light front approaching her and Bob is moving away from the light front approaching him. On the other hand, if frame F' is moving toward the left instead of the right, then Chris sees Bob start his clock first. So not only do the events not appear simultaneous to Chris, but their order is not even fixed.

Conclusion: There is no absolute simultaneity. That is, there is no absolute time. If we change from one reference frame to another, we have to modify time.

Now, let's consider vector transformations which are compatible with our axioms. In frame F , we have time t and a vector \vec{x} . If frame F' is moving toward the right (in the x -direction) with speed \vec{v}_0 , what is the vector \vec{x}' ?

We know that there is something special about time t and the direction of motion x . However, there is nothing special about the y and z directions. So we start with the

ansatz

$$\begin{aligned}x' &= Ax + Bt \\y' &= y \\z' &= z \\t' &= Cx + Dt.\end{aligned}$$

Recall the Michelson-Morley condition

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2.$$

Plugging in x', y', z', t' from our ansatz gives us

$$c^2t^2 - x^2 = c^2(Cx + Dt)^2 - (Ax + Bt)^2.$$

Rearranging and gathering like terms gives us

$$0 = c^2t^2 \left(-1 + D^2 - \frac{B^2}{c^2} \right) + x^2 (1 + C^2c^2 - A^2) + 2xt (c^2CD - AB).$$

The left side is zero, so the right side must be zero. This should be true for all x and t . This implies that the quantities in each pair of parentheses must be zero

$$\begin{aligned}-1 + D^2 - \frac{B^2}{c^2} &= 0 \\1 + C^2c^2 - A^2 &= 0 \\c^2CD - AB &= 0.\end{aligned}$$

Now we have three equations in four unknowns. We need one more equation. Consider the motion of the origin of F' . This is the point $x' = 0$. Since F' is moving in the x direction at speed v_0 with respect to frame F , we know that $x' = 0 = Ax + Bt$ where $x = v_0t$. This implies

$$B = -v_0A.$$

When we solve this system of four equations in four unknowns, we end up with

$$\begin{aligned}A &= \gamma \\B &= -\beta c\gamma \\C &= -\frac{\beta}{c}\gamma \\D &= \gamma,\end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}},$$

is the **Lorentz factor**, and

$$\beta = \frac{v_0}{c}.$$

So our ansatz becomes

$$\begin{aligned}x' &= \gamma(x - \beta ct) \\y' &= y \\z' &= z \\t' &= \gamma\left(-\frac{\beta}{c}x + t\right).\end{aligned}$$

This is the **Lorentz transformation** or **Lorentz boost**.

Note that

$$\beta c = v_0.$$

The directions perpendicular to the boost are not affected. Only the parallel components change during a Lorentz boost.

Discussion:

- If we do an expansion for $v_0 \ll c$ (i.e. $\beta \ll 1$), we get the Galilean transformation

$$\begin{aligned} x' &= x - v_0 t \\ t' &= t. \end{aligned}$$

We can think in terms of a Taylor expansion and add 1st-order correction terms to the Galilean transformation. The first-order relativistic corrections are $\mathcal{O}(\beta^2)$ for x' and $\mathcal{O}(\beta)$ for t' . So the time component is affected much more strongly by relativistic effects than the spatial components.

- To get the inverse Lorentz transformations, just switch the primes and let $\beta \rightarrow -\beta$ (or $v_0 \rightarrow -v_0$)
- Consider the limit $v_0 \rightarrow c$. Then γ becomes undefined. So the speed of light c is a limiting speed.

The above is all for a Lorentz boost in the x -direction. We can extend this to a Lorentz boost in a general direction $\vec{\beta} = \vec{v}_0/c$. For a generic $x = (ct, \vec{x})$, we want the Lorentz transformed 4-vector $x' = (ct', \vec{x}')$. The time component now transforms as

$$ct' = \gamma (ct - \vec{\beta} \cdot \vec{x}).$$

Notice that we have replaced βx in the equation with $\vec{\beta} \cdot \vec{x}$. For the spatial part, we know that only the components parallel to the direction of relative motion $\hat{\beta} = \vec{\beta}/\beta$ will change. The perpendicular components are unchanged. We can break the spatial part into parallel and perpendicular components

$$\begin{aligned} \vec{x} &= \vec{x}_\perp + \vec{x}_\parallel \\ \vec{x}' &= \vec{x}'_\perp + \vec{x}'_\parallel. \end{aligned}$$

Then since the perpendicular part doesn't change,

$$\vec{x}'_\perp = \vec{x}_\perp.$$

The parallel part transforms as

$$\vec{x}'_\parallel = \gamma (\vec{x}_\parallel - \vec{\beta} ct).$$

So combining the parallel and perpendicular parts, the spatial vector transforms as

$$\vec{x}' = \vec{x}_\perp + \gamma (\vec{x}_\parallel - \vec{\beta} ct).$$

We can write the parallel and perpendicular parts as

$$\begin{aligned} \vec{x}_\parallel &= (\vec{x} \cdot \hat{\beta}) \hat{\beta} = (\vec{x} \cdot \vec{\beta}) \frac{\vec{\beta}}{|\vec{\beta}|^2} \\ \vec{x}_\perp &= \vec{x} - \vec{x}_\parallel = \vec{x} - (\vec{x} \cdot \vec{\beta}) \frac{\vec{\beta}}{|\vec{\beta}|^2}. \end{aligned}$$

Altogether then, our Lorentz transformation

$$x = (ct, \vec{x}) \longrightarrow x' = (ct', \vec{x}'),$$

for an arbitrary boost $\vec{\beta}$, is

$$\begin{aligned} ct' &= \gamma (ct - \vec{\beta} \cdot \vec{x}) \\ \vec{x}' &= \vec{x} - \gamma \vec{\beta} ct + (\gamma - 1) (\vec{x} \cdot \vec{\beta}) \frac{\vec{\beta}}{|\vec{\beta}|^2}. \end{aligned}$$

Transformation of Velocity

Suppose we have a velocity \vec{v} in frame F . What is \vec{v}' as observed from the frame F' which is moving in the x -direction with speed v_0 ?

In frame F , the velocity components are

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}.$$

In frame F' , the velocity components are

$$v'_x = \frac{dx'}{dt'}, \quad v'_y = \frac{dy'}{dt'}, \quad v'_z = \frac{dz'}{dt'}.$$

We want to relate the components in the different frames.

By the Lorentz transformation equations, we know that

$$\begin{aligned} dx' &= \gamma (dx - \beta c dt) = \gamma (v_x - \beta c) dt \\ dt' &= \gamma \left(-\frac{\beta}{c} dx + dt \right) = \gamma \left(-\frac{\beta}{c} v_x + 1 \right) dt. \end{aligned}$$

Here, we used the fact that $v_x = \frac{dx}{dt}$. Plugging these into the velocity components for frame F' gives us

$$\begin{aligned} v'_x &= \frac{dx'}{dt'} = \frac{\gamma (v_x - \beta c) dt}{\gamma \left(-\frac{\beta}{c} v_x + 1 \right) dt} = \frac{v_x - v_0}{1 - \frac{v_0 v_x}{c^2}} \\ v'_y &= \frac{dy'}{dt'} = \frac{dy}{\gamma \left(-\frac{\beta}{c} v_x + 1 \right) dt} = \frac{v_y}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)} \\ v'_z &= \frac{dz'}{dt'} = \frac{dz}{\gamma \left(-\frac{\beta}{c} v_x + 1 \right) dt} = \frac{v_z}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)}. \end{aligned}$$

So given

$$\vec{v} = (v_x, v_y, v_z),$$

in frame F , then in frame F' , which is moving in the positive x -direction with respect to F , the Lorentz transformed velocity is

$$\vec{v}' = (v'_x, v'_y, v'_z) = \left(\frac{v_x - v_0}{1 - \frac{v_0 v_x}{c^2}}, \frac{v_y}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)}, \frac{v_z}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)} \right).$$

Discussion:

- For $\beta \ll 1$, we get the Galilean velocity transformations

$$\begin{aligned}v'_x &= v_x - v_0 \\v'_y &= v_y \\v'_z &= v_z.\end{aligned}$$

- The speed of light c is a limiting speed. For example, if $\vec{v} = (c, 0, 0)$, then the transformed velocity is $\vec{v}' = (c, 0, 0)$. So even though the velocity c and the velocity v_0 of the relative motion between the frames should “add”, the final velocity is never larger than c .
- To get the inverse transformation $\vec{v}' \rightarrow \vec{v}$, just exchange the primes and let $\vec{v}_0 \rightarrow -\vec{v}_0$
- We have experimental evidence, for example in decays

$$\pi^0 \longrightarrow \gamma + \gamma,$$

that relativistic velocities really do add in the manner that we derived.

Transformation of Acceleration

In frame F , we have acceleration components

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt}.$$

In frame F' , the components of the acceleration can be found by using the chain rule and plugging in what we found for the velocity components and for dt'/dt

$$\begin{aligned}a'_x &= \frac{dv'_x}{dt'} = \frac{dv'_x}{dt} \cdot \frac{dt}{dt'} = \frac{d}{dt} \left(\frac{v_x - v_0}{1 - \frac{v_0 v_x}{c^2}} \right) \cdot \left[\frac{1}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)} \right] \\&= \frac{a_x}{\gamma^3 \left(1 - \frac{v_0 v_x}{c^2} \right)^3} \\a'_y &= \frac{dv'_y}{dt'} = \frac{dv'_y}{dt} \cdot \frac{dt}{dt'} = \frac{d}{dt} \left(\frac{v_y}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)} \right) \cdot \left[\frac{1}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)} \right] \\&= \frac{a_y}{\gamma^2 \left(1 - \frac{v_0 v_x}{c^2} \right)^2} + \frac{\frac{v_0 v_y}{c^2} a_x}{\gamma^2 \left(1 - \frac{v_0 v_x}{c^2} \right)^3} \\a'_z &= \frac{dv'_z}{dt'} = \frac{dv'_z}{dt} \cdot \frac{dt}{dt'} = \frac{d}{dt} \left(\frac{v_z}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)} \right) \cdot \left[\frac{1}{\gamma \left(1 - \frac{v_0 v_x}{c^2} \right)} \right] \\&= \frac{a_z}{\gamma^2 \left(1 - \frac{v_0 v_x}{c^2} \right)^2} + \frac{\frac{v_0 v_z}{c^2} a_x}{\gamma^2 \left(1 - \frac{v_0 v_x}{c^2} \right)^3}.\end{aligned}$$

Notice that all three components of \vec{a}' depend on a_x .

Discussion:

- In the limit $\beta = v_0/c \rightarrow 0$, we get the expected Galilean transformations for the acceleration

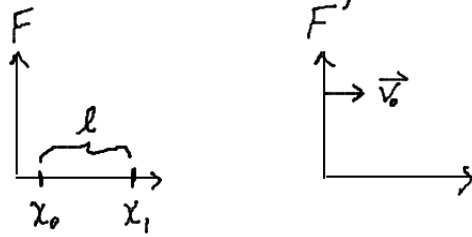
$$a'_x = a_x, \quad a'_y = a_y, \quad a'_z = a_z.$$

- Relativistic mechanics is a lot more complicated than Newtonian mechanics, because we have a mixing of the components of acceleration when one changes reference frames.

2.3 Length Contraction and Time Dilation

Lorentz Contraction

Consider an object of length ℓ that is at rest in frame F . What is its length in frame F' ?



Tip

Lorentz contraction tells us that a moving scale always appears shorter.

Tip

Time dilation tells us that moving clocks run slower.

By applying the inverse Lorentz transformation to both x_0 and x_1 , we get

$$\ell = x_1 - x_0 = \gamma(x'_1 + \beta ct') - \gamma(x'_0 + \beta ct') = \gamma(x'_1 - x'_0) = \gamma\ell'.$$

Thus,

$$\ell' = \frac{\ell}{\gamma}, \quad \text{where always } \gamma > 1.$$

Note that t' is the same for both x_1 and x_0 because we are assuming that both positions are measured at the same time in F' .

Discussion:

- The *same* shortening effect is seen if the change of frames is reversed. Notice that the relation between the lengths in the two frames is completely independent of the sign of v_0 .
- Contraction happens only in the direction of the boost. I.e., a moving volume is contracted only in the x -direction. The volume elements in the different frames are related by

$$dV' = dx' dy' dz' = \left(\frac{1}{\gamma} dx\right) (dy) (dz) = \frac{1}{\gamma} dV.$$

Notice that there is only γ instead of γ^3 since the contraction only occurs along one direction.

- A real observer would also have to take light propagation into account to really explain the appearance of a fast-moving object.
- There is no direct experimental evidence for length contraction simply due to how difficult it would be to measure this effect.

Time Dilation

Consider a time interval τ in F . Then in F' ,

$$\tau' = t'_2 - t'_1 = \gamma(t_2 - t_1) = \gamma\tau.$$

Thus,

$$\tau' = \gamma\tau, \quad \text{where always } \gamma > 1.$$

That is, the time interval appears longer in F' .

Discussion:

1. The same effect is seen for change of frames in the reverse direction. I.e. *all* moving clocks run slower.

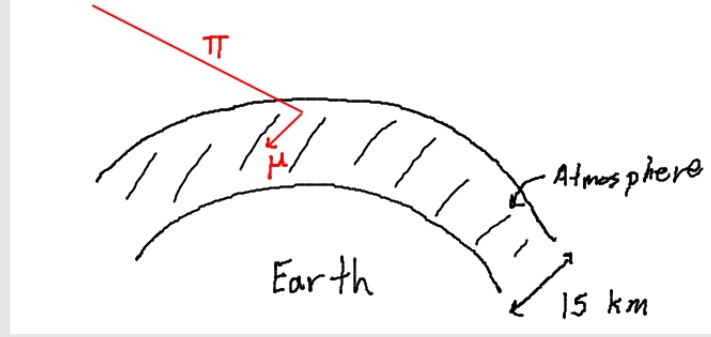
2. Twin paradox: To be able to compare their ages, one of the traveling twins has to return to the other. Only the one that didn't accelerate was always in an inertial frame. The twin who returns, and therefore accelerates and travels in a noninertial frame, is younger.

Example 2.3.1: Relativistic Muons

Primary cosmic rays such as pions, produce μ^\pm particles at high speed in our atmosphere. However, these muons decay quickly via the following mechanisms:

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$$

$$\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu.$$



In their rest frame, muons have a lifetime of

$$\tau_\mu = 2.2 \times 10^{-6} \text{ s.}$$

So classically, they travel a distance of

$$d = v_0 \tau_\mu \approx c \tau_\mu = 6.6 \times 10^2 \text{ m,}$$

assuming they enter the atmosphere at essentially the speed of light. This is much less than the 15 km thickness of our atmosphere, so we would expect most muons to decay before reaching the surface of Earth.

However, if we take into account relativistic effects, we know that time dilation means the muon lives a lot longer (and therefore travels further) in its frame. The time dilation equation tells us that

$$\tau_{\text{earth}} = \gamma \tau_\mu.$$

Then the actual distance traveled by the muon, as measured on Earth, is closer to

$$d_{\text{relativistic}} = v_0 \tau_{\text{Earth}} = v_0 \gamma \tau_\mu \simeq 10 \times d_{\text{classical}},$$

where we've taken $\gamma = 10$, which corresponds to about 99% the speed of light. Now, we get a value of

$$d \approx 6.6 \times 10^3 \text{ m} = 6.6 \text{ km.}$$

This factor of 10 increase in the lifetime of the muon makes a significant difference in the number of observed muons. For example, the exponential decay formula gives us the number $N(t)$ of muons expected to remain after time t

$$N(t) = N_0 e^{-t/\tau},$$

where N_0 was the initial number of particles. For example, if we wait 23 lifetimes, then

$$N(23\tau) = N_0 e^{-23} \approx \frac{N_0}{10^{10}}.$$

So the number of muons has been reduced by a factor of 10^{10} . This is a very large number, so we expect to see zero particles remaining. However, if we take the factor of 10 from relativistic effects into account, we get

$$N(2.3\tau) = N_0 e^{-2.3} \approx \frac{N_0}{10},$$

which is just a reduction by a factor of 10 in the number of remaining particles. We should definitely still see particles.

In practice, relativistic muons have been used to verify γ with a fractional error of 2×10^{-3} .

2.4 Minkowski Space

4-vectors and the Metric Tensor

In classical mechanics, a point in space is represented by a 3D vector $\vec{x} = (x, y, z)$, which has a norm-squared of $||\vec{x}||^2 = \vec{x} \cdot \vec{x}$, and general scalar product

$$\vec{x} \cdot \vec{y} = \sum_{i,j=1}^3 x_i \delta_{ij} x_j.$$

Scalar products are useful because they are invariant under orthogonal transformations such as rotations and reflections. The space of these 3D vectors is called “Euclidean space”.

In special relativity, we have rotations as well as Lorentz transformations. The norm-squared is

$$s^2 = c^2 t^2 - x^2 - y^2 - z^2.$$

This quantity is invariant under the Lorentz group which includes rotations and Lorentz transformations. More explicitly,

$$s^2 = \begin{bmatrix} ct & x & y & z \end{bmatrix} g_{\mu\nu} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix},$$

where

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} = \text{diag}(1, -1, -1, -1),$$

is the **metric tensor**. The space of these 4-vectors is called “Minkowski space”.

In 3D Euclidean space, the norm-squared of a vector is positive definite. This is not true in Minkowski space. In Minkowski space, the norm of a 4-vector can be positive, zero, or even negative.

For some event in spacetime, we define a **contravariant** 4-vector as

$$x^\mu = \begin{bmatrix} ct & x & y & z \end{bmatrix}^T.$$

Note that we use Greek (or “Lorentz”) indices for 4-vectors (i.e. $\mu = 0, 1, 2, 3$), and we use Latin indices for 3-vectors (i.e. $i = 1, 2, 3$).

We can write

$$s^2 = \sum_{\mu, \nu=0}^3 x^\mu g_{\mu\nu} x^\nu.$$

We will use the **Einstein summation** convention where repeated indices implies a sum over those indices unless noted otherwise. Using Einstein notation, we write the above simply as

$$s^2 = x^\mu g_{\mu\nu} x^\nu,$$

with the sum being implied.

We define the **covariant** 4-vector as

$$x_\mu = g_{\mu\nu} x^\nu.$$

The scalar/inner product of two 4-vectors is

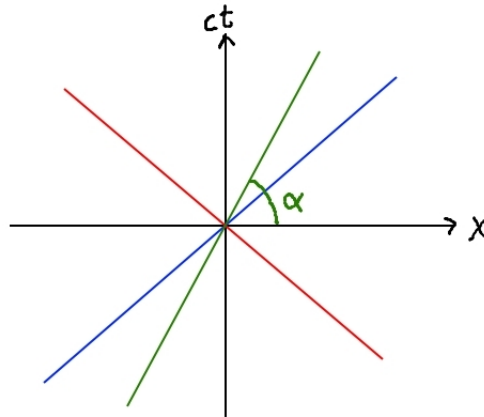
$$x \cdot y = x_\mu y^\mu = x^\mu g_{\mu\nu} y^\nu = x^0 y^0 - \vec{x} \cdot \vec{y}.$$

The scalar product is invariant under

- Normal rotations
- Lorentz boosts
- Parity ($\vec{x} \rightarrow -\vec{x}$)
- Time reversal ($t \rightarrow -t$)

Minkowski Diagrams

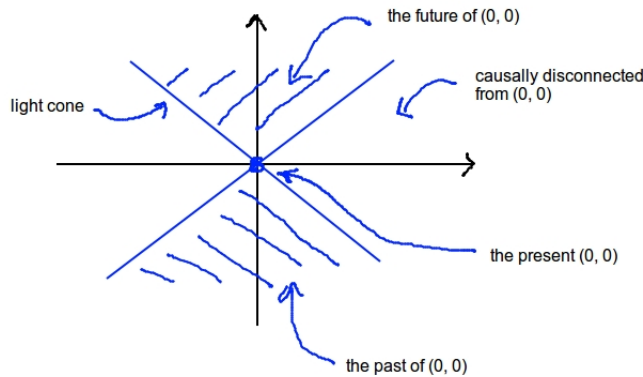
To visualize motion in 4D Minkowski spacetime, we draw 1 + 1 dimensional diagrams. We ignore the y and z components and draw the x and t dimensions, as shown in the example below.



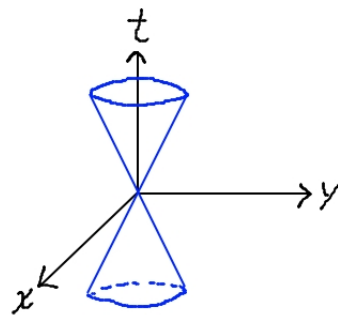
The blue line is at 45° and represents the worldline of a particle moving at constant speed c in the $+x$ direction. The red line represents the worldline of a particle moving at constant speed c in the $-x$ direction. The green line represents the worldline of a particle moving slower than c ($|\vec{v}| < c$). Note that

$$\tan \alpha = \frac{ct}{x} = \frac{c}{v}.$$

The origin of the Minkowski diagram, $(0, 0)$, represents the *present*. Anything in the cone above this point is the *future* of the spacetime event $(0, 0)$. Anything in the cone below this point is the *past* of the spacetime event $(0, 0)$.



The light cone separates causally disconnected regions of spacetime. If we add a second spatial dimension as shown below, it becomes more obvious why we call it a “cone”.



There are three possible cases for s^2 :

- If $s^2 > 0$, then the relation between the two events is called *time-like*. The two events are causally connected.
- If $s^2 = 0$, then the relation is *light-like*
- If $s^2 < 0$, then the relation is *space-like* (i.e. causally disconnected)

In general, for a 4-vector x , determine whether its time-like, light-like, or space-like by looking at the sign of $x \cdot x = x^\mu x_\mu$.

Note: Different authors use different conventions. For example, some use the convention that $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Another convention is $g_{\mu\nu} = \mathbb{1} = \text{diag}(1, 1, 1, 1)$, and then include the imaginary unit in the time part of a 4-vector as in $x^\mu = (ict, x, y, z)$ or even $x^\mu = (x, y, z, ict)$. Which convention that we use determines, for example, whether $s^2 > 0$ is time-like or space-like.

Rapidity

We can write a Lorentz transformation for motion along the x -direction with speed β as a matrix-vector multiplication

$$x'^\mu = \Lambda^\mu_\nu x^\nu.$$

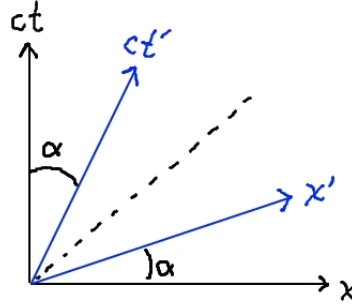
Fully written out, this is

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

Doing the multiplication gives us the familiar Lorentz transformation

$$\begin{aligned} x'^0 &= \gamma x^0 - \beta\gamma x^1 & \implies ct' &= \gamma ct - \beta\gamma x \\ x'^1 &= -\beta\gamma x^0 + \gamma x^1 & \implies x' &= -\beta\gamma ct + \gamma x \\ x'^2 &= x^2 & \implies y' &= y \\ x'^3 &= x^3 & \implies z' &= z. \end{aligned}$$

Consider the frame F and the Lorentz boosted frame F' on the same spacetime diagram as shown below.



Along the boosted time axis ct' , we know that $x' = 0$. This implies that

$$ct = \frac{1}{\beta}x.$$

Along the boosted space axis x' , we know that $t' = 0$. This implies that

$$ct = \beta x.$$

Thus, the angle α between ct and ct' is the same as the angle between x and x' . This angle is related to the relative speed between the frames via

$$\beta = \tan \alpha.$$

The Lorentz matrix (Λ^μ_ν) is a symmetric matrix with

$$\det(\Lambda^\mu_\nu) = 1.$$

We can also write it using the “hyperbolic” angle ζ , called the **rapidity** as

$$(\Lambda^\mu_\nu) = \begin{bmatrix} \cosh \zeta & -\sinh \zeta & 0 & 0 \\ -\sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

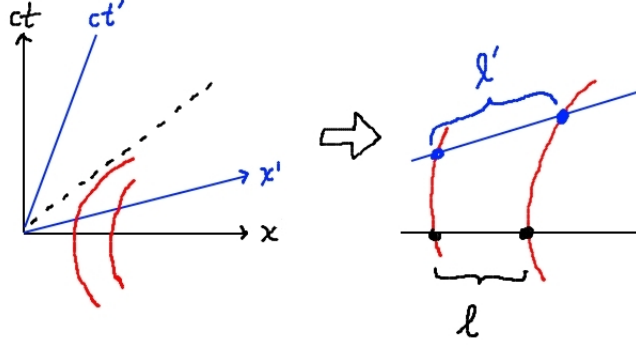
Comparing this with the regular Lorentz transformation matrix, we see that

$$\cosh \zeta = \gamma, \quad \tanh \zeta = \beta.$$

Suppose we draw frames F and F' on a single Minkowski diagram as shown below on the left. Then to scale between the two frames, we use a Lorentz invariant like

$$s^2 = c^2 t^2 - x^2 = c t'^2 - x'^2 = \text{const.}$$

This equation is a hyperbola with the light lines as asymptotes. Then, we can relate a length scale ℓ in frame F to the length scale ℓ' in frame F' with the Lorentz factor.



Doppler Effect and the Aberration of Light

Recall that Maxwell's equations allow plane wave solutions for \vec{E} and \vec{B} of the form

$$\vec{E}(\vec{x}, t) = \vec{E}_0 \cos(-\vec{k} \cdot \vec{x} + \omega t).$$

Maxwell's equations we know are already relativistic, so the phase $-\vec{k} \cdot \vec{x} + \omega t$ should already be Lorentz invariant. For example, whether or not two light waves interfere with each other should not depend on the reference frame.

First, consider the constant phase situation. We choose the constant to be zero for convenience. Then

$$0 = -\vec{k} \cdot \vec{x} + \omega t = \omega t - kx,$$

for propagation in the x -direction. This implies that

$$\frac{x}{t} = \frac{\omega}{k} = c,$$

where $|\vec{k}| = \omega/c$. We want the phase to be Lorentz invariant, so we write it as a 4-vector. Since $ct = x^0$ and $\omega/c = k^0$, we can write

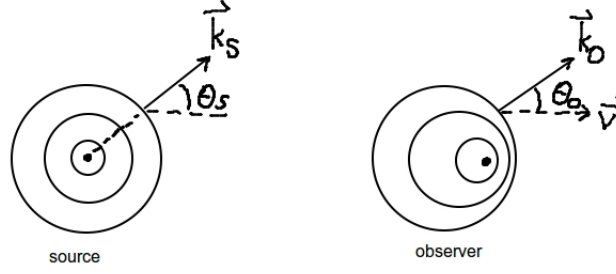
$$ct \cdot \frac{\omega}{c} - \vec{x} \cdot \vec{k} = x^\mu g_{\mu\nu} k^\nu = x \cdot k,$$

where

$$k^\mu = \begin{bmatrix} \frac{\omega}{c} \\ \vec{k} \end{bmatrix},$$

is the **wave 4-vector**. It transforms like x^μ under Lorentz transformations.

Consider a light source at rest in frame F . Now consider an observer moving in the x -direction with speed \vec{v} in frame F' .



How are \vec{k}_s and \vec{k}_o related, and how are θ_s and θ_o related?

In the source frame,

$$k_s^\mu = \left(\frac{\omega_s}{c}, \vec{k}_s \right) = \frac{\omega_s}{c} (1, \cos \theta_s, \sin \theta_s, 0).$$

In the observer frame,

$$k_o^\mu = \left(\frac{\omega_o}{c}, \vec{k}_o \right) = \frac{\omega_o}{c} (1, \cos \theta_o, \sin \theta_o, 0).$$

They are related by the Lorentz transformation

$$k_s^\mu = \Lambda^\mu_\nu k_o^\nu, \quad \vec{\beta} = \frac{\vec{v}}{c},$$

or in full form,

$$\frac{\omega_s}{c} \begin{bmatrix} 1 \\ \cos \theta_s \\ \sin \theta_s \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{\omega_o}{c} \begin{bmatrix} 1 \\ \cos \theta_o \\ \sin \theta_o \\ 0 \end{bmatrix}.$$

For the zero (i.e. frequency) component, we see that

$$\frac{\omega_s}{c} = \frac{\omega_o}{c} (\gamma - \beta\gamma \cos \theta_o),$$

or

$$\boxed{\omega_o = \frac{\omega_s}{\gamma(1 - \beta \cos \theta_o)}}.$$

This is the **Doppler effect**. Note that

- If $\theta_o = 0$, then the frequency is blue-shifted

$$\omega_o = \frac{\omega_s}{\gamma(1 - \beta)} = \omega_s \sqrt{\frac{1 + \beta}{1 - \beta}} > \omega_s.$$

- If $\theta_o = \pi$, then the frequency is red-shifted

$$\omega_o = \frac{\omega_s}{\gamma(1 + \beta)} < \omega_s.$$

Next, we look at the k^1 (i.e. x) component.

$$\frac{\omega_s}{c} \cos \theta_s = \frac{\omega_o}{c} \gamma (\cos \theta_o - \beta),$$

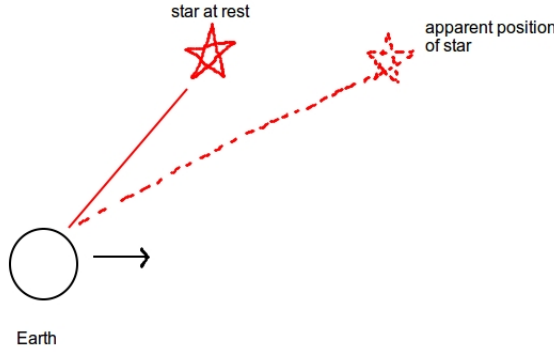
or

$$\cos \theta_s = \frac{\cos \theta_o - \beta}{1 - \beta \cos \theta_o}.$$

Equivalently,

$$\cos \theta_o = \frac{\cos \theta_s + \beta}{1 + \beta \cos \theta_s}.$$

So in addition to a shift in the frequency, we also have a shift in the direction of light propagation when we transform from one frame to another. This phenomenon is called the **aberration of light**.



2.5 Relativistic Kinematics

4-momentum

Classically, momentum is defined as

$$\vec{p} = m \frac{d\vec{x}}{dt}.$$

We want a “momentum” that transforms as a Lorentz 4-vector. This classical definition doesn’t even work as the spatial part of a 4-vector. While m is invariant and $d\vec{x}$ transforms under Lorentz transformation in the way we want, dt does not.

We know that

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

For real massive particles, which follow timelike trajectories, $d\tau^2 > 0$. So we can rewrite this as

$$d\tau = \frac{1}{\gamma} dt = \sqrt{1 - \frac{v^2}{c^2}} dt.$$

This is the **proper time**, which is the time in the particle’s rest frame. We know that this quantity $d\tau$ is Lorentz invariant since we constructed it (i.e. obtained it by taking the square root) from a scalar product. So if we define

$$p^\mu = m \frac{dx^\mu}{d\tau},$$

then we have a quantity that transforms as a 4-vector under Lorentz transformations. I.e., scalar products of this quantity with other 4-vectors are Lorentz invariant. We call this quantity the 4-momentum.

We can also write

$$p^\mu = m u^\mu,$$

where

$$u^\mu = \frac{dx^\mu}{d\tau},$$

is the **4-velocity**. However, we will rarely use this quantity.

We can also write

$$p^\mu = m \frac{1}{\sqrt{1 - v^2/c^2}} \begin{bmatrix} c \\ \vec{v} \end{bmatrix} = m\gamma \begin{bmatrix} c \\ \vec{v} \end{bmatrix},$$

where \vec{v} is the ordinary 3-velocity.

We can now name the temporal and spatial components of the 4-momentum

$$p^\mu = \begin{bmatrix} E/c \\ \vec{p} \end{bmatrix}, \quad \vec{p} = \gamma m \vec{v}.$$

Note that \vec{p} here is not the classical 3-momentum, but rather, it is γ times the classical 3-momentum. Technically, we can call this \vec{p} the *relativistic* 3-momentum. In general, from now on, \vec{p} will mean this relativistic 3-momentum rather than the classical 3-momentum.

Now the 4-velocity can be written in terms of components as

$$u^\mu = \gamma \begin{bmatrix} c \\ \vec{v} \end{bmatrix}.$$

Discussion:

- The first component of the 4-momentum tells us that

$$E = \gamma mc^2.$$

In the $\beta \rightarrow 0$ limit, this becomes

$$E = mc^2 + \frac{1}{2}mv^2 + \dots$$

Notice that the first term on the right is the particle's rest energy, and the second term is its classical kinetic energy.

- The spatial components of the 4-momentum are

$$\vec{p} = \gamma m \vec{v}.$$

In the $\beta \rightarrow 0$ limit, this becomes

$$\vec{p} = m \vec{v} + \dots$$

So in the non-relativistic limit, \vec{p} becomes the classical momentum.

- The scalar product of p with itself is

$$p^2 = p \cdot p = p_\mu p^\mu = \frac{E^2}{c^2} - \vec{p} \cdot \vec{p}.$$

In the rest frame of the particle, $\vec{v} = 0$, and so $\vec{p} = 0$. Then we just have

$$\frac{E^2}{c^2} = \frac{(mc^2)^2}{c^2} = m^2 c^2.$$

But p^2 is invariant, so if

$$p^2 = m^2 c^2,$$

in the rest frame, then it must always be true. So we can plug this into the energy to write

$$E = \sqrt{m^2 c^4 + \vec{p}^2 c^2}.$$

Remember, \vec{p} is the *relativistic* 3-momentum.

- The 4-vector

$$p^\mu = \begin{bmatrix} E/c \\ \vec{p} \end{bmatrix},$$

works also for massless particles which have $p^2 = m^2 c^2 = 0$. Thus, for massless particles, we find that

$$E = |\vec{p}|c.$$

Tip

Whenever there is a “ p ” without the vector arrow above it, then it is implied to be the Lorentz 4-momentum.

Scattering and Decay Kinematics

We will now consider decay and scattering problems like

$$e^+ + e^- \longrightarrow \mu^+ + \mu^-.$$

The key to solving such problems is to utilize the various conservation laws implied by the symmetries of the problem.

In all decay and scattering problems, the sum of the initial state momenta equals the sum of the final state momenta. That is,

$$\sum_i p_i^\mu = \sum_f p_f^\mu,$$

where the sum on the left is over the initial particles, and the sum on the right is over the final particles. Note that for the 0-components, this implies energy conservation, and for the other three components, this implies momentum conservation.

Consider the decay of a massive particle A into two particles B and C

$$A \longrightarrow B + C.$$

Then the 4-momentums are

$$p_A = p_B + p_C.$$

Note, we are suppressing the Lorentz indices here. We want to know, what is the energy of particle B in the rest frame of particle A ?

Note: We can set $c = 0$ in the beginning and then just add it back in at the end (if we want it there) by checking the units. The initial state 4-momentum, in its own rest frame (i.e. $\vec{p}_A = 0$), is

$$p_A = \begin{bmatrix} E_A \\ \vec{0} \end{bmatrix}.$$

The final state 4-momenta are

$$p_B = \begin{bmatrix} E_B \\ \vec{p}_B \end{bmatrix}, \quad p_C = \begin{bmatrix} E_C \\ \vec{p}_C \end{bmatrix}.$$

Since the initial and final momenta have to be equal, we know that

$$\begin{bmatrix} E_A \\ \vec{0} \end{bmatrix} = \begin{bmatrix} E_B \\ \vec{p}_B \end{bmatrix} + \begin{bmatrix} E_C \\ \vec{p}_C \end{bmatrix}.$$

We are trying to calculate E_B . We see that conservation of 4-momentum requires also conservation of the relativistic 3-momentum. In this case, $\vec{p}_C = -\vec{p}_B$. Next, we use the general relation $E = \sqrt{m^2 + \vec{p}^2}$ to simplify E_A and E_C . We leave E_B since that is what we’re solving for.

$$\begin{bmatrix} m_A^2 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} E_B \\ \vec{p}_B \end{bmatrix} + \begin{bmatrix} \sqrt{m_C^2 + \vec{p}_B^2} \\ -\vec{p}_B \end{bmatrix}.$$

This implies the energy conservation

$$m_A^2 = E_B + \sqrt{m_C^2 + \vec{p}_B^2}.$$

Moving E_B to the left side, squaring both sides, and then solving for \vec{p}_B^2 gives us

$$\vec{p}_B^2 = m_A^2 - 2m_A E_B + E_B^2 - m_C^2.$$

Then from the general formula $E_B = \sqrt{m_B^2 + \vec{p}_B^2}$, we get that

$$\begin{aligned} E_B^2 &= m_B^2 + \vec{p}_B^2 \\ E_B^2 &= m_B^2 + m_A^2 - 2m_A E_B + E_B^2 - m_C^2, \end{aligned}$$

which gives us

$$E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A}.$$

That was the long way of calculating E_B . An easier way is to work with invariants (i.e. scalar products). For example, starting from the equation of 4-momenta $p_A = p_B + p_C$, we can write the square

$$p_C^2 = (p_A - p_B)^2 = p_A^2 - 2p_A \cdot p_B + p_B^2.$$

Just keep in mind that these are 4-vectors. We know that $p_A^2 = m_A^2$, $p_B^2 = m_B^2$, and $p_C^2 = m_C^2$ (Recall the rule that $p^2 = m^2 c^2$). So

$$m_C^2 = m_A^2 - 2p_A \cdot p_B + m_B^2.$$

For the remaining scalar product, we have

$$p_A \cdot p_B = E_A E_B - \vec{p}_A \cdot \vec{p}_B = m_A E_B,$$

since $E_A = m_A$ and $\vec{p}_A = 0$. Plugging this in and rearranging, we get

$$E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A},$$

as before.

Discussion:

- If we wanted to calculate something in a different frame, we could calculate in the frame of particle A and then boost to the desired frame. Typically, this would involve calculating angles as well.
- The example above was for a particle decay. The same procedures and rules apply for scattering problems.

2.6 Charges in External EM Fields

Our goal is to construct a relativistic theory for the interaction of a charged particle with electric and magnetic fields. We will start from the action principle with some action S . Then we obtain the equations of motion from the condition that the action is stationary (i.e. $\delta S = 0$) at the physical point. We assume we will be able to describe \vec{E} and \vec{B} by some 4-vector A^μ . Our building blocks will be $d\tau$ and $A_\mu dx^\mu$, both of which are Lorentz invariant. In the end we will be able to show Lorentz invariance by the contraction of tensors.

We define the action

$$S = \int_a^b (-mc^2 d\tau - qA_\mu dx^\mu),$$

where m is the mass of the particle, and q is the coupling or “charge” of the particle. This is in SI units, and q can be positive, negative, or zero. We can write

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt = \frac{1}{\gamma} dt,$$

and

$$dx^\mu = u^\mu d\tau = \gamma \frac{c}{\vec{v}} \frac{1}{\gamma} dt,$$

where u^μ is the 4-velocity. Then the action can be written

$$S = \int_{t_1}^{t_2} dt \left[-mc^2 \frac{1}{\gamma} - q \left(A_0 c - \vec{A} \cdot \vec{v} \right) \right].$$

In standard classical mechanics, the action has the form $S = \int_{t_1}^{t_2} dt L(t, \vec{x}, \vec{v})$, where $L(t, \vec{x}, \vec{v})$ is the Lagrangian. In our case, the **Lagrangian** is

$$L(t, \vec{x}, \vec{v}) = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - q \left(A_0 c - \vec{A} \cdot \vec{v} \right).$$

Keep in mind that $A_0 = A_0(t, \vec{x})$ could depend on time and space.

The Euler-Lagrange equations of motion are given by

$$0 = \frac{d}{dt} \frac{\partial L}{\partial v_i} - \frac{\partial L}{\partial x_i}.$$

Taking the relevant derivatives, we get

$$\begin{aligned} \frac{\partial L}{\partial v_i} &= m\gamma v_i + qA_i = p_i + qA_i \\ \frac{d}{dt} \frac{\partial L}{\partial v_i} &= \dot{p}_i + q \frac{\partial A_i}{\partial t} + q \frac{\partial A_i}{\partial x_j} \frac{\partial x_j}{\partial t} \\ \frac{\partial L}{\partial x_i} &= -qc \frac{\partial A_0}{\partial x_i} + q \frac{\partial}{\partial x_i} (\vec{A} \cdot \vec{v}). \end{aligned}$$

Remember, for partial derivatives, we only care about *explicit* dependence. Plugging these into the Euler-Lagrange equation and vectorizing, gives us

$$0 = \frac{d}{dt} \vec{p} + q \frac{\partial}{\partial t} \vec{A} + q (\vec{v} \cdot \vec{\nabla}) \vec{A} + qc \vec{\nabla} A_0 - q \vec{\nabla} (\vec{A} \cdot \vec{v}).$$

Using a vector identity, we can write

$$\vec{\nabla} (\vec{A} \cdot \vec{v}) = (\vec{A} \cdot \vec{\nabla}) \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{A} + \vec{A} \times (\vec{\nabla} \times \vec{v}) + \vec{v} \times (\vec{\nabla} \times \vec{A}).$$

The first and third terms on the right are zero. Using this result, our Euler-Lagrange equation becomes the **Lorentz force**

$$\frac{d}{dt} \vec{p} = q\vec{E} + q\vec{v} \times \vec{B},$$

where the fields are

$$\begin{aligned} \vec{E} &= -c \vec{\nabla} A_0 - \frac{\partial}{\partial t} \vec{A} \\ \vec{B} &= \vec{\nabla} \times \vec{A}. \end{aligned}$$

Tip

The vector potential \vec{A} transforms under Lorentz transformations like the spatial part of a 4-vector, since it is. The fields \vec{E} and \vec{B} do not.

The 4-vector A^μ is the **4-potential**

$$A^\mu = \begin{bmatrix} A_0 \\ \vec{A} \end{bmatrix} = \begin{bmatrix} \phi/c \\ \vec{A} \end{bmatrix},$$

where ϕ and \vec{A} are the well-known scalar and vector potentials.

We can rewrite the Lorentz force in the manifestly covariant form

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu,$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

is the **electromagnetic field tensor**. It has components

$$(F^{\mu\nu}) = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}.$$

The **dual field tensor** is defined as

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}.$$

The partial derivatives are defined as

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \begin{bmatrix} \frac{\partial}{c\partial t} \\ -\vec{\nabla} \end{bmatrix}$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \begin{bmatrix} \frac{\partial}{c\partial t} \\ \vec{\nabla} \end{bmatrix}.$$

If we transform the 4-potential as

$$A^\mu \longrightarrow A^\mu - \partial^\mu \Lambda(\vec{x}, t),$$

where $\Lambda(\vec{x}, t)$ is some smooth function, then the field tensor does not change

$$F^{\mu\nu} \longrightarrow \partial^\mu (A^\nu - \partial^\nu \Lambda) - \partial^\nu (A^\mu - \partial^\mu \Lambda) = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}.$$

This transformation is called a **gauge transformation** and the field tensor is **gauge invariant**.

The Lorentz transform of the electric and magnetic fields, separated by components parallel and perpendicular to the direction of the relative motion of the frames, is

$$\begin{aligned} \vec{E}'_{\parallel} &= \vec{E}_{\parallel} \\ \vec{B}'_{\parallel} &= \vec{B}_{\parallel} \\ \vec{E}'_{\perp} &= \gamma (\vec{E}_{\perp} + \vec{v} \times \vec{B}) \\ \vec{B}'_{\perp} &= \gamma \left(\vec{B}_{\perp} - \frac{1}{c^2} \vec{v} \times \vec{E} \right). \end{aligned}$$

2.7 Groups and Tensors

The group of transformations which leave

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

invariant is called the **Poincare group**. It is the semidirect product of the group of translations

$$x^\mu \longrightarrow \Lambda^\mu_\nu x^\nu + a^\nu,$$

and the **Lorentz group** which consists of:

- Boosts (i.e. Lorentz transformations)
- Rotations

$$\begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix},$$

where R is a 3×3 rotation matrix with $\det R = 1$.

- Parity (or space reversal)

$$\begin{bmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{bmatrix},$$

where $\mathbb{1}$ is the 3×3 identity matrix.

- Time reversal

$$\begin{bmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{bmatrix}.$$

A **tensor** of rank n is a quantity with n Lorentz indices. Examples of rank-0 tensors (also called “scalars”) include numbers like 1 and scalar products of 4-vectors like $x \cdot x$. Examples of rank-1 tensors (also called “4-vectors”) are the 4-vectors x^μ , p^μ , and A^μ . Examples of rank-2 tensors (also called “matrices”) include $g_{\mu\nu}$ and $F_{\mu\nu}$. An example of a rank-4 tensor is the totally antisymmetric tensor $\varepsilon^{\mu\nu\kappa\lambda}$ defined with the convention $\varepsilon^{0123} = -\varepsilon_{0123} = \pm 1$.

Every Lorentz index transforms like that of a space-time 4-vector. For example,

$$\begin{aligned} x^\mu &\longrightarrow \Lambda^\mu_{\mu'} x^{\mu'} \\ F^{\mu\nu} &\longrightarrow \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} F^{\mu'\nu'}. \end{aligned}$$

Fully contracted products of tensors are Lorentz invariant or **covariant** because of

$$\Lambda^T g \Lambda = g.$$

For example, to transform a scalar product like

$$x_\mu y^\mu = x^T g y,$$

we can write

$$x'_\mu y^{\mu'} = (\Lambda x)^T g (\Lambda y) = x^T \Lambda^T g \Lambda y = x^T g y = x_\mu y^\mu.$$

In physics, fields are represented by tensors. For example:

symbol	rank	realized for
ϕ	0	Higgs field
A^μ	1	EM photons
$h^{\mu\nu}$	2	graviton

2.8 Maxwell's Equations

Now our goal is to derive the equations of motion for the electromagnetic fields.

In general, the action has the form

$$S = S_{matter} + S_{interactions} + S_{fields}.$$

In the last section, we considered only the first two terms

$$S_{matter} = \int d\tau (-mc^2)$$

$$S_{interactions} = \int d\tau (-qu_\mu A^\mu).$$

In general, S_{fields} is a function of the degrees of freedom which in this case are A^μ and $\partial^\nu A^\mu$

$$S_{fields} = S_{fields}[A^\mu, \partial^\nu A^\mu] = \int_a^b dt L_{fields}.$$

If our system consisted of a finite number of particles, then we would sum over a finite number of degrees of freedom. Now we are dealing with fields, and we have degrees of freedom at every point in space. Some key differences between dealing with particles and dealing with fields are:

<u>particles</u>	<u>fields</u>
\sum_i	$\longrightarrow \int d^3x$
x^μ	$\longrightarrow A^\mu(\vec{x}, t)$
$\frac{d}{dt}x^\mu$	$\longrightarrow \partial^\nu A^\mu(\vec{x}, t)$
$L = \sum_i$	$\longrightarrow L = \int d^3x \mathcal{L}[A^\mu(\vec{x}, t), \partial^\nu A^\mu(\vec{x}, t)]$

Note that \mathcal{L} is the **Lagrangian density**.

The action

$$S[\mathcal{L}] = \int d^4x \mathcal{L},$$

is now a *functional* of \mathcal{L} , hence the square brackets $S = S[...]$. We want S to be covariant, i.e., Lorentz invariant. We know that d^4x is already Lorentz invariant, so we just need to ensure that \mathcal{L} is Lorentz invariant.

We also want the whole thing to be gauge invariant, i.e., invariant under the transformation

$$A^\mu \longrightarrow A^\mu - \partial^\mu \Lambda.$$

We know that $F^{\mu\nu}$ is gauge invariant. The scalar product $A_\mu A^\mu$ is not gauge invariant.

We consider only variations of fields. We write the Lagrangian density as

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu,$$

where the first term on the right is the field part of \mathcal{L} , and the second term is the interaction part. The quantity $j_\mu A^\mu$ is Lorentz invariant, and $F_{\mu\nu} F^{\mu\nu}$ is both Lorentz and gauge invariant. Recall that previously we had

$$L_{interactions} = \int d^3x \mathcal{L}_{interactions} = -qu_\mu A^\mu.$$

Now we have $\mathcal{L}_{interactions} = -j_\mu A^\mu$. The new quantity j^μ is the **4-current density** defined as

$$j^\mu = \begin{bmatrix} c\rho \\ \vec{j} \end{bmatrix},$$

where ρ is the regular charge density, and \vec{j} is the regular current density. We treat j^μ here as an external (i.e. non-varying) source.

The action should be stationary, so

$$\begin{aligned} 0 = \delta S &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial A^\mu} \delta A^\mu + \frac{\partial \mathcal{L}}{\partial (\partial^\nu A^\mu)} \delta (\partial^\nu A^\mu) \right) \\ &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial A^\mu} \delta A^\mu + \frac{\partial \mathcal{L}}{\partial (\partial^\nu A^\mu)} \partial^\nu (\delta A^\mu) \right) \\ &= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial (\partial^\nu A^\mu)} \right) \delta A^\mu. \end{aligned}$$

This implies that the quantity in parentheses is zero, so

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial^\mu A^\nu)} - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0.$$

This is the Euler-Lagrange equation of motion for fields.

In our case, we can write the Lagrangian density in terms of the degrees of freedom A^ν and $\partial^\mu A^\nu$ as

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \\ &= -\frac{1}{2\mu_0} \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu \right) - j_\mu A^\mu \\ &= -\frac{1}{2\mu_0} \left(\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha \right) - j_\mu A^\mu. \end{aligned}$$

Then

$$\frac{\partial \mathcal{L}}{\partial (\partial^\mu A^\nu)} = -\frac{1}{2\mu_0} \left(2g^\alpha_\mu g^\beta_\nu \partial_\alpha A_\beta - 2g^\beta_\mu g^\alpha_\nu \partial_\alpha A_\beta \right) = -\frac{1}{\mu_0} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) = -\frac{1}{\mu_0} F_{\mu\nu}.$$

Note that

$$g^\alpha_\mu = \delta^\alpha_\mu = \begin{cases} 1 & \text{if } \alpha = \mu \\ 0 & \text{else} \end{cases}.$$

Then

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial^\mu A^\nu)} = -\frac{1}{\mu_0} \partial^\mu F_{\mu\nu}.$$

The other term in the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -j_\mu g^\mu_\nu = -j_\nu.$$

Plugging both terms into the Euler-Lagrange equation and simplifying gives us

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu.$$

Let's consider the $\nu = 0$ component of this equation. We find that

$$\vec{\nabla} \cdot \frac{\vec{E}}{c} = \mu_0 (c\rho),$$

where $c\rho = J^0$. Then, defining

$$c^2 \equiv \frac{1}{\mu_0 \epsilon_0},$$

we get

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

For the $\nu = 1$ component, we find

$$-\frac{1}{c^2} \frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 j_x.$$

We find similar equations for $\nu = 2$ and $\nu = 3$. Putting them all together and writing them in vector form gives us

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}.$$

Thus, the relativistic equation $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ gives us the familiar inhomogeneous Maxwell equations.

Recall that

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}.$$

If we wanted to work with $F^{\mu\nu}$ instead of A^μ , then we would look at

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} (\partial_\mu \partial_\alpha A_\beta - \partial_\mu \partial_\beta A_\alpha) = 0.$$

We know this is zero because we have a symmetric quantity (the quantity in parentheses) contracted with an antisymmetric quantity ($\epsilon^{\mu\nu\alpha\beta}$). Thus,

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0.}$$

In components, we get Maxwell's homogeneous equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0. \end{aligned}$$

Earlier, we wrote the interaction part of the action as

$$S_{int} = -q \int dx^\mu A_\mu = -q \int dt \frac{1}{\gamma} u^\mu A_\mu(\vec{x}_0, t).$$

We also had

$$S_{int} = - \int d^4x j^\mu A_\mu = - \int dt \int d^3x j^\mu(\vec{x}, t) A_\mu(\vec{x}, t).$$

By comparison,

$$q \frac{1}{\gamma} u^\mu A_\mu(\vec{x}_0, t) = \int d^3x j^\mu A_\mu(\vec{x}, t).$$

This implies, after writing $u^\mu/\gamma = c/\vec{v}$, that the **4-current** for a **point charge** at \vec{x}_0 is

$$j^\mu = \begin{bmatrix} c \\ \vec{v} \end{bmatrix} \delta^{(3)}(\vec{x} - \vec{x}_0).$$

While this doesn't look relativistic, the delta-function ensures that it is.

Continuity Equation

From the general equation

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu,$$

we can take the partial derivative to get

$$\partial_\nu \partial_\mu F^{\mu\nu} = \mu_0 \partial_\nu j^\nu.$$

We know that $\partial_\nu \partial_\mu$ is symmetric in μ and ν , and we know that $F^{\mu\nu}$ is antisymmetric. Since a symmetric quantity contracted with an antisymmetric quantity is zero, this implies

$$\partial_\mu j^\mu = 0.$$

In components, this gives us the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0,$$

of charge conservation.

2.9 Summary: Special Relativity

Skills to Master

- Be able to apply time dilation and length contraction to solve problems
- Be able to apply Lorentz transformations to calculate boosted 4-vectors
- Use Minkowski diagrams to understand problems graphically
- Be able to calculate the Doppler effect and the aberration of light
- Use invariant quantities and conservation laws to solve particle decay and scattering problems
- Be able to derive the EM field tensor $F^{\mu\nu}$ components from its definition in terms of A^μ
- Be able to apply Lorentz transformation to electric and magnetic fields

Relativity

We will often utilize the picture of a stationary frame F and a frame F' that is moving with velocity v_0 relative to frame F . Unless noted otherwise, v_0 will be along the x direction, and $t = t' = 0$ when the origins of F and F' coincide. All primed quantities will refer to quantities measured in the moving frame F' , and all unprimed quantities will refer to those measured in the stationary frame F .

Special relativity follows from two axioms:

1. The laws of physics are form invariant
2. The speed of light in vacuum has the same value c in every inertial (i.e. non-accelerating) frame

One consequence is that there is no absolute simultaneity. Even the time-ordering of events can change depending on your frame of reference.

If a light is flashed at the origin of frame F at time $t = 0$ (this also means it flashes at the origin of frame F' at $t' = 0$), then the expanding light front in both frames is the surface of a sphere. Since c is the same in both frames, the points on this surface satisfy the Michelson-Morley condition

$$c^2 t^2 - (x^2 + y^2 + z^2) = c^2 t'^2 - (x'^2 + y'^2 + z'^2).$$

Lorentz Transformations

If t , x , y , and z are measured in frame F , then the Lorentz transformation or “boost” gives us the quantities as measured in frame F'

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z, \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v_0}{c}.$$

Notice that the directions perpendicular to the boost are not affected. To get the inverse transformation ($F' \rightarrow F$), just switch the primes and let $\beta \rightarrow -\beta$.

For the transformation of velocity; if in frame F , $\vec{v} = (v_x, v_y, v_z)$, then in frame F' ,

$$\vec{v}' = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'} \right).$$

Then we can get dx' , dy' , dz' , and dt' by taking the differentials of the Lorentz transformation. After plugging those in and simplifying, we get

$$\vec{v}' = \left(\frac{v_x - v_0}{1 - \frac{v_0 v_x}{c^2}}, \frac{v_y}{\gamma \left(1 - \frac{v_0 v_x}{c^2}\right)}, \frac{v_z}{\gamma \left(1 - \frac{v_0 v_x}{c^2}\right)} \right).$$

To get the inverse transformation $\vec{v}' \rightarrow \vec{v}$, just exchange the primes and let $\vec{v}_0 \rightarrow -\vec{v}_0$.

For the transformation of acceleration, we have in frame F , $\vec{a} = (a_x, a_y, a_z)$, then in frame F' , we use the chain rule to write

$$\vec{a}' = \left(\frac{dv'_x}{dt} \frac{dt}{dt'}, \frac{dv'_y}{dt} \frac{dt}{dt'}, \frac{dv'_z}{dt} \frac{dt}{dt'} \right).$$

Then we plug in dt'/dt and differentiate each of the velocity components v'_i with respect to t and plug those in.

An object of length ℓ at rest in frame F and lying along the x -axis, has length

$$\ell' = \frac{\ell}{\gamma}, \quad \text{where always } \gamma > 1,$$

in frame F' . This is *length contraction*. In general, a moving length always appears shorter. Keep in mind that this contraction only happens along the direction of the boost. For example, a moving volume will only be contracted along one dimension—not along all three dimensions.

If two events at the origin of frame F are separated by a time interval τ , then in the moving frame F' , those events will be separated by a time interval

$$\tau' = \gamma\tau, \quad \text{where always } \gamma > 1,$$

This is *time dilation*. In general, a moving clock runs slower.

Consider a particle moving with speed v_0 . If τ is the (proper) time coordinate in the particle's rest frame, and t is the time coordinate in your frame, then

$$d\tau = \frac{1}{\gamma} dt = \sqrt{1 - \frac{v_0^2}{c^2}} dt.$$

This can also be derived from the relation $c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$.

Four-Vectors

A *contravariant* 4-vector is a 4-component vector of the form

$$a^\mu = \begin{bmatrix} a_0 \\ \vec{a} \end{bmatrix},$$

where \vec{a} is a regular 3-vector, and the Lorentz index is $\mu = 0, 1, 2, 3$. The corresponding *covariant* 4-vector is

$$a_\mu = \begin{bmatrix} a_0 \\ -\vec{a} \end{bmatrix}.$$

They are related via

$$a^\mu = g^{\mu\nu} a_\nu, \quad a_\mu = g_{\mu\nu} a^\nu,$$

where

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix},$$

is the metric tensor of Minkowski space. To raise or lower a single Lorentz index of a 4-vector or tensor, multiply the object by the metric tensor. The contraction of the two objects will result in the three spatial components associated with the index receiving a negative sign.

The scalar product of two 4-vectors is

$$a \cdot b = a_\mu b^\mu = a^\mu g_{\mu\nu} b^\nu = a^T g b = a_0 b_0 - \vec{a} \cdot \vec{b}.$$

In general, we use Einstein summation for repeated indices. The scalar product of 4-vectors is invariant under rotations, parity, time reversal, and most importantly—Lorentz transformations. Therefore, the scalar product of two 4-vectors as measured in frames under relative motion will be equal

$$a \cdot b = a' \cdot b'.$$

Some useful 4-vectors include

$x^\mu = \begin{bmatrix} ct \\ \vec{x} \end{bmatrix},$	\leftarrow position
$u^\mu = \gamma \begin{bmatrix} c \\ \vec{v} \end{bmatrix},$	\leftarrow velocity
$p^\mu = m \frac{dx^\mu}{d\tau} = m\gamma \begin{bmatrix} c \\ \vec{v} \end{bmatrix} = \begin{bmatrix} E/c \\ \vec{p} \end{bmatrix},$	\leftarrow momentum
$k^\mu = \begin{bmatrix} \omega/c \\ \vec{k} \end{bmatrix},$	\leftarrow wave vector
$A^\mu = \begin{bmatrix} \phi/c \\ \vec{A} \end{bmatrix},$	\leftarrow EM potential
$j^\mu = \begin{bmatrix} c\rho \\ \vec{j} \end{bmatrix},$	\leftarrow current density
$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \begin{bmatrix} \frac{1}{c} \frac{\partial}{\partial t} \\ -\vec{\nabla} \end{bmatrix},$	\leftarrow derivative.

Note, that \vec{p} , wherever it appears, is defined to be the *relativistic* 3-momentum

$$\vec{p} = \gamma m \vec{v},$$

which is the ordinary 3-momentum multiplied by γ .

Since p^2 is invariant, including in a particle's rest frame, we have the general relation

$$p^2 = m^2 c^2.$$

The relativistic energy is

$$E = \gamma m c^2 = \sqrt{m^2 c^4 + \vec{p}^2 c^2}.$$

For massless particles,

$$E = |\vec{p}|c.$$

The 4-current for a point charge at \vec{x}_0 is

$$j^\mu = q \begin{bmatrix} c \\ \vec{v} \end{bmatrix} \delta^{(3)}(\vec{x} - \vec{x}_0).$$

The Lorentz transformation of a 4-vector a is

$$a'^\mu = \Lambda^\mu_\nu a^\nu.$$

So if a is the 4-vector in frame F , then a' is that 4-vector in F' . The Λ^μ_ν are the elements of the Lorentz transformation matrix. If the relative motion is occurring only along the x -direction, then

$$\Lambda = [\Lambda^\mu_\nu] = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then we can write the transformation as the matrix-vector equation $a' = \Lambda a$.

Given a 4-vector a in frame F , get the transformed vector a' by $a' = \Lambda a$. Given a' (i.e. the 4-vector in the moving frame), then $a = \Lambda^{-1} a'$. The only difference between Λ and Λ^{-1} is that the minuses in Λ are turned to pluses in Λ^{-1} .

Alternatively, we can write the Lorentz matrix as

$$(\Lambda^\mu_\nu) = \begin{bmatrix} \cosh \zeta & -\sinh \zeta & 0 & 0 \\ -\sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where ζ is the *rapidity* defined as

$$\cosh \zeta = \gamma, \quad \tanh \zeta = \beta.$$

Note that rapidities add directly. For two consecutive boosts in the x -direction with rapidities ζ_1 and ζ_2 , the overall rapidity is $\zeta_1 + \zeta_2$. This is unlike the complicated equation for the addition of velocities.

In general, the Lorentz matrix (Λ^μ_ν) is a symmetric matrix with

$$\det(\Lambda^\mu_\nu) = 1.$$

In general,

$$\Lambda^T g \Lambda = g.$$

This forces fully contracted products of tensors to be Lorentz invariant. For example, for the scalar product,

$$x'_\mu y'^\mu = (\Lambda x)^T g (\Lambda y) = x^T \Lambda^T g \Lambda y = x^T g y = x_\mu y^\mu.$$

Applications

If x is the spacetime interval between two events, then the sign of x^2 tells you if the two events could be causally connected

- If $x^2 > 0$, then the relation between the two events is called *time-like*
- If $x^2 = 0$, then the relation is *light-like*
- If $x^2 < 0$, then the relation is *space-like* (i.e. causally disconnected)

We can draw frames F and F' on a single Minkowski diagram. Then to scale between the two frames, we use a Lorentz invariant like

$$c^2 t^2 - x^2 = c t'^2 - x'^2 = \text{const.}$$

This is the equation of a hyperbola with the light lines as asymptotes. Then, we can relate a unit length ℓ (or unit time) in frame F to the length scale ℓ' in frame F' by tracing along the hyperbolae.

Consider a light source at rest in frame F . At a point on the light front, the wave is propagating outward with some wave vector k^μ whose spatial part \vec{k} makes some angle θ with the x -axis. In frame F' moving in the x -direction with speed β relative to F , the same point on the wave front has wave vector k'^μ and makes an angle θ' with the x -direction. The two wave vectors are related by the Lorentz transformation $k'^\mu = \Lambda^\mu_\nu k^\nu$. Write the wave vectors in terms of ω and θ and ω' and θ' . Remember that $|\vec{k}| = \omega/c$ for a light wave. Then the zeroth component of the Lorentz transformation gives us

$$\omega' = \frac{\omega}{\gamma(1 - \beta \cos \theta)}.$$

This shift of the light's frequency is called the *Doppler effect*. Similarly, the one-component of the Lorentz transformation gives us

$$\cos \theta' = \frac{\cos \theta + \beta}{1 + \beta \cos \theta}.$$

This change in the direction of the light's propagation as seen from different frames is called the *aberration of light*.

To solve scattering problems and decays, we use conservation laws and invariant quantities. In general, total momentum is conserved,

$$\sum_i p_i^\mu = \sum_f p_f^\mu.$$

General approaches include:

- Write down the conservation of momentum equation in terms of vectors. The first component implies energy conservation. The other components imply conservation of relativistic 3-momentum. Simplify the energies using $E = \sqrt{m^2 c^4 + \vec{p}^2 c^2}$
- Write down the conservation of momentum equation. Rearrange (if necessary), and square both sides to get the invariant quantity p^2 . Simplify using $p^2 = m^2 c^2$ and $p_A \cdot p_B = E_A E_B / c^2 - \vec{p}_A \cdot \vec{p}_B$.

Charged Particles in EM Fields

In special relativity, electromagnetic fields \vec{E} and \vec{B} , are ultimately derived from the 4-potential A^μ .

The relativistic Lagrangian for a charged particle in electromagnetic fields is

$$L(t, \vec{x}, \vec{v}) = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - q \left(A_0 c - \vec{A} \cdot \vec{v} \right).$$

Plugging this into the action $S = \int dt L$ and minimizing, yields the Euler-Lagrange equation, which gives us

the Lorentz force

$$\frac{d}{dt}\vec{p} = q\vec{E} + q\vec{v} \times \vec{B},$$

where the fields are

$$\begin{aligned}\vec{E} &= -c\vec{\nabla}A_0 - \frac{\partial}{\partial t}\vec{A} \\ \vec{B} &= \vec{\nabla} \times \vec{A}.\end{aligned}$$

The Lorentz force can be written in the form

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu,$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

is the electromagnetic field tensor. It is antisymmetric and has components

$$(F^{\mu\nu}) = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}.$$

It transforms under Lorentz transformation as

$$F'^{\alpha\beta} = \Lambda^\alpha_\mu \Lambda^\beta_\nu F^{\mu\nu} \iff F' = \Lambda F \Lambda^T.$$

The dual field tensor is defined as

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}.$$

The field tensor, and thereby the fields, are gauge invariant. If we make the gauge transformation

$$A^\mu \longrightarrow A^\mu - \partial^\mu \Lambda(\vec{x}, t),$$

where $\Lambda(\vec{x}, t)$ is some smooth function, then the field tensor does not change.

The Lorentz transform of the electric and magnetic fields is

$$\begin{aligned}\vec{E}'_\parallel &= \vec{E}_\parallel \\ \vec{B}'_\parallel &= \vec{B}_\parallel \\ \vec{E}'_\perp &= \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}) \\ \vec{B}'_\perp &= \gamma\left(\vec{B}_\perp - \frac{1}{c^2}\vec{v} \times \vec{E}\right).\end{aligned}$$

This comes from the Lorentz transformation $F'^{\mu\nu} = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} F^{\mu'\nu'}$, which is equivalent to the matrix multiplication $F' = \Lambda F \Lambda^T$. Notice that the components parallel to the direction of the relative motion of the frames do not change.

Maxwell's Equations

For the electromagnetic fields (instead of a particle within the fields), the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu.$$

Then the Lagrangian is $L = \int d^3x \mathcal{L}$, and the action is $S = \int dt L$. Minimizing the action and solving the resulting Euler-Lagrange equation gives us

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu.$$

This gives us the inhomogeneous Maxwell equations. The $\nu = 0$ component of this gives us the scalar equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

The $\nu = 1, 2, 3$ components give us the vector equation

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}.$$

If we use the dual tensor $\tilde{F}^{\mu\nu}$ in the Lagrangian instead of $F^{\mu\nu}$, then we get

$$\partial_\mu \tilde{F}^{\mu\nu} = 0.$$

The components of this gives us Maxwell's homogeneous equations

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0.\end{aligned}$$

Note that

$$c^2 \equiv \frac{1}{\mu_0 \epsilon_0}.$$

If we differentiate $\partial_\nu(\partial_\mu F^{\mu\nu}) = \partial_\nu(\mu_0 j^\nu)$, we get

$$\partial_\mu j^\mu = 0.$$

This is the continuity equation, and in components,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0.$$

Chapter 3

Electrostatics

Recall Maxwell's equations

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} &= \mu_0 \vec{J}.\end{aligned}$$

Consider now the simplified case where the current and charge densities do not change in time

$$\frac{\partial \rho}{\partial t} = \frac{\partial \vec{J}}{\partial t} = 0,$$

the electric and magnetic fields do not change in time

$$\frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{B}}{\partial t} = 0,$$

and everything is in a vacuum. Then, Maxwell's equations simplify to the equations of **electrostatics**

$$\boxed{\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} &= 0.\end{aligned}}$$

and **magnetostatics**

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J}.\end{aligned}$$

We will study magnetostatics in the next chapter.

3.1 Gauss's Law

The first of Maxwell's equations of electrostatics is

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

If we integrate this over some volume V ,

$$\int_V \vec{\nabla} \cdot \vec{E} dV = \frac{1}{\epsilon_0} \int_V \rho dV,$$

then we can rewrite the left-hand side by using the divergence theorem

$$\boxed{\int_{A(V)} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int_V \rho dV = \frac{Q_{enc}}{\epsilon_0}.}$$

This is Gauss's law. The integral on the left is over the surface $A(V)$ of the volume V . Note that $d\vec{A} = \hat{n} dA$, where dA is an elemental area of the surface and \hat{n} is the unit vector normal to the surface at that point. The second integral is the integral of the charge density ρ over the volume, so it gives the total charge enclosed in the volume V .

Note that Gauss's law is only useful for simple distributions with some kind of symmetry. If you can solve a problem using Gauss's law, then that is usually the fastest way of doing it.

Similarly, we can integrate the second equation of electrostatics over some surface A

$$\int_A (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = 0.$$

Then we can rewrite the left-hand side using Stoke's theorem to get

$$\oint \vec{E} \cdot d\vec{\ell} = 0,$$

where the closed line integral is over the closed path along the border of the surface A .

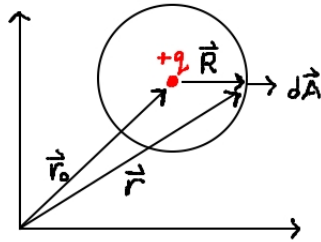
Point Charges

What is the electric field \vec{E} produced by a point charge q at \vec{r}_0 ?

The charge density for a single point charge q at \vec{r}_0 is

$$\rho(\vec{r}) = q\delta^{(3)}(\vec{r} - \vec{r}_0).$$

Note that the particle sits at \vec{r}_0 in our coordinate system, and we are measuring the field from the position \vec{r} . We will denote the vector pointing from \vec{r}_0 to \vec{r} as $\vec{R} = \vec{r} - \vec{r}_0$. To use Gauss's law, we consider a sphere of radius $R = |\vec{R}|$ centered on the particle at \vec{r}_0 . Now the measurement point \vec{r} is somewhere on the surface of the sphere. The surface element $d\vec{A}$ is a vector with magnitude dA that points outward normally to the surface of the sphere at point \vec{r} .



Since there's only a single charge at the center of the sphere, we know that $\vec{E} \parallel d\vec{A}$, and we know the field is radially symmetric, so we can write it in the form

$$\vec{E} = f(R) \frac{\vec{R}}{|\vec{R}|},$$

where $f(R)$ is a unitless function that gives the magnitude of the field, and $\vec{R}/|\vec{R}|$ is included to get the right units. Then Gauss's law reduces to

$$\int dA f(R) = \frac{q}{\epsilon_0}.$$

The function $f(R)$ is independent of position on the surface of the Gaussian sphere, so we can pull it outside of the integral, then $\int dA$ is just the surface area of the sphere

$$f(R) \int dA = 4\pi R^2 f(R) = \frac{q}{\epsilon_0}.$$

Thus, we find that the field of a point charge q at \vec{r}_0 is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3}.$$

If we have multiple point charges, then we can add their fields using the **principle of superposition**

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3},$$

where i goes over the particles.

For a smooth charge distribution, i.e. for macroscopic objects, consider the charge $dq = \rho dV$ in a small volume dV . Then

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}. \quad (3.1)$$

This is the formula for the “direct integration” of the field \vec{E} generated by a continuous charge distribution ρ .

Charge Distributions

For a **point charge** at the origin, the charge distribution in Cartesian coordinates is

$$\rho(\vec{r}) = \delta^{(3)}(\vec{r}) = \delta(x)\delta(y)\delta(z).$$

We will also have to consider **surface charge** distributions when we have infinitesimally thin surfaces. For example, if we have an infinitesimally thin sheet of charge in the xy -plane, then the charge distribution has the form

$$\rho(\vec{r}) = G(x, y) \delta(z),$$

where $G(x, y)$ is the Cartesian surface charge distribution. If we are in coordinates other than Cartesian coordinates, then the form of $\rho(\vec{r})$ will look different. However, we can always parametrize it to get it in the above form so that we can read off the Cartesian surface charge distribution $G(x, y)$.

For a line of charge along the z -axis, we would write

$$\rho(\vec{r}) = \lambda(z) \delta(x) \delta(y),$$

where $\lambda(z)$ is the **line charge** distribution in Cartesian coordinates.

Example 3.1.1

Write down the charge distribution $\rho(\vec{r})$ in cylindrical and in spherical coordinates if the charge distribution is a ring of radius R , lying in the xy -plane, centered on the z -axis, and carrying a line charge density $\lambda(\phi)$.

For spherical coordinates, the temptation might be to write

$$\rho(\vec{r}) = \lambda(\phi)\delta(r - R)\delta(\theta - \pi/2).$$

But this is wrong! Notice that the units are incorrect. The left-hand side has units of charge over length-cubed. Recalling that the units of the delta function is the reciprocal of its argument, we see that the right-hand side has units of charge over length-squared. In spherical coordinates, $\delta^3(\vec{r}) \neq \delta(r)\delta(\theta)\delta(\phi)$. Rather,

$$\delta^3(\vec{r}) = \delta(r) \frac{\delta(\theta)}{r} \frac{\delta(\phi)}{r \sin \theta}.$$

The correct charge distribution in our case is

$$\rho(\vec{r}) = \frac{\lambda(\phi)\delta(r - R)\delta(\theta - \pi/2)}{r \sin \theta}.$$

In cylindrical coordinates, we have

$$\rho(\vec{r}) = \lambda(\phi)\delta(s - R)\delta(z).$$

Given the charge distribution $\rho(\vec{r})$, we can obtain the electric field by direct integration via Eq. (3.1). Given the electric field, we can compute the charge density by rearranging one of the Maxwell equations

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}.$$

Note, whenever you have a field that behaves as

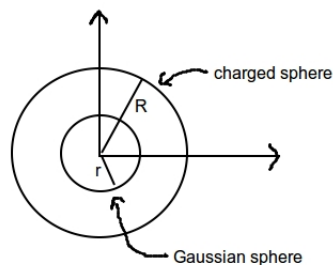
$$\vec{E} \approx \text{const} \cdot \frac{\vec{r}}{|\vec{r}|^3} = \text{const} \cdot \frac{\hat{r}}{r^2},$$

when $\vec{r} \approx 0$, then this behavior is being generated by a point charge at the origin. I.e., its generation requires a $\delta^{(3)}(\vec{r})$ distribution. When you see that, it is typically helpful to treat the point at the origin separately.

Charged Sphere

Consider a charged sphere of radius R with a spherically symmetric charge distribution $\rho(\vec{r}) = \rho(r)$.

We will consider a Gaussian sphere of radius r , where $r \leq R$ to calculate the field inside the charged sphere or $r \geq R$ to calculate the field outside the charged sphere.



We then apply Gauss's law

$$\int_{A(V)} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int_V dV \rho.$$

The radial symmetry of the charge distribution implies that $\vec{E}(\vec{r}) = E(r) \hat{r}$ and $d\vec{A} = dA \hat{r}$. Then

$$\int dA E(r) = E(r) \int dA = E(r) 4\pi r^2 = \frac{1}{\epsilon_0} Q(r),$$

where

$$Q(r) = \int_{V(r)} \rho dV,$$

is the charge contained within the Gaussian sphere of radius r . So the electric field of the charged sphere is

$$\vec{E}(\vec{r}) = E(r) \hat{r} = \frac{1}{4\pi\epsilon_0} Q(r) \frac{\hat{r}}{r^2}.$$

For a *uniformly* charged sphere, we can write the charge distribution as

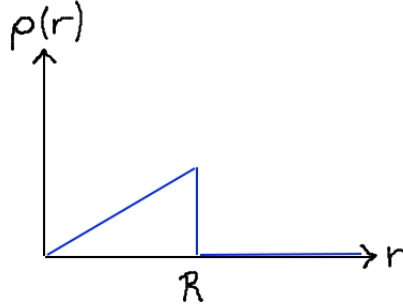
$$\rho(r) = \rho_0 \theta(R - r),$$

where the Heaviside step function $\theta(R - r)$ is +1 for $r \leq R$, and 0 for $r > R$.

An example of a non-uniformly charged sphere is

$$\rho(r) = \rho_0 \frac{r}{R} \theta(R - r).$$

A plot of this charge distribution is shown below:



Then if $r \leq R$, the charge enclosed is

$$Q(r) = \int d^3r' \rho(\vec{r}') = \int d\Omega \int_0^r dr' r'^2 \rho_0 \frac{r'}{R} \theta(R - r') = 4\pi \frac{\rho_0}{R} \int_0^r r'^3 dr' = \frac{\pi\rho_0}{R} r^4.$$

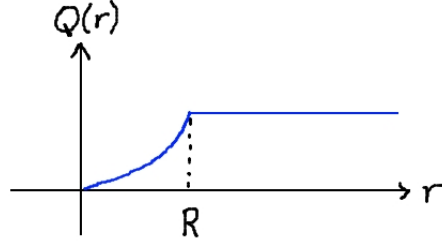
To get the total charge of the whole charged sphere, we just plug in R to get

$$Q_{tot} = Q(R) = \pi\rho_0 R^3.$$

Notice that in general, we can write

$$Q(r) = Q_{tot} \left(\frac{r}{R} \right)^4.$$

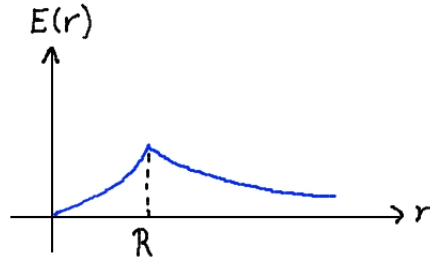
For $r \geq R$, i.e. outside the sphere, the charge enclosed is just $Q(r) = Q_{tot}$ since the θ -function cuts off the integral at that point.

**Tip**

Whenever you compute quantities, make a habit of checking the units of your final quantity.

For the electric field, we get

$$\vec{E}(\vec{r}) = \begin{cases} \frac{Q_{tot}}{4\pi\epsilon_0} \frac{1}{r^2} \left(\frac{r}{R}\right)^4 \hat{r} & \text{for } r < R \\ \frac{Q_{tot}}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} & \text{for } r > R \end{cases}$$



3.2 Electrostatic Potential

Helmholtz Theorem

The Helmholtz theorem gives us an alternative way to obtain \vec{E} from ρ . It states that for any smooth function $\vec{C}(\vec{r})$,

$$\vec{C}(\vec{r}) = -\frac{1}{4\pi} \vec{\nabla} \int d^3r' \frac{\vec{\nabla}' \cdot \vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{1}{4\pi} \vec{\nabla} \times \int d^3r' \frac{\vec{\nabla}' \times \vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|}.$$

This theorem is valid whenever these two integrals are well defined. Note that $\vec{\nabla}'$ acts on \vec{r}' (not \vec{r}).

We apply this theorem to the electrostatic field

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi} \vec{\nabla} \int d^3r' \frac{\vec{\nabla}' \cdot \vec{E}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{1}{4\pi} \vec{\nabla} \times \int d^3r' \frac{\vec{\nabla}' \times \vec{E}(\vec{r}')}{|\vec{r} - \vec{r}'|}.$$

We know that $\vec{\nabla}' \times \vec{E}(\vec{r}') = 0$, so the second integral is zero in this case. We also know that $\vec{\nabla}' \cdot \vec{E}(\vec{r}') = \rho(\vec{r}')/\epsilon_0$, so the above simplifies to

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi} \vec{\nabla} \int d^3r' \frac{\rho(\vec{r}')/\epsilon_0}{|\vec{r} - \vec{r}'|}.$$

So we can write the electric field as the gradient of a scalar function

$$\boxed{\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})}, \quad (3.2)$$

where

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} + \text{const.}$$

We choose the constant to be zero. Otherwise, we get contributions from all distant charges since a constant does not go to zero as $r \rightarrow \infty$. By setting the constant to zero, we get $\phi \rightarrow 0$ as $r \rightarrow \infty$. So our **scalar potential** is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|}. \quad (3.3)$$

Note, for a point charge at \vec{r}_0 , we find the scalar potential

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_0|}.$$

What happens if we take the gradient of this potential? We find that

$$\begin{aligned} \vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left((x - x')^2 + (y - y')^2 + (z - z')^2 \right)^{-1/2} \\ &= \left(-\frac{2(x - x')}{2|\vec{r} - \vec{r}'|^3}, -\frac{2(y - y')}{2|\vec{r} - \vec{r}'|^3}, -\frac{2(z - z')}{2|\vec{r} - \vec{r}'|^3} \right) \\ &= -\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}. \end{aligned}$$

Using this, we can take the gradient of the potential given in Eq. (3.3) then plug it into Eq. (3.2), and we get

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}.$$

This is the same result we got earlier for the electric field of a continuous charge distribution.

3.3 Electric Force and Work

The force due to an electrostatic field \vec{E} on a test charge q is

$$\vec{F} = q\vec{E}.$$

A “test” charge means that it probes the field but does not change it.

The mechanical work done to bring a charge from \vec{r}_i to \vec{r}_f along a path C , is

$$W_{\vec{r}_i \rightarrow \vec{r}_f} = \int_C \vec{F} \cdot d\vec{\ell} = q \int_C \vec{E} \cdot d\vec{\ell} = -q \int_C (\vec{\nabla} \phi) \cdot d\vec{\ell}.$$

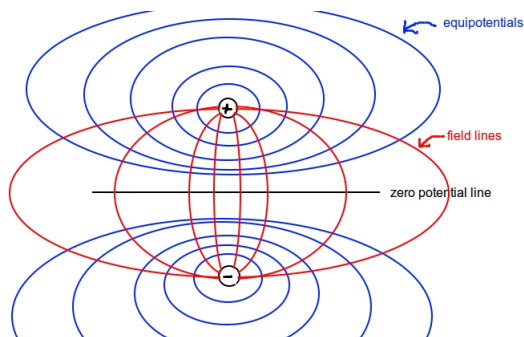
Thus,

$$W_{\vec{r}_i \rightarrow \vec{r}_f} = -q \left(\phi(\vec{r}_f) - \phi(\vec{r}_i) \right).$$

This is path-independent, which means the electric force is conservative.

The equation $W = q \int \vec{E} \cdot d\vec{\ell}$ implies that no work is done for motion perpendicular to \vec{E} . So along those lines, the electric potential ϕ is constant. We call them “equipotential surfaces”.

In the image below, the red lines illustrate a few of the field lines for an electric dipole, and the blue lines show the equipotential surfaces.



3.4 Electric Moments

Electric Monopole

Recall that for a point charge q , the potential is

$$\phi_{mono} = \frac{q}{4\pi\epsilon_0} \frac{1}{r},$$

and it goes as

$$\phi_{mono} \sim \frac{1}{r}.$$

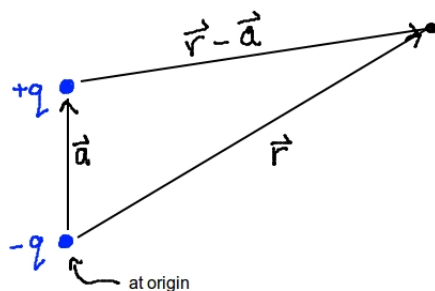
For a point charge q , the monopole is just the charge q . For a general charge distribution $\rho(\vec{r}')$, the monopole is the total charge

$$Q = \int d^3r' \rho(r').$$

The monopole is also called the *zeroth moment*.

Electric Dipole

Consider an electric dipole with a charge $-q$ at the origin, a charge $+q$ at \vec{a} , and the observer at \vec{r} , as shown below.



Then the potential is

$$\phi = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{|\vec{r} - \vec{a}|} - \frac{1}{|\vec{r}|} \right).$$

We want to consider the large distance behavior. That is, we want to look at the $a \ll r$ limit.

In general, for a Taylor expansion for small a , we can write

$$f(\vec{r} + \vec{a}) \approx f(\vec{r}) + a_x \frac{\partial f}{\partial x} \Big|_{\vec{r}} + a_y \frac{\partial f}{\partial y} \Big|_{\vec{r}} + a_z \frac{\partial f}{\partial z} \Big|_{\vec{r}} + \dots \approx f(\vec{r}) + \vec{a} \cdot \vec{\nabla} f.$$

In our case,

$$\frac{1}{|\vec{r} - \vec{a}|} \approx \frac{1}{|\vec{r}|} - \vec{a} \cdot \vec{\nabla} \frac{1}{|\vec{r}|} + \dots$$

So

$$\phi = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{|\vec{r}|} - \vec{a} \cdot \vec{\nabla} \frac{1}{|\vec{r}|} + \dots - \frac{1}{|\vec{r}|} \right) \approx \frac{1}{4\pi\epsilon_0} q \vec{a} \cdot \frac{\hat{r}}{r^2}.$$

If we write

$$\boxed{\vec{p} = q\vec{a}},$$

for the **dipole moment**, then

$$\boxed{\phi_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}},$$

is the potential of a point dipole. Notice that this formula is independent of the coordinate-system, and that it goes as

$$\phi_{dip} \sim \frac{1}{r^2}.$$

Note that $\vec{p} = q\vec{a}$ is the dipole moment for a pair of point charges separated by a distance a . For a general charge distribution $\rho(\vec{r}')$, the dipole moment is

$$\boxed{\vec{p} = \int d^3r' \rho(r') \vec{r}'}.$$

This is also called the *first moment*.

The electric field of the dipole is

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3}}.$$

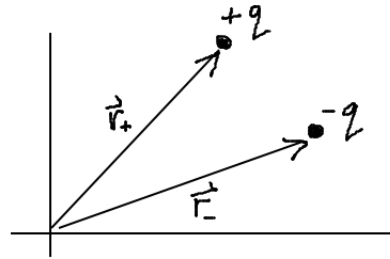
Note, this is for an ideal point dipole. It can be used to approximate the electric fields of non-ideal dipoles at large distances.

Next, consider the force on a dipole that is in an *external* field \vec{E} . The force on the minus charge is

$$\vec{F}_- = -q\vec{E}(\vec{r}_-),$$

and the force on the positive charge is

$$\vec{F}_+ = q\vec{E}(\vec{r}_+).$$



If we define $\vec{r} \equiv \vec{r}_-$, and $\vec{r}_+ = \vec{r} + \vec{a}$, then

$$\vec{F}_- = -q\vec{E}(\vec{r}),$$

and

$$\vec{F}_+ = q\vec{E}(\vec{r} + \vec{a}) \approx q\vec{E}(\vec{r}) + q\vec{a} \left(\vec{\nabla} \cdot \vec{E}(\vec{r}) \right).$$

Then the total force on the dipole is

$$\vec{F} = \vec{F}_- + \vec{F}_+ = q\vec{a} \left(\vec{\nabla} \cdot \vec{E}(\vec{r}) \right) = \vec{p} \left(\vec{\nabla} \cdot \vec{E}(\vec{r}) \right).$$

This can be written as

$$\vec{F}_{dip} = \vec{p} \left(\vec{\nabla} \cdot \vec{E} \right) = \vec{\nabla} \left(\vec{p} \cdot \vec{E} \right),$$

since \vec{p} is constant and \vec{E} is curl-less.

The potential energy of the dipole is

$$U = - \int \vec{F} \cdot d\vec{\ell} = - \vec{p} \cdot \vec{E}.$$

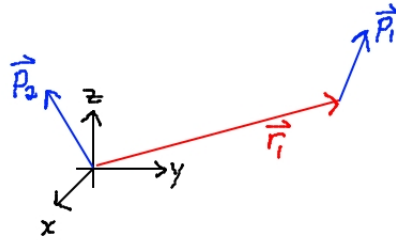
The torque on the dipole is

$$\begin{aligned} \vec{\tau} &= \vec{\tau}_- + \vec{\tau}_+ = \vec{r}_- \times \vec{F}_- + \vec{r}_+ \times \vec{F}_+ \\ &= -q\vec{r} \times \vec{E}(\vec{r}) + q(\vec{r} + \vec{a}) \times \left(\vec{E}(\vec{r}) + \vec{a} \left(\vec{\nabla} \cdot \vec{E}(\vec{r}) \right) \right) \\ &\approx (\vec{r} + \vec{p})(\vec{\nabla} \cdot \vec{E}) + \vec{p} \times \vec{E}. \end{aligned}$$

Thus,

$$\vec{\tau} = \vec{p} \times \vec{E} + \vec{r} \times \vec{F}.$$

Now, consider the two dipoles \vec{p}_1 and \vec{p}_2 as shown in the image below.



The dipole \vec{p}_2 is at the origin of our coordinate system, and \vec{r}_1 gives the position of dipole \vec{p}_1 . There is no external electric field. However, each dipole has its own electric field, and this field exerts a force on the other dipole. The potential energy of this interaction is

$$U = -\vec{p}_1 \cdot \vec{E}_2(\vec{r}_1) = -\vec{p}_2 \cdot \vec{E}(\vec{0}),$$

where, for example, $\vec{E}_2(\vec{r}_1)$ is the field due to \vec{p}_2 at the position \vec{r}_1 . We can write this as

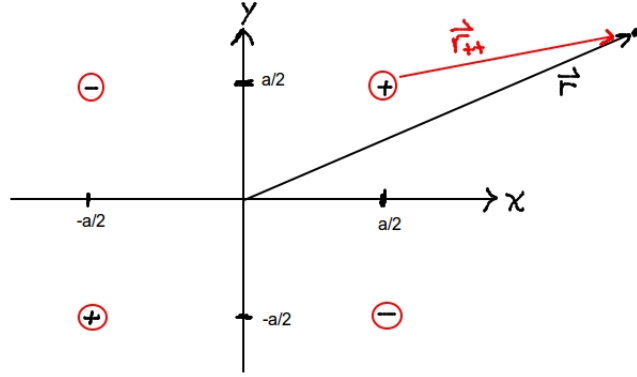
$$U = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_2 \cdot \hat{r})(\vec{p}_1 \cdot \hat{r})}{r^3}.$$

Electric Quadrupole

Consider the following square arrangement of two positive charges $+q$ and two negative charges $-q$. The sides of the square are a . We want to know the potential due to these charges at some distant test point located at \vec{r} .

Tip

The interaction force between two dipoles is in general not a central force.



This arrangement of charges has zero net charge and zero dipole moment. We will denote, for example, the vector from the charge in the positive x and positive y quadrant to the test point \vec{r} by \vec{r}_{++} . Similarly, the vector from the charge in the bottom right quadrant to the test point is \vec{r}_{+-} . Then

$$\begin{aligned}\vec{r}_{++} &= \vec{r} - \frac{a}{2}(1, 1, 0) \\ \vec{r}_{+-} &= \vec{r} - \frac{a}{2}(1, -1, 0) \\ \vec{r}_{-+} &= \vec{r} - \frac{a}{2}(-1, 1, 0) \\ \vec{r}_{--} &= \vec{r} - \frac{a}{2}(-1, -1, 0).\end{aligned}$$

Then the potential at \vec{r} is

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_{++}} + \frac{1}{r_{--}} - \frac{1}{r_{+-}} - \frac{1}{r_{-+}} \right).$$

Next, we expand each term at large distance $r \gg a$. To do a Taylor expansion of a multivariable function about the point \vec{r} , we use

$$\begin{aligned}f(\vec{r} + \delta\vec{r}) &= f(\vec{r}) + \left(\partial_i f(\vec{r}) \right) \delta r_i + \frac{1}{2} \left(\partial_i \partial_j f(\vec{r}) \right) \delta r_i \delta r_j + \dots \\ &= f(\vec{r}) + \left(\vec{\nabla} f(\vec{r}) \right) \delta\vec{r} + \frac{1}{2} \left(\partial_i \partial_j f(\vec{r}) \right) \delta r_i \delta r_j + \dots\end{aligned}$$

Note, summation should be performed over repeated indices.

For example, to expand $1/r_{++}$, we use $\vec{r}_{++} = \vec{r} + \delta\vec{r}$, where $\delta\vec{r} = -(a/2)(1, 1, 0)$. Then

$$\begin{aligned}\frac{1}{r_{++}} &= \frac{1}{r} + \vec{\nabla} \left(\frac{1}{r} \right) \cdot \left(-\frac{a}{2} \right) (1, 1, 0) + \frac{1}{2} \left(\frac{a}{2} \right)^2 (\partial_x \partial_x + 2\partial_x \partial_y + \partial_y \partial_y) \frac{1}{r} + \dots \\ &= \frac{1}{r} + \frac{a}{2r^2} \hat{r} \cdot (1, 1, 0) + \frac{a^2}{8} (\partial_x \partial_x + 2\partial_x \partial_y + \partial_y \partial_y) \frac{1}{r} + \dots\end{aligned}$$

For the other three terms, we get

$$\begin{aligned}\frac{1}{r_{--}} &= \frac{1}{r} + \frac{a}{2r^2} \hat{r} \cdot (-1, -1, 0) + \frac{a^2}{8} (\partial_x \partial_x + 2\partial_x \partial_y + \partial_y \partial_y) \frac{1}{r} + \dots \\ -\frac{1}{r_{+-}} &= -\frac{1}{r} - \frac{a}{2r^2} \hat{r} \cdot (1, -1, 0) - \frac{a^2}{8} (\partial_x \partial_x - 2\partial_x \partial_y + \partial_y \partial_y) \frac{1}{r} + \dots \\ -\frac{1}{r_{-+}} &= -\frac{1}{r} - \frac{a}{2r^2} \hat{r} \cdot (-1, 1, 0) - \frac{a^2}{8} (\partial_x \partial_x - 2\partial_x \partial_y + \partial_y \partial_y) \frac{1}{r} + \dots\end{aligned}$$

When we add all four results together, the first term in each cancels due to there being zero net charge. The second term in each cancels due to there being zero net dipole. In the third term, only the mixed derivatives don't cancel. We are left with

$$\phi = \frac{q}{4\pi\epsilon_0} a^2 \left(\partial_x \partial_y \frac{1}{r} \right).$$

Taking the partial derivative with respect to x (we are free to change their order),

$$\partial_x \frac{1}{r} = \partial_x \frac{1}{\sqrt{x^2 + y^2 + z^2}} = -\frac{x}{r^3}.$$

Taking the derivative with respect to y ,

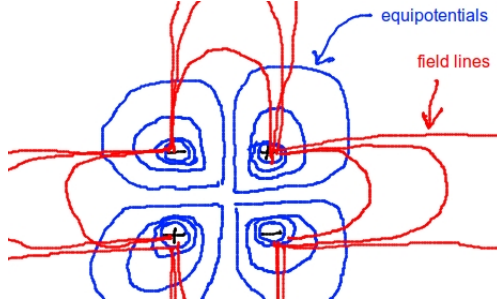
$$\partial_y \partial_x \frac{1}{r} = -x \partial_y \frac{1}{r^3} = \frac{3xy}{r^5}.$$

So we get the quadrupole potential

$$\phi_{quad} = \frac{q}{4\pi\epsilon_0} a^2 \frac{3xy}{r^5}.$$

Note that the quadrupole potential goes as

$$\phi_{quad} \sim \frac{1}{r^3}.$$



For a general charge distribution $\rho(\vec{r}')$, the quadrupole moment is defined by the quadrupole tensor

$$Q_{ij} = \int d^3r' \rho(r') (3r'_i r'_j - r'^2 \delta_{ij}).$$

This is also called the *second moment*. Note that the quadrupole tensor can be thought of as a 3×3 matrix. The indices i and j go over x , y , and z .

Some important properties of the quadrupole tensor include:

- It is traceless

$$Q_{ij} \delta_{ij} = 0.$$

Note, we are summing over repeated indices here.

- It is symmetric

$$Q_{ij} = Q_{ji}.$$

This implies that it can be diagonalized using orthogonal matrices (i.e. rotations). In practice, Q_{ij} can be diagonalized by choosing a coordinate system aligned with the principal axis (i.e. a symmetry axis) of Q_{ij} .

Tip

A symmetry axis is a principal axis, so if the coordinate system is aligned with a symmetry axis, then Q_{ij} is diagonal.

Tip

Be careful that you interpret $\hat{r}_i = (\hat{\mathbf{r}})_i$ correctly.

If Q_{ij} is diagonalized by the proper choice of coordinate system, then we only need to know three components to know the whole thing

$$\begin{bmatrix} Q_{xx} & & \\ & Q_{yy} & \\ & & Q_{zz} \end{bmatrix}.$$

Then, since we know it is also traceless, we can reduce this to two required components by writing

$$Q_{zz} = -Q_{xx} - Q_{yy}.$$

If we have an additional symmetry in the problem, we can reduce the number of components to calculate even further. For example, if our problem has cylindrical symmetry (rotational symmetry about the z -axis), then the x and y components are the same

$$Q_{yy} = Q_{xx}.$$

Then we only need to calculate the single component Q_{xx} to know the whole quadrupole tensor.

For general charge distributions, the quadrupole potential is

$$\phi_{quad} = \frac{1}{4\pi\epsilon_0} \frac{1}{2} Q_{ij} \frac{\hat{r}_i \hat{r}_j}{r^3}.$$

For a discrete charge distribution,

$$Q_{ij} = \sum_a q_a (3r_i r_j - r^2 \delta_{ij}),$$

where the sum is over the charges.

Example 3.4.1

Calculate the quadrupole tensor for the distribution of four charges shown in the beginning of this section.

We have a discrete distribution of four charges, so we use the formula

$$Q_{ij} = \sum_{a=1}^4 q_a (3r_i r_j - r^2 \delta_{ij}).$$

All of the charges are a distance $r = \sqrt{2}a/2$ from the origin, so $r^2 = a^2/2$.

For example,

$$\begin{aligned} Q_{xx} &= \sum_{a=1}^4 q_a \left(3x^2 - \frac{a^2}{2} \right) \\ &= q \left(3 \left(\frac{a}{2} \right)^2 - \frac{a^2}{2} \right) - q \left(3 \left(\frac{a}{2} \right)^2 - \frac{a^2}{2} \right) + q \left(3 \left(\frac{a}{2} \right)^2 - \frac{a^2}{2} \right) \\ &\quad - q \left(3 \left(\frac{a}{2} \right)^2 - \frac{a^2}{2} \right) \\ &= 0. \end{aligned}$$

Similarly, for the other diagonal elements, we get zero, $Q_{yy} = Q_{zz} = 0$.

For the off-diagonal elements, the Kronecker delta is zero. Then the terms containing z are zero: $Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$. A nonzero term is

$$\begin{aligned} Q_{xy} &= \sum_a q_a (3xy) \\ &= q \left(3\frac{a^2}{4} \right) - q \left(-3\frac{a^2}{4} \right) + q \left(3\frac{a^2}{4} \right) - q \left(-3\frac{a^2}{4} \right) \\ &= 3qa^2. \end{aligned}$$

Since the tensor is symmetric, we know that $Q_{yx} = 3qa^2$. So the full quadrupole tensor is

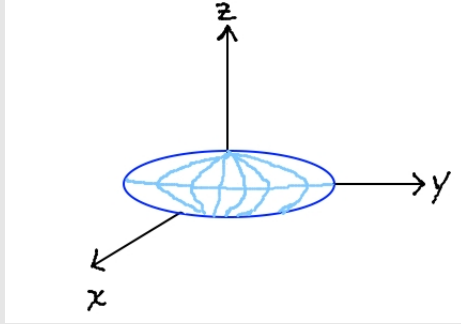
$$Q_{ij} = \begin{bmatrix} 0 & 3qa^2 & 0 \\ 3qa^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Tip

The quadrupole tensor has 9 components. However, you can use its properties and the symmetries of the problem to reduce the number of components that have to be calculated.

Example 3.4.2

Consider a spheroidal and homogeneous charge distribution ρ with semimajor axis a and semiminor axis b . This is a crude model of the charge distribution of a nucleus.



We can use scaled spherical coordinates

$$\begin{aligned} x &= a\eta \sin \theta \cos \phi \\ y &= a\eta \sin \theta \sin \phi \\ z &= b\eta \cos \theta, \end{aligned}$$

where $\eta \in [0, 1]$, $\phi \in [0, 2\pi]$, and $\theta \in [0, \pi]$.

Then

$$Q_{zz} = \int d^3r' \rho(r') (3z'^2 - \delta_{zz'} r'^2).$$

The Jacobian to go from x, y, z to η, θ, ϕ is

$$J = a^2 b \eta^2 \sin \theta.$$

Then

$$\begin{aligned} 3z'^2 - r'^2 &= 3z'^2 - (x'^2 + y'^2 + z'^2) \\ &= 2z'^2 - x'^2 - y'^2 \\ &= 2b^2 \eta^2 \cos^2 \theta - a^2 \eta^2 \sin^2 \theta. \end{aligned}$$

Then

$$Q_{zz} = \int a^2 b \eta^2 \sin \theta \, d\eta \, d\theta \, d\phi \, \rho_0 (2b^2 \eta^2 \cos^2 \theta - a^2 \eta^2 \sin^2 \theta).$$

This reduces to

$$Q_{zz} = \frac{2}{5} \rho_0 V (b^2 - a^2),$$

where the volume is

$$V = \frac{4}{3} a^2 b \pi.$$

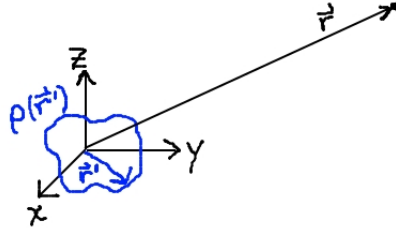
Rotational symmetry about the z -axis implies that Q_{ij} is diagonal, and $Q_{xx} = Q_{yy}$. The tracelessness of Q_{ij} implies that $Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz}$. So our quadrupole tensor is

$$Q_{ij} = \frac{2}{5} \rho_0 V (b^2 - a^2) \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since there is net charge, we know that Q_{ij} is not independent of the choice of origin.

3.5 Multipole Expansion

Suppose we have some general charge distribution $\rho(\vec{r}')$ which is localized near the origin of our coordinate system. What is the potential at large distance?



The potential is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \rho(r') \frac{1}{|\vec{r} - \vec{r}'|}.$$

We want to expand this for large r .

We can Taylor expand

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{|\vec{r}|} + r'_i \frac{\partial}{\partial r'_i} \frac{1}{|\vec{r} - \vec{r}'|} \Big|_{\vec{r}'=0} + \frac{1}{2} r'_i r'_j \frac{\partial}{\partial r'_i} \frac{\partial}{\partial r'_j} \frac{1}{|\vec{r} - \vec{r}'|} \Big|_{\vec{r}'=0} + \dots$$

We can write these derivatives as

$$\frac{\partial}{\partial r'_i} \frac{1}{|\vec{r} - \vec{r}'|} \Big|_{\vec{r}'=0} = -\frac{\partial}{\partial r_i} \frac{1}{|\vec{r} - \vec{r}'|} \Big|_{\vec{r}'=0} = -\partial_i \frac{1}{r},$$

and then

$$\frac{\partial}{\partial r'_i} \frac{\partial}{\partial r'_j} \frac{1}{|\vec{r} - \vec{r}'|} \Big|_{\vec{r}'=0} = \partial_i \partial_j \frac{1}{r}.$$

Then

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} - r'_i \partial_i \frac{1}{r} + \frac{1}{2} r'_i r'_j \partial_i \partial_j \frac{1}{r} + \dots$$

If we plug this back into the integral, we get

$$\begin{aligned}\phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(r') \left[\frac{1}{r} - r'_i \partial_i \frac{1}{r} + \frac{1}{2} r'_i r'_j \partial_i \partial_j \frac{1}{r} + \dots \right] \\ &= \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \dots\end{aligned}$$

where $\phi^{(0)}$ is the monopole term, and so on. At large r , only the leading term (i.e. the first nonzero term) $\phi^{(i)}$ dominates.

Recall that the monopole potential is

$$\phi^{(0)} = \frac{1}{4\pi\epsilon_0} \frac{Q_{tot}}{r},$$

where the monopole is

$$Q_{tot} = \int d^3r' \rho(r').$$

Recall that the dipole potential is

$$\phi^{(1)} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2},$$

where the dipole is

$$\vec{p} = \int d^3r' \rho(r') \vec{r}'.$$

The quadrupole potential is

$$\phi^{(2)} = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(r') \frac{1}{2} r'_i r'_j \partial_i \partial_j \frac{1}{r}.$$

We can write

$$\partial_i \partial_j \frac{1}{r} = \partial_i \left(-\frac{r_j}{r^3} \right) = \left(-\frac{\delta_{ij}}{r^3} + \frac{3r_i r_j}{r^5} \right). \quad (3.4)$$

This expression is traceless. That is,

$$\sum_{i,j=1}^3 \delta_{ij} \partial_i \partial_j \frac{1}{r} = 0.$$

From Eq. (3.4), we have that

$$r'_i r'_j \partial_i \partial_j \frac{1}{r} = (r'_i r'_j) \left(-\frac{\delta_{ij}}{r^3} + \frac{3r_i r_j}{r^5} \right).$$

Since the expression in parentheses is already traceless, if we contract this quantity with another term containing δ_{ij} , we don't change anything. So we can write

$$r'_i r'_j \partial_i \partial_j \frac{1}{r} = \left(r'_i r'_j - \frac{1}{3} r'^2 \delta_{ij} \right) \left(-\frac{\delta_{ij}}{r^3} + \frac{3r_i r_j}{r^5} \right).$$

We have not changed anything, since the inclusion of the term $-\frac{1}{3} r'^2 \delta_{ij}$ has only the effect of adding zero. The prefactors $-\frac{1}{3} r'^2$ are there for later convenience. Notice that, up to a factor of 3, the quantity in the left pair of parentheses is the integrand of the quadrupole tensor Q_{ij} , which we defined earlier. We know that this is also a traceless quantity, which means the $-\frac{\delta_{ij}}{r^3}$ in the second parentheses now has only the effect of adding a zero. So we can now remove it and write

$$r'_i r'_j \partial_i \partial_j \frac{1}{r} = \left(r'_i r'_j - \frac{1}{3} r'^2 \delta_{ij} \right) \left(\frac{3r_i r_j}{r^5} \right).$$

Tip

A given moment is independent of the choice of origin only if all lower moments are zero.

Tip

Be careful that you interpret $\hat{r}_i = (\hat{\mathbf{r}})_i$ correctly.

Thus, putting it all back together, we get the quadrupole potential

$$\phi^{(2)} = \frac{1}{4\pi\epsilon_0} \frac{1}{2} Q_{ij} \frac{\hat{r}_i \hat{r}_j}{r^3}.$$

In general, any moment is independent of the choice of origin only if all lower moments are zero. For example, if the monopole and dipole moments of a system are both zero, then the quadrupole moment of the system is independent of the choice of origin. Then we could perform a system translation $\vec{\mathbf{r}}_i' \rightarrow \vec{\mathbf{r}}_i' + \vec{\mathbf{a}}_i$ without affecting the quadrupole moment.

The multipole expansion of the potential of a point charge at the origin yields a single moment—the monopole. All higher multipoles are zero. Similarly, two oppositely charged particles centered on the origin will yield a nonzero dipole term, but a zero quadrupole term. However, the quadrupole term depends on the choice of origin since the dipole term is nonzero. If the two charges are shifted away from the origin, then there will be a nonzero quadrupole term.

We did the multipole expansion in Cartesian coordinates. These are not ideal for rotationally-symmetric systems. For those systems, spherical coordinates are more convenient. However, spherical coordinates are a *local* coordinate system in that the unit vectors change with position. So when we do the expansion in spherical coordinates, we have to account for that.

All together, we write the multipole expansion of the potential as

$$\phi = \frac{1}{4\pi\epsilon_0} \left(\frac{Q_{tot}}{r} + \frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{2!} Q_{ij} \frac{\hat{r}_i \hat{r}_j}{r^3} + \frac{1}{3!} Q_{ijk} \frac{\hat{r}_i \hat{r}_j \hat{r}_k}{r^4} + \dots \right).$$

We can calculate the electric field of the charge distribution using $\vec{\mathbf{E}} = -\vec{\nabla}\phi$. Then

$$\begin{aligned} \vec{\mathbf{E}}^{(0)} &= \frac{1}{4\pi\epsilon_0} Q_{tot} \frac{\hat{\mathbf{r}}}{r^2} \\ \vec{\mathbf{E}}^{(1)} &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \left[3(\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \vec{\mathbf{p}} \right]. \end{aligned}$$

For the quadrupole contribution, we can calculate the k -th component of the field as

$$E_k^{(2)} = \frac{1}{4\pi\epsilon_0} \frac{1}{2!} \frac{Q_{ij}}{r^4} \left(\delta_{ik} \hat{r}_j + \delta_{jk} \hat{r}_i - 5(\hat{r}_i \hat{r}_j) \hat{r}_k \right).$$

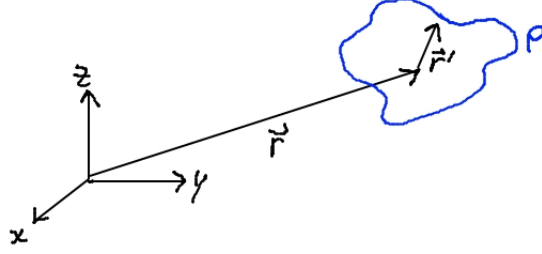
NOTE: Beware of how you interpret the meaning of something like \hat{r}_i . This is a scalar—not a vector. For example,

$$\hat{r}_x \equiv (\hat{\mathbf{r}})_x = |\hat{\mathbf{r}}| \sin \theta \cos \varphi = \sin \theta \cos \varphi.$$

Note: In general there are infinitely many higher order multipoles. Even for the simplest dipole consisting of two opposite charges (whose dipole moment incidentally is independent of the choice of origin since the net charge is zero) there will be infinitely many higher order multipoles. Unlike for the monopole, there is probably no physically realizable ideal dipole such that all higher order multipoles are zero.

3.6 Multipoles in External Field

Consider a charge distribution ρ that is localized around $\vec{\mathbf{r}}$. This charge distribution is in some external field with potential ϕ_{ext} that varies slowly in the region of $\vec{\mathbf{r}}$.



The interaction energy of the distribution ρ with the external field can be found by expanding for $\vec{r}' \ll \vec{r}$

$$\begin{aligned} U(\vec{r}) &= \int d^3r' \rho(r') \phi_{ext}(\vec{r} + \vec{r}') \\ &= \int d^3r' \rho(r') \left[\phi_{ext}(\vec{r}) + r'_i \partial_i \phi_{ext}(\vec{r}) + \frac{1}{2} r'_i r'_j \partial_i \partial_j \phi_{ext}(\vec{r}) + \dots \right] \\ &= U^{(0)} + U^{(1)} + U^{(2)} + \dots, \end{aligned}$$

where

$$\begin{aligned} U^{(0)} &= \int d^3r' \rho(r') \phi_{ext}(\vec{r}) = Q_{tot} \phi_{ext}(\vec{r}) \\ U^{(1)} &= \int d^3r' \rho(r') r'_i \partial_i \phi_{ext}(\vec{r}) = p_i \partial_i \phi_{ext} \\ U^{(2)} &= \int d^3r' \rho(r') \frac{1}{2} r'_i r'_j \partial_i \partial_j \phi_{ext}(\vec{r}) \\ &= \frac{1}{6} \partial_i \partial_j \phi_{ext} \int d^3r' \rho(r') (3r'_i r'_j - r'^2 \delta_{ij}) = \frac{1}{6} \partial_i \partial_j \phi_{ext} Q_{ij}. \end{aligned}$$

In the last line, we moved the partial derivatives and the potential outside of the integral since they don't depend on r' . We also added a term $-r'^2 \delta_{ij}$. We can do this because this term contributes nothing since

$$\delta_{ij} \partial_i \partial_j \phi_{ext} = -\vec{\nabla} \cdot \vec{E} = -\frac{\rho_{ext}}{\epsilon_0},$$

and ρ_{ext} is assumed to be effectively zero in the region where our test charges ρ are.

So all together,

$$U(\vec{r}) = Q_{tot} \phi(\vec{r}) + p_i \partial_i \phi(\vec{r}) + \frac{1}{6} Q_{ij} \partial_i \partial_j \phi(\vec{r}) + \dots,$$

where $\phi \equiv \phi_{ext}$. We can also write this in terms of \vec{E} as

$$U(\vec{r}) = Q_{tot} \phi(\vec{r}) - \vec{p} \cdot \vec{E} - \frac{1}{6} Q_{ij} \partial_i \partial_j \phi(\vec{r}) + \dots$$

The force on ρ is

$$\vec{F} = -\vec{\nabla} U = Q_{tot} (-\vec{\nabla} \phi) + p_i \partial_i (-\vec{\nabla} \phi) + \frac{1}{6} Q_{ij} \partial_i \partial_j (-\vec{\nabla} \phi) + \dots,$$

which we can write in vector form as

$$\vec{F} = Q_{tot} \vec{E} + \vec{\nabla} (\vec{p} \cdot \vec{E}) + \frac{1}{6} Q_{ij} \partial_i \partial_j \vec{E}(\vec{r}) + \dots$$

Notice that

- The monopole part of the distribution reacts to the external field
- The dipole reacts to the gradient of the field
- The quadrupole reacts to the curvature of the field

3.7 Laplace Equation in Spherical Coordinates

By substituting $\vec{E} = -\vec{\nabla}\phi$, we can write the Maxwell equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0},$$

as

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}.$$

This is **Poisson's equation**. It is typically written in the form

$$\Delta \phi = -\frac{\rho}{\epsilon_0},$$

where $\Delta \equiv \vec{\nabla} \cdot \vec{\nabla} = \nabla^2$. If there are no charges in the considered region, then $\rho = 0$, and we get the **Laplace equation**

$$\Delta \phi = 0.$$

In Cartesian coordinates, the Laplace operator is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In spherical coordinates, where θ is the polar angle and φ is the azimuthal angle, the transformation equations are

$$\begin{aligned} x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \theta. \end{aligned}$$

Then the Laplace operator is

$$\Delta \phi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\phi) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}.$$

Keep in mind that $\phi = \phi(r, \theta, \varphi)$ is the potential and φ is an angle. In the Laplace equation, this whole thing is zero.

We want to consider a solution suitable for a large distance expansion. We will try a product ansatz of the form

$$\phi(r, \theta, \varphi) = \frac{U(r)}{r} \cdot P(\theta) \cdot Q(\varphi).$$

Note, a product ansatz will not always solve all PDEs. It will only solve those which are *separable*. The most general solution of a PDE is not a product ansatz, but rather a linear combination of products.

Plugging the ansatz into the Laplace equation gives us

$$PQ \frac{d^2 U}{dr^2} + \frac{UQ}{r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + \frac{UP}{r^2 \sin^2 \theta} \frac{d^2 Q}{d\varphi^2} = 0.$$

Multiplying both sides by $r^2 \sin^2 \theta$ and dividing by UQP gives us

$$r^2 \sin^2 \theta \left[\frac{1}{U} \frac{d^2 U}{dr^2} + \frac{1}{Pr^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) \right] + \frac{1}{Q} \frac{d^2 Q}{d\varphi^2} = 0.$$

We can write this as

$$F(r, \theta) + G(\varphi) = 0,$$

where

$$F(r, \theta) = r^2 \sin^2 \theta \left[\frac{1}{U} \frac{d^2 U}{dr^2} + \frac{1}{Pr^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) \right]$$

$$G(\varphi) = \frac{1}{Q} \frac{d^2 Q}{d\varphi^2}.$$

If the differential equation is valid for all r , θ , and φ , then both $F(r, \theta)$ and $G(\varphi)$ must be constant. In other words, since $F(r, \theta)$ is independent of φ , we could take the derivative of everything with respect to φ , then we would see that $G(\varphi)$ must also be independent of φ . So we know that $F(r, \theta)$ and $G(\varphi)$ are both equal to constants, and in order to sum to zero, one of these constants must be the negative of the other.

We choose the constant such that

$$\frac{1}{Q} \frac{d^2 Q}{d\varphi^2} = -m^2.$$

This is a differential equation of the form $Q'' = -m^2 Q$. I.e., it is the equation of motion of a simple harmonic oscillator. Its solution $Q(\varphi)$ can be written in terms of sines/cosines or exponentials. The condition that $Q(\varphi + 2\pi) = Q(\varphi)$ implies that $m \in \mathbb{Z}$. Then we can write the solution as

$$Q(\varphi) = e^{\pm im\varphi}.$$

Similarly, we set $F(r, \theta) = +m^2$, then we can rearrange the remaining differential equation as

$$r^2 \frac{1}{U} \frac{d^2 U}{dr^2} + \frac{1}{P \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0.$$

Notice that the first term is a function only of r , and the remainder is a function only of θ . This implies that these two pieces are separately constant. We choose our constants to be $\ell(\ell + 1)$ and $-\ell(\ell + 1)$, then this differential equation can be separated as

$$\frac{d^2 U}{dr^2} - \frac{\ell(\ell + 1)}{r^2} U = 0$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + \left[\ell(\ell + 1) - \frac{m^2}{\sin^2 \theta} \right] P = 0.$$

For the $U(r)$ differential equation, we try the power ansatz $U(r) = r^\alpha$. Then the first two derivatives are

$$U'(r) = \alpha r^{\alpha-1}$$

$$U''(r) = \alpha(\alpha - 1) r^{\alpha-2}.$$

Plugging this back into the $U(r)$ differential equation gives us

$$\alpha(\alpha - 1) r^{\alpha-2} - \ell(\ell + 1) r^{\alpha-2} = 0,$$

for all r in the region of interest. This implies that

$$\alpha(\alpha - 1) - \ell(\ell + 1) = 0,$$

and thus, the possible solutions are those in which $\alpha = -\ell$ or $\alpha = \ell + 1$. Since it is a second-order differential equation, we only require two independent solutions, thus our general solution for $U(r)$ is

$$U(r) = C_1 \frac{1}{r^\ell} + C_2 r^{\ell+1}.$$

The remaining ODE for $P(\theta)$ is

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP}{d\theta} \right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2 \theta} \right] P = 0.$$

With $x \equiv \cos \theta$, we can write the θ -derivative as

$$\frac{d}{d\theta} = \frac{d(\cos \theta)}{d\theta} \frac{d}{d(\cos \theta)} = -\sin \theta \frac{d}{dx}.$$

We can also write $\sin^2 \theta = 1 - \cos^2 \theta = 1 - x^2$. Plugging everything in, our ODE becomes

$$\boxed{\frac{d}{dx} \left([1-x^2] \frac{dP}{dx} \right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2} \right) P = 0.}$$

We now have a differential equation in terms of $P = P(\cos \theta) = P(x)$. This is the **Legendre equation**.

Consider the case when $m = 0$. This occurs whenever the system has azimuthal symmetry. Then the ODE becomes

$$\frac{d}{dx} \left([1-x^2] \frac{dP}{dx} \right) + \ell(\ell+1)P = 0.$$

Now we try a power series ansatz

$$P(x) = x^\beta \sum_{j=0}^{\infty} a_j x^j.$$

The factor of x^β serves as an offset so that we can start the sum at $j = 0$ regardless of where the powers of x actually start. Using the same procedure as when we solved the $U(r)$ equation, we differentiate this $P(x)$, and then plug $P(x)$ and dP/dx back into the ODE and equate coefficients. When we do that, we get a second-order recursion relation that relates the coefficient a_{j+2} with the coefficient a_j

$$a_{j+2} = \frac{(j-1)(j+\ell+1)}{(j+1)(j+2)} a_j.$$

This relation shows us that for large j , $a_{j+2} \simeq a_j$, and since $x \equiv \cos \theta \in [-1, 1]$, the series will only converge if $|x| < 1$ or there exists some j_{max} such that the series terminates. In this case, for example, $j_{max} = \ell$. For this to work, ℓ must be an integer. It turns out that we can rename/redefine ℓ to be non-negative.

For the series to be finite, we must have either $a_0 = 0$ (if ℓ is odd) or $a_1 = 0$ (if ℓ is even). Thus, our solutions will be polynomials containing only even or odd terms, so we have the relation

$$P_\ell(-x) = (-1)^\ell P_\ell(x).$$

Our solutions $P_\ell(x)$ are special polynomials called **Legendre polynomials of the first kind**. They are normalized such that $P_0(x) = 1$. The first several of them are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(-1 + 3x^2) \\ &\vdots \end{aligned}$$

There is another solution to the differential equation that involves the *associated* Legendre polynomials. These are polynomials in $\ln(1-x)$ and $\ln(1+x)$. These solutions are less often required than the ones involving regular Legendre polynomials.

Recalling that we substituted $x = \cos \theta$, our solutions for the θ part of the Laplace equation, assuming azimuthal symmetry, are the Legendre polynomials of $\cos \theta$

$$P(\theta) = P_\ell(\cos \theta).$$

Multiplying our three solutions together, our general solution for the Laplace equation with azimuthal symmetry (i.e. no φ -dependence) is

$$\phi_\ell(r, \theta, \varphi) = \frac{U(r)}{r} \cdot P(\theta) \cdot Q(\varphi) = \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta),$$

where A_ℓ and B_ℓ are constants, which may depend on ℓ . Note that the solution $Q(\varphi)$ for the φ part is equal to 1 since $m = 0$ for azimuthal symmetry. The general solution is then a linear combination of these with different ℓ

$$\phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta),$$

Legendre Polynomials

From the previous section, recall that the first several Legendre polynomials $p_\ell(x)$ are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(-1 + 3x^2) \\ &\vdots \end{aligned}$$

They are normalized such that $P_0(x) = 1$.

These polynomials have the parity relation

$$P_\ell(-x) = (-1)^\ell P_\ell(x),$$

so if ℓ is even then P_ℓ is even, and if ℓ is odd then P_ℓ is odd.

They can be generated using the **Rodrigues formula**

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell.$$

Example 3.7.1: Basis for Function Spaces

Consider an arbitrary vector \vec{r} in \mathbb{R}^3 . We can write it as the linear combination

$$\vec{r} = r_1 \vec{e}_1 + r_2 \vec{e}_2 + r_3 \vec{e}_3,$$

where the basis vectors \vec{e}_i are orthonormal

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}, \quad \forall i, j,$$

and they are *complete* meaning the basis vectors span \mathbb{R}^3 . Then the coefficients are $r_i = \vec{r} \cdot \vec{e}_i$.

Example 3.7.2: Square Integrable Functions

Consider the space of square-integrable functions defined on the interval $[0, 2\pi]$. This space is denoted $L^2([0, 2\pi])$. We can expand such a function f as the linear combination

$$f = \sum_{m=-\infty}^{\infty} f_m e_m,$$

where the basis functions are

$$e_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi},$$

with $m \in \mathbb{Z}$. So our expansion is just a Fourier series.

We can also write

$$f = \sum_{m=-\infty}^{\infty} \langle e_m | f \rangle e_m,$$

where the inner product is defined as

$$\langle f | g \rangle = \int_0^{2\pi} f^*(\varphi) g(\varphi) d\varphi.$$

The e_m form an orthonormal basis

$$\langle e_m | e_{m'} \rangle = \delta_{mm'}.$$

There are countably infinite basis vectors.

With the Legendre polynomials, we can write the basis functions

$$e_\ell = \sqrt{\frac{2\ell+1}{2}} P_\ell(x).$$

These form a basis with respect to $L^2([-1, 1])$. We normalize the basis functions like this instead of just using the $P_\ell(x)$ directly, so that we get nicer expressions for the norm. Now the orthonormality condition is

$$\int_{-1}^1 dx e_\ell^* e_\ell = \delta_{\ell\ell'}.$$

Any piecewise continuous function f can be expanded as

$$f(x) = \sum_{\ell=0}^{\infty} f_\ell P_\ell(x),$$

where the coefficients are

$$f_\ell = \int_{-1}^1 dx e_\ell^*(x) f(x).$$

The set of e_ℓ is complete. The completeness relation can be written as

$$\sum_{\ell=0}^{\infty} e_\ell(x) e_\ell^*(y) = \delta(x - y),$$

then

$$f(x) = \int_{-1}^1 dy f(y) \delta(x - y) = \sum_{\ell=0}^{\infty} \int_{-1}^1 dy e_\ell^*(y) f(y) e_\ell(x) = \sum_{\ell=0}^{\infty} f_\ell e_\ell(x).$$

Applications with Azimuthal Symmetry

Example 3.7.3: Point Charge on z -axis

Consider a point charge q at \vec{r}' on the z -axis. What is the potential at \vec{r} ?

We already know the potential of a point charge, and we can simply write it down

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}'|}.$$

We want to relate this to what we learned above.

Since we have azimuthal symmetry, we know that up to a prefactor our solution is

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta).$$

What are the coefficients?

First, let us assume that $|\vec{r}| \equiv r < r' \equiv |\vec{r}'| = z$. Using the law of cosines, we can write

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2\vec{r} \cdot \vec{r}'} = \sqrt{r^2 + z^2 - 2rz \cos \theta},$$

where θ is the angle between \vec{r} and \vec{r}' . Since \vec{r}' is along the z -axis, this corresponds to the polar angle of spherical coordinates. We can write this as

$$|\vec{r} - \vec{r}'| = z \sqrt{1 + \left(\frac{r}{z}\right)^2 - 2\left(\frac{r}{z}\right) \cos \theta}.$$

Since this must hold for all θ , we look at $\theta = 0$. Then $\cos \theta = 1$, and we find

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{z \left(1 - \frac{r}{z}\right)}.$$

Using the formula for a geometric series, we can expand the right side as

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{z} \sum_{\ell=0}^{\infty} \left(\frac{r}{z}\right)^{\ell} = \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{(r')^{\ell+1}}.$$

So for $r < z$, this is our expansion of the left side in powers of r . This must be the r part of the solution since we set $\theta = 0$. Putting the θ dependence back in,

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r^{\ell}}{(r')^{\ell+1}} P_{\ell}(\cos \theta), \quad \text{for } r < r'.$$

We can do a similar thing for $r > r'$. Then we find that

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{(r')^{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta), \quad \text{for } r > r'.$$

We can combine these two to write a single general solution

$$\boxed{\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} P_{\ell}(\cos \theta),}$$

where $r_<$ means the smaller of r and r' and $r_>$ means the larger of r and r' . This formula works as a generating function of the Legendre polynomials. It is often written in the form

$$\frac{1}{\sqrt{1-2hx+x^2}} = \sum_{\ell=0}^{\infty} h^{\ell} P_{\ell}(x).$$

If you expand the left side, then the coefficients on the right are the Legendre polynomials.

Example 3.7.4: Conducting Sphere in a Uniform Field

Consider a conducting sphere in a uniform electric field \vec{E}_0 that is pointing in the z direction. The field is uniform far away, however, near the sphere it is distorted by the sphere. Find the potential.

We have azimuthal symmetry so the general solution is

$$\phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta).$$

The first step is to determine the boundary conditions. Since the sphere is a conductor, we know that it has constant potential on its surface. Second, far away we must have $\phi = -E_0 z$, since then $\vec{E} = -\vec{\nabla} \phi = E_0 \hat{z}$ as described in the problem description. So our boundary conditions are

$$\begin{aligned} \phi(R, \theta) &= \phi_0 \\ \phi(r \rightarrow \infty, \theta) &= -E_0 z. \end{aligned}$$

Applying the boundary condition at $r = R$, we have

$$\phi_0 = \sum_{\ell=0}^{\infty} \left(A_{\ell} R^{\ell} + \frac{B_{\ell}}{R^{\ell+1}} \right) P_{\ell}(\cos \theta).$$

We can multiply the left side by $1 = P_0(\cos \theta)$. Then we can expand the right side and equate coefficients of the Legendre polynomials. Since the left side has only $P_0(\cos \theta)$, we find that

$$\begin{aligned} \phi_0 &= A_0 + \frac{B_0}{R} \\ 0 &= A_{\ell} R^{\ell} + \frac{B_{\ell}}{R^{\ell+1}}, \quad \text{for } \ell > 0. \end{aligned}$$

Next, we apply the second boundary condition. At $r \rightarrow \infty$, the B terms go to zero due to the powers of r in the denominator. Then we are left with

$$-E_0 z = \sum_{\ell=0}^{\infty} A_{\ell} r^{\ell} P_{\ell}(\cos \theta).$$

Now, we replace $z = r \cos \theta = r P_1(\cos \theta)$ on the left and equate coefficients as before. We find that

$$\begin{aligned} A_0 &= 0 \\ A_1 &= -E_0 \\ A_{\ell} &= 0, \quad \text{for } \ell > 1. \end{aligned}$$

Thus, our solution is

$$\phi(r, \theta) = \phi_0 \frac{R}{r} - E_0 r \cos \theta + E_0 \frac{R^3}{r^2} \cos \theta.$$

The first term is the monopole, the second is due to the external field, and the third is the induced dipole.

Spherical Harmonics

In the previous sections, we assumed azimuthal symmetry ($m = 0$). Now we consider the more general case where there is no azimuthal symmetry. We go back to the *associated Legendre equation*

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_\ell^m}{dx} \right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2} \right] P_\ell^m = 0,$$

whose solutions $P_\ell^m(x)$ are the **associated Legendre polynomials**. They can be generated from the Legendre polynomials $P_\ell(x)$ by

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x), \quad m > 0, m \in \mathbb{N}.$$

It can be shown that regularity requires that

$$m = -\ell, \dots, \ell$$

is an integer.

The closed solution for $m \geq 0$ and $m < 0$ is

$$P_\ell^m(x) = \frac{1}{2^\ell \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - \ell)^\ell.$$

For $-m$

$$P_\ell^{-m}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x).$$

The orthogonality condition is

$$\int_{-1}^1 dx P_\ell^m(x) P_{\ell'}^m(x) = \delta_{\ell\ell'} \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}.$$

For $m \neq m'$, the orthogonality condition is more complicated.

We can combine our solutions $Q(\varphi)$ with these $P_\ell^m(\cos \theta)$, to get the **spherical harmonics**

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{2\ell+1}{2}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) \frac{e^{im\varphi}}{\sqrt{2\pi}}.$$

The first several, given in spherical and Cartesian coordinates, are

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}} \\ Y_{11} &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi} = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r} \\ Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \\ Y_{1-1} &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x-iy}{r} \end{aligned}$$

The orthogonality condition is

$$\int_{-1}^1 d(\cos \theta) \int_0^{2\pi} d\varphi Y_{\ell m}^*(\theta, \varphi) Y_{\ell' m'}(\theta, \varphi) = \delta_{\ell \ell'} \delta_{m m'}.$$

The completeness relation is

$$\delta^{(2)}(\Omega - \Omega') = \delta(\cos \theta - \cos \theta') \delta(\varphi - \varphi') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi).$$

A general function f on the unit sphere can be expanded as

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\theta, \varphi),$$

where the coefficients are

$$c_{\ell m} = \int_{-1}^1 d(\cos \theta) \int_0^{2\pi} d\varphi Y_{\ell m}^*(\theta, \varphi) f(\theta, \varphi).$$

We also have the parity relation

$$Y_{\ell, -m}(\theta, \varphi) = (-1)^m Y_{\ell m}^*(\theta, \varphi).$$

Under the action of the parity operator \hat{P} ,

$$\hat{P} Y_{\ell m}(\theta, \varphi) = Y_{\ell m}(\pi - \theta, \varphi + \pi) = (-1)^{\ell} Y_{\ell m}(\theta, \varphi).$$

So the $Y_{\ell m}$ are even or odd under spatial inversion. That is, they have definite parity.

If our problem does not have azimuthal symmetry, then the general solution to $\Delta\phi = 0$ in spherical coordinates is

$$\phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \varphi).$$

3.8 Spherical Multipole Moments

When we performed a multipole expansion of the potential

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|},$$

we did so in terms of Cartesian multipole moments. We want to repeat that but derive the spherical multipole moments.

Suppose \vec{r}' goes over the charge distribution, \vec{r} is the observation point, and γ is the angle between the two vectors. Keep in mind that we are no longer assuming azimuthal symmetry, and \vec{r}' is not generally on the z -axis. Then the cosine law gives us

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2rr' \cos \gamma}.$$

We could write $\cos \gamma = \hat{r} \cdot \hat{r}'$ and then write this in terms of spherical variables r, θ, φ and r', θ', φ' which are all mixed together. But this leads to a nasty integral for ϕ .

We are rescued by the “addition theorem” which tells us that

$$P_{\ell}(\cos \gamma) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi).$$

This is useful because it allows us to separate the angles. Each term on the right contains a product of two functions—one a function of the angles of $\hat{\mathbf{r}}'$, and the other a function of the angles of $\hat{\mathbf{r}}$. This will allow us to integrate term-by-term. The left side is in terms of γ —the relative angle of $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$. Now

$$\begin{aligned} \frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} &= \sum_{\ell=0}^{\infty} \frac{r_{>}^{\ell}}{r_{>}^{\ell+1}} P_{\ell}(\cos \gamma) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_{>}^{\ell}}{r_{>}^{\ell+1}} Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi). \end{aligned}$$

So the potential is

$$\phi(r, \theta, \varphi) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta, \varphi) \int d^3 r' \rho(r', \theta', \varphi') Y_{\ell m}^*(\theta', \varphi') \frac{r_{>}^{\ell}}{r_{>}^{\ell+1}}.$$

Example 3.8.1

Consider a sphere of radius R and $\rho(\vec{\mathbf{r}}') = \rho_0 = \text{constant}$ inside the sphere. What is the potential inside and outside the sphere?

We know that in general,

$$\phi(r, \theta, \varphi) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta, \varphi) \int d^3 r' \rho(r', \theta', \varphi') Y_{\ell m}^*(\theta', \varphi') \frac{r_{>}^{\ell}}{r_{>}^{\ell+1}}.$$

Since $\rho(\vec{\mathbf{r}}')$ is independent of the angles, we can use the fact that

$$1 = \sqrt{4\pi} Y_{00}(\theta', \varphi'),$$

to simplify the angular part as

$$\int d\Omega' Y_{\ell m}^*(\theta', \varphi') = \sqrt{4\pi} \int d\Omega' Y_{\ell m}^*(\theta', \varphi') Y_{00}(\theta', \varphi') = \sqrt{4\pi} \delta_{\ell 0} \delta_{m 0}.$$

Thus, our double sum expression for ϕ simplifies to a single term with $\ell = 0$ and $m = 0$.

$$\phi(r, \theta, \varphi) = \frac{\rho_0}{\epsilon_0} \int dr' \frac{r'^2}{r_{>}}.$$

In the region $r > R$, we have $r_{>} \equiv r$, then

$$\phi(r, \theta, \varphi) = \frac{\rho_0}{\epsilon_0} \frac{1}{r} \int_0^R dr' r'^2 = \frac{\rho_0}{3\epsilon_0} \frac{R^3}{r}.$$

In the region $r < R$, we have to be careful since now the larger of r and r' (i.e. $r_{>}$) could be either r or r' . We have to split the integral into two pieces to get

$$\begin{aligned} \phi(r, \theta, \varphi) &= \frac{\rho_0}{\epsilon_0} \int_0^r dr' \frac{r'^2}{r_{>}} + \frac{\rho_0}{\epsilon_0} \int_r^R dr' \frac{r'^2}{r_{>}} \\ &= \frac{\rho_0}{\epsilon_0} \frac{1}{r} \int_0^r dr' r'^2 + \frac{\rho_0}{\epsilon_0} \int_r^R dr' r' \\ &= \frac{\rho_0}{3\epsilon_0} \left(\frac{3}{2} R^2 - \frac{1}{2} r^2 \right). \end{aligned}$$

In the example above, we were probing inside the charge distribution. Now we look at the long distance behavior. In the long distance expansion, we know that $r > r'$ outside of a localized charge distribution, so

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi).$$

This is the **spherical multipole expansion** of $\phi(\vec{r})$. The coefficients $q_{\ell m}$ are the **spherical multipole moments**

$$q_{\ell m} = \int d^3r' \rho(\vec{r}') Y_{\ell m}^*(\theta', \varphi') r'^{\ell}.$$

Notice that the spherical harmonic in $q_{\ell m}$ is the complex conjugate. There are alternative definitions of the spherical multipole moments in which the complex conjugate is left in $\phi(\vec{r})$.

For each level ℓ , notice that

$$\phi \sim \frac{1}{r^{\ell+1}}.$$

Keep in mind that

$$m = -\ell, \dots, \ell,$$

so for a given ℓ , there are $2\ell + 1$ moments $q_{\ell m}$. We also have the simple parity relation

$$q_{\ell, -m} = (-1)^m q_{\ell m}^*.$$

For $\ell = 0$, we get the spherical monopole moment. It and its relation to the Cartesian monopole moment Q_{tot} is

$$q_{00} = \int d^3r' \rho(\vec{r}') \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{4\pi}} Q_{tot}.$$

For $\ell = 1$, we get the spherical dipole moments. It and its relation to the Cartesian dipole $\vec{p} = (p_x, p_y, p_z)$ are

$$\begin{aligned} q_{10} &= \int d^3r' \rho(\vec{r}') \sqrt{\frac{3}{4\pi}} z' = \sqrt{\frac{3}{4\pi}} p_z \\ q_{11} &= -\sqrt{\frac{3}{8\pi}} (p_x - ip_y) \\ q_{1-1} &= \sqrt{\frac{3}{8\pi}} (p_x + ip_y). \end{aligned}$$

If the $q_{\ell m}$ are defined in terms of the non-complex conjugated spherical harmonics, then the signs relating the spherical and Cartesian multipole moments will be different.

Similarly, for the quadrupole moments

$$\begin{aligned} q_{22} &= \frac{1}{12} \sqrt{\frac{15}{2\pi}} (Q_{11} - 2iQ_{12} - Q_{22}) \\ q_{21} &= -\frac{1}{3} \sqrt{\frac{15}{8\pi}} (Q_{13} - iQ_{23}) \\ q_{20} &= \frac{1}{2} \sqrt{\frac{5}{4\pi}} Q_{33} \\ q_{2-2} &= q_{22}^* \\ q_{2-1} &= -q_{21}^*. \end{aligned}$$

The ℓ encodes the radial fall-off of the potential, and within each ℓ , the m encodes the spatial information. Using spherical multipoles instead of Cartesian multipoles makes it easier to read off the symmetry properties and to add things.

The spherical multipoles moments are closely related to the orbital angular momentum of quantum mechanics. Recall from quantum mechanics that the orbital angular momentum operator is

$$\vec{L} = \vec{x} \times \vec{p} = \vec{x} \times \left(\frac{\hbar}{i} \vec{\nabla} \right).$$

Its components satisfy the commutation relation

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k.$$

Furthermore, its components commute with its square

$$[L^2, L_i] = 0, \quad \forall i,$$

and this implies that we can find simultaneous eigenstates for L^2 and L_i , and these can be used as a basis. The eigenvalue equations are

$$L^2 \psi_{\ell m} = \hbar^2 \ell(\ell+1) \psi_{\ell m}, \quad L_z \psi_{\ell m} = m\hbar \psi_{\ell m},$$

where the algebra and the normalizability requirement imply that $\ell \geq 0$ and $m = -\ell, \dots, \ell$. Then those simultaneous eigenstates are precisely the same spherical harmonics

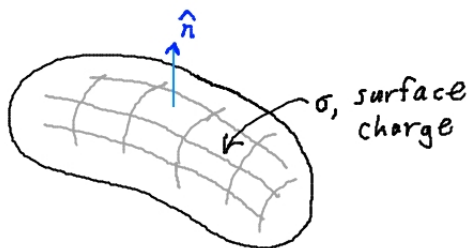
$$\psi_{\ell m} = Y_{\ell m}(\theta, \varphi),$$

which we are using in electrostatics.

3.9 Poisson Equation and General Boundary Conditions

To deal with complicated boundary conditions, we have to use Green's functions. We will look at those in a future section.

Consider a boundary surface (e.g. a conducting surface with surface charge σ) that separates two regions.

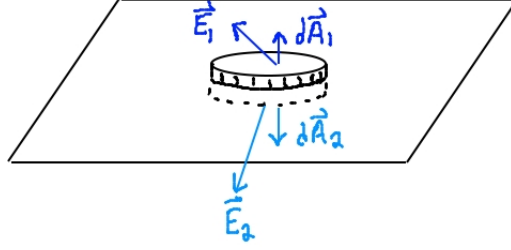


At a given point near the surface, we can write the electric field as the sum of normal and tangential components

$$\vec{E} = \vec{E}_n + \vec{E}_t.$$

What happens as we cross the surface from outside to inside?

Consider a Gaussian cylinder straddling the surface as shown below. The cylinder is much smaller than the region in consideration, and its radius is much larger than its height.



Gauss's law tells us

$$\oiint_{\partial \text{cylinder}} \vec{E} \cdot d\vec{A} = \oiint_{\text{top}} \vec{E} \cdot d\vec{A} + \oiint_{\text{bottom}} \vec{E} \cdot d\vec{A} + \oiint_{\text{side}} \vec{E} \cdot d\vec{A}.$$

If we let the height of the cylinder go to zero, then the third term vanishes. We can assume that \vec{E} is approximately constant along the top and the bottom of the cylinder. Then

$$\begin{aligned} \oiint_{\partial \text{cylinder}} \vec{E} \cdot d\vec{A} &\simeq \oiint_{\text{top}} \vec{E}_1 \cdot d\vec{A}_1 + \oiint_{\text{bottom}} \vec{E}_2 \cdot d\vec{A}_2 \\ &= \oiint_{\text{top}} \vec{E}_1 \cdot d\vec{A}_1 - \oiint_{\text{bottom}} \vec{E}_2 \cdot d\vec{A}_1 \\ &= \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \int d^3r \rho \\ &= \frac{1}{\epsilon_0} \int \sigma dx dy \simeq \frac{1}{\epsilon_0} \sigma \cdot dA_1. \end{aligned}$$

In the last step, we used the fact that $\sigma \simeq \text{const}$ in a small region.

This implies that

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{n} = \frac{1}{\epsilon_0} \sigma,$$

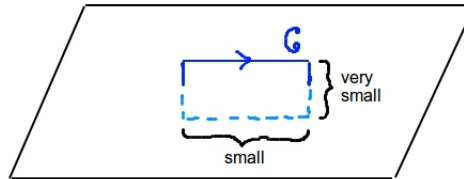
or

$$\boxed{E_{n,+} - E_{n,-} = \frac{\sigma}{\epsilon_0}},$$

where $E_{n,+}$ means the normal component of the field on the outside. So the normal component of \vec{E} jumps by σ/ϵ_0 . It is not continuous at a charged surface.

Next we apply Stoke's law by considering a rectangular loop going through the surface as shown below. By Stoke's law, we know that

$$\oint_C \vec{E} \cdot d\vec{\ell} = 0.$$



Then

$$\oint_C \vec{E} \cdot d\vec{\ell} \approx \vec{E}_1 \cdot d\vec{\ell}_1 + \vec{E}_2 \cdot d\vec{\ell}_2 = (\vec{E}_1 - \vec{E}_2) \cdot \hat{t} \cdot \text{const} = 0.$$

Thus,

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{t} = 0.$$

I.e. the tangential components of \vec{E} are continuous when crossing a charged surface. We can write this as

$$E_{t,+} = E_{t,-}.$$

where the left side is the tangential component of the electric field on the outside, and the right-hand side is the tangential component on the inside.

For the potential,

$$\frac{\partial\phi_-}{\partial n} - \frac{\partial\phi_+}{\partial n} = \frac{\sigma}{\epsilon_0},$$

where

$$\frac{\partial\phi}{\partial n} \equiv \hat{n} \cdot (\vec{\nabla}\phi).$$

So the boundary surface puts constraints on the derivatives of the potential ϕ .

3.10 Conductors

A simple model of a conductor is as a material containing positive and negative charges that are free (zero resistance) to move. Two important properties of conductors in this model are:

1. The field on the inside is zero

$$\vec{E}_{inside}(\vec{r}) = 0.$$

If there *were* a field on the inside then the free charges would move, and we would no longer be in the electrostatic case. We only care about the electrostatic case right now, so charges are not moving by definition.

2. The potential is constant

$$\phi_{inside} = \text{const.}$$

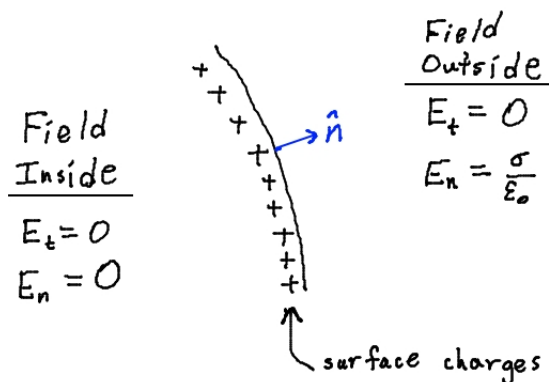
Consider a region within an arbitrary conductor. If we apply Gauss's law to this region, then using the fact that $\vec{E}_{inside}(\vec{r}) = 0$, we find that

$$Q_{enc} = 0.$$

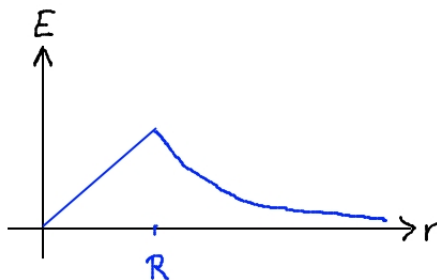
Thus, for any charged conductor, all of the charge must reside on its surface.

On the inside of a conductor, the tangential and normal components of the field are $E_t = 0$ and $E_n = 0$. Immediately on the outside of a conductor with surface charge σ , the tangential and normal components of the field must be

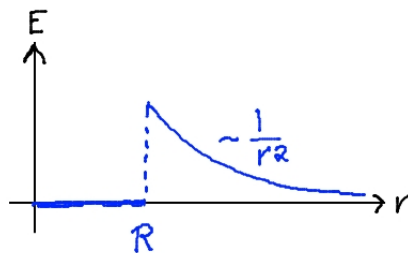
$$E_t = 0, \quad E_n = \frac{\sigma}{\epsilon_0}.$$



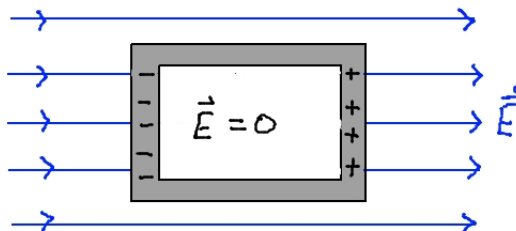
Recall the electric field of a homogeneously charged sphere looks like:



Now, the electric field of a conducting sphere looks like:

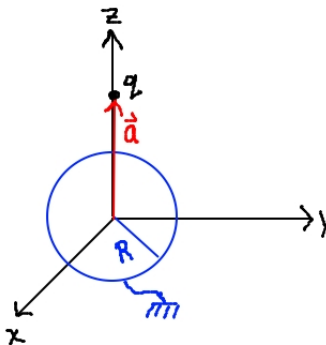


Since a region within a conductor contains no electric field, we could hollow out that region, and it would still contain no electric field, regardless of whether or not there is an electric field outside the conductor. This is the concept of a **Faraday cage**. If a Faraday cage is placed in an external field, then negative charge gathers on one side of the surface and positive charge gathers on the other side such that the field on the inside remains zero.



3.11 Image Charges

Consider a grounded conducting sphere of radius R . There is a charge q at a position \vec{a} above the sphere. What is the potential in the region outside the sphere?



Since the sphere is *grounded*, we have the boundary condition

$$\phi(r \leq R) = 0.$$

We also have the boundary condition

$$\phi(\vec{r}) \rightarrow 0, \text{ as } r \rightarrow \infty.$$

The Poisson equation is

$$\nabla\phi = -\frac{\rho}{\epsilon_0} = -\frac{q}{\epsilon_0}\delta(\vec{r} - \vec{a}).$$

Without loss of generality, we can orient our coordinate system such that \vec{a} is on the z -axis. Then we have the ansatz solution

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{A_{\ell}}{r^{\ell+1}} P_{\ell}(\cos\theta) + \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{a}|}.$$

For convenience, we separated out the part of the potential due to the point charge at \vec{a} . The first term solves the Laplace equation $\Delta\phi = 0$. The second term solves the Poisson equation $\Delta\phi = -\rho/\epsilon_0$.

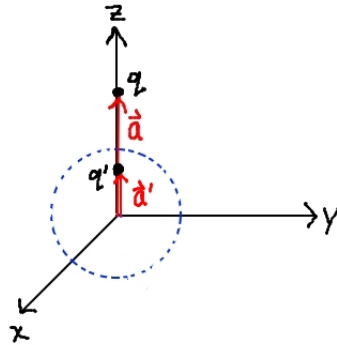
To fix the boundary condition at $r = R$, it is useful to expand $1/|\vec{r} - \vec{a}|$ in terms of $P_{\ell}(\cos\theta)$ as we did before. We then apply the boundary condition $\phi(r = R) = 0$, and fix the coefficients A_{ℓ} . Then we recognize that we can write the solution in the closed form

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{a}|} + \frac{1}{4\pi\epsilon_0} \frac{q'}{|\vec{r} - \vec{a}'|}, \quad r \gg z,$$

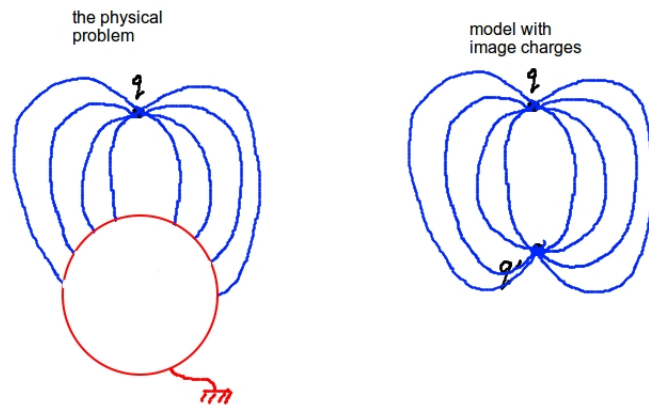
where

$$q' = -q\frac{R}{a}, \quad a' = \frac{R^2}{a}.$$

The second term in the potential looks like the potential of a charge q' sitting at \vec{a}' . That is, we interpret q' as an image charge (with opposite sign of the charge q) sitting at \vec{a}' .



In the region outside the sphere, the solution $\phi(\vec{r})$ looks exactly the same as the potential due to two charges— q at \vec{a} and q' at \vec{a}' .

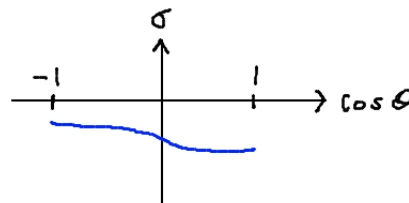


Note, the potential calculated above is only valid in the region outside the sphere.

On the surface of the conducting sphere, there is an induced surface charge density

$$\sigma = -\epsilon_0 \frac{\partial \phi_{\text{outside}}}{\partial n},$$

where $\hat{n} = \hat{r}$ in this case. If we plot it, it looks something like:



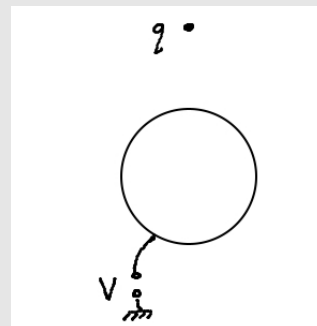
The total induced surface charge is

$$Q_{\text{ind}} = \oint \sigma dA.$$

The utility of the image charge method is that if you have a good guess for the solution, and it satisfies all the constraints, then just write it down because the solution is unique.

Example 3.11.1

Now instead of a grounded sphere, we repeat the problem but for a conducting sphere held at some constant potential V . Given the charge q at \vec{a} , what is the potential outside the sphere?



We have the same setup as the example above. The only difference is that the potential on the surface of the sphere is changed from zero to V . We can achieve this by adding a (image) charge q'' at the center of the sphere. Now, the solution has the form

$$\phi = \phi_1 + \frac{1}{4\pi\epsilon_0} \frac{q''}{|\vec{r} - \vec{r}''|},$$

where ϕ_1 is the solution for the problem with the grounded sphere. Since the potential is a constant V at constant radius R , we want q'' to be at the center of the sphere. Thus, $\vec{r}'' = 0$, and

$$\phi(R) = \phi_1(R) + \frac{1}{4\pi\epsilon_0} \frac{q''}{R} = V.$$

Since $\phi_1(R) = 0$, this implies that

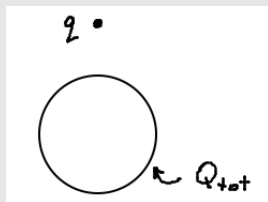
$$q'' = 4\pi\epsilon_0 VR.$$

Our final solution is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{a}|} + \frac{1}{4\pi\epsilon_0} \frac{-qR/a}{|\vec{r} - \vec{a}'|} + \frac{1}{4\pi\epsilon_0} \frac{4\pi\epsilon_0 VR}{|\vec{r} - 0|},$$

Example 3.11.2

We have the same problem, but now instead of knowing anything about the potential on the surface of the sphere, we only know that it has some total charge Q_{tot} distributed over its surface.



This example is really the same as the previous example, but now we don't know V beforehand. Now,

$$Q_{tot} = q' + q'',$$

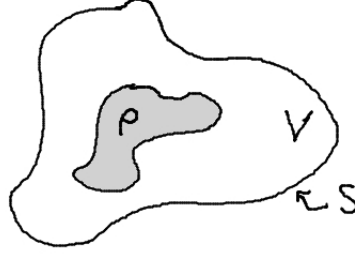
and we use

$$V = \frac{1}{R} (Q_{tot} - q').$$

3.12 Green's Theorem

Green's theorem is useful in applications involving conductors.

For the following discussion, we consider an arbitrary region of charge density ρ contained within an arbitrary volume V that is bounded by a surface S .



Recall the divergence theorem

$$\int_V \vec{\nabla} \cdot \vec{A} d^3r = \oint_S \vec{A} \cdot \hat{n} dA.$$

Consider two arbitrary scalar fields ϕ and ψ . Then

$$\vec{\nabla} \cdot (\phi \vec{\nabla} \psi) = \phi \Delta \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi.$$

We can write

$$(\phi \vec{\nabla} \psi) \cdot \hat{n} = \phi \frac{\partial \psi}{\partial n}.$$

Then the divergence theorem implies that

$$\int_V (\phi \Delta \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi) d^3r = \oint_S \phi \frac{\partial \psi}{\partial n} dA.$$

This is called **Green's first identity**. We can repeat the same thing with ϕ and ψ interchanged and then subtract one from the other to get the result

$$\boxed{\int_V (\phi \Delta \psi - \psi \Delta \phi) d^3r = \oint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dA.}$$

This is Green's second identity, but more commonly known as **Green's theorem**.

Now we choose

$$\psi = \frac{1}{|\vec{r} - \vec{r}'|},$$

where \vec{r} is the observation point, and \vec{r}' is the integration variable. Up to a prefactor of $1/4\pi\epsilon_0$, this is the potential of a point charge. Then we know from previous work that

$$\Delta \psi = -4\pi\delta(\vec{r} - \vec{r}').$$

We also choose ϕ to mean the potential. Then we know from Poisson's equation that

$$\Delta \phi = -\frac{\rho}{\epsilon_0}.$$

Plugging ϕ , $\Delta \phi$, ψ , and $\Delta \psi$ into Green's theorem gives us

$$\int_V \left[-4\pi\phi(\vec{r}')\delta(\vec{r} - \vec{r}') + \frac{1}{\epsilon_0} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \right] d^3r' = \oint_S \left[\phi \frac{\partial}{\partial n'} \frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \phi}{\partial n'} \right] dA'.$$

If \vec{r} is not in the volume V , then the delta function term vanishes. On the other hand, if \vec{r} is in the volume V , then after integrating the delta function term, we can rearrange the result as

$$\boxed{\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r' + \frac{1}{4\pi} \oint_S \left[\frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \phi}{\partial n'} - \phi \frac{\partial}{\partial n'} \frac{1}{|\vec{r} - \vec{r}'|} \right] dA'.$$

Notice that the volume integral is just the potential due to the explicit charges. The surface integral contains two terms. The first term gives the surface charge (a discontinuity in \vec{E}), and the second term is the “dipole layer term” (a discontinuity in the potential). The dipole layer corresponds to the potential of two charged surfaces in the limit that the two surfaces come together.

If $S \rightarrow \infty$, then the electric field \vec{E} on the surface S falls off faster than $1/R$, and the surface integral vanishes. So in the infinite-volume limit, we only have the potential due to the explicit charges.

For a charge-free volume, i.e. if $\rho(\vec{r}') = 0$, then the volume integral is absent. Then we get a formula for ϕ in terms of ϕ and $\frac{\partial\phi}{\partial n}$ on the boundary S . That is, we get an integral equation for ϕ . We can't get an explicit solution since ϕ and $\frac{\partial\phi}{\partial n}$ cannot be independently chosen.

Suppose there exist two solutions ϕ_1 and ϕ_2 with correct boundary conditions on the surface S . I.e., ϕ_1 and ϕ_2 satisfy Poisson's equation. Now, let

$$U = \phi_1 - \phi_2,$$

be the difference between the two solutions. Then we know that their Laplacian is zero: $\Delta U = 0$. We also know that either

$$U|_S = 0, \quad \text{or} \quad \frac{\partial U}{\partial n}|_S = 0,$$

where it's the first if we have Dirichlet boundary conditions and the second if we have Neumann boundary conditions. Let $\phi = \psi = U$, then Green's first identity gives us

$$\int_V \left(U \Delta U + |\vec{\nabla} U|^2 \right) d^3 r' = \int_S U \frac{dU}{dn} dA.$$

Since we know that $\Delta U = 0$ and either $U = 0$ on the surface or $\frac{dU}{dn} = 0$ on the surface, this simplifies to

$$\int_V |\vec{\nabla} U|^2 d^3 r' = 0.$$

This implies that

$$U = \text{const.}$$

For Dirichlet boundary conditions, $U|_S = 0$, which implies that $U = 0$. In any case, we know that ϕ_1 and ϕ_2 are the same, up to an overall constant (zero for Dirichlet), if we have pure Dirichlet or Neumann boundary conditions.

Above, we chose $\psi = 1/|\vec{r} - \vec{r}'|$. What if we made a different choice? We want to make a choice that simplifies our problem as much as possible. The requirement for the function $\psi = G(\vec{r}, \vec{r}')$ is that

$$\Delta G(\vec{r}, \vec{r}') = -4\pi\delta(\vec{r} - \vec{r}').$$

Then the function $G(\vec{r}, \vec{r}')$, called the **Green's function**, must have the form

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}'), \quad \text{where } \Delta F = 0 \text{ in the volume } V.$$

Note that Green's function is a function of six variables (\vec{r} and \vec{r}').

Think of Green's function G as an impulse response of the inhomogeneous differential equation with boundary conditions. I.e., it is effectively the potential of a point charge but takes into account the reaction of the system and the boundary conditions. The quantity $F/4\pi\epsilon_0$ corresponds to the potential of the image charge.

Tip

When using these formulae, remember that the normal direction n' points *outward* from the volume of interest. For example, if the problem is to find the potential in the half-space $z \geq 0$, then the normal direction would be $n' = -z'$.

The idea is to choose F such that boundary constraints become easy to implement. Now when we write Green's theorem, we interpret ϕ as the potential and $\psi = G$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}') G(\vec{r}, \vec{r}') d^3r' + \frac{1}{4\pi} \oint_S \left[G(\vec{r}, \vec{r}') \frac{\partial\phi}{\partial n'} - \phi(\vec{r}') \frac{\partial G}{\partial n'} \right] dA'.$$

Case: (Dirichlet B.C.s) For Dirichlet boundary conditions, we choose F such that G is zero on the boundary

$$G_D(\vec{r}, \vec{r}') \Big|_{\vec{r}' \in S} = 0.$$

Then the first term in the surface integral is zero and

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}') G_D(\vec{r}, \vec{r}') d^3r' - \frac{1}{4\pi} \int_S \phi(\vec{r}') \frac{\partial G_D}{\partial n'} dA'.$$

In practice, the difficulty is in finding an F such that $G = 0$ on the boundary S . Remember that $\rho(\vec{r}')$ is the *explicit* charges in the volume V . To construct G , we place a test charge at \vec{r}' and identify the image charge. Then

$$G = \frac{1}{|\vec{r} - \vec{r}'|} + F,$$

where the first term is the potential of the test charge (up to the prefactor of $1/4\pi\epsilon_0$), and F is the potential of the image charge (up to the prefactor), where the image charge is placed such that the boundary conditions are satisfied. So the problem of finding G is reduced to that of finding the potential of an image charge given a test charge at an arbitrary location in the system.

Case: (Neumann B.C.s) For Neumann boundary conditions, one's first guess might be to choose an F such that $\frac{\partial G}{\partial n'} = 0$ on the boundary. However, this does not work since $\int_S \frac{\partial G}{\partial n'} dA' = -4\pi$. The simplest allowed choice is to choose F such that

$$\frac{\partial G_N}{\partial n'}(\vec{r}, \vec{r}') \Big|_{\vec{r}' \in S} = -\frac{4\pi}{S},$$

where S is the total surface area. Now we get

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}') G_N(\vec{r}, \vec{r}') d^3r' + \frac{1}{4\pi} \int_S G_N(\vec{r}, \vec{r}') \frac{\partial\phi}{\partial n'} dA' + \langle\phi\rangle_S,$$

where $\langle\phi\rangle_S$ is the average of ϕ over the surface S . In typical applications where $S \rightarrow \infty$, this term vanishes. An example of Neumann boundary conditions would be a conducting sphere in a constant, uniform field \vec{E} . The boundary conditions are Neumann if we are given the field \vec{E} which is the derivative of the potential.

For Dirichlet boundary conditions,

$$G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r}).$$

I.e. the variables can be swapped. For Neumann boundary conditions, one can choose G such that the same is true.

Example 3.12.1

Consider the case of a conducting sphere of radius R with the potential $\phi(r = R, \theta, \varphi)$ on the surface given (and it could be complicated), and the potential $\phi(r \rightarrow \infty) = 0$ at infinity. These are Dirichlet boundary conditions. What is the potential outside the sphere?

In this example, we have a boundary surface at $r = R$ (the surface of the sphere), and at $r = \infty$. The volume V of interest is between these two bounding surfaces. That is, V is the entirety of the outside of the sphere.

We can use the image charge method to write down the Green's function. We want

$$\Delta G_D = -4\pi\delta(\vec{r} - \vec{r}'),$$

with

$$G_D(\vec{r}, \vec{r}') \Big|_{\vec{r}' \in S} = 0.$$

Suppose the potential on the surface is zero. We solved this case earlier. To find the Green's function, we place a point test charge at \vec{r}' , and consider the response of the system. We know that the potential on the outside can now be found by replacing the conducting sphere with an image charge. We find that^a

$$G_D(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + \frac{q'/q}{|\vec{r} - \vec{a}'|} = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{R/r'}{|\vec{r} - \frac{R^2}{r'^2}\vec{r}'|}.$$

^aThe Green function in spherical coordinates can also be computed without reference to image charges. See Jackson, for example.

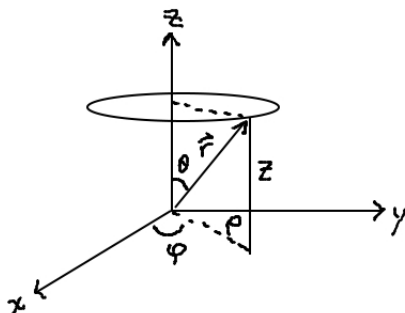
3.13 Laplace Equation in Cylindrical Coordinates

We want to solve Laplace's equation

$$\Delta\Phi = 0,$$

in cylindrical coordinates

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$



We assume a separable solution

$$\Phi(\rho, \varphi, z) = R(\rho)Q(\varphi)Z(z),$$

then using separation of variables, we get the three ODEs

$$\begin{aligned} \frac{d^2 Z}{dz^2} - k^2 Z &= 0 \\ \frac{d^2 Q}{d\varphi^2} + \nu^2 Q &= 0 \\ \frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R &= 0. \end{aligned}$$

This implies the solutions for Z and Q

$$\begin{aligned} Z(z) &= e^{\pm kz} \\ Q(\varphi) &= e^{\pm i\nu\varphi}. \end{aligned}$$

For the R ODE, there are three cases.

Case 1 ($k \in \mathbb{R}$ and $k > 0$): For this case, we write $x \equiv k\rho$, then our ODE becomes

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0.$$

This is the **Bessel differential equation**. It can be solved with a power series ansatz to get a recursion relation between the coefficients. This leads us to two solutions

$$\begin{aligned} J_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \nu + 1)} \left(\frac{x}{2}\right)^{2j} \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j - \nu + 1)} \left(\frac{x}{2}\right)^{2j}, \end{aligned}$$

called **Bessel functions of the first kind**. These are linearly independent if $\nu \notin \mathbb{Z}$. Away from localized charges, we require $Q(\varphi + 2\pi) = Q(\varphi)$, which implies $\nu \in \mathbb{Z}$. This means $J_\nu(x)$ and $J_{-\nu}(x)$ are not linearly independent, so we need to use a second independent solution like

$$N_\nu(x) = \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

which is called a **Bessel function of the second kind**. Also related are the **Hankel functions** of the first and second kinds:

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iN_\nu(x). \end{aligned}$$

Case 2 (k imaginary): If k is imaginary, then we get the ODE

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left(1 - \frac{\nu^2}{x^2}\right) R = 0,$$

where again $x \equiv k\rho$. The solutions are the **modified Bessel functions** of the first and second kinds:

$$\begin{aligned} I_\nu(x) &= i^{-\nu} J_\nu(ix) \\ K_\nu(x) &= \frac{1}{2} \pi i^{\nu+1} H_\nu^{(1)}(ix). \end{aligned}$$

Case 3 ($k = 0$): If $k = 0$, which occurs when the problem has no z -dependence, (i.e. $Z(z) = \text{const}$). This effectively reduces it to a 2D problem. Now the ρ equation is

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \frac{\nu^2}{\rho^2} R = 0.$$

Suppose that $R(\rho)$ is a solution. Then

$$\bar{R}(\rho) \equiv R(\lambda\rho),$$

is also a solution. This implies the ODE is homogeneous in ρ , which suggests that we try the solution $R = \rho^\alpha$. Then we get

$$(\alpha^2 - \nu^2) R = 0.$$

This should be valid for all ρ , which implies that

$$\alpha = \pm \nu.$$

So our solution is

$$R(\rho) = A\rho^\nu + B\rho^{-\nu}.$$

These are two linearly independent solutions provided that $\nu \neq 0$. In the special case that $\nu = 0$, we can directly integrate the ODE to get

$$R(\rho) = A + B \ln \rho.$$

Then, since there's no z -dependence, the general solution is

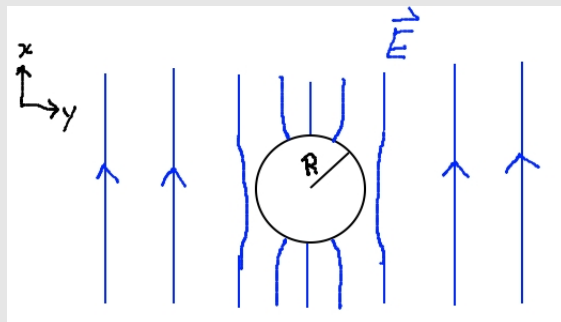
$$\begin{aligned} \Phi(\rho, \varphi) = & a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} \left\{ \rho^n \left[a_n \cos(n\varphi) + b_n \sin(n\varphi) \right] \right\} \\ & + \sum_{n=1}^{\infty} \left\{ \frac{1}{\rho^n} \left[c_n \cos(n\varphi) + d_n \sin(n\varphi) \right] \right\}. \end{aligned}$$

Tip

The choice of the solutions (oscillatory or exponential) depends on the boundary conditions of the given problem.

Example 3.13.1

Consider a long conducting cylinder of radius R lying along the z -axis. It is in an electric field that without the cylinder, would be uniform and $\vec{E} = E_0 \hat{x}$.



We are in cylindrical coordinates, and the setup is symmetric about the z -axis, so the general solution is

$$\begin{aligned} \Phi(\rho, \varphi) = & a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} \left\{ \rho^n \left[a_n \cos(n\varphi) + b_n \sin(n\varphi) \right] \right\} \\ & + \sum_{n=1}^{\infty} \left\{ \frac{1}{\rho^n} \left[c_n \cos(n\varphi) + d_n \sin(n\varphi) \right] \right\}. \end{aligned}$$

Since it is a conducting cylinder, we know that $\vec{E} = 0$ on the inside. We know the parallel components of \vec{E} do not jump as one crosses the surface from inside the cylinder to outside the cylinder, but the perpendicular component of the field does jump. So at the outer surface of the sphere, we know the electric field is perpendicular to the surface of the sphere.

As $\rho \rightarrow \infty$, we know that $\vec{E} \rightarrow E_0 \hat{x}$. This implies that $\Phi \rightarrow -E_0 x$ as $\rho \rightarrow \infty$. We can write this boundary condition as

$$\Phi \rightarrow -E_0 \rho \cos \varphi, \quad \text{as } \rho \rightarrow \infty.$$

Looking at the general solution, we see that as $\rho \rightarrow \infty$,

$$\Phi \rightarrow \sum_{n=1}^{\infty} \rho^n \left[a_n \cos(n\varphi) + b_n \sin(n\varphi) \right].$$

Equating this with the boundary condition at infinity, we see that

$$a_1 = -E_0, \quad a_n = 0 \text{ for } n > 1, \quad b_n = 0 \text{ for } n > 0.$$

Now our general solution is simplified to

$$\Phi(\rho, \varphi) = a_0 + b_0 \ln \rho - \rho E_0 \cos \varphi + \sum_{n=1}^{\infty} \left\{ \frac{1}{\rho^n} \left[c_n \cos(n\varphi) + d_n \sin(n\varphi) \right] \right\}.$$

At the surface of the cylinder, we have the boundary condition

$$\Phi(R) = \Phi_0.$$

I.e. at $\rho = R$, Φ is constant and there is no dependence on φ . Then

$$\Phi_0 = a_0 + b_0 \ln R - R E_0 \cos \varphi + \sum_{n=1}^{\infty} \left\{ \frac{1}{R^n} \left[c_n \cos(n\varphi) + d_n \sin(n\varphi) \right] \right\}.$$

Since it is independent of φ , the sum must cancel with the $-R E_0 \cos \varphi$ term. That is,

$$R E_0 \cos \varphi = \sum_{n=1}^{\infty} \left\{ \frac{1}{R^n} \left[c_n \cos(n\varphi) + d_n \sin(n\varphi) \right] \right\}.$$

Equating angular pieces, implies that $d_n = 0$ for all n and $c_n = 0$ for all $n > 1$, with c_1 determined by

$$R E_0 \cos \varphi = \frac{1}{R^1} c_1 \cos(1\varphi).$$

This implies that $c_1 = R^2 E_0$. Then the angular parts cancel each other, and we have

$$\Phi_0 = a_0 + b_0 \ln R,$$

which we can use to determine the coefficient a_0

$$a_0 = \Phi_0 - b_0 \ln R.$$

Incorporating these facts, our general solution is now

$$\Phi(\rho, \varphi) = \Phi_0 + b_0 \ln \frac{\rho}{R} - E_0 \left(1 - \frac{R^2}{\rho^2} \right) \rho \cos \varphi.$$

This leaves the b_0 coefficient undetermined, and it is related to the charge per unit length on the surface of the cylinder.

3.14 Summary: Electrostatics

Skills to Master

- Use Gauss's law to calculate the electric fields of simple charge distributions
- Given a potential ϕ , calculate the field \vec{E}
- Given a field \vec{E} , calculate the charge distribution ρ that generates that field
- Given a charge distribution ρ , calculate the total charge of the distribution, and the field
- Plot $\rho(r)$, $Q(r)$, and $E(r)$ for spherically-symmetric charge distributions
- Calculate the Cartesian electric monopole, dipole, and quadrupole moments for various discrete and continuous charge distributions
- Calculate the potential and field of an electric monopole, dipole, or quadrupole
- Calculate the force on, potential energy of, and torque on an electric dipole in an external field
- Solve the Laplace equation in Cartesian, spherical, and cylindrical coordinate systems for various systems of conductors
- Calculate the potential of some system using the method of images
- Calculate the potential of some system using Green's theorem
- Calculate spherical multipole moments for various discrete and continuous charge distributions

Maxwell's equations for electrostatics are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E} = 0.$$

Gauss's Law

If we integrate the first of Maxwell's equations over some volume V and then rewrite using the divergence theorem, we get *Gauss's law*

$$\int_{A(V)} \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int_V \rho dV = \frac{Q_{enc}}{\epsilon_0}.$$

The integral on the left is over the surface $A(V)$ of the volume V . Note that $d\vec{A} = \hat{n} dA$, where dA is an elemental area of the surface and \hat{n} is the unit vector normal to the surface at that point.

Using Gauss's law, we find that the field of a point charge q at \vec{r}_0 is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3}.$$

This can be extended immediately to multiple particles using the principle of superposition, and to continuous charge distributions with the integral

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}.$$

Charge Distributions

Given a field \vec{E} , the first of Maxwell's equations for electrostatics can be used to calculate the charge dis-

tribution that generates the field

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}.$$

Usually, we don't know \vec{E} , and we want to calculate it by writing down the charge distribution $\rho(\vec{r})$ and integrating.

In Cartesian coordinates, we can use the Dirac delta function to write the charge distribution for a point charge at the origin, a line charge along the z -axis, and a surface charge in the xy -plane

$$\begin{aligned} \rho(\vec{r}) &= \delta^{(3)}(\vec{r}) = \delta(x)\delta(y)\delta(z), & \text{point charge} \\ \rho(\vec{r}) &= \lambda(z) \delta(x) \delta(y), & \text{line charge} \\ \rho(\vec{r}) &= G(x, y) \delta(z), & \text{surface charge} \end{aligned}$$

Here, $G(x, y)$ is the Cartesian surface charge distribution, and $\lambda(z)$ is the line charge distribution.

Potential

Using the Helmholtz theorem, we can write the electric field as the gradient of a scalar function

$$\vec{E}(\vec{r}) = -\vec{\nabla} \phi(\vec{r}),$$

where the scalar function

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|},$$

is called the *electric potential*.

For a point charge

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_0|}.$$

Force and Work

The force on a test charge q due to an external electrostatic field \vec{E} is

$$\vec{F} = q\vec{E}.$$

Thus, the mechanical work done to bring a charge from \vec{r}_i to \vec{r}_f along a path C , is

$$W_{\vec{r}_i \rightarrow \vec{r}_f} = \int_C \vec{F} \cdot d\vec{\ell} = -q(\phi(\vec{r}_f) - \phi(\vec{r}_i)).$$

The work done is path-independent, which means the electric force is conservative.

Electric Moments

Monopole

The *monopole moment* of a system is just the total charge

$$Q = \int d^3r' \rho(r').$$

The monopole potential is the potential of a point charge Q

$$\phi_{\text{mono}} = \phi^{(0)} = \frac{Q}{4\pi\epsilon_0} \frac{1}{r}.$$

Note that $\phi \sim 1/r$ for a monopole.

Dipole

Two point charges $-q$ and q separated by a distance $|\vec{a}|$ where \vec{a} points from the negative charge to the positive charge form an ideal *dipole* in the limit $\vec{a} \rightarrow 0$. The *dipole moment* of such a system is

$$\vec{p} = q\vec{a}.$$

For a general charge distribution $\rho(\vec{r}')$, the dipole moment is

$$\vec{p} = \int d^3r' \rho(r') \vec{r}'.$$

The dipole potential is

$$\phi_{\text{dip}} = \phi^{(1)} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}.$$

Notice that it goes as $\phi \sim 1/r^2$. The electric field of an ideal dipole can be written in the coordinate-free form

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3}.$$

This can also be used to approximate the field of non-ideal dipoles at large distances. The force on a dipole in an external electric field \vec{E} is

$$\vec{F}_{\text{dip}} = \vec{p}(\vec{\nabla} \cdot \vec{E}) = \vec{\nabla}(\vec{p} \cdot \vec{E}),$$

its potential energy is

$$U = - \int \vec{F} \cdot d\vec{\ell} = -\vec{p} \cdot \vec{E},$$

and the torque on the dipole is

$$\vec{\tau} = \vec{p} \times \vec{E} + \vec{r} \times \vec{F},$$

where \vec{F} is some other external force (if it exists).

The interaction energy of two dipoles is

$$U = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{r^3}.$$

This can easily be derived by calculating the field \vec{E}_1 of dipole \vec{p}_1 and then plugging that into the formula for the potential energy of dipole \vec{p}_2 .

Quadrupole

For a general charge distribution $\rho(\vec{r}')$, the quadrupole moment is defined by the quadrupole tensor

$$Q_{ij} = \int d^3r' \rho(r') (3r'_i r'_j - r'^2 \delta_{ij}).$$

The indices i and j go over x, y , and z . Note, the r 's in the formula above all indicate distance of the charge from the origin—not distance between the charge and the observer. For a discrete charge distribution,

$$Q_{ij} = \sum_a q_a (3r_i r_j - r^2 \delta_{ij}),$$

where the sum is over the charges.

Remember, in cylindrical coordinates, $r^2 = s^2 + z^2$.

Some important properties of the quadrupole tensor include:

- It is traceless: $Q_{ij}\delta_{ij} = 0$
- It is symmetric: $Q_{ij} = Q_{ji}$. This implies that Q_{ij} can be diagonalized by choosing a coordinate system aligned with a principal axis

For general charge distributions, the quadrupole potential is

$$\phi_{\text{quad}} = \phi^{(2)} = \frac{1}{4\pi\epsilon_0} \frac{1}{2} Q_{ij} \frac{\hat{r}_i \hat{r}_j}{r^3}.$$

Notice that the quadrupole potential goes as $\phi \sim 1/r^3$.

Be careful how you interpret something like \hat{r}_i . This is a scalar—not a vector. It is a component of the unit vector in the \vec{r} direction. For example, $\hat{r}_x \equiv (\hat{r})_x = \sin\theta \cos\varphi$. It is not the same as $\hat{x} = (1, 0, 0)$.

In general, any moment is independent of the choice of origin only if all lower moments are zero.

Multipole Expansion

For a general charge distribution $\rho(\mathbf{r}')$ localized near the origin of our coordinate system, we can expand the potential for large r to get the *multipole* expansion

$$\begin{aligned}\phi(\mathbf{r}) &= \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \dots \\ &= \frac{1}{4\pi\epsilon_0} \left(\frac{Q_{tot}}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{2!} Q_{ij} \frac{\hat{r}_i \hat{r}_j}{r^3} + \dots \right).\end{aligned}$$

At large r , only the leading term dominates.

We can calculate the electric field of the charge distribution using $\vec{\mathbf{E}} = -\vec{\nabla}\phi$. Then

$$\begin{aligned}\vec{\mathbf{E}} &= \vec{\mathbf{E}}^{(0)} + \vec{\mathbf{E}}^{(1)} + \dots \\ &= \frac{1}{4\pi\epsilon_0} \left(Q_{tot} \frac{\hat{\mathbf{r}}}{r^2} + \frac{1}{r^3} \left[3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p} \right] + \dots \right).\end{aligned}$$

Laplace's Equation

Spherical Coordinates

By substituting $\vec{\mathbf{E}} = -\vec{\nabla}\phi$, into one of Maxwell's equations, we get Poisson's equation $\nabla^2\phi = \Delta\phi = -\frac{\rho}{\epsilon_0}$. If there are no charges in the considered region, then $\rho = 0$, and we get the Laplace equation

$$\nabla^2\phi = \Delta\phi = 0.$$

In spherical coordinates, this is

$$0 = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\phi) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \phi}{\partial \varphi^2}.$$

Assuming a separable solution, we try $\phi(r, \theta, \varphi) = \frac{U(r)}{r} \cdot P(\theta) \cdot Q(\varphi)$.

If our system is azimuthally symmetric, so that there is no dependence on φ , then the general solution to Laplace's equation is

$$\phi(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos\theta).$$

The $P_{\ell}(x)$ are Legendre polynomials. The first several are

$$\begin{aligned}P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(-1 + 3x^2) \\ &\vdots \\ &\vdots\end{aligned}$$

These polynomials have the parity relation

$$P_{\ell}(-x) = (-1)^{\ell} P_{\ell}(x),$$

so if ℓ is even then P_{ℓ} is even, and if ℓ is odd then P_{ℓ} is odd.

The $P_{\ell}(x)$ form a basis with orthonormality condition

$$\int_{-1}^1 dx P_{\ell}(x) P_m(x) = \frac{2}{2\ell+1} \delta_{\ell m}.$$

A useful trick when integrating a Legendre polynomial is to use the orthonormality condition and $1 = P_0(\cos\theta)$.

If integrating Legendre polynomials, it's often much faster to make the substitution $x = \cos\theta$. Then the Legendre polynomials are simply polynomials in x , which are much easier to integrate than polynomials in $\cos\theta$. For example,

$$\int_0^{\pi} \sin\theta d\theta P_{\ell}(\cos\theta) = \int_{-1}^1 dx P_{\ell}(x).$$

If a problem does not have azimuthal symmetry, then the general solution is

$$\phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \varphi),$$

where the $Y_{\ell m}(\theta, \varphi)$ are the spherical harmonics. The first several, given in spherical and Cartesian coordinates, are

$$\begin{aligned}Y_{00} &= \frac{1}{\sqrt{4\pi}} \\ Y_{11} &= -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r} \\ Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \\ Y_{1-1} &= \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x-iy}{r}\end{aligned}$$

The orthogonality condition is

$$\int d\Omega Y_{\ell m}^*(\theta, \varphi) Y_{\ell' m'}(\theta, \varphi) = \delta_{\ell\ell'} \delta_{mm'},$$

where the integral goes over the spherical angles. The spherical harmonics have definite parity, and

$$Y_{\ell, -m}(\theta, \varphi) = (-1)^m Y_{\ell m}^*(\theta, \varphi).$$

Remember that $m = -\ell, \dots, \ell$.

A useful trick when integrating a Legendre polynomial is to use the orthonormality condition and $1 = \sqrt{4\pi} Y_{00}(\theta, \varphi)$.

Cylindrical Coordinates

Laplace's equation $\Delta\Phi = 0$, in cylindrical coordinates is

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

We again assume a separable solution $\Phi(\rho, \varphi, z) = R(\rho)Q(\varphi)Z(z)$. In general, the solutions are Bessel functions or modified Bessel functions. However, if the problem has no z -dependence, then the general solution reduces to

$$\Phi(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} \left\{ \rho^n \left[a_n \cos(n\varphi) + b_n \sin(n\varphi) \right] \right\} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\rho^n} \left[c_n \cos(n\varphi) + d_n \sin(n\varphi) \right] \right\}.$$

General Procedure

The general procedure for solving Laplace's equation is:

1. Choose a coordinate system
2. Write down the general solution for Laplace's equation in that coordinate system. Is the problem azimuthally symmetric?
3. Identify the boundary conditions and write them in the most convenient form for your chosen coordinate system. For example, if a boundary condition contains z and you are in spherical coordinates, then write $z = r \cos \theta = r P_1(\cos \theta)$.
 - What is the potential at the given surfaces? Are they conductors?
 - What is the potential at infinity? Does it go to zero?
4. Apply the boundary conditions to identify the values of the nonzero coefficients in your general solution. Equate coefficients. Use the orthonormality condition.

Spherical Multipole Expansion

If we're far away from a charge distribution, we can expand the potential of the charge distribution as

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi),$$

where

$$q_{\ell m} = \int d^3r' \rho(\vec{r}') Y_{\ell m}^*(\theta', \varphi') r'^{\ell},$$

are the spherical multipole moments. For discrete charges,

$$q_{\ell m} = \sum_i q_i r_i^{\ell} Y_{\ell m}^*.$$

Notice that the spherical harmonic in $q_{\ell m}$ is the complex conjugate. Remember, if the system has azimuthal symmetry, then $m = 0$.

For each level ℓ , notice that $\phi \sim 1/r^{\ell+1}$. Keep in mind that $m = -\ell, \dots, \ell$, so for a given ℓ , there are $2\ell + 1$ moments $q_{\ell m}$. We also have the simple parity relation

$$q_{\ell, -m} = (-1)^m q_{\ell m}^*.$$

For $\ell = 0$, we get the spherical monopole moment.

and its relation to the Cartesian monopole moment Q_{tot} is

$$q_{00} = \int d^3r' \rho(\vec{r}') \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{4\pi}} Q_{tot}.$$

For $\ell = 1$, we get the spherical dipole moments, which can be related to the Cartesian dipole moment $\vec{p} = (p_x, p_y, p_z)$

$$\begin{aligned} q_{10} &= \int d^3r' \rho(\vec{r}') \sqrt{\frac{3}{4\pi}} z' = \sqrt{\frac{3}{4\pi}} p_z \\ q_{11} &= -\sqrt{\frac{3}{8\pi}} (p_x - ip_y) \\ q_{1-1} &= \sqrt{\frac{3}{8\pi}} (p_x + ip_y). \end{aligned}$$

Boundary Behavior

Consider a boundary surface with surface charge density σ . The normal component of the electric field is discontinuous across such a charged surface. In general,

$$E_{n,+} - E_{n,-} = \frac{\sigma}{\epsilon_0},$$

where $E_{n,+}$ means the normal component of the field on the outside. So the normal component of \vec{E} jumps by σ/ϵ_0 . The tangential components are continuous across a charged surface

$$E_{t,+} = E_{t,-}.$$

Since the electric field is the negative derivative of the potential, we can write these two results in terms of the potential. For example,

$$\frac{\partial\phi_-}{\partial n} - \frac{\partial\phi_+}{\partial n} = \frac{\sigma}{\epsilon_0}.$$

Conductors

We model conductors as materials containing positive and negative charges that are completely free to move. Some important facts about conductors are:

- The field inside a conductor is zero

$$\vec{E}_{inside}(\vec{r}) = 0.$$

- The potential inside a conductor is constant

$$\phi_{const}(\vec{r}) = \text{const.}$$

- Within a conductor, Gauss's law and $\vec{E} = 0$ imply that

$$Q_{enc} = 0.$$

Thus, any charge, if it exists, must reside on the surface of the conductor.

- On the inside of a conductor, the tangential and normal components of the field \vec{E} are zero. Immediately on the outside of a conductor with surface charge σ , these components are

$$E_t = 0, \quad E_n = \frac{\sigma}{\epsilon_0}.$$

These follow directly from the boundary conditions discussed earlier.

Image Charges

The solution to the Poisson equation is unique. Thus, if a solution can be guessed that satisfies all the boundary conditions, then it must be *the* solution to the problem. This is the idea behind image charges. The general procedure is:

1. Identify the boundary conditions and the region within which you want to solve Poisson's equation for the potential ϕ . Typically, this is the region outside of any conductors described in the problem statement
2. Remove all conductors from the problem. Put in image charges such that the potential due to all original charges plus all image charges satisfies the same boundary conditions as the original problem containing conductors
3. Write down the potential due to all original and image charges and verify explicitly that the original boundary conditions are satisfied. Since the solution to Laplace's equation is unique, this must be the solution to the original problem

Keep in mind:

- Never put image charges into the region for which you are calculating the potential

- The calculated potential is only valid in the region identified earlier. It is not valid in the region containing image charges

A charge near a conducting surface will induce a surface charge density

$$\sigma = \epsilon_0 E_n = -\epsilon_0 \frac{\partial \phi_{outside}}{\partial n},$$

on the conducting surface. *This is evaluated on the surface.* The total induced surface charge is

$$Q_{ind} = \oint \sigma dA,$$

where the integral goes over the entire surface.

Green's Theorem

Consider an arbitrary region of charge density ρ contained within an arbitrary volume V that is bounded by a surface S . Given two arbitrary scalar fields ϕ and ψ , Green's theorem says that

$$\int_V (\phi \Delta \psi - \psi \Delta \phi) d^3r = \oint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dA.$$

To get a version of this theorem that is particularly useful for electrodynamics, we choose ϕ to be the ordinary electric potential, and call ψ the Green's function $G(\vec{r}, \vec{r}')$, whose only requirement is that $\Delta G(\vec{r}, \vec{r}') = -4\pi\delta(\vec{r} - \vec{r}')$. This implies the form

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}'),$$

where $\Delta F = 0$ in the volume V . Note that \vec{r} is the observation point and \vec{r}' is the integration variable.

If the problem has Dirichlet boundary conditions, we choose F such that G is zero on the boundary

$$G_D(\vec{r}, \vec{r}') \Big|_{\vec{r}' \in S} = 0.$$

Then the Green's theorem can be written as

$$\begin{aligned} \phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}') G_D(\vec{r}, \vec{r}') d^3r' \\ &\quad - \frac{1}{4\pi} \int_S \phi(\vec{r}') \frac{\partial G_D}{\partial n'} dA'. \end{aligned}$$

In practice, the difficulty is in finding an F such that $G = 0$ on the boundary S . Remember that $\rho(\vec{r}')$ is the *explicit* charges in the volume V . To construct G , we place a test charge at \vec{r}' and identify the image charge. Then

$$G = \frac{1}{|\vec{r} - \vec{r}'|} + F,$$

where the first term is the potential of the test charge (up to the prefactor of $1/4\pi\epsilon_0$), and F is the potential

of the image charge (up to the prefactor), where the image charge is placed such that the boundary conditions are satisfied.

Remember, $G(\vec{r}, \vec{r}')$ is a function *only* of the geometry and does not care about the actual value the potential may or may not have.

If the problem has Neumann boundary conditions, we choose F such that

$$\left. \frac{\partial G_N}{\partial n'}(\vec{r}, \vec{r}') \right|_{\vec{r}' \in S} = -\frac{4\pi}{S},$$

where S is the total surface area. Now we get

$$\begin{aligned} \phi(\vec{r}) = & \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}') G_N(\vec{r}, \vec{r}') d^3r' \\ & + \frac{1}{4\pi} \int_S G_N(\vec{r}, \vec{r}') \frac{\partial \phi}{\partial n'} dA' + \langle \phi \rangle_S, \end{aligned}$$

where $\langle \phi \rangle_S$ is the average of ϕ over the surface S . In typical applications where $S \rightarrow \infty$, this term vanishes.

When using these formulae, remember that the normal direction n' points *outward* from the volume of interest.

Miscellaneous

Memorize at least the radial part of the gradient, and divergence in spherical coordinates.

$$\begin{aligned} \vec{\nabla} \phi &= \frac{\partial \phi}{\partial r} \hat{r} \\ \vec{\nabla} \cdot \vec{E} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r). \end{aligned}$$

Be careful when taking divergences. If you're taking the divergence of something that contains any \hat{r}/r^2 , then it is generally safer to fully expand the divergence of products of functions using the product rule

$$\vec{\nabla} \cdot (f \vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\nabla f),$$

and then using the fact

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^{(3)}(\vec{r}).$$

Whenever you have a field that behaves as $\vec{E} \sim \hat{r}/r^2$ when $\vec{r} \approx 0$, then this behavior is being generated by a point charge, i.e., the charge distribution contains a delta function.

The law of cosines may be useful:

$$|\vec{r} - \vec{r}'|^2 = r^2 + r'^2 - 2\vec{r} \cdot \vec{r}'.$$

To do Taylor expansion of a multivariable scalar function about the point \vec{r} , we use

$$\begin{aligned} f(\vec{r} + \delta\vec{r}) &= f(\vec{r}) + \left(\vec{\nabla} f(\vec{r}) \right) \delta\vec{r} \\ &\quad + \frac{1}{2} \left(\partial_i \partial_j f(\vec{r}) \right) \delta r_i \delta r_j + \dots \end{aligned}$$

If $\vec{F}(\vec{r} + \delta\vec{r})$ is a vector function, then

$$\vec{F}(\vec{r} + \delta\vec{r}) = \vec{F}(\vec{r}) + \left(\vec{\nabla} \cdot \vec{F}(\vec{r}) \right) \delta\vec{r} + \dots$$

Notice that for a vector function, the gradient becomes the divergence.

Chapter 4

Magnetostatics

Recall that Maxwell's equations decouple when ρ , \vec{j} , \vec{E} , and \vec{B} are independent of time. The two Maxwell's equations for magnetostatics are

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \mu_0 \vec{j} \\ \vec{\nabla} \cdot \vec{B} &= 0.\end{aligned}$$

4.1 Laws of Ampere and Biot-Savart

We can use the continuity equation $\partial_\mu j^\mu = 0$, and the time-independence of the charge density $\frac{\partial \rho}{\partial t} = 0$ to conclude that

$$\vec{\nabla} \cdot \vec{j} = 0,$$

in electrostatics and magnetostatics. Then we can use $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$ and Stoke's theorem

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} = \oint_C \vec{B} \cdot d\vec{\ell},$$

where S is the surface of some volume and C is the closed curve along the boundary of S , to get **Ampere's law**

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_C,$$

where

$$I_C = \int_S \vec{j} \cdot d\vec{A},$$

is the current passing through the surface S .

The other Maxwell equation, $\vec{\nabla} \cdot \vec{B} = 0$, implies that there exist no monopoles. Using the divergence theorem gives us

$$\oint_S \vec{B} \cdot d\vec{A} = 0,$$

where S is the surface enclosing some volume V . More generally,

$$\int_S \vec{B} \cdot d\vec{A} = \phi_M,$$

is the **flux** of the magnetic field through the surface S .

Similar to what we did in electrostatics, we can use the Helmholtz theorem to write a function $\vec{B}(\vec{r})$ in terms of the divergence and curl of \vec{B} . This leads us to the **Biot-Savart law**

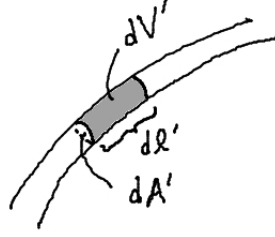
$$\vec{B}(\vec{r}) = \vec{\nabla} \times \frac{\mu_0}{4\pi} \int d^3 r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|},$$

which can be written as

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \vec{j}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}.$$

For a thin wire, we can write the Biot-Savart law as

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int d\vec{\ell}' \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}.$$



4.2 The Vector Potential

Recall that

$$\vec{B} = \vec{\nabla} \times \vec{A},$$

where \vec{A} is the vector potential. This equation holds in general—not just for magnetostatics.

By gauge invariance, we can choose a different \vec{A} resulting in the same field \vec{B} . For the 4-potential, the gauge transformation is

$$A^\mu \rightarrow A^\mu - \partial^\mu \Lambda,$$

for some function Λ . For the 3-potential,

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla} \Lambda.$$

Suppose we have some generic \vec{A}' , which we can transform via the gauge transformation

$$\vec{A} = \vec{A}' + \vec{\nabla} \Lambda.$$

Then

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}' + \Delta \Lambda.$$

We can always set this quantity to zero because we can always find a Λ such that

$$\Delta \Lambda = -\vec{\nabla} \cdot \vec{A}'.$$

In other words, we can always choose \vec{A}' such that

$$\vec{\nabla} \cdot \vec{A} = 0.$$

This is called the **Coulomb gauge**. There are other gauge choices such as the Lorentz gauge, but the Coulomb gauge is often the most convenient because then we get magnetostatics equations which are similar in form to the equations of electrostatics.

The Coulomb gauge is also valid in dynamical situations. It is not just for magnetostatics.

The Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ does not uniquely determine \vec{A} . Whenever $\Delta\Lambda = 0$, the resulting \vec{A} is the same. That is, we can have different choices Λ and Λ' , but if $\Delta\Lambda' = \Delta\Lambda$, then the resulting \vec{A} will be the same. On the other hand, if \vec{A} vanishes at infinity, then $\vec{\nabla} \cdot \vec{A} = 0$ does uniquely determine \vec{A} .

One drawback of the Coulomb gauge is that it hides causality and the Lorentz structure. Results in the Coulomb gauge can end up looking acausal, but once you go to the level of the fields, all acausalities are resolved.

In the Coulomb gauge,

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A} = \mu_0 \vec{j}.$$

Since we are in the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$, and we get

$$\Delta \vec{A} = -\mu_0 \vec{j}.$$

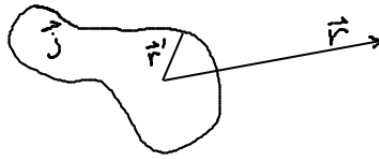
This is the Poisson equation for the vector potential. Keep in mind that this is a vector equation, so it is really three Poisson equations. From this, if we know \vec{j} , then we can calculate \vec{A} by integration

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}.$$

Taking the curl of this equation gives us the Biot-Savart law. This result for the vector potential is similar to the result in electrostatics for the scalar potential, but now we have three equations. This results in richer physical phenomena in magnetostatics than is found in electrostatics.

4.3 Magnetic Dipoles

Consider currents \vec{j} localized in some volume.



As in electrostatics, we can expand

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} - \vec{r}' \cdot \vec{\nabla} \frac{1}{r} + \dots,$$

for large \vec{r} . Then we can expand the components of the vector potential as

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{1}{r} \int d^3r' j_i(\vec{r}') + \frac{\vec{r}}{r^3} \cdot \int d^3r' \vec{r}' j_i(\vec{r}') + \dots \right].$$

The first term gives us the monopole contribution, and the second term gives us the dipole contribution.

Since the currents are localized, i.e. $\vec{j} = 0$ on the surface of some volume, then the divergence theorem implies that

$$\int d^3r' \vec{\nabla}' \cdot (r'_i \vec{j}) = 0.$$

We can expand the left side as

$$\int d^3r' \left(j_i(\vec{r}') + r'_i \vec{\nabla}' \cdot \vec{j} \right) = 0.$$

Since we're only considering the static case, we know that $\vec{\nabla}' \cdot \vec{j} = 0$, so the second term in the integral is zero. Thus,

$$\int d^3r' j_i(\vec{r}') = 0.$$

That is, there is no monopole contribution, and so

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{\vec{r}}{r^3} \cdot \int d^3r' \vec{r}' j_i(\vec{r}') + \dots \right].$$

For the dipole term, we again use charge localization and the divergence theorem to reason that

$$\int d^3r' \vec{\nabla}' \cdot (r'_i r'_j \vec{j}) = 0.$$

We can expand this as

$$\int d^3r' (j_i r'_j + j_j r'_i) = 0.$$

So we can write

$$\begin{aligned} \vec{r} \cdot \int d^3r' \vec{r}' j_i(\vec{r}') &= r_j \int d^3r' r'_j j_i \\ &= -\frac{1}{2} r_j \int d^3r' (r'_i j_j - r'_j j_i) \\ &= -\frac{1}{2} \varepsilon_{ijk} r_j \int d^3r' (\vec{r}' \times \vec{j})_k \\ &= -\frac{1}{2} \left(\vec{r} \times \int d^3r' (\vec{r}' \times \vec{j}) \right)_i. \end{aligned}$$

Remember, we are using Einstein summation notation. So we can write the vector potential as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} + \dots,$$

where

$$\vec{m} = \frac{1}{2} \int d^3r' \vec{r}' \times \vec{j}(\vec{r}'),$$

is the **magnetic dipole moment**.

Taking the cross product allows us to write the magnetic field in terms of the magnetic dipole moment as

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} + \dots \right].$$

The dipole term is typically the leading term. In fact, we will not do anything beyond the dipole term in this course.

Recall that the force on a moving point charge due to a magnetic field is

$$\vec{F} = q\vec{v} \times \vec{B}.$$

This implies that $\vec{F} \perp \vec{v}$, which implies that the magnetic field does no work on the particle. Now, suppose $q\vec{v}$ is a small current element, then we get the force on a current density due to a magnetic field

$$\vec{F} = \int d^3r' \vec{j} \times \vec{B}.$$

For a thin wire,

$$d\vec{F} = I d\vec{\ell} \times \vec{B}.$$

If we expand the magnetic field

$$\vec{B}(\vec{r}') = \vec{B}(0) + (\vec{r}' \cdot \vec{\nabla}) \vec{B}(0) + \dots,$$

and plug it into the integral formula for \vec{F} , we get

$$\vec{F}(\vec{r}) = (\vec{m} \times \vec{\nabla}) \times \vec{B},$$

which we can write as

$$\boxed{\vec{F}(\vec{r}) = \vec{\nabla} (\vec{m} \cdot \vec{B}) + \dots}$$

This is the force on a magnetic dipole \vec{m} due to \vec{B} .

This implies that the potential energy of a magnetic dipole \vec{m} in a magnetic field \vec{B} is

$$\boxed{U = -\vec{m} \cdot \vec{B} + \dots}$$

Note that this is not exact—hence the ellipses. Even an ideal current loop has higher orders.

The general formula for the torque on a dipole is

$$\vec{\tau} = \int d^3r' \vec{r}' \times (\vec{j} \times \vec{B}).$$

With multipole expansion,

$$\vec{\tau} = \int d^3r' \left[(\vec{r}' \cdot \vec{B}) \vec{j} - (\vec{r}' \cdot \vec{j}) \vec{B} \right].$$

This simplifies to

$$\boxed{\vec{\tau} = \vec{m} \times \vec{B} + \dots}$$

Does \vec{m} depend on the origin of the coordinate system? Given a magnetic moment

$$\vec{m} = \frac{1}{2} \int d^3r' \vec{r}' \times \vec{j}(\vec{r}'),$$

we can shift it by a constant vector \vec{r}_0 to get

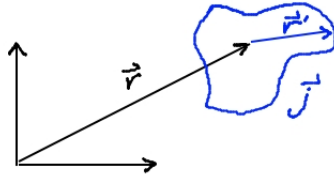
$$\vec{m} = \frac{1}{2} \int d^3r' (\vec{r}_0 + \vec{r}') \times \vec{j}(\vec{r}') = \frac{1}{2} \int d^3r' \vec{r}' \times \vec{j}(\vec{r}') + \frac{1}{2} \vec{r}_0 \times \int d^3r' \vec{j}(\vec{r}').$$

The first term on the right is the old expression for \vec{m} , and the second term is zero, so \vec{m} does not depend on the choice of origin.

Previously, we had \vec{j} localized at the origin, then

$$\vec{\tau} = \vec{m} \times \vec{B}(0).$$

Now we generalize to distributions away from the origin.



Now,

$$\begin{aligned}\vec{\tau} &= \int d^3r' (\vec{r} + \vec{r}') \times \left(\vec{j}(\vec{r} + \vec{r}') \times \vec{B}(\vec{r} + \vec{r}') \right) \\ &= \vec{r} \times \int d^3r' \left(\vec{j}(\vec{r} + \vec{r}') \times \vec{B}(\vec{r} + \vec{r}') \right) + \int d^3r' \vec{r}' \times \left(\vec{j}(\vec{r} + \vec{r}') \times \vec{B}(\vec{r} + \vec{r}') \right).\end{aligned}$$

The first term is \vec{r} cross the force $\vec{F}(\vec{r})$ and the second is the expression for $\vec{\tau}$ for \vec{j} localized at the origin. So

$$\vec{\tau}(\vec{r}) = \vec{r} \times \vec{F}(\vec{r}) + \vec{m} \times \vec{B}(\vec{r}) + \dots$$

4.4 Field of Moving Point Charge

For a moving particle with trajectory $\vec{r}_0(t)$,

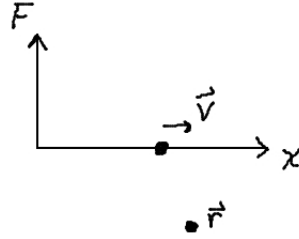
$$\vec{j}(\vec{r}, t) = q\vec{v}\delta^{(3)}(\vec{r} - \vec{r}_0).$$

To calculate \vec{A} , we can't just plug this into our previous integral formula for $\vec{A}(\vec{r})$ since that was derived assuming *magnetostatics*.

Consider a point charge q at rest at the origin in frame F' . Its potential in this frame is

$$\phi'(\vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}'|}, \quad \vec{A}'(\vec{r}') = 0.$$

Now consider a frame F wherein the charge (which is at rest in F') is moving in the $+x$ direction with velocity \vec{v} . Assume the charge is at the origin in F at time $t = 0$.



The potentials in F and F' are

$$\begin{aligned}F' : \quad A^{\mu'} &= \left(\frac{\phi'}{c}, \vec{A}' \right) = \left(\frac{\phi'}{c}, 0, 0, 0 \right) \\ F : \quad A^\mu &= \left(\frac{\phi}{c}, \vec{A} \right) = \left(\frac{\phi}{c}, A_x, A_y, A_z \right).\end{aligned}$$

Then we do a Lorentz boost $A^{\mu'} \rightarrow A^\mu$

$$A^\mu = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A^{\mu'}, \quad \vec{\beta} = \frac{\vec{v}}{c}.$$

This implies that

$$\phi = \gamma\phi',$$

and

$$A_x = \gamma\beta\frac{\phi'}{c}, \quad A_y = A_z = 0,$$

or

$$\vec{A} = \frac{\gamma}{c} \vec{\beta} \phi' = \frac{\vec{\beta}}{c} \phi.$$

We have

$$\phi(x', y', z') = \frac{1}{4\pi\epsilon_0} \frac{\gamma q}{\sqrt{x'^2 + y'^2 + z'^2}}.$$

We want $\phi(x, y, z)$, rather than $\phi(x', y', z')$, so we Lorentz transform the coordinates

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu.$$

This implies

$$x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z,$$

so

$$|\vec{r}'|^2 = \gamma^2(x - \beta ct)^2 + y^2 + z^2,$$

with $1/\gamma^2 = 1 - \beta^2$. Thus,

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{s},$$

where

$$s \equiv \sqrt{(x - \beta ct)^2 + (1 - \beta^2)(y^2 + z^2)}.$$

The electric field is

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$

Plugging in what we found,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q(1 - \beta^2)}{s^3} (x - vt, y, z).$$

More generally,

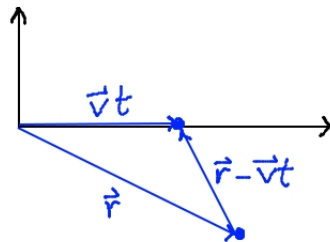
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q(1 - \beta^2)}{s^3} (\vec{r} - \vec{v}t).$$

For the magnetic field,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\frac{\vec{\beta}}{c} \phi \right) = (\vec{\nabla} \phi) \times \frac{\vec{\beta}}{c} + \phi \vec{\nabla} \times \frac{\vec{\beta}}{c} = \left(-\vec{E} - \frac{\partial \vec{A}}{\partial t} \right) \times \frac{\vec{\beta}}{c} = -\vec{E} \times \frac{\vec{\beta}}{c}.$$

Here we used the fact that $\vec{\nabla} \times \vec{\beta} = 0$ and that $\frac{\partial \vec{A}}{\partial t} \times \vec{\beta} = 0$ since $\vec{A} \parallel \vec{\beta}$. This tells us that a moving charge generates a magnetic field

$$\vec{B} = -\frac{1}{c^2} \vec{E} \times \vec{v}.$$



The fields refer to the charge's location at its *current* position—seemingly violating time dilation. However, it just appears to be wrong. It is really the field of the charge at a slightly earlier time (retarded position).

Keep in mind that this is no longer magnetostatics since \vec{j} is not time-independent. Consider the non-relativistic limit $\beta \ll 1$. In this limit,

$$\vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|} + \mathcal{O}(\beta^2).$$

This reproduces the Biot-Savart law for $\vec{j} = q\vec{v}\delta^{(3)}(\vec{r} - \vec{r}_0)$.

4.5 Larmor Precession

Consider a magnetic dipole

$$\vec{m} = \frac{1}{2} \int d^3r' \vec{r}' \times \vec{j}(\vec{r}'),$$

generated by point charges q_i moving non-relativistically with speed $v_{(i)} \ll c$. Then

$$\vec{j}_{(i)} = q_i \vec{v}_{(i)} \delta^{(3)}(\vec{r} - \vec{r}_{(i)}(t)).$$

Then

$$\begin{aligned} \vec{m} &= \sum_i \frac{1}{2} \int d^3r' \vec{r}' \times \left(q_i \vec{v}_{(i)} \delta^{(3)}(\vec{r} - \vec{r}_{(i)}(t)) \right) \\ &= \sum_i \frac{1}{2} q_i \vec{r}_{(i)} \times \vec{v}_{(i)} \\ &= \sum_i \frac{1}{2} \frac{q_i}{m_i} \vec{r}_{(i)} \times m_i \vec{v}_{(i)} \\ &= \sum_i \frac{1}{2} \frac{q_i}{m_i} \vec{L}_{(i)} \end{aligned}$$

If all charges have the same charge to mass ratio, then $q_i/m_i = q/m$ is a constant and

$$\boxed{\vec{m} = \frac{q}{2m} \vec{L}},$$

where

$$\vec{L} = \sum_i \vec{L}_{(i)} = \sum_i \vec{r}_{(i)} \times m_i \vec{v}_{(i)},$$

is the total angular momentum of the system.

Consider a dipole at the origin in a uniform magnetic field \vec{B} . Then

$$\vec{\tau} = \vec{m} \times \vec{B} = \frac{d\vec{L}}{dt}.$$

This implies

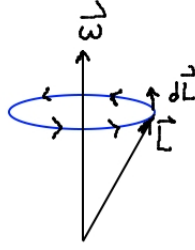
$$\frac{d\vec{L}}{dt} = \frac{q}{2m} \vec{L} \times \vec{B}.$$

Thus,

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L}},$$

where

$$\boxed{\vec{\omega} = -\frac{q}{2m} \vec{B}}.$$



So the angular momentum \vec{L} , and thus \vec{m} , precesses around $\vec{\omega}$ (or $-\vec{B}$) with **Larmor frequency** ω .

The spin of a particle is also an angular momentum and generates a magnetic dipole moment. Consider a particle with charge e

$$\vec{m} = g \frac{e}{2m} \vec{S},$$

for $g = 1$ for orbital momentum. For spin, it can have other values. For example, for a spin-1/2 electron, $|\vec{S}| = \hbar/2$ and Dirac theory predicts $g_e = 2$. In reality, g deviates from 2 due to quantum field effects. This *anomalous magnetic moment* of the electron is

$$a_e \equiv \frac{g_e - 2}{2} = \frac{\alpha}{2\pi} + \dots \neq 0,$$

where $\alpha \approx 1/137$ is the fine structure constant. The experimental value for the electron is

$$a_e^{exp} = 0.001\,159\,652\,180\,85(76).$$

This agrees very well with QED, and it's actually how we are able to measure the fine structure constant α to high precision.

4.6 Summary: Magnetostatics

Skills to Master

•

The two Maxwell's equations for magnetostatics are

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \mu_0 \vec{j} \\ \vec{\nabla} \cdot \vec{B} &= 0.\end{aligned}$$

Ampere and Biot-Savart Laws

Using $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$ and Stoke's theorem, we can derive Ampere's law

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_C,$$

which allows us to easily calculate the magnetic field for some simple symmetric current distributions. The current I_C through the closed loop C is given by

$$I_C = \int_S \vec{j} \cdot d\vec{A}.$$

The flux of the magnetic field \vec{B} through a surface S is

$$\int_S \vec{B} \cdot d\vec{A} = \phi_M.$$

This quantity is zero for any *closed* surface.

Using the Helmholtz theorem with the magnetic field gives us the Biot-Savart law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \vec{j}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3},$$

which allows us to calculate the magnetic field due to a general current distribution. For a thin wire, this simplifies to

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int d\vec{\ell}' \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3},$$

where I is the current through the wire.

The Vector Potential

Recall that the magnetic field can be written as the curl of a vector potential \vec{A}

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

Under a gauge transformation,

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda,$$

where Λ is some scalar function, the magnetic field is unchanged. This allows us always to choose the Coulomb gauge where

$$\vec{\nabla} \cdot \vec{A} = 0.$$

In the Coulomb gauge, replacing \vec{B} in the Maxwell equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$ with the equivalent $\vec{\nabla} \times \vec{A}$ gives us the Poisson equation for the vector potential

$$\Delta \vec{A} = -\mu_0 \vec{j}.$$

Since \vec{A} is a 3-vector, this is really a set of three Poisson equations. This Poisson equation implies that for a general current distribution,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}.$$

Magnetic Dipoles

If the current distribution \vec{j} is localized in some region, we can perform a long distance expansion of the integral formula for $\vec{A}(\vec{r})$. This allows us to write the vector potential as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} + \dots,$$

where

$$\vec{m} = \frac{1}{2} \int d^3r' \vec{r}' \times \vec{j}(\vec{r}'),$$

is the magnetic dipole moment. This is the first nonzero moment in the multipole expansion of $\vec{A}(\vec{r})$. As such, \vec{m} is always independent of the choice of origin.

Taking the cross product of $\vec{A}(\vec{r})$ gives us

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} + \dots \right].$$

The force on a current distribution, due to an external \vec{B} field is

$$\vec{F}(\vec{r}) = \vec{\nabla} (\vec{m} \cdot \vec{B}) + \dots$$

This is the force on a magnetic dipole \vec{m} due to \vec{B} . Its potential energy is

$$U = -\vec{m} \cdot \vec{B} + \dots$$

The torque on it is

$$\vec{\tau}(\vec{r}) = \vec{m} \times \vec{B}(\vec{r}) + \vec{r} \times \vec{F}(\vec{r}) + \dots$$

If \vec{j} is localized at the origin, then the second term is zero.

Field of Moving Point Charge

For a moving point charge with trajectory $\vec{r}_0(t)$, we have the “current”

$$\vec{j}(\vec{r}, t) = q\vec{v}\delta^{(3)}(\vec{r} - \vec{r}_0).$$

Consider the scalar and vector potentials in our frame F and in the particle's rest frame F' , which is moving in the $+x$ direction with velocity \vec{v} . After Lorentz transforming the 4-potential to get $\vec{A}(x', y', z')$ and $\phi(x', y', z')$ and then Lorentz transforming the coordinates to get them in terms of x, y, z , we find that the scalar and vector potentials are

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{s}$$

$$\vec{A} = \frac{\vec{\beta}}{c} \phi.$$

where

$$s \equiv \sqrt{(x - \beta ct)^2 + (1 - \beta^2)(y^2 + z^2)}.$$

Then the electric field of the moving particle is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q(1 - \beta^2)}{s^3} (\vec{r} - \vec{v}t).$$

For example, if the particle is moving in the x direction with speed v , then $\vec{r} - \vec{v}t = (x - vt, y, z)$. The associated magnetic field is

$$\vec{B} = -\frac{1}{c^2} \vec{E} \times \vec{v}.$$

Larmor Precession

If we have a group of point charges moving non-relativistically and they all have the same charge to mass ratio q/m , then the magnetic dipole moment they form can be written in terms of the total angular momentum of the system

$$\vec{m} = \frac{q}{2m} \vec{L}.$$

Recall that the total angular momentum of a system of particles is just the sum of their individual angular momenta

$$\vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i.$$

For a dipole \vec{m} at the origin in a uniform magnetic field \vec{B} , we can write the torque $\vec{\tau} = \vec{m} \times \vec{B}$ as the time derivative of the total angular momentum. Thus,

$$\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L},$$

where

$$\vec{\omega} = -\frac{q}{2m} \vec{B}.$$

This implies that the angular momentum vector \vec{L} (and also \vec{m}) precesses about $\vec{\omega}$ (or $-\vec{B}$) with the Larmor frequency

$$\omega \equiv |\vec{\omega}| = \frac{qB}{2m}.$$

Chapter 5

Electrodynamics

5.1 Plane Waves

Now we return to the full time-dependent Maxwell equations

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} &= \mu_0 \vec{j} \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0.\end{aligned}$$

Consider a region with no charge or currents, so $\rho = 0$ and $\vec{j} = \vec{0}$. Then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \Delta \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E}.$$

Here we used a vector calculus identity as well as the Maxwell equations with $\rho = 0$ and $\vec{j} = \vec{0}$. This implies that

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \vec{E} = 0.$$

This is the **wave equation** for the electric field \vec{E} . We define the **d'Alembert operator** as

$$\square \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta = \partial_\mu \partial^\mu,$$

then we can write the wave equation as

$$\square \vec{E} = 0.$$

Recall that

$$\partial^\mu = \begin{bmatrix} \frac{1}{c} \frac{\partial}{\partial t} \\ -\vec{\nabla} \end{bmatrix}, \quad \partial_\mu = \begin{bmatrix} \frac{1}{c} \frac{\partial}{\partial t} \\ \vec{\nabla} \end{bmatrix}.$$

We repeat the same thing for \vec{B} ,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\Delta \vec{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{E}.$$

This gives us the wave equation for the magnetic field \vec{B}

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \vec{B} = 0.$$

Consider solutions $f = f(z, t)$ which depend only on a single spatial variable. Such solutions are called plane waves. We can “factor” the PDE as

$$\begin{aligned} \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) f(z, t) &= 0 \\ \left(\frac{1}{c} \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) \left(\frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) f(z, t) &= 0. \end{aligned}$$

If we define the d'Alembert variables

$$\xi \equiv z + ct, \quad \eta \equiv z - ct,$$

then

$$\frac{\partial}{\partial \xi} = \frac{1}{2} \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right), \quad \frac{\partial}{\partial \eta} = \frac{1}{2} \left(\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} \right),$$

and

$$\frac{\partial^2}{\partial \xi \partial \eta} f(z, t) = 0.$$

Thus, we can treat $f(z, t)$ as independent of ξ , then

$$\frac{\partial}{\partial \eta} f(z, t) = g_2(\eta) \implies f(\xi, \eta) = \int d\eta g_2(\eta) + f_2(\xi).$$

Alternatively, we can treat $f(z, t)$ as independent of η , then

$$\frac{\partial}{\partial \xi} f(z, t) = g_1(\xi) \implies f(\xi, \eta) = \int d\xi g_1(\xi) + f_1(\eta).$$

This implies that

$$f(\xi, \eta) = f_1(\eta) + f_2(\xi),$$

which we can write as

$$\boxed{f(z, t) = f_1(z - ct) + f_2(z + ct).}$$

A function like $f_1(z - ct)$ gives some shape defined by f_1 that is moving to the right with speed c . A function like $f_2(z + ct)$ gives some shape defined by f_2 that is moving to the left with speed c .

Transverse EM Waves

We now consider transverse electromagnetic (TEM) plane waves involving $\vec{E} = \vec{E}(z, t)$ and $\vec{B} = \vec{B}(z, t)$, which depend only on a single spatial variable— z . Since $\rho = 0$, we know that $\vec{\nabla} \cdot \vec{E} = 0$, and this implies that

$$\frac{\partial E_z}{\partial z} = 0.$$

That is, E_z has no z -dependence. We also have

$$(\vec{\nabla} \times \vec{B}) \cdot \hat{z} = \left(\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) \cdot \hat{z} \implies \frac{\partial E_z}{\partial t} = c^2 (\vec{\nabla} \times \vec{B})_z = 0,$$

which implies that E_z also has no t -dependence.

$$\frac{\partial E_z}{\partial t} = 0.$$

Thus, $E_z(z, t)$ has neither z nor t dependence, so

$$E_z = \text{constant}.$$

We can set the constant to be zero. Thus, \vec{E} oscillates only in the xy -plane. We have two solutions, both of which are perpendicular to \hat{z}

$$\vec{E}_+(z, t) = \vec{f}_\perp(z + ct), \quad \vec{E}_-(z, t) = \vec{g}_\perp(z - ct).$$

Then

$$\frac{\partial}{\partial t} \vec{B}_+ = -\vec{\nabla} \times \vec{f}_\perp(z + ct) = \begin{bmatrix} \partial_z f_{\perp,y} \\ -\partial_z f_{\perp,x} \\ 0 \end{bmatrix} = -\hat{z} \times \frac{\partial \vec{f}_\perp}{\partial z} = -\hat{z} \times \left(\frac{1}{c} \frac{\partial}{\partial t} \vec{f}_\perp \right).$$

In the last equality we used the fact that the dependence of f_\perp on z and t has the form $z + ct$, so

$$\frac{\partial \vec{f}_\perp}{\partial z} = \frac{1}{c} \frac{\partial \vec{f}_\perp}{\partial t}.$$

So we found that,

$$\frac{\partial}{\partial t} \vec{B}_+ = -\hat{z} \times \left(\frac{1}{c} \frac{\partial}{\partial t} \vec{f}_\perp \right) = -\hat{z} \times \left(\frac{1}{c} \frac{\partial}{\partial t} \vec{E}_+ \right),$$

which implies

$$c\vec{B}_+ = -\hat{z} \times \vec{E}_+ + (\text{time-independent piece}).$$

This implies that

$$\vec{E}_+ \cdot \vec{B}_+ = 0 \implies \vec{E}_+ \perp \vec{B}_+.$$

Similarly, one can show that

$$\vec{E}_- \cdot \vec{B}_- = 0 \implies \vec{E}_- \perp \vec{B}_-.$$

In general,

$$\vec{E} = \vec{E}_+ + \vec{E}_-.$$

but

$$(\vec{E}_+ + \vec{E}_-) \cdot (\vec{B}_+ + \vec{B}_-) \neq 0.$$

For EM waves traveling in a generic but constant direction \hat{k} ,

$$\vec{E}(\vec{r}, t) = \vec{E}_\perp(ct - \hat{k} \cdot \vec{r}),$$

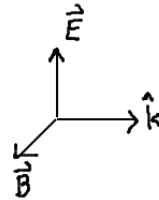
where \vec{E}_\perp is perpendicular to the direction of propagation \hat{k} . The associated magnetic field is

$$\boxed{c\vec{B}(\vec{r}, t) = \hat{k} \times \vec{E}(\vec{r}, t).}$$

The magnitudes of the field are related as

$$|\vec{E}| = c|\vec{B}|.$$

In general, for transverse EM waves, \vec{E} , \vec{B} , and \hat{k} form a right-handed mutually-perpendicular triad of vectors.



Monochromatic Plane Waves

For the wave equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right) f(z, t) = 0,$$

we assume a separable solution

$$f(z, t) = Z(z)T(t).$$

In the special case of *monochromatic* plane waves, we can write the solution in the form

$$\vec{E}_\perp(\phi) = \text{Re} \left(\vec{E}_{\perp,0} e^{i\phi} \right),$$

where $\phi \propto \hat{\mathbf{k}} \cdot \vec{\mathbf{r}} - ct$, and $\vec{E}_{\perp,0}$ is a complex constant vector in the xy -plane. That is,

$$\vec{E}_\perp(\vec{\mathbf{r}}, t) = \text{Re} \left(\vec{E}_{\perp,0} e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t)} \right),$$

with $\omega = c|\vec{\mathbf{k}}|$. We can also write it in 4-vector notation as

$$\boxed{\vec{E}_\perp(\vec{\mathbf{r}}, t) = \text{Re} \left(\vec{E}_{\perp,0} e^{-ik_\mu x^\mu} \right),}$$

where

$$k^\mu = \left(\frac{\omega}{c}, \vec{\mathbf{k}} \right),$$

is the wave 4-vector. For $\vec{\mathbf{B}}$ we have a similar structure.

The physical fields are always real. Remember that

$$\text{Re } e^{i\phi} = \cos \phi.$$

Typically people will write the fields without explicitly noting that the real part must be taken. For example, people will write

$$\vec{E}_\perp(\vec{\mathbf{r}}, t) = \vec{E}_{\perp,0} e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t)}.$$

This is fine as long as we only perform linear operations with the fields. However, taking products, magnitudes, etc. of the fields will cause problems unless we first take the real parts of the fields.

Polarization

At a fixed point $\vec{\mathbf{r}}$ in space, the fields are harmonic oscillators.

Consider a linear combination of two plane wave solutions with the same direction of propagation

$$\vec{E} = A \hat{\mathbf{e}}_1 \cos(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t + \delta_1) + B \hat{\mathbf{e}}_2 \cos(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t + \delta_2) = E_1 \hat{\mathbf{e}}_1 + E_2 \hat{\mathbf{e}}_2, \quad (5.1)$$

where all quantities are real and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are perpendicular to $\hat{\mathbf{k}}$ and to each other. At a fixed position $\vec{\mathbf{r}}$, \vec{E}_1 and \vec{E}_2 act like a pair of perpendicular harmonic oscillators, and the general shape traced out by the vector \vec{E} (at the fixed position $\vec{\mathbf{r}}$) is an ellipse. After some algebra, it can be shown that

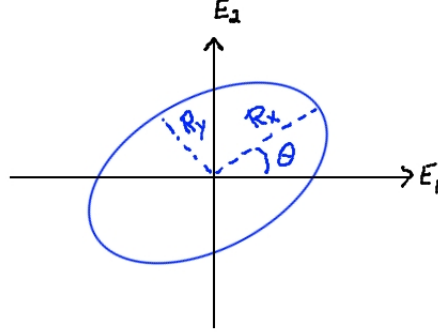
$$\left(\frac{E_1}{A} \right)^2 + \left(\frac{E_2}{B} \right)^2 - 2 \frac{E_1 E_2}{AB} \cos \delta = \sin^2 \delta,$$

where $\delta \equiv \delta_1 - \delta_2$. This is the equation of an ellipse centered at the origin and rotated through an angle

$$\theta = \frac{1}{2} \cot^{-1} \left[\frac{A^2 - B^2}{2AB \cos \delta} \right].$$

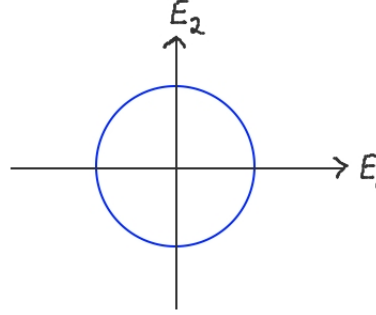
Elliptic Polarization

In general, for the EM wave given in Eq. (5.1), we have elliptical polarization



Circular Polarization

For the EM wave given in Eq. (5.1), if $A = B$ and $\delta = \delta_1 - \delta_2 = \pi/2 + m\pi$ for $m \in \mathbb{Z}$, then we have circular polarization.

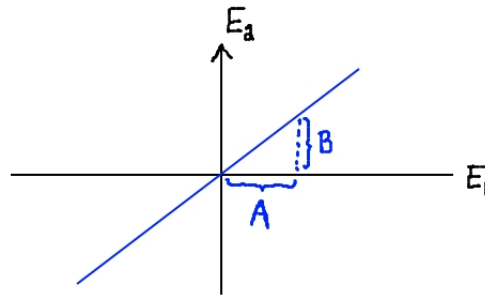


Circular polarization corresponds to the spin of the photon in or opposite the direction of motion. We can have left or right circular polarization. The convention in high energy physics, is that if the photon's spin is in the direction of motion, then it has right polarization. If its spin is opposite its direction of motion then it has left polarization.

A circular wave is *right*-polarized if when you wrap your fingers around the direction of rotation (i.e. right-hand rule) your thumb points in the direction of the wave's propagation. It is *left*-polarized if your thumb points opposite the direction of propagation.

Linear Polarization

For the EM wave given in Eq. (5.1), if $\delta = \delta_1 - \delta_2 = m\pi$ for $m \in \mathbb{Z}$, then we have **linear polarization**



Wave Packets

We can write the general superposition of solutions as the Fourier transform with respect to space

$$f(\vec{r}, t) = \text{Re} \left(\int d^3k f(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right),$$

where for electromagnetic waves in a vacuum,

$$\boxed{\omega(k) = ck},$$

is the dispersion relation. The Fourier coefficient is

$$f(\vec{k}) = \frac{1}{(2\pi)^3} \int d^3\vec{r} f(\vec{r}, t=0) e^{-i\vec{k} \cdot \vec{r}}.$$

Note, the “ $\text{Re}(\dots)$ ” ensures that we only need to include the $-i\omega t$ solutions and not also the $+i\omega t$ solutions since

$$\text{Re} \left(e^{i(\vec{k} \cdot \vec{r} + \omega t)} \right) = \cos(\vec{k} \cdot \vec{r} + \omega t) = \cos(-\vec{k} \cdot \vec{r} - \omega t) = \text{Re} \left(e^{i(-\vec{k} \cdot \vec{r} - \omega t)} \right).$$

That is, by integrating over $-\vec{k}$ as well as $+\vec{k}$, we are already including the $+i\omega t$ solutions.

In 1D, e.g. for a wave packet traveling in the z -direction,

$$\begin{aligned} f(z, t) &= \int_{-\infty}^{\infty} dk f(k) e^{i(kz - \omega t)} \\ f(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dz f(z, 0) e^{-ikz}. \end{aligned}$$

Taking the real part in the end is implied here.

Lorenz Gauge

Recall that

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$

Plugging this into Maxwell equation $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ gives us

$$\vec{\nabla} \cdot \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \right) = -\Delta\phi - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = \frac{\rho}{\epsilon_0}.$$

Adding and subtracting $\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$ between the two equality symbols allows us to write

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \phi - \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} \right) = \frac{\rho}{\epsilon_0}.$$

Similarly, one can derive

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \vec{A} + \vec{\nabla} \left(\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} \right) = \mu_0 \vec{J}.$$

We want to pick a gauge where the second term is zero

$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0.$$

This is called the **Lorenz gauge**, and in covariant notation, it is

$$\partial_\mu A^\mu = 0.$$

In the Lorenz gauge, the simplified equations for ϕ and \vec{A} can be written in covariant form as

$$\partial_\nu \partial^\nu A^\mu = \mu_0 j^\mu.$$

If A^μ is not in the Lorenz gauge, then we can make a gauge transformation

$$A'^\mu = A^\mu + \partial^\mu \Lambda.$$

Then we need a Λ such that

$$0 = \partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda.$$

This implies

$$\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu.$$

This is a wave equation, and it can always be solved for Λ .

One advantage of the Lorenz gauge is that $\partial_\mu A^\mu = 0$ is a scalar product of two Lorentz vectors, so it is Lorentz invariant and independent of the frame. A disadvantage of the Lorenz gauge is that it's not very restrictive. For monochromatic plane waves, there are six components of \vec{E} and \vec{B} , but there are really only two degrees of freedom since $\vec{E} \perp \vec{B}$, $\vec{E} \perp \vec{k}$ and $\vec{B} \perp \vec{k}$.

In a charge-free vacuum, it is possible to choose a frame with $\phi = 0$. Combine this with the Lorenz gauge condition $\partial_\mu A^\mu = 0$, and this implies $\vec{\nabla} \cdot \vec{A} = 0$. This is the **Coulomb gauge**. With this choice (and remember this is not generally true), $A^0 = 0$, and

$$\vec{A}(\vec{r}, t) = \text{Re} \int d^3k \vec{A}(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)},$$

where

$$\vec{A}(\vec{k}) = \frac{1}{(2\pi)^3} \int d^3r \vec{A}(\vec{r}, t=0) e^{-i\vec{k} \cdot \vec{r}}.$$

Then $\vec{\nabla} \cdot \vec{A} = 0$ implies that

$$0 = \text{Re} \vec{\nabla} \cdot \int d^3k \vec{A}(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \text{Re} \int d^3k i\vec{k} \cdot \vec{A}(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)},$$

and this implies that

$$\vec{k} \cdot \vec{A} = 0.$$

5.2 Energy and Momentum Conservation

Poynting's Theorem and Energy Conservation

Consider the work performed by the electric field \vec{E} on a charged particle (recall that \vec{B} does no work). The Lorentz force is

$$\vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right).$$

Then

$$d\mathcal{E}_{\text{mech}} = \vec{F} \cdot d\vec{S} = \vec{F} \cdot \vec{v} dt,$$

where $\mathcal{E}_{\text{mech}}$ is the mechanical energy. Recall that for a charged particle,

$$\vec{j} = q\vec{v}\delta(\vec{r} - \vec{r}_0(t)).$$

We can write

$$\frac{d\mathcal{E}_{mech}}{dt} = \int d^3r \, \vec{F} \cdot \frac{\vec{j}}{q} = \int d^3r \, \vec{E} \cdot \vec{j}.$$

Here, $\vec{E} \cdot \vec{j}$ is the “mechanical work density per unit time” or the “mechanical power density”.

If we plug in one of the Maxwell equations,

$$\vec{E} \cdot \vec{j} = \vec{E} \cdot \frac{1}{\mu_0} \left(\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right).$$

Using the vector identity

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}),$$

we can write

$$\begin{aligned} \vec{E} \cdot \vec{j} &= \frac{1}{\mu_0} \left[\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \right] - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \\ &= \frac{1}{\mu_0} \left[-\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \right] - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \\ &= -\frac{\partial}{\partial t} \left[\frac{1}{2} \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \right) \right] - \vec{\nabla} \cdot \left(\frac{\vec{E} \times \vec{B}}{\mu_0} \right) \\ &= -\frac{\partial u}{\partial t} - \vec{\nabla} \cdot \vec{S}, \end{aligned}$$

where

$$u = \frac{1}{2} \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \right),$$

is the energy density of the electromagnetic fields, and

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0},$$

is the **Poynting vector** with units of energy per area per unit time. It is a measure of the energy flowing in the \vec{S} direction.

A non-vanishing \vec{S} on the surface of a volume V does not necessarily imply that there is an energy exchange through the surface of V . There *could* be energy exchange, but \vec{S} could also be parallel to the surface implying no energy exchange through the surface.

Then

$$\vec{E} \cdot \vec{j} + \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0,$$

which implies Poynting’s theorem of energy conservation

$$\frac{d}{dt} (\mathcal{E}_{mech} + \mathcal{E}_{fields}) + \int_{surface} \vec{S} \cdot d\vec{A} = 0,$$

where

$$\frac{d}{dt} \mathcal{E}_{mech} = \int_V d^3r \, \vec{E} \cdot \vec{j},$$

and

$$\mathcal{E}_{fields} = \int_V d^3r \, u.$$

This all assumes that no particles are leaving the volume.

Momentum Conservation

We know that

$$\frac{d}{dt}\vec{p}_{mech} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

This implies

$$\boxed{\frac{d}{dt}\vec{p}_{mech} = \int d^3r (\rho\vec{E} + \vec{j} \times \vec{B})},$$

where

$$\rho\vec{E} + \vec{j} \times \vec{B},$$

is the “mechanical momentum density per unit time”.

Using Maxwell’s equations to eliminate all references to the charges ρ and \vec{j} , it can be shown that

$$\rho\vec{E} + \vec{j} \times \vec{B} = -\epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \frac{\partial}{\partial x^i} \sigma_{ij},$$

where

$$\sigma_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} |\vec{E}|^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} |\vec{B}|^2 \right),$$

are the components of **Maxwell’s stress tensor**. Then we can write

$$\left(\rho\vec{E} + \vec{j} \times \vec{B} + \frac{\partial \vec{g}}{\partial t} \right)_j = \frac{\partial}{\partial x^i} \sigma_{ij},$$

where

$$\boxed{\vec{g} = \epsilon_0 \vec{E} \times \vec{B} = \frac{1}{c^2} \vec{S}},$$

is the **momentum density** of the fields. If we integrate, then

$$\boxed{\left[\int_V dV (\rho\vec{E} + \vec{j} \times \vec{B}) + \frac{d}{dt} \int_V dV \vec{g} \right]_j = \int_S dA \hat{n}_i \sigma_{ij}.$$

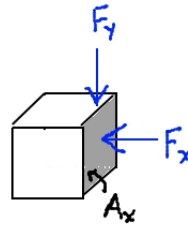
This is *momentum conservation*. The quantity on the right side is a force. Specifically, $\sigma_{ij} dA_i$ is a force in the j th direction and applied across the area dA pointing in the i th direction. That is,

$$\sigma_{ij} dA_i = \frac{\text{force in the } j\text{th direction}}{\text{area in the } i\text{th direction}}.$$

For example

$$\sigma_{xx} = \frac{F_x}{A_x}, \quad (\text{compression})$$

$$\sigma_{xy} = \frac{F_y}{A_x}, \quad (\text{shear}).$$



In 4D, we can write energy-momentum conservation together as

$$\boxed{\partial_\nu T^{\mu\nu} + f^\mu = 0},$$

where f^μ is the force density and

$$T^{\mu\nu} = \begin{bmatrix} u & S_x/c & S_y/c & S_z/c \\ g_x c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ g_y c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ g_z c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix} = \begin{bmatrix} u & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}.$$

Notice that $\mu = 0$ gives us energy conservation, and $\mu = i$ gives us momentum conservation.

5.3 Retarded Green's Function

We now look at Green's function for time-dependent potentials.

Recall

$$\partial_\nu \partial^\nu A^\mu = \mu_0 j^\mu \iff \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) A^\mu = \mu_0 j^\mu.$$

Recall that for the Poisson equation $\Delta\phi = -\rho/\epsilon_0$ with boundaries at infinity we can use Green's function. For a point charge distribution, $G = 1/|\vec{r} - \vec{r}'|$, then $\Delta G = -4\pi\delta(\vec{r} - \vec{r}')$. For a general charge distribution,

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') G(\vec{r}, \vec{r}').$$

We repeat this but now for the time-dependent Green's function. We start with

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) G(\vec{r}, t, \vec{r}', t') = 4\pi\delta^{(3)}(\vec{r} - \vec{r}')\delta(t - t').$$

This implies

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' dt' \rho(\vec{r}', t') G(\vec{r}, t, \vec{r}', t').$$

We can show that this time-dependent potential solves $\partial_\mu \partial^\mu \phi = \rho/\epsilon_0$. Note that if we calculate $\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \phi$, then the operator $\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right)$ acts only on \vec{r} and t and not on \vec{r}' or t' .

Now we want to construct an appropriate Green's function $G(\vec{r}, t, \vec{r}', t')$. As for boundary conditions, we will assume the volume is bounded at infinity.

Our first assumption is that G is invariant under translations in time and space. That is, we want G to be invariant under

$$\begin{aligned} \vec{r}, \vec{r}' &\longrightarrow \vec{r} + \vec{a}, \vec{r}' + \vec{a} \\ t, t' &\longrightarrow t + b, t' + b, \end{aligned}$$

where \vec{a} and b are arbitrary constants. Notice that under this transformation,

$$\vec{r} - \vec{r}' \longrightarrow \vec{r} - \vec{r}'.$$

So we can write

$$G(\vec{r}, t, \vec{r}', t') = G(\vec{R}, \tau),$$

where

$$\vec{R} \equiv \vec{r} - \vec{r}', \quad \tau \equiv t - t'.$$

Our second assumption is that $G \rightarrow 0$ at infinity.
The Fourier transform is

$$G(\vec{R}, \tau) = \int d^3k d\omega \tilde{G}(\vec{k}, \omega) e^{i\vec{k} \cdot \vec{R} - i\omega\tau}.$$

Notice that

$$4\pi\delta(\vec{R})\delta(\tau) = \frac{4\pi}{(2\pi)^4} \int d^3k d\omega e^{i\vec{k} \cdot \vec{R} - i\omega\tau}. \quad (5.2)$$

From the Fourier definition of G ,

$$\partial_\mu \partial^\mu G = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) G = \int d^3k d\omega \left(\frac{(-i\omega)^2}{c^2} - (i\vec{k})^2 \right) e^{i\vec{k} \cdot \vec{R} - i\omega\tau}.$$

We want this to be equal to the negative of Eq. (5.2). So we require

$$\left(\frac{(-i\omega)^2}{c^2} - (i\vec{k})^2 \right) \tilde{G} = \frac{4\pi}{(2\pi)^4}.$$

This implies

$$\tilde{G}(\vec{k}, \omega) = -\frac{4\pi}{(2\pi)^4} \frac{c^2}{\omega^2 - \vec{k}^2 c^2}.$$

Then

$$G(\vec{R}, \tau) = -\frac{4\pi c^2}{(2\pi)^4} \int d^3k d\omega \frac{1}{\omega^2 - \vec{k}^2 c^2} e^{i\vec{k} \cdot \vec{R} - i\omega\tau}.$$

This is the Green's function G that solves

$$\partial_\mu \partial^\mu G = -4\pi\delta^{(3)}(\vec{R})\delta(\tau).$$

We can perform the angular integrals using

$$\int d^3k = \int_0^\infty dk k^2 \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\varphi.$$

Writing $\vec{k} \cdot \vec{R} = kR \cos\theta$,

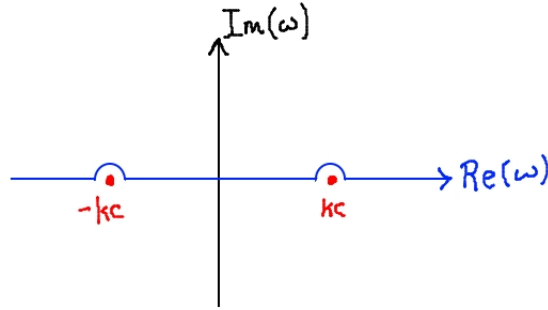
$$\begin{aligned} G(\vec{R}, \tau) &= -\frac{4\pi c^2}{(2\pi)^4} \int d^3k d\omega \frac{1}{\omega^2 - \vec{k}^2 c^2} e^{i\vec{k} \cdot \vec{R} - i\omega\tau} \\ &= -\frac{4\pi c^2}{(2\pi)^4} \int d\omega e^{-i\omega\tau} \int_0^\infty dk k^2 \frac{1}{\omega^2 - k^2 c^2} \int_{-1}^1 d(\cos\theta) e^{ikR \cos\theta} \int_0^{2\pi} d\varphi \\ &= -\frac{4\pi c^2}{(2\pi)^3} \int d\omega e^{-i\omega\tau} \int_0^\infty dk k^2 \frac{1}{\omega^2 - k^2 c^2} \int_{-1}^1 d(\cos\theta) \frac{1}{ikR} (e^{ikR} - e^{-ikR}) \\ &= -\frac{4\pi c^2}{(2\pi)^3} \frac{1}{iR} \int d\omega e^{-i\omega\tau} \int_0^\infty dk k \frac{1}{\omega^2 - k^2 c^2} (e^{ikR} - e^{-ikR}) \\ &= -\frac{4\pi c^2}{(2\pi)^3} \frac{1}{iR} \int d\omega e^{-i\omega\tau} \int_{-\infty}^\infty dk k \frac{1}{\omega^2 - k^2 c^2} e^{ikR} \\ &= -\frac{4\pi c^2}{(2\pi)^3} \frac{1}{iR} \int_{-\infty}^\infty dk k \int_{-\infty}^\infty d\omega \frac{1}{\omega^2 - k^2 c^2} e^{i(kR - \omega\tau)}. \end{aligned}$$

Next, we want to do the ω -integral. Using partial fraction decomposition, we can write

$$\int_{-\infty}^\infty d\omega \frac{1}{\omega^2 - k^2 c^2} e^{i(kR - \omega\tau)} = \int_{-\infty}^\infty d\omega \frac{1}{2kc} \left(\frac{1}{\omega - kc} - \frac{1}{\omega + kc} \right) e^{i(kR - \omega\tau)}.$$

The problem is that we have poles at $\omega = kc$ where the integral over ω is undefined. We must define it more carefully. To resolve this, we redefine the integral using a regularizing parameter ε , which is taken to zero in the end. This is a standard approach in quantum field theory.

The idea is to deform the integration contour into the complex ω -plane. When we integrate along the real axis, we avoid the poles at $\pm kc$ by integrating along semi-circles of radius ε .



Note, this choice of integration path is not unique. We could also choose the semi-circles to extend downward, or for either one to extend upwards and the other one downwards.

To evaluate the integral, we use **Cauchy's theorem**, which states that

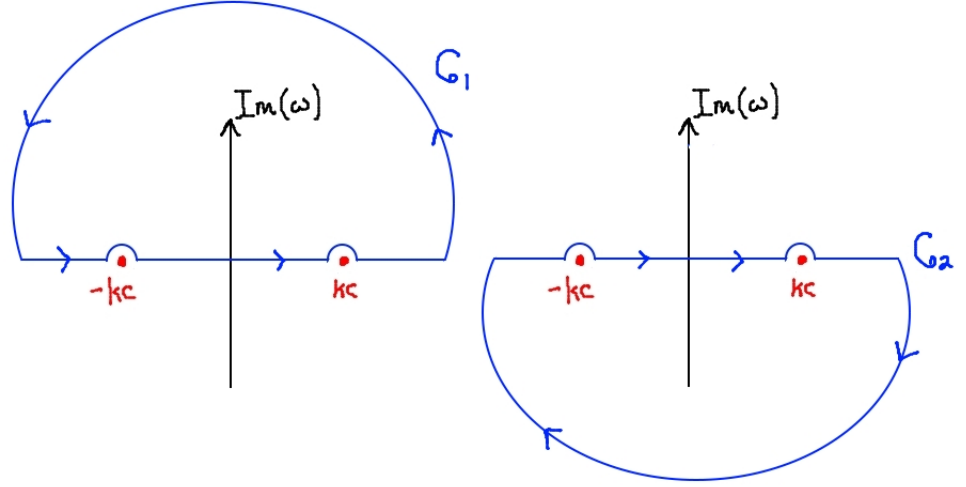
$$\oint_{\mathcal{C}} dz f(z) = 2\pi i \sum_{z_i} \text{Res } f(z_i).$$

On the left, we have the counter-clockwise integral of a complex function $f(z)$ over a closed curve \mathcal{C} . On the right, we have a sum over the residues of the poles z_i , contained inside \mathcal{C} , of the function $f(z)$. Cauchy's theorem is valid for functions $f(z)$ which are analytic except for a finite number of poles. To obtain the residue of a pole, we can use a Laurent series expansion. For example, for a pole at $z = 0$, we would expand $f(z)$ about $z = 0$

$$f(z) = \frac{c_{-n}}{z^n} + \cdots + \frac{c_{-1}}{z} + c_0 + \cdots$$

Then the residue of the pole is the coefficient of the $1/z$ term. In this case, the residue is c_{-1} .

We can close the contour (remember, we require a closed loop to apply Cauchy's theorem), either by going upward (\mathcal{C}_1) or downward (\mathcal{C}_2) in a semi-circle of radius $R \rightarrow \infty$. In this limit, the contribution from the semi-circle part of the contour must go to zero in order for us to recover the integral along only the real axis, which is what we want.



The contour we choose depends on the sign of $\tau = t - t'$.

If $\tau < 0$, we use the upper semi-circle. That is, we use the contour C_1 , which contains no poles. So by Cauchy's theorem,

$$\oint_{C_1} d\omega \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} = 0.$$

We can break the contour into the real line and the upper semi-circle,

$$\oint_{C_1} d\omega \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} = \int_{-\infty}^{\infty} \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} + \int_{\text{semi-circle}} d\omega \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2}.$$

It can be shown that the second integral goes to zero in the limit in which the semi-circle radius goes to infinity, thus implying that

$$\int_{-\infty}^{\infty} \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} = \oint_{C_1} d\omega \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} = 0, \quad \text{for } \tau < 0.$$

If $\tau > 0$, we use the lower semi-circle. That is, we use the contour C_2 , which contains the two poles. Calculating the residue of each pole and then applying Cauchy's theorem, we get

$$\oint_{C_2} d\omega \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} = 2\pi i \frac{1}{2kc} (e^{-ikc\tau} - e^{ikc\tau}).$$

A slight complication is that this result is valid for a counter-clockwise oriented circle, and we want a clockwise oriented circle so that the integral along the real line goes in the right direction. So when we break the contour into two pieces—the real line plus the lower semi-circle, we have to insert a negative sign.

$$-\oint_{C_2} d\omega \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} = \int_{-\infty}^{\infty} \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} + \int_{\text{semi-circle}} d\omega \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2}.$$

Again, it can be shown that the second integral goes to zero in the limit in which the semi-circle radius goes to infinity, thus implying that

$$\int_{-\infty}^{\infty} \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} = -\oint_{C_2} d\omega \frac{e^{-i\tau\omega}}{\omega^2 - k^2c^2} = -2\pi i \frac{1}{2kc} (e^{-ikc\tau} - e^{ikc\tau}), \quad \text{for } \tau > 0.$$

Now that we've completed the ω -integral, our Green's function is simplified to

$$G(\vec{R}, \tau) = 0,$$

for $\tau < 0$. For $\tau \geq 0$,

$$\begin{aligned} G(\vec{R}, \tau) &= \frac{1}{2\pi} \frac{c}{R} \int_{-\infty}^{\infty} dk \left(e^{-ikc\tau} - e^{ik\tau} \right) e^{ikR} \\ &= \frac{c}{R} \left(\delta(R - c\tau) - \delta(R + c\tau) \right). \end{aligned}$$

The quantity $R + c\tau$ is always positive for the case when $\tau \geq 0$, so the second delta-function does not contribute. We can then combine both $\tau < 0$ and $\tau \geq 0$ cases in the single function

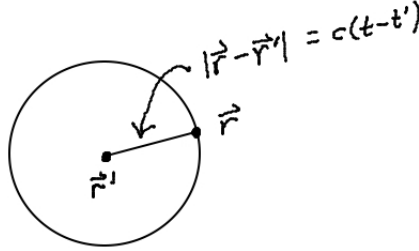
$$G(\vec{R}, \tau) = \frac{1}{R} \delta(\tau - R/c),$$

or

$$G(\vec{r}, t, \vec{r}', t') = \frac{1}{|\vec{r} - \vec{r}'|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right).$$

If we have a source at \vec{r}' , then points on the wave front satisfy the retarded time relation

$$|\vec{r} - \vec{r}'| = c(t - t').$$



A change of source at \vec{r}' and time t' changes the potential at another point \vec{r} only at the later (retarded) time

$$t = t' + \frac{|\vec{r} - \vec{r}'|}{c}.$$

In the limit where $c \rightarrow \infty$, we get the “instantaneous” Green’s function

$$G = \frac{1}{|\vec{r} - \vec{r}'|} \delta(t - t').$$

We can also consider the *advanced* (as opposed to the *retarded*) Green’s function. However, this seems to be unphysical since the reaction at \vec{r} occurs before the change in source at \vec{r}' .

From

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' dt' \rho(\vec{r}', t') G(\vec{r}', t', \vec{r}, t),$$

we get the **retarded potential**

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho\left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c}\right)}{|\vec{r} - \vec{r}'|}.$$

Similarly,

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}\left(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c}\right)}{|\vec{r} - \vec{r}'|}.$$

5.4 Radiation

Radiation of Oscillating Sources

Consider an oscillating source

$$\vec{j}(\vec{r}, t) = e^{-i\omega t} \vec{j}(\vec{r}).$$

In general we can decompose a current density in a Fourier decomposition

$$\vec{j}(\vec{r}, t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \vec{j}(\vec{r}, \omega),$$

and treat each component separately. Then the oscillating source gives us the vector potential

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3r' \vec{j}(\vec{r}') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|},$$

where $k = \omega/c$.

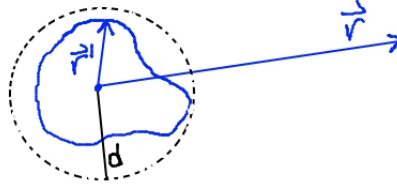
There are three relevant length scales:

- We assume that the source is localized such that $|\vec{r}'| \lesssim d$ for some d . Also, we assume the source is at the origin
- We have the observer at \vec{r}
- We have the wavelength

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega/c},$$

of the emitted radiation

So our three relevant length scales are d , \vec{r} , and λ . In general, we assume the observer is outside the source distribution (i.e. $r \gg d$) and the sources are non-relativistic (i.e. $\lambda \gg d$)



We have three zones to consider:

- Near (static) zone: $d \ll r \ll \lambda$
- Intermediate (induction) zone: $d \ll r \sim \lambda$
- Far (radiation) zone: $d \ll \lambda \ll r$

Near Zone

In the near zone, we have $\lambda \gg r$ which implies $\omega, k \sim 0$. That is,

$$k|\vec{r} - \vec{r}'| \approx kr \ll 1 \implies e^{ik|\vec{r}-\vec{r}'|} \approx 1.$$

Then

$$\vec{A}(\vec{r}, t) = e^{-i\omega t} \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}.$$

This is just $e^{-i\omega t}$ times the static vector potential. This is a quasistationary result. I.e., the source varies so slowly that the static solution is good.

Intermediate Zone

In the intermediate or induction zone, we have $d \ll r \sim \lambda$. In this zone we have to keep all powers of kr , making this region complicated to deal with. We will not do anything in this zone.

Far Zone

In the far or radiation zone, we have $d \ll \lambda \ll r$. For this zone we can do a simultaneous expansion of r'/r and kr' . We can write

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2\vec{r} \cdot \vec{r}'} = r - \frac{\vec{r}}{r} \cdot \vec{r}' + \dots \simeq r - \hat{r} \cdot \vec{r}'.$$

Thus,

$$\frac{1}{|\vec{r} - \vec{r}'|} \approx \frac{1}{r} + \mathcal{O}\left(\frac{d^2}{r^2}\right),$$

so

$$e^{ik|\vec{r} - \vec{r}'|} = e^{ikr} e^{-ik(\hat{r} \cdot \vec{r}')} + \mathcal{O}(d/r).$$

So the vector potential can be written as

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int d^3r' \vec{j}(\vec{r}') e^{-ik(\hat{r} \cdot \vec{r}')}.$$

At large distances, $e^{i(kr - \omega t)}/r$, is an outgoing spherical wave. At *very* large distances, the spherical wave front looks locally like a plane wave.

From $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = c\vec{B} \times \hat{r}$ and with \vec{A} in spherical coordinates, we can write (after some algebra)

$$\begin{aligned} \vec{B} &= \frac{\partial \vec{A}}{\partial t} \times \hat{r} = i\vec{k} \times \vec{A} \\ \vec{E} &= c\vec{B} \times \hat{r}. \end{aligned}$$

The energy carried away by radiation is given by the Poynting vector which reduces to

$$\vec{S} = \frac{c}{\mu_0} |\vec{B}|^2 \hat{r}.$$

The power is given by

$$P = \frac{d\mathcal{E}}{dt} = \int \vec{S} \cdot d\vec{A} = \int |\vec{S}| r^2 d\Omega.$$

The r^2 in the integrand is cancelled by a factor of $1/r^2$ in $|\vec{B}|^2$. The power radiated in some solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} = |\vec{S}| r^2 = \frac{c}{\mu_0} |\vec{B}|^2 r^2.$$

We can expand the remaining oscillating factor as

$$e^{-ik(\hat{r} \cdot \vec{r}')} = 1 - ik(\hat{r} \cdot \vec{r}') - \frac{1}{2}(k\hat{r} \cdot \vec{r}')^2 + \dots$$

Each term is suppressed by $kd \sim v/c$. Typically, we keep only the leading term. The first term in this expansion corresponds to electric dipole radiation, and the second term corresponds to magnetic dipole or electric quadrupole radiation.

Electric Dipole Radiation

For the dipole approximation, we keep only the first term in the $k \cdot d$ expansion

$$e^{-ik(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')} \approx 1.$$

Then the vector potential can be written

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{-i\omega t'}}{r} \int d^3r' \vec{j}(\vec{r}'),$$

where $t' = t - r/c$. Since $\vec{j}(\vec{r}, t) = e^{i\omega t} \vec{j}(\vec{r})$, we can write this as

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \int d^3r' \vec{j}(\vec{r}', t').$$

Integrating by parts (integrating '1' and differentiating \vec{j}), and then applying the continuity equation, we can write

$$\begin{aligned} \int d^3r' \vec{j}(\vec{r}', t') &= - \int d^3r' \vec{r}' \left(\vec{\nabla} \cdot \vec{j}(\vec{r}', t') \right) + \text{surface term} \\ &= - \int d^3r' \vec{r}' \left(-\frac{\partial \rho}{\partial t} \right) \\ &= \frac{d}{dt} \int d^3r' \rho(\vec{r}', t') \vec{r}' \\ &= \dot{\vec{p}}(t'). \end{aligned}$$

where \vec{p} is the electric dipole moment. Note that $d/dt = d/dt'$ here. Therefore,

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \dot{\vec{p}}(t') \Big|_{t'=t-r/c}.$$

The magnetic field is

$$\vec{B}(\vec{r}, t) = \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \times \hat{\mathbf{r}},$$

which becomes

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi c} \frac{1}{r} \ddot{\vec{p}}(t') \times \hat{\mathbf{r}} \Big|_{t'=t-r/c}.$$

Only the magnitude of \vec{B} enters into the formula for the power. Suppose \vec{p} is along the z -axis, and θ is the angle between \vec{p} and the observer at \vec{r} , then since the time derivative doesn't change the direction of \vec{p} , we can write

$$\left| \ddot{\vec{p}} \times \hat{\mathbf{r}} \right| = \left| \ddot{\vec{p}} \right| \sin \theta.$$

Therefore,

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} \left| \ddot{\vec{p}}(t') \right|^2 \sin^2 \theta \Big|_{t'=t-r/c}.$$

This is the power at a moment in time. It gives the general angular distribution of radiated power for electric dipole radiation. In general, the angle between the dipole and the observer can be time-dependent, and \vec{p} may not be in the z -direction, so we should really write

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} \left| \ddot{\vec{p}}(t') \right|^2 \sin^2(\alpha(t')) \Big|_{t'=t-r/c},$$

where $\alpha(t')$ is the angle between \vec{p} and \vec{r} . Note that $dP/d\Omega$ does not depend on the distance r to the observer, because the solid angle does not depend on the distance to the observer. However, the total power received by a receiver with fixed surface area will fall off as $1/r^2$ since the solid angle subtended by the receiver falls off as $1/r^2$.

Consider the dependence in $dP/d\Omega$ of r , ω , and the angles:

- We know that $\vec{B} \propto 1/r^2$, but this cancels in $dP/d\Omega$. Then the only r -dependence in $dP/d\Omega$ is in $t' = t - r/c$, which is averaged out. So $dP/d\Omega$ is effectively constant in r for a given $d\Omega$.
- This form of the equation for $dP/d\Omega$ is independent of ω —it works for all modes. For a single mode, we would have $dP/d\Omega \propto \omega^4$.
- The only angular dependence in $dP/d\Omega$ is $\sin^2 \theta$. We see that no radiation is projected in the forward or backward directions relative to \vec{p} . In fact, most of the radiation is orthogonal to the direction of \vec{B} .

To get the total power radiated, we have to integrate

$$\int d\Omega \sin^2 \theta = 2\pi \int_{-1}^1 d(\cos \theta) (1 - \cos^2 \theta) = \frac{8}{3}\pi.$$

So we get

$$P = \frac{2}{3} \frac{\mu_0}{4\pi c} \left| \ddot{\vec{p}}(t') \right|^2.$$

Suppose

$$\vec{p} = \vec{p}_0 \cos \left(\omega t - \omega \frac{r}{c} \right).$$

The time average of $\cos^2(\omega t - \omega r/c)$ over a single period is

$$\frac{1}{T} \int_0^T dt \cos^2(\omega t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos^2 \phi = \frac{1}{2}.$$

When averaging, we can ignore the phase shift r/c in the cosine since a phase shift does not change the average value. Then the time average of the radiated power is

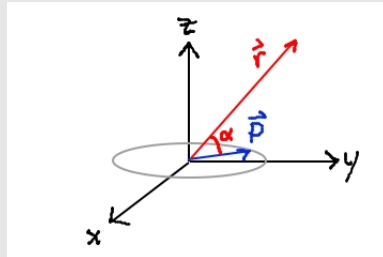
$$\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{\mu_0}{32\pi^2 c} |\vec{p}_0|^2 \omega^4 \sin^2 \theta.$$

The time average of the total power radiated is

$$\langle P \rangle_t = \frac{1}{6} \frac{\mu_0}{4\pi c} |\vec{p}_0|^2 \omega^4.$$

Example 5.4.1

Suppose you have a dipole moment \vec{p} that rotates in the xy -plane with angular velocity $\vec{\omega} = \omega \hat{z}$.



The dipole moment is

$$\vec{p} = (\cos(\omega t - \varphi), \sin(\omega t - \varphi), 0),$$

and so

$$\ddot{\vec{p}} = -\omega^2 \vec{p}.$$

Then plugging it into the formula,

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} \omega^4 |\vec{p}|^2 \sin^2(\alpha(t')) \Big|_{t'=t-r/c}.$$

Note that the angle α between \vec{p} and the observer at \vec{r} is now time-dependent. We can write the cosine of the angle as a dot product $\cos \alpha = \hat{p} \cdot \hat{r}$. Since we will be taking the time-average anyway, we are free to choose \hat{r} to be in the xz -plane at a given time. So

$$\cos \alpha = \hat{p} \cdot \hat{r} = (\cos(\omega t - \varphi), \sin(\omega t - \varphi), 0) \cdot (\sin \theta, 0, \cos \theta) = \cos(\omega t - \varphi) \sin \theta.$$

Then

$$\begin{aligned} \langle \sin^2 \alpha(t) \rangle_t &= \langle 1 - \cos^2 \alpha(t) \rangle_t \\ &= 1 - \langle \cos^2 \alpha(t) \rangle_t \\ &= 1 - \langle \cos^2(\omega t - \varphi) \rangle_t \sin^2 \theta \\ &= 1 - \frac{1}{2} \sin^2 \theta = \frac{1 + \cos^2 \theta}{2}. \end{aligned}$$

Therefore,

$$\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{\mu_0}{16\pi^2 c} \omega^4 |\vec{p}|^2 \frac{1 + \cos^2 \theta}{2}.$$

Keep in mind that θ is the polar angle to the observation point. Notice that the radiation is stronger in the directions perpendicular to the plane in which \vec{p} is rotating.

For a point charge q with mass m at \vec{r} , the dipole moment is simply

$$\vec{p} = q\vec{r} \implies \ddot{\vec{p}} = q\ddot{\vec{r}}.$$

If the particle is accelerated by some force $\vec{F} = ma = m\ddot{\vec{r}}$, then

$$\ddot{\vec{p}} = qa.$$

Then it radiates power

$$P = \frac{2}{3} \frac{\mu_0}{4\pi c} \left| \ddot{\vec{p}}(t') \right|^2 = \frac{2}{3} \frac{\mu_0}{4\pi c} q^2 a^2.$$

This is the rate at which the particle loses energy due to dipole radiation. This is the **Larmor formula**. It is generally valid for non-relativistic particles with speed $v \ll c$.

Suppose instead of being given \vec{p} , we are given the current distribution. Then we know from earlier work that

$$\dot{\vec{p}}(t) = \int d^3 r' \vec{j}(\vec{r}', t).$$

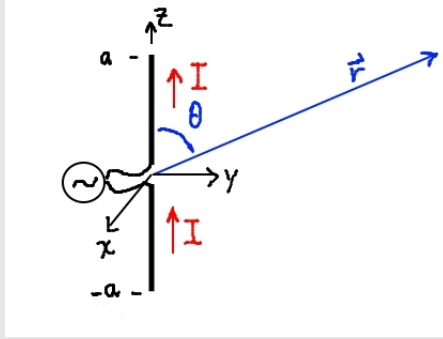
From this,

$$\ddot{\vec{p}}(t) = \frac{d}{dt} \int d^3 r' \vec{j}(\vec{r}', t).$$

Example 5.4.2

Consider a center-fed linear antenna as shown below. It has length $2a$, is centered at the origin, and lies along the z -axis. The current in the antenna is

$$I(t) = I_0 \left(1 - \frac{|z|}{a} \right) \cos(\omega t).$$



In this example,

$$\dot{\vec{p}} = \int d^3r \vec{j} = \int_{-a}^a dz I \hat{z} = I_0 \cos(\omega t) \hat{z} \int_{-a}^a dz \left(1 - \frac{|z|}{a} \right) = a I_0 \cos(\omega t) \hat{z}.$$

Then

$$|\ddot{\vec{p}}|^2 = a^2 I_0^2 \omega^2 \sin^2(\omega t).$$

The total power radiated is

$$P = \frac{2}{3} \frac{\mu_0}{4\pi c} |\ddot{\vec{p}}(t')|^2 = \frac{2}{3} \frac{\mu_0}{4\pi c} (a I_0 \omega)^2 \sin^2(\omega t).$$

Then

$$\langle P \rangle_t = \frac{2}{3} \frac{\mu_0}{4\pi c} |\ddot{\vec{p}}(t')|^2 = \frac{1}{3} \frac{\mu_0}{4\pi c} (a I_0 \omega)^2.$$

Magnetic Dipole and Electric Quadrupole Radiation

Suppose we have a current loop carrying an AC current $I = I_0 \sin \omega t$. This system has no electric dipole, so there is no electric dipole radiation. We have to go to the next term in the expansion

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} e^{i(kr - \omega t)} \int d^3r' \vec{j}(\vec{r}') \left[1 - ik\hat{r} \cdot \vec{r}' + \dots \right].$$

We can write this expansion in the form

$$\vec{A}(\vec{r}, t) = \vec{A}_0 + \vec{A}_1 + \dots,$$

where \vec{A}_0 is the dipole term that we already derived. Note that

$$\frac{1}{c} \frac{d}{dt'} \vec{j}(\vec{r}', t') = -ik e^{i(kr - \omega t)} \vec{j}(\vec{r}').$$

After some algebra and with the use of vector identities, the new term can be written

$$\vec{A}_1(\vec{r}, t) = \vec{A}_{1A}(\vec{r}, t) + \vec{A}_{1B}(\vec{r}, t),$$

where

$$\begin{aligned}\vec{A}_{1A}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \frac{1}{cr} \hat{r} \times \frac{d}{dt} \frac{1}{2} \int d^3r' \vec{r}' \times \vec{j}(\vec{r}', t') \\ \vec{A}_{1B}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \frac{1}{cr} \frac{d}{dt} \int d^3r' \left[\frac{1}{2} (\hat{r} \cdot \vec{j}) \vec{r}' + (\hat{r} \cdot \vec{r}') \vec{j} \right].\end{aligned}$$

The first can be written as

$$\vec{A}_{1A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{cr} \hat{r} \times \dot{\vec{m}}.$$

Using the continuity equation and integrating by parts, the components of the second one can be written as

$$(\vec{A}_{1B})_i = \frac{\mu_0}{4\pi} \frac{1}{cr} \hat{r}_k \frac{1}{6} \frac{d^2}{dt^2} \int d^3r' 3\rho r'_k r'_i.$$

Double-check this result. This is almost the electric quadrupole moment Q_{ki} . Recall that

$$Q_{ki} = \int d^3r' (3\rho r'_k r'_i - \delta_{ki} r'^2).$$

Note that

$$\hat{r}_k \delta_{ki} = \hat{r}_i.$$

The result is that

$$\vec{A}_{1B} = \vec{A}_{1B}' = \frac{\mu_0}{4\pi} \frac{1}{6cr} \ddot{\vec{Q}},$$

where $(\ddot{\vec{Q}})_i \equiv \hat{r}_k Q_{ki}$. Then

$$\vec{B}_1 = \frac{\mu_0}{4\pi c} \left[\frac{1}{cr} (\ddot{\vec{m}} \times \hat{r}) \times \hat{r} + \frac{1}{6rc} \ddot{\vec{Q}} \times \hat{r} \right].$$

Compare this with the result from electric dipole radiation

$$\vec{B}_0 = \frac{\mu_0}{4\pi c} \frac{1}{r} (\ddot{\vec{p}} \times \hat{r}).$$

We have the same $1/r$ behavior in both, but in \vec{B}_1 we have a factor of $1/c$. This velocity suppression is the result of this being a higher-order term in the kd expansion.

For the electric field, we get

$$\vec{E}_1 = \frac{\mu_0}{4\pi} \left[\frac{1}{cr} (\ddot{\vec{m}} \times \hat{r}) + \frac{1}{6rc} (\ddot{\vec{Q}} \times \hat{r}) \times \hat{r} \right].$$

Compare this with the result from electric dipole radiation

$$\vec{E}_0 = \frac{\mu_0}{4\pi} \frac{1}{r} (\ddot{\vec{p}} \times \hat{r}) \times \hat{r}.$$

The electric dipole, magnetic dipole, and electric quadrupole contributions to multipole radiation are

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi} \left[\frac{1}{r} \dot{\vec{p}} + \frac{1}{rc} \left(\dot{\vec{m}} \times \hat{r} + \frac{1}{6} \ddot{\vec{Q}} \right) + \dots \right] \\ \vec{B} &= \frac{\mu_0}{4\pi c} \left[\frac{1}{r} (\ddot{\vec{p}} \times \hat{r}) + \frac{1}{rc} \left((\ddot{\vec{m}} \times \hat{r}) \times \hat{r} + \frac{1}{6} (\ddot{\vec{Q}} \times \hat{r}) \right) + \dots \right] \\ \vec{E} &= \frac{\mu_0}{4\pi} \left[\frac{1}{r} (\ddot{\vec{p}} \times \hat{r}) \times \hat{r} + \frac{1}{rc} \left(\ddot{\vec{m}} \times \hat{r} + \frac{1}{6} (\ddot{\vec{Q}} \times \hat{r}) \times \hat{r} \right) + \dots \right].\end{aligned}$$

The total power radiated is

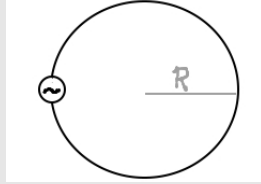
$$P_{tot} = \frac{\mu_0}{4\pi c} \left[\frac{2}{3} |\ddot{\vec{p}}|^2 + \frac{2}{3c^2} |\ddot{\vec{m}}|^2 + \frac{1}{180c^2} \ddot{\vec{Q}}^{ij} \ddot{\vec{Q}}^{ji} + \dots \right].$$

Example 5.4.3

Consider a circular antenna of radius R carrying an alternating current

$$I(t) = I_0 \sin(\omega t),$$

along the wire. What is the total power radiated?



In vector form, we write the current along the wire as

$$\vec{I}(t) = I_0 \sin(\omega t) \hat{\phi}.$$

For the electric dipole, we know that

$$\begin{aligned} \dot{\vec{p}}(t) &= \int d^3r' \vec{j}(\vec{r}', t) = I_0 \sin(\omega t) \int d\ell \hat{\phi} \\ &= I_0 \sin(\omega t) R \int_0^{2\pi} d\varphi (-\sin \varphi \hat{x} + \cos \varphi \hat{y}) = 0. \end{aligned}$$

So there is no electric dipole contribution to the radiation.

Recall that the magnetic dipole moment of a flat loop of current is $\vec{m} = I\vec{a}$. In this case,

$$\vec{m} = I_0 \pi R^2 \sin(\omega t) \hat{z}.$$

Then

$$\ddot{\vec{m}} = -\omega^2 I_0 \pi R^2 \sin(\omega t) \hat{z}.$$

The power radiated is

$$P_{tot} = \frac{\mu_0}{4\pi c} \frac{2}{3c^2} |\ddot{\vec{m}}|^2 = \frac{\mu_0}{4\pi c} \frac{2}{3c^2} \omega^4 I_0^2 \pi^2 R^4 \sin^2(\omega t).$$

The time average is

$$\langle P_{tot} \rangle_t = \frac{\mu_0}{4\pi c} \frac{\omega^4 I_0^2 \pi^2 R^4}{3c^2}.$$

5.5 Summary: Electrodynamics

Skills to Master

•

Plane Waves

Consider a region with no charge ($\rho = 0$) or currents ($\vec{j} = 0$). If we take the curl of two of Maxwell's equations and plug in the other two equations, we can derive the wave equations for the electric and magnetic fields

$$\begin{aligned} \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \vec{E} &= 0 \\ \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \vec{B} &= 0. \end{aligned}$$

In the special case of *monochromatic* plane waves traveling in the \vec{k} direction, we can write the solutions in the complex form

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_0 e^{-ik_\mu x^\mu} \\ \vec{B}(\vec{r}, t) &= \frac{1}{c} \hat{k} \times \vec{E}(\vec{r}, t), \end{aligned}$$

where $\omega = c|\vec{k}|$ and $k^\mu = (\omega/c, \vec{k})$ is the wave 4-vector. In general, the coefficient \vec{E}_0 is a complex vector whose real part is perpendicular to \vec{k} . The physical fields are always real. The implication is that one takes the real part of the complex fields given above.

Consider a linear combination of two plane wave solutions with the same direction of propagation

$$\begin{aligned} \vec{E} &= A \hat{e}_1 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta_1) \\ &\quad + B \hat{e}_2 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta_2) \\ &= E_1 \hat{e}_1 + E_2 \hat{e}_2, \end{aligned}$$

where all quantities are real and \hat{e}_1 and \hat{e}_2 are perpendicular to \hat{k} and to each other. At a fixed position \vec{r} , \vec{E}_1 and \vec{E}_2 act like a pair of perpendicular harmonic oscillators, and the general shape traced out by the vector \vec{E} (at the fixed position \vec{r}) is an ellipse. After some algebra, it can be shown that

$$\left(\frac{E_1}{A} \right)^2 + \left(\frac{E_2}{B} \right)^2 - 2 \frac{E_1 E_2}{AB} \cos \delta = \sin^2 \delta,$$

where $\delta \equiv \delta_1 - \delta_2$. Then

- In general, we have *elliptic* polarization
- If $A = B$ and $\delta = \delta_1 - \delta_2 = \pi/2 + m\pi$ for $m \in \mathbb{Z}$, then we have *circular* polarization

- $\delta = \delta_1 - \delta_2 = m\pi$ for $m \in \mathbb{Z}$, then we have *linear polarization*

We can write the general superposition of plane wave solutions as the Fourier transform with respect to space

$$\begin{aligned} f(\vec{r}, t) &= \int d^3k f(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ f(\vec{k}) &= \frac{1}{(2\pi)^3} \int d^3\vec{r} f(\vec{r}, t=0) e^{-i\vec{k} \cdot \vec{r}}, \end{aligned}$$

where for electromagnetic waves in a vacuum,

$$\omega(k) = ck,$$

is the dispersion relation. Taking the real part in the end is implied here. In 1D, e.g. for a wave packet traveling in the z -direction,

$$\begin{aligned} f(z, t) &= \int_{-\infty}^{\infty} dk f(k) e^{i(kz - \omega t)} \\ f(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dz f(z, 0) e^{-ikz}. \end{aligned}$$

In the *Lorenz gauge*,

$$\partial_\mu A^\mu = 0.$$

Then ϕ and \vec{A} can be written in covariant form as $\partial_\nu \partial^\nu A^\mu = \mu_0 j^\mu$. If A^μ is not in the Lorenz gauge, then we can always make a gauge transformation $A'^\mu = A^\mu + \partial^\mu \Lambda$ that leads to the wave equation for \vec{A} : $\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu$, which can always be solved for Λ . One advantage of the Lorenz gauge is that $\partial_\mu A^\mu = 0$ is a scalar product of two Lorentz vectors, so it is Lorentz invariant and independent of the frame.

In a charge-free vacuum, it is possible to choose a frame with $\phi = 0$. Combine this with the Lorenz gauge condition $\partial_\mu A^\mu = 0$, and this implies the *Coulomb gauge*

$$\vec{\nabla} \cdot \vec{A} = 0.$$

With this choice (and remember this is not generally true), $A^0 = 0$, and

$$\begin{aligned} \vec{A}(\vec{r}, t) &= \text{Re} \int d^3k \vec{A}(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ \vec{A}(\vec{k}) &= \frac{1}{(2\pi)^3} \int d^3r \vec{A}(\vec{r}, t=0) e^{-i\vec{k} \cdot \vec{r}}. \end{aligned}$$

Then $\vec{\nabla} \cdot \vec{A} = 0$ implies that $\vec{k} \cdot \vec{A} = 0$.

Energy and Momentum

Poynting's theorem of energy conservation states that

$$\frac{d}{dt}(\mathcal{E}_{mech} + \mathcal{E}_{fields}) + \int_{surface} \vec{S} \cdot d\vec{A} = 0,$$

where the time rate of change of the mechanical energy is

$$\frac{d}{dt}\mathcal{E}_{mech} = \int_V d^3r \vec{E} \cdot \vec{j},$$

and the energy stored in the fields is the integral over the volume of the fields

$$\mathcal{E}_{fields} = \int_V d^3r u,$$

of the energy density of the fields

$$u = \frac{1}{2} \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \right).$$

The Poynting vector

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0},$$

with units of energy per area per unit time, is a measure of the energy flowing in the \vec{S} direction. The flux of energy through a surface is

$$\int \vec{S} \cdot d\vec{A}.$$

The statement of momentum conservation is given by

$$\left[\int_V dV \left(\rho \vec{E} + \vec{j} \times \vec{B} \right) + \frac{d}{dt} \int_V dV \vec{g} \right]_j = \int_S dA \hat{n}_i \sigma_{ij},$$

where $\rho \vec{E} + \vec{j} \times \vec{B}$ is the mechanical momentum density per unit time,

$$\vec{g} = \epsilon_0 \vec{E} \times \vec{B} = \frac{1}{c^2} \vec{S},$$

is the momentum density of the fields, and $\sigma_{ij} dA_i$ is a force in the j th direction applied across the area dA pointing in the i th direction.

In 4D, we can write energy-momentum conservation together as

$$\partial_\nu T^{\mu\nu} + f^\mu = 0,$$

where f^μ is the force density and

$$T^{\mu\nu} = \begin{bmatrix} u & S_x/c & S_y/c & S_z/c \\ g_x c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ g_y c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ g_z c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}.$$

Notice that $\mu = 0$ gives us energy conservation, and $\mu = i$ gives us momentum conservation.

Retarded Green's Function

In electrostatics, we were able to write the potential as an integral over the charge distribution times a Green's function $G(\vec{r})$ that satisfied $\Delta G = -4\pi\delta^{(3)}(\vec{r}-\vec{r}')$. For the time-dependent case, we can derive the *retarded* Green's function

$$G(\vec{r}, t, \vec{r}', t') = \frac{1}{|\vec{r}-\vec{r}'|} \delta\left(t-t' - \frac{|\vec{r}-\vec{r}'|}{c}\right),$$

which satisfies

$$\partial_\mu \partial^\mu G = -4\pi\delta^{(3)}(\vec{r}-\vec{r}')\delta(t-t').$$

If we have a source at \vec{r}' , then points on the wave front satisfy the retarded time relation $|\vec{r}-\vec{r}'| = c(t-t')$. A change of source at \vec{r}' and time t' changes the potential at another point \vec{r} only at the later (retarded) time

$$t = t' + \frac{|\vec{r}-\vec{r}'|}{c}.$$

The retarded potentials are

$$\begin{aligned} \phi(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \\ \vec{A}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}. \end{aligned}$$

Radiation

Consider an oscillating source

$$\vec{j}(\vec{r}, t) = e^{-i\omega t} \vec{j}(\vec{r}).$$

This implies the vector potential

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3r' \vec{j}(\vec{r}') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|},$$

where $k = \omega/c$.

In general, we assume the source \vec{j} is localized and small such that both the wavelength λ of the emitted radiation and the observer distance r are much larger than the extent of the source distribution.

In the “near” zone where $r \ll \lambda$, we have $e^{ik|\vec{r}-\vec{r}'|} \approx 1$, then we get the quasistationary result

$$\vec{A}(\vec{r}, t) = e^{-i\omega t} \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}.$$

In the “far” or “radiation” zone where $r \gg \lambda$, we can do a simultaneous expansion of r'/r and kr' to get

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int d^3r' \vec{j}(\vec{r}') e^{-ik(\hat{r} \cdot \vec{r}')}.$$

From now on we will assume that we are in the radiation zone. We can expand the remaining oscillating factor as

$$e^{-ik(\hat{r} \cdot \vec{r}')} = 1 - ik(\hat{r} \cdot \vec{r}') - \frac{1}{2}(k\hat{r} \cdot \vec{r}')^2 + \dots$$

The first term corresponds to electric dipole radiation, and the second term corresponds to magnetic dipole or electric quadrupole radiation. Typically, we keep only the leading term.

From $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = c\vec{B} \times \hat{r}$ and with \vec{A} in spherical coordinates, we can derive the fields

$$\begin{aligned} \vec{B} &= \frac{\partial \vec{A}}{\partial t} \times \hat{r} = i\vec{k} \times \vec{A} \\ \vec{E} &= c\vec{B} \times \hat{r}. \end{aligned}$$

The energy carried away by radiation is given by the Poynting vector which reduces to

$$\vec{S} = \frac{c}{\mu_0} |\vec{B}|^2 \hat{r}.$$

The power radiated in some solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} = |\vec{S}|r^2 = \frac{c}{\mu_0} |\vec{B}|^2 r^2.$$

E1 Radiation

For electric dipole (E1) radiation,

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \int d^3r' \vec{j}(\vec{r}', t').$$

This leads us to

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi c} \frac{1}{r} \ddot{\vec{p}}(t') \times \hat{r} \Big|_{t'=t-r/c},$$

where $\ddot{\vec{p}}(t')$ is the second time derivative of the electric dipole moment \vec{p} .

Recall that for discrete point charges,

$$\vec{p} = \sum_i e_i \vec{r}_i.$$

More generally,

$$\vec{p}(t') = \int d^3r' \rho(\vec{r}', t') \vec{r}'.$$

If given a current distribution, the first time derivative of the dipole moment can be calculated (after integrating by parts and applying the continuity equation) as

$$\dot{\vec{p}}(t') = \int d^3r' \vec{j}(\vec{r}', t').$$

If $\alpha(t')$ is the (possibly time-dependent) angle between the dipole moment \vec{p} and the observation point \vec{r} , then the angular distribution of the power radiated at a given moment in time is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} \left| \ddot{\vec{p}}(t') \right|^2 \sin^2(\alpha(t')) \Big|_{t'=t-r/c}.$$

Note that $dP/d\Omega$ does not depend on the distance r to the observer, because the solid angle does not depend on the distance to the observer. However, the total power received by a receiver with fixed surface area will fall off as $1/r^2$ since the solid angle subtended by the receiver falls off as $1/r^2$. To get the total power P , we integrate $dP/d\Omega$ over the spherical angles. Typically we want a time average of P or $dP/d\Omega$ in which case it is helpful to remember that the time average of $\sin^2 \theta$ and $\cos^2 \theta$ is simply $1/2$.

For a point charge q with mass m at \vec{r} , the dipole moment is simply $\vec{p} = q\vec{r}$, which implies that $\ddot{\vec{p}} = q\ddot{\vec{r}} = q\vec{a}$, where \vec{a} is the acceleration of the particle. Then it radiates power

$$P = \frac{2}{3} \frac{\mu_0}{4\pi c} \left| \ddot{\vec{p}}(t') \right|^2 = \frac{2}{3} \frac{\mu_0}{4\pi c} q^2 a^2.$$

This is the Larmor formula for the rate at which the particle loses energy due to dipole radiation.

In summary, for electric dipole radiation,

$$\begin{aligned} \vec{A} &= \frac{\mu_0}{4\pi} \frac{\dot{\vec{p}}}{r} \\ \vec{B} &= \frac{\mu_0}{4\pi} \frac{1}{c} \frac{(\ddot{\vec{p}} \times \hat{r})}{r} \\ \vec{E} &= \frac{\mu_0}{4\pi} \frac{(\ddot{\vec{p}} \times \hat{r}) \times \hat{r}}{r} \\ P_{tot} &= \frac{\mu_0}{4\pi c} \frac{2}{3} |\ddot{\vec{p}}|^2. \end{aligned}$$

M1 Radiation

If $\vec{p} = 0$ as is the case for a current loop, then to calculate the radiation fields and radiated power, we have to go to higher order such as to magnetic dipole (M1) or electric quadrupole (E2).

For M1 radiation,

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi} \frac{1}{rc} \left(\dot{\vec{m}} \times \hat{r} \right) \\ \vec{B} &= \frac{\mu_0}{4\pi} \frac{1}{rc^2} \left(\ddot{\vec{m}} \times \hat{r} \right) \times \hat{r} \\ \vec{E} &= \frac{\mu_0}{4\pi} \frac{1}{rc} \left(\ddot{\vec{m}} \times \hat{r} \right) \\ P_{tot} &= \frac{\mu_0}{4\pi} \frac{2}{3c^3} |\ddot{\vec{m}}|^2.\end{aligned}$$

Procedure

If given a time-dependent current $I(t)$, calculate the radiation as follows:

1. Write the current in vector form $I(t) \rightarrow \vec{I}(t)$. For example, if it is along the z -axis, attach a \hat{z} . If

it forms a circle in the xy -plane, attach a $\hat{\phi}$

2. Calculate the second time-derivative of the electric dipole moment

$$\ddot{\vec{p}}(t') = \frac{d}{dt} \int d^3r' \vec{j}(\vec{r}', t') \rightarrow \frac{d}{dt} \int d\ell \vec{I}.$$

When you integrate, remember to also integrate the unit direction vector e.g. $(\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y})$

3. Plug this result into the E1 formulae for the fields and radiated power
4. If the electric dipole moment is zero, calculate the magnetic dipole moment. For a flat loop of current, $\vec{m} = I\vec{a}$.
5. Plug this result into the M1 formulae for the fields and radiated power

