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Dissertation

Graph complexes and the Moduli space of Riemann surfaces

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Abstract

In this thesis we compare several combinatorial models for the Moduli space of open-closed cobordisms and their compactifications. More precisely, we study Godin's category of admissible fat graphs, Costello's chain complex of black and white graphs, and Bödigheimer's space of radial slit configurations. We use Hatcher's proof of the contractibility of the arc complex to give a new proof of a result of Godin, which states that the category of admissible fat graphs is a model of the mapping class group of open-closed cobordisms. We use this to give a new proof of Costello's result, that the complex of black and white graphs is a homological model of this mapping class group. Beyond giving new proofs of these results, the methods used give a new interpretation of Costello's model in terms of admissible fat graphs, which is a more classical model of the Moduli space. This connection could potentially allow to transfer constructions in fat graphs to the black and white model. Moreover, we compare Bödigheimer's radial slit configurations and the space of metric admissible fat graphs, producing an explicit homotopy equivalence using a "critical graph" map. This critical graph map descends to a homeomorphism between the Unimodular Harmonic compactification and the space of Sullivan diagrams, which are natural compactifications of the space of radial slit configurations and the space of metric admissible fat graphs, respectively. Finally, we use experimental methods to compute the homology of the chain complex of Sullivan diagrams of the topological type of the disk with up to seven punctures, and we give explicit generators for the non-trivial groups. We use these experimental results to show that the first and top homology groups of the chain complex of Sullivan diagrams of the topological type of the punctured disk are trivial; and to give two infinite families of non-trivial classes of the homology of Sullivan diagrams which represent non-trivial string operations.

Resumé

In denne afhandling sammenligner vi adskillige kombinatoriske modeller for Moduli-rummet af ben-lukkede kobordismer og deres kompaktifikationer. Mere prcist undersger vi Godins kategori af tilladte tykke grafer, Costellos kdekompleks af sort-hvide grafer og Bödigheimers rum af radialspalte-konfigurationer. Vi benytter Hatchers bevis for at buekomplekset er kontraktibelt, til at give et nyt bevis for et resultat af Godin, der siger at kategorien af tilladte tykke grafer er en model for afbildningsklasse-gruppen for ben-lukkede kobordismer. Dette anvender vi til at give et nyt bevis for Costellos resultat at komplekset af sort-hvide grafer er en homologisk model for denne afbildningsklasse-gruppe. Udover at give nye beviser for disse resultater giver de anvendte metoder en ny fortolkning af Costellos model i form af tilladte tykke grafer, hvilket er en mere klassisk model for Moduli-rummet. Denne sammenhng kan potentielt give en overfrsel af klassiske konstruktioner blandt tykke grafer til den sort-hvide model. Endvidere sammenligner vi Bödigheimers radialspalte-konfigurationer med rummet af metriske tilladte tykke grafer og producerer en eksplicit homotopikvivalens ved brug af en “kritisk graf”-afbildning. Denne kritisk graf-afbildning fres over i en homeomorfi mellem den Unimodulre Harmoniske kompaktifikation og rummet af Sullivan-diagrammer, som er naturlige kompaktifikationer af hhv. rummet af radialspalte-konfigurationer og rummet af metriske tilladte tykke grafer. Endelig benytter vi eksperimentielle metoder til at udregne homologien af kdekomplekset af Sullivan-diagrammer som har topologisk type af en disk med op til syv huller, og vi angiver eksplicitte frembringere for de ikke-trivielle grupper. Vi anvender disse eksperimentielle resultater til at vise at den frste og den verste homologigruppe for kdekomplekset af Sullivan-diagrammer vis topologiske typer er en punkteret diske, er trivielle; og til at angive to uendelige familier af ikke-trivielle klasser i homologien af Sullivan-diagrammer der reprsenterer ikke-trivielle streng-operationer.

Contents

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Part I

Introduction and Summary

CHAPTER 1

Introduction

1. Motivation

1.1. Surfaces, cobordisms and moduli spaces. The problem of classifying certain mathematical objects up to some notion of equivalence is prevalent in all areas of mathematics; and for centuries, surfaces and their classification have played an important role in topology, geometry and algebraic geometry. A *Riemann surface* X is an oriented surface together with a complex structure. There are several equivalent ways of defining a complex structure on X. One is to state that X is a complex manifold of dimension one i.e. the charts of X map an open neighbourhood of each point in X to an open subspace of $\mathbb{C}$ and all transition maps are holomorphic. Another is to give a conformal structure on X, which is loosely speaking a structure which allows the measurement of angles on X. Up to homeomorphism, Riemann surfaces are classified by their genus, number of boundary components, and number of punctures. However, this classification only remembers the topology of the surface and completely ignores the complex structure. One way of studying the geometric classification of Riemann surfaces is by the theory of moduli. A moduli space is a geometric object which solves a geometric classification problem in the sense that it parametrizes the objects we wish to study up to a notion of equivalence. With this in mind, the *moduli space of an oriented surface* S is a space that parametrizes all compact Riemann surfaces of topological type S up to complex-analytic isomorphism. The theory of moduli spaces of surfaces is very extensive. We give a short account on some of the major results in this area, mainly following [**FM11**]. One way to construct the Moduli space of Riemann surfaces involves two main objects: the mapping class group and Teichmüller space. We describe them, their main properties, and their relation to the Moduli space of surfaces.

1.1.1. *The mapping class group.* Let S_g denote the closed oriented surface of genus g and $S_{g,n}$ denote the compact oriented surface of genus g with n boundary components obtained from S_g by cutting out n open disks. In general, let S denote a compact oriented surface with or without boundary. The *mapping class group of* S, which we denote by $\mathrm{Mod}(S)$, is the group of components of the topological group of orientation preserving self-diffeomorphisms which fix the boundary point-wise i.e., it is the group $\pi_0(\mathrm{Diff}^+(S, \partial S))$. This definition is equivalent to several others, namely

$$\mathrm{Mod}(S) \cong \pi_0(\mathrm{Homeo}^+(S, \partial S)) \cong \mathrm{Diff}^+(S, \partial S)/\sim_i \cong \mathrm{Homeo}^+(S, \partial S)/\sim_h$$

where $\sim_i$ and $\sim_h$ denote the isotopy and homotopy relations respectively. The mapping class group and the diffeomorphism group are closely related. In the '60s, Earle and Eells showed that for $g \geq 2$ the diffeomorphism group $\mathrm{Diff}(S_g)$ has contractible components. Thus, their classifying spaces $B\mathrm{Diff}(S_g)$ and $B\mathrm{Mod}(S_g)$ are homotopy equivalent. These groups are connected with surface bundles, since the classifying space of the diffeomorphism group classifies isomorphism classes of surface bundles. More precisely, for a paracompact, Hausdorff space B there is a one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ S-\text{bundles over B} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{maps } B \to B\mathrm{Diff}(S) \end{array} \right\}.$$

Properties and invariants of the mapping class group have been investigated over the years. To understand this group we study its elements, and we do so by analysing their action on simple closed curves. The simplest elements of the mapping class groups were introduced by Max Dehn and are called *Dehn twists*. Consider a simple closed curve α in a surface S, by cutting the curve from the surface we obtain a surface with two extra boundary components. Intuitively, a Dehn twist on S along α is represented by the diffeomorphism obtained by cutting the surface along that curve performing a full rotation on one of the boundaries and glueing the boundaries back together by the identity map. Figure 1 gives a local picture of a Dehn twist. Note that in particular, Dehn twist are elements with infinite order. A fundamental theorem on the theory of mapping class groups is due to Dehn in 1938 and states that $\mathrm{Mod}(S_g)$ is generated by finitely many Dehn twists.

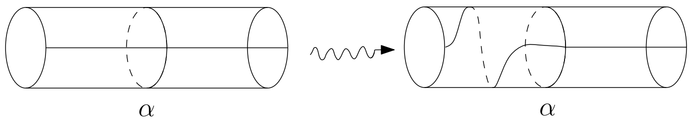

FIGURE 1. A local picture of a Dehn twist along α

Much work has been done on computing the (co)homology of the mapping class group. However, at the same time very much and very little is known about it. The combined works of Mumford, Birman and Powell in the '60s and '70s showed that that for $g \geq 3$ we have that

$$H_1(\mathrm{Mod}(S_g); \mathbb{Z}) = 0.$$

Later on, in the '80s, Harer showed that for $g \geq 4$

$$H_2(\mathrm{Mod}(S_g); \mathbb{Z}) \cong \mathbb{Z}.$$

On the other hand, by glueing a genus one surface with two boundary components to the unique boundary of the genus g surface with one boundary component we obtain maps:

$$\mathrm{Mod}(S_{1,1}) \to \mathrm{Mod}(S_{2,1}) \to \dots \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(S_{g+1,1}) \to \dots \tag{1.1}$$

In [**Har85**], Harer showed that these maps induce an isomorphisms in (co)homology in a range of dimension increasing with the genus g. More precisely, let $\mathrm{Mod}(S_\infty)$ denote the direct limit of (1.1), Harer showed that:

$$H_k(\mathrm{Mod}(S_{g,1})) \cong H_k(\mathrm{Mod}(S_\infty)) \quad H^k(\mathrm{Mod}(S_{g,1})) \cong H^k(\mathrm{Mod}(S_\infty)) \quad \text{for } g \geq 2k+1$$

In 1983 Mumford conjectured that over the rationals the stable cohomology is a polynomial algebra. This is known as the *Mumford conjecture*, and using the work of Tillmann [**Til97**] and Madsen and Tillmann [**MT01**] , Madsen and Weiss proved it in [**MW07**]. Moreover, the stable mod-p cohomology was described by Galatius in [**Gal04**]. Thus, the stable cohomology of the mapping class group is quite well understood. However, little is known about the unstable cohomology. Explicit calculations have been done using combinatorial models of moduli space and are mentioned in later sections.

1.1.2. *Teichmüller space and the Moduli space.* A *marked complex structure* on S is a tuple (X, φ), where X is a Riemann surface and $\varphi : S \to X$ is an orientation preserving diffeomorphism. The *Teichmüller space of* S, which we denote $\mathcal{T}(S)$, is the space of all marked complex structures on S up to isotopy. More precisely, two marked complex structures (X, φ) and (X', φ') are *equivalent* if there is a biholomorphic map $f : X \to X'$ such that $f \circ \varphi$ and φ' are isotopic. Teichmüller space has a natural topology which makes it homeomorphic to an open ball.

The mapping class group of S acts on Teichmüller space by precomposition with the marking. The Moduli space of S, which we denote $\mathcal{M}(S)$, is the quotient of Teichmüller space by this action i.e., it is defined by

$$\mathcal{M}(S) := \mathcal{T}(S)/\mathrm{Mod}(S).$$

Fricke showed that this action is properly discontinuous, and thus $\mathcal{M}(S)$ is an orbifold. Moreover, when S has boundary, the action of the mapping class group is free, and the homology of the Moduli space coincides with the homology of the classifying space of the mapping class group. However, when S has no boundary, the action is not free, but it can be shown that:

$$H_*(\mathcal{M}(S); \mathbb{Q}) \cong H_*(\mathrm{Mod}(S); \mathbb{Q}).$$

1.1.3. *Extension to open-closed cobordisms.* These concepts can be extended in a natural way to 2-dimensional open-closed cobordisms, which have applications in string topology, that we will discuss later. An *open-closed cobordism* $S_{g,p+q}$ is an oriented surface of genus g with p_1 incoming circles, p_2 incoming intervals, q_1 outgoing circles and q_2 outgoing intervals, where $p = p_1 + p_2$ and $q = q_1 + q_2$ (see Figure 2). More precisely, $S_{g,p+q}$ is an oriented surface with boundary together with

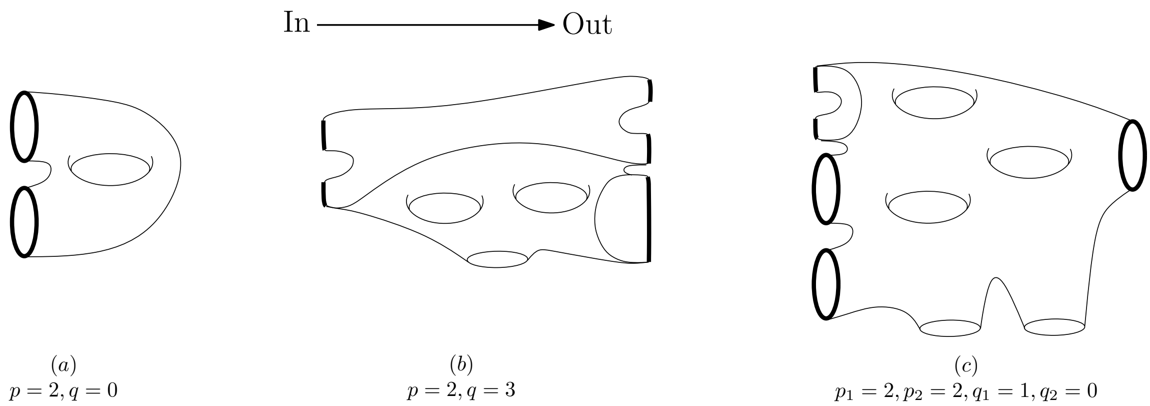

FIGURE 2. Examples of open-closed cobordisms, the incoming and outgoing boundaries are marked with thick lines. (a) A closed cobordism (b) An open cobordism (c) An open-closed cobordism

a partition of the boundary into three parts $\partial_{in}S$, $\partial_{out}S$ and $\partial_{free}S$, parametrizing diffeormorphisms

$$N_{in} := (\textstyle\bigsqcup_{i=1}^{p_1} S^1) \bigsqcup (\bigsqcup_{i=1}^{p_2} I) \to \partial_{in}S \qquad N_{out} := (\bigsqcup_{i=1}^{q_1} S^1) \bigsqcup (\bigsqcup_{i=1}^{q_2} I) \to \partial_{out}S \ ,$$

and an ordering of the components of N_{in} and N_{out}, where I denotes the unit interval. Since the surface $S_{g,p+q}$ is oriented, giving parametrizing diffeomorphisms is equivalent, up to orientation preserving homeomorphism, to fixing a marked point in each component of $\partial_{in}S \cup \partial_{out}S$ and giving an ordering of these. So we can think of an open-closed cobordims as an oriented surface with marked points at the boundary which are ordered and labelled as either: (a) incoming or outgoing and (b) open or closed, such a closed marked point is the only marked point on a boundary component. When $p_2 = q_2 = 0$ we call $S_{g,p+q}$ a *closed cobordism*, and when $p_1 = q_1 = 0$ we call $S_{g,p+q}$ an *open cobordism*. Up to a homeomorphism that respects the parametrization of the boundary, 2-dimensional open-closed cobordisms are classified by their genus, number of boundary components and the combinatorial data given by the decorations at the boundary.

The notions of Teichmüller space, the Moduli space and mapping class groups are extended in a natural way. More precisely, the Teichmüller space of $S_{g,p+q}$, which we denote $\mathcal{T}(S_{g,p+q})$, is the space of all equivalence classes of marked complex structures, where the equivalence relation is given by the action of diffeomorphisms that are isotopic to the identity and respect the decorations at the boundary. The *mapping class group of* $S_{g,p+q}$ is

$$\mathrm{Mod}(S_{g,p+q}) := \pi_0(\mathrm{Diff}^+(S_{g,p+q}, \partial_{in}S \cup \partial_{out}S))$$

where $\mathrm{Diff}^+(S_{g,p+q}, \partial_{in}S \cup \partial_{out}S)$ is the space of orientation preserving diffeomorphisms that fix $\partial_{in}S \cup \partial_{out}S$ point-wise.

On a related note, a surface with decorations, is a surface with marked points in its interior. The mapping class group of such a surface is the group of components of the topological group of orientation preserving diffeormorphisms that send decorations to decorations. We can think of this as a special case of an open-closed cobordism. To see this, note that elements of the mapping class group do not need to fix points that belong to the free boundary components. Then, by collapsing each free boundary circle of the surface to a point i.e., collapsing the boundary components of the surface which do not contain a marked point, we get a map from the mapping class group of open-closed cobordisms to the mapping class group of surfaces with boundaries and decorations and this map is a homeomorphism.

1.2. String Topology. The theory of moduli spaces also has applications in the field of string topology, which we briefly describe in this section. Inspired by physics, string topology studies the algebraic structures of the spaces of paths and loops in manifolds. This field started in 1999 when Chas and Sullivan described some algebraic structures of the equivariant and non-equivariant homology of the free loop space of manifolds [**CS99**]. In particular, they constructed a *loop product*

$$H_i(LM) \otimes H_j(LM) \to H_{i+j-d}(LM)$$

where M is a closed, oriented manifold of dimension d and LM is the free loop space of M. After this, many authors gave different constructions of the loop product and generalizations of it. Among these, in [**CG04**], Cohen and Godin extended the Chas Sullivan loop product and constructed operations on the homology of the loop space of M parametrized over $H_0(\mathcal{M}_{\mathrm{closed}})$ where $\mathcal{M}_{\mathrm{closed}}$ is the disjoint union of the Moduli spaces of closed cobordisms. More precisely, they constructed operations

$$\mu_{S_{g,p+q}} : H_*(LM)^{\otimes p} \longrightarrow H_{*-\chi(S_{g,p+q})}(LM)^{\otimes q}$$

where $\mu_{S_{g,p+q}}$ depends only on the topological type of the closed cobordism $S_{g,p+q}$. These operations are compatible with glueing surfaces along the parametrized boundaries and form what is called a topological quantum field theory. The operation for the pair of pants, that is when $p = 2$ and $q = 1$, coincides with the Chas and Sullivan loop product. However, in [**Tam10**], Tamanoi shows that these operations are trivial for $g > 0$ or $q \geq 3$. Thus, to describe string operations in such a manner one should parametrize them over a richer space.

With this in mind, in [**God07a, Kup11**], Godin and Kupers construct higher string operations, which are operations parametrized over the twisted homology of the Moduli space of open-closed cobordisms. More precisely, they construct operations of the form

$$H_*(\mathcal{M}(S_{g,p+q}), \mathcal{L}^{\otimes d}) \otimes H_*(LM)^{\otimes p_1} \otimes H_*(LM)^{\otimes p_2} \to H_*(LM)^{\otimes q_1} \otimes H_*(LM)^{\otimes q_2}$$

where $\mathcal{M}(S_{g,p+q})$ is the Moduli space of the open-closed cobordism $S_{g,p+q}$ and $\mathcal{L}^{\otimes d}$ is a local coefficient system. These operations are compatible with glueing cobordisms along their parametrized boundary and form what is called a homological conformal field theory. Moreover, these operations coincide on H_0 with the ones constructed by Cohen and Godin mentioned earlier.

On the other hand, for M a simply connected manifold, with coefficients in a field there is an isomorphism $H^*(LM) \cong HH_*(C^*(M), C^*(M))$ where $HH_*(A, A)$ is the Hochschild homology of an algebra A (cf. [**Jon87**]). Moreover, it can be shown that rationally, $C^*(M)$ is quasi-isomorphic (as an algebra) to a commutative Frobenius algebra (cf. [**LS07**]). So

$$H^*(LM) \cong HH_*(A(M)^*, A(M)^*) \tag{1.2}$$

for some commutative Frobenius algebra $A(M)^*$. Given an open-closed cobordism $S_{g,p+q}$, Tradler and Zeinalian define a chain complex of Sullivan diagrams $\mathscr{SD}(p, q)$ (which is a complex generated by graphs which we describe later), and show that this complex acts on the Hochschild chains of symmetric Frobenius algebras [**TZ06**]. In [**WW11, Wah12**], Wahl and Westerland give a general method to construct operations on the Hochschild homology of algebras with a given structure e.g. associative, commutative, Frobenius. In particular, given a Frobenius algebra A and for each class in the homology of the Moduli space of an open-closed cobordism $S_{g,p+q}$, they construct a natural operation of the form

$$C_*(A, A)^{\otimes p_1} \otimes C_*(A, A)^{\otimes p_2} \to C_*(A, A)^{\otimes q_1} \otimes C_*(A, A)^{\otimes q_2}$$

where $C_*(A, A)$ are the Hochschild chains of an algebra A. This construction behaves well under the isomorphism (1.2) and thus the operations on the Hochschild homology of $A(M)^*$ give operations on $H^*(LM)$. Furthermore, Wahl studies the chain complex of all such natural operations (cf. [**Wah12**]). She shows that there is a complex of so called *formal operations* which we denote $Nat(p, q)$ which approximates the chain complex of all natural operations. In particular, in the case of symmetric Frobenius algebras, she shows that there is an inclusion

$$\mathscr{SD}(p, q) \hookrightarrow Nat(p, q)$$

and this inclusion is a split quasi-isomorphism; showing that the operations given by Tradler and Zeinalian are all the formal operations in the symmetric Frobenius case. Moreover, she uses this quasi-isomorphism to find two infinite families of non-trivial classes in the homology of $\mathscr{SD}(p, q)$, which represent non-trivial string operations. These operations correspond to open-closed cobordisms with an arbitrary number of boundary components and arbitrary genus, which contrast with the triviality result of Tamanoi. Following these ideas, Klamt studies the case of commutative

Frobenius algebras in [**Kla13**]. She constructs a chain complex of *looped diagrams* denoted $l\mathcal{D}$, together with a map from $l\mathcal{D}$ to the chain complex of formal operations. Thus, looped diagrams give operations on the Hochschild homology of commutative Frobenius algebras. Moreover, she gives a chain map from Sullivan diagrams to looped diagrams. Therefore, the chain complex $l\mathcal{D}$ recovers all operations that come from Sullivan diagrams in the commutative Frobenius case. Finally, $l\mathcal{D}$ recovers other known operations in commutative Frobenius algebras, in particular it recovers Loday's lambda operations (cf. [**Lod89**]).

In order to find non-trivial string operations through the constructions above we are interested in finding non-trivial classes in the homology of $\mathcal{M}$, $\mathscr{S}\mathscr{D}$ and $l\mathcal{D}$. Moreover, it is also of interest to understand what the underlying spaces of $\mathscr{S}\mathscr{D}$ and $l\mathcal{D}$ are and what their relation to moduli space is.

2. Combinatorial models of the Moduli Space of surfaces

In order to study the homology of the Moduli space of surfaces or to construct string operations parametrized over these, many combinatorial models of the Moduli space of surfaces have been built. Although these models are all abstractly homotopy equivalent, they where developed through very different methods and thus the direct connections between them is not obvious. Furthermore, one problem in string topology is the contrast between the many different constructions and the lack of comparisons between these constructions. It is the goal of this thesis to build direct connections between these different models in the hope of transferring information, applications, and comparing constructions between them. In this section we briefly describe the combinatorial models we study.

2.1. Fat graphs. A *combinatorial graph* is a finite, 1-dimensional CW complex. The 0-cells are called *vertices* and the 1-cells are called *edges*. The vertices which are connected to exactly one edge are called *leaves* and all other vertices are called *inner vertices*. The number of edges attached at a vertex is called the *valence of the vertex*. Informally, a *fat graph* or *ribbon graph* is a combinatorial graph together with a cyclic ordering of the edges incident at each vertex. From a fat graph we can construct a surface by fattening the edges to strips and glueing them together at vertices according to the cyclic ordering. This surface is well defined up to topological type, and we call this the *topological type of the graph* (see Figure 1).

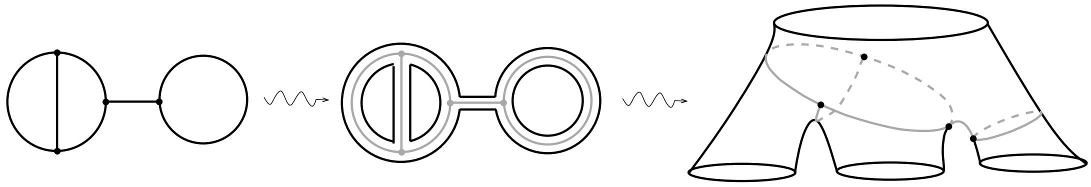

FIGURE 1. A fat graph and its associated surface obtained by fattening the edges. The cyclic structure at the vertices is given by the orientation of the plane.

The fat structure on the graph defines *boundary cycles*, which are sequences of half edges of the graph corresponding to the boundary components of the surface. More precisely, to describe a boundary cycle, we first choose an edge on the graph and an orientation of it. We follow this edge from its start vertex to its end vertex and then continue with the next edge emanating from this vertex according to the cyclic structure and orient it as starting on the end vertex of the previous edge. We continue this procedure until we reach the first edge with its original orientation (see Figure 2). This procedure gives a map from the circle to Γ which is well defined up to homeomorphism.

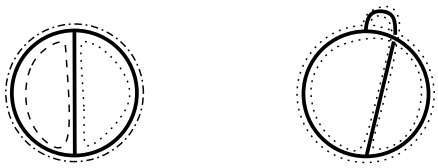

FIGURE 2. Two different fat graphs with the same underlying combinatorial graph. Their cyclic structure is the one induced by the orientation of the plane. The boundary cycles are marked with dotted lines. The fat graph on the left has three boundary cycles, while the fat graph on the right has only one.

Following the ideas of Strebel [**Str84**], Penner, Bowditch and Epstein gave a triangulation of Teichmüller space of surfaces with decorations, which is equivariant under the action of its corresponding mapping class group (cf. [**Pen87, BE88**]). In this triangulation, simplices correspond to equivalence classes of marked fat graphs, where a marking of a fat graph is an isotopy class of embeddings of the graph in its corresponding surface, which is a homotopy equivalence. The quotient of this triangulation by the mapping class group gives a combinatorial model of the Moduli space of surfaces with decorations which is rationally equivalent to the Moduli space of compact surfaces. In [**Har86**], Harer generalizes Strebel's ideas to the case of surfaces with punctures and boundary components. Finally, in [**Pen88, Kon94**] both Penner and Kontsevich construct a chain complex generated by equivalence classes of fat graphs which rationally computes the homology of the Moduli space of surfaces with punctures.

These ideas have been used to construct categorical models of moduli space. The nature of this models allows the use of category theoretic and homotopy theoretic arguments to prove things about moduli space. In [**Igu02**], Igusa constructs a category $\mathcal{F}at$, where the objects are fat graphs with vertices of valence at least 3, and the morphisms homotopy equivalences that respect the fat structure. He shows that this category models the mapping class groups of punctured surfaces. More precisely,

$$|\mathcal{F}at| \simeq \amalg B\mathrm{Mod}(S_g^m)$$

where S_g^m denotes the oriented surface of genus g with m punctures and the disjoint union runs over all topological types of such surfaces.

Following these ideas, Godin constructs a category $\mathcal{F}at^b$ where the objects are isomorphism classes of bordered fat graphs, which are fat graphs with exactly one leaf on each boundary cycle and where all other vertices are at least trivalent. Furthermore, she shows that this category models the mapping class groups of bordered surfaces [**God07b**]. The combinatorial nature of the model allowed Godin to use a computer to calculate the unstable homology of the moduli space of surfaces with boundary for low genus and small number of boundary components, which had been achieved earlier by Ehrenfried and which was published in [**ABE08**]. In order to construct higher string operations, Godin generalized these ideas to the case of open-closed cobordisms [**God07a**]. She constructs a category $\mathcal{F}at^{oc}$ with objects isomorphism classes of fat graphs with leaves, in which all inner vertices are at least trivalent and where the leaves are ordered and labelled as: (a) incoming or outgoing and (b) open or closed. From such a graph one can construct an open-closed cobordism, well-defined up to topological type, where the underlying surface is constructed by the fattening procedure described above and the marked points in the boundary are given by the leaves. In this paper Godin shows that $\mathcal{F}at^{oc}$ is a model for the mapping class group of open-closed cobordisms, that is

$$|\mathcal{F}at^{oc}| \simeq \amalg B\mathrm{Mod}(S_{g,p+q})$$

where the disjoint union runs over all topological types of open-closed cobordisms in which not all the boundary is free.

In the same paper, Godin gives the notion of an *admissible fat graph*, which is a fat graph where the boundary cycles corresponding to the incoming closed leaves are disjoint embedded circles in the graph. More precisely, the maps from the circle to Γ marking the boundary components where the incoming closed leaves belong to are disjoint embeddings (see Figure 3). Furthermore she defines a category $\mathcal{F}at^{ad}$ which is a full subcategory of $\mathcal{F}at^{oc}$, on objects admissible fat graphs and shows that this subcategory is also a model of the mapping class group, that is

$$|\mathcal{F}at^{ad}| \simeq \amalg B\mathrm{Mod}(S_{g,p+q})$$

where the disjoint union runs over all topological types of open-closed cobordisms in which not all the boundary is incoming closed or free.

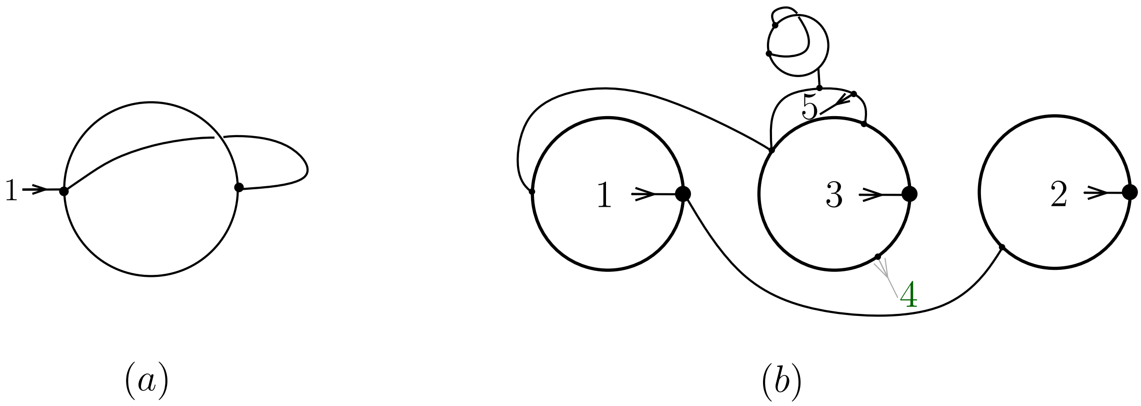

FIGURE 3. (a) An open-closed graph that is not admissible. (b) An admissible open-closed graph. Ingoing and outgoing leaves are marked with arrows. Open leaves are given in grey and close leaves in black.

Closely related to these categorical models are the spaces of metric fat graphs. A *metric fat graph* is a fat graph whose underlying space is a metric space. Equivalently, a metric fat graph is a fat graph together with a map from the set of edges of the graph to $\mathbb{R}^+$, which we can think of as assigning lengths to the edges of the graph. The space of metric fat graphs have been given a topology in [**Har88, Pen87, Igu02**]. A path in this space is given by continuously changing the lengths of the edges of a graph. In particular, Igusa constructs a space of metric fat graphs, as a simplicial space where simplices are indexed by equivalence classes of fat graphs and shows that this space is homotopy equivalent to the geometric realization of $\mathcal{F}at$.

The *space of metric admissible fat graphs* $\mathcal{MFat}^{ad}$ is the subspace of the space of metric fat graphs where the underlying graphs are admissible. The *space of Sullivan diagrams*, which we denote $\mathcal{SD}$, is a quotient space of $\mathcal{MFat}^{ad}$ modulo an equivalence relation of slides along edges that do not belong to the embedded circles. Figure 4 shows an example of this relation. A point in $\mathcal{SD}$ is, loosely speaking, a

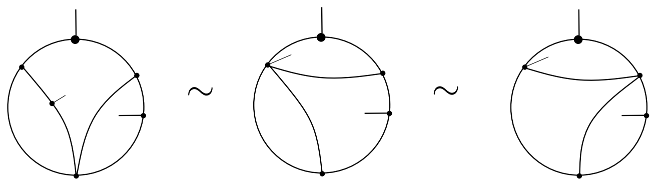

FIGURE 4. An example of the slide relation on admissible fat graphs

metric admissible fat graph in which we consider all edges that do not belong to the embedded circles to be of length zero. This space has a canonical CW-complex structure and its cellular chain complex is the complex $\mathscr{SD}$ constructed by Tradler and Zeinalian. The generators are non-metric Sullivan diagrams and the differential is given by collapsing edges on the admissible cycles. Recall that Wahl showed that up to a split quasi-isomorphism, this complex is the complex of formal operations of the Hochschild complex of symmetric Frobenius algebras. (see 1.2). Therefore, we can think of $\mathcal{SD}$ as the space which parametrizes all formal operations on the Hochschild complex of symmetric Frobenius algebras.

It is important to remark, that the term Sullivan diagram and space of Sullivan diagrams should be handled with caution, since different inequivalent definitions of Sullivan diagrams have been used in several papers by different authors. In [**CG04**], Cohen and Godin use a space $\mathcal{CF}$, which they call the space of Sullivan chord diagrams, to construct string operations. However, $\mathcal{CF}$ is actually a subspace of $\mathcal{MFat}^{ad}$ and this space is not homotopy equivalent to either $\mathcal{MFat}^{ad}$ or $\mathcal{SD}$. Also motivated by string topology, in [**PR11**], Poirier and Rounds construct a space which they call the space of chord diagrams and denote it $\overline{SD}$ which is a subspace of $\mathcal{MFat}^{ad}$. They also define a quotient space $\overline{SD}/\sim$ where the equivalence relation is given by slides away from the admissible boundaries, and the space $\overline{SD}/\sim$ is homeomorphic to $\mathcal{SD}$.

2.2. The chain complex of Black and White graphs. In order to describe an action of the chains of the moduli space of surfaces on the Hochschild homology of any $\mathscr{A}_\infty$ Frobenius algebra, Costello constructs a chain complex which we denote $\mathscr{BW}-Graphs$, that models the homology of moduli space of open-closed cobordisms, that is

$$H_*(\mathscr{BW}-Graphs) \cong H_*(\amalg \mathcal{M}(S_{g,p+q}))$$

where the disjoint union runs over all open-closed cobordisms in which not all the boundary is outgoing closed [**Cos06a, Cos06b**]. In [**WW11**], Wahl and Westerland describe this chain complex in terms of fat graphs with two types of vertices, which they denote black and white fat graphs. More precisely, a *black and white fat graph* is a fat graph in which all vertices are either black or white, and all white vertices have a choice of start edge attached to it i.e., the edges incident at a white vertex have an ordering not only a cyclic ordering.

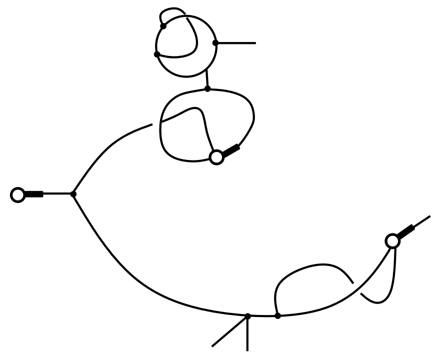

FIGURE 5. An example of a black and white graph. The cyclic ordering at the vertices is given by the orientation of the plane and the start edge of the white vertices are marked by thick edges.

In order to prove this result, Costello studies the moduli space of open-closed cobordisms with possibly nodal boundary $\overline{\mathcal{M}}_{g,p+q}$, which is a partial compactification of $\mathcal{M}(S_{g,p+q})$ and he shows that the inclusion $\mathcal{M}(S_{g,p+q}) \hookrightarrow \overline{\mathcal{M}}_{g,p+q}$ is a homotopy equivalence. He defines a space $D_{g,p+q}$ which is a subspace of $\overline{\mathcal{M}}_{g,p+q}$ and shows that the inclusion $D_{g,p+q} \hookrightarrow \overline{\mathcal{M}}_{g,p+q}$ is a weak homotopy equivalence. Moreover, the space $D_{g,p+q}$ has a canonical cellular structure. The cellular complex of $D_{g,p+q}$ is generated by configurations of disks and annuli glued together at points in their boundary. More precisely, the generators are given by glueing together at marked points two types of components. The components are:

- annuli with a parametrization of the inner boundary and marked points at the outer boundary
- disks with marked points at the boundary

Figure 6 shows an example of a generator. Note that in particular, there are two types of annuli, the ones that have the parametrization point of the inner boundary aligned to a marked point and the ones that do not.

The dual picture of these building blocks gives an interpretation of such a configuration in terms of black and white fat graphs (see Figure 6). More precisely, for each disk, we fix a black vertex in the interior of the disk and a half edge connecting the inner vertex to each marked point in the boundary. The orientation of the disk gives a cyclic structure of the half edges incident at that vertex. On the other hand, we radially connect with half edges the inner boundary of the annulus to the marked points of the outer boundary. In the case where the parametrization point of the inner boundary is radially aligned to a marked point, we mark its corresponding half edge as special. In the case where the parametrization point of the inner boundary is not radially aligned to a marked point we connect the parametrization point radially with the outer boundary with an additional half edge and mark it as special. Finally, we regard the inner boundary of the annulus as a white vertex. The orientation of the annulus gives a cyclic ordering of the half edges attached at the white vertex and the special half edge is the start half edge at the white vertex.

A black and white graph $\hat{G}$ is called a *blow-up* of a black and white graph G, if G can be obtained from $\hat{G}$ by collapsing an edge that is not connected to a leaf and that does not connect two white vertices. The chain complex of black and white

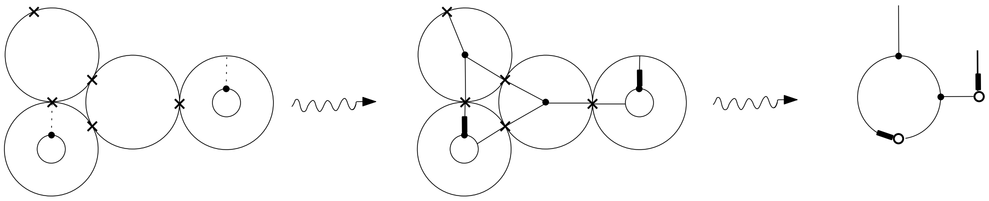

FIGURE 6. On the left a degenerate surface built by glueing disks and annuli, it corresponds to the moduli space of cobordisms whose underlying surface is of genus zero and have four boundary components. On the right the process to obtain the black and white graph that represents this configuration. The start half edges are marked with a thick line.

graphs $\mathscr{BW} - Graphs$ is the chain complex generated by such graphs where the differential of a graph is the sum of all its possible blow-ups.

2.3. Radial Slit Configurations. In order to study the homology of moduli space of surfaces with punctures and inspired by Hilbert's uniformization theorem, Bödigheimer constructs a space of so called parallel slit domains and shows it is homeomorphic to the moduli space of surfaces with punctures [**Böd90**]. This space has a nice combinatorial presentation which makes direct computations of some homology groups possible. He later extends this for the case of closed cobordisms with at least one incoming and one outgoing boundary component by using radial slit configurations [**Böd06**]. The idea of the radial slit model is that any Riemann surface with p incoming boundary circles and q outgoing boundary circles can be obtained from p annuli in different complex planes by a cut and glue procedure. The inner boundaries of the p annuli correspond to the incoming boundaries of the Riemann surface. To obtain the surface, we radially cut slits on the annuli and glue them together along pairs of slits. Figure 7 shows this procedure for the pair of pants.

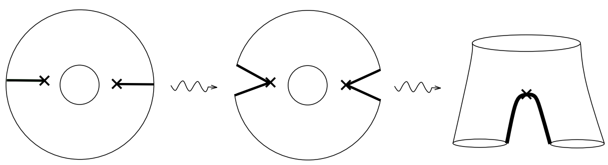

FIGURE 7. A pair of pants built from a radial slit configuration

To this effect, Bödigheimer constructs a space $\overline{\mathfrak{Rad}}$ which consists of all configurations of slits in annuli, together with the combinatorial information of how to glue these slits together. Moreover he defines a subspace $\mathfrak{Rad} \subset \overline{\mathfrak{Rad}}$, which consists of configurations such that the glueing procedure gives a non-degenerate surface. He shows that this space is a non-compact manifold which is homeomorphic a so called space of harmonic potentials H and constructs a vector bundle $H \to \mathcal{M}$ showing that there is a homotopy equivalence

$$\mathfrak{Rad} \simeq \mathcal{M}$$

where $\mathcal{M}$ is the disjoint union of the moduli spaces of closed cobordisms. The description of a point in $\mathfrak{Rad}$ and its interpretation as a Riemann surface with boundary is completely explicit. This allows to transparently define a composition operation on the components of $\mathfrak{Rad}$, which gives an operad structure to $\mathfrak{Rad}$ and corresponds to glueing cobordisms along their parametrized boundary. Moreover, the space $\mathfrak{Rad}$ is dense in $\overline{\mathfrak{Rad}}$, and the latter space is compact. Thus, $\mathfrak{Rad}$ has a natural notion of compactification, called the harmonic compactification. Geometrically, $\overline{\mathfrak{Rad}}$ is the compactification in which surfaces can have boundary circles or handles degenerate to radius zero as long as there is a path in the surface from the incoming boundary to the outgoing boundary that does not go through a degeneration. The combinatorial nature of $\mathfrak{Rad}$ has allowed for computations of the unstable homology of $\mathcal{M}$ for cases of low genus and small number of boundary components given by Ehrenfried in his thesis and published in [**ABE08**], which were later confirmed by Godin using categories of fat graphs as mentioned earlier.

CHAPTER 2

Summary of Results

Paper A

In this paper we compare the categories of fat graphs introduced by Godin in [**God07a**] and the chain complex of black and white graphs introduced by Costello in [**Cos06b**]. First, we give a new proof, more geometric in nature, that shows that the categories $\mathcal{F}at^{ad}$ and $\mathcal{F}at^{oc}$ are models for the mapping class groups as given in Theorem A.

Theorem A. *The categories of open closed fat graphs and admissible fat graphs are models for the classifying spaces of mapping class groups of open closed cobordisms. More specifically there is a homotopy equivalence*

$$|\mathcal{F}at^{oc}| \to \coprod_{[S_{g,p+q}]} B\mathrm{Mod}(S_{g,n+m})$$

where the disjoint union runs over all topological types of open closed cobordisms where not all the boundary is free. Moreover, the this map restricts on the subcategory of admissible fat graphs to a homotopy equivalence

$$|\mathcal{F}at^{ad}| \to \coprod_{[S_{g,p+q}]} B\mathrm{Mod}(S_{g,n+m})$$

where the disjoint union runs over all topological types of open closed cobordisms where not all the boundary is free or outgoing closed.

In [**God07a**], Godin proves this result by comparing fibrations. However, there is a step missing in the proof which we do not know how to complete. Instead, following her ideas in [**God07b**], we construct categories $\mathcal{EF}at^{oc}$ and $\mathcal{EF}at^{ad}$ of marked metric fat graphs i.e. fat graphs together with an embedding into their corresponding surface, which is a homotopy equivalence. These categories project onto $\mathcal{F}at^{oc}$ and $\mathcal{F}at^{ad}$ by forgetting the marking. The categories of fat graphs and marked fat graphs split into connected components given by their topological type as open closed cobordisms. We show that the subcategories corresponding to the open closed cobordisms $S_{g,p+q}$ fit in the following commutative square

$$\begin{array}{ccc} \mathcal{EF}at^{ad}_{g,n+m} & \hookrightarrow & \mathcal{EF}at^{oc}_{g,p+q} \\ \downarrow & & \downarrow \\ \mathcal{F}at^{ad}_{g,n+m} & \hookrightarrow & \mathcal{F}at^{oc}_{g,p+q} \end{array}$$

where the horizontal maps are inclusions. We show directly, that there is a free action of $\mathrm{Mod}(S_{g,n+m})$ on $\mathcal{EF}at^{oc}_{g,p+q}$ with quotient $\mathcal{F}at^{oc}_{g,p+q}$ and similarly for the admissible case. Finally, we use Hatcher's proof of the contractibility of the arc complex to show that the categories of marked metric fat graphs are contractible.

In the second section, we use admissible fat graphs to give a new proof of a theorem originally proved by Costello in [**Cos06b, Cos06a**] by very different methods. More precisely we show

Theorem B. *The chain complex of black and white graphs is a model for the classifying spaces of mapping class groups of open closed cobordisms. More specifically there is an isomorphism*

$$\mathrm{H}_*(\mathscr{BW} - Graphs) \cong \mathrm{H}_*\left(\coprod_{[S_{g,p+q}]} B\mathrm{Mod}(S_{g,n+m})\right)$$

where the disjoint union runs over all topological types of open closed cobordisms where there is at least one boundary component which is not outgoing closed and not all the boundary is free.

We prove this using Theorem A. More precisely, we construct a filtration

$$\mathcal{F}at^{ad} \ldots \supset \mathcal{F}at^{n+1} \supset \mathcal{F}at^{n} \supset \mathcal{F}at^{n-1} \ldots \mathcal{F}at^{1} \supset \mathcal{F}at^{0}$$

that gives a cell-like structure on $|\mathcal{F}at^{ad}|$ where the quasi-cells are indexed by black and white graphs i.e. $|\mathcal{F}at^{n}|/|\mathcal{F}at^{n-1}| \simeq \vee S^n$ where the wedge sum is indexed by black and white graphs of degree n. Besides proving Theorem B, the structure of the proof gives a direct connection between the admissible fat graph model and the black and white graph model, which we expect to be useful (see Chapter 3).

Paper B

This paper is joint work with Alexander Kupers. In this paper we make a direct connection between the space of radial slit configurations $\mathfrak{Rad}$, the category of admissible fat graphs $\mathcal{F}at^{ad}$ and their compactifications: we give a zigzag of maps connecting $\mathfrak{Rad}$ and the realization of $\mathcal{F}at^{ad}$, that descends to a homotopy equivalence between the harmonic compactification $\overline{\mathfrak{Rad}}$ and the space of Sullivan diagrams $\mathcal{SD}$.

To make this concrete, we give an explicit definition of the topology of the space of metric admissible fat graphs $\mathcal{MF}at^{ad}$ and show that this space has a homotopy equivalent subspace which is homeomorphic to the realization of the category of admissible fat graphs, i.e. there is a sequence of maps

$$|\mathcal{F}at^{ad}| \xrightarrow{\cong} \mathcal{MF}at^{ad}_1 \xrightarrow{\simeq} \mathcal{MF}at^{ad}$$

where the first maps is a homemorphism and the second map is an inclusion. The key ingredient to build the connection between fat graphs and $\mathfrak{Rad}$, is that any radial slit configuration has a naturally associated metric admissible fat graph, called the critical graph of the configuration. However, the association of the critical graph is not continuous. We resolve this issue by constructing a blow-up of $\mathfrak{Rad}$, which we denote $\mathfrak{Rad}^{\sim}$ together with maps making the following diagram commute

$$\begin{array}{ccccc} \mathfrak{Rad} & \xleftarrow{\simeq} & \mathfrak{Rad}^{\sim} & \xrightarrow{\simeq} & \mathcal{MFat}^{ad} \\ \downarrow & & & & \Big\downarrow \\ \overline{\mathfrak{Rad}} & & & & \Big\downarrow \\ \simeq\downarrow & & & & \twoheaddownarrow \\ \overline{\mathfrak{UMRad}} & & \xrightarrow[\cong]{\hspace{6em}} & & \mathcal{SD} \end{array}$$

where $\overline{\mathfrak{UMRad}}$ is the homotopy equivalent subspace of $\overline{\mathfrak{Rad}}$ in which all slits have the same length as given in [**Böd06**]. All maps decorated by $\simeq$ are homotopy equivalences and all maps decorated by $\cong$ are homeomorphisms.

Paper C

In this paper we give both some experimental results and some general results about the homology of the chain complex of Sullivan diagrams $\mathscr{SD}$ of the topological type of a disk with punctures or additional boundary components with one admissible cycle. Let $\mathscr{SD}_{D^c}$ denote the chain complex of Sullivan diagrams corresponding to the disk with c punctures and $\mathscr{SD}_{P_c}$ denote the chain complex of Sullivan diagrams with one admissible cycle corresponding to the generalized pair of pants with c legs i.e. a genus 0 surface with $c+1$ boundary components. In this paper we give a way of representing a generator of $\mathscr{SD}_{D^c}$ in terms of a tuple of natural numbers and a non-crossing partition, and we give an algorithm that describes how to get all generators for a given number of punctures. We implement this algorithm and compute the homology of $\mathscr{SD}_{D^c}$ for $1 \leq c \leq 7$ (see Table 0.1) and provide explicit generators of the non-trivial homology groups.

c	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}
2	$\mathbb{Z}$	$\mathbb{Z}$	0										
3	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0								
4	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0						
5	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	0	0				
6	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0		
7	$\mathbb{Z}$	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0

TABLE 0.1. Homology of the chain complex of Sullivan diagrams of topological type a disk with c punctures.

Furthermore, using this description we show that the top homology group of $\mathscr{SD}_{D^c}$ is trivial, and that for $c > 2$, the first homology group is trivial. Finally, we lift the generators obtained by the experimental computation to cycles on the homology of $\mathscr{SD}_{P_c}$. By this procedure, we find two infinite families of non-trivial classes of the homology of $\mathscr{SD}_{P_c}$ which represent non-trivial string operations. One of this families is homologous to one described in [**Wah12**].

CHAPTER 3

Perspectives

The work presented in this thesis raises some questions for potential future research. In this section we outline some of these questions and present some ideas on how to attack them.

Glueing of Black and White Graphs

The chain complex of black and white graphs was originally built from degenerate surfaces, and as such there are certain constructions which are not natural in this context, as for example the glueing of surfaces along boundary components. Paper A gives a new point of view of black and white graphs, relating them directly to admissible fat graphs which are a more classical model of moduli space. In particular, we believe glueing constructions would be easier to understand in this context. More precisely, let $S := S_{g,p+q}$ and $S' := S_{g',p'+q'}$ be open closed cobordisms and say that we have an identification of the outgoing boundary of S with the incoming boundary of S'. Then, we can glue $\partial_{out}S$ to $\partial_{in}S'$ along their boundary parametrizations and obtain a surface $S\#S'$. This induces a map:

$$BMod(S) \times BMod(S') \longrightarrow BMod(S\#S'). \tag{0.1}$$

Let $\mathscr{BW}_S$ denote the chain complex of black and white graphs of topological type S. In [**Cos06b**], Costello shows that if S does not have any outgoing closed boundary i.e. if $q_1 = 0$, then the map (0.1) is modelled by a chain map:

$$\circ_{BW} : \mathscr{BW}_S \otimes \mathscr{BW}_{S'} \longrightarrow \mathscr{BW}_{S\#S'} \tag{0.2}$$

which sends $G \otimes G' \mapsto G \circ G'$, where $G \circ G'$ is the graph obtained by glueing the i-th outgoing leaf of G to the i-th incoming open leave of G'. However, there is no natural extension of this map to the case of a general open closed cobordism.

Using the ideas of Kaufmann, Livernet, and Penner [**KLP03**], we believe we can construct a map on metric admissible fat graphs

$$\circ_{\mathcal{MF}at^{ad}} : \mathcal{MF}at^{ad}_S \times \mathcal{MF}at^{ad}_{S'} \longrightarrow \mathcal{MF}at^{ad}_{S\#S'} \tag{0.3}$$

which models (0.1). Moreover, we can identify the space of metric fat graphs as the realization of the category of admissible fat graphs, by using the barycentric coordinates to determine the lengths of the edges of the graphs. Paper A gives a cell-like structure on $\mathcal{MF}at^{ad}$ where the quasi-cells are indexed by black and white graphs. Therefore, an element $G \otimes G' \in \mathscr{BW}_S \otimes \mathscr{BW}_{S'}$ represents a product of quasi cells, say $\mathscr{E}_G \times \mathscr{E}_{G'}$ and the restriction of (0.3) gives a map

$$\circ_{\mathcal{MF}at^{ad}} : \mathscr{E}_G \times \mathscr{E}_{G'} \longrightarrow \mathcal{MF}at^{ad}_{S\#S'}$$

We believe we can use this map to extend (0.2) to a chain map for arbitrary open closed cobordisms, showing that black and white graphs give a model for the cobordism category of surfaces, and that this map coincides with the chain map given on [**WW11**] for the construction of natural operations in Hochschild homology.

On the homology of the Moduli space of surfaces

Godin and Ehrenfried used different combinatorial models of moduli space to obtain computations of the moduli space of bordered surfaces. However, the size of such models is restrictive. They where both able to perform such computations only for the cases of closed cobordisms of the type $S_{g,p+q}$ where $2g + p + q \leq 5$. In Paper B we give a projection map $\pi : \mathfrak{Rad}^{\sim} \to \mathcal{MFat}^{ad}$ which is a homotopy equivalence. This map is not surjective, since there are many admissible fat graphs that correspond to radial slit configurations of degenerate surfaces. It would be interesting to know if there is a nice combinatorial description of the image of π, and if we can define a deformation retraction

$$\mathcal{MFat}^{ad} \times I \longrightarrow Im(\pi).$$

If such construction exists, then we might find an even smaller chain complex which models moduli space of bordered surfaces. This might allow for new computations of the homology of the mapping class group via experimental methods.

On the homology of Sullivan diagrams

From the computations of Paper C, we conjecture that the lower homology groups of the chain complex of Sullivan diagrams of topological type of the disk with c punctures, for $c > 2$ is given by:

$$\tilde{H}_*(\mathscr{S}\mathscr{D}_{D^c}) = \begin{cases} 0 & \text{if } i \leq c-2 \\ 0 & \text{if } i = c-1 \text{ and } c \text{ is odd} \\ \mathbb{Z} & \text{if } i = c-1 \text{ and } c \text{ is even} \\ \mathbb{Z} & \text{if } i = c \text{ and } c \text{ is odd} \end{cases}$$

A potential approach to prove part of this conjecture, is to study the chain complex of Sullivan diagrams using topological combinatorics. More precisely, using the explicit description of the generators in terms of a tuple of natural numbers and a non-crossing partition, we believe we might be able to define a Morse flow on the $(c-1)$-skeleton of $\mathscr{S}\mathscr{D}_{D^c}$ onto a subspace with trivial homology, showing that the first $c-2$ homology groups are trivial.

Moreover, if the conjecture is correct, this might indicate the the chain complexes of Sullivan diagrams have homological stability for the case of surfaces with punctures. We believe that by attaching chords we might be able to give well defined chain maps

$$\mathscr{S}\mathscr{D}_{S_{g,3}} \to \mathscr{S}\mathscr{D}_{S_{g,4}} \to \ldots \to \mathscr{S}\mathscr{D}_{S_{g,n}} \to \mathscr{S}\mathscr{D}_{S_{g,n+1}} \to \ldots$$

$$\mathscr{S}\mathscr{D}_{S_{0,2}} \to \mathscr{S}\mathscr{D}_{S_{1,2}} \to \ldots \to \mathscr{S}\mathscr{D}_{S_{g,2}} \to \mathscr{S}\mathscr{D}_{S_{g+1,2}} \to \ldots$$

and study homological stability for such maps. Moreover, the homemorphism between $\mathcal{SD}$ and $\overline{\mathfrak{UMRad}}$ might allow to proof such statements using radial slit configurations and operations on moduli space studied by Bödigheimer.

Another aspect we can explore further is to extend the algorithm given in Paper C for the genus 0 case with leaves or for the genus 1 case. The representation we have for a Sullivan diagram in terms of a tuple of natural numbers and a non-crossing partition extends naturally to the case with leaves by listing all possible places in which a leave can be attached. Furthermore, it is possible to use a similar idea to describe a genus one Sullivan diagram in the circle by triples of numbers representing: the euler characteristic, the number of legs and the genus, together with and a 1-crossing partition or a non-crossing partition. As far as we know, the homology of Sullivan diagrams in these cases is unknown, and just as in the case of Paper C, a few experimental results might lead to the first general statements about the homology of these complexes.

String topology

Recall that Klamt [**Kla13**] introduced a chain complex $l\mathcal{D}$, which gives operations of commutative Frobenius algebras and that this complex recovers the operations coming from Sullivan diagrams in the commutative Frobenius case. Loosely speaking a loop diagram is a Sullivan diagram in which we remember the admissible cycles, but forget the cyclic ordering at all the other vertices, and we add loops that start at each leaf of the diagram. In particular, the map $\mathscr{S}\mathscr{D} \to l\mathcal{D}$ is given by taking a Sullivan diagram, forgetting part of its cyclic structure and adding a loop for each boundary cycle that is connected to a leaf.

We are interested in studying the underlying underlying space of the chain complex $l\mathcal{D}$. This space gives string topology operations and seems to be the space of string topology operations. Moreover, these operations have a notion of glueing and assemble together nicely in some sort of "field theory". However, we would like field theories to have a geometric flavour, and as such we would like to interpret this space as some sort of compactification of some geometric space of surfaces. Such a space has a different number of connected components than both moduli space and the space of Sullivan diagrams, since the commutative nature of $l\mathcal{D}$ allows diagrams to change "genus". Besides the Harmonic compactification, Bödigheimer's radial slit configuration model has a different compactification, which uses filtrations of symmetric groups. In this compactification, configurations are allowed to change genus and number of boundary components. We wonder, if this space, or possibly this space with decorations, is a first candidate for an underlying space of the chain complex $l\mathcal{D}$.

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Part II

Scientific papers

Paper A

COMPARING FAT GRAPH MODELS OF MODULI SPACE

DANIELA EGAS

Abstract. Godin introduced the categories of open closed fat graphs $\mathcal{F}at^{oc}$ and admissible fat graphs $\mathcal{F}at^{ad}$ as models of the mapping class group of open closed cobordism. Similarly, Costello introduced a chain complex of black and white graphs $\mathcal{BW}-Graphs$, as a homological model of this mapping class group. We use the contractibility of the arc complex to give a new proof of Godin's result that $\mathcal{F}at^{ad}$ is a model of the mapping class group of open closed cobordisms and use this result to give a new proof of Costello's result that $\mathcal{BW}-Graphs$ is a homological model of this mapping class group. The nature of this proof also provides a direct connection between both models which were previously only abstractly equivalent with potential applications.

1. Introduction

1.1. Cobordisms and their Moduli. The study of surfaces and their structure has been a central theme in many areas of mathematics. One approach to study the genus g closed oriented surface S_g, is by the *moduli space of* S_g which we denote $\mathcal{M}_g$, which is a space that classifies all compact Riemann surfaces of genus g up to complex-analytic isomorphism. We recall some the concepts involved in this field mainly following [FM11, Ham13]. A *marked metric complex structure* on S_g, is a tuple (X, φ), where X is a Riemann surface and $\varphi : S \to X$ is an orientation preserving diffeomorphism. Two complex structures (X, φ) and (X', φ') are *equivalent* if there is a biholomorphic map $f : X \to X'$ such that $f \circ \varphi$ and φ' are isotopic. The *Teichmüller space of* S_g which we denote $\mathcal{T}_g$, is the space of all equivalence classes of marked metric complex structures. It is a contractible manifold of dimension $6g - 6$. The *mapping class group of* S_g, which we denote $\mathrm{Mod}(S_g)$, is the group of components of the group of orientation preserving self-diffeomorphisms of the surface i.e. $\pi_0(\mathrm{Diff}^+(S_g))$. One can show that this definition is equivalent to many others namely

$$\mathrm{Mod}(S_g) \cong \pi_0(\mathrm{Homeo}^+(S_g)) \cong \mathrm{Diff}^+(S_g)/\sim_i \cong \mathrm{Homeo}^+(S_g)/\sim_h$$

where $\sim_i$ and $\sim_h$ denote the isotopy and homotopy relations respectively. The mapping class group acts on Teichmüller space by precomposition with the marking and the moduli space of S_g is the quotient of Teichmüller space by this action i.e. $\mathcal{M}_g := \mathcal{T}_g/\mathrm{Mod}(S_g)$.

These definitions can be extended to surfaces with additional structure. We will study the case of open closed cobordisms, which has applications in string topology and topological field theories. An *open closed cobordism* $S_{g,p+q}$ is an oriented surface with boundary together with a partition of the boundary into three parts $\partial_{in}S$, $\partial_{out}S$ and $\partial_{free}S$ and parametrizing diffeormorphisms

$$\partial_{in}S \to N_{in} \qquad\qquad \partial_{out}S \to N_{out}$$

where N_{in} is a space with $p = p_1+p_2$ ordered connected components, p_1 of these components are circles which mark the incoming closed boundaries and p_2 of them are intervals which mark the incoming open boundaries. Similarly, N_{out} is a space with $q = q_1 + q_2$ ordered connected components, q_1 of these components are circles which mark the outgoing closed boundaries and q_2 of them are intervals which mark the outgoing open boundaries. The parametrizing diffeormorphisms give an ordering of the incoming and outgoing boundary components, see Figure 1.1. Note that since the surface $S_{g,p+q}$ is oriented, then up to homotopy, to give the parametrizing diffeomorphisms is equivalent to fixing a marked point in each component of $\partial_{in}S \cup \partial_{out}S$ and giving an ordering of these.

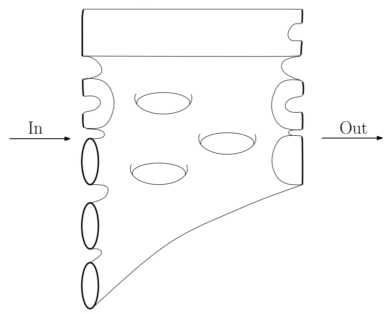

FIGURE 1.1. An open closed cobordism whose underlying surface has genus 3 and 7 boundary components. There are 3 incoming closed boundaries and no outgoing closed boundaries. There are 3 incoming open boundaries and 5 outgoing open boundaries.

As in the case of surfaces, 2-dimensional open closed cobordisms $S_{g,p+q}$ and $S'_{g,p+q}$ have the same *topological type as open closed cobordisms* if there is an orientation preserving homeomorphism $h : S_{g,p+q} \to S'_{g,p+q}$ that respects the parametrization diffeomorphisms, or equivalently that sends the i-th marked point on the incoming (resp. outgoing) boundary component of $S_{g,p+q}$ to the i-th marked point in the incoming (resp. outgoing) boundary component of $S'_{g,p+q}$.

The notions of Teichmüller space, Moduli Space and mapping class groups are extended in a natural way. More precisely, a *marked metric complex structure* on $S_{g,p+q}$ is a tuple (φ, X), where X is a Riemann surface with boundary parametrizations and $\varphi : S \to X$ is an orientation preserving diffeomorphism that respects the boundary parametrizations. Two complex structures (X, φ) and (X', φ') are *equivalent* if there is a biholomorphic map $f : X \to X'$ that respects the boundary parametrizations such that $f \circ \varphi$ and φ' are isotopic. The *Teichmüller space of* $S_{g,p+q}$ which we denote $\mathcal{T}_{g,p+q}$, is the space of all equivalence classes of marked metric complex structures. The *mapping class group of* $S_{g,p+q}$ is

$$\mathrm{Mod}(S_{g,p+q}) := \pi_0(\mathrm{Diff}^+(S_{g,p+q}, \partial_{in}S \cup \partial_{out}S))$$

where $\mathrm{Diff}^+(S_{g,p+q}, \partial_{in}S \cup \partial_{out}S)$ is the space of orientation preserving diffeomorphisms that fix $\partial_{in}S \cup \partial_{out}S$ point wise. The mapping class group acts on Teichmüller space by precomposition with the marking and $\mathcal{M}_{g,p+q} := \mathcal{T}_{g+q}/\mathrm{Mod}(S_{g,p+q})$. When there is at least one marked point in a the boundary of $S_{g,p+q}$, the action of $\mathrm{Mod}(S_{g,p+q})$ is free and thus $\mathcal{M}g, p+q$ is a classifying space of $\mathrm{Mod}(S_{g,p+q})$.

1.2. **Admissible Fat graphs.** Informally, a fat graph is a graph in which each vertex has a cyclic ordering of the edges that are attached to it, see Definition 2.4 for a precise definition. This cyclic ordering allows us to fatten the graph to obtain a surface. In [Pen87], Penner constructs a triangulation of the decorated Teichmúller space of surfaces with punctures, which is equivariant under the action of the mapping class group, giving a model of the decorated Moduli space of punctured surfaces. In [Igu02], Igusa constructs a category $\mathcal{F}at$, with objects fat graphs whose vertices have valence greater or equal to three. He shows that this category models the mapping class groups of punctured surfaces. Following these ideas, in [God07b], Godin constructs a category $\mathcal{F}at^b$ of fat graphs with leaves and shows that this category models the mapping class groups of bordered surfaces. In [God07a] she extends this construction and gives a category $\mathcal{F}at^{ad}$ of open closed fat graphs and a full subcategory $\mathcal{F}at^{oc}$ of admissible fat graphs which model the mapping class groups of open closed cobordisms. In this paper, we give a new proof of this result, shown in Theorem A, which is more geometric in nature, by using the contractibility of the arc complex.

Theorem A. *The categories of open closed fat graphs and admissible fat graphs are models for the classifying spaces of mapping class groups of open closed cobordisms. More specifically there is a homotopy equivalence*

$$|\mathcal{F}at^{oc}| \to \coprod_{S_{g,p+q}} B\mathrm{Mod}(S_{g,p+q})$$

where the disjoint union runs over all topological types of open closed cobordisms where not all the boundary is free. Moreover, this map restricts on the subcategory of admissible fat graphs to a homotopy equivalence

$$|\mathcal{F}at^{ad}| \to \coprod_{S_{g,p+q}} B\mathrm{Mod}(S_{g,p+q})$$

where the disjoint union runs over all topological types of open closed cobordisms where not all the boundary is free or outgoing closed.

1.3. **Black and White graphs.** In [Cos06a, Cos06b], Costello constructs a modular space of degenerate surfaces and shows that this space is weakly equivalent to the Moduli space of Riemann surfaces. This space has a natural CW-structure and the generators of its cellular complex are given by disks and annuli glued at the boundary. Using a dual representation of the discs and annuli, in [WW11], Wahl and Westerland describe this chain complex as a complex of fat graphs with two types of vertices, black vertices corresponding to the center of the disks and white vertices corresponding to the inner boundary of the annuli. Following the terminology of [WW11], we denote this complex the complex of black and white graphs. Costello gives a geometric proof of the following theorem, using the moduli space of surfaces with possibly nodal boundary.

Theorem B. *The chain complex of black and white graphs is a model for the classifying spaces of mapping class groups of open closed cobordisms. More specifically there is an isomorphism*

$$\mathrm{H}_*(\mathscr{BW} - Graphs) \cong \mathrm{H}_*\left(\coprod_{S_{g,p+q}} B\mathrm{Mod}(S_{g,p+q})\right)$$

where the disjoint union runs over all topological types of open closed cobordisms where there is at least one boundary component which is not outgoing closed.

In this paper, we give a new proof of this theorem using Theorem A. More precisely, we construct a filtration

$$\mathcal{F}at^{ad} \ldots \supset \mathcal{F}at^{n+1} \supset \mathcal{F}at^{n} \supset \mathcal{F}at^{n-1} \ldots \mathcal{F}at^{1} \supset \mathcal{F}at^{0}$$

that gives a cell-like structure on $\mathcal{F}at^{ad}$ where the quasi-cells are indexed by black and white graphs i.e. $|\mathcal{F}at^{n}|/|\mathcal{F}at^{n-1}| \simeq \vee S^n$ where the wedge sum is indexed by black and white graphs of degree n. Besides proving Theorem B, the structure of the proof gives a direct connection between the admissible fat graph model and the black and white graph model, which we expect to be useful, since there are certain constructions which are not natural in the black and white picture which might be easier to understand in the admissible fat graph setting. In particular, we believe this connection could provide a notion of glueing cobordisms along their boundary in terms of the black and white graphs.

The organization of the paper is as follows. Section 1 gives preliminary definitions of fat graphs, their morphisms and their fattening to a surface. Section 2 describes the categorical models of fat graphs and gives the proof of Theorem A. Section 3 describes the chain complex of black and white graphs and gives the proof of Theorem B.

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2. Preliminary Definitions

We give the basic definitions regarding fat graphs, their realizations and morphisms.

Definition 2.1. A *combinatorial graph* G is a tuple $G = (V, H, s, i)$, with a finite set of vertices V, a finite set of half edges H, a map $s : H \to V$ and an involution with no fixed points $i : H \to H$.

The map s ties each half edge to its source vertex and the involution i attaches half edges together. An edge of the graph is an orbit of i. The valence of a vertex $v \in V$ is the cardinality of the set $s^{-1}(v)$ and a *leave* of a graph is a univalent vertex.

Definition 2.2. The *geometric realization* of a combinatorial graph G is the CW-complex $|G|$ with one 0-cell for each vertex, one 1-cell for each edge and attaching maps given by s.

Definition 2.3. A *tree* is a graph whose geometric realization is a contractible space and a *forest* is a graph whose geometric realization is the disjoint union of contractible spaces.

Definition 2.4. A *fat graph* $\Gamma = (G, \sigma)$ is a combinatorial graph together with a cyclic ordering σ_v of the half edges incident at each vertex v. The *fat structure* of the graph is given by the data $\sigma = (\sigma_v)$ which is a permutation of the half edges. Figure 2.1 shows some examples of fat graphs.

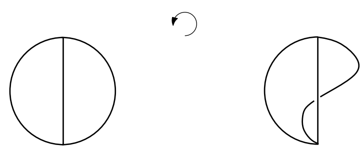


Figure 2.1. Two different fat graphs which have the same underlying combinatorial graph. The fat structure is given by the orientation of the plane.

Definition 2.5. The *boundary cycles* of a fat graph are the cycles of the permutation of half edges given by $\omega = \sigma \circ i$. Each boundary cycle c gives a list of half edges and determines a list of edges of the fat graph Γ, those edges containing the half edges listed in c. The *boundary cycle sub-graph* corresponding to c is the subspace of $|\Gamma|$ given by the edges determined by c which are not leaves. When clear from the context we will refer to a boundary cycle sub-graph simply as boundary cycle.

Remark 2.6. From a fat graph $\Gamma = (G, \sigma)$ one can construct a surface with boundary Σ_Γ by fattening the edges. More explicitly, one can construct this surface by replacing each edge with a strip and glueing these strips at a vertex according to the fat structure. Notice that there is a strong deformation retraction of Σ_Γ onto $|G|$ so one can think of $|G|$ as the skeleton of the surface. The fat structure of Γ is completely determined by ω. Moreover, one can show that the boundary cycles of a fat graph $\Gamma = (G, \omega)$ correspond to the boundary components of Σ_Γ [God07b]. Therefore, the surface Σ_Γ is completely determined by the combinatorial graph and its fat structure.

Definition 2.7. A *morphism of combinatorial graphs* $\varphi : G \to \tilde{G}$ is a map of sets $\varphi : V_G \coprod H_G \to V_{\tilde{G}} \coprod H_{\tilde{G}}$ such that

- For every vertex $v \in V_{\tilde{G}}$ the preimage $\varphi^{-1}(v)$ is a tree in G.
- For every half edge $A \in H_{\tilde{G}}$ the preimage $\varphi^{-1}(A)$ contains exactly one half edge of G.
- The following diagrams commute

$$\begin{array}{ccc} V_G \coprod H_G & \xrightarrow{\tilde{s}_G} & V_G \coprod H_G \\ \varphi \downarrow & & \downarrow \varphi \\ V_{\tilde{G}} \coprod H_{\tilde{G}} & \xrightarrow[\tilde{s}_{\tilde{G}}]{} & V_{\tilde{G}} \coprod H_{\tilde{G}} \end{array} \qquad \begin{array}{ccc} V_G \coprod H_G & \xrightarrow{\tilde{i}_G} & V_G \coprod H_G \\ \varphi \downarrow & & \downarrow \varphi \\ V_{\tilde{G}} \coprod H_{\tilde{G}} & \xrightarrow[\tilde{i}_{\tilde{G}}]{} & V_{\tilde{G}} \coprod H_{\tilde{G}} \end{array}$$

where $\tilde{i}$, respectively $\tilde{s}$, is the extension of the involution i, respectively the source map s, to $V \coprod H$ by the identity on V.

Definition 2.8. A *morphism of fat graphs* $\varphi : (G, \omega) \to (\tilde{G}, \tilde{\omega})$ is a morphism of combinatorial graphs which respects the fat structure i.e. $\varphi(\omega) = \tilde{\omega}$.

Remark 2.9. Note that, if two fat graphs Γ, $\tilde{\Gamma}$ are isomorphic and they have at least one leaf in each connected component, and these leaves are labelled by $\{1, 2, 3 \ldots k\}$ i.e. the leaves are ordered, then there is unique morphism of graphs that realizes this isomorphism while respecting the labelling of the leaves. Thus, a fat graph Γ that has at least one labelled leaf in each connected component has no automorphisms besides the identity morphism.

Remark 2.10. Note that a morphism of combinatorial graphs induces a simplicial, surjective homotopy equivalence on geometric realizations and does not change the number of boundary cycles. Thus, if there is a morphism of fat graphs $\varphi : \Gamma \to \tilde{\Gamma}$ then the surfaces Σ_Γ and $\Sigma_{\tilde{\Gamma}}$ are homeomorphic.

3. Categories of Fat Graphs

3.1. **The Definition.** We now construct the basic objects and morphisms that form the categories of fat graphs.

Definition 3.1. An *open-closed fat graph* is a triple $\Gamma^{oc} = (\Gamma, In, Closed)$ where Γ is a fat graph with ordered leaves and In and $Closed$ are subsets of the set of leaves of Γ. This subsets give a labelling of the leaves of Γ as incoming or outgoing and as closed or open. The triple Γ^{oc} should be given such that the following hold:

- All inner vertices are at least trivalent
- A closed leaf must be the only leaf in its boundary cycle

We allow degenerate graphs which are a corolla with 1 or 2 leaves. Figure 3.1 shows an example of an open-closed fat graph.

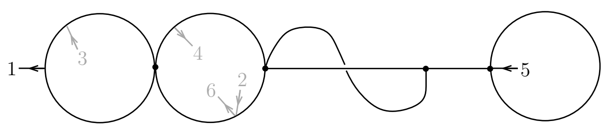


FIGURE 3.1. An example of a closed fat graph which is not admissible. The incoming and outgoing leaves are marked by incoming or outgoing arrows. The closed leaves are depicted in black and the open ones in grey.

Definition 3.2. An *admissible fat graph* $\Gamma^{ad} = (\Gamma, In, Closed)$ is an open-closed fat graph in which all outgoing closed boundary cycles are disjoint embedded circles in Γ. Figure 3.2 shows an example of admissible fat graphs while Figure 3.1 shows an example of an open-closed fat graph which is not admissible.

Note that an open-closed fat graph can not be an admissible fat graph if all the boundary is outgoing closed.

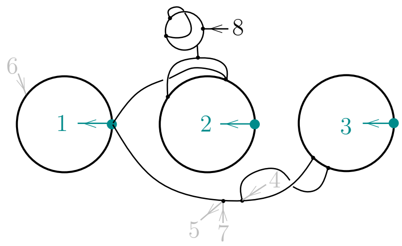


FIGURE 3.2. An example of admissible fat graphs. The admissible leaves (outgoing closed) are pictured in green.

Notation 3.3. When it is clear from the context we will simply write Γ instead of Γ^{oc} or Γ^{ad}

Definition 3.4. A *morphism of open-closed fat graphs* is a morphism of fat graphs which respects the labelling of the leaves. Two morphisms $\varphi_i : \Gamma_i \to \tilde{\Gamma_i}$ for $i = 1, 2$ are equivalent if there are isomorphisms which make the following diagram commute

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\varphi_1} & \tilde{\Gamma}_1 \\ \cong\downarrow & & \downarrow\cong \\ \Gamma_2 & \xrightarrow[\varphi_2]{} & \tilde{\Gamma}_2 \end{array}$$

Remark 3.5. Let $[\Gamma]$ and $[\Gamma']$ be two isomorphism classes of open-closed fat graphs. One can show that all morphisms $\varphi : [\Gamma] \to [\tilde{\Gamma}]$ can be realized uniquely as a collapse of a sub-forest of Γ which does not contain any leaves. The argument is exactly the same as the one given in [God07b] for the case where all leaves are incoming closed.

Definition 3.6. The category of open-closed fat graphs $\mathcal{F}at^{oc}$ is the category with objects isomorphism classes of open-closed fat graphs with at least one leaf on each component and morphisms equivalences classes of morphisms. The category of admissible fat graphs $\mathcal{F}at^{ad}$ is the full subcategory of $\mathcal{F}at^{oc}$ on objects isomorphism classes of admissible fat graphs.

Remark 3.7. These categories are slightly different than the ones given in [God07a] since there are no leaves for the free boundary components. However, the exact same argument given in [God07b] shows that these categories are well defined. More precisely, composition is well defined since as given in Remark 2.9, there is a unique isomorphism of open-closed fat graphs between two open-closed fat graphs with at least one leaf on each component and an open-closed fat graph of such kind has no automorphisms besides the identity morphism.

From an open-closed fat graph one can construct an open-closed cobordism $S_{g,p+q}$. First construct a bordered oriented surface Σ_Γ as for a regular fat graph. Now, divide the boundary by the following procedure. For a boundary component corresponding to a closed leave, label the entire boundary component as incoming or outgoing according to the labelling of the leaf and choose a marked point on the boundary. For a boundary component corresponding to one or more open leaves assign to each leaf a small part of the boundary (homeomorphic to the unit interval) such that none of these intervals intersect and such that they respect the cyclic ordering ordering of the leaves on the corresponding boundary cycle. Then label such intervals as incoming or outgoing according to their corresponding leaves and choose a marked point in each interval. Label the rest of the boundary as free. Finally order the marked points at the boundary according to the ordering of their corresponding leaves. This gives and open-closed cobordism $S_{g,p+q}$ well defined up to topological type.

3.2. **Fat graphs as models for the mapping class group.** The categories $\mathcal{F}at^{oc}$ and $\mathcal{F}at^{ad}$ are introduced by Godin in [God07a]. In this paper, she show that both categories are models of the classifying space of the mapping class group by comparing a sequence of fibrations. However, there is a step missing in the proof which we do not know how to complete. More precisely, Godin proves this by comparing certain fiber sequences, but a map connecting them is not explicitly constructed and we do not know how to construct such map. In this section we give a new proof, more geometric in nature, that shows that these categories model mapping class groups, following the ideas of [God07b].

Theorem 3.8. *The categories of open-closed fat graphs and admissible fat graphs are models for the classifying spaces of mapping class groups of open-closed cobordisms. More specifically there is a homotopy equivalence*

$$|\mathcal{F}at^{oc}| \to \coprod_{S_{g,p+q}} B\mathrm{Mod}(S_{g,p+q})$$

where the disjoint union runs over all topological types of open-closed cobordisms where not all the boundary is free. Moreover, the restriction of this map to the subcategory of admissible fat graphs induces a homotopy equivalence

$$|\mathcal{F}at^{ad}| \to \coprod_{S_{g,p+q}} B\mathrm{Mod}(S_{g,p+q})$$

where the disjoint union runs over all topological types of open-closed cobordisms where not all the boundary is free or outgoing closed.

Let $\mathcal{F}at^{oc}_{g,p+q}$ denote the full subcategory with objects open-closed fat graphs whose realization give an open-closed cobordism of topological type $S_{g,p+q}$, and define $\mathcal{F}at^{ad}_{g,p+q}$ similarly. Note that a morphism of open-closed fat graphs induces a homotopy equivalence on realizations which respects the structure that determines the topological type of the graph as an open-closed cobordism. Therefore we have the following isomorphisms:

$$\mathcal{F}at^{oc} \cong \coprod_{S_{g,p+q}} \mathcal{F}at^{oc}_{g,p+q} \qquad\qquad \mathcal{F}at^{ad} \cong \coprod_{S_{g,p+q}} \mathcal{F}at^{ad}_{g,p+q}$$

The idea of the proof is to show there is a homotopy equivalence on each connected component by constructing coverings of $\mathcal{F}at^{oc}_{g,p+q}$ and $\mathcal{F}at^{ad}_{g,p+q}$ which are contractible and admit a free transitive action of their corresponding mapping class group.

Notation 3.9. For each topological type of open-closed cobordism choose and fix a representative $S_{g,p+q}$ and let x_k denote the marked point in the k-th incoming boundary for $1 \leq k \leq p$ and x_{p+k} denote the marked point on the k-th outgoing boundary $1 \leq k \leq q$. Given an open-closed fat graph Γ^{oc}, let $v_{in,k}$ denote the k-th incoming leaf and $v_{out,k}$ denote the k-th outgoing leave.

Definition 3.10. A *marking* of an open-closed fat graph is an isotopy class of embeddings $H : |\Gamma^{oc}| \hookrightarrow S_{g,p+q}$ such that $H(v_{in,k}) = x_k$, $H(v_{out,k}) = x_{p+k}$ and the fat structure of Γ^{oc} coincides with the one induced by the orientation of the surface. We will call the pair $([\Gamma^{oc}], [H])$ a marked open-closed fat graph.

Remark 3.11. Given a marking $H : |\Gamma| \hookrightarrow S_{g,p+q}$, by definition it holds that, $\pi_1(\Gamma) \cong \pi_1(S_{g,p+q})$, and that H is given such that the induced map on π_1 sends the ith boundary cycle of Γ to the ith boundary component of $S_{g,p+q}$. Moreover, since the fat structure of Γ coincides with the one induced by the orientation of the surface we can thicken Γ inside $S_{g,p+q}$ to a subsurface of the same topological type as $S_{g,p+q}$. Thus, there is a deformation retraction of $S_{g,p+q}$ onto this subsurface and onto Γ, showing that the embedding H is a homotopy equivalence.

Remark 3.12. Let Γ be an admissible fat graph, F be a forest in Γ which does not contain any leaves of Γ and H be a representative of a marking $[H]$ of Γ. Since $[H]$ is a marking, the image of $H|_F$ (the restriction of H to $|F|$) is contained in a disjoint union of disks away from the boundary. Therefore, the marking H induces a marking $H_F : |\Gamma/F| \hookrightarrow S_{g,p+q}$ given by collapsing each of the trees of F to a point of the disk in which their image is contained. Note that H_F is well defined up to isotopy and it makes the following diagram commute up to homotopy

$$\begin{array}{ccc} |\Gamma| & \twoheadrightarrow & |\Gamma/F| \\ & \overset{H}{\searrow} & \downarrow H_F \\ & & S_{g,p+q} \end{array}$$

Definition 3.13. Define the category $\mathcal{EFat}^{oc}$ to be the category with objects marked open-closed fat graphs $([\Gamma^{oc}], [H])$ and morphisms given by morphisms in $\mathcal{Fat}^{oc}$ where the map acts on the marking as stated in the previous remark. Define $\mathcal{EFat}^{ad}$ to be the full subcategory of $\mathcal{EFat}^{oc}$ with objects $([\Gamma^{ad}], [H])$ marked admissible fat graphs.

Proof of Theorem 3.8. There are natural projections from the categories $\mathcal{EFat}^{oc}$ and $\mathcal{EFat}^{ad}$ onto the categories $\mathcal{Fat}^{oc}$ and $\mathcal{Fat}^{ad}$ by forgetting the marking. It is enough to show the result in each connected component. Let $\mathcal{EFat}^{oc}_{g,p+q}$ and $\mathcal{EFat}^{ad}_{g,p+q}$ be the full subcategories of $\mathcal{EFat}^{oc}$ and $\mathcal{EFat}^{ad}$ corresponding to open-closed cobordisms of topological type $S_{g,p+q}$. These subcategories fit in the following commutative square

$$\begin{array}{ccc} \mathcal{EFat}^{ad}_{g,p+q} & \hookrightarrow & \mathcal{EFat}^{oc}_{g,p+q} \\ \downarrow & & \downarrow \\ \mathcal{Fat}^{ad}_{g,p+q} & \hookrightarrow & \mathcal{Fat}^{oc}_{g,p+q} \end{array}$$

where the horizontal maps are inclusions. We show that there is a free action of $\mathrm{Mod}(S_{g,p+q})$ on $\mathcal{EFat}^{oc}_{g,p+q}$ with quotient $\mathcal{Fat}^{oc}_{g,p+q}$ i.e., we show that $\mathrm{Mod}(S_{g,p+q})$ acts on $|\mathcal{EFat}^{oc}_{g,p+q}|$ and we show that this action is free and transitive on the fibers by showing that it is free and transitive on the 0-simplices.

The mapping class group acts on $\mathcal{EFat}^{oc}$ by composition with the marking. Thus, it is enough to show that this group acts freely and transitively on the markings i.e. for any two markings $[H_1]$ and $[H_2]$ there is a unique $[\varphi] \in \mathrm{Mod}(S_{g,p+q})$ such that $[\varphi \circ H_1] = [H_2]$. Given two such markings, we will construct a homeomorphism $f : S_{g,p+q} \to S_{g,p+q}$ such that $[f \circ H_1] = [H_2]$ which we can approximate by a diffeomorphism by Nielsen's approximation theorem. By remark 3.11 $S_{g,p+q} \setminus H_1(\Gamma)$ has $p+q+f$ connected components where f is the number of free boundary components of $S_{g,p+q}$, say $S_{g,p+q} \setminus H_1(\Gamma) := \sqcup_i S_i$ for $1 \leq i \leq p+q+f$. Moreover, each component S_i is of one of the following forms:

i If there is exactly one leaf in a boundary cycle, then S_i is a disc bounded by the image under H_1 of the given boundary cycle and by the corresponding boundary component.

ii If there is more than one leaf on a boundary cycle, then S_i is a disc bounded by the image under H_1 of part of the boundary cycle and part of the corresponding boundary component (the sections bounded by consecutive leaves).

iii If there is no leaf in a boundary cycle, then S_i is an annulus with boundaries the image of H_1 of the given boundary cycle and the corresponding boundary component.

We construct f by defining homeomorphisms in each component which can be glued together consistently. Order S_i according to the ordering of the incoming and outgoing leaves and a chosen ordering of the free boundary components. Notice first that if S_i is of the types (i) or (ii) then the corresponding boundary component of the surface is not free. So the restriction of f to such component should give a map $f_i : S_i \to S_i$. In this case, define $\tilde{f}_i : \partial S_i \to \partial S_i$ to be the identity on the boundary section and to be $H_2 \circ H_1^{-1}$ on the image of the boundary cycle. Since S_i is homeomorphic to a disk, we can extend $\tilde{f}_i$ to a map $f_i : S_i \to S_i$ which is uniquely defined up to homotopy. Moreover, if S_i is of type (iii) then the corresponding boundary component is free and thus the restriction of f should give a map $f_{ij} : S_i \to S_j$ where S_j also corresponds to a free boundary component and it could be that $i = j$. In this case, define $\tilde{f}_{ij} : \partial S_i \to \partial S_j$ to be $H_2 \circ H_1^{-1}$ on the image of the boundary cycle and a homeomorphism homotopic to the identity on the boundary of the surface. This morphism can be extended though not uniquely to a map $f_{ij} : S_i \to S_j$, choose any extension of such map. These maps can be glued together giving the desired map f which we can approximate by a diffeomorphism φ. Moreover, two non-homotopic extensions of f_{ij} differ only by powers of a Dehn twists around the free boundary and thus give the same element in the mapping class group i.e. $[\varphi]$ is determined uniquely in $\mathrm{Mod}(S_{g,p+q})$. This argument restricts to the subcategory $\mathcal{EF}at^{ad}_{g,p+q}$.

Propositions 3.19, 3.20, 3.22 and 3.24 in the next subsection show that $|\mathcal{EF}at^{oc}_{g,p+q}|$ and $|\mathcal{EF}at^{ad}_{g,p+q}|$ are contractible, which finishes the proof. □

3.2.1. *The categories of marked fat graphs are contractible.* In this section we will show that $\mathcal{EF}at^{oc}_{g,p+q}$ and $\mathcal{EF}at^{ad}_{g,p+q}$ are contractible categories by using Hatcher's proof of the contractibility of the arc complex. This section is self contained and can be skipped if desired.

Definition 3.14. Let S be an orientable surface and V a finite set of marked points in S.

- An *essential arc* α_0, is an embedded arc in S that starts and ends at V, intersects $\partial S \cup V$ only at its endpoints and it is not boundary parallel i.e. α_0 does not separate S into two components one of which is a disk that intersects V only at the endpoints of α_0.
- An *arc set* α in S is a collection of arcs $\alpha = \{\alpha_0, \alpha_1, \cdots \alpha_n\}$ such that their interiors are pairwise disjoint and no two arcs are ambient isotopic relative to V.
- An *arc system* $[\alpha]$ in S is an ambient isotopy class of arc sets of S relative to V.
- An arc system is *filling* if it separates S into polygons that do not contain a marked point in its interior.

The following definition and result is originally due to Harer in [Har86] to which later on Hatcher gives a very beautiful and simple proof in [Hat91]

Definition 3.15. Let S be an orientable surface and V a finite set of marked points in S. The *arc complex*, $\mathscr{A}(S,V)$, is the complex with vertices isotopy classes of essential arcs $[\alpha_0]$, k simplices arc systems of the form $[\alpha] = [\alpha_0, \alpha_1, \cdots \alpha_k]$ and faces obtained by passing to subcollections.

Theorem 3.16 ([Har86], [Hat91])**.** *Let S be an orientable surface and V a finite, non-empty set of marked points in S. The complex $\mathscr{A}(S,V)$ is contractible whenever S is not a disk or an annulus with V contained in one connected component of ∂S.*

Definition 3.17. Let $S_{g,p+q}$ be an open cobordism and let Υ denote the set of marked points on the boundary i.e. $\Upsilon = \{x_1, \dots x_{p+q}\}$. We assign a second set Δ of marked points on such cobordism as follows. On each closed boundary choose a marked point $y_i \neq x_i$, where x_i is the marked point corresponding to the closed boundary. On the boundary components with open incoming or outgoing boundaries choose a marked point y_i in each free section of the boundary component. Using this we make the following definitions.

- An *essential arc* α_0 *in the cobordism* $S_{g,p+q}$, is an embedded arc that starts and ends at $\Delta \cup \partial_{free} S$, it intersects $\partial S \cup \Delta$ only at its endpoints and it is not boundary parallel i.e. α_0 does not separate $S_{g,p+q}$ into 2 components one of which is a disk that intersects Δ only at the endpoints of α_0.
- An *arc set* α *in the cobordism* $S_{g,p+q}$, is a collection of arcs $\alpha = \{\alpha_0, \alpha_1, \cdots \alpha_n\}$ such that their interiors are pairwise disjoint and no two arcs are ambient isotopic relative to Δ.
- An *arc system* $[\alpha]$ in the cobordism $S_{g,p+q}$ is an ambient isotopy class of arc sets of $S_{g,p+q}$ relative to Δ.

We now make the connection between the arc complex and the category of fat graphs.

Definition 3.18. Let $\mathscr{A}_0(S_{g,p+q}, \Delta)$ denote the poset category of filling arc systems ordered by inclusion. In the case where $S_{g,p+q}$ is a disk with $p+q \geq 1$ the surface is already a polygon with no marked vertices in its interior and thus we consider the empty set to be a filling arc system.

Proposition 3.19. *There is an isomorphism of categories* $\mathscr{A}_0(S_{g,p+q}, \Delta)^{op} \cong \mathcal{EF}at^{oc}_{g,p+q}$

Proof. It is enough to show this for a connected cobordism. Throughout the proof the cobordism $S_{g,p+q}$ will be fixed, so for simplicity we will denote it S and we will denote the category $\mathcal{EF}at^{oc}_{g,p+q}$ as $\mathscr{E}$. Using the sets Υ and Δ defined previously we will construct contravariant inverse functors

$$\Phi : \mathscr{A}_0 \leftrightarrows \mathscr{E} : \Psi$$

We first define the functor Φ on objects. Let $[\alpha] = [\alpha_0, \dots, \alpha_k]$ be a filling arc system. Choose a representative arc set $\alpha = \{\alpha_0, \dots, \alpha_k\}$. Then, $S \backslash \alpha = \coprod_i T_i$ is a disjoint union of polygons. Construct a fat graph Γ on the surface S by setting a vertex v_i in each T_i. If T_i and T_j are bordering components separated by an arc α_{ij} connect v_i with v_j with an edge E_{ij} that crosses only α_{ij} and crosses it exactly once. Moreover, if the marked point $x_j \in T_i$ connect v_i with x_j via an edge L_j. Make all edges non-intersecting on the surface. Each polygon T_i has an induced orientation coming from S, this gives a cyclic ordering of the edges incident at v_i. Note that the x_j's are leaves of Γ. Moreover, by construction Γ comes with a natural marking $[H]$ on S. So set $\phi(\alpha) = ([\Gamma], [H])$. Note that all the polygons T_i have at least three bounding arcs or are of the form shown in Figure 3.3 with at least one marked point y_i bounded by an essential arc. Thus, all inner vertices in Γ are at least trivalent, so $\phi(\alpha)$ is an object of $\mathscr{E}$. Moreover, setting $\Phi([\alpha]) = \phi(\alpha)$ is well defined since two representatives α and β of the arc system $[\alpha]$ are ambient isotopic so they split the surface in the same number of connected components giving isomorphic underlying fat graphs. Moreover, we can use the ambient isotopy connecting both representatives to show that they induce the same marking on $[\Gamma]$.

To define Φ on morphisms, let $[\beta]$ be a face of $[\alpha]$. We can find representatives such that $\alpha = \beta \cup \{\alpha_0, \dots, \alpha_n\}$. Note that if the edges corresponding to $\{\alpha_0, \dots, \alpha_n\}$ form a cycle on $\phi(\alpha)$ then β is not filling. Therefore this image must be a forest and there is a uniquely defined morphism obtained from collapsing such forest which gives the map $\Phi([\alpha]) \to \Phi([\beta])$. This construction behaves well with composition.

We now define the functor Ψ on objects. Let $([\Gamma], [H])$ be an object of $\mathscr{E}$. Then for representatives (Γ, H), the complement $S \backslash H(\Gamma)$ is a disjoint union of connected components, say $\coprod_i S_i$. By construction the component S_i contains exactly one marked point of Δ (in

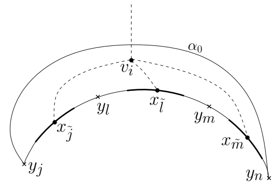


FIGURE 3.3. If α_0 is essential, then there must be at least one y_i bounded by it on the boundary

which case S_i is a polygon) or a free boundary component (in which case S_i is an annulus). Define an arc set $\psi(\Gamma, H)$ as follows. If there is an edge E_{ij} whose image under H separates S_i and S_j, then $\psi(\Gamma, H)$ has an arc α_{ij} crossing only E_{ij}. The arc α_{ij} starts in S_i either at the marked point of Δ if there is one or at any point in the free boundary component if there is none, and it ends in S_j in a similar fashion. Notice that it might be that $i = j$ i.e. the arc starts and ends at the same point. Now pull all arcs tight to make them all non-intersecting and discard the arcs that crossed the leaves. By construction this arc set is filling; thus, let $\Psi([\Gamma], [H]) = [\psi(\Gamma, H)]$. As before this functor is well defined on objects and it is defined on morphisms in the a similar was as for Φ. Finally, the functors Φ and Ψ are clearly inverses of each other. □

Proposition 3.20. *The category $\mathscr{A}_0(S_{g,p+q}, \Delta)$ is contractible.*

Proof. We follow a similar proof to the one given by Giansiracusa on [Gia10] on a similar poset. On the cobordism $S_{g,p+q}$ collapse each free boundary component to a point z_i. This gives a new cobordism $\tilde{S}_{g,p+q}$ with no free boundary and a new set of marked points $V = \Delta \cup \{z_1, \ldots, z_f\}$ where f is the number of free boundary components of $S_{g,p+q}$. Notice that there is an equivalence of categories $\mathscr{A}_0(S_{g,p+q}, \Delta) \cong \mathscr{A}_0(\tilde{S}_{g,p+q}, V)$. Thus, it is enough to show that for a cobordism $S_{g,p+q}$ with no free boundary and a set of marked points V which is the union of Δ and a finite set of marked points $\{z_1, \ldots, z_f\}$ in the interior of $S_{g,p+q}$ the category $\mathscr{A}_0(S_{g,p+q}, V)$ is contractible. We will do this by induction on the complexity of the cobordism namely on the tuple $k = (g, n, f, p+q)$ ordered lexicographically, where g is the genus of the surface, n is the number of boundary components, f is the number of marked points in the interior of the surface and p and q are the number of incoming, respectively outgoing boundaries of the cobordism. For simplicity, for the rest of the proof denote the tuple $(S_{g,p+q}, V)$ by $(S, c(S) = k)$.

We start the induction with $k = (0, 1, 0, p+q)$ for any $p + q \geq 1$. In this case, the category $\mathscr{A}_0(S, c(S) = k)$ is contractible since it has the empty set as initial element. Now let $k = (g, n, f, p+q) > (0, 1, 0, r)$ for any $r \geq 1$ and assume contractibility holds for all $k' < k$. Let $\mathscr{P}(\mathscr{A}(S, c(S) = k))$ be the poset category obtained from $\mathscr{A}(S, c(S) = k)$ by barycentric subdivision and let ι denote the inclusion:

$$\iota : \mathscr{A}_0(S, c(S) = k) \hookrightarrow \mathscr{P}(\mathscr{A}(S, c(S) = k))$$

For an object $[\alpha]$ in $\mathscr{P}(\mathscr{A}(S, c(S) = k))$, consider the comma category $[\alpha] \setminus \iota$ which in this case is the full subcategory of $\mathscr{A}_0(S, c(S) = k)$ with objects:

$$\mathrm{Ob}([\alpha] \setminus \iota) = \{[\beta] \in \mathscr{A}_0(S, c(S) = k) | [\beta] \geq [\alpha]\}$$

Note first the set of objects is not empty, since every arc system can be extended to a filling arc system. Let α be a representative of $[\alpha]$. Then, $(S, c(S) = k) \setminus \alpha$ is a disjoint union of cobordisms $\coprod_{i=1}^{m} (S_i, c(S_i) = k_i)$ and there is an isomorphism of categories

$$\Phi : [\alpha] \setminus \iota \cong \prod_{i=1}^{m} \mathscr{A}_0(S_i, c(S_i) = k_i) : \Psi$$

The mutually inverse functors are given as follows. Let $[\beta]$ be an object of $[\alpha]\setminus\iota$. We can choose a representative such that $\beta = \alpha \cup \{\beta_0, \dots \beta_n\}$. Each essential arc β_i is completely contained in some S_j. Denote $\{\beta_{0_i} \cdots \beta_{l_i}\}$ the arc set contained in S_i; this set fills S_i and it is possibly empty if S_i is already a disk not containing marked points in its interior. Set $\Phi([\beta]) = \prod_{i=1}^{m}[\beta_{0_i} \cdots \beta_{l_i}]$ with the natural map on morphisms. This is a well defined functor with inverse $\Psi(\prod_{i=1}^{m}[\beta_{0_i} \cdots \beta_{l_i}]) = [\alpha] \cup_{i=1}^{m} [\beta_{0_i} \cdots \beta_{l_i}]$ and the natural map on morphisms. Now, since all arcs of α are essential, then $k_i < k$ for all $1 \leq i \leq m$; so by the induction hypothesis $\mathscr{A}_0(S_i, c(S_i) = k_i)$ is contractible and thus $[\alpha]\setminus\iota$ is a contractible category. Then, Quillen Theorem A gives that ι is a homotopy equivalence. Finally, since $k \geq (0, 1, 0, p+q)$ then then either $f > 0$, $n > 1$ or $g > 0$ so $(S, c(S) = k)$ is neither a disk, nor a cylinder with all the marked points contained in one boundary component. Therefore, by Theorem 3.16 $\mathscr{A}(S, c(S) = k)$ is contractible, which finishes the proof. □

We now look into the case of the admissible fat graphs, and give a geometric interpretation for such condition.

Definition 3.21. Let $S_{g,p+q}$ be a cobordism with $p + q \geq 1$. Let $\partial_{i_1}S, \partial_{i_2}S \dots \partial_{i_k}S$ be the boundary components which are outgoing closed and α be an arc set in the cobordism $S_{g,p+q}$ as given in definition 3.17. We say that α is an *admissible arc set* if the following conditions hold.

- *i* Either α has a subset of arcs which cut $S_{g,p+q}$ into k components such that the j-th component contains in its interior all the arcs starting at $\partial_{i_j}S$ for all $1 \leq j \leq k$, or α is a face of such an arc set.
- *ii* α does not contain an arc that starts and ends at $\partial_{i_j}S$ for any $1 \leq j \leq k$.
- *iii* For all $1 \leq j \leq k$, let y_{i_j} be the marked point of Δ on $\partial_{i_j}S$, and $\alpha_{j_1}, \alpha_{j_2} \dots \alpha_{j_r}$ be the arcs in α that start or end at y_{i_j}. The arc set α also contains arcs $\beta_{j_1}, \beta_{j_2} \dots \beta_{j_{\tilde{r}}}$ such that the subspace $(\cup_l \alpha_{j_l}) \cup (\cup_l \beta_{j_l}) - y_{i_j}$ is connected.

Note that conditions $i - iii$ are well defined for arc systems. We define $\mathscr{B}(S_{g,p+q}, \Delta)$ to be the *subcomplex of* $\mathscr{A}(S_{g,p+q}, \Delta)$ *of admissible arc systems.* Similarly we define $\mathscr{B}_0(S_{g,p+q}, \Delta)$ to be the *subposet of* $\mathscr{A}_0(S_{g,p+q}, \Delta)$ *of filling admissible arc systems.*

Proposition 3.22. *Let $[\alpha]$ be an arc system in the cobordism $S_{g,p+q}$ and let $([\Gamma_\alpha], [H_\alpha])$ be its corresponding open-closed marked fat graph under the isomorphism of Theorem 3.19. The arc system $[\alpha]$ is admissible if and only if the graph $[\Gamma_\alpha]$ is admissible.*

Before proving the proposition we will state an immediate corollary

Corollary 3.23. *There is an isomorphism of categories $\mathscr{B}_0(S_{g,p+q}, \Delta)^{op} \cong \mathcal{EF}at^{ad}_{g,p+q}$*

Proof. This isomorphism is just a restriction of the isomorphism of theorem 3.19, which is well defined by the proposition above. □

Proof of Proposition 3.22. Let $\partial_{i_1}S, \partial_{i_2}S \dots \partial_{i_k}S$ be the boundary components of $S_{g,p+q}$ which are outgoing closed, let α and Γ_α be representatives of the arc system and fat graph of the theorem, and let $C_1, C_2 \dots C_k$ denote the boundaries cycles of Γ_α which are outgoing closed. Recall that Γ_α is an admissible fat graph if for all $1 \leq j \leq k$, the induced maps

$$c_j : S^1 \to C_j \to |\Gamma_\alpha|$$

are disjoint embeddings. We will show that condition i is equivalent to saying that all c_j's are disjoint and conditions ii and iii are equivalent to saying that each c_j is an embedding.

Note that condition i does not hold for α if and only if for some i_j and i_s with $j \neq s$ at least one of the following hold

- (a) There is an arc in α, say $\alpha_{j,s}$, connecting $\partial_{i_j}S$ and $\partial_{i_s}S$.
- (b) There is a component in $S_{g,p+q} - \alpha$, say $T_{j,s}$, which has an arc starting at $\partial_{i_j}S$, say α_j, and and an arc starting at $\partial_{i_s}S$, say α_s, as part of its boundary. Let $v_{j,s}$ denote a marked point in the interior $T_{j,s}$.

If a holds, then Γ_α must have an edge $E_{j,s}$ constructed by crossing $\alpha_{j,s}$. The edge $E_{j,s}$, belongs to the i_j-th and i_s-th boundary cycles i.e. c_j and c_s intersect at the edge $E_{j,s}$. If b holds, then there must be an edge E_j (respectively E_s) constructed by crossing the boundary of $T_{j,s}$ at α_j (respectively α_s) and connecting to $v_{j,s}$. Moreover, the edges E_j (resp. E_s) belongs to the i_j-th (resp. i_s-th) boundary cycles i.e. c_j and c_s intersect at the point $v_{j,s}$. Finally, notice that if two outgoing boundary cycles of Γ_α intersect at an edge (respectively at a point) then condition a (respectively b) hold on α. Therefore condition i is equivalent to saying that all c_j's are disjoint.

We will show now that condition ii is equivalent to saying that the map c_j does not intersect itself at an edge. If ii does not hold, then there must be and arc α_j in α that starts and ends at $\partial_{i_j}S$. Let E_j be its corresponding edge on Γ_α. Recall that α_j and E_j cross exactly once. Then α_j starts on one side of E_j crosses to the other side at the intersection point and then returns to the initial side without any additional crossing. This means that both sides of E_j belong to the same boundary cycle i.e. c_j intersects itself on the edge E_j. The inverse assertion follows similarly.

Assume condition ii holds in α for some $1 \leq j \leq k$. Then c_j does not intersect itself at an edge, but it could still intersect itself at a point. Let $\alpha_{j_1}, \alpha_{j_2} \dots \alpha_{j_r}$ be the arcs in α that start or end at y_{i_j}, the marked point of Δ on $\partial_{i_j}S$. Since condition ii holds, each of these arcs must start and end at a different boundary component. The orientation of the surface gives a cyclic ordering of the arcs. Assume that the labelling given above respects this order. Let $A_1, A_2 \dots A_r$ denote the areas of the surface between these arcs in that given order, see Figure 3.4 below. Let Ξ be the smallest subset of α such that: $\alpha_{j_1}, \alpha_{j_2} \dots \alpha_{j_r} \subset \Xi \subset \alpha$

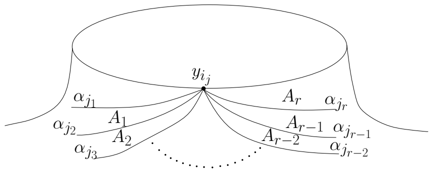


FIGURE 3.4. Local picture at the outgoing closed boundary $\partial_{i_j}S$ for an arc system α in which condition ii of Definition 3.21 holds.

and $\Xi - y_{i_j}$ has a minimum number of connected components. If condition iii holds then $\Xi - y_{i_j}$ is connected. The areas A_s and A_l for $1 \leq s \neq l \leq r$ belong to the same connected component in $S_{g,p+q} - \alpha$, if and only if there is a path in $S_{g,p+q}$ connecting them that does not intersect with α, and this happens if and only if $\Xi - y_{i_j}$ is not connected. Therefore, if iii holds each area A_s contains a different vertex v_s of Γ_α. Moreover, the vertex v_r is connected to the i_j-th leave of Γ_α. Let E_s denote the edge in Γ_α that crosses α_{j_s}, see Figure 3.5. Then the i_j-th boundary corresponds to one side of the edges E_s for $1 \leq s \leq r$ i.e. c_j is an embedding.

If condition iii does not hold then $\Xi - y_{i_j}$ is not connected. Assume for simplicity first that $\Xi - y_{i_j}$ has 2 connected components. Then Ξ must fall in one of the two following cases

(a) The two components of $\Xi - y_{i_j}$ are next to each other in $S_{g,p+q}$ i.e. there is an t such that $\alpha_{j_1}, \alpha_{j_2} \dots \alpha_{j_t}$ belong to one component and $\alpha_{j_{t+1}}, \alpha_{j_{t+2}} \dots \alpha_{j_r}$ to the other (see Figure 3.6.) Then by the argument above each A_s for $1 \leq s \leq t-1$ or $t+1 \leq s \leq r-1$ contains a different vertex v_s of Γ_α. However, A_t and A_r belong to the same connected component in $S_{g,p+q} - \alpha$ so they both contain only one vertex, say v, of Γ_α which is connected to $\partial_{i_j}S$. As before, the i_j-th boundary corresponds to one side of the edges E_s for $1 \leq s \leq r$ but these edges intersect at the point v.

(b) The two components of $\Xi - y_{i_j}$ are nested in $S_{g,p+q}$ i.e. there are $t < l$ such that $\alpha_{j_1}, \alpha_{j_2} \dots \alpha_{j_t - 1}, \alpha_{j_{l+1}}, \alpha_{j_{l+2}} \dots \alpha_{j_r}$ belong to one component and $\alpha_{j_t}, \alpha_{j_{t+1}} \dots \alpha_{j_l}$

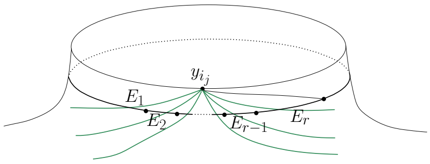


FIGURE 3.5. Local picture at the outgoing closed boundary $\partial_{i_j} S$ for an arc system α in which conditions *ii* and *iii* of Definition 3.21 hold. The arcs are shown in green and their corresponding edges in black.

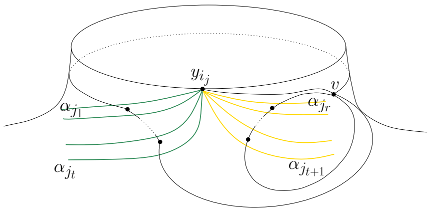


FIGURE 3.6. Local picture at the outgoing closed boundary $\partial_{i_j} S$ for an arc system α in which conditions *ii* holds but condition *iii* does not. The arcs are shown in green and yellow to distinguish the connected components they belong to in Ξ. The picture represents case *a* in which the components are next to each other. The edges corresponding to the arcs are shown in black.

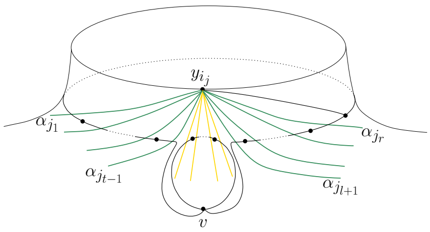


FIGURE 3.7. Local picture at the outgoing closed boundary $\partial_{i_j} S$ for an arc system α in which conditions *ii* holds but condition *iii* does not. The arcs are shown in green and yellow to distinguish the connected components they belong to in Ξ. The picture represents case *b* in which the components are nested. The edges corresponding to the arcs are shown in black.

to the other (see Figure 3.7). Then similarly, each A_s contains a different vertex v_s of Γ_α except for $s = t-1$ and $s = l$, since A_{t-1} and A_l belong to the same connected component in $S_{g,p+q} - \alpha$. So they both contain only one vertex say v of Γ_α. Then as before, the i_j-th boundary cycle intersects itself at v.

The case for more connected components is a combination of these two cases giving that the i_j-th boundary cycle intersects itself in multiple points. Therefore conditions *ii* and *iii* together are equivalent to saying that the map c_j is an embedding, which finishes the proof. □

Proposition 3.24. *Let $S_{g,p+q}$ be an open-closed cobordism that is not a disk, and whose boundary is not completely outgoing closed. The the complex $\mathscr{B}(S_{g,p+q}, \Delta)$ is contractible.*

Before proving the proposition we will state a corollary.

Corollary 3.25. *If $S_{g,p+q}$ is an open-closed cobordism whose boundary is not completely outgoing closed, then poset category $\mathscr{B}_0(S_{g,p+q},\Delta)$ is contractible.*

Proof. For the case of one boundary component, the contractibility of $\mathscr{B}_0(S_{g,p+q},\Delta)$ follows immediately, since this just reduces to the case of $\mathscr{A}_0(S_{g,p+q},\Delta)$. The general case, follows by induction just as in the proof of Proposition 3.20 from the contractibility of $\mathscr{B}(S_{g,p+q},V)$. □

Proof of Proposition 3.24. As in the proof of 3.20, by collapsing all the free boundary components to a point, we can consider $S_{g,p+q}$ to be a cobordism with no free boundary components with a set of marked points V which contains a finite number of marked points in the interior of $S_{g,p+q}$. This implies that $(S_{g,p+q},V)$ is neither a disk with V contained in the boundary nor an annulus with V contained in one boundary component. If $S_{g,p+q}$ has no outgoing boundary component, then the proposition follows directly from the contractibility of the arc complex $\mathscr{A}(S_{g,p+q},V)$. The proof for the general case follows directly as a reduction of Hatcher's proof of the contractibility of the arc complex in [Hat91], so we just give a sketch of this proof.

We consider first the case where $S_{g,p+q}$ has at most one marked point in each boundary component. Hatcher writes a flow of the arc complex $\mathscr{A}(S,V)$ onto the star of a vertex. We will sketch the construction of this flow and see that it restricts to $\mathscr{B}(S_{g,p+q},V)$ if one chooses the vertex correctly; which finishes the proof in this special case since the closure of the star of a vertex is contractible. Since not all the boundary is outgoing closed, we can find an essential arc β that starts and ends at a boundary component which is not outgoing closed. Hatcher construction gives a continuous flow

$$\mathscr{B}(S_{g,p+q},V) \to \overline{Star([\beta])}$$

In order to construct this flow let $\tilde{\sigma}_1 = [\alpha]$ be a $k-$simplex of $\mathscr{B}(S_{g,p+q},V)$ and choose a representatives $\{\alpha_0 \dots \alpha_k\}$ with minimal intersection with β. Let $x_1,\dots,x_l$ denote the intersection points of α and β occurring in that order. The first intersection point x_1 corresponds to an arc α_i. Let $\tilde{\alpha}_{i_1}$ and $\tilde{\alpha}_{i_2}$ be the arcs obtained by sliding α_i along β all the way to the boundary of β, see figure 3.8.

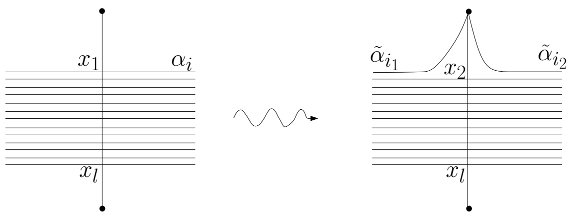


FIGURE 3.8. The arcs $\tilde{\alpha}_{i_1}$ and $\tilde{\alpha}_{i_2}$ obtained by sliding α_i along β

Define σ_2 to be the simplex given by $\tilde{\sigma_1} \cup \tilde{\alpha}_{i_1} \cup \tilde{\alpha}_{i_2}$ and $\tilde{\sigma}_2$ to be the simplex given by replacing α_i with $\tilde{\alpha}_{i_1} \cup \tilde{\alpha}_{i_2}$ in $\tilde{\sigma}_1$. If one of these new arcs is boundary parallel we just discard it, but notice that since there is at most one marked point in each boundary component then at least one of these two arcs is not boundary parallel. Since β doesn't intersect with any outgoing closed boundary component, we can see that this construction preserves conditions *i-iii* of definition 3.21 i.e. σ_2 and $\tilde{\sigma}_2$ are admissible arc sets. Furthermore, σ_2 contains $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ as faces and this last simplex intersects β only at $x_2,\dots,x_l$. In this way we can define a sequence of simplices in $\mathscr{B}(S_{g,p+q}, V)$

$$\begin{array}{ccccccccccc} & & \sigma_2 & & \sigma_3 & & & & \sigma_l & & \\ & \nearrow & & \nwarrow \; \nearrow & & \nwarrow & \cdots\cdots & \nearrow & & \nwarrow & \\ \tilde{\sigma_1} & & & \tilde{\sigma_2} & & \tilde{\sigma_3} & & \tilde{\sigma_l} & & & \tilde{\sigma}_{l+1} \end{array}$$

where σ_j contains $\tilde{\sigma}_{j-1}$ and $\tilde{\sigma}_j$ as faces and this last simplex intersects β only at $x_j, \ldots, x_l$. Finally, by construction $\tilde{\sigma}_{l+1}$ is in the closure of the star of β. Thus, we can define a flow $\mathscr{B}(S_{g,p+q}, V) \times I \to \mathscr{B}(S_{g,p+q}, V)$ by use of barycentric coordinates which flows linearly along this finite sequence of simplices and when restricted to a face corresponds to the flow of the face. Moreover, we can also show this flow is well defined. This finishes the proof in the special case. Now, to consider the case where there is a boundary component with more than one marked point. It is enough to consider what happens when we add a marked point to the boundary. Let $V' = V \cup p$, where $p \in \partial S_{g,p+q}$. This additional marked point p can not be added to an outgoing closed boundary by the way we have constructed Δ. By using a similar argument as for the case with at most one marked point in the boundary component we can show that if $\mathscr{B}(S_{g,p+q}, V)$ is n connected then $\mathscr{B}(S_{g,p+q}, V')$ is $n+1$ connected. Wahl describes this argument in detail in [Wah08] and we can see that her argument restricts to $\mathscr{B}(S_{g,p+q}, V)$ in a similar way as for the special case. $\square$

4. The Chain Complex of Black ad White Graphs

4.1. The Definition. In [Cos06b], Costello gives a complex which models the mapping class group of open-closed cobordisms. In [WW11], Wahl and Westerland rewrite this complex in terms of fat graphs. In this section we describe this complex as it is defined in [WW11].

Definition 4.1. A *generalized black and white graph* G is a tuple $G = (\Gamma, V_b, V_w, In, Out, Open, Closed)$ where Γ is a fat graph, and V_b, V_w are subsets of V the set of vertices of Γ. We call V_b the set of black vertices and V_w the set of white vertices. The sets, In, Out, $Open$ and $Closed$ are subsets of Γ_L, the set of leafs of Γ. In the tuple G the following must hold

- $V = V_b \sqcup V_w$, all vertices are either black or white
- The subsets $Open$ and $Closed$ are disjoint
- The subsets In and Out are disjoint
- $Out \subset Open$, all the outgoing leaves are open and $In \subset Open \sqcup Closed$ all incoming leaves are either open or closed.
- $In \sqcup Out = Open \sqcup Closed$
- $V_w \cap (Open \sqcup Closed) = \emptyset$
- All black inner vertices are at least trivalent, white vertices are allowed to have valence 1 or 2

Additionally,

- The white vertices are labelled $1, 2, \ldots |V_w|$
- Each white vertex has a choice of a start half edge i.e. the half edges incident at a white vertex are totally ordered not only cyclically ordered.
- The incoming and outgoing leaves are labelled $1, 2, \ldots |In \sqcup Out|$
- A closed leaf is the only labelled leaf on its boundary cycle

We allow degenerate graphs which are either the empty graph, or a corolla with 1 or 2 leaves.

Definition 4.2. A *black and white graph* is a generalized black and white graph in which all the leaves are labelled, except possibly the leaves which are connected to the start of a white vertex, which are allowed to be unlabelled. Figure 4.1 shows an example of a black and white graph.

Remark 4.3. Note that a black and white fat graph with no white vertices is just an open-closed fat graph with no outgoing closed leaves.

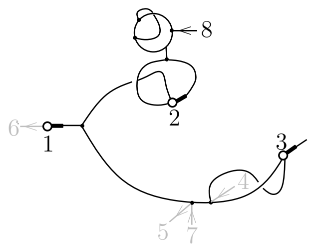


FIGURE 4.1. An example of a black and white fat graph. The incoming and outgoing leaves are marked with arrows. Leaves in black are closed and leaves in grey are open. The start half edges of the white vertices are thickened.

As for open-closed fat graphs, from a black and white fat graph G we can construct an open-closed cobordism $S_{g,p+q}$. First construct a bordered oriented surface Σ_G. To do this, we thicken the edges of G to strips and glue them together at black vertices according to the cyclic ordering. Then, we thicken each white vertex to an annulus and glue to its outer boundary the strips corresponding to the edges attached to it according to the cyclic ordering. We label the inner boundary of the annuli as outgoing closed and order these components by the ordering of the white vertices. We label and order the rest of the boundary of Σ_G in the same way as for open-closed fat graphs. This construction gives and open-closed cobordism well defined up to topological type (see Figure 4.2).

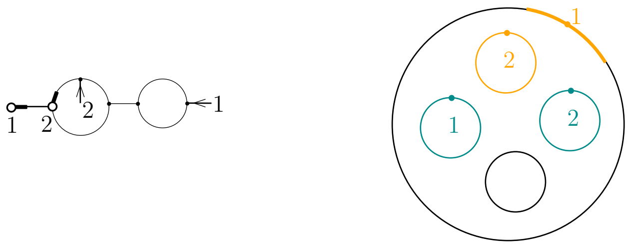


FIGURE 4.2. A black and white fat graph and its corresponding open-closed cobordism. The incoming boundary is shown in yellow and the outgoing boundary in green.

Definition 4.4. An *orientation* of a fat graph Γ is a unit vector in $\det(\mathbb{R}(V \sqcup H))$, this is equivalent to an ordering of the set of vertices and an orientation for each edge.

Definition 4.5. A generalized black and white graph G has an *underlying black and white graph* $\lfloor G \rfloor$ defined as $\lfloor G \rfloor = G$ if G is already a black and white graph i.e. G has no unlabelled leaves that are not connected to the start of a white vertex. On the other hand, let G be a graph with an unlabelled leaf l which is not connected to the start of a white vertex, let e_l denote the edge of l and v_l the other vertex to which e_l is attached. If $|v_l| > 3$, then $\lfloor G \rfloor$ is the empty graph, where $|v_l|$ denotes the valence of v_l. If $|v_l| = 3$, then $\lfloor G \rfloor$ is the graph obtained by forgetting l, e_l and v_l.

If G has an orientation, it induces an orientation on $\lfloor G \rfloor$ which we only need to describe in the case where $\lfloor G \rfloor$ is not G or the empty graph. In this case, let l, e_l and v_l be given as above and let $e_l = \{h_l, \tilde{h}_l\}$ where $s(h_l) = v_l$ and $s(\tilde{h}_l) = l$. Let $s^{-1}(v_l) = (h_1, h_2, h_l)$ occurring in that cyclic ordering. Rewrite the orientation of G as $v_l \wedge h_1 \wedge h_2 \wedge h_l \wedge \tilde{h}_l \wedge l \wedge x_1 \wedge \ldots \wedge x_k$. The induced orientation in $\lfloor G \rfloor$ is $x_1 \wedge \ldots \wedge x_k$.

Definition 4.6 (Edge Collapse)**.** Let G be a (generalized) black and white graph, and e be an edge of G which is neither a loop nor does it connect two white vertices. The *set of edge collapses* G/e is the collection of (generalized) black and white graphs obtained by collapsing e in G and identifying its two end vertices. If both vertices are black we declare the new vertex to be black. If one of the vertices is white, we declare the new vertex to be white with the same label as the white vertex of e.

- *Fat structure* The collapse of e induces a well defined cyclic structure of the half edges incident at the new vertex.
- *Start half edge* If e does not contain the start half edge of a white vertex, then there is a unique black and white fat graph obtained by collapsing e. If e contains the start half edge of a white vertex, there is a finite collection of black and white graphs obtained by collapsing e. Each graph in this collection corresponds to a choice of placement of the start half edge among the half edges incident at the collapsed black vertex. See Figure 4.3 an example of this collection.
- *Orientation* An orientation of G induces an orientation of the elements of G/e as follows. Let $e := \{h_1, h_2\}$, $s(h_1) = v_1$, and $s(h_2) = v_2$. Write the orientation of G as $v_1 \wedge v_2 \wedge h_1 \wedge h_2 \wedge x_1 \wedge \ldots \wedge x_k$. Then the induced orientation of an element of G/e is given by $v \wedge x_1 \wedge \ldots \wedge x_k$.

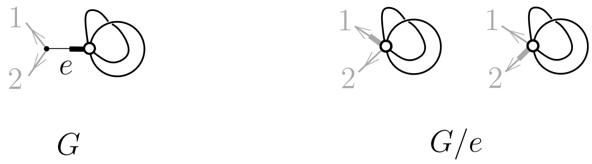


FIGURE 4.3. On the left a black and white fat graph G and to the right its edge collapse set G/e.

Definition 4.7. Let G and $\tilde{G}$ be generalized black and white graphs. We say $\tilde{G}$ is a *blow-up* of G if there is an edge e of $\tilde{G}$ such that $G \in \tilde{G}/e$.

Remark 4.8. Note that the blow-up of a black and white graph is not necessarily a black and white graph again, since the blow might contain unlabelled leaves which are not the start of a white vertex. See Figure 4.4 for an example.

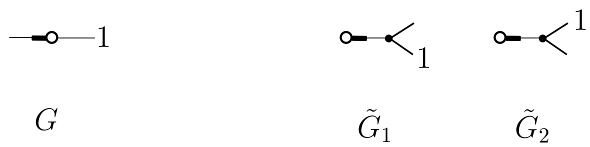


FIGURE 4.4. An example of a black and white fat graph G with blow-ups $\tilde{G}_1$ and $\tilde{G}_2$ that have unlabelled leaves that are not the start edge of a white vertex.

Definition 4.9 (The chain complex of Black and White Graphs)**.** The chain complex of black and white fat graphs $\mathscr{BW} - Graphs$ is the complex generated as a $\mathbb{Z}$ module by isomorphism classes of oriented black and white graphs modulo the relation where -1 acts by reversing the orientation. The degree of a black and white graph G is

$$\deg(G) := \sum_{v\in V_b}(|v|-3) + \sum_{v\in V_w}(|v|-1)$$

where $|v|$ is the valence of the vertex v. The differential of a black and white graph G is

$$d(G) := \sum_{\substack{(\tilde{G},e)\\ G\in\tilde{G}/e}} \lfloor\tilde{G}\rfloor$$

where the sum runs over all isomorphism classes of generalized black and white graphs which are blow-ups of G. Figure 4.5 gives some examples of the differential.

Remark 4.10. In [WW11], it is shown that d is indeed a differential. Note that, since the number of white vertices, and the number of boundary cycles remain constant under blow-ups and edge collapses, the chain complex $\mathscr{BW}-Graphs$ splits into finite chain complexes each of which corresponds to a topological type of open-closed cobordism.

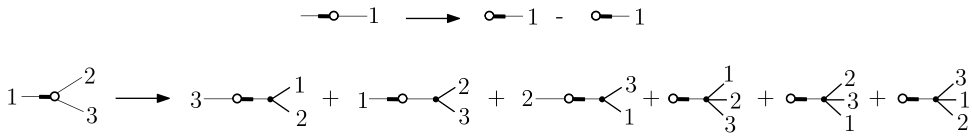


FIGURE 4.5. Differential for two black and white fat graphs. All the labelled leaves are incoming open.

4.2. **Black and white graphs as models for the mapping class group.** Using a partial compactification of the moduli space of open-closed cobordisms, Costello proves that the chain complex $\mathscr{BW}-Graphs$ is a model for the mapping class groups of open-closed cobordisms (cf.[Cos06a, Cos06b]). We give a new proof this result by showing that $\mathscr{BW}-Graphs$ is a chain complex of $|\mathcal{F}at^{ad}|$. In [God07b], Godin gives a CW structure on $|\mathcal{F}at^{oc}|$ which restricts to $|\mathcal{F}at^{ad}|$ in which each p-cell is given by a fat graph $[\Gamma]$ of degree p where

$$\deg([\Gamma]) := \sum_v(|v|-3)$$

and the sum ranges over all inner vertices of $[\Gamma]$ and $|v|$ denotes the valence of v. From this structure, she constructs a chain complex which is the complex generated as a $\mathbb{Z}$ module by isomorphism classes of oriented fat graphs modulo the relation where -1 acts by reversing the orientation. The differential of a fat graph $[\Gamma]$ is

$$d([\Gamma]) := \sum_{\substack{([\tilde{\Gamma}],e)\\ [\Gamma]=[\tilde{\Gamma}/e]}} [\tilde{\Gamma}]$$

While working with Sullivan diagrams, a quotient of $\mathscr{BW}-Graphs$, Wahl gives a natural association that constructs a black and white graph from an admissible fat graph by collapsing the admissible boundary to a white vertex and using the leaf marking the admissible boundaries to mark the start half edge [WW11]. This construction is only well defined in special kind of admissible fat graphs.

Definition 4.11. Let Γ be an admissible fat graph. Γ is *essentially trivalent at the boundary*, if every vertex on the admissible cycles of Γ is trivalent or it has valence 4 and is attached to the leaf marking the admissible cycle.

Remark 4.12. There is a bijection between the set of isomorphism classes of black and white graphs and the set of isomorphism classes of admissible fat graphs which are essentially trivalent at the boundary. To see this, let G be a black and white graph. Construct an admissible fat graph Γ_G by blowing up each white vertex to an admissible cycle. The start half edge of the white vertex gives the position of the leaf marking its corresponding admissible cycle. That is, if the start half edge is an unlabelled leaf, then the leaf of the corresponding admissible cycle in Γ_G is a attached to a trivalent vertex. Otherwise, the leaf corresponding to the admissible cycle is attached to the same vertex to which the start half edge is attached to. Label all the admissible leaves using the labelling of the white vertices in G. The fat graph Γ_G is by construction an admissible fat graph which is essentially trivalent at the boundary. Figure 4.6 shows an example of this construction. In the other direction, given an admissible fat graph Γ which is essentially trivalent at the boundary, construct a black and white fat graph G_Γ by collapsing the admissible boundaries to white vertices and placing the start half edge according to the position of the admissible leaves in Γ. These constructions are clearly inverse to each other.

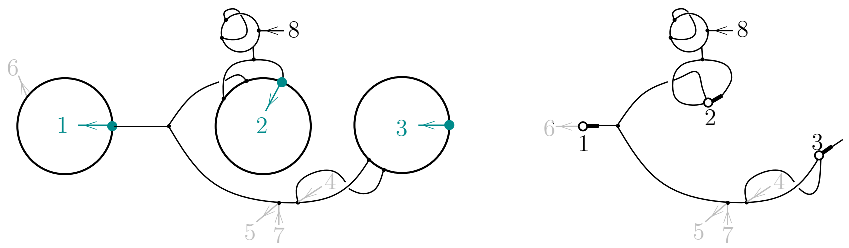


FIGURE 4.6. On the left an admissible fat graph that is essentially trivalent at the boundary and on the right its corresponding black and white graph

However, this natural association does not give a chain map between $\mathscr{BW} - Graphs$ and the chain complex constructed by Godin. To realize this, note that by blowing up white vertices to admissible cycles on a black and white graph, all black vertices remain unchanged i.e. a black vertex of degree n is sent to a black vertex of degree n. However, a white vertex of degree n is sent to an admissible cycle with $n+1$ edges where the sum of the degrees of its vertices is at most 1. Instead of giving a chain map we will construction a filtration

$$\mathcal{F}at^{ad} \ldots \supset \mathcal{F}at^{n+1} \supset \mathcal{F}at^{n} \supset \mathcal{F}at^{n-1} \ldots \mathcal{F}at^{1} \supset \mathcal{F}at^{0}$$

that gives a cell-like structure on $\mathcal{F}at^{ad}$ where the quasi-cells are indexed by black and white graphs i.e. $|\mathcal{F}at^{n}|/|\mathcal{F}at^{n-1}| \simeq \vee S^n$ where the wedge sum is indexed by isomorphism classes of black and white graphs of degree n.

4.2.1. *The Filtration.* In order to give such a filtration we use a mixed degree on $\mathcal{F}at^{ad}$ that considers the vertices away from the admissible cycles as well as the edges on the admissible cycles.

Definition 4.13. Let Γ be an admissible fat graph with k admissible cycles. Let E_a denote the set of edges on the admissible cycles, V_b the set of vertices that do not belong to the admissible cycles, V_a the set of vertices on the admissible cycles which are not attached to an admissible leaf, and $V_{a,*}$ be the set of vertices on the admissible cycles which are attached to an admissible leaf. The *mixed degree of* Γ is

$$\deg^{\mathrm{m}}(\Gamma) := |E_a| - k + \sum_{v \in V_a \cup V_b} (|v| - 3) + \sum_{v \in V_{a,*}} (\max\{0, |v| - 4\})$$

Figure 4.7 shows some examples of admissible fat graphs of mixed degree two.

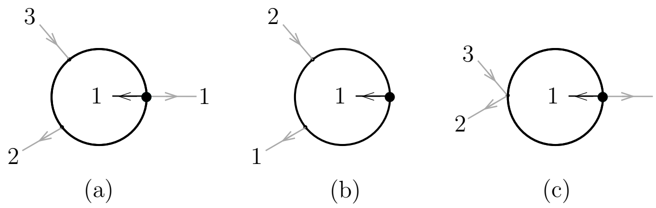


FIGURE 4.7. Three different admissible fat graphs all of mixed degree two. In particular (a) is l_3 and (b) is $\tilde{l}_3$.

Notice that the mixed degree is well defined for isomorphism classes of admissible fat graphs. We will use this degree to describe a filtration of $\mathcal{F}at^{ad}$

Definition 4.14. $\mathcal{F}at^n$ is the full subcategory of $\mathcal{F}at^{ad}$ on objects isomorphism classes of admissible fat graphs $[\Gamma]$ s.t. $\deg^{\mathrm{m}}([\Gamma]) \leq n$.

4.2.2. *The Quasi-cells.* We now describe the quasi-cell corresponding to a black and white graph G.

Definition 4.15. An admissible fat graph $\tilde{\Gamma}$ is a *blow-up* of an admissible fat graph Γ if there is an edge e of $\tilde{\Gamma}$ such that $\Gamma = \tilde{\Gamma}/e$. Furthermore, $\tilde{\Gamma}$ is a *blow-up away from the admissible boundary* if e does not belong to an admissible cycle in $\tilde{\Gamma}$. If e contains a vertex on an admissible cycle but does not belong to one we say $\tilde{\Gamma}$ is obtained from Γ by *pushing away from the admissible cycles.* Finally, $\tilde{\Gamma}$ is a *blow-up at the admissible boundary* if e belongs to an admissible cycle in $\tilde{\Gamma}$.

Definition 4.16. A white vertex on a black and white graph is called *generic* if all its leaves are labelled and *suspended* otherwise. Similarly, an admissible cycle C in a graph which is essentially trivalent at the boundary is called *generic* if the vertex connected to the admissible leaf has valence at least 4 and *suspended* otherwise.

Definition 4.17. We define the following full subcategories of $\mathcal{F}at^{ad}$

- For $n \geq 3$, $\mathscr{T}_n$ is the full subcategory o $\mathcal{F}at^{ad}$ on objects trees with n leaves $\{1, 2, \ldots n\}$ occurring in that cyclic order.
- Let l_n be the admissible fat graph of mixed degree $n-1$ with one admissible boundary cycle which consists of n edges, together with n leaves labelled $\{1, 2 \ldots n\}$ attached to it in that cyclic order, such that leaf 1 is attached to the vertex connected to the admissible leaf, see Figure 4.7 (a). $\mathscr{L}_n$ is the full subcategory of $\mathcal{F}at^{ad}$ on objects l_n and all admissible fat graphs $[\Gamma]$ obtained from l_n by collapsing edges at the admissible cycles and blow-ups away from the admissible cycles. See Figure 4.8 for an example.
- Let $\tilde{l}_n$ be the admissible fat graph of mixed degree $n-1$ with one admissible boundary cycle which consists of n edges and $n-1$ leaves labelled $\{1, 2 \ldots n\}$ attached to it in that cyclic ordering such that there is no leaf attached to the vertex connected to the admissible leaf, see Figure 4.7 (b). $\widetilde{\mathscr{L}_n}$ is the full subcategory of $\mathcal{F}at^{ad}$ on objects $\tilde{l}_n$ and all admissible fat graphs $[\Gamma]$ obtained from $\tilde{l}_n$ by collapsing edges at the admissible cycles and blow-ups away from the admissible cycles. See Figure 4.9 for an example.

Definition 4.18. Let G be a black and white graph, V_b be the set of its black vertices, V_g be the set of generic white vertices and V_s be the set of suspended white vertices. The *quasi-cell* of G is the category

$$\mathscr{E}_G \cong \prod_{v \in V_b} \mathscr{T}_{|v|} \times \prod_{v \in V_g} \mathscr{L}_{|v|} \times \prod_{v \in V_s} \widetilde{\mathscr{L}}_{|v|}$$

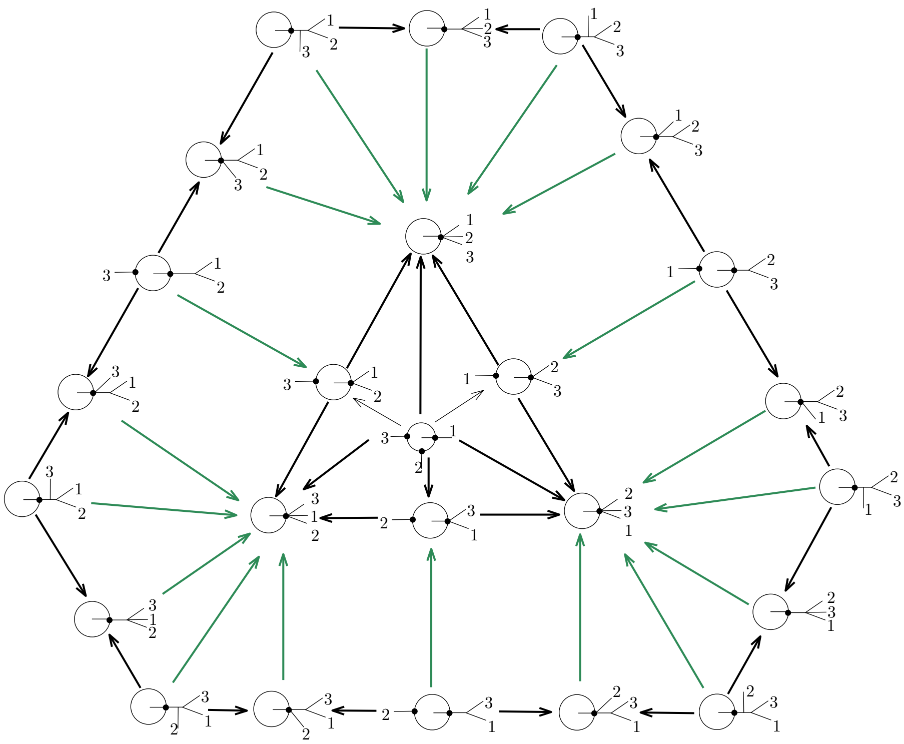


FIGURE 4.8. The category $\mathscr{L}_3$. The arrows in green indicate the deformation retraction onto the core $\mathscr{C}_3$

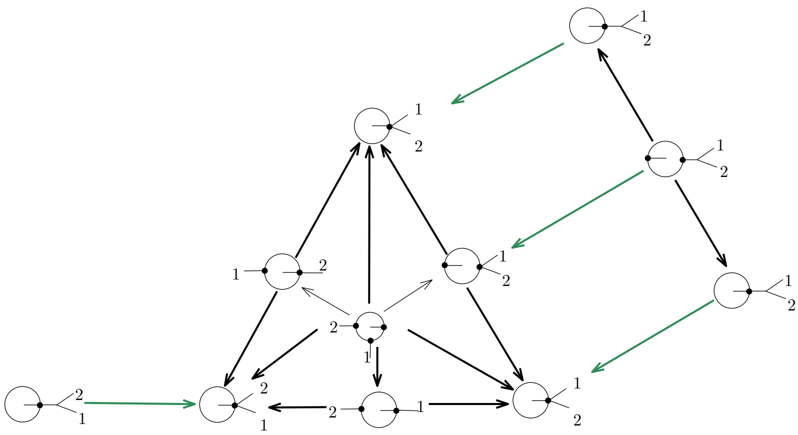


FIGURE 4.9. The category $\widetilde{\mathscr{L}_3}$. The arrows in green indicate the deformation retraction onto the core $\widetilde{\mathscr{C}_3}$

Remark 4.19. In [God07b], Godin shows that $|\mathscr{T}_n|$ is homeomorphic to a disk D^{n-3}. In fact by choosing a root of the trees in $\mathscr{T}_n$ we can show that $|\mathscr{T}_n|$ is a Stasheff polyhedron or associahedron whose vertices are given by different ways in which we can bracket a product of $n-1$ variables.

Definition 4.20. The *core of* $\mathscr{L}_n$ which we denote $\mathscr{C}_n$ is the full subcategory of $\mathscr{L}_n$ on objects obtained from l_n by edge collapses. Similarly the *core of* $\widetilde{\mathscr{L}_n}$ which we denote $\widetilde{\mathscr{C}_n}$ is the full subcategory of $\widetilde{\mathscr{L}_n}$ on objects obtained from $\tilde{l}_n$ by edge collapses.

Remark 4.21. Note that $\mathscr{C}_n$ is the full subcategory of $\mathscr{L}_n$ on objects admissible fat graphs of mixed degree n.

Definition 4.22.

- The *boundary of* $\mathscr{C}_n$ *(respectively* $\widetilde{\mathscr{C}_n}$*)*, which we denote $\partial\mathscr{C}_n$ (respectively $\partial\widetilde{\mathscr{C}_n}$), is the full subcategory of $\mathscr{C}_n$ (resp. $\widetilde{\mathscr{C}_n}$) on objects different from l_n (respectively $\tilde{l}_n$).
- The *interior of the realization of* $\mathscr{C}_n$, is the subspace
$$int(|\mathscr{C}_n|) = |\mathscr{C}_n| - |\partial\mathscr{C}_n|$$
similarly
$$int(|\widetilde{\mathscr{C}_n}|) = |\widetilde{\mathscr{C}_n}| - |\partial\widetilde{\mathscr{C}_n}|$$

Lemma 4.23. *The nerve* $N\mathscr{C}_n$ *is isomorphic to the barycentric subdivision of* $\Delta[n-1]$*, thus* $|\mathscr{C}_n|$ *is homeomorphic to* Δ^{n-1}*. The interior of the realization of* $\widetilde{\mathscr{C}_n}$ *is homeomorphic to the interior of* Δ^{n-1} *i.e.* $int(|\widetilde{\mathscr{C}_n}|) \cong int(\Delta^{n-1})$*.*

Proof. Note first that the fat structure together with the admissible leaf induce an ordering of the vertices on the admissible cycles of l_n and $\tilde{l}_n$, where the first vertex is the vertex connected to the admissible leaf. In the case of $\mathscr{C}_n$, for $0 \leq i \leq n-1$, let e_i denote the edge connecting the vertices $(i-1)$ and i. Let $[n]$ denote the set $\{0, 1, \dots n\}$. It is enough to show that $\mathscr{C}_n$ is isomorphic to the poset category $\mathscr{P}([n-1])$. Note that the fat structure and the labelling of the leaves gives that for any object $[\Gamma]$ in $\mathscr{C}_n$ there is a unique morphism $l_n \to [\Gamma]$. Therefore $\mathscr{C}_n$ is isomorphic to the undercategory $l_n/\mathscr{L}_n$. An object $\beta : l_n \to [\Gamma]$ in $l_n/\mathscr{L}_n$, is uniquely determined by a set of edges on the admissible cycle $\zeta_\beta := \{e_{\beta_1}, e_{\beta_2}, \dots e_{\beta_r}\}$ whose union is not the entire boundary cycle. For the object given by the identity, the set ζ_{id} is the empty set. We define a functor $\Phi : l_n/\mathscr{L}_n \to \mathscr{P}([n-1])$ on objects by $\Phi(\beta) := \{0, 1, \dots, n-1\} - \{\beta_1, \beta_2, \dots \beta_r\}$ this induces a natural map on morphisms and it is easy to see that it is an isomorphism.

In the case of $\widetilde{\mathscr{C}_n}$, for $0 \leq i \leq n-1$ let e_i denote the edge connecting the vertices i and $(i+1)$. Then the argument above shows that $\tilde{l}_n/\widetilde{\mathscr{C}_n}$ is isomorphic to $\mathscr{P}[n-1]$. However, the forgetful functor $F : \tilde{l}_n/\widetilde{\mathscr{C}_n} \to \widetilde{\mathscr{C}_n}$ is injective on morphisms but not on objects. To see this, let $\zeta_{\beta_1} := \{e_1, e_2, \dots, e_{n-1}\}$ and let $\zeta_{\beta_2} := \{e_0, e_1, \dots, e_{n-2}\}$, then $F(\beta_1) = F(\beta_2)$. Therefore the realization of $\widetilde{\mathscr{C}_n}$ is not homeomorphic to the simplex. However, the geometric realization of F induces a map $|F| : |\tilde{l}_n/\widetilde{\mathscr{C}_n}| \cong |\mathscr{P}[n-1]| = \Delta^{n-1} \to |\widetilde{\mathscr{C}_n}|$ which is injective on the interior of the simplex. □

Definition 4.24. Let Γ be an admissible fat graph, V_a be the set of vertices on the admissible cycles which are not attached to an admissible leaf, and $V_{a,*}$ be the set of vertices on the admissible cycles which are attached to an admissible leaf. Let ξ_Γ be the set
$$\xi_\Gamma := \{v \in V_a | |v| > 3\} \cup \{v \in V_{a,*} | |v| > 4\}$$
We can construct from Γ, an admissible fat graph which essentially trivalent at the boundary, which we denote $\hat{\Gamma}$, by pushing out all the vertices of ξ_Γ i.e. by blow-ups away from the admissible boundary given by a single edge on each of the vertices of ξ_Γ. We call this procedure *making the graph* Γ *essentially trivalent*. Note that his procedure is well defined on isomorphism classes of fat graphs.

Definition 4.25. The *black and white degree of an admissible fat graph* $[\Gamma]$ is

$$\deg^{\mathrm{bw}}([\Gamma]) := \deg(G_{\hat{\Gamma}})$$

where $[\hat{\Gamma}]$ is the graph obtained by making $[\Gamma]$ essentially trivalent, $G_{\hat{\Gamma}}$ is the black and white graph corresponding to $\hat{\Gamma}$ under the isomorphism given in 4.12 by collapsing admissible boundaries to white vertices, and deg is the degree on black and white graphs.

We define a few special subcategories of the building blocks of a quasi-cell.

Definition 4.26.

- The *boundary of* $\mathscr{T}_n$, $\mathscr{L}_n$ *and* $\widetilde{\mathscr{L}_n}$ which we denote $\partial\mathscr{T}_n$, $\partial\mathscr{L}_n$ and $\partial\widetilde{\mathscr{L}_n}$, are the full subcategories of respectively $\mathscr{T}_n$, $\mathscr{L}_n$ and $\widetilde{\mathscr{L}_n}$ on objects of mixed degree $k < n$.
- The *thick boundary of* $\mathscr{T}_n$, $\mathscr{L}_n$ *and* $\widetilde{\mathscr{L}_n}$ which we denote $\eth\mathscr{T}_n$, $\eth\mathscr{L}_n$ and $\eth\widetilde{\mathscr{L}_n}$, are the full subcategories of respectively $\mathscr{T}_n$, $\mathscr{L}_n$ and $\widetilde{\mathscr{L}_n}$ on objects of black and white degree $k < n$.

Remark 4.27. Note that $\eth\mathscr{T}_n = \partial\mathscr{T}_n$. Moreover, note that $|\eth\mathscr{L}|$ intersects $|\mathscr{L}_n|$ exactly at the boundary of the core $|\partial\mathscr{C}_n|$, and similarly, $|\eth\widetilde{\mathscr{L}}|$ intersects $|\widetilde{\mathscr{L}_n}|$ exactly at the boundary of the core $|\partial\widetilde{\mathscr{C}_n}|$.

We now construct functors $P : \mathscr{L}_n \to \mathscr{C}_n$ and $\widetilde{P} : \widetilde{\mathscr{L}_n} \to \widetilde{\mathscr{C}_n}$. For an object $[\Gamma]$ of $\mathscr{L}_n$, let F_Γ denote the subforest of all edges that are not on the admissible cycles and are not connected to a leaf. We define the functor P on objects by $[\Gamma] \mapsto [\Gamma/F_\Gamma]$. This induces a natural map on morphisms. To see this, let $\psi_F : [\Gamma] \to [\Gamma/F]$ be a morphism in $\mathscr{L}_n$ and note that $[\Gamma/(F \cup F_\Gamma)] = [(\Gamma/F)/(F_{\Gamma/F})]$. We define $\widetilde{P}$ similarly, see Figures 4.8 and 4.9.

Lemma 4.28. *The functors P and $\widetilde{P}$ induce maps $|P| : (|\mathscr{L}_n|, |\partial\mathscr{L}_n|) \to (|\mathscr{C}_n|, |\partial\mathscr{C}_n|)$ and $|\widetilde{P}| : (|\widetilde{\mathscr{L}_n}|, |\partial\widetilde{\mathscr{L}_n}|) \to (|\widetilde{\mathscr{C}_n}|, |\partial\widetilde{\mathscr{C}_n}|)$ which is are homotopy equivalences of pairs.*

Proof. In this proof we always use isomorphism classes of graphs, but we exclude the brackets from the notation, to avoid clutter. Note that the objects of $\mathscr{C}_n$ have no edges which are not on the admissible cycles or connected to a leaf, thus P is the identity on objects of the core. Therefore, P restricts to a functor $p := P| : \partial\mathscr{L}_n \to \partial\mathscr{C}_n$. We show first that $|P|$ is a homotopy equivalence. Let ι denote the inclusion functor $\iota : \mathscr{C}_n \hookrightarrow \mathscr{L}_n$. It is clear that $P \circ \iota = id_{\mathscr{C}_n}$. On the other hand, we have a natural transformation $\eta : id_{\mathscr{L}_n} \implies \iota \circ P$ given by $\eta_\Gamma : \Gamma \to \Gamma/F_\Gamma$. So $|P|$ is a homotopy equivalence. Note that, $\eta_\Gamma = id_\Gamma$ for $\Gamma \in \mathscr{C}_n$. Therefore, $|\eta|$ is a strong deformation retraction of $\mathscr{L}_n$ onto its core. This argument depends only on the fact that there is a unique morphism $\Gamma \to P(\Gamma)$. We will this idea several times in what comes next.

The functor p, pushes $\partial\mathscr{L}_n$ onto $\partial\mathscr{C}_n$. We define a notion of depth, and show that p is the composition of $n-1$ functors which sequentially push in the graphs according to their depth and that each functor induce a homotopy equivalence on realizations. Let Γ be an object of $\mathscr{L}_n$. The *depth of* Γ is

$$\mathrm{depth}(\Gamma) := |E_a|$$

where E_a is the set of edges on the admissible cycle. Recall that $|\mathscr{C}_n|$ is the barycentric subdivision of $\Delta[n-1]$, and thus we can interpret an object Γ in $\mathscr{C}_n$ as representing a face of $\Delta[n-1]$ of a certain dimension. We call this the *dimension of* Γ and denote it $\dim(\Gamma)$. For $1 \le i \le n$ we define a category X_i to be the full subcategory of $\eth\mathscr{L}_n$ on objects:

- $\Gamma \in \partial\mathscr{L}_n$ such that $\mathrm{depth}(\Gamma) \ge i$
- $\Gamma \in \partial\mathscr{C}_n$ such that Γ represents a face of $|\mathscr{C}_n|$ of dimension $\le n-2$

Note that for $\Gamma \in \partial\mathscr{L}_n$, it holds that $1 \le \mathrm{depth}(\Gamma) \le n-1$. Therefore, $X_1 = \partial\mathscr{L}_n$ and $X_n = \partial\mathscr{C}_n$. For $1 \le i \le n-1$ we define a functors $\psi_i : X_i \to X_{i+1}$ on objects by:

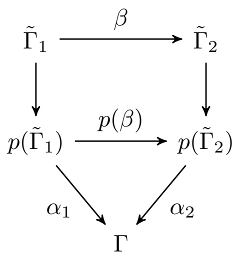
$$\psi_i(\Gamma) := \begin{cases} p(\Gamma) & \Gamma \in \partial\mathscr{L}_n,\ \text{depth}(\Gamma) = i, \\ \Gamma & \text{else} \end{cases}$$

with the natural map induced on morphisms. Thus, we have a sequence of functors

$$\partial\mathscr{L}_n = X_1 \xrightarrow{\psi_1} X_2 \xrightarrow{\psi_2} \dots Xn - 1 = \xrightarrow{\psi_{n-1}} X_n = \partial\mathscr{C}_n$$

and it clearly holds that $p := \psi_{n-1} \circ \dots \psi_2 \circ \psi_1$.

The comma category ψ_i/Γ has objects $(\tilde{\Gamma}, \alpha)$ where $\tilde{\Gamma} \in X_i$ and $\alpha : p(\tilde{\Gamma}) \to \Gamma$ is a morphism in X_{i+1}. Morphisms from $(\tilde{\Gamma_1}, \alpha_1)$ to $(\tilde{\Gamma_2}, \alpha_2)$ in ψ_i/Γ are given by morphisms β in X_i such that the bottom triangle in diagram 4.1 commutes.



(4.1)

We separate ψ_i/Γ into three different cases

If $\Gamma \in \partial\mathscr{L}_n$: Morphisms of fat graphs are given by collapsing edges. Thus, $\Gamma \in \partial\mathscr{L}_n$, all the graphs and arrows in diagram 4.1 are be objects and morphisms in X_i. Therefore $\psi_i/\Gamma = X_i/\Gamma$ which is a contractible category.

If $\Gamma \in \partial\mathscr{C}_n$, $\dim(\Gamma) \leq i - 2$: For $j = 1, 2$, the graph $p(\tilde{\Gamma_j})$ is a blow-up away from the admissible boundary of the graph $\tilde{\Gamma}_j$. Moreover, the condition on the dimension of Γ implies that $\Gamma \in X_i$. Thus, the existence of morphisms α_j implies that there are morphisms $\tilde{\alpha}_j : \tilde{\Gamma}_j \to \Gamma$ in X_i. Then the category ψ_i/Γ is contractible, since the object (Γ, id_Γ) is terminal.

If $\Gamma \in \partial\mathscr{C}_n$, $\dim(\Gamma) = i - 1$: In this case, Γ is not an object in X_i. However, by the case above, we can see that the objects of ψ/Γ are all blow-ups of Γ together with a map to Γ in X_{i+1}. These collapse maps onto Γ are unique. Therefore, ψ/Γ is the full subcategory of X_i on objects that are are blow-ups of Γ. Let $\mathscr{D}_1$ denote the full subcategory of X_i on objects that are obtained from Γ by blow-ups away from the admissible cycle. Similarly, let $\mathscr{D}_2$ denote the full subcategory of X_i on objects that are obtained from $\hat{\Gamma}$ by blow-ups away from the admissible cycle, where $\hat{\Gamma}$ is the graph obtained by making Γ essentially trivalent at the boundary. Then, we have inclusions of categories

$$\mathscr{D}_2 \hookrightarrow \mathscr{D}_1 \hookrightarrow \psi_i/\Gamma$$

Let $\tilde{\Gamma}$ be an object in ψ_i/Γ which is not an object in $\mathscr{D}_1$. There is a unique morphism $\gamma_{\tilde{\Gamma}}$ in X_{i+1} of the form $\gamma_{\tilde{\Gamma}} : p(\tilde{\Gamma}) \to \Gamma$ and this morphism is given by collapsing edges on the admissible cycle. Note that $\tilde{\Gamma}$ and $p(\tilde{\Gamma})$ have the same structure on the admissible cycle, in particular they have the same number of edges on the admissible cycle. Thus, the map $\gamma_{\tilde{\Gamma}}$ lifts to a unique map $\gamma_{\tilde{\Gamma}*} : \tilde{\Gamma} \to \Gamma'$ where Γ' is an object in $\mathscr{D}_1$. More precisely, the morphism $\gamma_{\tilde{\Gamma}*}$ is given by collapsing the same edges on the admissible cycles of $\tilde{\Gamma}$ that $\gamma_{\tilde{\Gamma}}$ collapses on the admissible cycles of $p(\tilde{\Gamma})$. This defines a functor $G_1 : \psi_i/\Gamma \to \mathscr{D}_1$ that is the identity on objects of $\mathscr{D}_1$ and on all other objects it is given by $\tilde{\Gamma} \mapsto \gamma_{\tilde{\Gamma}*}(\tilde{\Gamma})$. Note that since $\gamma_{\tilde{\Gamma}*}$ is uniquely defined, the same argument used to show that P induces a homotopy equivalence shows that G_1 induces a homotopy equivalence on realizations. Similarly, define a functor $G_2 : \mathscr{D}_1 \to \mathscr{D}_2$ that it is given on objects by $\Gamma \mapsto \widehat{\Gamma'}$ where $\widehat{\Gamma'}$ is the graph obtained from Γ' by making it essentially trivalent at the boundary. Note that G_2 is the identity on objects of $\mathscr{D}_2$ and that there is a unique morphism $\widehat{\Gamma'} \to \Gamma'$. Thus the same argument shows that G_2 induces a homotopy equivalence on realizations.

Finally, we show that $\mathscr{D}_2$, the subcategory on objects that are obtained from $\hat{\Gamma}$ by blow-ups away from the admissible cycle, has a contractible realization. Let $v_1, v_2 \dots v_r$ denote the vertices on the admissible cycle of Γ and let $k_1, k_2, \dots k_r$ denote the number of leaves that are attached at each vertex. Consider the functor Φ

$$\Phi : \mathscr{D}_2 \longrightarrow \prod_{j=1}^{r} \mathscr{T}^{k_j+1}$$

that it is given on objects by $\tilde{\Gamma} \to (T_1, T_2, \dots, T_r)$, where T_j is the tree attached to the vertex v_j of $\tilde{\Gamma}$ and the map on morphisms is defined in a natural way. It is easy to see that Φ induces an isomorphisms of categories. The inverse functor is give by reattaching the trees at the vertices of the admissible cycle. Then by Remark 4.19, $\mathscr{D}_2$ is a contractible category and thus so is ψ_i/Γ

Then by Quillen's Theorem A, each ψ_i induce a homotopy equivalence and therefore so does p. The proof for $\widetilde{P}$ follows exactly the same way.

□

We define subcategories and subspaces of the quasi-cell of a black and white graph G.

Definition 4.29. The *core of the quasi-cell of* G is

$$\overline{\mathcal{e}}_G := \prod_{v \in V_b} (\mathscr{T}_{|v|}) \times \prod_{v \in V_g} (\mathscr{C}_{|v|}) \times \prod_{v \in V_s} (\widetilde{\mathscr{C}}_{|v|})$$

The *boundary of the core of the quasi-cell of* G is

$$\partial\overline{\mathcal{e}}_G := \prod_{v \in V_b} (\partial\mathscr{T}_{|v|}) \times \prod_{v \in V_g} (\partial\mathscr{C}_{|v|}) \times \prod_{v \in V_s} (\partial\widetilde{\mathscr{C}}_{|v|})$$

The *boundary of the quasi-cell of* G is

$$\partial\mathscr{E}_G \cong \prod_{v \in V_b} \partial\mathscr{T}_{|v|} \times \prod_{v \in V_g} \partial\mathscr{L}_{|v|} \times \prod_{v \in V_s} \partial\widetilde{\mathscr{L}}_{|v|}$$

The *thick boundary of the quasi-cell of* G is

$$\eth\partial\mathscr{E}_G \cong \prod_{v \in V_b} \eth\mathscr{T}_{|v|} \times \prod_{v \in V_g} \eth\mathscr{L}_{|v|} \times \prod_{v \in V_s} \eth\widetilde{\mathscr{L}}_{|v|}$$

The *open quasi-cell of* G is

$$\mathcal{e}_G := \prod_{v \in V_b} int(|\mathscr{T}_{|v|}|) \times \prod_{v \in V_g} int(|\mathscr{C}_{|v|}|) \times \prod_{v \in V_s} int(|\widetilde{\mathscr{C}}_{|v|}|)$$

Corollary 4.30. *There is a functor* $P_G : \mathscr{E}_G \to \overline{\mathcal{e}}_G$ *that after realization, induces a homotopy equivalence of pairs*

$$|P_G| : (|\mathscr{E}_G|, |\partial\mathscr{E}|) \to (|\overline{\mathcal{e}}_G|, \partial|\overline{\mathcal{e}}_G|)$$

Proof. This follows immediately from Lemma 4.28. The functor P_G is obtained by using P and $\widetilde{P}$ on the components of $\mathscr{E}_G$. □

Remark 4.31. Let Γ_G denote the the fat graph corresponding to a black and white graph G. Consider l_n as a black and white graph. It is easy to see that for any G in the differential of l_n, the graph Γ_G, is obtained from l_n by collapsing m consecutive edges in the admissible cycle for $1 \leq m \leq n-1$ and then making the graph essentially trivalent. Similarly, consider $\tilde{l}_n$ as a black and white graph. It is easy to see that for any G in the differential of $\tilde{l}_n$, Γ_G is obtained from $\tilde{l}_n$ by collapsing m consecutive edges in the admissible cycle that do not contain the admissible leaf for $1 \leq m \leq n-2$ and then making the graph essentially trivalent or by collapsing an edge that contains the admissible leaf.

Remark 4.32. We have shown that $|\mathscr{L}_n|$ is an $n-1$ disk whose boundary is a sphere which is given by quasi-cells corresponding to the black and white graphs G in the differential of l_n. In an analogous way than for the category $\mathscr{T}_n$, we can interpret the graphs in the differential of l_n as meaningful bracketings on n variables arranged in a circle using one parenthesis. Thus $\mathscr{L}_n$ is a realization of the cyclohedron.

4.2.3. *The Cell-like structure on Admissible Fat Graphs.* We now use the quasi-cells described in the previous subsection to give a cell like structure on $\mathcal{F}at^{ad}$.

Definition 4.33. Let G be a black and white graph of degree n. We will define a functor

$$\varphi_g : \mathscr{E}_G \to \mathcal{F}at^n$$

Let H denote the set of half edges of G and V_b the set of black vertices. Choose a fixed but arbitrary ordering of V_b, and for each $v \in V_b$ choose a fixed but arbitrary start half edge. Then we can describe H as $H := \amalg_{1\leq i\leq |V_b|} H_i$, where H_i is the subset of half edges attached at the i-th vertex. Note that the cyclic ordering and the start half edges give a total ordering of the sets H_i. Let $v_{l_1}, v_{l_2}, \dots v_{l_s}$ denote the generic white vertices of G ordered by their labelling and $v_{j_1}, v_{j_2}, \dots v_{j_t}$ denote the suspended white vertices of G ordered by their labelling. Cut in half all the edges of G and complete each half edge $h \in H_i$ to a leaf labelled by the label of h in the total ordering of H_i. This gives a disjoint union of corollas on black and white vertices and m chords, where the chords correspond to the leaves of G. Blow up the white vertices to admissible cycles. This gives a tuple of graphs

$$\alpha_G := (T_{G_1}, T_{G_2}, \dots T_{G_{|V_b|}}, \Gamma_{G_{l_1}}, \Gamma_{G_{l_2}}, \dots \Gamma_{G_{l_s}}, \Gamma_{G_{j_1}}, \Gamma_{G_{j_2}}, \dots \Gamma_{G_{j_t}})$$

where T_{G_i} is the corolla corresponding to i-th black vertex, $\Gamma_{G_{l_i}}$ is $l_{|v_{l_i}|}$, and $\Gamma_{G_{j_i}}$ is $\tilde{l}_{|v_{j_i}|}$. Note that α_G is an object of $\mathscr{E}_G$. Let (i,j) denote the j-th leave of the i-th graph of α_G and let $\{(i_1,j_1),(i_2,j_2),\dots(i_m,j_m)\}$ be the leaves of α_G that correspond to leaves in G. This procedure gives an involution

$$\iota : \bigcup_{i,j}(i,j) - \bigcup_{l=1}^{m}(i_l,j_l) \to \bigcup_{i,j}(i,j) - \bigcup_{l=1}^{m}(i_l,j_l)$$

given by the involution in H which attaches its half edges and a bijection

$$g : \{1,2,\dots m\} \to \bigcup_{l=1}^{m}(i_l,j_l)$$

given by the labelling of the leaves of G. Let

$$\alpha := (T_1, T_2, \dots T_{|V_b|}, \Gamma_{l_1}, \Gamma_{l_2}, \dots \Gamma_{l_s}, \Gamma_{j_1}, \Gamma_{j_2}, \dots \Gamma_{j_t})$$

be an object in $\mathscr{E}_G$. Then we define $\varphi_G(\alpha)$ to be the graph obtained from α by glueing together the leaves of α according to ι and then forgetting the attaching vertex so that the graph obtained has inner vertices of valence at least 3, and then label the remaining leaves of α according to g. Notice that $\varphi_G(\alpha)$ has mixed degree at most n and that $\varphi_G(\alpha_G)$ is the admissible fat graph obtained from G by blowing up its white vertices as shown in 4.12. The functor is naturally defined on morphisms since morphisms in $\mathscr{E}_G$ and $\mathcal{F}at^n$ are given by collapses of inner forests that do not contain any leaves.

Lemma 4.34. *Let $\mathcal{F}at^{ad}_{g,p+q}$ and $\mathcal{F}at^{n}_{g,p+q}$ denote the full subcategories of $\mathcal{F}at^{ad}$ and $\mathcal{F}at^{n}$ on fat graphs of topological type $S_{g,p+q}$. Since the category $\mathcal{F}at^{ad}_{g,p+q}$ is finite, then there is an N such that $\mathcal{F}at^{N}_{g,p+q} = \mathcal{F}at^{ad}_{g,p+q}$. If $n \leq N$, $\mathcal{F}at^{n}_{g,p+q}$ is covered by quasi-cells of dimension n i.e. $\bigcup_G |\mathrm{Im}(\varphi_G)| = |\mathcal{F}at^n|$ where the union runs over all isomorphism classes of black and white graphs of degree n and of topological type $S_{g,p+q}$.*

Proof. Let $[\Gamma]$ be an object in $\mathcal{F}at^{n}_{g,p+q}$, we will show there is a G of degree n such that $[\Gamma] \in Im(\varphi_G)$. If $[\Gamma]$ is an admissible fat graph of mixed degree n which is essentially trivalent at the boundary, then $[\Gamma] \in Im(\varphi_{G_\Gamma})$ where G_Γ is the black and white graph corresponding to $[\Gamma]$ as given in 4.12. If $[\Gamma]$ is an admissible fat graph of mixed degree $k < n$ which is essentially trivalent at the boundary. Then, since $n \leq N$, by collapsing edges that do not belong to the admissible cycles and blow ups at the admissible cycles of $[\Gamma]$, we can obtain an graph $[\tilde{\Gamma}]$ which is essentially trivalent at the boundary and of degree n. Then $[\Gamma] \in Im(\varphi_{G_{\tilde{\Gamma}}})$ where $G_{\tilde{\Gamma}}$ is the black and white graph corresponding to $[\tilde{\Gamma}]$. Note that this argument also shows that $Im(\varphi_{G_\Gamma}) \subset Im(\varphi_{G_{\tilde{\Gamma}}})$ where G_Γ is the black and white graph corresponding to $[\Gamma]$. Finally, assume $[\Gamma]$ is not essentially trivalent at the boundary. Note that collapsing an edge on a generic admissible boundary does not change the mixed degree of the graph. Similarly, collapsing an edge on a suspended admissible boundary that does not contain the admissible leave does not change the mixed degree of the graph. Equivalently, blow-ups at an admissible boundary that do not separate the admissible leave do not change the mixed degree of the graph. Therefore, we can blow up $[\Gamma]$ at the admissible boundary to an admissible fat graph $[\tilde{\Gamma}]$ of degree at most n which is essentially trivalent at the boundary. Note that $[\Gamma] \in Im(\varphi_{G_{\tilde{\Gamma}}})$ and we are done on objects by the argument above.

Now we show that given a morphism $\psi_e : [\Gamma] \to [\Gamma/e]$ in $\mathcal{F}at^n$, then $\psi_e \in Im(\varphi_G)$ for some black an white graph G of degree n. If e doe not belong to an admissible cycle, then $\deg^{\mathrm{m}}([\Gamma]) < \deg^{\mathrm{m}}([\Gamma/e])$. Then by the procedure described above, we can construct a graph $[\widetilde{\Gamma/e}]$ such that ψ_e is a morphism in the image of $\mathscr{E}_{G_{\widetilde{\Gamma/e}}}$. Similarly, if e is an edge on an admissible cycle then $\deg^{\mathrm{m}}([\Gamma]) \geq \deg^{\mathrm{m}}([\Gamma/e])$ and thus there is a graph $[\tilde{\Gamma}]$ such that ψ_e is a morphism in the image of $\mathscr{E}_{G_{\tilde{\Gamma}}}$. Similarly, for a general k-simplex $\xi := [\Gamma_0] \to [\Gamma_1] \ldots \to [\Gamma_k]$, we choose a vertex of ξ say $[\Gamma_l]$ such that it has maximum degree in ξ, this is not a unique choice. Then by the procedure described above, we can construct a graph $[\widetilde{\Gamma_l}]$ such that ξ is contained in the image of $\mathscr{E}_{G_{\widetilde{\Gamma_l}}}$. □

Remark 4.35. Let G be a black and white graph of degree n and let Γ_G be its corresponding admissible fat graph. By remark 4.31, for any $\tilde{G}$ in the differential of G its corresponding admissible fat graph $\Gamma_{\tilde{G}}$ is obtained from Γ_G by one of the following procedures:

- A blow-up at a vertex that does not belong to an admissible cycle
- Collapsing consecutive edges on an admissible cycle that do not contain a trivalent vertex connected to the admissible leaf, and then making the graph essentially trivalent.
- Collapsing an edge on an admissible cycle that contains a trivalent vertex connected to the admissible leaf.

Note then that each Γ_{G_i} is an admissible fat graph of mixed degree $n-1$ which is essentially trivalent at the boundary which is obtained from Γ by collapses at the admissible cycles and expansions away from the admissible cycles. Notice moreover, that any graph Γ' of mixed degree $k < n$ that is obtained from Γ by collapses at the admissible cycles and expansions away from the admissible cycles can be obtained in this way from some Γ_{G_i}. Therefore, the argument of the proof of the lemma above gives

$$|\varphi_G(\partial \mathscr{E}_G)| = \bigcup_i |\varphi_{G_i}(\mathscr{E}_{G_i})|$$

We know that $|\mathcal{F}at^n|$ is covered by quasi-cells of dimension n, now we want to show that they sit together nicely inside this space. Recall that $\boldsymbol{e}_G$ is the interior of the core of the quasi-cell $\mathscr{E}_G$.

Lemma 4.36. *Let G and G' be different isomorphism classes of black and white graphs of degree n. Then the following hold*

- *The restriction $\varphi_G|_{\boldsymbol{e}_G} : \boldsymbol{e}_G \to |Fat^n|$ is injective*
- *The image of $\boldsymbol{e}_G$ is disjoint from the image of $\boldsymbol{e}_{G'}$ i.e.* $\mathrm{Im}(\varphi_G|_{\boldsymbol{e}_G}) \cap \mathrm{Im}(\varphi_{G'}|_{\boldsymbol{e}_{G'}}) = \emptyset$

Proof. Note that the functor $\varphi_G : \mathscr{E}_G \to \mathcal{F}at^n$ is not necessarily injective on objects. Let $[\Gamma]$ be an object in $\mathcal{F}at^n$ of mixed degree n which is essentially trivalent at the boundary. By the bijection of 4.12, there is a unique black an white graph G_Γ corresponding to $[\Gamma]$, and thus $[\Gamma]$ lies only on the image of $\mathscr{E}_{G_\Gamma}$. Moreover, there is a unique object of $\mathscr{E}_{G_\Gamma}$ in the preimage of $[\Gamma]$, namely α_{G_Γ}, where α_{G_Γ} is given by cutting edges of G_Γ as given in definition 4.33. Consider the map induced by φ on the k-nerve of the core i.e. the map $N_k\varphi : N_k\overline{\boldsymbol{e}}_G \longrightarrow N_k\mathcal{F}at^n$ which sends $\zeta := (\alpha_0 \to \ldots \to \alpha_k) \mapsto \xi := ([\Gamma_0] \to \ldots \to [\Gamma_k])$. If the simplex ξ intersects the image of $\boldsymbol{e}_G$, then there is an $l \leq k$ such that $[\Gamma_l]$ is essentially trivalent at the boundary and $\deg^{\mathrm{m}}([\Gamma_k]) = n$. This implies that α_l is in the interior of the core, and since the interior of the core is a disk, there is a unique simplex defined by $(\alpha_l \to \alpha_{l+1} \to \ldots \to \alpha_k)$ which maps to the simplex $([\Gamma_l] \to [\Gamma_{l+1}] \to \ldots \to [\Gamma_k])$. Moreover, the image of the simplex defined by $\alpha_0 \to \alpha_1 \ldots \to \alpha_{l-1}$ does not intersect the image of $\boldsymbol{e}_G$. Therefore the map $\varphi_G|_{\boldsymbol{e}_G}$ is injective. The image of $\boldsymbol{e}_G$ is disjoint from the image of $\boldsymbol{e}_{G'}$ for any G' different than G by the same argument. □

Remark 4.37. The functor $\varphi_G : \mathscr{E}_G \to \mathcal{F}at^n$ is not necessarily injective on objects. If φ_G is not injective on objects of mixed degree $k \leq n-1$, then $|\varphi_G(\mathscr{E}_G)|$ intersects itself at the boundary of the quasi-cell. On the other hand if $|\mathrm{Im}(\varphi_G)|$ self intersects on the interior, then it must self intersect already at the boundary of the core i.e. there must be α_1, α_2 in $\mathscr{E}_G$ such that $\varphi_G(\alpha_1) = \varphi_G(\alpha_2)$ and $\deg^{\mathrm{m}}(\varphi_G(\alpha_1)) = n$. If this happens, then α_1 and α_2 are in a way symmetric, in the sense that they only differ from each other on the numbering of their leaves, since the same graph is obtained from both configurations by attaching their leaves through the functor φ. Therefore, for each morphism in the thick boundary $\psi_{i_1} : \alpha_{i_1} \to \alpha_1$ in $\eth\mathscr{E}_G$ there is exactly one morphism $\psi_{i_2} : \alpha_{i_2} \to \alpha_2$ in $\eth\mathscr{E}_G$ such that $\varphi_G(\psi_{i_1}) = \varphi_G(\psi_{i_2})$. That is if $|\mathrm{Im}(\varphi_G)|$ self intersects on the interior, then it self intersects at vertices of the boundary of the core and simplicially on all simplices on the thick boundary containing such vertices. The same argument show that if $|\mathrm{Im}(\varphi_G)|$ and $|\mathrm{Im}(\varphi'_G)|$ intersect on their interior, then they intersect at vertices of the boundary of their cores and simplicially on all simplices on the thick boundary containing such vertices.

The following theorem is originally proved by Costello in [Cos06a, Cos06b] by very different methods.

Theorem 4.38. *The chain complex of black and white graphs is a model for the classifying spaces of mapping class groups of open-closed cobordisms. More specifically there is an isomorphism*

$$\mathrm{H}_*(\mathscr{BW} - Graphs) \cong \mathrm{H}_*\left(\coprod_{[S_{g,p+q}]} B\mathrm{Mod}(S_{g,p+q}) \right)$$

where the disjoint union runs over all topological types of open-closed cobordisms where there is at least one boundary component which is not outgoing closed.

Proof. It is enough to show that $\mathscr{BW} - Graphs$ is a chain complex of $|\mathcal{F}at^{ad}|$ since by 3.8, $\mathcal{F}at^{ad}$ is a model for the classifying space of the mapping class group.

We define a chain complex $\mathrm{C}_*^{\mathrm{quasi}}$ using the filtration on $\mathcal{F}at^{ad}$ given by the mixed degree of the graphs i.e. we define $\mathrm{C}_n^{\mathrm{quasi}} := H_n(|\mathcal{F}at^n|, |\mathcal{F}at^{n-1}|)$. Since the quasi-cells of dimension n cover $\mathcal{F}at^n$ and their boundaries cover $\mathcal{F}at^{n-1}$ we have that

$$H_*(|\mathcal{F}at^n|, |\mathcal{F}at^{n-1}|) = H_*\left(\bigcup_G |\varphi_G(\mathscr{E}_G)|, \bigcup_G |\varphi_G(\partial\mathscr{E}_G)|\right)$$

Using Corollary 4.30 we get a functor $\Pi_n : \amalg_G \mathscr{E}_G \to \amalg_G \overline{\boldsymbol{e}}_G$ that induces a homotopy equivalence of pairs

$$|\Pi_n| : (\amalg_G |\mathscr{E}_G|, \amalg_G |\partial\mathscr{E}|) \longrightarrow (\amalg_G |\overline{\boldsymbol{e}}_G|, \amalg_G |\partial\overline{\boldsymbol{e}}_G|)$$

Recall that Π_n is the identity on objects of the core. Then, since the images of the quasi-cells intersect nicely on the thick boundary as mentioned in 4.37, the map $|\Pi_n|$ descends to a map

$$|\pi_n| : (\bigcup_G |\varphi(\mathscr{E}_G)|, \bigcup_G |\varphi(\partial\mathscr{E}|)) \longrightarrow (\bigcup_G |\varphi(\overline{\boldsymbol{e}}_G)|, \bigcup_G |\varphi(\partial\overline{\boldsymbol{e}}_G)|)$$

which is a homotopy equivalence of pairs. Since these are a CW pairs we have that

$$\tilde{H}_*(\bigcup_G |\varphi(\overline{\boldsymbol{e}}_G)|, \bigcup_G |\varphi(\partial\overline{\boldsymbol{e}}_G)|) \cong \tilde{H}_*\left(\frac{\bigcup_G |\varphi(\overline{\boldsymbol{e}}_G)|}{\bigcup_G |\varphi(\partial\overline{\boldsymbol{e}}_G)|}\right)$$

Recall that the interior of the associahedron and the cores are disks as given in 4.19 and 4.23. Therefore, the interior of the core of a quasi-cell $\boldsymbol{e}_G$ is a open disk of dimension n where n is the degree of G as a black and white graph. Moreover, the image of the interiors of the cores of the quasi-cells are non-intersecting in $\mathcal{F}at^n$ as given in Lemma4.36. Therefore,

$$\tilde{H}_*(|\mathcal{F}at^n|, |\mathcal{F}at^{n-1}|) \cong \tilde{H}_*\left(\frac{\bigcup_G |\varphi(\overline{\boldsymbol{e}}_G)|}{\bigcup_G |\varphi(\partial\overline{\boldsymbol{e}}_G)|}\right) \cong \tilde{H}_*(\vee_G S^n)$$

Thus, $\mathrm{C}_n^{\mathrm{quasi}}$ is the free group generated by black and white graphs of degree n. The differential $\mathrm{d}_n^{\mathrm{quasi}} : H_n(|\mathcal{F}at^n|, |\mathcal{F}at^{n-1}|) \to H_{n-1}(|\mathcal{F}at^{n-1}|, |\mathcal{F}at^{n-2}|)$, is given by the connecting homomorphism of the long exact sequence of the triple $(|\mathcal{F}at^n|, |\mathcal{F}at^{n-1}|, |\mathcal{F}at^{n-2}|)$. We can show, see for example [God07b], that a choice of orientation of a black and white graph corresponds to a compatible choice of orientations of the simplices that correspond to its quasi-cell. Thus the differential takes a generator given by an n dimensional quasi-cell, to its boundary in $\mathcal{F}at^{n-1}$ and by 4.35 the boundary of a quasi-cell is given by the union of the quasi-cells corresponding to the differential of G. So the chain complex $\mathrm{C}_*^{\mathrm{quasi}}$ is the chain complex of black and white graphs $\mathscr{BW}-Graphs$.

On the other hand, the same argument that shows that cellular homology is isomorphic to singular homology, gives that $H_n(\mathrm{C}_*^{\mathrm{quasi}}) \cong H_n(|\mathcal{F}at^{ad}|)$ (cf. [Mcc00, 4.13]). We give a brief sketch of this argument. Consider the spectral sequence arising from the filtration of $\mathcal{F}at^{ad}$. The first page is given by $E^1_{p,q} = H_{p+q}(|\mathcal{F}at^p|, |\mathcal{F}at^{p-1}|)$. Since the quotients in the filtration are wedges of spheres we have that

$$H_{p+q}(|\mathcal{F}at^p|, |\mathcal{F}at^{p-1}|) = \begin{cases} \mathrm{C}_p^{\mathrm{quasi}} & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Moreover the d^1 differential is given by the $\mathrm{d}^{\mathrm{quasi}}$ and thus by definition

$$E^2_{p,q} = \begin{cases} H_p(\mathrm{C}_*^{\mathrm{quasi}}) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Since all the terms of E^2 are concentrated on the row $q = 0$ all higher differentials are trivial and $E^2_{p,q} = E^\infty_{p,q}$. Finally, for this spectral sequence $E^\infty_{p,q} \cong H_p(|\mathcal{F}at^{ad}|)$. The easiest way to show that is by considering the argument in each connected component where $\mathcal{F}at^{ad}_{g,p+q}$ is a finite complex and thus the filtration is finite. □

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Paper B

COMPARING COMBINATORIAL MODELS OF MODULI SPACE AND THEIR COMPACTIFICATIONS

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Abstract. In this paper we compare two combinatorial models for the moduli space of two-dimensional cobordisms: Bödigheimer's radial slit configurations and Godin's admissible fat graphs, producing an explicit homotopy equivalence using a "critical graph" map. We also discuss natural compactifications of these two models, the unimodular harmonic compactification and Sullivan diagrams respectively, and prove that the homotopy equivalence induces a homeomorphism between these compactifications.

Contents

1. Introduction

In this paper we compare two combinatorial models of the moduli space of cobordisms: Bödigheimer's radial slit configurations and Godin's admissible fat graphs. We start this section with an introduction to the moduli space, giving a conformal description of it. After that we describe various combinatorial models and how they relate to each other, which includes our main result, Theorem 1.1. Finally we describe two possible applications.

1.1. **The moduli space of cobordisms.** Mathematicians have been interested in surfaces and their properties for centuries. An integral part of this, the study of families of surfaces – known as "moduli theory" – goes back to the nineteenth century. This study can proceeds along many different paths; one can use algebraic geometry, hyperbolic geometry, complex geometry, conformal geometry or group theory and the interplay of these techniques led to large amounts of interesting mathematics. One of the main points of this theory is the construction of *moduli space*. Intuitively the moduli space of a surface is the space of all surfaces isomorphic to a given one, characterized by the property that equivalence classes of maps into it should classify equivalence classes of families of surfaces.

There are more types of surfaces one might consider than closed surfaces of genus $g \geq 1$. For modern applications to field theories, one family of surfaces that is of particular interest is that of two-dimensional oriented cobordisms. Two-dimensional oriented cobordisms S are oriented surfaces Σ with parametrized boundary $\partial\Sigma$ divided into an incoming and outgoing part. More precisely, there is a pair of maps

$$\iota_{\text{in}} : \bigsqcup_{i=1}^{n} S^1 \to \partial\Sigma \qquad \text{and} \qquad \iota_{\text{out}} : \bigsqcup_{j=1}^{m} S^1 \to \partial\Sigma$$

such that $\iota_{\text{in}} \sqcup \iota_{\text{out}}$ is a homeomorphism.

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Studying operations indexed by isomorphism classes of such two-dimensional oriented cobordisms leads to the definition of two-dimensional TQFT's. One might be interested in more refined structure and hence want to consider the moduli space of such cobordisms, not just their connected components. We will now give the analogue of the conformal definition of the moduli space of these cobordisms, following section 2 of [Böd06].

Let S be an isomorphism class of two-dimensional oriented cobordism that is connected and such that both the incoming and outgoing boundary are non-empty. Up to isomorphism we can think of the parametrizations of the boundary components as being given by a point in each boundary component. So $S = S_{g,n+m}$ is a connected oriented surface of genus g with $n+m$ boundary components each containing a single points p_i for $1 \leq i \leq n+m$. The marked points are ordered and divided into two sets: the incoming set (which contains n marked points) and the outgoing set (which contains m marked points).

To define moduli space we start by considering metrics g on S. Two metrics are said to be conformally equivalent if they are equal up to a pointwise rescaling by a continuous function. This is equivalent to having the same notion of angle. A conformal class of metrics $[g]$ is an equivalence class of metrics under this relation of conformal equivalence. A diffeomorphism $f : S_1 \to S_2$ between two-dimensional manifolds $(S_1, [g]_1)$, $(S_2, [g]_2)$ with conformal classes of metrics is said to be conformal diffeomorphism if $f^*[g]_2 = [g]_1$. We want to restrict our attention to those conformal classes of metrics on S that have the following property: each incoming boundary component has a neighborhood that is conformally diffeomorphic to a neighborhood of the boundary in $\{z \in \mathbb{C} \,|\, ||z|| \geq 1\}$ and each outgoing boundary component has a neighborhood that is conformally diffeomorphic to a neighborhood of the boundary in $\{z \in \mathbb{C} \,|\, ||z|| \leq 1\}$. We will call these conformal classes the conformal classes with good boundary.

The moduli space $\mathcal{M}_g(n,m)$ will have as underlying set the set of conformal classes of metrics on S with good boundary modulo the relation of conformal diffeomorphism fixing the points p_i. We will now define the Teichmüller metric on this set, with respect to which two elements are close together if they are related by a diffeomorphism that is conformal up to a small error.

To define this one looks at all images K of $[0,1]^2$ in S under embeddings. By the Riemann mapping theorem each of these is actually also the image of an rectangle $[0,a] \times [0,b]$ with well-defined modules $\mathrm{Mod}(K) := a/b$ under a conformal embedding. Let $f : (S, [g]_1) \to (S, [g]_2)$ be a diffeomorphism, not necessarily preserving the conformal class. It is said to be quasiconformal of constant C if for all rectangle K we have

$$\frac{1}{C}\mathrm{Mod}_1(K) \leq \mathrm{Mod}_2(f(K)) \leq C\mathrm{Mod}_1(K)$$

A diffeomorphism is quasiconformal if it is quasiconformal of some constant C. In that case we can define the dilatation $\mathrm{Dil}(f)$ of a quasiconformal diffeomorphism f to be the infimum over all C. If $QC([g]_1, [g]_2)$ denotes the set of all quasiconformal diffeomorphisms between $(S, [g]_1)$ and $(S, [g]_2)$ fixing the points p_i, then we can define the Teichmüller metric

$$d((S, [g]_1), (S, [g]_2)) = \log \inf\{\mathrm{Dil}(f) \,|\, f \in QC([g]_1, [g]_2)\}$$

This completes the conformal definition of the moduli space of two-dimensional oriented cobordisms isomorphic to S

$$\mathcal{M}_g(n,m) = \left(\frac{\text{conformal classes of metrics on } S \text{ with good boundary}}{\text{conformal diffeomorphisms fixing the points } p_i}, \text{ Teichmüller metric}\right)$$

For non-connected S, we simply take the product of these spaces over all components. As long as ∂S is non-empty for each connected component of S, an alternative definition of these spaces is as the quotient of Teichmüller space (the space of quasiconformal maps modulo conformal equivalence) by the action of the mapping class group $\Gamma_{S;\partial S}$ (the connected components of the diffeomorphism group $\mathrm{Diff}(S;\partial S)$). This is a free proper action on a contractible space and $\mathcal{M}_g(n,m) \simeq B\Gamma_{S;\partial S}$. In that case also all connected components of $\mathrm{Diff}(S;\partial S)$ are contractible and we conclude that

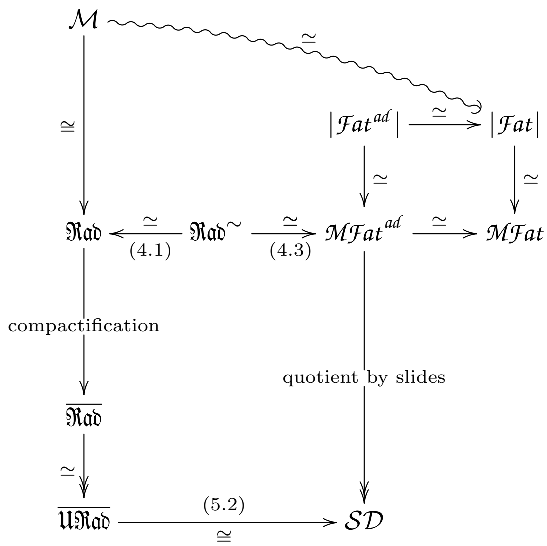
$$\mathcal{M}_g(n,m) \simeq B\Gamma_{S;\partial S} \simeq B\mathrm{Diff}(S;\partial S)$$

The last term makes clear why $\mathcal{M}_g(n,m)$ is a model moduli space of two-dimensional oriented cobordisms; any bundle of cobordisms over a paracompact space B with transition functions diffeomorphisms can be obtained by pulling back a certain universal bundle from $\mathcal{M}_g(n,m)$ along a map $B \to \mathcal{M}_g(n,m)$.

We end this subsection by describing this universal bundle. The moduli space $\mathcal{M}_g(n,m)$ is the quotient of the space of conformal classes of metric $[g]_1$ by the diffeomorphisms. The universal bundle can be constructed similarly: one starts with the space consisting of pairs of a conformal class of metric and a point $x \in S$, and takes the quotient by conformal diffeomorphisms acting diagonally.

1.2. **An overview of combinatorial models of moduli space.** The goal of this paper is to discuss several combinatorial models of the moduli space of cobordisms with non-empty incoming and outgoing boundary and their compactifications. Such models are relevant to explicit constructions of field theories like string topology.

In this introduction we start by discussing a larger collection of combinatorial models and the relations between these, before focusing on two particular models. The relations between all the various models are spelled out in the following diagram. An arrow is a continuous map; if decorated by $\simeq$ it is homotopy equivalence and if decorated by $\cong$ it is a homeomorphism, and a squiggle decorated by $\simeq$ is an abstract homotopy equivalence. We fix g, n and m and drop them from the notation:



Let us summarize the objects that are appear in this diagram.

Moduli space $\mathcal{M}$: This is the archetypical "space of cobordisms", a conformal model of which was discussed in Section 1.1. It consists of conformal classes of metrics modulo conformal diffeomorphisms, with the Teichmüller metric.

The radial slit configurations $\mathfrak{Rad}$: This is a model for $\mathcal{M}$ due to Bödigheimer, consisting of glueing data to construct a conformal class of metric by glueing together annuli in $\mathbb{C}$. The main theorem of [Böd06] is that there exist maps $\mathcal{M} \to \mathfrak{Rad}$ and $\mathfrak{Rad} \to \mathcal{M}$ which are mutually inverse homeomorphisms. This and related models will be discussed in detail in Section 2, and $\mathfrak{Rad}$ will be defined in Definition 2.11.

The blow-up of the radial slit configurations $\mathfrak{Rad}^\sim$: To discuss the relation between $\mathfrak{Rad}$ and $\mathcal{MF}at$, in this paper we introduce $\mathfrak{Rad}^\sim$ as a blow-up of $\mathfrak{Rad}$ by including resolutions of the critical graph for non-generic radial slit configurations. This is done in Subsection 4.1.

The harmonic compactification $\overline{\mathfrak{Rad}}$: Naturally $\mathfrak{Rad}$ arises as an open subspace of a compact space $\overline{\mathfrak{Rad}}$. In this compactification handle or outgoing boundary components are allowed to degenerate to points in certain situations. It is defined in Definition 2.9.

The unimodular harmonic compactification $\overline{\mathfrak{URad}}$: The space $\overline{\mathfrak{URad}}$ is a deformation retract of $\overline{\mathfrak{Rad}}$ obtained by collapsing some contractible data. It is defined in Definition 2.17.

The fat graphs $\mathcal{F}at$: Fat graphs are graphs with the additional structure of a cyclic ordering of the edges going into each vertex and data encoding the parametrization of its "boundary components". Taking as morphisms maps of fat graphs that collapse a disjoint union of trees one can define a category of fat graphs $\mathcal{F}at$. The space $|\mathcal{F}at|$ is the geometric realization of this category. This and related models will be discussed in detail in Section 3, and $\mathcal{F}at$ will be defined in Definition 3.7.

The admissible fat graphs $\mathcal{F}at^{ad}$: A fat graph is said to be admissible if its incoming boundary graph embeds in it. The space $|\mathcal{F}at^{ad}|$ is the geometric realization of the full subcategory on the admissible fat graphs. It is defined in Definition 3.7.

The metric fat graphs $\mathcal{MF}at$: Closely related to $\mathcal{F}at$ is the space of metric fat graphs $\mathcal{MF}at$. In this case one also includes the data of the lengths of edges of the fat graphs.

The admissible metric fat graphs $\mathcal{MF}at^{ad}$: Just like $\mathcal{F}at^{ad}$ is a subcategory of $\mathcal{F}at$ consisting of fat graphs that are admissible, $\mathcal{MF}at^{ad}$ is a subspace of $\mathcal{MF}at$. It is defined in Definition 3.12.

The Sullivan diagrams $\mathcal{SD}$: Sullivan diagrams are the quotient of $\mathcal{MF}at^{ad}$ by the equivalence relation of slides away from the admissible boundary. This space has a canonical CW-complex structure and its cellular chain complex is the complex of (cyclic) Sullivan chord diagrams introduced by Tradler and Zeinalian and used by them and afterwards by Wahl and Westerland in order to construct operations on the Hochschild chains of symmetric Frobenius algebras (cf. [TZ06, WW11]). They are defined in Definition 3.18.

In this article we will focus on the bottom square; that is, the relations between radial slit configurations, admissible metric fat graphs and their compactifications. Our main results is the following:

Theorem 1.1. *We define a space* $\mathfrak{Rad}^{\sim}$ *and maps* (4.17), (4.26) *and* (5.1). *in the square*

$$\begin{array}{ccccc}
\mathfrak{Rad} & \xleftarrow[(4.17)]{\simeq} & \mathfrak{Rad}^{\sim} & \xrightarrow[(4.26)]{\simeq} & \mathcal{MF}at_0^{ad} \\
\downarrow & & & & \downarrow \\
\overline{\mathfrak{Rad}} & & & & \\
\simeq \downarrow (2.18) & & & & \\
\overline{\mathfrak{URad}} & & \xrightarrow[\cong]{(5.1)} & & \mathcal{SD}_0
\end{array}$$

which make the diagram commute, where all maps that are decorated by $\simeq$ are homotopy equivalences and all maps decorated by $\cong$ are homeomorphisms.

There exist other combinatorial models related to the moduli space of cobordisms which are not discussed in detail in this paper. We will describe three such models in the following remarks.

Remark 1.2. In order to describe an action of the chains of the moduli space of surfaces on the Hochschild homology of $\mathscr{A}_\infty$ Frobenius algebras, Costello constructs a chain complex that models the homology of the moduli space ([Cos06a, Cos06b]). In [WW11], Wahl and Westerland describe this chain complex in terms of fat graphs with two types of vertices, which they denote *black and white fat graphs*. There is an equivalence relation of black and white graphs given by slides away from the white vertices. The quotient complex is the complex of Sullivan diagrams whose underlying space is $\mathcal{SD}$.

Remark 1.3. In [CG04] Ralph Cohen and Veronique Godin define Sullivan chord diagrams of genus g with p incoming and q outgoing boundary components. These chord diagrams were also used in [FT09]. These are fat graphs obtained from glueing trees to circles. These fit together into a space $\mathcal{CF}(g;p,q)$ of metric chord diagrams and this space is a subspace of $\mathcal{F}at^{ad}$. They are thus *not* the same as Sullivan diagrams, here defined in Definition 3.18, though they do admit a quotient map to $\mathcal{SD}$. It is known that the space of metric chord diagrams is not homotopy equivalent to moduli space, see remark 3 of [God07a].

Remark 1.4. In her thesis [Poi10], Kate Poirier defines a space $\overline{SD}(g,k,l)/\sim$ of "string diagrams modulo slide equivalence" of genus g with k incoming and l outgoing boundary components and more generally she defines "string diagrams with many levels modulo slide equivalence" $\overline{LD}(g,k,l)/\sim$.

Proposition 2.3 of [Poi10] says that $\overline{SD}(g,k,l)/\sim\ \simeq \overline{LD}(g,k,l)/\sim$ She also defines a subspace $SD(g,k,l)$ of $\overline{SD}(g,k,l)$.

Both $\overline{SD}(g,k,l)$ and $SD(g,k,l)$ are subspaces of $\mathcal{MFat}^{ad}$ and by counting components one can see that these inclusions can't be homotopy equivalences. However, the quotient map $\overline{SD}(g,k,l)/\sim \to \mathcal{SD}$ is a homeomorphism.

1.3. **Applications of these models.**

We will next explain two of the applications of combinatorial models for moduli space.

1.3.1. *Explicit computations of the homology of moduli spaces.* We will see that combinatorial models provide cell decompositions for moduli space. This make an explicit computation of the (co)homology of moduli space using cellular (co)homology possible. Instead of studying $\mathcal{M}_g(n,m)$, it turns out to be more convenient to study the closely related moduli space $\mathcal{M}_g^{1,n}$ of surfaces of genus g with one parametrized boundary component and permutable n punctures. There are variations of $\mathfrak{Rad}$ and $\mathcal{MFat}^{ad}$ that are models for $\mathcal{M}_g^{1,n}$.

Simultaneously, much is known about the homology of $\mathcal{M}_g^{1,n}$ and much is unknown about it. For example, Harer stability tells us $H_*(\mathcal{M}_g^{1,n})$ stabilizes as $g \to \infty$ or $n \to \infty$ (cf. [Har85, Wah08]) and the Madsen-Weiss theorem tells us what it stabilizes to (cf. [MW07]). On the other hand, we know almost nothing of the homology outside of the stable except that there has to be an enormous amount of it. In such a world, explicit computations of all the homology of $\mathcal{M}_g^{1,n}$ for low g and n is helpful to inform and test conjectures about the general structure.

An example of an explicit computation of the homology of moduli spaces using fat graphs is given in [God07b]. Here Godin computes the integral homology of the $\mathcal{M}_g^{1,0}$ for $g = 1, 2$ and $\mathcal{M}_g^{2,0}$ for $g = 1$.

The computation of the homology of moduli spaces using radial slit configurations, or the closely related parallel slit configurations, is a long-term project of Bödigheimer and his students. See for example [ABE08] for the computation of the integral homology of $\mathcal{M}_g^{1,n}$ for $2g + n \leq 5$ using parallel slits. Many of the results of this program are at the moment only available in PhD theses or notes from talks.

1.3.2. *Two-dimensional field theories, in particular string topology.* Combinatorial models of moduli space have been an important tool in the study of two-dimensional field theories for a long time. Maybe the first application was Kontsevich's proof of the Witten conjecture using fat graphs in [Kon92]. Since then people have used fat graphs to get a grip on field theories, one relevant example of which is Costello's classification so-called classical conformal field theories in [Cos06b] which served as inspiration for the sketch of the proof of the cobordism hypothesis [Lur09].

More concretely combinatorial models for the moduli space of cobordisms have played a big role in the construction of string operations; operations $H_*(\mathcal{M}_g(n,m); \mathcal{L}^{\otimes d}) \otimes H_*(LM)^{\otimes n} \to H_*(LM)^{\otimes m}$ for compact oriented manifolds M. Chas and Sullivan already thought of the pair of pants cobordisms as a figure-eight graph [CS99], and a large part of the constructions of string operations since have used graphs. One relevant example is Godin's work [God07a], which uses the space $\mathcal{Fat}^{ad}$. Using Costello's model for moduli space together with a Hochschild homology model for the cohomology of the free loop space, Wahl and Westerland [WW11, Wah12] not only constructed similar string operations, but showed that they can be extended to $\mathcal{SD}$. One can also use radial slit configurations to construct string operations, see [Kup11].

One problem in string topology is the contrast between the large amount different constructions and the lack of comparisons between these constructions. We think that the critical graph equivalence of 4 might make it possible to compare constructions involving fat graphs and Sullivan diagrams, to constructions involving radial slit configurations and the harmonic compactification.

1.4. **Outline of paper.**

In Sections 2 and 3 we define radial slit configurations, (metric) fat graphs and their compactifications in detail. In Section 4 we prove that the critical graph of a radial slit configuration allows one to construct a homotopy equivalence between $\mathfrak{Rad}$ and $\mathcal{MFat}^{ad}$. In Section 5 we prove that this homotopy equivalence descends to a homeomorphism between $\overline{\mathfrak{URad}}$ and $\mathcal{SD}$.

1.5. **Acknowledgements.**

This paper grew out of discussions at the String Topology and Related Topics at the Center for Symmetry and Deformation at the University of Copenhagen. The authors would like to thank Nathalie Wahl and Carl-Friedrich Bödigheimer for helpful conversations and comments. The first author was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

2. Radial slit configurations and the harmonic compactification

2.1. **The definition.** In this subsection we introduce Bödigheimer's radial slit configuration model for the moduli space of two-dimensional cobordisms with non-empty incoming and outgoing boundary. The idea is that any such cobordism and any conformal class of metric on it can be obtained by taking annuli, making cuts in these annuli and then glueing along these cuts. References for the radial slit configurations and related models include [Böd90], [Böd06], [ABE08], [Ebe05] and section 2 of [Kup11].

Before giving a rigorous definition of the radial slit configuration space $\mathfrak{Rad}$ we will explain how to arrive at this definition given that one wants to build cobordisms from glueing annuli along cuts. The reader may skip this introduction and go directly to Definition 2.2, the definition of the possibly degenerate radial slit preconfigurations.

To arrive at moduli space, one would want such a construction of cobordisms out of annuli to be continuous in the data, in a way we make precise later, and result in as many conformal classes as possible, indeed of all them. This leads to the following two guidelines in our discussion; (i) try to topologize all parts of data that reasonably can be topologized and (ii) try to make the broadest possible definition. With these guidelines in mind, we can start our discussion.

The simplest cobordism with non-empty incoming and outgoing boundary is arguably the cylinder, with one incoming boundary component and one outgoing boundary component. Using for example complex analysis or the theory of harmonic functions, one can see that each such cylinder is conformally equivalent to one of the following annuli for $R \in (1, \infty)$:

$$\mathbb{A}_R = \{z \in \mathbb{C} \,|\, 1 \leq |z| \leq R\}$$

and we will therefore take these annuli as our basic building blocks. Each of these annuli has an inner boundary $\partial_{\text{in}}\mathbb{A}_R = \{z \in \mathbb{C} \,|\, |z| = 1\}$ and outer boundary $\partial_{\text{out}}\mathbb{A}_R = \{z \in \mathbb{C} \,|\, |z| = R\}$. Note that they also come equipped with a canonical metric, being subsets of $\mathbb{C}$.

Suppose we are interested in creating a cobordism with n incoming boundary components, then we start with n annuli $\mathbb{A}_{R_i}$, whose inner boundaries are going to be the incoming boundary of our cobordism. To construct our cobordism we will make cuts radially inward from the outer boundaries of the annuli. Such cuts are uniquely specified by points $\zeta \in \sqcup_{i=1}^n \mathbb{A}_{R_i}$, which we will call slits. They need not be disjoint. As will become clear, the number of slits must always be an even number $2h$ and we thus number them $\zeta_1, \ldots, \zeta_{2h}$. It turns that for a total genus g cobordism with n incoming and m outgoing boundary components we will need $2h = 2(2g-2+n+m)$ slits.

We want to glue the different sides of the cuts together to get back a surface. To get a metric on the surface from the metric on the cut annuli, necessarily two cuts that we glue together must be of the same length. For the orientations to work out, we must glue a side clockwise from a cut to one counterclockwise from a cut. To avoid singularities, if one side of the cut corresponding to ζ_i is glued to a side of the cut corresponding to ζ_j, the same must be true for the other two sides. From this we see that our glueing procedure should be described by a pairing on $\{1, \ldots, 2h\}$, encoded by a permutation $\lambda : \{1, \ldots, 2h\} \to \{1, \ldots, 2h\}$ consisting of h cycles of length 2. We should furthermore demand that if ζ_i lies of the annulus $\mathbb{A}_{R_j}$ and $\zeta_{\lambda(j)}$ lies on the annulus $\mathbb{A}_{R_{j'}}$, then $R_j - |\zeta_i| = R_{j'} - |\zeta_{\lambda(i)}|$. See Figure 2.1 for a simple example.

However, there are several problematic situations that could occur. Firstly, if two slits ζ_i and ζ_j lie on the same radial segment (a subset of the annulus $\mathbb{A}_{R_j}$ of the form $\{z \in \mathbb{A}_{R_j} \mid \arg(z) = \theta\}$ for some θ), then our cutting and glueing procedure is not well-defined. We still need to keep track of whether ζ_i is clockwise or counterclockwise from ζ_j. To do this we also keep track of a successor permutation $\omega : \{1, \ldots, 2h\} \to \{1, \ldots, 2h\}$. This has n cycles, corresponding to the n annuli, and we should demand that each cycle contains the numbers of the slits in one of the annuli and is compatible with the weak cyclic ordering on these coming from the argument of the slits. In a sense the successor permutation keeps track of the fact that when two slits coincide, one is actually supposed to be infinitesimally counterclockwise from the other. See Figure 2.2.

This still isn't enough if all slits on an annulus lie on the same radial segment, because in these cases we can only deduce the ordering of the slits up to a cyclic permutation. To fix this we add additional data, which will turn out to be completely superfluous except in the case that all the slits

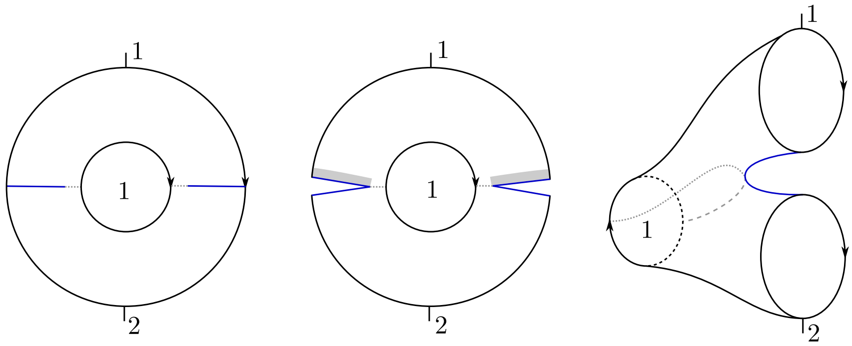


FIGURE 2.1. An example of the way that cutting and glueing slits in annuli leads to a cobordism. In this case we start with the annulus on the left, cut along the blue lines to obtain the cut annulus in the middle, and finally glue the grey sides of the cuts and the white sides of the cuts together respectively to get the cobordism on the right. In this particularly simple example both the pairing λ and the successor permutation ω are uniquely determined.

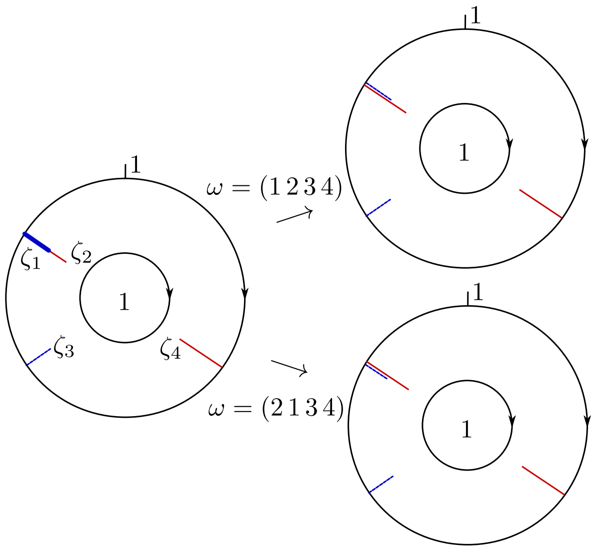


FIGURE 2.2. An example of a radial slit preconfiguration with a two slits on the same angular segment; ζ_1 is the shorter thick blue slit and ζ_2 is the longer thin red slit. The successor permutation ω allows us to think of ζ_1 as either infinitesimally clockwise or counterclockwise from ζ_2.

on an annulus coincide. The data that we add is the angular distance $r_i \in [0, 2\pi]$ in counterclockwise direction from ζ_i to $\zeta_{\omega(i)}$. In almost all cases one can deduce this from the locations of the ζ_i and ω, but in the case where all slits on an annulus lie on the same radial segment, one of them will have $r_i = 2\pi$, while the others will have $r_j = 0$. This allows one to determine the ordering of the slits, since the slit ζ_i with $r_i = 2\pi$ should be first in clockwise direction from the angular gap between the slits.

We have almost described enough data to construct a cobordism now. We can build a surface, which has as boundary components the inner boundaries of the annuli and some other boundary components. Since we wanted m outgoing boundary components we restrict to the subset of data that gives us m boundary components in addition to the inner boundaries of the annuli. The inner boundary of the annuli come with a canonical parametrization, but the outer ones do not have such a parametrization yet, though they do have a canonical orientation coming from the orientation of the outer boundary of the annuli. Hence we add one point P_i in each of the outgoing boundary components, m in total. Finally, we will need to include these new parametrization points in ω and the r_i's. To do this, we write $\xi_i = \zeta_i$ for $1 \leq i \leq 2h$ and $\xi_{2h+i} = P_i$ for $1 \leq i \leq m$, and expand our definition of ω to an permutation $\tilde{\omega}$ of $2h+m$ elements and add additional $r_{2h+i} \in [0, 2\pi]$ for $1 \leq i \leq m$.

We will collect all this data and the conditions we put on it in a definition; these will be the possibly degenerate radial slit preconfigurations. It turns out that we haven't excluded all possibilities of getting a degenerate surface yet and similarly we have yet to identify points leading to cobordisms that are conformally equivalent. Since definition will involve the permutations λ and ω, we require a notation for symmetric groups.

Notation 2.1. Let $\mathfrak{S}_k$ denote the symmetric group on k elements.

Definition 2.2. The space of *possibly degenerate radial slit preconfigurations* $\overline{\mathfrak{PRad}}_h(n,m)$ is given by a subspace of $(\bigsqcup_{i=1}^n \mathbb{C})^{2h} \times \mathfrak{S}_{2h} \times \mathfrak{S}_{2h+m} \times [0,2\pi]^{2h+m} \times (1,\infty)^n \times (\bigsqcup_{i=1}^n \mathbb{C})^m$ whose elements we denote by $L = (\vec{\zeta}, \lambda, \tilde{\omega}, \vec{r}, \vec{R}, \vec{P})$. The ζ_i are called the *slits*, λ the *slit pairing*, ω the *successor permutation*, the r_i the *angular distances*, R_j the *outer radii* and the P_i the *parametrization points*. For notation, let $\xi \in (\bigsqcup_{i=1}^n \mathbb{C})^{2h+m}$ be given by $\xi_i = \zeta_i$ for $1 \leq i \leq 2h$ and $\xi_{i+2h} = P_i$ for $1 \leq i \leq m$ and let $\omega \in \mathfrak{S}_{2h}$ be the restriction of $\tilde{\omega}$ to the set $\{1, 2 \ldots 2h\}$. The subspace $\overline{\mathfrak{PRad}}_h(n,m)$ consists of such data subject to the following six conditions:

(i) Each slit ζ_j lies in $\bigsqcup_{i=1}^n \mathbb{A}_{R_i} \subset \bigsqcup_{i=1}^n \mathbb{C}$ and each parametrization point P_i lies in $\bigsqcup_{i=1}^n \partial_{out}\mathbb{A}_{R_i}$.
(ii) The slit pairing λ consists of h 2-cycles. We demand for all $1 \leq i \leq 2h$ that if ζ_i lies on the annulus $\mathbb{A}_{R_j}$ and $\zeta_{\lambda(j)}$ lies on the annulus $\mathbb{A}_{R_{j'}}$ we have that $R_j - |\zeta_i| = R_{j'} - |\zeta_{\lambda(i)}|$.
(iii) The successor permutation $\tilde{\omega}$ consists of a disjoint union of n cycles and these cycles consist exactly of the indices of the ξ_i lying on one of the annuli. We demand that permutation action of $\tilde{\omega}$ on these ξ_i preserves the weakly cyclic ordering which comes from the argument (as usual taken in counterclockwise direction).
(iv) The *boundary component permutation* $\lambda \circ \omega$ consists of m cycles of varying length. It will turn out that cycles correspond to the outgoing boundary components.
(v) We demand that P_i lies in the subset O_i of $\bigsqcup_{i=1}^n \partial_{\mathrm{out}}\mathbb{A}_R \subset \bigsqcup_{i=1}^n \mathbb{C}$ which we will now define. The m cycles of $\lambda \circ \omega$ partition the outer boundaries of the annuli into a collection of m subsets, overlapping only in isolated points. We demand that each of these contains a P_i and denote that subset O_i. To be precise, each O_i is the union of the parts in the outer boundary between the radial segments through ζ_j and $\zeta_{\omega(j)}$ respectively, for all j in a cycle of $\lambda \circ \omega$. Once we discuss the universal surface bundle, we can rephrase this condition in a more intuitive way.
(vi) The angular distances r_i must be compatible with the location of the ξ_i and the successor permutation $\tilde{\omega}$ in the following sense. If ξ_i does not lie an annulus with all slits and parametrization points coinciding, then r_i is equal to the angular distance in counterclockwise direction from ξ_i to $\xi_{\tilde{\omega}(i)}$. If ξ_i lies on an annulus with all slits and parametrization points coinciding, then r_i is equal to either 0 or 2π and exactly one ξ_j on that annulus has $r_j = 2\pi$.

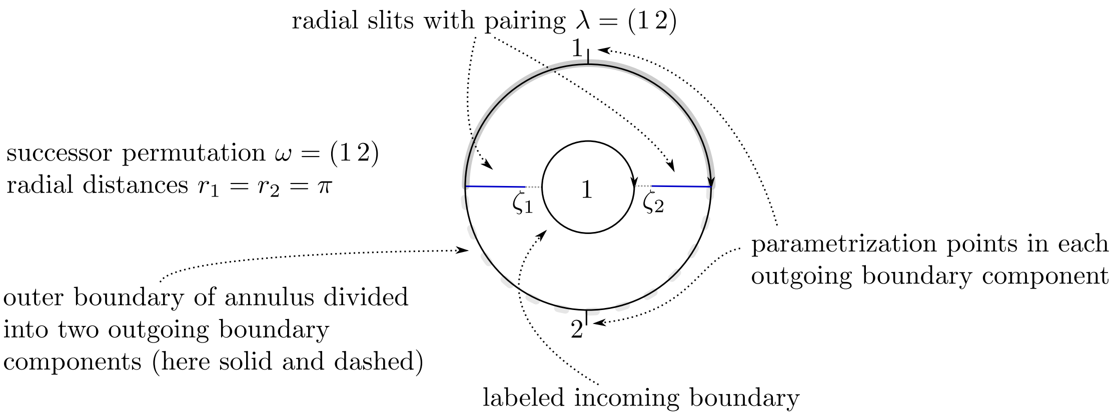


FIGURE 2.3. The configuration of Figure 2.1 with all its data pointed out.

Let us now give the construction of the possibly degenerate cobordism $S(L)$ out of $L \in \overline{\mathfrak{PRad}}_h(n,m)$, making precise the informal discussion above. In the next subsection we will describe how this construction leads to a surface bundle over $\mathfrak{Rad}$ and why this is the universal surface bundle over moduli space.

Given a configuration L we first need to define the sector space $\tilde{\Sigma}(L)$, a precise definition of the pieces used in the glueing construction. It turns out to be more convenient to complete the cuts all the way to the inner boundary of the annuli and reglue them later, a slight departure from our informal discussion preceding definition 2.2. See Figure 2.4 for examples of the different types of sectors.

Definition 2.3. Let r be the number of annuli containing no slits. Then $\tilde{\Sigma}(L)$ will be a disjoint union of $2h+r$ subsets of annuli. These come in four types:

Ordinary sectors: If $\arg(\zeta_i) \neq \arg(\zeta_{\omega(i)})$ and ζ_i lies on the annulus $\mathbb{A}_{R_j}$, then we set

$$F_i = \{z \in \mathbb{A}_{R_j} \mid \arg(\zeta_i) \leq \arg(z) \leq \arg(\zeta_{\omega(i)})\}$$

Thin sectors: If $\arg(\zeta_i) = \arg(\zeta_{\omega(i)})$, $r_i = 0$ and ζ_i lies on the annulus $\mathbb{A}_{R_j}$, then we set

$$F_i = \{z \in \mathbb{A}_{R_j} \mid \arg(\zeta_i) = \arg(z)\}$$

Full sectors: If $\arg(\zeta_i) = \arg(\zeta_{\omega(i)})$, $r_i = 2\pi$ and ζ_i lies on the annulus $\mathbb{A}_{R_j}$, then we set F_i to be equal to the annulus $\mathbb{A}_{R_j}$ cut open along the segment $\arg(z) = \arg(\zeta_i)$, with that segment doubled so that it is homeomorphic to a closed rectangle.

Entire sectors: If an annulus $\mathbb{A}_{R_j}$ doesn't contain any slits and is j'th in the induced ordering on the r annuli that don't contain any slits, we set $F_{2h+j'} = \mathbb{A}_{R_j}$.

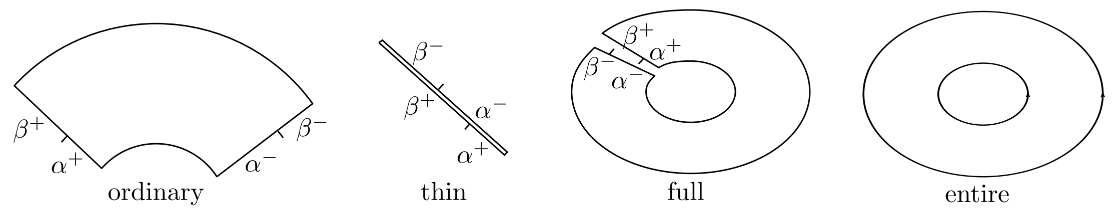


FIGURE 2.4. Examples of the four different types of radial sectors, with the subsets $\alpha^\pm$ and $\beta^\pm$ pointed out.

The surface $\Sigma(L)$ underlying the cobordism $S(L)$ will be obtained as a quotient space of the sector space by an equivalence relation that makes identifications among radial segments in the boundary of the sectors. We will now define the subsets involved in those identifications.

Definition 2.4. If F_i is an ordinary or thin sector corresponding to the slit ζ_i on the annulus $\mathbb{A}_{R_j}$, then we define the following subspaces of F_i:

$$\begin{aligned}
\alpha_i^+ &= \{z \in \mathbb{A}_{R_j} \mid \arg(z) = \arg(\zeta_{\omega(i)}) \text{ and } ||z|| \leq ||\zeta_{\omega(i)}||\} \\
\alpha_i^- &= \{z \in \mathbb{A}_{R_j} \mid \arg(z) = \arg(\zeta_i) \text{ and } ||z|| \leq ||\zeta_i|| \\
\beta_i^+ &= \{z \in \mathbb{A}_{R_j} \mid \arg(z) = \arg(\zeta_{\omega(i)}) \text{ and } ||z|| \geq ||\zeta_{\omega(i)}||\} \\
\beta_i^- &= \{z \in \mathbb{A}_{R_j} \mid \arg(z) = \arg(\zeta_i) \text{ and } ||z|| \geq ||\zeta_i||
\end{aligned}$$

If F_i is a full sector then our definitions have to be slightly different, because now the two radial segments in the boundary have the same argument. Let S_i^+ be the radial segment bounding F_i in counterclockwise direction and S_i^- be the radial segment bounding it in clockwise direction, then we define the following subspaces of F_i:

$$\begin{aligned}
\alpha_i^+ &= \{z \in S_i^+ \mid ||z|| \leq ||\zeta_{\omega(i)}||\} \\
\alpha_i^- &= \{z \in S_i^- \mid ||z|| \leq ||\zeta_i|| \\
\beta_i^+ &= \{z \in S_i^+ \mid ||z|| \geq ||\zeta_{\omega(i)}||\} \\
\beta_i^- &= \{z \in S_i^- \mid ||z|| \geq ||\zeta_i||
\end{aligned}$$

We can now define the equivalence relation $\approx_L$ and the surface $\Sigma(L)$.

Definition 2.5. The equivalence relation $\approx_L$ on $\tilde{\Sigma}(L)$ is the one generated by

(i) We identify $z \in \alpha_i^+$ with $z \in \alpha_{\omega(i)}^-$.
(ii) We identify $z \in \beta_i^+$ with $z \in \beta_{\lambda(i)}^-$.

We define the surface $\Sigma(L)$ to be $\tilde{\Sigma}(L)/\approx_L$. We will now describe it as a cobordism.

Note that $\Sigma(L)$ has a map from each inner boundary $\partial_{\mathrm{in}}\mathbb{A}_{R_j}$

$$\iota_j^{\mathrm{in}} : S^1 \cong \partial_{\mathrm{in}}\mathbb{A}_{R_j} \to \Sigma(L)$$

These are the inclusions of subspaces if none of the slits lie on the inner boundary of the annuli. One can also define the outgoing boundary components. In particular, consider the intersection of the outer boundary of the annuli with the sectors. These gives us a subspace of $\Sigma(L)$. For each cycle in $\lambda \circ \omega$ these images form a circle with canonical orientation and canonical starting point P_k. This gives us for the cycle $\lambda \circ \omega$ containing P_k a map

$$\iota_k^{\mathrm{out}} : S^1 \to \Sigma(L)$$

These are the inclusions of subspaces if none of the slits on the outer boundary of the annuli.

Definition 2.6. We define the cobordism $S(L)$ as follows: the underlying possibly degenerate surface with boundary is $\Sigma(L)$. The parametrization of the incoming boundary is by the maps

$$\bigsqcup_{j=1}^{n} \iota_j^{\mathrm{in}} : \bigsqcup_{j=1}^{n} S^1 \cong \bigsqcup_{j=1}^{n} \partial_{\mathrm{in}}\mathbb{A}_{R_j} \to \Sigma(L)$$

The parametrization of the outgoing boundary is by the maps

$$\bigsqcup_{k=1}^{m} \iota_k^{\mathrm{out}} : \bigsqcup_{k=1}^{m} S^1 \to \Sigma(L)$$

We mentioned before that this definition might lead to degenerate cobordisms for some L and involves some conformal classes of cobordisms more than once. It is easy to see that we in fact get each conformal class of cobordisms at least $(2h)!$ times; the labeling on the slits doesn't matter for the surface one constructs. For the degenerate surfaces, consider the example in Figure 2.5. In the remaining part of this section we will explain the extent of both issues and how to resolve them.

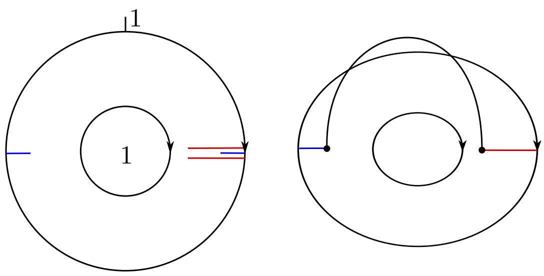


FIGURE 2.5. An example of a radial slit preconfiguration leading to a degenerate surface. The black arc connecting two on the surface on the right original was the line segment between the two red slits.

We have already explained one way in which one might obtain the same surface more than once, by having different labels. It turns out that are only two additional identifications we have to make to generate the correct equivalence relation.

For the first additional identification, think instead of doing all the cutting and glueing simultaneously, doing it in order of increasing modulus of the slits. This results in the same cobordism but it becomes clear that if ζ_i lies on the same radial segment as ζ_j and satisfies $|\zeta_i| \geq |\zeta_j|$, it might as well be on the other side of $\zeta_{\lambda(j)}$. That is, it might as well have "jumped" over the slit ζ_j to $\zeta_{\lambda(j)}$. For the second additional identification, note that if a parametrization point similarly "jumps" over a slit, this doesn't change the parametrization of the outgoing boundary.

These will turn out to be all required identifications, and we now combine them into a single equivalence relation on $\overline{\mathfrak{PRad}}_h(n,m)$.

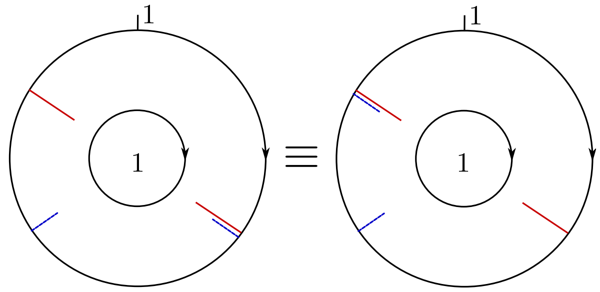


FIGURE 2.6. A jump of a slit. The pairing λ is given by the colors, but is uniquely determined by the configuration.

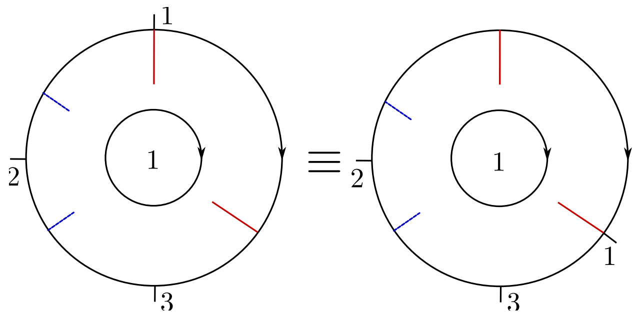


FIGURE 2.7. A jump of a parametrization point.

Definition 2.7. Let $\equiv$ be the equivalence relation on $\overline{\mathfrak{PRad}}_h(n,m)$ generated by

Relabelling of the slits: We identify two preconfigurations if they can be obtained from each other by relabeling the slits. More precisely for every permutation $\sigma \in \mathfrak{S}_{2h}$ we can obtain from $L = (\vec{\zeta}, \lambda, \tilde{\omega}, \vec{r}, \vec{R}, \vec{P})$ of $\overline{\mathfrak{PRad}}_h(n,m)$ a new element $\sigma(L) = (\vec{\zeta^\sigma}, \sigma \circ \lambda \circ \sigma^{-1}, \tilde{\omega}^\sigma, \vec{r^\sigma}, \vec{R}, \vec{P})$ of $\overline{\mathfrak{PRad}}_h(n,m)$, where $\vec{\zeta^\sigma}$ is the collection of slits defined by $(\zeta^\sigma)_i = \zeta_{\sigma(i)}$, $\vec{r^\sigma}$ is defined by $(r^\sigma)_i = r_{\tilde{\sigma}(i)}$, where $\tilde{\sigma} \in \mathfrak{S}_{2h+m}$ is the permutation induced by extending σ by the identity, and finally $\tilde{\omega}^\sigma = \tilde{\sigma} \circ \tilde{\omega} \circ \tilde{\sigma}^{-1}$. In this case we set $L \equiv L'$.

Slit jumps: We identify two preconfigurations if they can be obtained from each other by a slit jump, see Figure 2.6. More precisely if we are given preconfiguration L and a pair i and $j = \omega(i)$ such that $r_i = 0$ and $|\zeta_i| \geq |\zeta_j|$, then we can obtain a new preconfiguration L' as follows. We replace ζ_i by the point $\zeta_i' = (|\zeta_{\lambda(j)}| + |\zeta_i| - |\zeta_j|)\frac{\zeta_{\lambda(j)}}{|\zeta_{\lambda(j)}|}$ and keep all the other slits the same. We then put i after of $\lambda(j)$ in $\tilde{\omega}$ to obtain $\tilde{\omega}'$ and set $r_i' = r_{\lambda(j)}$ and $r'_{\lambda(j)} = 0$. The rest of the data remains the same. In this case we set $L \equiv L'$.

Parametrization point jumps: We identify two preconfigurations if they can be obtained from each other by a jump of a parametrization point, see Figure 2.7. More precisely, if we are given a preconfiguration L in which there is a P_i such that $j = \tilde{\omega}(i+2h)$ for some j and $r_{i+2h} = 0$, then we can obtain a new preconfiguration L' by keeping all the data the same except replacing P_i with P_i' lying at the radial segment through $\zeta_{\lambda(j)}$ and setting $r'_{i+2h} = r_{\lambda(j)}$ and $r'_{\lambda(j)} = 0$. In this case we set $L \equiv L'$.

Definition 2.8. A radial slit preconfiguration is said to be *generic* if it does not play a role in any slit or parametrization point jumps.

We can now define the harmonic compactification, whose name we will justify later.

Definition 2.9 (Harmonic compactification)**.** The harmonic compactification $\overline{\mathfrak{Rad}}_h(n,m)$ is the quotient space $\overline{\mathfrak{PRad}}_h(n,m)/\equiv$.

We now have to deal with the problem that some of the points in $\overline{\mathfrak{Rad}}_h(n,m)$ lead to cobordisms with degenerate underlying surface. Bödigheimer gave a criterion for when a configuration leads to degenerate surface.

Proposition 2.10. *The surface underlying the cobordism $\Sigma(L)$ constructed out of a preconfiguration L is degenerate if and only if it is equivalent under $\equiv$ to a preconfiguration satisfying one of the following three conditions:*

- **Slit hitting inner boundary:** *There is a slit ζ_i with $|\zeta_i| = 1$.*
- **Slit hitting outer boundary:** *There is a slit ζ_i on an annulus $\mathbb{A}_{R_j}$ with $|\zeta_i| = R_j$.*
- **Slits are "squeezed":** *There is a pair $i, j = \lambda(i)$ such that ζ_i and ζ_j lie on the same annulus, $\zeta_i = \zeta_j$ and such that all for k between i and j in the cyclic ordering we have that $|\zeta_k| \leq |\zeta_i| = |\zeta_j|$ (see Figure 2.5 for an example). If all slits on the annulus containing ζ_i and ζ_j lie at the same point, we additionally should require that all of the k between i and j satisfy $r_k = 0$.*

Definition 2.11 (Radial slit configurations)**.** $\mathfrak{Rad}_h(n,m)$ is the subspace of $\overline{\mathfrak{Rad}}_h(n,m)$ of configurations that do not have a representative satisfying (i), (ii) or (iii) of proposition 2.10.

There is a subspace $\mathfrak{Rad}_h^{\mathrm{conn}}(n,m)$ of $\mathfrak{Rad}_h(n,m)$ of configurations $[L]$ such that every annulus contains at least one slit and such that the resulting cobordism $S(L)$ is connected. Bödigheimer proved in section 7.5 of [Böd06], with additional details in [Ebe05], that $\mathfrak{Rad}_h(n,m)$ is a model for moduli space:

Theorem 2.12 (Bödigheimer)**.** *The map that assigns to each $[L] \in \mathfrak{Rad}_h^{\mathrm{conn}}(n,m)$ the conformal class of the cobordism $S(L)$ gives us a homeomorphism*

$$\mathfrak{Rad}_h^{\mathrm{conn}}(n,m) \cong \mathcal{M}_g(n,m)$$

where g is determined by $h = 2g - 2 + n + m$.

The proof involves checking that $\mathfrak{Rad}_h(n,m)$ is a manifold of dimension $6h+n+m$ (see also [EF06] for remarks on the real-analytic structure). It sits as a dense open subset in the compact space $\overline{\mathfrak{Rad}}_h(n,m)$. In this way we can think of $\overline{\mathfrak{Rad}}_h(n,m)$ as a particular compactification of $\mathfrak{Rad}_h(n,m)$. Geometrically, one can think of it as the compactification where handles or boundary components can degenerate to radius zero, as long as there is always a path from each incoming to an outgoing boundary component that doesn't pass through any degenerate handles or boundary components. Colloquially, "the water must always be able to leave the tap".

Using the previous theorem, the following is easy to deduce (see e.g. section 2.4 of [Kup11]):

Corollary 2.13. *There is a homotopy equivalence*

$$\mathfrak{Rad}_h(n,m) \simeq \bigsqcup_{[\Sigma]} B\mathrm{Diff}(\Sigma, \partial\Sigma)$$

where the disjoint union is over all isomorphism classes of two-dimensional cobordisms with n incoming boundary components, m outgoing boundary components and total genus g determined by $h = 2g - 2 + n + m$.

Remark 2.14. One can make sense of glueing of cobordisms on the level of radial slits. This gives the spaces $\mathfrak{Rad}_h(n,m)$ the structure of a prop in topological spaces. One of the advantages of the radial slit configurations over fat graphs is the ease with which one can describe the prop structure.

Note that instead of first identifying preconfigurations and then taking out the configurations leading to degenerate surfaces, we could have done this the other way around. This leads to a different intermediate space.

Definition 2.15. The space of *radial slit preconfigurations* $\mathfrak{PRad}_h(n,m)$ is the subspace of $\overline{\mathfrak{PRad}}_h(n,m)$ obtained as the preimage of $\mathfrak{Rad}_h(n,m)$ under the quotient map $\overline{\mathfrak{PRad}}_h(n,m) \to \overline{\mathfrak{Rad}}_h(n,m)$.

There are two closely related spaces we would like to define for use in later sections. The first should be thought of as a space of representatives of radial slit configurations up to reordering of the slits.

Definition 2.16 (Unlabeled radial slit configurations)**.** The space of *unlabeled radial slit configurations* $\mathfrak{QRad}_h(n,m)$ is the quotient of $\mathfrak{PRad}_h(n,m)$ under the equivalence relation $\equiv'$ generated by just the relabelings of slits.

The second is a deformation retract of the harmonic compactification, where we have discarded homotopically irrelevant information.

Definition 2.17 (Unimodular harmonic compactification)**.** The *unimodular harmonic compactification* $\overline{\mathfrak{URad}}_h(n,m)$ is the subspace of $\overline{\mathfrak{Rad}}_h(n,m)$ of configurations satisfying $|\zeta_i| = 2$ for all $i \leq \{1,\ldots,2h\}$ and $R_j = 3$ for all $j \in \{1,\ldots,n\}$.

Note that apart from the inclusion $\iota : \overline{\mathfrak{URad}}_h(n,m) \hookrightarrow \overline{\mathfrak{Rad}}_h(n,m)$ there is also a projection $p : \overline{\mathfrak{Rad}}_h(n,m) \to \overline{\mathfrak{URad}}_h(n,m)$ which makes all slits have modulus 2 and all annuli have outer radius 3. This is a homotopy inverse to the inclusion.

Lemma 2.18. *The maps ι and p are mutually inverse up to homotopy.*

Proof. First note that $p \circ \iota$ is equal to the identity on $\overline{\mathfrak{URad}}$. For $\iota \circ p$, note that a homotopy from the identity on $\overline{\mathfrak{Rad}}$ to $\iota \circ p$ is given at time $t \in [0,1]$ sending each slit ζ_i to $\frac{(1-t)|\zeta_i|+2t}{|\zeta_i|}\zeta_i$ and each radius R_j to $(1-t)R_j + 3t$. □

The spaces constructed in this section fit together in the following diagram

$$\begin{array}{ccccc}
\mathfrak{PRad}_h(n,m) & \xrightarrow{\text{compactification}} & \overline{\mathfrak{PRad}}_h(n,m) & & \\
\downarrow & & \downarrow & & \\
\mathfrak{QRad}_h(n,m) & & & & \\
\downarrow & & & & \\
\mathfrak{Rad}_h(n,m) & \xrightarrow[\text{compactification}]{} & \overline{\mathfrak{Rad}}_h(n,m) & \overset{\simeq}{\rightleftarrows} & \overline{\mathfrak{URad}}_h(n,m)
\end{array}$$

2.2. **The universal surface bundle.** In the previous section, we motivated the definition of the space of radial slit configurations by explaining how each preconfiguration gives us exactly the data we need to construct a cobordism $S(L)$. Our choice of topology on the radial slit configurations was furthermore guided by the idea that this construction should in some sense be continuous. In this section we will make this precise, defining a surface bundle over $\mathfrak{Rad}$. This will turn out to be the universal surface bundle.

The first thing we note is that by definition of our equivalence relation $\equiv$ on $\mathfrak{PRad}_h(n,m)$, there is a canonical isomorphism between $S(L)$ and $S(L')$ if $L \equiv L'$. This allows us to make sense of the cobordism $S([L])$ for an equivalence class $[L]$.

The idea for constructing the universal surface bundle over $\mathfrak{Rad}_h(n,m)$ is that we will show how to make the construction of $S(L)$ continuous in L, in the sense that it gives us a space over $\mathfrak{PRad}_h(n,m)$, and then we will similarly identify cobordisms over equivalent points to get a surface bundle over $\mathfrak{Rad}_h(n,m)$. To check it is universal we will compare to the definition of the universal bundle over the conformal construction of moduli space.

We start by making sense of the radial sectors $\tilde{\Sigma}(L)$ as a space over $\overline{\mathfrak{PRad}}_h(n,m)$. This definition seems obvious; we think of the sectors as a subspace of a disjoint union of annuli for each L, so one is tempted to just state that $\tilde{\Sigma}(L)$ is the relevant subspace of $\overline{\mathfrak{PRad}}_h(n,m) \times \left(\bigsqcup_{j=1}^{n} \mathbb{A}_{R_j}\right)$. There are two problems with this: the full sectors aren't actually subspaces of annuli and the number of entire sectors is not constant over $\overline{\mathfrak{PRad}}_h(n,m)$. Both are relatively harmless problems and they are easily solved by firstly taking a suitable double cover of the annuli and secondly splitting $\overline{\mathfrak{PRad}}_h(n,m)$ into different components depending on the number of entire sectors. The technical details of this are covered in Lemma 2.5 of [Kup11], but we suffice here by saying that there exists a space $\tilde{\mathbb{A}}$ over $\overline{\mathfrak{PRad}}_h(n,m)$ whose fibers are essentially a constant disjoint union of annuli, such that there is a space $\tilde{\mathfrak{S}}_h(n,m) \subset \tilde{\mathbb{A}}$ whose fiber over L can be canonically identified with the sector space $\tilde{\Sigma}(L)$.

Recall that $\approx_L$ is the equivalence relation on $\tilde{\Sigma}(L)$ used to glue the sectors together and obtain a surface. We now define the equivalence relation $\sim$ by using the equivalence relation $\approx_L$ fiberwise:

Definition 2.19. Let $\sim$ be the equivalence relation on $\tilde{\mathfrak{S}}_h(n,m)$ generated by saying that a pair (L,z) where $L \in \overline{\mathfrak{PRad}}_h(n,m)$ and $z \in \tilde{\Sigma}(L) \subset \tilde{\mathfrak{S}}_h(n,m)$ and a pair (L',z') where $L' \in \overline{\mathfrak{PRad}}_h(n,m)$ and $z' \in \tilde{\Sigma}(L) \subset \tilde{\mathfrak{S}}_h(n,m)$ are equivalent if $L = L'$ and $z \approx_L z'$.

As mentioned before, there is a canonical isomorphism $\phi_{L,L'}$ between $\Sigma(L)$ and $\Sigma(L')$ if $L \equiv L'$. Using this we can define a version of $\equiv$ for $\tilde{\mathfrak{S}}_h(n,m)$.

Definition 2.20. Let $\cong$ be the equivalence relation on $\tilde{\mathfrak{S}}_h(n,m)$ generated by $\sim$ and by saying that (L,z) and (L',z') are equivalent if $L \equiv L'$ and $z' = \phi_{L,L'}(z)$.

We can now define the surface bundle.

Definition 2.21. We define $\mathfrak{S}_h(n,m)$ to be $\tilde{\mathfrak{S}}_h(n,m)/\cong$. This is a space over $\mathfrak{Rad}_h(n,m)$.

We will now sketch the proof that this space, a priori just a space over $\mathfrak{Rad}_h(n,m)$ with fibers having the structure of cobordisms, has the desired properties. That is, we will sketch how to proof it is a surface bundle and a universal one at that. This is implicit in [Böd06] but never explicitly stated there.

Proposition 2.22. *The space $\mathfrak{S}_h(n,m)$ over $\mathfrak{Rad}_h(n,m)$ is a universal surface bundle.*

Sketch of proof. It suffices to prove this for the space $\mathfrak{Rad}_h^{\mathrm{conn}}(n,m)$. Then Theorem 2.12 tells us that the assignment $[L] \mapsto [S[L],[g]_L]$ is a homeomorphism $\mathfrak{Rad}_h(n,m) \to \mathcal{M}_g(n,m)$. Pulling back the universal bundle over $\mathcal{M}_g(n,m)$ as defined as the end of subsection 1.1 exactly gives $\mathfrak{S}_h(n,m)$. $\square$

3. Admissible at graphs and string diagrams

3.1. **The definition.** Following the ideas of Strebel [Str84], Penner, Bowditch and Epstein gave a triangulation of Teichmüller space of surfaces with decorations, which is equivariant under the action of its corresponding mapping class group (cf. [Pen87, BE88]). In this triangulation, simplices correspond to equivalence classes of marked fat graphs and the quotient of this triangulation gives a combinatorial model of the moduli space of surfaces with decorations. These concepts were studied by Harer for the case of surfaces with punctures and boundary components (cf. [Har86]). These ideas where later used by Igusa to constructs a category of fat graphs that models the mapping class groups of punctured surfaces (cf. [Igu02]). Godin extends Igusa's construction for the cases of surfaces with boundary and for open-closed cobordisms (cf. [God07b, God07a]). In this section we will define a category of fat graphs and specific subcategories of it in the spirit of Godin. We also define the space of metric fat graphs as constructed by Harer and Penner and specific subspaces of it. These two ideas are closely related in the sense that the classifying spaces of these categories are homotopy equivalent to their corresponding spaces of metric fat graphs. In the end of the section we define the space of Sullivan diagrams which is a quotient of a certain subspace of the space of metric fat graphs.

Definition 3.1. A *combinatorial graph* G is a tuple $G = (V,H,s,i)$, with a finite set of *vertices* V, a finite set of *half edges* H, a *source map* $s : H \to V$ and an *edge pairing* involution with no fixed points $i : H \to H$.

The source map s ties each half edge to its source vertex, and the edge pairing involution i attaches half edges together. The set E of *edges* of the graph is given by the set of orbits of i. The *valence* of a vertex $v \in V$ is the cardinality of the set $s^{-1}(v)$. A *leaf* of a graph is a univalent vertex and an *inner vertex* is a vertex that is not a leaf. The *geometric realization* of a combinatorial graph G is the CW-complex $|G|$ with one 0-cell for each vertex, one 1-cell for each edge and attaching maps given by s. A *tree* is a graph whose geometric realization is a contractible space and a *forest* is a graph whose geometric realization is the disjoint union of contractible spaces.

Definition 3.2. A *fat graph* $\Gamma = (G,\sigma)$ is a combinatorial graph together with a cyclic ordering σ_v of the half edges incident at each vertex v. The *fat structure* of the graph is given by the data $\sigma = (\sigma_v)$ which is a permutation of the half edges.

From a fat graph $\Gamma = (G,\sigma)$ one can construct a surface with boundary Σ_Γ by thickening the edges. More explicitly, one can construct this surface by replacing each edge with a strip and glueing this strips at a vertex according to the fat structure. Notice that there is a strong deformation retraction of Σ_Γ onto $|G|$ so one can think of $|G|$ as the skeleton of the surface.

Definition 3.3. The *boundary cycles* of a fat graph are the cycles of the permutation of half edges given by $\omega = \sigma \circ i$.

Remark 3.4. Note that the fat structure of Γ is completely determined by ω. Moreover, one can show that the boundary cycles of a fat graph $\Gamma = (G,\omega)$ correspond to the boundary components of Σ_Γ (cf. [God07b]). Therefore, the surface Σ_Γ is completely determined up to topological type by the combinatorial graph and its fat structure.

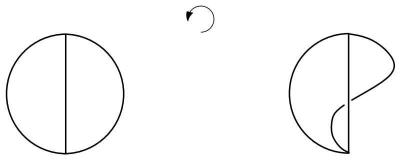


FIGURE 3.1. Two different fat graphs (where the fat structure is given by the orientation of the plane, here denoted by the circular arrow) which have the same underlying combinatorial graph.

We now construct the basic objects and morphisms that form the main categories we will use.

Definition 3.5. A *closed fat graph* $\Gamma = (\Gamma, \mathrm{In}, \mathrm{Out})$ is a fat graph with a partition of the set of leaves into two sets In and Out, such that:

(i) All inner vertices are at least trivalent.
(ii) The leaves are ordered and either an element of In or an element of Out, called *incoming* or *outgoing* respectively.
(iii) There is exactly one leaf on each boundary cycle.

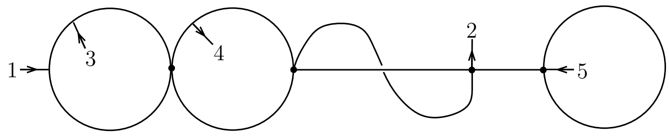


FIGURE 3.2. An example of a closed fat graph which is not admissible. The incoming and outgoing leaves are marked by incoming or outgoing arrows.

Definition 3.6. We call a closed fat graph Γ an *admissible fat graph* if the boundary cycles corresponding to the incoming closed leaves are disjoint embedded circles in Γ and we refer to these as *admissible cycles* (see Figure 3.3).

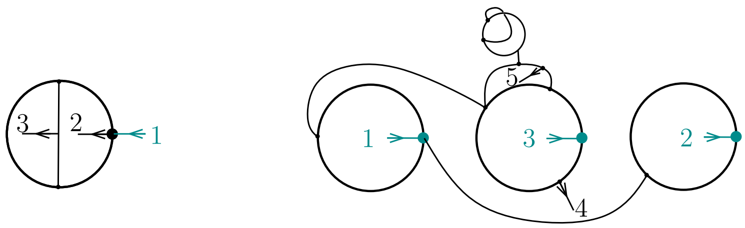


FIGURE 3.3. Two examples of admissible fat graphs. The graph on the left has the topological type of the pair of pants and the one on the right of a surface of genus 1 with 5 boundary components.

Definition 3.7. The category of closed fat graphs $\mathcal{F}at$ is the category with objects isomorphism classes of closed fat graphs and morphisms $[\Gamma] \to [\Gamma/F]$ given by collapses of subforests of Γ which do not contain any leaves. The category of admissible fat graphs $\mathcal{F}at^{ad}$ is the full subcategory of $\mathcal{F}at$ with objects isomorphism classes of admissible fat graphs.

The categories $\mathcal{F}at$ and $\mathcal{F}at^{ad}$ are well defined. In fact, the category $\mathcal{F}at$ was introduced by Godin in [God07b] and $\mathcal{F}at^{ad}$ is a slight variation of the category introduced by the same author in [God07a].

Remark 3.8. Note that the collapse of a subforest which does not contain any leaves is a surjective homotopy equivalence on geometric realizations which does not change the number of boundary cycles. Therefore, if there is a morphism $\varphi : [\Gamma] \to [\tilde{\Gamma}]$ between isomorphism classes of fat graphs then the surfaces $\Sigma_{[\Gamma]}$ and $\Sigma_{[\tilde{\Gamma}]}$ are homeomorphic.

From a closed fat graph we can construct a two-dimensional cobordism. The underlying surface of the cobordism is the oriented surface Σ_Γ. This gives an orientation of the incoming and outgoing boundary circles, so its enough to give a labelled marked point in each boundary component. Note that each of the boundary circles corresponds to exactly one leaf in the graph. We fix a marked point in each boundary circle and label it according to the labelling of its leaf. This gives a cobordism which is well defined up to isomorphism.

We now describe the space of metric fat graphs, several equivalent versions of this space and its dual concept (using weighted arc systems instead of fat graphs) have been studied by Harer, Penner, Igusa and Godin in [Har88, Pen87, Igu02, God04] respectively.

Definition 3.9. A *metric admissible fat graph* is a tuple (Γ, λ) where Γ is an admissible fat graph and λ is a *length function*, i.e. a function $\lambda : E_\Gamma \to \mathbb{R}^{\geq 0}$ where E_Γ is the set of edges of Γ and λ is given such that the following hold:

(i) $\lambda(e) = 1$ if e is a leaf.
(ii) $\lambda^{-1}(0)$ is a forest in Γ.
(iii) For any admissible cycle C in Γ it holds that $\sum_{e \in C} \lambda(e) = 1$.

We will call the value $\lambda(e)$ the *length* of the edge e in Γ.

Definition 3.10. Let Γ be an admissible fat graph with p admissible cycles and $(n_1, n_2 \dots n_p)$ be the number of edges on each admissible cycle and let $n := \sum_i n_i$. The *space of length functions on* Γ is given as a set by

$$\mathcal{M}(\Gamma) := \{\lambda : E_\Gamma \to \mathbb{R}^{\geq 0} | \lambda \text{ is a length function}\}$$

There is a natural inclusion

$$\mathcal{M}(\Gamma) \hookrightarrow \Delta^{n_1 - 1} \times \Delta^{n_2 - 1} \times \dots \times \Delta^{n_p - 1} \times (\mathbb{R}^{\geq 0})^{\sharp E_\Gamma - n}$$

we give $\mathcal{M}(\Gamma)$ the subspace topology via this inclusion.

Definition 3.11. Two metric admissible fat graphs (Γ, λ) and $(\tilde{\Gamma}, \tilde{\lambda})$ are called *isomorphic* if there is an isomorphism of admissible fat graphs $\varphi : \Gamma \to \tilde{\Gamma}$ such that $\lambda = \tilde{\lambda} \circ \varphi_*$, where φ_* is the map induced by φ on E_Γ.

Definition 3.12. The *space of metric admissible fat graphs* is defined as follows

$$\mathcal{MF}at^{ad} := \frac{\coprod_\Gamma \mathcal{M}(\Gamma)}{\sim}$$

where Γ runs over all admissible fat graphs and the equivalence relation $\sim$ is given by

$$(\Gamma, \lambda) \sim (\tilde{\Gamma}, \tilde{\lambda}) \iff (\Gamma/\lambda^{-1}(0), \lambda|_{E_\Gamma - \lambda^{-1}(0)}) \cong (\tilde{\Gamma}/\tilde{\lambda}^{-1}(0), \tilde{\lambda}|_{E_{\tilde{\Gamma}} - \tilde{\lambda}^{-1}(0)})$$

In other words, (i) we identify isomorphic admissible fat graphs with the same metric and (ii) a metric admissible fat graph with some edges of length 0 is identified with metric fat graph in which these edges are collapsed and all other edge lengths remain unchanged.

Notation 3.13. We denote by $\mathcal{MF}at_1^{ad}$, the subspace of metric admissible fat graphs whose edge length is at most 1.

Lemma 3.14. *The space of metric admissible fat graphs $\mathcal{MF}at^{ad}$ is homotopy equivalent to the classifying space of $\mathcal{F}at^{ad}$.*

Proof. We will define an inclusion $\iota : |\mathcal{F}at^{ad}| \hookrightarrow \mathcal{MF}at^{ad}$ giving an identification of the classifying space of $\mathcal{F}at^{ad}$ with the subspace of $\mathcal{MF}at_1^{ad}$ and then we give a strong deformation retraction of $\mathcal{MF}at^{ad}$ onto this subspace. A point $x \in |\mathcal{F}at^{ad}|$ is represented by $x = ([\Gamma_0] \to [\Gamma_1] \to \dots \to [\Gamma_k], s_0, s_1, \dots s_k) \in N_k \mathcal{F}at^{ad} \times \Delta^k$, where N_k denote the k-simplices of the nerve. Choose representatives Γ_i for $0 \leq i \leq k$ and for each i, let C_j^i denote the jth admissible cycle of Γ_i and n_j^i denote the number of edges in C_j^i. Each graph Γ_i naturally defines a metric admissible fat graph (Γ_0, λ_i) where λ_i is given as follows:

$$\begin{array}{rcl} \lambda_i : E_{\Gamma_0} & \longrightarrow & \mathbb{R}^{\geq 0} \\ e & \longmapsto & \left\{ \begin{array}{cl} 0 & \text{if } e \text{ is collapsed in } \Gamma_i \\ 1/n_j^i & \text{if } e \in C_j^i \\ 1 & \text{otherwise} \end{array} \right. \end{array}$$

Then define $\iota(x) := (\Gamma_0, \sum_{i=0}^{k} s_i\lambda_i)$. It is easy to show that this assignment is well defined and respects the simplicial relations of the geometric realization and thus defines a continuous map. This map is injective and its image is the subspace $\mathcal{MF}at_1^{ad}$ of metric admissible fat graphs whose edge length is at most 1.

We now construct a continuous map $r : \mathcal{MF}at^{ad} \times I \to \mathcal{MF}at^{ad}$ which is a strong deformation retraction of $\mathcal{MF}at^{ad}$ onto $\mathcal{MF}at_1^{ad}$. Since all the graphs we are considering are finite, we can define a continuous function g as follows:

$$\begin{array}{rcl} g : \mathcal{MF}at^{ad} & \longrightarrow & \mathbb{R}^{>0} \\ (\Gamma, \lambda) & \longmapsto & \max\{1, \max_{e\in E_\Gamma}\{\lambda(e)\}\} \end{array}$$

Then define r by linear interpolation as follows:

$$r((\Gamma,\lambda),t) := (\Gamma, (1-t)\lambda + t\lambda_g)$$

where λ_g is the rescaled length function given by:

$$\begin{array}{rcl} \lambda_g : E_\Gamma & \longrightarrow & \mathbb{R}^{\geq 0} \\ e & \longmapsto & \begin{cases} \lambda(e) & \text{if } e \text{ belongs to an admissible cycle} \\ \frac{\lambda(e)}{g(\Gamma,\lambda)} & \text{if } e \text{ does not belong to an admissible cycle} \end{cases} \end{array}$$

. □

Remark 3.15. The space $\mathcal{MF}at^{ad}$ and the category $\mathcal{F}at^{ad}$ splits into connected components given by the topological type of the graphs as two-dimensional cobordisms i.e.

$$\mathcal{MF}at^{ad} = \bigsqcup_{g,n,m} \mathcal{MF}at^{ad}_{g,n+m}$$

$$\mathcal{F}at^{ad} = \bigsqcup_{g,n,m} \mathcal{F}at^{ad}_{g,n+m}$$

where $\mathcal{MF}at^{ad}_{g,n+m}$ and $\mathcal{F}at^{ad}_{g,n+m}$ are the connected component corresponding to admissible fat graphs with n admissible cycles which are homotopy equivalent to a surface of genus g and $n+m$ boundary components.

We now define a quotient space of $\mathcal{MF}at^{ad}$, which we will see in section 5 is the analogue of the harmonic compactification for admissible fat graphs. To define this quotient space $\mathcal{SD}$ we first define an equivalence relation of metric admissible fat graphs $\sim_{SD}$.

Definition 3.16. We say $\Gamma_1 \sim_{SD} \Gamma_2$ if Γ_2 can be obtained from Γ_1 by:

Slides: Sliding vertices along edges that do not belong to the admissible cycles.

Forgetting lengths of non-admissible edge: Changing the lengths of the edges that do not belong to the admissible cycles.

Figure 3.4 shows some examples of equivalent admissible fat graphs.

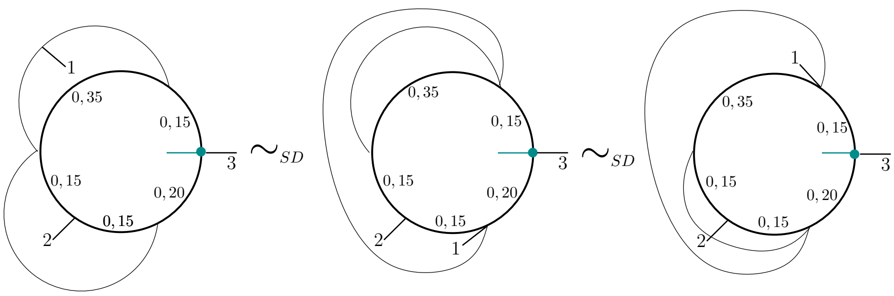


FIGURE 3.4. Three equivalent metric admissible fat graphs.

Definition 3.17. A *metric Sullivan diagram* is an equivalence class of metric admissible fat graphs under the relation $\sim_{SD}$.

We can think of a Sullivan diagram as an admissible fat graph where the edges not belonging to the admissible cycles are of length zero.

Definition 3.18. The *space of Sullivan diagrams* $\mathcal{SD}$ is the quotient space $\mathcal{SD} = \mathcal{MF}at^{ad}/\sim_{SD}$.

Remark 3.19. A path in $\mathcal{SD}$ is given by continuously moving the vertices on the admissible cycles and this space splits into connected components given by topological type.

Remark 3.20. In Section 5 we show that the space $\mathcal{SD}$ has canonical CW-complex structure. Its cellular chain complex is the complex of (cyclic) Sullivan chord diagrams introduced by Tradler and Zeinalian and used by them and afterwards by Wahl and Westerland to construct operations on the Hochschild chains of symmetric Frobenius algebras (cf. [TZ06, WW11]).

3.2. **The universal mapping class group bundle.** In this section we describe the universal mapping class group bundles over $\mathcal{F}at^{ad}$ and $\mathcal{MF}at^{ad}$. Recall that from an admissible fat graph we can construct an open closed cobordism well defined up to topological type. For each topological type choose and fix a representative surface $S_{g,n+m}$ of genus g with n incoming boundary components and m outgoing boundary components. Fix a marked point x_k in the kth incoming boundary for $1 \leq k \leq n$ and a marked point x_{k+n} in the kth outgoing boundary $1 \leq k \leq m$.

Definition 3.21. Let Γ be an admissible fat graph of topological type $S_{g,n+m}$ and let $v_{\text{in},k}$ denote the kth incoming leaf and $v_{\text{out},k}$ denote the kth outgoing leaf. A *marking* of Γ is an isotopy class of embeddings $H : |\Gamma| \hookrightarrow S_{g,n+m}$ such that $H(v_{\text{in},k}) = x_k$, $H(v_{\text{out},k}) = x_{k+n}$ and the fat structure of Γ coincides with the one induced by the orientation of the surface. We will call the pair $(\Gamma, [H])$ a *marked fat graph* and we denote by $\text{Mark}(\Gamma)$ the *space of markings of* Γ with the discrete topology.

Remark 3.22. Given a marking $H : |\Gamma| \hookrightarrow S_{g,n+m}$, by definition it holds that $\pi_1(\Gamma) \cong \pi_1(S_{g,n+m})$ and that H has the property that the induced map on π_1 sends the ith boundary cycle of Γ to the ith boundary component of $S_{g,n+m}$. Moreover, since the fat structure of Γ coincides with the one induced by the orientation of the surface we can thicken Γ inside $S_{g,n+m}$ to a subsurface of the same topological type as $S_{g,n+m}$. Thus, there is a deformation retraction of $S_{g,n+m}$ onto this subsurface and onto Γ, and the embedding H is a homotopy equivalence.

Remark 3.23. Let Γ be an admissible fat graph, F be a forest in Γ which does not contain any leaves of Γ and H be the representative of a marking $[H]$ of Γ. By the previous remark, the image of $H|_F$ (the restriction of H to $|F|$) is contained in a disjoint union of disks away from the boundary. Therefore, the marking H induces a marking $H_F : |\Gamma/F| \hookrightarrow S_{g,n+m}$ given by collapsing each of the trees of F to a point of the disk in which their image is contained. Note that H_F is well defined up to isotopy and it makes the following diagram commute up to homotopy

$$\begin{array}{ccc} |\Gamma| & \twoheadrightarrow & |\Gamma/F| \\ & {\scriptstyle H}\searrow & \big\downarrow {\scriptstyle H_F} \\ & & S_{g,n+m} \end{array}$$

In fact, up to isotopy, there is a unique embedding of a tree with a fat structure into a disk, in which the fat structure of the tree coincides with the one induced by the orientation of the disk and the endpoints are fixed points on the boundary. This can proven by induction. Start with the case where F is a single edge. Up to homotopy, there is a unique embedding of an arc in a disk where the end points of the arc are fixed points at the boundary. Then by [Feu66], there is also a unique embedding up to isotopy. For the induction step, let α be an arc embedded in the disk with its endpoints at the boundary and let a and b be fixed points in the boundary of a connected component of $D - \alpha$. Then we have a map

$$\text{Emb}^{a,b}(I, D - \alpha) \longrightarrow \text{Emb}^{a,b}(I, D)$$

where $\text{Emb}^{a,b}(I, D - \alpha)$ is the space of embeddings of a path in $D - \alpha$ which start at a and end at b, with the $\mathbb{C}^\infty$ topology, and similarly for $\text{Emb}^{a,b}(I, D)$. By [Gra73], this map induces injective maps in all homotopy groups, in particular in π_0, which gives the induction step.

It then follows that, given $[H_F]$ a marking of Γ/F there is a unique marking $[H]$ of Γ such that the above diagram commutes up to homotopy. Thus, there is a one to one correspondence between

Mark(Γ) and Mark(Γ/F). From now on, we will denote by $[H_F]$ the marking of Γ/F corresponding to the marking $[H]$ of Γ under this identification. This identification depends on the map connecting both graphs i.e. given $[H]$ a marking of Γ, if $\tilde{\Gamma} = \Gamma/F_1 = \Gamma/F_2$ then $[H_{F_1}]$ and $[H_{F_2}]$ can be different markings of $\tilde{\Gamma}$. Figure 3.5 gives an example of this for the case of the cylinder.

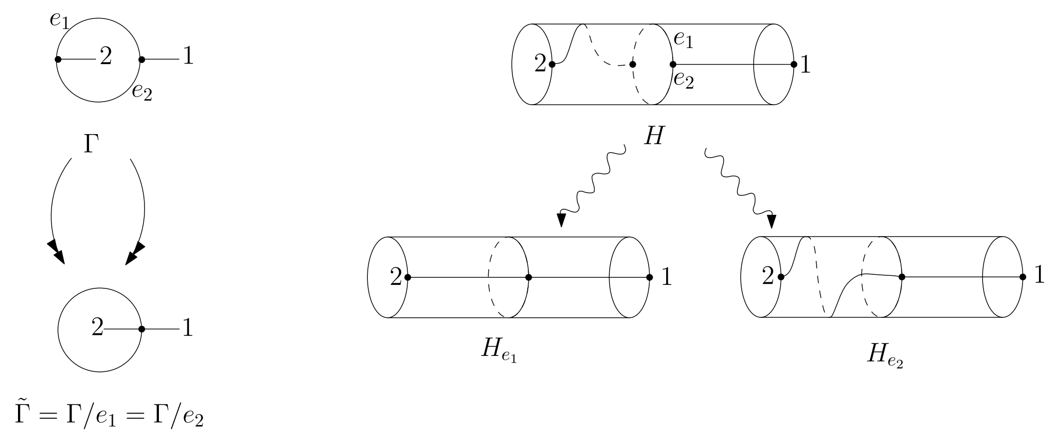


FIGURE 3.5. Two different embeddings of $\tilde{\Gamma}$ in the cylinder differing by a Dehn twist and corresponding to the same marking of Γ.

Definition 3.24. Define the category $\mathcal{EF}at^{ad}$ to be the category with objects isomorphism classes of marked admissible fat graphs $([\Gamma], [H])$ (where two marked admissible fat graphs are isomorphic if their underlying fat graphs are isomorphic and they have the same marking) and morphisms given by morphisms in $\mathcal{F}at^{ad}$ where the map acts on the marking as stated in the previous remark.

Remark 3.25. Let $\mathcal{EF}at^{ad}_{g,n+m}$ denote the full subcategory with objects marked admissible fat graphs whose thickening give a cobordism of topological type $S_{g,n+m}$. The mapping class group of $S_{g,n+m}$ denoted $\mathrm{Mod}(S_{g,n+m})$, acts on $\mathcal{EF}at^{ad}_{g,n+m}$ by composition with the marking.

Following the ideas in [God07b], the following result is proven in [Ega14]

Proposition 3.26. *The projection* $|\mathcal{EF}at^{ad}_{g,n+m}| \to |\mathcal{F}at^{ad}_{g,n+m}|$ *is a universal* $\mathrm{Mod}(S_{g,n+m})$*-bundle.*

In [Ega14] this result is given in more generality for a category modelling open closed cobordism and not only closed cobordisms. The idea of the proof is simple, and follows the original ideas of Igusa and Godin. Since all spaces involved are CW-complexes it is enough to shows that $|\mathcal{EF}at^{ad}_{g,n+m}|$ is contractible, which one can do directly by using the contractibility of the arc complex; and that the action of $\mathrm{Mod}(S_{g,n+m})$ on $\mathcal{EF}at^{ad}_{g,n+m}$ is free and transitive i.e. for any two markings $[H_1]$ and $[H_2]$ there is a unique $[\varphi] \in \mathrm{Mod}(S_{g,n+m})$ such that $[\varphi \circ H_1] = [H_2]$. We want to extend this universal bundle construction to a universal bundle over $\mathcal{MF}at^{ad}$.

Definition 3.27. The *space of marked metric admissible fat graphs* $\mathcal{EMF}at^{ad}$ is defined to be

$$\mathcal{EMF}at^{ad} := \frac{\coprod_\Gamma \mathcal{M}(\Gamma) \times \mathrm{Mark}(\Gamma)}{\sim_E}$$

where Γ runs over all admissible fat graphs and the equivalence relation is given by

$$(\Gamma, \lambda, [H]) \sim_E (\tilde{\Gamma}, \tilde{\lambda}, [\tilde{H}]) \iff (\Gamma, \lambda) \cong (\tilde{\Gamma}, \tilde{\lambda}) \text{ and } [H_{\lambda^{-1}(0)}] = [\tilde{H}_{\tilde{\lambda}^{-1}(0)}]$$

and ≅ denotes isomorphism of metric fat graphs.

By remark 3.23 we can see, that as a set $\mathcal{EMF}at^{ad}$ is given by $\{([\Gamma, \lambda], [H]) | [\Gamma, \lambda] \in \mathcal{MF}at^{ad}, [H] \in \mathrm{Mark}([\Gamma])\}$. As before, let $\mathcal{EMF}at^{ad}_{g,n+m}$ denote the subspace of marked metric admissible fat graphs whose thickening give an open closed cobordism of topological type $S_{g,n+m}$. Then again, $\mathrm{Mod}(S_{g,n+m})$ acts on $\mathcal{EMF}at^{ad}_{g,n+m}$ by composition with the marking.

Proposition 3.28. *The projection* $\mathcal{EMF}at^{ad}_{g,n+m} \to \mathcal{MF}at^{ad}_{g,n+m}$ *is a universal* $\mathrm{Mod}(S_{g,n+m})$*-bundle.*

Proof. It suffices to show that $\mathcal{EMFat}^{ad}_{g,n+m}$ is the pullback of a universal $\mathrm{Mod}(S_{g,n+m})$-bundle along a homotopy equivalence. Recall from the proof of Lemma 3.14 that we have constructed a homotopy equivalence $r(-,1) : \mathcal{MFat}^{ad} \to \mathcal{MFat}^{ad}_1$ which rescales each graph such that all edge lengths are at most 1 and a homeomorphism $\iota : |\mathcal{Fat}^{ad}| \to \mathcal{MFat}^{ad}_1$ by using the barycentric coordinates of the realization to define the edge lengths of the graphs. Using the restrictions of these maps to each connected component we construct the diagram below:

$$\begin{array}{ccccc} \mathcal{EMFat}^{ad}_{g,n+m} & \xrightarrow[\simeq]{r(-,1)|_{g,n+m}\times \mathrm{id}} & \mathcal{EM}^{ad}_{(g,n+m),1} & \xleftarrow[\cong]{\iota|_{g,n+m}\times \mathrm{id}} & |\mathcal{EFat}^{ad}_{g,n+m}| \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{MFat}^{ad}_{g,n+m} & \xrightarrow[\simeq]{r(-,1)|_{g,n+m}} & \mathcal{MFat}^{ad}_{(g,n+m),1} & \xleftarrow[\cong]{\iota|_{g,n+m}} & |\mathcal{Fat}^{ad}_{g,n+m}| \end{array}$$

It is clear by inspection that the diagram commutes and that the left square is a pullback diagram. Then by 3.26 the proposition holds. ◻

4. The critical graph equivalence between radial slit configurations and fat graphs

4.1. **The blowup of the radial slit configurations and the critical graph map.** In Bödigheimer's construction there is a natural admissible metric fat graph associated to a configuration; the unstable critical graph. This is the graph obtained by considering inner boundaries of the annuli and the complements of the slit segments and glueing them together according to the combinatorial data. The inner boundaries of the annuli give the admissible cycles of the graph and the incoming leaves are placed at the positive real line of each annuli. The outgoing leaves are built using marked points on the outgoing boundary components. This graph is a canonical fat graph inside the surface $S(L)$.

We now make this definition precise. To a radial slit configuration $L \in \mathfrak{QRad}$ we associate a space E_L defined as follows:

Definition 4.1. The space E_L is given by

$$E_L = \left(\bigsqcup_{1\le j\le n} \partial_{\mathrm{in}}\mathbb{A}_j\right) \sqcup \left(\bigsqcup_{1\le j\le 2h} E_j\right) \sqcup \left(\bigsqcup_{1\le j\le n} I_j\right) \sqcup \left(\bigsqcup_{1\le j\le m} E_{2h+j}\right)$$

where the terms are given by

(i) For $1 \le j \le 2h$ for each slit $\zeta_j \in \mathbb{A}_k$ we have that $E_j = \{z \in \mathbb{A}_k |\, \arg(z) = \arg(\zeta_j), |z| \le |\zeta_j|\}$. Recall that we have the notation of $\xi_j = \zeta_j$ for $1 \le j \le 2h$ and $\xi_{j+2h} = P_j$ for $1 \le j \le m$, so we can similarly define E_{2h+j}.
(ii) For each annulus $\mathbb{A}_j$ we have that $I_j = \{z \in \mathbb{C}_j |\, \arg(z) = 0, 0 \le z \le 1\}$.
(iii) For $1 \le j \le m$ for each marked point $P_j \in \mathbb{A}_k$ we have that $E_{j+2h} = \{z \in \mathbb{A}_k |\, \arg(z) = \arg(P_j)\}$.

We define an equivalence relation $\sim_L$ on E_L as the equivalence relation generated by:

(i) We have that $(1 \in I_j) \sim_L (1 \in \partial_{in}\mathbb{A}_j)$ for $j = 1, 2, \dots n$.
(ii) For $r \in \partial_{\mathrm{in}}\mathbb{A}_k$ and $e \in E_j$, we set $r \sim_L e$ if and only if $r = e$.
(iii) For $e \in E_j$ and $e' \in E_k$, we set $e \sim_L e'$ if and only if $e = \zeta_j$, $e' = \zeta_k$, and $j = \lambda(k)$.
(iv) For $e \in E_j$ and $e' \in E_{\tilde\omega(j)}$, we set $e \sim_L e'$ if and only if ξ_j and $\xi_{\tilde\omega(j)}$ lie on the same radial segment and $|e| = |e'| \le \min\left\{|\xi_j|, |\xi_{\tilde\omega(j)}|\right\}$.

Definition 4.2. For $L \in \mathfrak{QRad}$ the corresponding *critical graph* Γ_L is the underlying graph of the quotient space $E_L/\!\sim_L$ (see Figure 4.1).

Note that the quotient space Γ_L is invariant under the slit jump relation. Thus for a configuration $[L] \in \mathfrak{Rad}$ there is a well defined graph $\Gamma_{[L]}$. This is actually an isomorphism class of a graph since the half edges are not labeled, but we will write $\Gamma_{[L]}$ for simplicity since the choice of a representative will not affect any of the constructions below.

Furthermore, this graph is naturally embedded in the surface $\Sigma_{[L]}$ and thus it has a fat structure induced by the orientation of the surface. Moreover, this graphs is also naturally endowed with a metric $\lambda_{[L]}$ given by the standard metric in $\mathbb{C}$. This association is such that the incoming leaves always have length 1 and the outgoing leaves always have a strictly positive length. Because for our purposes the lengths of the outgoing leaves is superfluous information, we set $\lambda_{[L]}(e)$ to be given by the standard metric in $\mathbb{C}$ if e is not a leaf and $\lambda_{[L]}(e) = 1$ if e is a leaf. This makes $(\Gamma_{[L]}, \lambda_{[L]})$ a metric admissible fat graph. We will just write Γ_L, when it is clear from that context that we are talking about the critical metric graph. Figure 4.1 shows some examples of critical metric graphs for simple radial slit configurations.

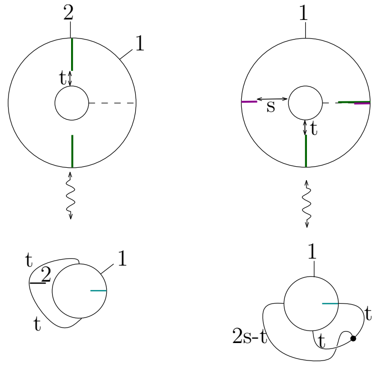


FIGURE 4.1. Critical graphs for different configurations.

This construction gives a natural association $\mathfrak{Rad} \to \mathcal{MFat}^{ad}$ given by $[L] \mapsto (\Gamma_{[L]}, \lambda_{[L]})$. However, this map is not continuous. The discontinuity occurs at non-generic configurations. To see this consider for example a path in $\mathfrak{Rad}$ given by continuously changing the the argument of a single slit as shown in Figure 4.2. When the moving slit reaches a neighbour slit the associated metric graph jumps. To solve this problem we need to separate the different representatives of a non-generic configuration and connect them without changing the homotopy type of $\mathfrak{Rad}$. We will do this by constructing a space $\mathfrak{Rad}^{\sim}$ in which we enlarge $\mathfrak{Rad}$ at non-generic configurations by a contractible space.

To define $\mathfrak{Rad}^{\sim}$ we will define a smaller equivalence relation $\sim_t$ on E_L for $L \in \mathfrak{QRad}$. To do this, we first need to introduce some useful notiation.

Notation 4.3. Given $L \in \mathfrak{QRad}$, recall that we have denoted $\xi_j = \zeta_j$ for $1 \leq j \leq 2h$ and $\xi_{j+2h} = P_j$ for $1 \leq j \leq m$. The ξ_i's define l distinct radial segments where $l \leq 2h + m$. These can be ordered lexicographically using the pairs (k, θ) where k is the number of the annulus to which the radial segment corresponds and θ is its argument, giving a totally ordered list of radial segments $S_1, S_2 \dots S_l$. The ξ_j's that lie on S_i can be totally ordered using $\tilde{\omega}$ and $\vec{r}$. We denote by ξ_{i_j} the j-th

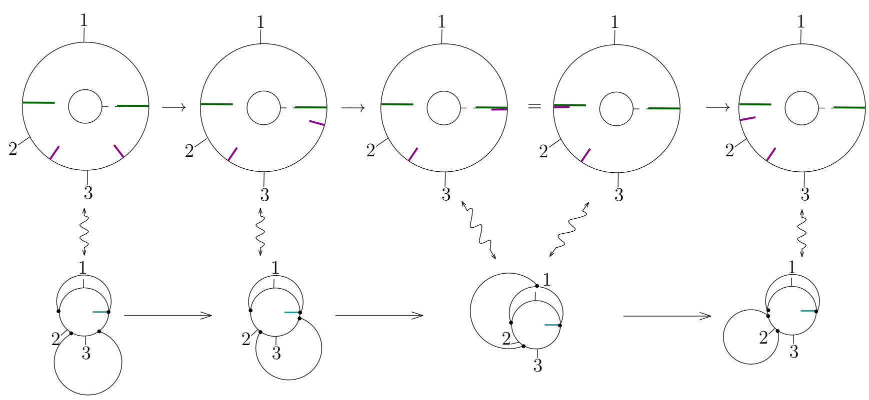


FIGURE 4.2. An example of a path in $\mathfrak{Rad}$ which has an associated non continuous path in $\mathcal{MFat}^{ad}$.

slit or parametrization point lying on S_i according to this order, i.e. the slits and parametrization points lying on S_i are $\xi_{i_1}, \xi_{i_2} \dots \xi_{i_{s_i}}$ where s_i be the number of slits and marked points that lie on S_i. Note that this notation is independent of the labelling of the slits.

Definition 4.4. Let $d_i(L) = \sum_{j=1}^{i-1} (s_j - 1)$, and $d(L) = \sum_{i=1}^{l} d_i(L)$ and let $t \in I^{d(L)}$ where I is the unit interval. We define an equivalence relation $\sim_t$ on the space $E_L = \left(\bigsqcup_{1\leq j\leq n} \partial_{\text{in}}\mathbb{A}_j\right) \sqcup \left(\bigsqcup_{1\leq j\leq 2h} E_j\right) \sqcup \left(\bigsqcup_{1\leq j\leq n} I_j\right) \sqcup \left(\bigsqcup_{1\leq j\leq m} E_{j+2h}\right)$ to be generated by:

- (i) $(1 \in I_j) \sim (1 \in \partial_{in}\mathbb{A}_j)$ for $j = 1, 2, \dots n$.
- (ii) For $r \in \partial_{in}\mathbb{A}_j$ and $e \in E_j$ we have that $r \sim_t e$ if and only if $r = e$.
- (iii) For $1 \leq j, k \leq 2h$, $e \in E_j$ and $e' \in E_k$ we have that $e \sim_t e'$ if and only if $e = \zeta_j$, $e' = \zeta_k$, and $j = \lambda(k)$.
- (iv) For $1 \leq i \leq l$, $1 \leq j \leq s_i - 1$, $e \in E_{i_j}$ and $e' \in E_{i_{j+1}}$ we have that $e \sim_t e'$ if and only if $|e| = |e'| \leq t_{d_i+j}(\min\left\{|\xi_{i_j}|, |\xi_{i_{j+1}}|\right\}) + (1 - t_{d_i+j})$. Notice that the conditions imply that ξ_{i_j} and $\xi_{i_{j+1}}$ lie on the same radial segment, namely S_i.

Definition 4.5. We define $\Gamma_{L,t}$ to be the underlying graph of the quotient space $E_L/\sim_t$. If $\alpha = (0, 0, 0 \dots 0)$ we will call this the *unfolded graph of* L and denote it $\Gamma_{L,0}$ (see Figure 4.3).

Notice that as before $\Gamma_{L,t} \in \mathcal{F}at^{ad}$ since it has a naturally associated fat structure. For $t = (1, 1, 1, \dots, 1)$ we have that $\Gamma_{L,t}$ is the critical graph Γ_L which is invariant under slit and parametrization points jumps. However, for any other t, the graph $\Gamma_{L,t}$ is not invariant under slit jumps, so it is not well defined for $[L] \in \mathfrak{Rad}$. As for the critical graph $\Gamma_{L,t}$ has a natural metric making $(\Gamma_{L,t}, \lambda_{L,t})$ an admissible metric fat graph. Figure 4.3 shows examples of unfolded and partially unfolded metric admissible fat graphs of a specific configuration.

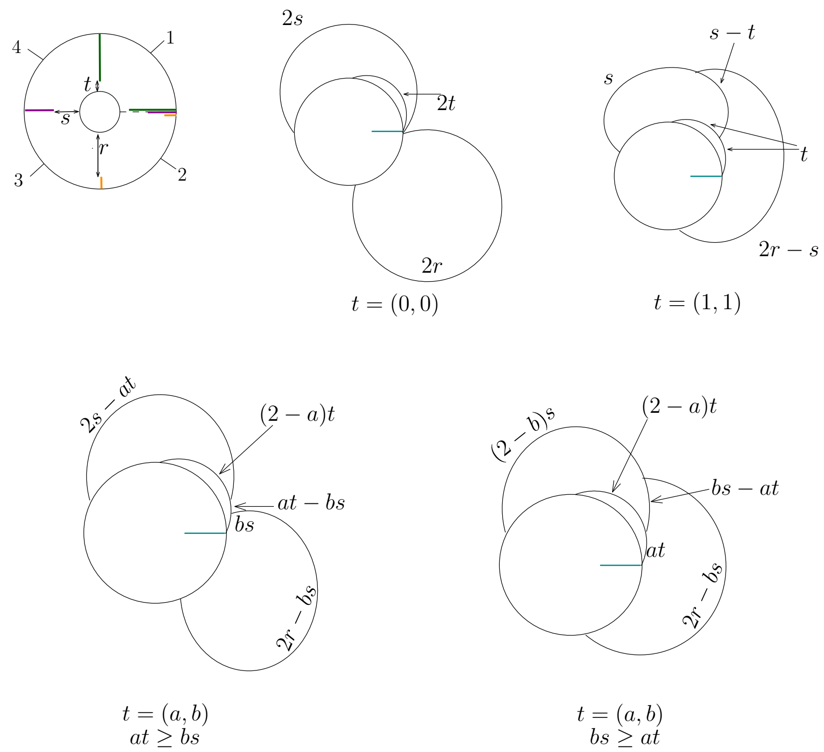


FIGURE 4.3. A configuration $[L]$ on the top left, and several unfolded graphs of it for different t. The leaves have been omitted from the unfolded graphs to make them more readable, but in this case they are all located along the admissible cycles according to the positions of the marked points in $[L]$.

Definition 4.6. Let $[L] \in \mathfrak{Rad}$, we define a subspace of $\mathcal{MF}at^{ad}$

$$\mathcal{G}([L]) := \{[\Gamma_{L_i,t}, \lambda_{L_i,t}] | [L] = [L_i], t \in I^{d(L_i)}\}.$$

We define the blow up of $\mathfrak{Rad}$ to be the space

$$\mathfrak{Rad}^{\sim} = \{([L], [\Gamma, \lambda]) \in \mathfrak{Rad} \times \mathcal{MFat}^{ad} | [\Gamma, \lambda] \in \mathcal{G}([L])\}.$$

For simplicity, we will just write $\Gamma_{L_i,t}$ or Γ when it is clear from the context that we are talking about metric graphs.

We will show that $\mathfrak{Rad}^{\sim}$ is constructed by blowing up $\mathfrak{Rad}$ at $[L]$ by a contractible space, $\mathcal{G}([L])$, which is a family of graphs that "interpolates" between the critical graph of $[L]$ and the unfolded graphs of the different representatives $L_1, L_2, \ldots L_k$ of $[L]$ in $\mathfrak{QRad}$.

Definition 4.7. For L in $\overline{\mathfrak{PRad}}$ the radial segments of the slits, the parametrization points and the positive real lines, divide the annuli in which the pre-configuraiton L sits into different areas which we will call *chambers* (see Figure 4.5). Moreover, each slit ζ_i in L defines a circle of radius $|\zeta_i|$ on the annulus where it sits. These circles, divide the annuli into *radial sectors* (see Figure 4.4). Given L and L' in $\overline{\mathfrak{PRad}}$, we say they have the same combinatorial data if L' can be obtained from L by continuously moving the slits and parametrization points without collapsing any chamber or radial sector. This defines an equivalence relation on $\overline{\mathfrak{PRad}}$. A *combinatorial type* $\mathcal{L}$ is an equivalence class of pre-configurations under this relation. Intuitively, this is the data carried over by the picture of a pre-configuration without remembering the precise placement of the slits. Notice that if L is a degenerate (respectively non degenerate) pre-configuration then so is any pre-configuration of the same combinatorial type. Thus, we can talk about a degenerate or non degenerate combinatorial type. Moreover, this definition passes to the quotient. Thus one can talk about $[\mathcal{L}]$ a combinatorial type of configurations.

Notice that two pre-configurations with the same combinatorial type have the same (non-metric) admissible fat graphs but with different length functions. Thus it makes sense to talk about $\Gamma_{\mathcal{L},t}$ which is a (non-metric) admissible fat graph. Similarly, it makes sense to talk about the critical graph of a combinatorial type of a configuration which we denote $\Gamma_{[\mathcal{L}]}$.

Lemma 4.8. *The subspace $\mathcal{G}([L])$ is a contractible finite CW-complex.*

Proof. Let $L_1, L_2, \ldots L_k$ be the different representatives of $[L]$ in $\mathfrak{QRad}$ and let $\mathcal{L}_1, \mathcal{L}_2, \ldots \mathcal{L}_k$ be their combinatorial types. We define a category $\mathsf{G}([\mathcal{L}])$ which is a full subcategory of $\mathcal{Fat}^{ad}$ on objects $\Gamma_{\mathcal{L}_i,t}$ for $t \in \{0,1\}^{d([L])}$. Recall that for any $1 \leq i \leq k$ we have that $\Gamma_{[\mathcal{L}]} = \Gamma_{\mathcal{L}_i,(1,1,\ldots,1)}$. Therefore, for any $\Gamma \in \mathsf{G}([\mathcal{L}])$ there is a morphism $\Gamma_{[\mathcal{L}]} \to \Gamma$. Moreover, let $F_{[\mathcal{L}]}$ be the subgraph of $\Gamma_{[\mathcal{L}]}$ consisting of the union of all edges collapsed in a morphism $\Gamma_{[\mathcal{L}]} \to \Gamma$ in $\mathsf{G}([\mathcal{L}])$. Then, we can check by the construction of the unfolded graphs that $F_{[\mathcal{L}]}$ is in fact a subforest of $\Gamma_{[\mathcal{L}]}$ and thus the morphisms $\Gamma_{[\mathcal{L}]} \to \Gamma$ are unique. Therefore, $\Gamma_{[\mathcal{L}]}$ is initial in $\mathsf{G}([\mathcal{L}])$ which shows that this is a contractible and finite category. We will give a homeomorphism $f : |\mathsf{G}([\mathcal{L}])| \to \mathcal{G}([L])$ by using a barycentric coordinates as in the proof of 3.14, which finishes the proof. A point $x \in |\mathsf{G}([\mathcal{L}])|$ is given by a tuple

$$((\Gamma_{\mathcal{L}_{i_0},t_0} \to \Gamma_{\mathcal{L}_{i_1},t_1} \to \ldots \to \Gamma_{\mathcal{L}_{i_n},t_n}), (s_0, s_1, \ldots, s_n)) \in N_n\mathsf{G}([\mathcal{L}]) \times \Delta^n$$

We define $f(x) := (\Gamma_{L_{i_0},t_0}, \sum_{j=1}^{n} s_j \lambda_{L_{i_j},t}) \in \mathcal{G}([L])$. Note that any graph in $\mathcal{G}([L])$ has an underlying (non-metric) graph of the form $\Gamma_{L_i,t}$ for $t \in \{0,1\}^{d([L])}$ and that all possible metrics are given by linear interpolations between $\lambda_{L_i,t}$ for $t \in \{0,1\}^{d([L])}$. Thus, it is easy to see that f is a well defined continuous bijection. □

The blow up of $\mathfrak{Rad}$ splits into connected components given by the topological type of the cobordism they describe. That is

$$\mathfrak{Rad}^{\sim} := \coprod_{h,p,q} \mathfrak{Rad}_h^{\sim}(p,q)$$

Moreover, this space comes with two natural projections

$$\mathfrak{Rad} \xleftarrow{\pi_1} \mathfrak{Rad}^{\sim} \xrightarrow{\pi_2} \mathcal{MFat}^{ad}$$

We call π_1 the *blow-down map* and π_2 the *critical graph map*. In the remaining subsections we will show that these projections are homotopy equivalences.

4.2. **The blow-down map is a homotopy equivalence.** We want to prove that the blow-down map $\pi_1 : \mathfrak{Rad}^\sim \to \mathfrak{Rad}$ is a homotopy equivalence. The idea of the proof is as follows: both $\mathfrak{Rad}^\sim$ and $\mathfrak{Rad}$ are nice spaces and π_1 is a nice map with contractible fibers, so it is a homotopy equivalence. To make this statement precise, we need to replace "nice" with actual mathematical content. The precise statement is: $\mathfrak{Rad}$ and $\mathfrak{Rad}^\sim$ are *absolute neighborhood retracts* (henceforth ANR's), and $\pi_1 : \mathfrak{Rad}^\sim \to \mathfrak{Rad}$ is a proper cell-like map. We can then use the following result of Lacher, more precisely Lemma 2.1 of [Lac68].

Theorem 4.9 (Lacher)**.** *A proper cell-like map between ANR's is a proper homotopy equivalence.*

We will now define the terms that appear in this theorem.

Definition 4.10. A space X is an *ANR* if it has the property that if X is a closed subspace of a metric space Y, then X is a neighborhood retract of Y.

Definition 4.11. (i) A subset A of a manifold M is *cellular* is it is the intersection $\bigcap_n E_n$ of a nested countable sequence $E_1 \subset E_2 \subset \ldots$ of n-cells E_i in manifold M, i.e. subsets homeomorphic to D^n.
(ii) A space X is *cell-like* if there is an embedding (i.e. continuous map that is an homomorphism onto its image) $\phi : X \to M$ of X into a manifold, such that $\phi(X)$ is cellular.
(iii) A map $f : X \to Y$ is *cell-like* if for all $y \in Y$ the point inverse $f^{-1}(\{y\})$ is a cell-like space.

Both of these definitions are sufficiently abstract that it is hard to apply them directly. Our main reference for ANR's is [vM89] and our main reference for cell-like spaces is [Lac68]. We will now give two propositions stating the properties of ANR's and cell-like spaces. This will use Borsuk's notion of shape for compacta in the Hilbert cube [Bor68].

Proposition 4.12. *The following are properties of ANR's:*
(i) For all $n \geq 0$, the closed n-disk is an ANR.
(ii) An open subset of an ANR is an ANR.
(iii) Let X be a space with an open cover by ANR's, then X is an ANR.
(iv) If X and Y are compact ANR's, $A \subset X$ is a compact ANR and $f : A \to Y$ is continuous, then $X \cup_f Y$ is an ANR.
(v) Any locally finite CW-complex is an ANR.
(vi) The shape type of a compact ANR is well-defined and equal to its homotopy type.

Proof. Property (i) follows from Corollary 5.4.6 of [vM89], property (ii) is Theorem 5.4.1, property (iii) is theorem 5.4.5, property (iv) is Theorem 5.6.1. Together these can combined to prove property (v), by noting that by (ii) and (iii) one can reduce to the case of finite CW-complex and since by definition these can be obtained by glueing closed n-disks together, (i) and (iv) prove that finite CW-complexes are ANR's. Property (vi) is Theorem 2.1 of [Bor68]. ☐

Proposition 4.13. *The following are properties of cell-like spaces:*
(i) A finite-dimensional metric space is cell-like if and only if has the shape type of a point.
(ii) Finite contractible CW-complexes are cell-like.

Proof. Property (i) is Theorem 1 of [Lac68] and property (ii) follows by combining this with properties (v) and (vi) of 4.12. ☐

Our next goal is to check that the spaces $\mathfrak{Rad}$, $\mathfrak{Rad}^\sim$ are ANR's and that the map $\pi_1 : \mathfrak{Rad}^\sim \to \mathfrak{Rad}$ is proper and cell-like.

Proposition 4.14. *The space $\mathfrak{Rad}$ is an ANR.*

Proof. It is a manifold and by Morse theory every manifold is a locally finite CW-complex. These are ANR's by property (v) of Proposition 4.12. Alternatively one can argue that $\mathfrak{Rad}$ is an open subspace of the finite CW-complex $\overline{\mathfrak{Rad}}$ and use properties (ii) and (v) of Proposition 4.12. ☐

To prove that $\mathfrak{Rad}^\sim$ is an ANR and that π_1 is a proper cell-like map, we will write $\mathfrak{Rad}^\sim$ as an open subspace of a space obtained by glueing together finitely many compact ANR's.

Proposition 4.15. *The space $\mathfrak{Rad}^\sim$ is an ANR.*

Proof. It is enough to show this in each connected component. In this proof we fix g, h, n, and m and we drop them from the notation. Note that $\overline{\mathfrak{Rad}}\backslash\mathfrak{Rad}$ is a finite CW-complex, being a subcomplex of over $\overline{\mathfrak{Rad}}$. Consider the following subspace of $\overline{\mathfrak{Rad}} \times \mathcal{MFat}^{ad}$:

$$(\overline{\mathfrak{Rad}})^{\sim} = \begin{cases} ([L], \Gamma, \lambda) & \text{if } [L] \in \mathfrak{Rad} \text{ and } (\Gamma, \lambda) \in \mathcal{G}(L) \\ ([L], \Gamma, \lambda) & \text{if } [L] \in \overline{\mathfrak{Rad}}\backslash\mathfrak{Rad} \text{ and } (\Gamma, \lambda) \in \mathcal{MFat}^{ad} \text{ with all edges of length} \leq \max\{R_j\} \end{cases}$$

This space is homeomorphic to one obtained by glueing together $\overline{\mathfrak{Rad}}\backslash\mathfrak{Rad}\times\mathcal{MFat}_1^{ad}$ and $\overline{\mathfrak{Rad}_{\mathcal{L}}}\times\mathcal{G}([\mathcal{L}])$ for all combinatorial types $\mathcal{L}$ along $\partial\overline{\mathfrak{Rad}_{\mathcal{L}}} \times \mathcal{G}([\mathcal{L}])$. Both $\overline{\mathfrak{Rad}}\backslash\mathfrak{Rad} \times \mathcal{MFat}_1^{ad}$ and the $\overline{\mathfrak{Rad}_{\mathcal{L}}} \times \mathcal{G}([\mathcal{L}])$ are finite CW-complexes and thus compact ANR's. Using induction over the dimension of the cells $\overline{\mathfrak{Rad}_{\mathcal{L}}}$, one proves that $\partial\overline{\mathfrak{Rad}_{\mathcal{L}}} \times \mathcal{G}([\mathcal{L}])$ and $\overline{\mathfrak{Rad}}\backslash\mathfrak{Rad} \times \mathcal{MFat}_1^{ad} \cup \left(\bigcup_{\dim\leq k} \overline{\mathfrak{Rad}_{\mathcal{L}}} \times \mathcal{G}([\mathcal{L}])\right)$ are ANR's by repeatedly applying property (iv) of Proposition 4.12. We conclude that $(\overline{\mathfrak{Rad}})^{\sim}$ is also an ANR.

Finally $\mathfrak{Rad}^{\sim}$ is an open subspace of $(\overline{\mathfrak{Rad}})^{\sim}$ and by property (ii) of Proposition 4.12 we conclude it is an ANR. □

Proposition 4.16. *The map $\pi_1 : \mathfrak{Rad}^{\sim} \to \mathfrak{Rad}$ is proper and cell-like.*

Proof. We note that π_1 extends to a continuous map $\bar{\pi}_1 : (\overline{\mathfrak{Rad}})^{\sim} \to \overline{\mathfrak{Rad}}$. Let $K \subset \mathfrak{Rad}$ be compact, then it is also compact considered as a subset of $\overline{\mathfrak{Rad}}$ and thus closed. This means that $\bar{\pi}_1^{-1}(K)$ is closed in $(\overline{\mathfrak{Rad}})^{\sim}$ and since the latter is a compact space it must compact. But $\bar{\pi}_1^{-1}(K) \subset \mathfrak{Rad}^{\sim}$ and $\bar{\pi}_1^{-1}(K) \cap \mathfrak{Rad}^{\sim} = \pi_1^{-1}(K)$, so that π_1 is proper.

That π_1 is cell-like is a consequence of Lemma 4.8, which says that the point inverses of π_1 are contractible finite CW-complexes, and property (ii) in Lemma 4.13, which says that contractible finite CW-complexes are cell-like. □

Corollary 4.17. *The projection $\pi_1 : \mathfrak{Rad}^{\sim} \to \mathfrak{Rad}$ is a homotopy equivalence.*

Proof. Apply Theorem 4.9 to Propositions 4.14, 4.15 and 4.16. □

4.3. **The critical graph map is a homotopy equivalence.** We now show that the critical graph map $\mathfrak{Rad}^{\sim} \to \mathcal{MFat}^{ad}$ is a homotopy equivalence by using the relation between the universal bundles over $\mathfrak{Rad}$ and $\mathcal{MFat}^{ad}$. We start by recalling some results regarding universal bundles.

Given a 2-dimensional cobordism $S_{g,n+m}$ and a paracompact base space B, there is a one-to-one correspondence between isomorphism classes of smooth $S_{g,n+m}$-bundles over B, i.e. the transition functions lie in $\mathrm{Diff}(S_{g,n+m})$, and isomorphism classes of principal $\mathrm{Diff}(S_{g,n+m})$-bundles over B.

To see this consider a principal $\mathrm{Diff}(S_{g,n+m})$-bundle $p : W \to B$. Its corresponding $S_{g,n+m}$-bundle is given by taking $S_{g,n+m} \times_{\mathrm{Diff}(S_{g,n+m})} W$. To go in the other direction, suppose that $\pi : E \to B$ is a smooth $S_{g,n+m}$-bundle. Each fiber $E_b := \pi^{-1}(b)$ is a Riemman surface with boundary, together with a marked point in each boundary component. These marked points are ordered and labelled as incoming or outgoing. Let x_k^b denote the marked point in the kth incoming boundary component for $1 \leq k \leq n$ and x_{k+n}^b denote the marked point in the kth outgoing boundary $1 \leq k \leq m$. Its corresponding $\mathrm{Diff}(S_{g,n+m}, \partial S_{g,n+m})$-bundle is given by taking fiberwise orientation-preserving diffeomorphisms i.e. it is the bundle $p : W \to B$ whose fibers are given by

$$W_b := p^{-1}(b) = \{\varphi : S_{g,n+m} \to E_b | \varphi \text{ is a diffeomorphism, } \varphi(x_i) = x_i^b\}$$

These constructions are mutually inverse.

Furthermore, each connected component of $\mathrm{Diff}(S_{g,n+m})$ is contractible, so taking π_0 is a homotopy equivalence and thus there is a one-to-one correspondence between principal $\mathrm{Diff}(S_{g,n+m})$-bundles and principal $\mathrm{Mod}(S_{g,n+m})$-bundles, where one can obtain the $\mathrm{Mod}(S_{g,n+m})$-bundle corresponding to $p : W \to B$ by taking π_0.

We now construct a space $E\mathfrak{Rad}$ that maps onto $\mathfrak{Rad}$ and use the relations above to show that $E\mathfrak{Rad} \to \mathfrak{Rad}$ is a universal $\mathrm{Mod}(S_{g,n+m})$-bundle. To construct this space we use the same idea as for $\mathcal{EMFat}^{ad}$ that is, as a set

$$E\mathfrak{Rad} := \{([L], [H]) | [L] \in \mathfrak{Rad}, [H] \text{ is a marking of } \Gamma_{[L]}\}$$

The topology on $E\mathfrak{Rad}$ must be such that a path in $E\mathfrak{Rad}$ is given by a path in $\mathfrak{Rad}$ say $\gamma : t \to [L(t)]$ together with a sequence of markings $H_t : \Gamma_{[L(t)]} \hookrightarrow S_{g,n+m}$ which are completely determined by H_{t_0} and the path γ. To make this more precise we first give a cover of $\mathfrak{Rad}$ by using the notion of combinatorial type.

Definition 4.18. Let $[\mathcal{L}]$ be a non degenerate combinatorial type, we define a subspace of $\mathfrak{Rad}$ as follows:

$$\mathfrak{Rad}_{[\mathcal{L}]} := \{[L] \in \mathfrak{Rad} | \text{the combinatorial type of } [L] \text{ is } [\mathcal{L}]\}$$

From this definition it is clear that $\{\mathfrak{Rad}_{[\mathcal{L}]}\}_{[\mathcal{L}]}$ where the set runs over all non degenerate combinatorial types $[\mathcal{L}]$ is a cover of $\mathfrak{Rad}$. To make the topology of $E\mathfrak{Rad}$ precise, we will give a procedure for which given a combinatorial type $[\mathcal{L}]$, a marking of $\Gamma_{[\mathcal{L}]}$ and a configuration $[\tilde{L}] \in \partial\overline{\mathfrak{Rad}_{[\mathcal{L}]}}$ we obtain a well defined marking of $\Gamma_{[\tilde{\mathcal{L}}]}$ where $[\tilde{\mathcal{L}}]$ is the combinatorial type of $[\tilde{L}]$. To do this, notice that if $[\mathcal{L}]$ and $[\tilde{\mathcal{L}}]$ are related in this manner then $[\tilde{\mathcal{L}}]$ must be obtained from $[\mathcal{L}]$ by collapsing chambers and radial sectors. We will analyse these cases separately first.

Definition 4.19. Let $[\mathcal{L}]$ and $[\mathcal{L}']$ be two non degenerate combinatorial types such that $[\mathcal{L}']$ can be obtained from $[\mathcal{L}]$ by collapsing radial sectors say $R_{i_1}, R_{i_2} \dots R_{i_k}$ and let $R := \cup_i R_i$. We will define a map in $\mathcal{F}at^{ad}$

$$\rho : \Gamma_{[\mathcal{L}]} \to \Gamma_{[\mathcal{L}']}$$

which we will call the *radial sector collapse map* (see Figure 4.4).

Choose a representative $[L]$ of $[\mathcal{L}]$. Then following the construction of $\Gamma_{[L]}$ we can define a subgraph F_R which is given by the intersection of E_L and R. The subgraph F_R must be a forest inside $\Gamma_{[L]}$. To see this, assume there is a loop in F_R, then there must be a loop in $\Gamma_{[L]}$, this means that there are two paired slits ζ_i, $\zeta_{\lambda(i)}$ which lie on the same radial segment. Since $[L]$ is non degenerate the there must be slits $\zeta_{i_1}, \zeta_{i_2} \dots \zeta_{i_j}$ such that $i_j \geq 1$ and $|\zeta_{i_l}| < |\zeta_i|$ for all i_l. Finally, since the loop is in F_R, then R must contain the radial segment between ζ_i and ζ_{i_l} for some i_l, but then collapsing R will give a degenerate configuration and we assumed $[\mathcal{L}']$ is non degenerate. Therefore F_R is a forest in $\Gamma_{[L]}$ and since $\Gamma_{[L]} = \Gamma_{[\mathcal{L}]}$ this description gives a well defined subforest of $\Gamma_{[\mathcal{L}]}$ giving with a well defined map on $\mathcal{F}at^{ad}$.

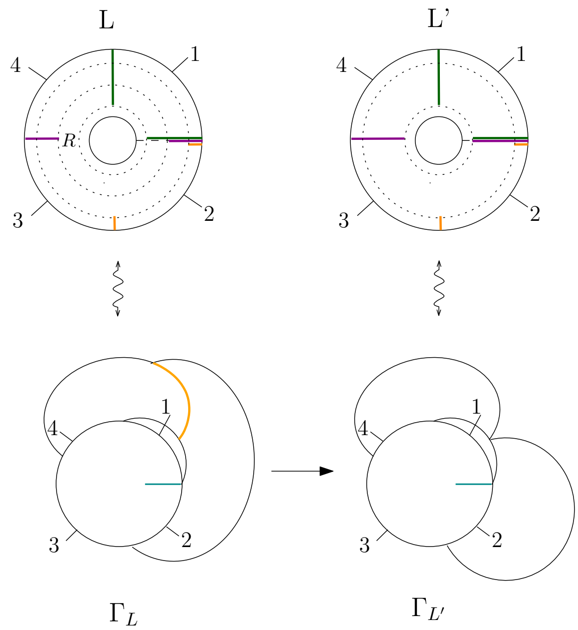


FIGURE 4.4. An example of the radial collapse map. The radial sectors are marked with dotted lines. The radial sector R is collapsed in L and the radial collapse map is given by collapsing the edge shown in orange.

Definition 4.20. Let $[\mathcal{L}]$ and $[\mathcal{L}'']$ be two non degenerate combinatorial types such that $[\mathcal{L}'']$ can be obtained from $[\mathcal{L}]$ by collapsing chambers. We will define an admissible fat graph $\Gamma([\mathcal{L}], [\mathcal{L}''])$ together with a zigzag in $\mathcal{F}at^{ad}$

$$\Gamma_{[\mathcal{L}]} \xrightarrow{\tau_1} \Gamma([\mathcal{L}], [\mathcal{L}'']) \xleftarrow{\tau_2} \Gamma_{[\mathcal{L}']}$$

which we will call the *chamber collapse zigzag* (see Figure 4.5).

Choose a representative $L \in \mathfrak{QRad}$ of combinatorial type $[\mathcal{L}]$ and let $L'' \in \mathfrak{QRad}$ be the preconfiguration of combinatorial type $[\mathcal{L}'']$ obtained by collapsing chambers onto their centers. We will call the radial segments onto which the chambers have been collapsed the *special radial segments*. Notice that L'' is well defined up to a choice of L, and slit jumps and parametrization point jumps away from the special radial segments. Thus the idea is to define $\Gamma([\mathcal{L}],[\mathcal{L}''])$ as a partially unfolded graph of L'' which is unfolded at the special radial slit segments and folded everywhere else. This would give a well defined isomorphism class of admissible fat graphs. To make this precise, let $S_{k_1}, S_{k_2} \dots S_{k_r}$ denote the special radial segments of L''. We define $\Gamma([\mathcal{L}],[\mathcal{L}'']) = \Gamma_{L'',t}$ where $t \in I^{d(L'')}$ is defined as follows:

$$t_\alpha := \begin{cases} 0 & \text{if } \alpha = k_i + j \text{ for } 1 \leq i \leq r \text{ and } 1 \leq j \leq s_{k_i} - 1 \\ 1 & \text{else} \end{cases}$$

This is a well defined isomorphism class of admissible fat graphs, since the graph is folded in all radial segments in which jumps are allowed. Let F_L be the subgraph of Γ_L obtained by the intersection of E_L with the collapsing chambers. Then $\tau_1 : \Gamma_{[\mathcal{L}]} = \Gamma_L \to \Gamma_L/F_L = \Gamma([\mathcal{L}],[\mathcal{L}''])$ is a well defined map in $\mathcal{F}at^{ad}$. Similarly let $F_{L''}$ be the subgraph of $\Gamma_{L''}$ obtained from the intersection of $E_{L''}$ and the special radial segments. Then $\tau_2 : \Gamma_{[\mathcal{L}'']} = \Gamma_{L''} \to \Gamma_{L''}/F_{L''} = \Gamma([\mathcal{L}],[\mathcal{L}''])$ is a well defined map in $\mathcal{F}at^{ad}$.

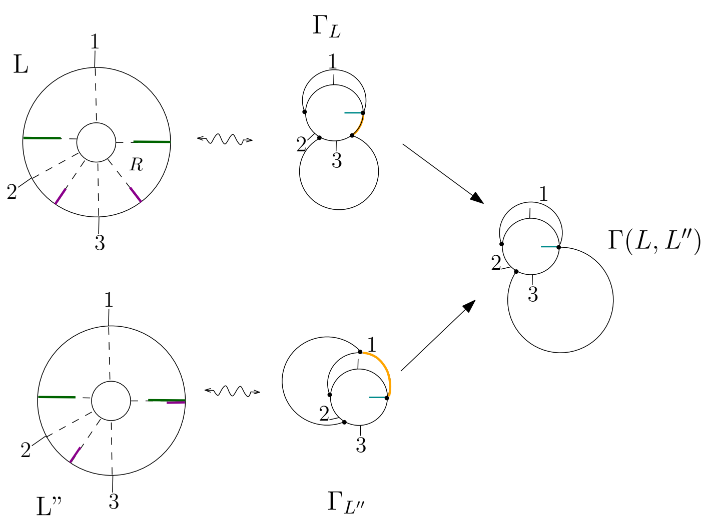


FIGURE 4.5. An example of the chamber collapse zigzag. The chambers are marked with dotted lines. The chamber R is collapsed in L and the chamber collapse zigzag is given by collapsing the edges shown in orange.

For the general case consider $[\tilde{L}] \in \partial\overline{\mathfrak{Rad}_{[\mathcal{L}]}} \cap \mathfrak{Rad}_{[\tilde{\mathcal{L}}]}$, then $[\tilde{\mathcal{L}}]$ is obtained from $[\mathcal{L}]$ by collapsing radial segments and chambers. Let $[\mathcal{L}']$ be the configuration obtained from collapsing only the radial segments. Then the construction above gives a well defined zigzag in $\mathcal{F}at^{ad}$.

(4.1)
$$\begin{array}{ccccc} & & \Gamma_{[\mathcal{L}']} & \xrightarrow{\tau_1} & \Gamma([\mathcal{L}'],[\mathcal{L}]) & & \\ & \overset{\rho}{\nearrow} & & & & \overset{\tau_2}{\nwarrow} & \\ \Gamma_{[\mathcal{L}]} & & & & & & \Gamma_{[\mathcal{L}']} \end{array}$$

If $[\tilde{\mathcal{L}}]$ is obtained only by collapsing radial segments then $\tau_1 = \tau_2 = id$ and if $[\tilde{\mathcal{L}}]$ is obtained only by collapsing chambers then $\rho = id$.

Definition 4.21. We define the space $E\mathfrak{Rad}$ as follows

$$E\mathfrak{Rad} := \frac{\bigsqcup_{[\mathcal{L}]} \mathfrak{Rad}_{[\mathcal{L}]} \times \text{Mark}(\Gamma_{[\mathcal{L}]})}{\sim}$$

where the disjoint union runs over all non degenerate combinatorial types $[\mathcal{L}]$ and the equivalence relation $\sim$ is generated by:

(i) For $[\tilde{L}] \in \partial\overline{\mathfrak{Rad}_{[\mathcal{L}]}} \cap \mathfrak{Rad}_{[\tilde{\mathcal{L}}]}$, $[H] \in \mathrm{Mark}(\Gamma_{[\mathcal{L}]})$, $[\tilde{H}] \in \mathrm{Mark}(\Gamma_{[\tilde{\mathcal{L}}]})$. If

$$[\tilde{H}] = (\tau_{2*})^{-1} \circ (\tau_{1*}) \circ \rho_*([H])$$

then $([\tilde{L}], [H]) \sim ([\tilde{L}], [\tilde{H}])$, where ρ, τ_1 and τ_2 are given as in diagram 4.1 and the induced maps are the ones constructed in Remark 3.23.

Proposition 4.22. *The projection $E\mathfrak{Rad} \to \mathfrak{Rad}$ is a the universal $\mathrm{Mod}(S_{g,n+m})$-bundle over $\mathfrak{Rad}$.*

Proof. It is enough to show that $E\mathfrak{Rad} \to \mathfrak{Rad}$ is the $\mathrm{Mod}(S_{g,n+m})$-bundle corresponding to the universal surface bundle $p : \mathfrak{S}_h(p, q) \to \mathfrak{Rad}$. Recall that the universal surface bundle has fibers $p_{[L]} = S([L])$ a Riemman surface with boundary, together with a marked point in each boundary component. These marked points are ordered and labelled as incoming or outgoing. Let x^L_k denote the marked point in the k-th incoming boundary component for $1 \leq k \leq n$ and x^L_{k+n} denote the marked point in the k-th outgoing boundary $1 \leq k \leq m$. Following the description in the beginning of this subsection, the $\mathrm{Diff}(S_{g,n+m})$-bundle, say $W \to \mathfrak{Rad}$, corresponding to the universal surface bundle is given by taking fiberwise orientation preserving diffeomorphisms i.e.

$$W_{[L]} := \{\varphi : S_{g,n+m} \to S([L]) \,|\, \varphi \text{ is orientation-preserving diffeomorphism s.t. } \varphi(x_i) = x^L_i\}$$

Furthermore, its corresponding $\mathrm{Mod}(S_{g,n+m})$-bundle, say $Q \to \mathfrak{Rad}$, has fibers $Q_{[L]} := W_{[L]}/\text{isotopy}$. Notice that $Q_{[L]}$ is discrete, and thus by the description of $E\mathfrak{Rad}$ it is enough to show that there is a one to one correspondence between $\mathrm{Mark}(\Gamma_{[L]})$ and $Q_{[L]}$. We define inverse maps

$$\Phi : Q_{[L]} \leftrightarrows \mathrm{Mark}(\Gamma_{[L]}) : \Psi$$

By construction, there is a canonical embedding $H_{[L]} : \Gamma_{[L]} \hookrightarrow S([L])$ and this embedding is a marking of $\Gamma_{[L]}$ in $S([L])$. Given $[\varphi] \in Q_{[L]}$ we define $\Phi([\varphi]) := [\varphi^{-1} \circ H_{[L]}]$, this is a well defined map.

To go the other way around, let $[H] \in \mathrm{Mark}(\Gamma_{[L]})$ and choose a representative $H : \Gamma_{[L]} \hookrightarrow S_{g,n+m}$. We will construct an orientation preserving homeomorphism $f : S_{g,n+m} \to S([L])$ such that $[f \circ H] = [H_{[L]}]$; which we can approximate by a diffeomorphism φ, by Nielsen's approximation theorem [Nie24]. By 3.22, the complement $S_{g,n+m} - H(\Gamma - \text{leaves of } \Gamma)$ is a disjoint union of $n+m$ cylinders. For all $1 \leq i \leq n+m$, one of the boundary components of the ith cylinder consists of the ith boundary of $S_{g,n+m}$. The other boundary component consists of the image of the ith boundary cycles of Γ under H. Finally, the leaf corresponding to the ith boundary component is embedded in the cylinder and connects both boundary components. Therefore, $S_{g,n+m} - H(\Gamma_{[L]}) = \amalg_{i=1}^{n+m} D_i$ where each D_i is a disc. Let x_i denote the marked point of the ith boundary component of $S_{g,n+m}$. Then, the boundary of D_i has two copies of x_i and connecting them on one side is the ith boundary component of $S_{g,n+m}$ and on the other the embedded image of the ith boundary cycle of $\Gamma_{[L]}$. The orientation of the i-th boundary component of $S_{g,n+m}$ allows us to order the two copies of x_i and label them as $x_{i,1}$ for the first one and $x_{i,2}$ for the second. Similarly, $S([L]) - H_{[L]}(\Gamma_{[L]}) = \amalg_{i=1}^{n+m} \tilde{D}_i$ where each $\tilde{D}_i$ is a disc. Let $x^L_{i,j}$ for $j = 1, 2$ denote the two copies of the marked point of the i-th boundary component of $S([L])$ that lie on the boundary of $\tilde{D}_i$. Define $f_i|_{\partial D_i} : \partial D_i \to \partial \tilde{D}_i$ to be an orientation preserving homeomorphism such that $f(x_{i,j}) = x^L_{i,j}$ for $j = 1, 2$ and let f_i be an extension of $f_i|_{\partial D_i}$ to the entire disc. Moreover, one can choose the maps $f_i|_{\partial D_i}$ consistently so that they glue together to a homeomorphism $f : S_{g,n+m} \to S([L])$. Since the maps f_i are uniquely defined up to homotopy then f is also uniquely defined up to homotopy. We define $\Psi([H]) = [\varphi]$, where φ is a diffeomorphism approximating f. The map Ψ is well-defined and by construction it is inverse to Φ. □

We now want to extend this to $\mathfrak{Rad}^{\sim}$.

Definition 4.23. We define a blow-up of $E\mathfrak{Rad}$ as follows

$$E\mathfrak{Rad}^{\sim} := \{(([L], [H]), [\Gamma, \lambda, \tilde{H}]) \in E\mathfrak{Rad} \times \mathcal{EMF}at^{ad} \,|\, [\Gamma, \lambda] \in \mathcal{G}([L])\}$$

where $\mathcal{G}([L])$ is the space given in Definition 4.6 with which we blow up $\mathfrak{Rad}$ at a configuration $[L]$.

Corollary 4.24. *The projection $E\mathfrak{Rad}^{\sim} \to \mathfrak{Rad}^{\sim}$ is the universal $\mathrm{Mod}(S_{g,n+m})$-bundle over $\mathfrak{Rad}^{\sim}$*

Proof. This is clear since the diagram below is pullback diagram and π_1 is a homotopy equivalence by 4.17.

$$\begin{array}{ccc} E\mathfrak{Rad}^{\sim} & \xrightarrow{\pi_1\times \mathrm{id}} & E\mathfrak{Rad} \\ \downarrow & & \downarrow \\ \mathfrak{Rad}^{\sim} & \xrightarrow[\pi_1]{\simeq} & \mathfrak{Rad} \end{array}$$

To see that it is a pullback, let $\mathsf{G}[\mathcal{L}]$ be the full subcategory of $\mathcal{F}at^{ad}$ on objects the underlying graphs of $\mathcal{G}([L])$ given in Definition 4.6. Recall we that for any $[\Gamma] \in \mathsf{G}[\mathcal{L}]$ there is a contractible choice of zigzags in $G[L]$, from $[\Gamma_{[L]}]$ to $[\Gamma]$ by the proof of Lemma 4.8. Therefore, by Remark 3.23, a marking of $[\Gamma_{[L]}]$, uniquely determines a marking of $[\Gamma]$ and vice versa. Thus,for $[\Gamma, \lambda] \in \mathcal{G}([L])$ it is equivalent to give a tuple $(([L], [H]), [\Gamma, \lambda, \tilde{H}]) \in E\mathfrak{Rad} \times \mathcal{EMF}at^{ad}$ than either a triple $(([L], [H]), [\Gamma, \lambda])$ or a triple $([L], [\Gamma, \lambda, \tilde{H}])$. Showing the the diagram above is a pullback. □

We now describe a general result on universal bundles.

Proposition 4.25. *Let $E \to B$ and $E' \to B'$ be universal principal G-bundles with B and B' paracompact spaces. Let $f : B \to B'$ be a continuous map. If $f^*(E')$ is isomorphic to E as a bundle over B, then f is a homotopy equivalence.*

Proof. For any space X we can build a diagram

$$\begin{array}{ccc} [X, B] & \xrightarrow{\cong} & \{\text{Principal } G \text{ bundles over } X\} \\ {\scriptstyle f\circ -}\downarrow & \nearrow{\scriptstyle \cong} & \\ [X, B'] & & \end{array}$$

This diagram commutes since $f^*(E') \cong (E)$. For $X = B'$ one gets that there is a $[g] \in [B', B]$ such that $[f \circ g] = [id_{B'}]$. Then, $g^*(E) \cong g^*(f^*(E')) = E'$, so we can repeat the argument and obtain that there is an $h \in [B, B']$ such that $[g \circ h] = [id_B]$. Finally, since $[h] = [f \circ g \circ h] = [f]$ then f and g are mutually inverse homotopy equivalences. □

We can now conclude that π_2 is a homotopy equivalence.

Corollary 4.26. *The projection $\pi_2 : \mathfrak{Rad}^{\sim} \to \mathcal{MF}at^{ad}$ is a homotopy equivalence.*

Proof. This follows directly from the proposition by the same argument given in the proof of 4.24, since the following diagram is a pullback.

$$\begin{array}{ccc} \mathfrak{Rad}^{\sim} & \xrightarrow{\pi_2\times \mathrm{id}} & \mathcal{EMF}at^{ad} \\ \downarrow & & \downarrow \\ \mathfrak{Rad}^{\sim} & \xrightarrow[\pi_2]{} & \mathcal{MF}at^{ad} \end{array}$$

□

5. Sullivan Diagrams and the harmonic compactification

We now compare the harmonic compactification of radial slit configurations $\overline{\mathfrak{Rad}}$ and the quotient space of metric admissible fat graphs $\mathcal{SD}$, as defined in Definitions 2.9 and 3.18 respectively:

Proposition 5.1. *The space $\mathcal{SD}$ is homotopy equivalent to the harmonic compactification of the space of radial slit configurations $\overline{\mathfrak{Rad}}$ and is in fact homeomorphic to the unimodular harmonic compactification $\overline{\mathfrak{URad}}$.*

Proof. It is enough to show this for connected cobordisms. Recall that the harmonic compactification of the space of radial slit configurations $\overline{\mathfrak{Rad}}$ is homotopy equivalent to the space of unimodular radial slit configurations $\overline{\mathfrak{URad}}$, so it suffices to prove a stronger statement: $\mathcal{SD}$ and $\overline{\mathfrak{URad}}$ are homeomorphic CW-complexes where the cells are indexed by the combinatorial data.

Since in $\overline{\mathfrak{UMad}}$ all annuli have the same outer and inner radius and all slits have the same modulus, the radial sections are superfluous information, thus, the combinatorial type of a univalent configuration is determined only by its chamber configuration. More precisely, two univalent configurations $[L]$ and $[L']$ have the same combinatorial type if and only if they differ from each other only by the angular size of the chambers. Finally, the orientation of the complex planes together with the positive real line induce a total ordering of the chambers on each annulus.

Similarly, on a Sullivan diagram, the leaves of the boundary cycles and the fat structure at the vertices where they are attached give a total ordering of the edges on the admissible cycles. We say two Sullivan diagrams $[\Gamma]$ and $[\Gamma']$ have the same combinatorial data if they differ from each other only on the lengths of the edges on the admissible cycles. A *(non-metric) Sullivan diagram G* is an equivalence class of Sullivan diagrams under this relation. We will first show that a radial slit configuration and a Sullivan diagram carry over the same combinatorial data. That is, that there is a bijection

$$\begin{gathered}\Upsilon := \{\text{Combinatorial types of univalent radial slit configurations}\} \\ \updownarrow \\ \Lambda := \{\text{non-metric Sullivan diagrams}\}\end{gathered}$$

We define a map $f : \Upsilon \to \Lambda$ by $[\mathcal{L}] \mapsto G_{[\mathcal{L}],0}$ where $G_{[\mathcal{L}],0}$ is the underlying (non metric) Sullivan diagram of the unfolded graph of $[\mathcal{L}]$. This map is well defined, since a slit or a parametrization point jumping along another slit corresponds to a slide of a vertex along an edge not belonging to the admissible cycle. For example the configurations in Figure 5.1 are mapped to the graphs in Figure 5.2.

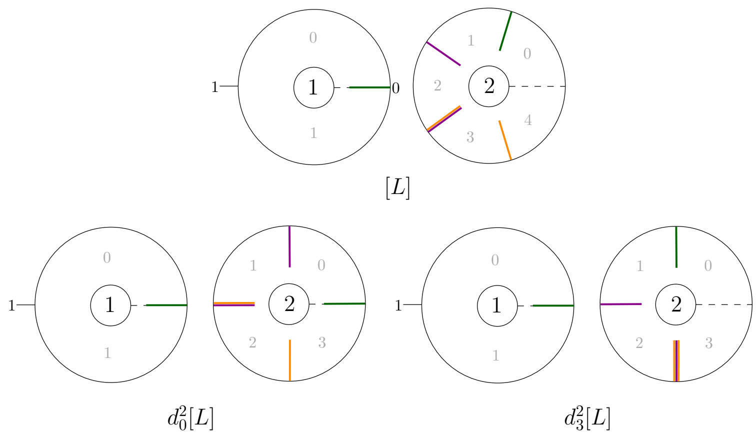


FIGURE 5.1. A 5-cell which is a product $\Delta^1 \times \Delta^4$ simplices in $\overline{\mathfrak{UMad}}$ and part of its boundary. The chambers are numbered in grey.

We no construct the inverse map $g : \Lambda \to \Upsilon$. Notice that any non metric Sullivan diagram has a canonically associated metric Sullivan diagram by assigning all the edges in an admissible cycle the same length. Moreover any Sullivan diagram has a fat graph representative with all its vertices on the admissible cycles. A representative of a metric Sullivan diagram with all its vertices on the admissible cycles is given by the following data:

(i) A collection of p parametrized circles, $C_1, C_2, \ldots, C_p$ which are disjoint, ordered, and of length 1.
(ii) A finite number of chords $l_1, l_2, \ldots, l_s$ where a chord is a graph which consist of two vertices connected by an edge. Let V denote the set of vertices of such chords.
(iii) A subset $\widetilde{V} \subset V$ such that, $\widetilde{V}$ contains at least one vertex of each chord and $|V - \widetilde{V}| = m$.
(iv) An assignment $\alpha : \widetilde{V} \to \sqcup_i C_i$ which will indicate how to attach the chords onto the p-circles. Two or more chords may be attached on the same circle and even on the same point. The assignment α should attach at least one chord in on each circle.
(v) For each x in the image of α, an ordering of the subset of chords attached to x, that is, an ordering of the set $\alpha^{-1}(x)$.

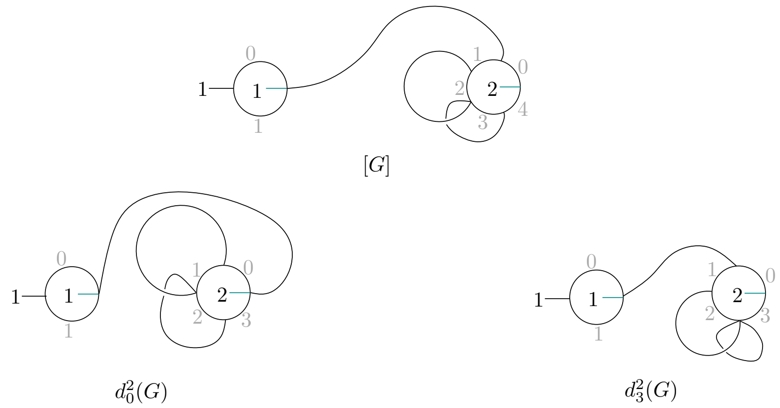


Figure 5.2. A 5-cell which is a product $\Delta^1 \times \Delta^4$ simplices in $\mathcal{SD}$ and part of its boundary. The edges are numbered in grey.

From this data one can construct a well defined metric fat graph with inner vertices of valence greater or equal to 3. The chords are attached onto the p circles using α. This gives the circles the structure of a graph by considering the attaching points as vertices and the intervals between them as edges. It just remains to give a fat structure at the attaching points. To do this let x be in the image of α. The parametrization of the circles gives a notion of incoming and outgoing half edges on x say e_x^- and e_x^+ respectively. Moreover there is an ordering of the chords attached on x say $(l_{x,1}, l_{x,2}, \ldots, l_{x,s})$. The cyclic ordering at x is given by $(e_x^-, l_{x,1}, l_{x,2}, \ldots, l_{x,s}, e_x^+)$ as it is shown in Figure 5.3. Informally, this is to say all chords are attached on the outside of the circles according to the order given by the data. The chords that are attached only at one vertex give the leaves of the Sullivan diagram.

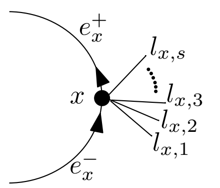


Figure 5.3. Fat structure induced at vertex x where the cyclic ordering is given by the orientation on the plane.

From this it is clear what the inverse map g should be. Given a Sullivan diagram G, its associated metric Sullivan diagram gives the data (i) to (v) listed above. Then, $g(G) = (\zeta, \lambda, \tilde{\omega}, \vec{r}, \vec{P})$ where ζ is given by α on the chords attached at both ends, λ is given by those chords (i.e. $\lambda(i) = k$ if and only if there is a chord attached on both ends connecting i and k), $\vec{P}$ is given by α on the chords attached only at one vertex, and $\tilde{\omega}$ and $\vec{r}$ are completely determined by the ordering of the chords at each attaching point. This map is well defined since slides along chords correspond to jumps along slits. Moreover, this map is clearly inverse to f.

We will show that $\overline{\mathfrak{URad}}$ and $\mathcal{SD}$ have homeomorphic CW structures, where the cells are indexed by Υ. A combinatorial type $[\mathcal{L}]$ indexes a cell of dimension n, where n is the number of chambers of $[\mathcal{L}]$ minus the number of annuli. To make this precise, let $[\mathcal{L}]$ be a combinatorial type with p annuli and for every i let $n_i + 1$ be the number of chambers of the ith annulus. We denote by $d_j^i([\mathcal{L}])$ to be the combinatorial type obtained by collapsing the jth chamber of the i-th annulus (see Figure 5.1). We define $e_{[\mathcal{L}]}$ to be the product of simplices $\Delta^{n_1} \times \Delta^{n_2} \times \ldots \Delta^{n_p}$. We will construct homeomorphisms

$$\overline{\mathfrak{URad}} \xleftarrow{\ \varphi\ } \frac{\bigsqcup_{[\mathcal{L}]\in\Upsilon} e_{[\mathcal{L}]}}{\sim} \xrightarrow{\ \psi\ } \mathcal{SD}$$

where the equivalence relation is given by

$$(e_{[\mathcal{L}]}, (t_{10} \ldots t_{1n_1} \ldots t_{i(j-1)}, 0, t_{i(j+1)} \ldots t_{pn_p})) \sim (e_{d_j^i([\mathcal{L}])}, (t_{10} \ldots t_{1n_1} \ldots t_{i(j-1)}, t_{i(j+1)} \ldots t_{pn_p}))$$

In other words, the j-th face of the i-th simplex of $e_{[\mathcal{L}]}$ is identified with $e_{d^i_j([\mathcal{L}])}$.

The homeomorphism φ is clear once one notices that any configuration $[L]$ in $\overline{\mathfrak{URad}}$ is completely and uniquely determined by its underlying combinatorial type $[\mathcal{L}]$ and a tuple $(t_{10} \ldots t_{1n_1} \ldots t_{ij} \ldots t_{pn_p})$ where t_{ij} is the relative angular length of the jth chamber of the ith annulus. To construct the map ψ one must first notice that similarly, any Sullivan diagram $[\Gamma]$ in $\mathcal{SD}$ is completely and uniquely determined its non metric underlying Sullivan diagram G and a tuple $(t_{10} \ldots t_{1n_1} \ldots t_{ij} \ldots t_{pn_p})$ where t_{ij} is the length of the j-th edge of the i-th admissible cycle. Using this we can define $\psi(e_{[\mathcal{L}]}, (t_{10} \ldots t_{pn_p})) = [\Gamma] = (f([\mathcal{L}]), (t_{10} \ldots t_{pn_p}))$.

It is easy to show that the maps φ and ψ are continuous and define CW-structures on $\overline{\mathfrak{URad}}$ and $\mathcal{SD}$ respectively.

□

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Paper C

ON THE HOMOLOGY OF SULLIVAN DIAGRAMS

DANIELA EGAS

Abstract. We show that the first and top homology groups of the chain complex of Sullivan diagrams of the topological type of the punctured disk are trivial. We compute all the homology groups of the chain complex of Sullivan diagrams of the topological type of the disk with up to seven punctures and we give generators for the non trivial groups. We use these generators to give two infinite families of non trivial classes of the homology of Sullivan diagrams of topological type the generalized pair of pants.

1. Introduction

Let S_g denote the genus g closed oriented surface, and let $S_{g,n}$ denote the compact surface with boundary obtained by cutting out n disjoint open disks from S_g. Finally, let $S_{g,n}^m$ denote the oriented surface with n boundary components and m punctures obtained by cutting out m point of $S_{g,n}$. The study of surfaces, their classification and their properties has been a central theme in mathematics for centuries. One approach to study this subject is the theory of moduli. The moduli space of $S_{g,n}^m$ which we denote $\mathcal{M}_{g,n}^m$, is loosely speaking, the space of all conformal classes of metrics on $S_{g,n}^m$, modulo the relation of conformal diffeomorphism fixing the boundary components pointwise. By glueing a genus one surface with two boundary components to the unique boundary of $S_{g,1}$ we obtain a map

$$\mathcal{M}_{1,1} \to \mathcal{M}_{2,1} \to \dots \mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1} \to \dots$$

In [Har90], Harer showed that these maps induce an isomorphisms in a range of dimension increasing with the genus g and the stable homology has been completely described in [MW05, Til97, Gal04].

However, little is known about the unstable homology of $\mathcal{M}_{g,n}$. Ehrenfried in [ABE08] and Godin in [God07b] have computed $H_*(\mathcal{M}_{g,n})$ for low genus and small number of boundary components by using combinatorial models. However, the size of these models is restrictive. Ehrenfried uses a model of Moduli Space developed by Bödigheimer [Böd06]. In this model, Bödigheimer constructs a space $\mathfrak{Rad}_{g,n}$, a point in this space is described by n annuli in n different complex planes with a configuration of marked points in their interior. The main idea is that any surface can be obtained by taking the annuli and cutting slits on them according to the marked points and then glue the annuli along these cuts. This model comes with a natural notion of compactification of moduli space which Bödigheimer calls the Harmonic compactification and we denote $\overline{\mathfrak{Rad}}_{g,n}$.

On the other hand, Godin uses the ideas of Penner, and Igusa to construct a space of fat graphs which is a model of Moduli Space [God07b, God07a]. A fat graph, is a graph in which each vertex has a cyclic ordering of the edges that are attached to it. The idea is that every such graph can be fattened to obtain a surface. The space of fat graphs has a homotopy equivalent subspace, the space of admissible fat graphs which we denote $\mathfrak{Fat}_{g,n}^m$ [EK14]. This subspace has a natural quotient which is the space of Sullivan diagrams $\mathfrak{sD}_{g,n}^m$ A point in this space consists of a fat graph obtained from a finite number of parametrized circles of length 1 embedded on the plane (this embedding gives a notion of outside and inside of the circle) to which one attaches a number of chords from the inside. We think of these chords as being of length zero, thus vertices can slide along the chords of a diagram. The space of Sullivan diagrams, $\mathfrak{sD}_{g,n}^m$, has a canonical CW-structure and its cellular complex is the chain complex of Sullivan diagrams $\mathscr{SD}^m_{g,n}$, originally defined by Tradler Zeinalian in [TZ06] to study operations on the Hochschild homology of symmetric Frobenius algebras. This chain complex is completely defined combinatorially, see Definition 2.9, and it is significantly smaller than the chain complexes used in the calculations of Ehrenfried and Godin. Finally, the spaces $\mathfrak{sD}_{g,n}$ and $\overline{\mathfrak{Rad}}_{g,n}$ are homotopy equivalent [EK14]. We summarize this statements in diagram below.

$$\begin{array}{ccccc} \mathfrak{Fat}_{g,n} & \simeq & \mathcal{M}_{g,n} & \simeq & \mathfrak{Rad}_{g,n} \\ \downarrow & & & & \downarrow \\ \mathfrak{sD}_{g,n} & & \xrightarrow{\simeq} & & \overline{\mathfrak{Rad}}_{g,n} \end{array} \tag{1.1}$$

Thus, studying the homology homology of Sullivan diagrams could give further insight into the unstable homology of Moduli space. The main goal of this paper is to give an algorithm to compute the homology of Sullivan diagrams in the genus 0 case, and use it to get computations of the homology of $\mathfrak{sD}$ on special cases, which then lead to a few first general computations of the homology groups.

The study of $H_*(\mathcal{M}^m_{g,n})$ and $H_*(\mathfrak{sD}^m_{g,n})$ is an interesting topic in itself. However, it is also interesting from the perspective of string topology, which studies algebraic structures on the homology of free loop spaces. Let S_{g,n_1+n_2} be a 2-dimensional cobordism with n_1 incoming boundary components and n_2 outgoing boundary components whose underlying surface is $S_{g,n}$ where $n = n_1 + n_2$. For LM the free loop space of a manifold M, Cohen and Godin construct operations

$$H_{n_1}(LM)^{\otimes n_1} \longrightarrow H_{n_2}(LM)^{\otimes n_2} \tag{1.2}$$

parametrized by isomorphism classes of surfaces S^m_{g,n_1,n_2} which behave well with glueing surfaces along their boundaries [CG04]. However, it was shown by Tamanoi that most of these operations are trivial [Tam09]. Thus, to study string operations one should consider a richer space over which to parametrize them.

With this in mind, Godin and Kupers [God07a, Kup11] define higher string operations, which are operations as in 1.2 parametrized over $H_*(\mathcal{M}^m_{g,n_1+n_2})$. On the other hand, when M is simply connected, with coefficients in a field there is an isomorphism $HH_*(C^{-*}(M), C^{-*}(M)) \cong H^*(LM)$, where $HH_*(A, A)$ denotes the Hochschild homology of an algebra A [Jon87]. In [Wah12], Wahl studies natural operations on the Hochschild homology of algebras. She defines the chain complex of all natural operations on the Hochschild homology of algebras with a given structure e.g. Frobenius, Commutative. She also defines the chain complex of all formal operations, which loosely speaking is an approximation of the chain complex of all natural operations. Finally, she shows that the chain complex of formal operations on the Hochschild homology of Frobenius algebras is quasi-isomorphic to the chain complex of Sullivan diagram. She uses this identification to give classes in the homology of Sullivan diagrams which at the same time represent non trivial string operations on the homology of the free loop space of the sphere LS^n. Thus, determining $H_*(\mathfrak{sD}^m_{g,n})$ may allows us to find more non trivial string operations. In [CG04] Cohen and Godin construct string operations using chords diagrams. We should note that although the concepts are very closely related, the chain complex of this space is not the same complex as $\mathscr{SD}$, nor their underlying spaces are homotopy equivalent. On the other hand, Poirier and Rounds construct string operations using a different space of chord diagrams and they describe a quotient of this space $\overline{SD}/\sim$ which is homeomorphic to $\mathfrak{sD}$.

The paper is organized as follows. In Section 2 we define the chain complex of Sullivan diagrams. In Section 3 we give a way of representing a Sullivan diagram in terms of a tuple of natural numbers and a non-crossing partition and we give an algorithm that describes how to get all Sullivan diagrams of the disk with c punctures i.e. $\mathfrak{sD}^c_{0,1}$. In Section 4 we give the results obtained with the computer program on the homology of $\mathfrak{s}\mathfrak{D}^c_{0,1}$. In section 5 we show the first and top homology groups of $\mathfrak{s}\mathfrak{D}^c_{0,1}$ are trivial. Note that by collapsing all but one boundary components to a point we get a map

$$\mathcal{M}_{g,n} \longrightarrow \mathcal{M}^{n-1}_{g,1}$$

In Section 6 we give two infinite families of non trivial classes of the homology of $\mathfrak{s}\mathfrak{D}_{0,n}$ which we obtain by lifting the classes obtained in Section 4 along this map.

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2. The Definition

Definition 2.1. A *combinatorial graph* G is a tuple $G = (V, H, s, i)$, with a finite set of vertices V, a finite set of half edges H, a map $s : H \to V$ and an involution with no fixed points $i : H \to H$.

The map s ties each half edge to its source vertex and the involution i attaches half edges together. Thus an edge of the graph is an orbit of i. The valence of a vertex $v \in V$ is the cardinality of the set $s^{-1}(v)$ and a *leave* of a graph is a univalent vertex.

Definition 2.2. The *geometric realization* of a combinatorial graph G is the CW-complex $|G|$ with one 0-cell for each vertex, one 1-cell for each edge and attaching maps given by s.

Definition 2.3. A *fat graph* $\Gamma = (G, \sigma)$ is a combinatorial graph in which all inner vertices are at least trivalent, all leaves are ordered, and there is a cyclic ordering σ_v of the half edges incident at each vertex v. The *fat structure* of the graph is given by the data $\sigma = (\sigma_v)$ which is a permutation of the half edges. Figure 2.1 shows some examples of fat graphs.

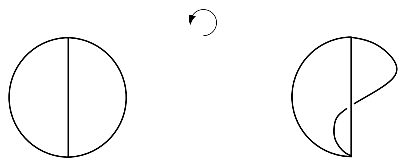


Figure 2.1. Two different fat graphs (where the fat structure is given by the orientation of the plane) which have the same underlying combinatorial graph.

Definition 2.4. The *boundary cycles* of a fat graph are the cycles of the permutation of half edges given by $\omega = \sigma \circ i$. Each boundary cycle c gives a list of half edges and determines a list of edges of the fat graph Γ, those edges containing the half edges listed in c. The *boundary cycle sub-graph* corresponding to c is the subspace of $|\Gamma|$ given by the edges determined by c which are not leaves. When clear from the context we will refer to a boundary cycle sub-graph simply as boundary cycle.

Definition 2.5. A *p-admissible fat graph* is a fat graph in which p of its boundary cycles are disjoint embedded circles. These boundary cycles are labelled $1, 2, \dots p$ and we will refer to them as *admissible cycles.* Furthermore, each admissible cycle has exactly one leaf and we refer to these leaves as the *admissible leaves.* Figure 2.2 shows an example of an admissible fat graphs.

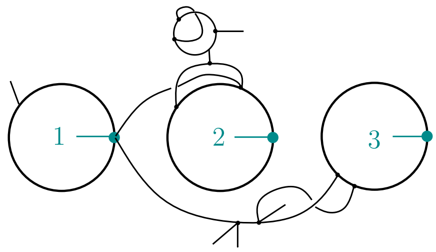


FIGURE 2.2. An example of a 3-admissible fat graph.

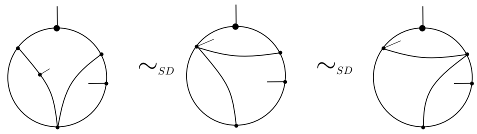


FIGURE 2.3. Three equivalent 1-admissible fat graphs.

Definition 2.6. Let $[\Gamma_1]$ and $[\Gamma_2]$ be p-admissible fat graphs. We say $[\Gamma_1] \sim_{SD} [\Gamma_2]$ if $[\Gamma_2]$ can be obtained from $[\Gamma_1]$ by sliding vertices along edges that do not belong to the admissible cycles. Figure 2.3 shows some examples of equivalent admissible fat graphs.

Remark 2.7. Notice that this is equivalent to saying that $[\Gamma_2]$ and $[\Gamma_1]$ are connected by a zigzag of edges collapses, on edges that do not belong to the admissible cycles.

It is easy to see that $\sim_{SD}$ is an equivalence relation.

Definition 2.8. A *p-Sullivan diagram G* is an equivalence class of p-admissible fat graphs under the relation $\sim_{SD}$.

One can think of a Sullivan diagram as an admissible fat graph where the edges not belonging to the admissible cycles are of length zero. We use this objects to define a chain complex of Sullivan diagrams, which was originally defined by Tradler and Zeinalian in [TZ06].

Definition 2.9. The *chain complex of Sullivan Diagrams* $\mathscr{SD}$, is the complex generated as a $\mathbb{Z}$ module by isomorphism classes of Sullivan diagrams. The degree of a p-Sullivan diagram G is

$$\deg(G) := |E_a| - p$$

where E_a is the set of edges that belong to the admissible cycles. The fat structure together with the leave at the admissible cycles give a natural ordering of the edges that belong to the admissible cycles $e_0, e_1, \ldots e_{|E_a|-1}$. The differential of a Sullivan diagram G is

$$d(G) := \sum_{i=1}^{|E_a|} (-1)^i G/e_i$$

where G/e_i is the Sullivan diagram obtained by collapsing the edge e_i. Note that G/e_i is well defined since we are only collapsing edges on the admissible cycles. It is easy to check that d is indeed a differential. Figure 2.4 gives and example of the differential.

Remark 2.10. From a fat graph Γ we can construct a surface with punctures, boundaries and marked points at the boundaries by a fattening procedure. Construct this surface by replacing each edge with a strip, glueing these strips at a vertex according to the fat structure and collapsing to a puncture each boundary component which is not connected to a leaf. Note that there is a strong deformation retraction of Σ_Γ onto $|G|$ so one can think of

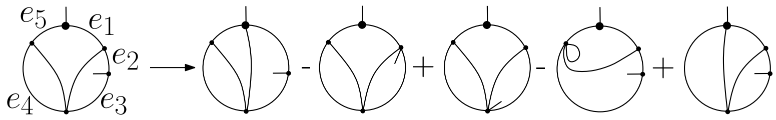


FIGURE 2.4. The differential of a 1-Sullivan diagram of degree 4.

$|G|$ as the skeleton of the surface. The fat structure of Γ is completely determined by ω. Moreover, one can show that the boundary cycles of the fat graph Γ correspond to the boundary components and punctures of Σ_Γ [God07b]. Therefore, the surface Σ_Γ is completely determined by the combinatorial graph and its fat structure. Finally, on each boundary component that is connected to m leaves choose m non intersecting marked points and label them according to the labelling of their corresponding leaves in a way that the cyclic ordering of the marked points at a boundary component coincides with the cyclic ordering at which the leaves occur at their corresponding boundary cycle. This surface with decoration is well defined up to a homemorphism that respects the decorations. When two surfaces are connected by such a homeomorphism, we say they have the same *topological type*. This is an equivalence relation, and the equivalence classes are the topological types of surfaces with puntures, boundary and decorations at the boundary.

Let e be an edge of Γ which is not a loop or a leaf. Note that collapsing e is a homotopy equivalence on geometric realizations and does not change the number of boundary cycles. Thus, the surfaces Σ_Γ and $\Sigma_{\Gamma/e}$ have the same topological type. Since the equivalence relation $\sim_{SD}$ and the differential on Sullivan diagrams are given by collapsing edges which are not loops, the chain complex $\mathscr{SD}$ splits into finite chain complexes each of which consist of Sullivan diagrams that fatten to a surface of a given topological type.

Remark 2.11. Note that the chain complex of 1-Sullivan diagrams, is the chain complex of a Δ-set (or semi-simplicial set), where the k simplices are degree k Sullivan diagrams and the i-th face of a 1-Sullivan diagram G is $d_i(G) := G/e_i$. This space splits into connected components given by topological type.

3. The Algorithm

We describe the data that gives a Sullivan diagram in an alternative way, and use this to give a unique representative for each Sullivan diagram of the punctured disk. We then provide a way of listing such representatives and determining their differential.

Definition 3.1. Let G be a p-Sullivan diagram, and for $1 \leq i \leq p$, let v_i be the vertex in the i-th admissible cycle which is connected to the admissible leaf. The Sullivan diagram G is called *standard* if for all i, $|v_i| \geq 4$, where $|v_i|$ is the valence of the vertex v_i i.e. the admissible leaves are not isolated.

Note that any Sullivan diagram G, can be obtained from a standard Sullivan diagram $\hat{G}$ by sliding some, possibly all, of the admissible leaves along the admissible cycles against the ordering of the edges on the admissible cycles i.e. $\hat{G}$ is the Sullivan diagram obtained from G by collapsing the first edge of each admissible cycle in which the admissible leaf is isolated. Thus, in order to list all Sullivan diagrams of a given topological type, it is enough to list all standard Sullivan diagrams and then obtain all non standard diagrams by moving the admissible leaves as stated above.

Definition 3.2. We call an admissible fat graph *essentially trivalent at the boundary* if all the vertices on the admissible cycles have valence 3, except possibly the vertex that is connected to the admissible leaf which can have valence 4.

Remark 3.3. Note that any Sullivan diagram has a representative which is essentially trivalent at the boundary, by sliding higher valence vertices away of the admissible cycles. Now, recall that by thickening a fat graph we obtain a surface with decorations well defined up to topological type. This defines an equivalence relation on the set of fat graphs. Note that two fat graphs which are of the same topological type are connected by a zigzag of edge collapses. In particular, if two fat graphs say G and $\widetilde{G}$ correspond to the same topological type, they must have the same number of leaves say $l_1, l_2, \ldots l_k$ and $\tilde{l}_1, \tilde{l}_2, \ldots \tilde{l}_k$. Moreover, l_j and $\tilde{l}_j$ must lay on the same boundary cycle for all j and if there is more than one leave on a given boundary cycle then their cyclic ordering coincides in G and $\widetilde{G}$. Thus, given an equivalence class of fat graphs $[G]$ there is a well defined set of leaves $\{l_1, l_2, \ldots l_k\}$ corresponding to it. Therefore, a standard p Sullivan diagram with p leaves is given by the following data:

- A collection of p circles, $C_1, C_2 ... C_p$ which are ordered and disjointly embedded on the plane.
- A finite number of equivalence classes of fat graphs $[G_1], [G_2] ... [G_k]$ where the equivalence relation is given by their topological type. Let L denote the set of leaves of such graphs i.e. $L = \{l_1^1, l_2^1, \ldots l_{q_1}^1, l_1^2, l_2^2, \ldots l_{q_2}^2, \ldots l_1^k, l_2^k, \cdots l_{q_k}^k\}$, and let n denote the total number of leaves i.e. $n = |L|$.
- An ordered partition of n into p summands i.e. a p-tuple of natural numbers n_i such that $n = n_1 + n_2 + \ldots n_p$.
- An injective assignment $\alpha : L \to \amalg_{i=1}^{p} \{0, 1, 2, \ldots n_i - 1\}$

By choosing representatives of the attached graphs $G_1, G_2, \ldots G_k$, we can construct a fat graph representing a Sullivan diagram. On each circle C_i, fix n_i marked points labelled $0, 1, 2 \ldots n_i - 1$ in clockwise order and attach the graphs onto the p-circles using α. Notice that this gives the circles the structure of a graph by considering the attaching points, as vertices and the intervals between them as edges. It just remains to give a fat structure at the attaching points and to add the admissible leaves. Let x be an attaching point on the circles, the embedding of the circles give the notion incoming and outgoing half edges on x in clockwise direction, say e_x^- and e_x^+ respectively. The cyclic ordering at x is given by $(e_x^+, \alpha^{-1}(x), e_x^-)$. Informally, this is to say all leaves are attached on the inside of the circles. Following the same idea, on each circle attach an admissible leaf at the marked points 0 from the outside, see Figure 3.1. Note that since different choices of representatives G_i are connected by a zigzag of edge collapses, they all give the same Sullivan diagram.

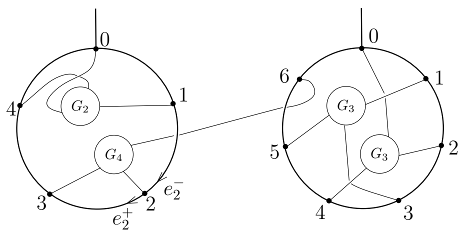
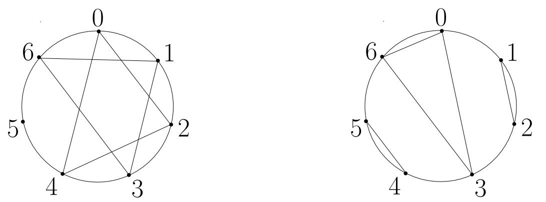


FIGURE 3.1. A representative of a 2-Sullivan diagram which is essentially trivalent at the boundary and it constructed by attaching graphs onto 2 embedded circles.

This representation of a Sullivan diagram is not unique, since, permuting graphs with the same topological type or cyclically permuting the labelling of the leaves, does not change the Sullivan diagram obtained under this construction.

3.1. **Writing the generators.** Now we focus on the case of the disk with c punctures for $c > 1$. Let G be a Sullivan diagram corresponding to a punctured disk. Then G, must be a 1-Sullivan diagram and any fat graph representative of G is an admissible fat graph with only one leaf, the leaf of the admissible cycle, and no crossings i.e. it can be embedded on the plane. So any representative of G is built by attaching graphs with no crossings to the admissible cycle in a non-crossing manner. In order to describe this attachment we recall a classical concept from combinatorics

Definition 3.4. Let $[n]$ denote the set $\{0, 1, 2, \dots n\}$. A *partition of* $[n]$, is a collection of pairwise disjoint subsets of $[n]$ say $\{\Omega_1, \Omega_2 \dots \Omega_k\}$ such that $\bigcup_{i=1}^{k} \Omega_i = [n]$. The subsets Ω_i are denoted the *blocks* of the partition. Any partition of $[n]$ can be represented graphically, by considering a circle with $n+1$ marked points labelled $0, 1, 2 \dots n$ in that cyclic order and joining circularly successive elements of each block by chords, see Figure 3.2. A *non-crossing partition of* $[n]$ is a partition of $[n]$ such that its graphical representation is planar i.e. the chords representing the partition intersect only at the marked points on the circle.



$\{0, 2, 4\}\{1, 3, 6\}\{5\}$ $\{1, 2\}\{0, 3, 6\}\{4, 5\}$

FIGURE 3.2. On the left a crossing partition and on the right a non crossing partition of [6]

Since all graphs attached to construct G must have no crossings, they can be completely determined, up to topological type, by their euler characteristic and their number of legs. This leads to the following definition.

Definition 3.5. A *spider* is a pair of natural numbers (j, l) that belongs to the set

$$\{(j, 0) | j \in \mathbb{N}, j \geq 2\} \bigcup \{(j, l) | j, l \in \mathbb{N}, j \geq 1, l \geq 1\}$$

We call j the *number of legs of the spider* and l the *number of loops of the spider*. A *spider configuration with* k *spiders* is a k-tuple of spiders $[(j_1, l_1), (j_2, l_2), \dots (j_k, l_k)]$

The spiders represent the graphs we will attach to the embedded circles. We interpret a spider (j, l) as a graph with at most 1 inner vertex, j leaves and l loops. More precisely, the spider $(2, 0)$ is a chord i.e. a graph with 2 vertices and one edge connecting them, for $j \geq 3$ the spider $(j, 0)$ is a corolla with j leaves, and the spider (j, l) is a graph with l loops and j leaves joint at a single vertex. We do not consider $(1, 0)$ as a spider, since attaching such a graph would create extra leaves.

Recall that any Sullivan diagram of the punctured disk can be thickened to a punctured disk and thus has a planar representation where the cyclic ordering at each vertex is induced by the orientation of the plane. Therefore, by remark 3.3 any standard Sullivan diagram of the punctured disk of degree n is uniquely represented by a non-ordered tuple

$$[(l_1, \Omega_1), (l_2, \Omega_2), \dots, (l_k, \Omega_k)]$$

where $[\Omega_1, \Omega_2, \dots \Omega_k]$ is a non crossing partition of $\{0, 1, 2 \dots n\}$ and $[(|\Omega_1|, l_1), (|\Omega_2|, l_2), \dots (|\Omega_k|, l_k)]$ is a spider configuration. To see this, fix $n+1$ marked points on the circle labelled $0, 1, 2 \dots n$ in clockwise order. For each $1 \leq i \leq k$ we attach the spider $(|\Omega_i|, l_i)$ on the points of Ω_i. Notice that since the legs of the spider are not labelled, there is a unique way of attaching the spider to the points of Ω_i in the circle in a non crossing way. Thus, to give a standard Sullivan diagram of the punctured disk of degree n is equivalent to give a spider configuration, a matching non crossing partition, in the sense that the size of the blocks match the number of legs of the spiders, and a way to pair these two together. Note that the pairing is also part of the data. To see this, consider a spider configuration with two different spiders with the same number of legs. Then a matching partition would have two blocks of the same size and thus there are two different ways for pairing the partition and the spider configuration, each of which gives a different Sullivan diagram.

3.2. **Listing all possible spiders.** We describe how to list all possible spider configurations that give rise to a Sullivan diagram of the topological type of the punctured disk. We first establish some notation

Notation 3.6. We denote by c the number of punctures of the disk, by k the number of spiders in a spider configuration, by m the total number of legs of a spider configuration (i.e. the number of attaching points on the admissible cycle) and by l the total number of loops of a spider configuration. Finally, we denote by s the number of spiders with only one leg in a spider configuration. We refer to s as the number of singletons in the configuration. In particular, for a spider configuration $[(j_1, l_1), (j_2, l_2), \dots (j_k, l_k)]$, we have that $m := \sum_i j_i$ and $l := \sum_i l_i$

Observation 3.7. We denote the number of puncture of the disk c, because any Sullivan diagram of the disk with c punctures, is a configuration of chords in the circle which divides its interior into c chambers.

Fix the number of punctures of the disk. A simple Euler characteristic argument gives that:

$$1 \leq m \leq 2c - 2 \tag{3.1}$$

$$l = c + k - m - 1 \tag{3.2}$$

The idea is to first find all possible triples $\xi := (k, l, m)$ such that 3.1 and 3.2 hold and then find all possible spider configurations corresponding to each ξ. Note first, that $\xi := (1, c-1, 1)$ is a valid triple, and it corresponds to the only 1-Sullivan diagram of degree 0 given by attaching the spider $[1, c-1]$ at a vertex. We find all other triples by a simple recursion given in algorithm 1

Algorithm 1 Find all valid tuples ξ

```
K := 1, L := c − 1, M := 1
while M ≤ 2c − 2 do
  if L > 0 then
    M := M + 1
    L := L − 1
    ξ := (K, L, M)
  else
    M := M + 1
    K := K + 1
    ξ := (K, L, M)
  end if
  m := M, k := K, l := L
  while k < m do
    K := K + 1
    L := L + 1
    ξ := (k, l, m)
  end while
end while
```

Given a valid triple $\xi := (k, l, m)$, we describe a way to list all possible spider configurations corresponding to ξ. This is more involved, but we give a brief description of the procedure. This is given by a case by case analysis as follows

If $l = 0$:
Since there are not loops, this is equivalent to listing all possible distributions of m legs into k spiders. Moreover, since there are no spiders with 1 leg and 0 loops, no singletons are allowed in the configuration i.e. $s = 0$. Thus, to list all spider configurations corresponding to ξ is equivalent to list all partitions of the set with m elements into k blocks where all blocks are of size at least 2, or equivalently to list all partitions of the set of $m - 2k$ elements into at most k blocks.

If $l \neq 0$:
If there are loops in the configuration then singletons are allowed, each singleton must have at least 1 loop. Let $f := l - s$, we will refer to f as the number of free loops. We consider two distinct cases:

- **If $k = m$:** In this case, every spider is a singleton i.e. $s = k = m$. Thus, listing all spider configurations, amounts to listing all ways of distributing the free loops among the singletons i.e. listing all partitions of the set with f elements into at most k blocks.
- **If $k < m$:** In this case, the minimum number of singletons is $s_{\min} := \max\{0, 2k - m\}$ and the maximum number of singletons $s_{\max} := \min\{k-1, l\}$. For each s such that $s_{\min} \leq s \leq s_{\max}$, we first find all the possible distribution of legs on the spider configuration. This is given by listing all partitions of the set with $m - s$ elements into $k - s$ blocks of size at least 2. A similar procedure gives all possible distributions of the free loops along the spiders with given number of legs.

3.3. **Attaching spiders to the admissible cycle.** Non crossing partitions are a classic object of study in combinatorics, we recall another classical object from combinatorics

Definition 3.8. An *n-Dyck path* is a monotonic path in the $n \times n$ grid from $(0,0)$ to (n,n) consisting of n up steps of the form $(1,0)$ and n horizontal steps of the form $(0,1)$ that never goes below the diagonal.

It is well known that the number of non-crossing partitions and the number of Dyck paths are counted by the Catalan number. In [uM09], Črepinšek and Mernik, give an efficient algorithm to list all n Dyck paths and in [Pro83], Prodinger gives a bijection from the set of non-crossing partitions of a set with n elements to the set of n-Dyck paths. We use these elements to efficiently list all non crossing partitions of a set with n elements.

Definition 3.9. Let $[(j_1, l_1), (j_2, l_2), \dots (j_k, l_k)]$ be a spider configuration with l legs. A non crossing partition of the set $\{0, 1, 2, \dots l-1\}$ *matches the spider configuration*, if it has k blocks say $\Omega_1, \Omega_2, \dots \Omega_k$ and there is a permutation $\sigma \in \Sigma_k$ such that $|\Omega_i| = j_{\sigma(i)}$. Note that this permutation need not to be unique, each permutation with this characteristic would give a different pairing.

The procedure in 3.2 lists all possible spider configurations that give rise to a Sullivan diagram of topological type a punctured disk with c punctures. Now given a valid spider configuration, we find all matching non crossing partitions following [uM09] and pair the spider configuration to each partition in all possible ways. This gives a list of all standard Sullivan diagrams. Each standard Sullivan diagram gives rise to a non standard Sullivan diagram by isolating the admissible leaf in counter-clockwise direction. This gives and exhaustive list of all Sullivan diagrams of the topological type of the disk with c punctures.

3.4. **Computing the differential.** The differential can be easily determined in this representation. Let G be a Sullivan diagram of degree $m - 1$ given by

$$G := [(l_1, \Omega_1), (l_2, \Omega_2), \dots, (l_k, \Omega_k)]$$

where m is the number of attaching points in the circle. To determine the differential of G is enough to determine the Sullivan diagram G/e_i for $0 \leq i \leq m$. There are two different cases:

If $i, i+1 \in \Omega_j$: Then for $1 \leq r \leq k$ let

$$\tilde{\Omega}_r := \{x \in \Omega_r | x \leq i\} \bigcup \{x-1 | x \in \Omega_r, x \geq i+2\}$$

Then the $\tilde{\Omega}_r$'s give a partition of the set $\{0, 1, \ldots m-2\}$ with k blocks and G/e_i is given by

$$G/e_i := [(l_1, \tilde{\Omega}_1), \ldots, (l_j + 1, \tilde{\Omega}_j), \ldots, (l_k, \tilde{\Omega}_k)]$$

If $i \in \Omega_j, i+1 \in \Omega_s, j \neq r$: Then for $1 \leq r \leq k, r \neq j, s$ let

$$\tilde{\Omega}_r := \{x \in \Omega_r | x \leq i\} \bigcup \{x-1 | x \in \Omega_r, x \geq i+2\}$$

and let

$$\tilde{\Omega}_{j,s} := \{x \in \Omega_j \cup \Omega_s | x \leq i\} \bigcup \{x-1 | x \in \Omega_j \cup \Omega_s, x \geq i+2\}$$

Then the $\tilde{\Omega}_r$'s and $\tilde{\Omega}_{j,s}$ give a partition of the set $\{0, 1, \ldots m-2\}$ with $k-1$ blocks and G/e_i is given by

$$G/e_i := [(l_1, \tilde{\Omega}_1), \ldots, (l_j + l_s, \tilde{\Omega}_{j,s}), \ldots, (l_k, \tilde{\Omega}_k)]$$

4. Results

We used the procedure described in the previous section using Magma and obtained the results listed in Table 4.1.

c	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}
2	$\mathbb{Z}$	$\mathbb{Z}$	0										
3	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0								
4	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0						
5	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	0	0				
6	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0		
7	$\mathbb{Z}$	0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0

Table 4.1. Homology of the chain complex of Sullivan diagrams of topological type a disk with c punctures.

In Figures 4.1 and 4.2 we provide generators for the non trivial homology groups.

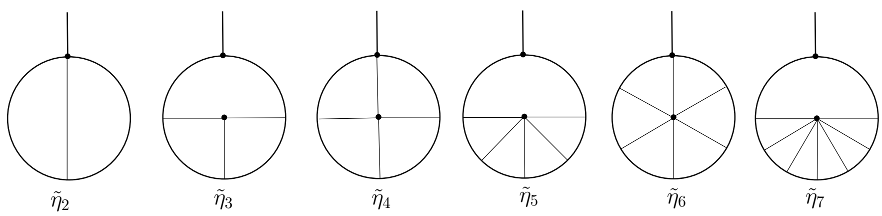


Figure 4.1. The Sullivan diagram η_c is the generator of the first non trivial homology group of the chain complex of Sullivan diagrams of the topological type of the disk with c punctures.

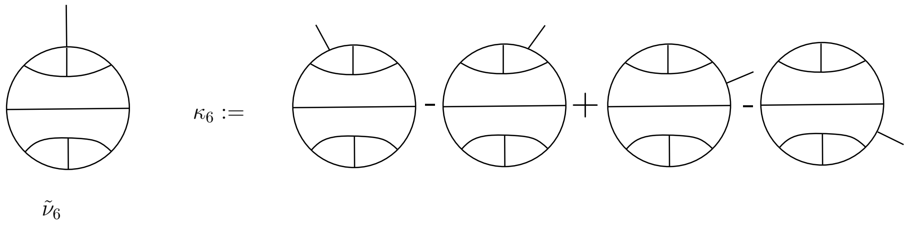


FIGURE 4.2. On the left ν_6 a generator of $H_6(\mathscr{S}\mathscr{D}_{D^6})$ and on the right $\tilde{\nu}_6$ a generator for $H_7(\mathscr{S}\mathscr{D}_{D^6})$

5. First and top homology of Sullivan Diagrams of the Punctured Disk

We use the presentation of Sullivan diagrams described in the previous section to show that the first and top homology of Sullivan diagrams of the punctured disk are trivial.

Definition 5.1. Let X be a Δ-set, and let σ be a simplex of X. We say σ is a *free face* if it appears exactly once as a face of a top dimensional simplex.

Remark 5.2. Let $\tilde{\sigma}$ be an n simplex and a top dimensional cell of a Δ-set X and let $\sigma := d_k(\tilde{\sigma})$. Let Λ^n_k denote the k horn of Δ^n i.e. the union of all the faces of Δ^n except the k-th face. There is a deformation retraction of Δ^n onto Λ^n_k, and if σ is a free face this deformation extends to X.

Proposition 5.3. *Let $\mathcal{SD}_{D^c}$ be the chain complex of Sullivan diagrams of topological type of a disk with c punctures. The top homology of $\mathcal{SD}_{D^c}$ is trivial i.e. $H_{2c-2}(\mathcal{SD}_{D^c}) = 0$ and if $c \geq 3$ then the first homology group is also trivial i.e. $H_1(\mathcal{SD}_{D^c}) = 0$.*

Proof. Let $\mathfrak{sD}^c_{0,0}$ be the space of Sullivan diagrams of topological type a of disk with c punctures. More precisely, $\mathfrak{sD}^c_{0,0}$ is the Δ-set described in 2.11 whose chain complex is $\mathcal{SD}_{D^c}$. By remark 5.2, it is enough to show that every top dimensional simplex has a free face. To see this, notice that a top dimensional cell corresponds to a maximally expanded Sullivan diagram and such a diagram is constructed by attaching chords onto a ground circle and attaching an isolated leaf. Therefore, a maximally expanded Sullivan diagram, say G, is represented by a non crossing partition of the set with $2c-1$ elements into c blocks, where one block is of size 1 corresponding to the leaf of the diagram, and all other blocks have size 2 corresponding to the chord attachments. Such a partition always has a block of the form $\{i, i+1\}$. Then, G/e_i is a face of G and a free face of $\mathfrak{sD}^c_{0,0}$. So there is a deformation retraction of $\mathfrak{sD}^c_{0,0}$ onto its $(2c-1)$-skeleton.

For the statement about the first homology, let $c \geq 3$. The 0-chains, 1-chains and 2 chains of $\mathcal{SD}_{D^c}$ are shown in Figure 5.1. Thus, up to degree 2 $\mathcal{SD}_{D^c}$ is given by

$$\mathbb{Z}^{3c-4+p(c,3)} \xrightarrow{d_2} \mathbb{Z}^c \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

where $p(c,3)$ is the number of ordered partitions of c into 3 summands $c = s+t+(c-s-t)$ such that $s, t, c-s-t \geq 1$. From this description it is easy to see that d_2 is surjective. □

6. Classes of the Homology of Sullivan Diagrams of the Generalized Pair of Pants

Let P_c denote the genus 0 surface with $c+1$ boundary components and exactly one marked point in each boundary component. The marked points are ordered. We refer to P_c as a generalized pair of pants with c legs. Let D^c denote the disk with c punctures. By collapsing the last c boundary components of P_c and forgetting their marked points we get a map $P_c \to D^c$ which induces a map

$$\mathscr{S}\mathscr{D}_{P_c} \longrightarrow \mathscr{S}\mathscr{D}_{D^c}$$

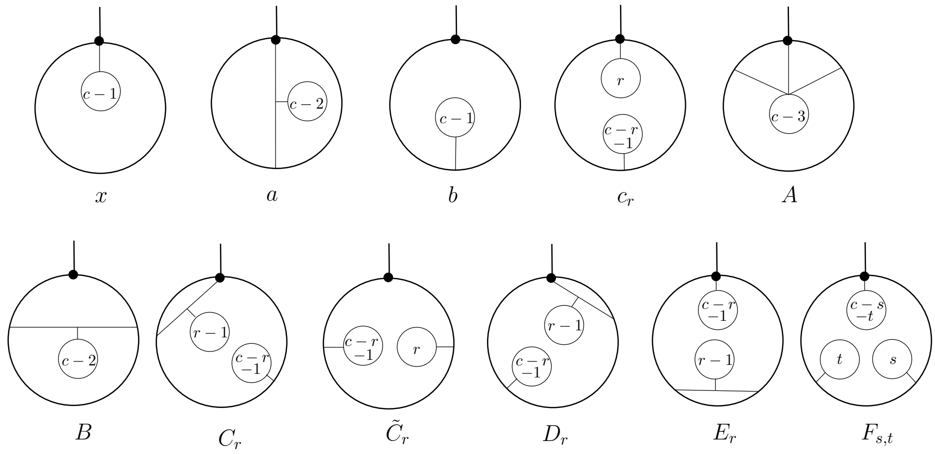


FIGURE 5.1. A list of all Sullivan diagrams of the topological type of the disk with c punctures and of degree at most 2. The annotations inside the circles represent the number of loops attached at that vertex and the conditions on r, s and t are the following: $1 \leq r \leq c-2$, $s,t \geq 1$ and $c-1 \geq s+t$. Note that $d(A) = a$, $d(B) = 2a - b$ and $d(C_r) = c_r$.

which is give by forgetting all the leaves except the admissible leave. We lift generators of the homology of Sullivan diagrams of topological type D^c given in section 4 to cycles of $\mathscr{S}\mathscr{D}_{P_c}$ by symmetrically placing leaves along all boundary cycles. Note that this lift is in no way unique. We show that these cycles are classes by following the ideas outlined in [Wah12]. We give a brief sketch of this idea. For any Frobenius A we can study operations of the form

$$CC_*(A,A)^{\otimes p} \longrightarrow CC_*(A,A)^{\otimes q}$$

where $CC_*(A,A)$ are the Hochschild chains of A. Tradler and Zeinalian in [TZ06], describe an action of a Sullivan diagram on the Hochschild Homology of any finite dimensional, unital Frobenius algebra A. In [WW11], Wahl and Westerland give a recipe of how to read a Sullivan diagram as an operation on an algebra A. All natural operations of Frobenius algebras form a chain complex, and in [Wah12], Wahl introduces a chain complex $Nat(p,q)$ of formal operations, which are an approximation of the chain complex of natural operations. Let $\mathscr{S}\mathscr{D}(p,q)$ denote the chain complex of Sullivan diagrams with p admissible circles and $p+q$ leaves, exactly one in each boundary cycle. Wahl shows that there is an inclusion

$$\mathscr{S}\mathscr{D}(p,q) \hookrightarrow Nat(p,q)$$

and this inclusion is a split quasi-isomorphism. Therefore, a cycle in $\mathscr{S}\mathscr{D}(p,q)$ which acts non trivially on $HH_*(A)$ for some A is a class in $H_*(\mathscr{S}\mathscr{D}(p,q))$. Since the classes obtained in section 4 follow a very clear pattern, this allows us to give 2 infinite families of classes in the homology of Sullivan diagrams of topological type P_c of increasing degree.

Definition 6.1. We define η_c and ν_c which are chains of $\mathscr{S}\mathscr{D}(1,c)$.

η_c**:** For c even, let $\tilde{\eta}_c$ be the degree $c-1$ Sullivan diagram obtained by attaching the spider $(c,0)$ onto an embedded circle according to the partition $\{0,1,\ldots c-1\}$. Note that this attachment divides the interior of the embedded circle into c chambers, see Figure 4.1. Let η_c be the Sullivan diagram of degree $2c-1$ obtained by attaching an isolated leaf labelled $i+1$ at the admissible cycle between i and $(i+1)_{\text{ mod } c}$, for each $0 \leq i \leq c-1$. For c odd, let $\tilde{\eta}_c$ be the degree c Sullivan diagram obtained by attaching the spider $(c,0)$ to an embedded circle according to the partition $\{1,2,\ldots c\}$, see Figure 4.1. Let $\eta_{c,1}$ be the Sullivan diagram of degree $2c-1$ obtained by attaching an isolated leaf labelled $i+1$ at the admissible cycle between i and $(i+1)$, for each $1 \leq i \leq c-2$ and a leaf labelled 1 at the vertex 0. Let $\eta_{c,2}$ be the Sullivan diagram of degree $2c-1$ obtained by attaching an isolated leaf labelled $i+1$ at the admissible cycle between i and $(i+1)$, for each $1 \leq i \leq c-2$ and a leaf labelled 1 at the inner vertex of the spider. We define η_c to be the difference $\eta_c = \eta_{c,2} - \eta_{c,1}$. See Figure 6.1 for examples of η_c.

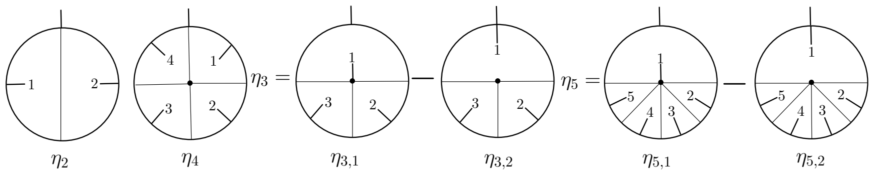


FIGURE 6.1. Non trivial classes on the homology of Sullivan diagrams of the topological type of the generalized pair of pants

ν_c: For c even and $c \geq 6$, let $\tilde{\nu}_c$ be the Sullivan diagram of degree $c+1$ obtained by attaching the spiders $(2,0), (\frac{c}{2},0), (\frac{c}{2},0)$ according to the partition $\{0, \frac{c}{2}+1\}$, $\{1,2,\dots \frac{c}{2}\}$, $\{\frac{c}{2}+2, \frac{c}{2}+3, \dots c+2\}$, see Figure 4.2. We define 4 Sullivan diagrams diagrams of degree $2c-1$ in $\mathscr{S}\mathscr{D}(1,c)$, which we denote $\nu_{c,i}$ for $i = 1,2,3,4$. Each $\nu_{c,i}$ is constructed by first attaching an isolated leaf labelled $i+1$ at the admissible cycle between i and $(i+1)$, for each $1 \leq i \leq \frac{c}{2}-1$ and an isolated leaf labelled $i-1$ for each $\frac{c}{2}+2 \leq i \leq c$. Then $\nu_{c,1}$ is obtained by attaching a leaf labelled 1 at vertex 1 and a leaf labelled c at vertex 0 such that there is exactly on leaf on each boundary cycle. The chain $\nu_{c,2}$ is obtained by attaching a leaf labelled 1 at vertex 1 and a leaf labelled c at vertex $c+1$ such that there is exactly on leaf on each boundary cycle. The chain $\nu_{c,3}$ is obtained by attaching a leaf labelled 1 at vertex 0 and a leaf labelled c at vertex 0 such that there is exactly on leaf on each boundary cycle and the cyclic ordering of the leaves corresponds to the one in $\nu_{c,1}$ and $\nu_{c,2}$. The chain $\nu_{c,4}$ is obtained by attaching a leaf labelled 1 at vertex 0 and a leaf labelled c at vertex $c-1$ such that there is exactly on leaf on each boundary cycle. We define ν_c to be alternative sum $\nu_c = \sum_{i=1}^{4}(-1)^{i+1}\nu_{c,i}$. See Figure 6.2 for an example of ν_c.

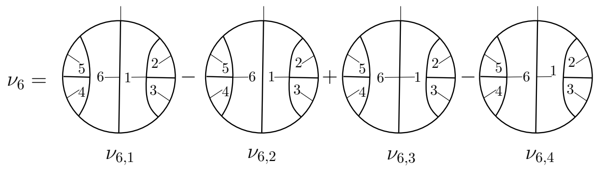


FIGURE 6.2. Non trivial class on the homology of Sullivan diagrams of the topological type of the generalized pair of pants

Consider the Frobenius algebra $A = \mathbb{Z}[x]/(x^2)$ with $|x| = 1$, where the coproduct is given by $\nu(1) = 1 \otimes x - x \otimes 1$ and $\nu(x) = x \otimes x$. The reduced Hochschild complex of A is generated by elements $1 \otimes x^{\otimes n}$ in total degree 0 and by elements $x \otimes x^{\otimes n}$ in total degree 1. It is easy to check that the differential is trivial and therefore, $\overline{C}_*(A,A) = \overline{HH}_*(A,A)$. In [Wah12, 4.2], Whal shows that for c even, the η_c's are non-trivial classes of the homology of $\mathscr{S}\mathscr{D}$ by showing that these are cycles that give non trivial operations on A. We use the same approach to show that all of chains defined are classes on the homology $\mathscr{S}\mathscr{D}$.

Proposition 6.2. *For $c \geq 3$ and odd, η_c is a non trivial class of the homology of $\mathscr{S}\mathscr{D}$. For $c \geq 6$ and even, ν_c is a non trivial class in the homology of $\mathscr{S}\mathscr{D}$.*

Proof. A direct calculation gives that η_c and ν_c are cycles in $\mathscr{SD}$. In order to see these cycles are non trivial classes in the homology we read them as operations on the Hochschild homology of A. We follow the recipe given in WW6.2. Then for $i = 1, 2$ and $j = 1, 2, 3, 4$, we have operations

$$\eta_{c,i*} : \oplus_{k_0+k_1+\ldots k_c=c}(HH_{k_0}(A) \otimes HH_{k_1}(A) \otimes \ldots HH_{k_c}(A)) \longrightarrow HH_{2c-1}(A)$$

$$\nu_{c,j*} : \oplus_{k_0+k_1+\ldots k_c=c}(HH_{k_0}(A) \otimes HH_{k_1}(A) \otimes \ldots HH_{k_c}(A)) \longrightarrow HH_{2c-1}(A)$$

We test this operations on the tuple $(x, x, \ldots, x)$. We choose representatives of $\eta_{c,1}$ and $\eta_{c,2}$ in which all vertices have valence 3. We place the x's on the leaves that do not belong to the admissible cycle and "read" the graph minus the admissible cycle as a composition of operations in A. This gives that

$$\eta_{c,1*}(x, x, \ldots, x) = 1 \otimes x \otimes x \ldots \otimes x$$

$$\eta_{c,2*}(x, x, \ldots, x) = 1 \otimes 1 \otimes x \ldots \otimes x + 1 \otimes x \otimes 1 \ldots \otimes x + \ldots + 1 \otimes x \otimes x \ldots \otimes x \otimes 1 = 0$$

Then $\eta_{c*}(x, x, \ldots, x) = 1 \otimes x \otimes x \ldots \otimes x$ and it is therefore a non trivial operation, which implies that η_c is a non trivial class of the homology of $\mathscr{SD}$.

Similarly, we choose representatives the $\nu_{c,i}$'s in which all vertices have valence 3 except for the vertex that is connected to the admissible leaf which has valence 4. We place the x's on the leaves that do not belong to the admissible cycle and "read" the graph minus the admissible cycle as a composition of operations in A. This gives that

$$\nu_{c,2*}(x, x, \ldots, x) = 1 \otimes x \otimes x \ldots \otimes x$$

$$\nu_{c,1*}(x, x, \ldots, x) = \nu_{c,3*}(x, x, \ldots, x) = \nu_{c,4*}(x, x, \ldots, x) = 0$$

Then $\nu_{c*}(x, x, \ldots, x) = -1 \otimes x \otimes x \ldots \otimes x$ and it is therefore a non trivial operation, which implies that ν_c is a non trivial class of the homology of $\mathscr{SD}$. □

Remark 6.3. A is actually the cohomology algebra of S^1 i.e. $A := H^*(S^1)$. Moreover, up to a degree shift and signs it is also the cohomology algebra of S^n for $n \geq 2$. Since signs and degrees didn't play a role in the proof of 6.2, the same argument of [Wah12, p. 29] shows that η_c and ν_c give non trivial string operations on $H_*(LS^n)$, where LS^n is the free loop space of S^n.

Remark 6.4. The classes η_c and ν_c have the same degree and topological type as the classes μ_{c-1} given by Wahl. A simple argument shows that $\mu_{c-1} - \eta_c$ is a boundary and thus μ_{c-1} and η_c are homologous. We show this graphically in Figure 6.3. The general case, follows in exactly the same way. On the other hand, we do not know if μ_{c-1} and ν_c are homologous.

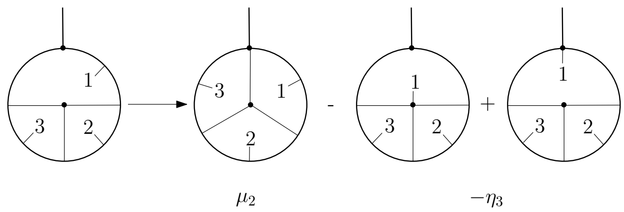


FIGURE 6.3. A Sullivan diagram whose boundary is $\mu_2 - \eta_3$

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