GRAVITATIONAL WAVES FROM BINARY NEUTRON STARS AND TEST PARTICLE INSPIRALS INTO BLACK HOLES

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2008

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As ground-based gravitational wave detectors are searching for gravitational waves at their design sensitivity and plans for future space-based detectors are underway, it is important to have accurate theoretical models of the expected gravitational waves to be able to detect potential signals and extract information from the measured data. This thesis contains work on developing theoretical tools for modeling the expected gravitational waves from two different classes of sources, which are key targets for current and future gravitational wave detectors. The work is based on four papers in collaboration with Éanna Flanagan. (i) We show that ground-based gravitational wave detectors may be able to constrain the nuclear equation of state using the early, relatively clean portion of the signal of detected neutron star neutron star inspirals.

(ii) The second class of gravitational wave source we consider are radiation - reaction driven inspirals of test particles into much more massive black holes. Chapter 5 contains our work on developing a rigorous formalism based on two-timescale expansions for treating the evolving orbit. Our results provide a clarification of the existing prescription for computing the leading order orbital motion and resolve the difficulties with previous approaches for going beyond leading order.

(iii) In Chapter 6, we analyze the effect of gravitational radiation reaction on generic orbits around a body with an axisymmetric mass quadrupole moment Q to linear order in Q, to linear order in the mass ratio and in the weak-field limit.

In addition we consider a system of two point masses where one body has a single mass multipole or current multipole. We show that within our approximations the motion is not integrable (except for the cases of spin and mass quadrupole).

(iv) Chapter 7 gives an alternative derivation of the result of Sago for an explicit expression for the time-averaged rate of change of the Carter constant (a third constant of geodesic motion around a rotating black hole in addition to energy and axial angular momentum) in the adiabatic limit which is formulated in terms of sums over modes and can be used for numerically computing leading order waveforms.

BIOGRAPHICAL SKETCH

Tanja obtained the high school diploma “Abitur” from the Main-Taunus-Schule in Hofheim, Germany and the BA from the University of Colorado at Boulder. She went to Cornell University for her graduate studies, where she worked with Éanna Flanagan.

ACKNOWLEDGEMENTS

I thank Éanna Flanagan for being a superb adviser. With his enthusiasm about science, his ability to communicate and present physics in a clear and well - organized way, his friendliness and willingness to take the time to provide extra explanations, discussions, help and advice, he has taught me a lot about how to be a better scientist and has made working with him a great experience.

I thank my family for all their love, support and encouragement.

I thank everyone I met, students, faculty, postdocs, and staff for many interesting conversations, valuable discussions, fun experiences and educational lectures, and for their friendship and support.

I gratefully acknowledge support from the John and David Boochever Prize Fellowship in Theoretical Physics at Cornell.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Gravitational Wave Astronomy

To introduce the work in this thesis on theoretical tools for analyzing sources of gravitational waves, we first give some well - known background material that can be found in textbooks such as [1, 2].

1.1.1 Gravitational Waves

Almost a century ago, Einstein's theory of general relativity radically changed the notion of space and time: they are not just the stage upon which events occur; instead spacetime is a dynamic entity which curves, expands and contracts around matter and energy. The theory of general relativity predicts the existence of transverse distortions of spacetime curvature, called gravitational waves, which, as a consequence of causality, propagate at the speed of light (since information about the changing gravitational field cannot reach distant observers faster than light). However, scientists at the time concluded that gravitational radiation would not be observable because it is produced only in extremely small quantities in everyday and atomic processes. For example, the probability for an electron transition of energy $E \sim 1$eV between two atomic states by gravitational radiation rather than electromagnetic radiation is of order the ratio of the square of the dimensionless "coupling constants" for the gravitational and electromagnetic interactions [1]: $\sim (G/c^5)(E^2/\hbar)/(e^2/\hbar c) \sim 10^{-54}$, which reflects how weakly gravitational waves interact with matter fields.

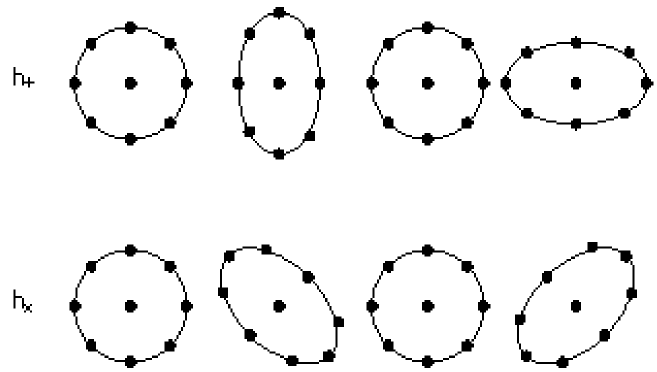

Figure 1.1: The effect of a gravitational wave passing down the $z-$axis on a ring of test particles is an oscillatory stretching and squeezing of space along orthogonal axes.

Nevertheless, in the 1960's, scientists started to look for gravitational radiation emitted coherently by the bulk motion of matter and energy in violent astrophysical processes, for which the prospects of detection were better. One characteristic of a gravitational waves' spacetime warpage is an oscillatory stretching and squeezing of space. Test particles in the presence of a passing gravitational wave will experience gravitational tidal forces that alternately stretch and squeeze along orthogonal axes in the plane perpendicular to the direction of propagation. The tidal deformations preserve the area enclosed by a ring of test particles, so a measure of the strength is the relative fractional deformation, or dimensionless strain amplitude, $h = 2\Delta L/L$, where L is the length and ΔL is the change in length. Just as electromagnetic waves, gravitational waves have two polarizations, commonly called h_+ and $h_\times$, however, they are rotated by 45^o with respect to one another as opposed to 90^o because they correspond to a spin-2 field. The effect of the two polarization fields on a ring of test particles is illustrated in Fig. (1.1).

The strain amplitude will typically be very small when waves from astrophysical sources reach the Earth. In the leading order approximation at large distances from the source, gravitational waves are produced by the time-changing mass quadrupole moment $Q_{ij}(t) \equiv \int d^3x \rho(\mathbf{x}, t)[x_i x_j - x^2 \delta_{ij}/3]$, where ρ is the density, since monopole waves would violate mass-energy conservation and dipole waves would violate momentum conservation. The dimensionless strain is of order [1]:

$$h \sim \frac{G}{c^4} \frac{1}{r} \frac{d^2}{dt^2} Q, \tag{1.1}$$

where r is the distance to source. The tiny factor of $(G/c^4) = 8 \times 10^{-45} \mathrm{s}^2 \mathrm{kg}^{-1} \mathrm{m}^{-1}$ reflects the fact that gravity is the weakest of the fundamental interactions. Only sources which are compact and highly dynamical can compensate for this factor. But even for large masses undergoing rapid variation, the expected strain from typical sources scientists hope to detect on earth is still very small:

$$h \sim 10^{-22} \left(\frac{M}{2.8 M_\odot} \right)^{5/3} \left(\frac{0.01\mathrm{s}}{\mathrm{P}} \right)^{2/3} \left(\frac{100\mathrm{Mpc}}{r} \right), \tag{1.2}$$

where the numbers correspond to typical binary neutron stars that are spiralling together with an orbital period P, and the symbol $M_\odot$ denotes the mass of the Sun, $\approx 2 \times 10^{30}$kg.

The gravitational waves from astrophysical sources have low frequencies (10^{-18}Hz - 10^3Hz) since the frequency is determined by the characteristic timescale for the source, and we expect that events involving large astrophysical bodies probably have timescales greater than a millisecond. Compare this to the high frequency of order 10^{15}Hz of visible light. For light, the wavelength is typically much smaller than the size of its source, so it can form images; this is not possible for gravitational waves whose wavelength is typically much larger than the size of source. The information contained in the waves is encoded in the time varying amplitudes of the two polarizations $h_+(t)$ and $h_\times(t)$, as for stereophonic sound. Gravitons are typically phase coherent, emitted by bulk mass motion, rather than phase incoherent superpositions of waves from atoms, molecules, and particles.

Gravitational waves have not yet been directly detected but compelling indirect evidence for their existence was the basis of the 1993 Nobel Prize in physics. Hulse and Taylor had monitored the orbital motion of the binary pulsar PSR1913+16 (two neutron stars orbiting each other) for 18 years from the Doppler shifting of radio signals emitted by the pulsar. General relativity predicts that gravitational radiation carries off energy and angular momentum and as a result the orbit shrinks. The prediction for the inspiral rate of 3mm per orbit agrees to $\sim 0.1\%$ with the observation, within the experimental uncertainty [3]. Today, astronomers are performing similar measurements on five more such double neutron star systems that have been discovered since then [4].

Scientists are now trying to detect gravitational waves directly, and to use them as a tool for astronomy to study phenomena that are likely not visible electromagnetically. Whereas electromagnetic signals from distant events are easily absorbed and scattered (for example by dust), gravitational waves pass through essentially unimpeded because they couple so weakly to matter.

The theoretical description of gravitational waves

We now discuss the regime in which the notion of "gravitational waves" makes sense. Within finite regions of space, gravitational waves cannot be defined at a fundamental level, one can only speak about time-varying gravitational fields. Gravitational waves can only be approximately defined in local regions in the special context when their wavelength $\lambda_{\rm GW}$ is much smaller than the characteristic scale $\mathcal{R}$ of the background curvature. This is analogous to the surface of a grapefruit, which has an overall, roughly spherical background curvature and dimples on small scales, analogous to the gravitational waves. For example, for $\sim$ 100Hz waves, the wavelength is $\lambda_{\rm GW} \sim$ 500km and on earth, the background curvature is $\mathcal{R} \sim 10^9$km, so this will be a good approximation [1]. Mathematically, one can describe gravitational radiation in this regime as approximately plane waves within a small region of space and, to linear order, define the background quantities such as the curvature and the distance rule to be the "coarse-grain " average value over lengthscales large compared to $\lambda_{\rm GW}$ but small compared to $\mathcal{R}$. The leftover, fluctuating pieces can be interpreted as effectively describing gravitational waves, which can then be treated as any other matter source. A meaningful concept is then the average energy density over spacetime volumes of dimensions larger than $\lambda_{\rm GW}$ but much smaller than $\mathcal{R}$, which must include the backreaction describing how the wave produces background spacetime curvature due to the nonlinear interactions with itself. Energy and momentum density cannot be localized at a point and are not defined on lengthscales shorter than the wavelength. A plane wave propagating in a flat background spacetime is completely described by its two dimensionless polarization amplitudes h_+ and $h_\times$. Taking the propagation direction to be along the $z-$axis, one finds that the energy flux T^{tz} in the gravitational wave is given by

$$T^{tz} = \frac{1}{16\pi}\frac{c^3}{G}\langle(\partial_t h_+)^2 + (\partial_t h_\times)^2\rangle, \tag{1.3}$$

where the angular brackets mean an average over several wavelengths. Assuming that the wave varies as $h_+ = h\cos(\omega t - \omega z)$, the energy flux is given by

$$T^{tz} = \frac{\pi}{4}\frac{c^3}{G^2} f^2 h^2 \approx 1.5 \times 10^{-3}\mathrm{Wm}^{-2}\left(\frac{h}{10^{-22}}\right)^2\left(\frac{f}{1\mathrm{kHz}}\right)^2, \tag{1.4}$$

where the numbers are for a supernova in the Virgo cluster of galaxies. Note that this flux is large by astronomical standards: it is comparable to the flux of reflected sunlight from a full moon. However, most gravitons pass through a detector (like neutrinos and unlike photons).

Interaction of gravitational waves with a detector

A simple way to see how the waves affect matter is to consider how two free particles in empty space react to the wave. Gravitational waves cause the *proper distance* between two freely falling particles to oscillate, even if the *coordinate separation* is constant. Consider a freely falling test particle and define a coordinate system that is chosen to be as nearly Newtonian as possible, i.e. distorted as little as possible by the gravitational waves, so that coordinate displacements are the same as proper separations to a good accuracy. In this coordinate system, consider another nearby test particle and let L^j be the components of the separation vector between the particles' worldlines initially. In the approximation that $L << \lambda_{\rm GW}$, a passing gravitational wave will produce a relative acceleration given by [1]

$$\frac{d^2L^i}{dt^2} = \frac{1}{2}\ddot{h}^{\rm TT}_{ij} L^j, \tag{1.5}$$

where the overdots indicate time derivatives and $h^{\rm TT}_{ij}$ is a symmetric spatial tensor which is transverse to the propagation direction and trace - free (the analog of the vector potential in Lorentz gauge in electromagnetism) and has non - zero components $h_{xx} = -h_{yy} = h_+(t-z)$ and $h_{xy} = h_{yx} = h_\times(t-z)$ (for propagation along the z - axis). It follows that the particles' separation changes by and amount

$$\delta L^i(t) = \frac{1}{2}h^{\rm TT}_{ij}(t) L^j, \tag{1.6}$$

where δL^i is the coordinate displacement produced by the passing wave.

Current interferometer detectors

The great challenge in detecting gravitational waves is the extraordinarily small effect the waves produce on a detector. As discussed above, even waves from violent astrophysical events have a very small amplitude when they reach the Earth, of order $h \sim 10^{-21}$. For an object 1 m in length, this means that its ends would move by 10^{-21}m relative to one other. This distance is about a millionth of the width of a proton. The most sensitive gravitational wave detectors today are Michelson-type interferometers, such as LIGO, the Laser Interferometer Gravitational wave Observatory with sites in Livingston, Louisiana and Hanford, Washington [5]. The LIGO detectors are part of a network of similar detectors around the world, most notably the French-Italian VIRGO, the British-German GEO, and the Japanese TAMA detector [6]. The cartoon-version of the detectors is illustrated in Fig. (1.2). Two mirrors, which act as test masses, hang far apart in a vacuum pipe (4 or 2 km) forming one "arm", and two more mirrors form a perpendicular arm. A laser beam is split in two after passing through a beam splitter located at the vertex of the perpendicular arms and each beam enters one of the arms. The light bounces between the mirrors repeatedly before recombining at the beam splitter and returning to the readout at the photodiode. A relative change in separation $\Delta L = \delta L_x - \delta L_y$ of the end mirrors and the beam splitter will produce a phase shift of the laser beams $\delta\phi = (4\pi/\lambda)\Delta L$, where λ is the wavelength of the laser, which results in a change in the intensity at the photodiode.

The detector sensitivity is limited by frequency-dependent noise of various kinds: For example, there are non-gravitational wave contributions to the time-varying spacetime curvature or tidal fields from near-zone sources such as due to the weather or human or seismic activity, which act as sources of noise in the detector output and dominate at frequencies below $\sim$ 10Hz. At higher frequencies, thermal noise (such as due to thermal motion of modes of vibration of the mirrors or of the suspension fibers) and photon shot noise are the limiting factors.

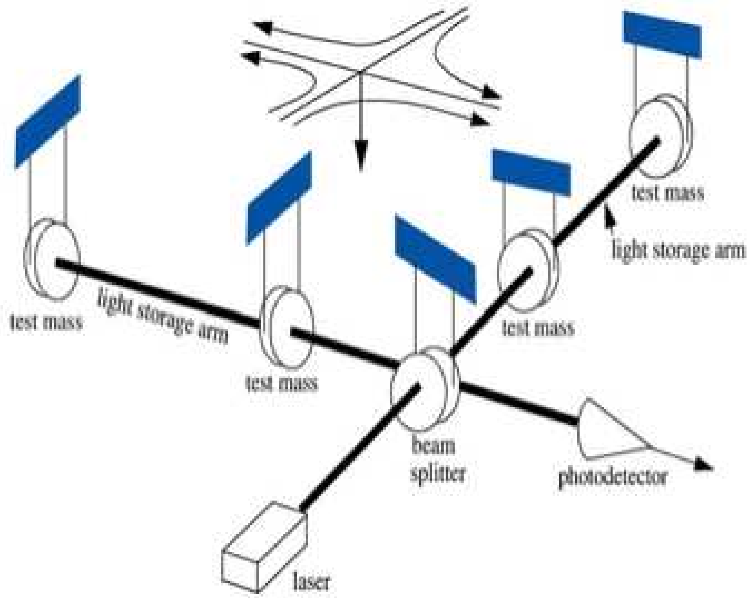

Figure 1.2: The principles of a laser interferometer detector. The top portion shows the forcelines at a given instant due to a gravitational wave propagating vertically downwards. Two mirrors in each of two perpendicular arms act as test masses. Laser light enters the arms simultaneously and is read out at the photodiode after traveling up and down the arms. The presence of the gravitational wave changes the proper separation of the mirrors, which results in a phase shift between the laser beams from the different arms, producing a change in the interference pattern at the diode. In general, the interferometer will measure some weighted combination of the two polarizations with the weights depending on the location of the source in the sky and its orientation relative to the detector. From K. Thorne.

Sources of gravitational waves

LIGO has gathered a full year of data at its design sensitivity, monitoring displacements a thousand times smaller than the size of a proton. Reaching this design sensitivity was a great achievement, and was aided by the formation of a large international collaboration of over 500 people from 35 institutions. LIGO's frequency band is $\sim 40 - 1000$Hz, which corresponds to the last few minutes of the inspiral of binary neutron stars or black holes of a few solar masses, visible to LIGO out to $\sim$ 15 megaparsecs. Astrophysical sources in this band besides compact object (neutron star, black hole) inspirals and mergers include spinning neutron stars in our Galaxy, supernovae, stochastic waves from processes in the early Universe (inflation, phase transitions, etc.) and the large discovery space of unexpected sources and effects in the universe. LIGO can observe neutron star binary inspirals out to a distance of $\sim$ 20Mpc $\sim 6 \times 10^{20}$km, which includes the thousands of galaxies in the Virgo cluster. The fact that no events have been seen yet has been used to place upper limits on the event rates. For binary neutron stars, statistical analyses based on the observed number of progenitor binary star systems indicated an event detection rate of between $(1/3000)\text{yr}^{-1}$ to $1/8\text{yr}^{-1}$.

LIGO is currently being upgraded and will explore a ten times larger volume of the Universe in a two-year run starting in 2009. After 2010, a new, improved detector will be installed, which will survey a volume a thousand-fold larger than initial LIGO. The expected event detection rate for neutron star inspirals is between 1yr^{-1} to 2day^{-1}.

Planned space-based interferometer detector

As discussed above, the sensitivity of ground based detectors is fundamentally limited at low frequencies because they cannot be shielded from time-varying curvature fluctuations due to the environment. This problem could be avoided by having a detector in space, such as the planned Laser Interferometer Space Antenna (LISA) [7], jointly sponsored by the European Space Agency (ESA) and NASA and hoped to launch in 2018. The mission consists of three drag-free spacecraft flying five million kilometers apart in an equilateral triangle whose center will follow Earth's orbit around the Sun. Each spacecraft carries instruments made up of two optical assemblies, which contain the main optics, lasers, and a free-falling gravitational reference sensor. The sensor is used to control the motion of the spacecraft and contains two small cubes, shielded from any disturbances and allowed to float freely within the spacecraft. The cubes, which act as test masses, are highly polished to enable them to reflect laser light and thus act as mirrors in an interferometer. The relative motion of these cubes on different spacecraft, five million km apart, are what will detect passing gravitational waves.

LISA will make its observations in a low-frequency band between $\sim 0.1 -$ 100mHz making it complementary to ground based detectors. Sources of gravitational waves detectable by LISA should include newly forming black holes, colliding massive black holes, inspirals of neutron stars or black holes into massive black holes and pairs of inspiralling white dwarf stars (these are guaranteed sources, with quite a few target binaries already catalogued by X-ray and optical studies) [8].

Other kinds of gravitational wave detectors

The oldest kind of detector is the bar detector first built by Weber in the 1960's. Bar detectors are typically massive cylinders of materials which have little damping (high quality factor) in their fundamental frequency of oscillation. The idea is that an impinging gravitational wave of the right frequency will set the fundamental mode into oscillations, and the bar's displacement can be detected by a sensor. The bar's resonant frequency must be in the range of frequencies of the incoming wave, so the bars operate as narrow-band detectors and measure only the Fourier component of the waveform at the resonant frequency. For a supernova explosion, the typical frequency is $\sim$ 1kHz but since they are broadband sources, bar detectors with resonant frequencies near 1kHz should be able to detect it. Tuning to this frequency range means bars with typical lengths of $\sim 1-4$m. Bars cooled to liquid helium temperatures can measure strains of order $h \sim 10^{-19}$.

Gravitational wave searches in space have been made for short periods by planetary missions with other primary science objectives. Some current missions are using microwave Doppler tracking to search for gravitational waves in the low-frequency ($\sim 10^{-2}-10^{-4}$Hz) gravitational wave band [9]. This is set by the $\sim 100s$ it takes for accurate clock readout and also by the fact that Earth's rotation prevents continuous tracking from the same site. In the Doppler method the earth and a distant spacecraft (at a separation of $\sim 1-10$AU) act as free test masses with a ground-based precision Doppler tracking system continuously monitoring the ratio $\Delta\nu/\nu_0$ of the Doppler shift in frequency $\Delta\nu$ to the earth-spacecraft radio link carrier frequency ν_0. A gravitational wave having strain amplitude h incident on the earth-spacecraft system causes perturbations of order h in the time series of $\Delta\nu/\nu$.

A technique to detect passing gravitational waves in the ultra low frequency band (f $\sim 10^{-9}$Hz) is by using pulsar timing observations. Pulsars are extremely stable clocks, and it is now possible to make timing observations of millisecond pulsars to a precision of $\sim$ 100 ns, which allows the pulsar parameters to be determined with great accuracy. The Parkes Pulsar Timing Array project aims to observe 20 millisecond radio pulsars over several years and to compare the observed arrival times of pulses with a model of the pulsars parameters. The differences between the actual arrival times and the predictions, i.e. the "timing residuals", indicate the presence of unmodelled effects such as calibration errors, (additional) orbital companions, spin-down irregularities and gravitational waves. For a given pulsar and gravitational wave source, the effect of a passing gravitational wave is only dependent upon the characteristic strain at the pulsar and at the Earth. The strains evaluated at the positions of multiple pulsars will be uncorrelated, whereas the component at the Earth will lead to a correlated signal in the timing residuals of all pulsars.

Sources include stochastic backgrounds from supermassive black holes, cosmic strings or relic gravitational waves from the big bang, the formation of supermassive black holes, and from cosmic string cusps.

1.1.2 Benefits of Theoretical Modeling of Gravitational Wave Sources

The detection and interpretation of a large class of gravitational wave signals is based on matched filtering, i.e. the noisy detector output is integrated against a bank of theoretical waveforms called templates, and the parameters of the template are varied to maximize the overlap integral. Schematically, the overlap integral of the template with the signal has the form $\sim \int [s(f)T^*(f)/S_n(f)]df$, where $s(f)$ is the Fourier transform of the signal, $T^*(f)$ the template, and $S_n(f)$ the detector noise power spectrum. The signal waveforms from compact object inspirals are oscillatory with many cycles (tens of thousands for LIGO, hundreds of thousands for LISA), so the template must capture the phase evolution with extremely high accuracy. If the waveforms slip by a fraction of a radian, it will be obvious in the cross-correlation and may impede detection. Therefore, the required theoretical accuracy is ~ 1 radian or better. Alternatively, the detection of a phase perturbation could give information about neutron star or black hole physics. Computing waveforms to this accuracy is a great theoretical challenge.

The work in this thesis focuses on developing the theoretical tools for describing the gravitational radiation from binary inspirals, so that we may answer questions such as: What is the nature of the gravitational waves generated? What information about these sources can be extracted from the measured signal? What is the effect of the loss of energy and angular momentum to the gravitational radiation on its source? This thesis studies theoretical aspects of two different classes of sources of gravitational radiation. The first kind of source is a system of two neutron stars orbiting one another and is discussed in Sec. (1.2) below and Chapters (2) and (3). The second class of sources are test body inspirals into massive black holes, which is presented in Sec. (1.3) below and Chapters (4), (5), and (6). For each class of sources, we first give some well - known background material in order to place the work in context.

1.2 Neutron stars

Our present understanding is the following. Neutron stars are produced when the degenerate cores of massive stars undergo gravitational collapse to nuclear densities, driving off the outer layers as a supernova explosion. If the Sun were a neutron star, all of its matter would be packed into a ball that could fit inside Crater Lake (in Oregon), with one teaspoon of its material having a mass over 5×10^{12} kg. Neutron stars are often described as a macroscopic nucleus of 10^{51} nucleons held together by gravity instead of the strong force. A neutron star's gravity is so intense that the escape velocity from the surface is half the speed of light; they are the most compact objects without event horizons known today. They are observed electromagnetically as X-ray sources and radio pulsars, and at present there are over 2000 known neutron stars in the Milky Way and the

Magellanic Clouds.

Neutron stars are very complicated objects whose internal structure remains poorly understood. For example, they are believed to have solid crusts and a heavy liquid mantle of free electrons, protons and neutrons. The neutrons are likely to be superfluid and the protons superconducting, which occurs at temperatures of $\gtrsim 10^9$ K and thus makes neutron stars the ultimate high-temperature superconductors. Little is known about the exact nature of the superdense matter in the core, at densities ~ 10 times the density of an atomic nucleus. It has also been suggested that neutron-star cores may contain unique forms of matter, for example Bose Einstein condensates of subatomic particles such as pions and kaons or deconfined quarks.

Learning about dense matter from neutron stars is challenging because observations only provide indirect information. One approach is to exploit the fact that the equation of state, or pressure-density relation $p = p(\rho)$ for a neutron star can be directly mapped onto relations that involve macroscopic quantities such as a mass-radius relation $M = M(R)$. Existing individual measurements of M and R can give some useful constraints, which we will review below, however, strong constraints on the equation of state would come from accurate measurements of M and R in a single neutron star.

The neutron star mass has a theoretical upper limit of at most $3M_\odot$, assuming causality. The existence of a maximum mass is a consequence of general relativity, and it reflects the stiffness of the equation of state at high densities of several times nuclear density. If high-density matter is very compressible, the star will be comparatively small for its mass. The presence of exotic matter (such as hyperons, Bose - condensates or quarks), which is especially compressible, also lowers the maximum stable mass for a neutron star. Observations of extremely massive neutron stars can therefore eliminate entire families of equations of state, and in particular the existence of exotic matter in a star's interior.

The neutron star radius is controlled primarily by properties of the nuclear force at densities in the immediate vicinity of the nuclear saturation density [10]. For the nearly pure neutron matter found in neutron stars, it is a direct measure of the density dependence of the nuclear symmetry energy (the symmetry energy is the change in nuclear energy associated with changing the neutron-proton asymmetry) [10].

We now review as background existing methods for determining neutron star masses and radii in order to place the new work in this thesis in context.

Existing mass measurements

Accurate measurements of neutron star masses are obtained from timing observations of the radio signals in binary pulsars. If at least two parameters characterizing relativistic effects such as Shapiro time delay, periastron advance or orbital decay due to gravitational radiation can be determined, the masses can be inferred to accuracies as high as 0.01%. Most of the neutron stars in such binaries have masses in the range of $M \sim 1.25 - 1.44 M_\odot$ [10].

For neutron stars with white dwarf or main sequence star companions, astronomers can estimate the neutron star mass if the companion mass can be determined from its electromagnetic spectrum. The range of masses in such binaries is from $1.1 - 2.2 M_\odot$, but with typical accuracies of only $\sim 10\%$ [10].

Mass estimates for neutron stars are also possible for some X-ray sources, which involve a neutron star accreting matter from a companion. Combining measurements of the X-ray pulse delays, X-ray eclipses and radial velocities give indications of a wide range of masses $1-2.4M_{\odot}$. However, the complicated properties of these sources make the mass estimates highly uncertain [10].

It may also be possible to constrain neutron star masses from observations of quasi-periodic oscillations of X-rays from gas accretion onto the neutron star once a reliable theoretical model of this process becomes available [10].

Existing radius constraints

A determination of the radius of a neutron star in addition to its mass would yield important information about the state of matter at nuclear densities. Different theoretical models for the nuclear equation of state predict, for a $1.4M_{\odot}$ neutron star, radii in the range of $R \sim 7-16$km. However, there is currently no accurate method of measuring radii. Some weak constraints can be inferred from electromagnetic observations, although these are highly dependent on the theoretical models used to interpret the observations.

A lower limit on the radius for a given mass can be inferred from pulsar glitches, which are sudden discontinuities in the spin-down of pulsars. One leading model supposes that the glitches involve the transfer of angular momentum from superfluid neutrons in the crust to the entire star, which is spinning down due to electromagnetic emission. For the Vela pulsar, this model implies that $R \geq 3.6 + 3.9(M/M_{\odot})$km [10].

Observations of the thermal radiation from isolated cooling neutron stars can potentially constrain the redshifted radius $R_{\infty} = R/\sqrt{1-2GM/Rc^2}$. This requires that the source's distance can be accurately assessed and the composition of the atmosphere and magnetic field can be modeled. The measured quantities are the flux and temperature of the radiation, both of which are redshifted as the radiation climbs out of the neutron star's potential well.

Neutron star seismology combined with tentative models limits the ratio of the thickness of the crust to the radius and can be used to place limits on the $M(R)$ parameter space. This comes from measurements of more than one frequency of oscillation, which can for example be due to torsional vibrations of the star's crust.

Gravitational light-bending suppresses the amplitude of variations of the pulsed emission of X-rays such as from rotating neutron stars since it allows an observer to see a larger part of the star than just the hemisphere facing towards him. Observations of pulsations in the emitted radiation can therefore constrain the ratio M/R. For Her X-1, with $M \sim 1.291.59 M_\odot$, this method implies a radius range of $10.1\text{km} < R < 13.1\text{km}$. However, this result depends on the model assumed, for example, for the magnetic field.

The effects of gravity cause the observed frequencies of the spectral lines to be shifted to lower values, by a factor of $1/(1+z) = [1 - 2GM/Rc^2]^{1/2}$, where z is the redshift. X-ray observations of EXO0748 676, a neutron star that is accreting gas from a lower-mass star, showed a pair of resonance scattering lines which were interpreted to be Fe XXV and XXVI, implying $z = 0.345$ if the spectral line identifications are correct. A few similar measurements have been performed, for example using data from the XMM-Newton satellite [11].

X-ray bursts, possibly due to thermonuclear reactions on neutron star surfaces, have peak fluxes comparable to the to the Eddington flux (when the radiation pressure equals the gravitational force on the gas) $F_{\rm Edd} = GMc/(\kappa d^2)[1-2GM/Rc^2]^{1/2}$, where κ is the opacity (which is modeled theoretically) and d the distance to the source. Many sources exhibit quiescent states between bursts, believed to involve the radiation of thermal emission with a cooling flux $F_{\rm cool} = \alpha(R^2/d^2)[1 - 2GM/Rc^2]^{-1}$, where α depends on the composition and temperature and is modeled theoretically. If in addition, spectral lines allow a determination of the redshift, these three observations can be combined to determine the distance, mass and the radius of a single star.

Analogous to the existence of a maximum mass is the existence of a maximum compactness GM/Rc^2, which is thought to be such that $R \gtrsim 2.8M$. This limits the minimum spin period before the star starts to shed its mass to be $\sim \sqrt{M_\odot/M}(R/10\text{km})^{3/2}$ms. The spin rate therefore sets an upper limit to the radius of a star of a given mass. The pulsar with the most rapid spin rate currently known is PSR J1748-2446ad with a frequency of 716 Hz, which, for a mass of $1.4M_\odot$ implies a radius of $R \lesssim 14.3$km.

The most relativistic binary neutron star currently known is PSR J0737-3039, for which a measurement of spinorbit coupling could eventually lead to a determination of the moment of inertia of one of the neutron stars within a few years. The moment of inertia, being roughly proportional to MR^2, is a sensitive measure of neutron star radius since the mass will be accurately known.

The radius could also potentially be constrained from quasi-periodic oscillations of X-rays from gas accretion onto the neutron star if the frequency of the innermost stable circular orbit for the gas can be determined from the shape of the peaks in the frequency spectrum. Potential future constraints on the radius could also come from neutrino observations from supernova signals, when the proto-neutron stars are formed.

Complementary to astrophysical observations, scientist use laboratory measurements of dense matter parameters such as the nuclear charge radii of neutron-rich heavy nuclei such as ^{208}Pb to place some constraints on the large parameter space of neutron star interiors.

In summary, neutron star masses can be determined accurately in some cases, radii are poorly constrained, and a few redshifts have been measured, but there are no accurate, model-independent measurements of M and R for the same star.

1.2.1 Potential gravitational wave measurements

Astronomical observations (such as from orbital motion, Doppler shifts of spectral lines, eclipsing X-ray signals, etc.) show that about two-thirds of stars have a gravitationally bound stellar companion; these are called binary stars. In binary systems consisting of compact objects (white dwarfs, neutron stars of black holes) the two bodies can approach one another closely without being disrupted by tidal forces. The lifetime of the binary is approximately $t_0 \sim 10^5 P(P/1s)^{5/3}$ [1], where P is the orbital period (three of the five double - neutron star systems known so far have orbits tight enough that the two neutron stars will merge within a Hubble time). If $P < 1/2$ day, the lifetime is less than the Hubble time. This is the population targeted by LIGO. The binary undergoes a long inspiral phase in which the orbit gradually shrinks due to gravitational wave backreaction. Only the last few minutes, at frequencies 10Hz$\leq f \leq$ 1000Hz will be within LIGO's sensitive frequency band.

In addition to being a key source for LIGO, neutron star binary inspirals are also the leading candidates for the source of a type of gamma - ray burst observed by astronomers, the so-called " short/hard" bursts, which refers to their duration and intensity. According to this hypothesis, the bursts are produced by the merger phase, which is very sensitive to the neutron star internal structure.

Observations of the gravitational waves from merger events could potentially yield the simultaneous direct determinations of the masses and radii. The adiabatic inspiral terminates either when the orbit becomes unstable (at which point two neutron stars are orbiting each other at hundreds of times per second) and the objects merge or, for some neutron star - black hole binaries, when the neutron star is tidally disrupted. In either case, a measurement of the gravity-wave frequency at this point can be used to constrain the neutron star radius. Fig. (1.3) shows the expected gravitational wave signal from a neutron star binary inspiral together with the LIGO noise curves. The signal terminates at the innermost stable circular orbit, when the gravitational wave frequency is of order 800 Hz.

The highly dynamical spacetime gives rise to gravitational radiation with a characteristic pattern (a "chirp"), with the amplitude and frequency both increasing with time. Fig. (1.4) shows qualitatively an expected inspiral waveform as a function of time.

Computing the dynamical spacetime for the binary is in general a very difficult task; however, there are certain regimes in the parameter space of the member's masses and orbital separation which admit analytical approximation methods. The main theoretical tool for modelling the early, low frequency part of the inspiral is

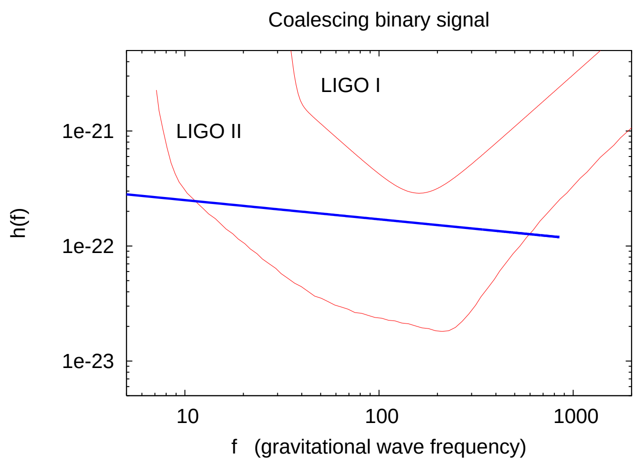

Figure 1.3: The noise curves $h_{\rm rms}(f) = \sqrt{f S_h(f)}$ for LIGO I and LIGO II are shown in red (thin lines). The thicker blue line shows the signal $h_c(f)$ for two $1.4 M_\odot$ neutron stars at a distance of 200Mpc. The signal terminates at the innermost stable circular orbit, where the gravitational wave frequency (twice the orbital frequency) is $f_{\rm isco} \sim 850$Hz assuming the stars have $R = 10$km, and pressure-density relation $p \propto \rho^2$. From Racine and Flanagan, 2006.

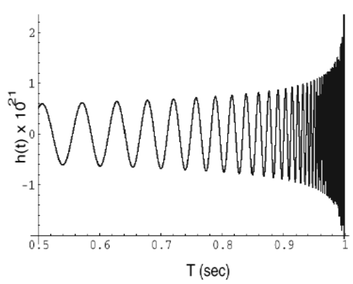

Figure 1.4: The form of an expected "chirp" signal from an inspiralling binary as a function of time. Both the frequency and amplitude increase as the inspiral progresses. From K. Thorne.

the post-Newtonian formalism, which assumes that the two bodies, treated as spinning point particles, are moving at slow velocities under their mutual gravitational influences. The expansion parameter is $v^2/c^2 \sim GM/(rc^2)$, where v is the orbital velocity and M the total mass. This approximation is very accurate during the early part of the inspiral, at frequencies below $\sim$ 400Hz and has been iterated to high orders. A point particle description of binaries involving neutron stars may not be adequate because finite-size effects could be non-negligible even during the early part of the inspiral, as will be discussed in Ch. 2.

Previous investigations of the possibility of obtaining constraints on the internal structure from the gravitational wave signal have focused on the very end of the inspiral and the coalescence phase. (i) A method for determining the compactness ratio GM/Rc^2 based on the observed deviation of the gravitational wave energy spectrum from point-mass behavior at the end of inspiral has been suggested [12]. (ii) For neutron star-black hole binaries, the frequency at which the neutron star is tidally disrupted strongly depends on the star's radius [13]. (iii) Several numerical simulations have studied the dependence of the gravitational wave spectrum on the radius during the coalescence phase (see, e.g. [14]). (iv) The quasinormal mode frequencies of a neutron star differ from those for a black hole [15].

However, there are a number of difficulties associated with trying to extract equation of state information from this late time regime after contact or innermost stable orbit, at frequencies $f \gtrsim$ 700Hz: (i) The highly complex behavior requires solving the full nonlinear equations of general relativity together with relativistic hydrodynamics. (ii) The signal depends on unknown quantities such as the spins of the stars. (iii) The signals from the hydrodynamic merger (at frequencies $\gtrsim$ 1000 Hz) are outside of LIGO's most sensitive band.

It would therefore be of great advantage if one could instead obtain information about the neutron star internal structure from the early, relatively clean part of the inspiral signal at frequencies $f \lesssim 400\text{Hz}$. We investigate the prospect of this possibility in the next section and in Ch. 2.

Our results suggest that there is a potential to obtain useful information from an analysis of this early portion of the gravitational wave signal, complementary to the (more studied) information in the late time signal.

Obtaining information about neutron star internal structure from the inspiral signal

In chapter (2), we show how model-independent constraints of the neutron star internal structure can be obtained instead from gravitational wave observations with LIGO using data only from the early part of the inspiral at frequencies $f \leq 400\text{Hz}$, where the signal is very clean and theoretical errors are well-understood. The stars can be accurately modeled as point particles, possibly spinning, with a small correction due to finite size effects. As discussed above, because of the matched-filtering based signal, if the accumulated phase shift due to the finite size corrections becomes of order unity or larger, it could corrupt the detection of signals or alternatively, detecting a phase perturbation could give information about the neutron star structure. The influence of the internal structure on the gravitational wave phase in this early regime of the inspiral is characterized by a single parameter, namely the ratio λ of the induced quadrupole to the perturbing tidal field due to the companion.

The ratio λ is related to the star's dimensionless tidal Love number k_2 by $k_2 = 3G\lambda R^{-5}/2$, where R is the star's radius. The Love number encodes information about the star's degree of central condensation. Stars that are more centrally condensed will have a smaller response to a tidal field, resulting in a smaller Love number. We computed the Love numbers for fully relativistic neutron stars for the first time and found that they differ from the Newtonian values that were used in previous analyses by up to $\sim 24\%$ for plausible approximate neutron star models (for simplicity, we modelled the pressure-density relation with a simple polytropic form $p = K\rho^{1+1/n}$, where p is the pressure and ρ is the rest mass density. The constant K describes how compressible the matter is, and the exponent $1 + 1/n$ is related to the degree of central concentration of the neutron star interior). In Ch. (2) we show that for an inspiral of two non-spinning $1.4M_\odot$ neutron stars at a distance of 50 Mpc, LIGO II detectors will be able to constrain λ to $\lambda \leq 2.01 \times 10^{37} \mathrm{g\,cm^2 s^2}$ with 90% confidence. This number is an upper limit on λ in the case that no tidal phase shift is observed. The corresponding constraint on radius would be $R \leq 13.6\,\mathrm{km}$ $(15.3\,\mathrm{km})$ for relevant fully relativistic neutron star models, for $1.4M_\odot$ neutron stars.

We now turn to the discussion of the second class of source of gravitational waves, namely test particles inspiralling into much more massive black holes. The work in this thesis focused on developing the mathematical formalism for treating this system, but we first give some relevant motivation and background information.

1.3 Extreme mass ratio inspirals

So far, we have only discussed some aspects of the two-body problem in the weak-field, slow motion regime valid for binaries at large orbital separation. Different computational techniques are necessary for binaries which are highly relativistic. For comparable masses at small separation, one must use numerical relativity. Numerical relativists have recently made spectacular progress: They can simulate the merger of two spinning black holes (see e.g. [16] and references therein) and make important astrophysical predictions such as the potentially large size of the kicks given to the black holes by the emitted gravitational waves [17, 18], which may recently have been observationally confirmed [19]. Numerical methods become increasingly difficult and computationally expensive as the mass ratio is decreased and as the separation is increased. However, one can instead use systematic analytical approximation methods that rely on identifying a small parameter to define a perturbation expansion. As discussed above, the main such theoretical tool that has been used for binaries at large separation, where the gravitational field is weak, are post-Newtonian expansions [20]. These methods have been very successful for modelling motion in the solar system and of binary pulsars [21] but break down in the highly relativistic regime. A theoretical understanding of binaries in the relativistic regime with one member much more massive than the other can be obtained by exploiting the fact that the mass ratio is small: the binary can be modeled as the spacetime of the larger mass with a perturbation due to the small mass.

Observational relevance

The highly relativistic, small mass ratio regime is now becoming observationally accessible: Compact objects spiraling into much larger black holes due to gravitational wave backreaction are expected to be a key source for both LISA and LIGO. Infrared and optical observations of stars and gas in the central regions of galaxies indicate the presence of dark central objects with mass more than a million times the mass of the sun confined to a very small region of space; these objects with masses in the range of $10^4 \leq M \leq 10^7 M_\odot$ are believed to be supermassive black holes [22]. Stellar mass compact objects get kicked by multibody gravitational deflection processes in the stellar cluster that surrounds these central objects and get captured into highly relativistic orbits. Most orbits will be highly eccentric, and the orbits will gradually shrink and become less eccentric due to gravitational wave backreaction. Such inspirals will be visible to LISA out to redshifts $z \approx 1$ [23, 24, 25]. It has been estimated [26, 27] that LISA should see about 50 such events per year, based on N-body simulations of stellar dynamics in galaxies' central cusps [28]. There are many uncertainties associated with the estimates for the LISA event rates, for example the populations of compact objects in galactic nuclei are not well known.

Inspirals of black holes or neutron stars into intermediate mass ($50 \leq M \leq 1000 M_\odot$) black holes would be visible to Advanced LIGO out to distances of several hundred Mpc [29], where the event rate could be about $3-30$ per year [29, 30]. Evidence for the existence of intermediate mass black holes comes for example from a class of X-ray sources discovered in recent years which seem to be too bright to be black holes of a few tens of solar masses but too dim to be supermassive black holes.

Science payoffs

For both types of sources discussed above, the small body will linger in the central object's strong curvature region for many thousands of wave cycles before merger; this will allow high precision studies. The gravitational waves will be rich with information. For example, one will be able, for the first time, to extract an accurate observational map of the large body's spacetime geometry, or equivalently the values of all its multipole moments. This will allow an unambiguous identification of the central object as a black hole or potentially lead to the discovery of non-black-hole central objects such as boson stars [31, 32] or naked singularities. Inferring the properties of the central object's spacetime geometry from the much smaller object's orbital evolution is analogous to what geodesy satellites such as the GRACE and CHAMP missions do for the Earth. The satellites' orbits probe the Earth's gravitational potential, which encodes an extremely precise map of the matter distribution of earth and is used to monitor climate changes such as the loss rate of the polar ice caps.

The gravitational waves also carry important astrophysical information. Observing many events and measuring the central object's mass and spin to high accuracy will provide a census of the properties of central black holes and can provide useful information about the hole's growth history [33]. Measuring the inspiralling objects‘ masses will teach us about the stellar population in the central parsec of galactic nuclei. A potential payoff for cosmology is that if the LISA event rate is large enough, one can measure the Hubble constant H_0 to about 1% [34], which would indirectly aid dark energy studies [35].

1.3.1 Modelling extreme mass ratio inspirals

To realize the science goals for these sources requires reliable theoretical models of the inspiral waveforms for matched filtering. The accuracy requirement is roughly that the theoretical template's phase must remain accurate to ~ 1 cycle over the many cycles of waveform in the highly relativistic regime ($\sim 10^2$ cycles for LIGO, $\sim 10^5$ for LISA). There has been a significant research effort within the general relativity community aimed at providing such accurate templates [36, 37, 38]. A theoretical understanding of binaries in the relativistic regime with one member much more massive than the other can be obtained by exploiting the fact that the mass ratio is small: the binary can be modeled as the spacetime of the larger mass with a perturbation due to the small mass.

On short timescales, the small object moves on a bound geodesic orbit of the black hole's spacetime, characterized by its conserved energy E, z-component of angular momentum L_z, and a third constant of the motion, the Carter constant Q (the relativistic analogue of the magnitude of the non-axial angular momentum). In contrast to Newtonian orbits, which are planar and have only a single frequency, strong field black hole geodesic orbits have three distinct orbital frequencies. The motion is confined within a toroidal region with three degrees of freedom. Despite being more complicated than the Newtonian analogue, the motion is still completely integrable and can be treated using the methods of Hamiltonian mechanics.

The small body's geodesic motion in the Kerr background is corrected by self-force and radiation reaction effects describing the body's interaction with its own spacetime distortion [39]. In the regime where the radiation reaction time is much longer than the orbital time, which is a good approximation for most of the inspiral for astrophysical binaries, the self-force effects cause the parameters E, L_z and Q to evolve adiabatically and the orbit to shrink.

The formal expression for the leading order gravitational self-interaction of a body was derived more than ten years ago. However, the practical implementation presents difficulties because the self-force is singular at the body's location and must be regularized. The full leading order self-force for practical implementations is not yet available for generic orbits around spinning black holes, although there has been great recent progress. Many researchers are now working on various approximate methods of computing the orbital motion and the gravitational waveform.

To compute just the leading order motion, one can sidestep the requirement of computing the full self-force and replace use its time averaged (actually averaged over the orbital torus), radiative piece instead, which is fairly simple to compute.

There are various theoretical difficulties associated with going to higher order, which we resolved in the work presented in Ch. 4.

Two-timescale expansion method

We have developed a new approximation scheme based on a two-timescale expansion, which resolves the difficulties with the standard perturbation formalism and is presented in chapter 4. We cast the equations describing binary inspiral in the extreme mass ratio limit in terms of action angle variables, and derive properties of general solutions using two-timescale expansions, which are a systematic method for studying the cumulative effect of a small disturbance on a dynamical system that is active over a long time. The method is based on the fact that the systems evolve adiabatically: the radiation reaction timescale is much longer than the orbital timescale. Our formalism applied to the orbital motion provides a rigorous derivation and clarification of the leading order, adiabatic approximation to Kerr inspirals and gives a systematic framework for computing post-adiabatic corrections needed for measurement templates. One of the key results of our analysis is the identification of which pieces of the self-forces are required to compute the adiabatic and post-adiabatic motions, which is of great practical importance as an explicit computational prescription currently exists only for a piece of the leading order self-force.

Analytical results for inspirals in the weak field regime

As discussed above, the leading order waveforms for extreme mass ratio inspirals can currently be computed for generic orbits. However, the calculations are computationally expensive, and they give only the leading order evolution. To complement these waveforms, it is desirable to have approximate waveforms that can be generated cheaply and quickly but which still capture the main features of true waveforms. These can also be useful to assess the accuracy of the leading order, adiabatic approximation since the self-force in the weak field regime is known to higher order. Different kinds of such weak field, approximate waveforms have already been used to scope out data analysis issues for LISA.

As discussed above, astronomical observations have established the existence of extremely compact, massive objects. Generally, these objects are thought to black holes as predicted by general relativity. Testing this hypothesis requires going beyond black holes, which is difficult because very few alternative theories of gravity make predictions for black holes that differ from those of general relativity. One can focus instead on the simpler task of considering spacetimes which are more general than the black hole spacetimes in general relativity, which does not require a priori knowledge of the corresponding theory of gravity. For any gravitating body that is stationary, axisymmetric, and reflection symmetric across the equator (which encompasses black holes plus a wide variety of perturbations and other objects) the exterior spacetime is fully specified by a pair of multipole moment families: the mass multipole moments and the current multipole moments. If the gravitating body is a black hole in general relativity, then the values of the mass and current moments are strongly restricted: the exterior spacetime is completely characterized by its two lowest multipole moments, the total mass and the spin angular momentum, all higher multipoles are completely determined by these two values; this is called the Kerr spacetime. More general spacetimes of a massive compact object have a different multipolar structure, which does not satisfy these strict constraints. Testing if the object is a black hole with just two independent multipole moments therefore requires that we be able to compare against objects with the wrong multipole structure. As discussed above, the spacetime's multipolar structure in encoded in the orbital motion of test bodies.

In Chapter (5), we consider the effects of multipole moments on inspiral waveforms, in particular the effects of the central object's quadrupole moment and of the leading order spin self interaction in the weak field regime. We examine the effect of an axisymmetric quadrupole moment Q of a central body on test particle inspirals, to linear order in Q, to the leading post-Newtonian order, and to linear order in the mass ratio. This system admits three constants of the motion in absence of radiation reaction: energy, angular momentum along the symmetry axis, and a third constant analogous to the Carter constant. We compute instantaneous and time-averaged rates of change of these three constants. Our result, when combined with an interaction quadratic in the spin (the coupling of the black hole's spin to its own radiation reaction field), gives the next to leading order evolution. The effect of the quadrupole is to circularize eccentric orbits and to drive the orbital plane towards antialignment with the symmetry axis.

In addition we consider a system of two point masses where one body has a single mass multipole or current multipole. To linear order in the mass ratio, to linear order in the multipole, and to the leading post-Newtonian order, we show that there does not exist an analog of the Carter constant for such a system (except for the cases of spin and a mass quadrupole). Thus, the existence of the Carter constant for a black hole in general relativity depends on interaction effects between the different multipoles. With mild additional assumptions, this result falsifies the conjecture that all vacuum, axisymmetric spacetimes possess a third constant of the motion for geodesic motion.

Evolution of the Carter constant in the adiabatic limit

As discussed above, the leading-order, adiabatic waveforms can be computed using only the time-averaged, radiative piece of the full first order self force. In practice, this means that it only requires computing the time - averaged time rates of change of the three constants of motion: the energy, axial angular momentum, and Carter constant. For the energy and angular momentum, one can compute the amounts radiated to infinity and the horizon using the well - known technique of black hole perturbation theory and impose global flux conservation to infer the time-averaged rates of change of the orbital constants. Incorporating radiation reaction for generic orbits requires in addition a method of computing the rate of change of the Carter constant, for which there is no currently known conservation law. The authors of Ref. [40] derived an explicit formula for the the time-averaged time derivative of the Carter constant in terms of a mode sum expansion for a particle coupled to a scalar field, and Ref. [41] extended this result to the tensor case. Chapter 7 contains a rederivation and extension of this result, giving more details on the derivation than previously available and a self-contained treatment in a unified notation. It also shows that the standard results are consistent with the two - timescale approximation at leading order.

CHAPTER 2

CONSTRAINING NEUTRON STAR TIDAL LOVE NUMBERS WITH GRAVITATIONAL WAVE DETECTORS

SUMMARY: Ground-based gravitational wave detectors may be able to constrain the nuclear equation of state using the early, low frequency portion of the signal of detected neutron star neutron star inspirals. In this early adiabatic regime, the influence of a neutron star's internal structure on the phase of the waveform depends only on a single parameter λ of the star related to its tidal Love number, namely, the ratio of the induced quadrupole moment to the perturbing tidal gravitational field. We analyze the information obtainable from gravitational wave frequencies smaller than a cutoff frequency of 400 Hz, where corrections to the internal-structure signal are less than 10%. For an inspiral of two nonspinning $1.4M_{\odot}$ neutron stars at a distance of 50 Megaparsecs, LIGO II detectors will be able to constrain λ to $\lambda \leq 2.0 \times 10^{37}\mathrm{gcm}^2\mathrm{s}^2$ with 90% confidence. Fully relativistic stellar models show that the corresponding constraint on radius R for $1.4M_{\odot}$ neutron stars would be $R \leq 13.6$ km (15.3 km) for a $n = 0.5$ ($n = 1.0$) polytrope with equation of state $p \propto \rho^{1+1/n}$.

Originally appeared in Phys. Rev. D **77** *021502(R), (2008), with É. Flanagan.*

2.1 Background and Motivation

Coalescing binary neutron stars are one of the most important sources for gravitational wave (GW) detectors [24]. LIGO I observations have established upper limits on the event rate [42], and at design sensitivity LIGO II is expected to detect inspirals at a rate of $\sim 2/\text{day}$ [43].

One of the key scientific goals of detecting neutron star (NS) binaries is to obtain information about the nuclear equation of state (EoS), which is at present fairly unconstrained in the relevant density range $\rho \sim 2 - 8 \times 10^{14}\text{g cm}^{-3}$ [44]. The conventional view has been that for most of the inspiral, finite-size effects have a negligible influence on the GW signal, and that only during the last several orbits and merger at GW frequencies $f \gtrsim 500$ Hz can the effect of the internal structure be seen.

There have been many investigations of how well the EoS can be constrained using these last several orbits and merger, including constraints from the GW energy spectrum [12], and, for black hole/NS inspirals, from the NS tidal disruption signal [13]. Several numerical simulations have studied the dependence of the GW spectrum on the radius [45]. However, there are a number of difficulties associated with trying to extract equation of state information from this late time regime: (i) The highly complex behavior requires solving the full nonlinear equations of general relativity together with relativistic hydrodynamics. (ii) The signal depends on unknown quantities such as the spins of the stars. (iii) The signals from the hydrodynamic merger (at frequencies $\gtrsim 1000$ Hz) are outside of LIGO's most sensitive band.

The purpose of this paper is to demonstrate the potential feasibility of instead obtaining EoS information from the early, low frequency part of the signal. Here, the influence of tidal effects is a small correction to the waveform's phase, but it is very clean and depends only on one parameter of the NS – its Love number [46].

2.2 Tidal interactions in compact binaries

The influence of tidal interactions on the waveform's phase has been studied previously using various approaches [47, 48, 49, 14, 15, 46]. We extend those studies by (i) computing the effect of the tidal interactions for fully relativistic neutron stars, i.e. to all orders in the strength of internal gravity in each star, (ii) computing the phase shift analytically without the assumption that the mode frequency is much larger that the orbital frequency, and (iii) performing a computation of how accurately the Love number can be measured.

The basic physical effect is the following: the $l = 2$ fundamental f-modes of the star can be treated as forced, damped harmonic oscillators driven by the tidal field of the companion at frequencies below their resonant frequencies. Assuming circular orbits they obey equations of motion of the form [50]

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = A(t)\cos[m\Phi(t)], \tag{2.1}$$

where $q(t)$ is the mode amplitude, ω_0 the mode frequency, γ a damping constant, m is the mode azimuthal quantum number, $\Phi(t)$ is the orbital phase of the binary, and $A(t)$ is a slowly varying amplitude. The orbital frequency $\omega(t) = \dot{\Phi}$ and $A(t)$ evolve on the radiation reaction timescale which is much longer than $1/\omega_0$. In this limit the oscillator evolves adiabatically, always tracking the minimum of its time-dependent potential. The energy absorbed by the oscillator up to time t is

$$E(t) = \frac{\omega_0^2 A(t)^2}{2(\omega_0^2 - m^2\omega^2)^2} + \gamma\int_{-\infty}^{t} dt' \frac{m^2\omega(t')^2 A(t')^2}{w_0^4 + m^2\omega(t')^2\gamma^2}. \tag{2.2}$$

The second term here describes a cumulative, dissipative effect which dominates over the first term for tidal interactions of main sequence stars. For NS-NS binaries, however, this term is unimportant due to the small viscosity [49], and the first, instantaneous term dominates.

The instantaneous effect is somewhat larger than often estimated for several reasons: (i) The GWs from the time varying stellar quadrupole are phase coherent with the orbital GWs, and thus there is a contribution to the energy flux that is linear in the mode amplitude. This affects the rate of inspiral and gives a correction of the same order as the energy absorbed by the mode [48]. (ii) Some papers [49, 47, 14] compute the orbital phase error as a function of orbital radius r. This is insufficient as one has to express it in the end as a function of the observable frequency, and there is a correction to the radius-frequency relation which comes in at the same order. (iii) The effect scales as the fifth power of neutron star radius R, and most previous estimates took $R = 10\,\mathrm{km}$. Larger NS models with e.g. $R = 16\,\mathrm{km}$ give an effect that is larger by a factor of ~ 10.

2.3 Tidal Love number

Consider a static, spherically symmetric star of mass m placed in a time-independent external quadrupolar tidal field $\mathcal{E}_{ij}$. The star will develop in response a quadrupole moment Q_{ij}. In the star's local asymptotic rest frame (asymptotically mass centered Cartesian coordinates) at large r the metric coefficient g_{tt} is given by (in units with $G = c = 1$) [51]:

$$\frac{(1 - g_{tt})}{2} = -\frac{m}{r} - \frac{3Q_{ij}}{2r^3}\left(n^i n^j - \frac{\delta^{ij}}{3}\right) + \frac{\mathcal{E}_{ij}}{2} x^i x^j + \dots \tag{2.3}$$

where $n^i = x^i/r$; this expansion defines the traceless tensors $\mathcal{E}_{ij}$ and Q_{ij}. To linear order, the induced quadrupole will be of the form

$$Q_{ij} = -\lambda \mathcal{E}_{ij}. \tag{2.4}$$

Here λ is a constant which we will call the tidal Love number (although that name is usually reserved for the dimensionless quantity $k_2 = \frac{3}{2} G \lambda R^{-5}$). The relation

(2.4) between Q_{ij} and $\mathcal{E}_{ij}$ defines the Love number λ for both Newtonian and relativistic stars. For a Newtonian star, $(1 - g_{tt})/2$ is the Newtonian potential, and Q_{ij} is related to the density perturbation $\delta\rho$ by $Q_{ij} = \int d^3x \delta\rho \left(x_i x_j - r^2\delta_{ij}/3\right)$.

We have calculated the Love numbers for a variety of fully relativistic NS models with a polytropic pressure-density relation $P = K\rho^{1+1/n}$. Most realistic EoS's resemble a polytrope with effective index in the range $n \simeq 0.5 - 1.0$ [52]. The equilibrium stellar model is obtained by numerical integration of the Tolman-Oppenheimer-Volkhov equations. We calculate the linear $l = 2$ static perturbations to the Schwarzschild spacetime following the method of [53]. The perturbed Einstein equations $\delta G_\mu{}^\nu = 8\pi\delta T_\mu{}^\nu$ can be combined into a second order differential equation for the perturbation to g_{tt}. Matching the exterior solution and its derivative to the asymptotic expansion (2.3) gives the Love number. For $m/R \sim 10^{-5}$ our results agree well with the Newtonian results of Refs. [47, 54]. Figure 1 shows the range of Love numbers for $m/R = 0.2256$, corresponding to the surface redshift $z = 0.35$ that has been measured for EXO0748-676 [55]. Details of this computation will be published elsewhere.

2.4 Effect on gravitational wave signal

Consider a binary with masses m_1, m_2 and Love numbers λ_1, λ_2. For simplicity, we compute only the excitation of star 1; the signals from the two stars simply add in the phase. Let ω_n, $\lambda_{1,n}$ and Q_{ij}^n be the frequency, the contribution to λ_1 and the contribution to Q_{ij} of modes of the star with $l = 2$ and with n radial nodes, so that $\lambda_1 = \Sigma_n \lambda_{1,n}$ and $Q_{ij} = \Sigma_n Q_{ij}^n$. Writing the relative displacement as

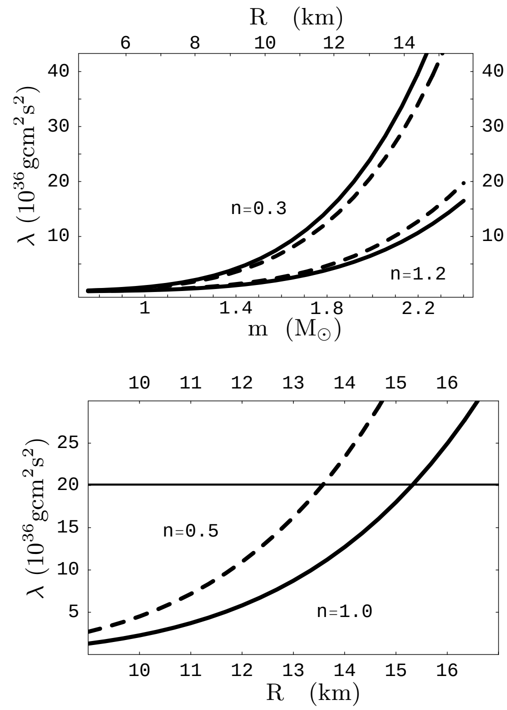

Figure 2.1: [Top] The solid lines bracket the range of Love numbers λ for fully relativistic polytropic neutron star models of mass m with surface redshift $z = 0.35$, assuming a range of $0.3 \leq n \leq 1.2$ for the adiabatic index n. The top scale gives the radius R for these relativistic models. The dashed lines are corresponding Newtonian values for stars of radius R. [Bottom] Upper bound (horizontal line) on the weighted average $\tilde{\lambda}$ of the two Love numbers obtainable with LIGO II for a binary inspiral signal at distance of 50 Mpc, for two non-spinning, $1.4 M_{\odot}$ neutron stars, using only data in the frequency band $f < 400$ Hz. The curved lines are the actual values of λ for relativistic polytropes with $n = 0.5$ (dashed line) and $n = 1.0$ (solid line).

$\mathbf{x} = (r\cos\Phi, r\sin\Phi, 0)$, the action for the system is

$$\begin{aligned} S &= \int dt \left[\frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\Phi}^2 + \frac{M\mu}{r}\right] - \frac{1}{2}\int dt Q_{ij}\mathcal{E}_{ij} \\ &\quad + \sum_n \int dt \frac{1}{4\lambda_{1,n}\omega_n^2}\left[\dot{Q}_{ij}^n\dot{Q}_{ij}^n - \omega_n^2 Q_{ij}^n Q_{ij}^n\right]. \end{aligned} \tag{2.5}$$

Here M and μ are the total and reduced masses, and $\mathcal{E}_{ij} = -m_2\partial_i\partial_j(1/r)$ is the tidal field. This action is valid to leading order in the orbital potential but to all orders in the internal potentials of the NSs, except that it neglects GW dissipation, because Q_{ij} and $\mathcal{E}_{ij}$ are defined in the star's local asymptotic rest frame [56].

Using the action (2.5), adding the leading order, Burke-Thorne GW dissipation terms, and defining the total quadrupole $Q_{ij}^{\mathrm{T}} = Q_{ij} + \mu x_i x_j - \mu r^2\delta_{ij}/3$ with $Q_{ij} = \Sigma_n Q_{ij}^n$, gives the equations of motion

$$\ddot{x}^i + \frac{M}{r^2}n^i = \frac{m_2}{2\mu}Q_{jk}\partial_i\partial_j\partial_k\frac{1}{r} - \frac{2}{5}x_j\frac{d^5Q_{ij}^{\mathrm{T}}}{dt^5}, \tag{2.6a}$$

$$\ddot{Q}_{ij}^n + \omega_n^2 Q_{ij}^n = m_2\lambda_{1,n}\omega_n^2\partial_i\partial_j\frac{1}{r} - \frac{2}{5}\lambda_{1,n}\omega_n^2\frac{d^5Q_{ij}^{\mathrm{T}}}{dt^5}. \tag{2.6b}$$

By repeatedly differentiating Q_{ij}^{T} and eliminating second order time derivative terms using the conservative parts of Eqs. (2.6), we can express $d^5Q_{ij}^{\mathrm{T}}/dt^5$ in terms of x^i, $\dot{x}^i$, Q_{ij}^n and $\dot{Q}_{ij}^n$ and obtain a second order set of equations; this casts Eqs. (2.6) into a numerically integrable form.

When GW damping is neglected, there exist equilibrium solutions with $r =$ const, $\Phi = \Phi_0 + \omega t$ for which Q_{ij}^{T} is static in the rotating frame. Working to leading order in $\lambda_{1,n}$, we have $Q_{11}^{\mathrm{T}} = \mathcal{Q}' + \mathcal{Q}\cos(2\Phi)$, $Q_{22}^{\mathrm{T}} = \mathcal{Q}' - \mathcal{Q}\cos(2\Phi)$, $Q_{12}^{\mathrm{T}} = \mathcal{Q}\sin(2\Phi)$, $Q_{33}^{\mathrm{T}} = -2\mathcal{Q}'$, where

$$\mathcal{Q} = \frac{1}{2}\mu r^2 + \sum_n \frac{3m_2\lambda_{1,n}}{2(1-4x_n^2)r^3}, \quad \mathcal{Q}' = \frac{1}{6}\mu r^2 + \sum_n \frac{m_2\lambda_{1,n}}{2r^3} \tag{2.7}$$

and $x_n = \omega/\omega_n$. Substituting these solutions back into the action (2.5), and into the quadrupole formula $\dot{E} = -\frac{1}{5}\langle \dddot{Q}_{ij}^{\mathrm{T}}\dddot{Q}_{ij}^{\mathrm{T}}\rangle$ for the GW damping, provides an effective description of the orbital dynamics for quasicircular inspirals in the adiabatic limit. We obtain for the orbital radius, energy and energy time derivative

$$
\begin{aligned}
r(\omega) &= M^{1/3}\omega^{-2/3}\left[1+\frac{3}{4}\sum_n \chi_n g_1(x_n)\right], &\text{(2.8a)}\\
E(\omega) &= -\frac{\mu}{2}(M\omega)^{2/3}\left[1-\frac{9}{4}\sum_n \chi_n g_2(x_n)\right], &\text{(2.8b)}\\
\dot{E}(\omega) &= -\frac{32}{5}M^{4/3}\mu^2\omega^{10/3}\left[1+6\sum_n \chi_n g_3(x_n)\right], &\text{(2.8c)}
\end{aligned}
$$

where $\chi_n = m_2\lambda_{1,n}\omega^{10/3}m_1^{-1}M^{-5/3}$, $g_1(x) = 1+3/(1-4x^2)$, $g_2(x) = 1+(3-4x^2)(1-4x^2)^{-2}$, and $g_3(x) = (M/m_2+2-2x^2)/(1-4x^2)$. Using the formula $d^2\Psi/d\omega^2 = 2\,(dE/d\omega)\,/\dot{E}$ for the phase $\Psi(f)$ of the Fourier transform of the GW signal at GW frequency $f=\omega/\pi$ [57] now gives for the tidal phase correction

$$
\begin{aligned}
\delta\Psi(f) &= -\frac{15m_2^2}{16\mu^2M^5}\sum_n \lambda_{1,n}\int_{v_i}^{v} dv'v'\left(v^3-v'^3\right)g_4(x_n'),\\
g_4(x) &= \frac{2M}{m_2(1-4x^2)}+\frac{22-117x^2+348x^4-352x^6}{(1-4x^2)^3}.
\end{aligned}
\tag{2.9}
$$

Here $v=(\pi Mf)^{1/3}$, v_i is an arbitrary constant related to the initial time and phase of the waveform, and $x_n' = (v')^3/(M\omega_n)$. In the limit $\omega \ll \omega_n$ assumed in most previous analyses [47, 49, 14, 46], we get

$$
\delta\Psi = -\frac{9}{16}\frac{v^5}{\mu M^4}\left[\left(11\frac{m_2}{m_1}+\frac{M}{m_1}\right)\lambda_1 + 1\leftrightarrow 2\right], \tag{2.10}
$$

which depends on internal structure only through λ_1 and λ_2. Here we have added the contribution from star 2. The phase (2.10) is formally of post-5-Newtonian (P5N) order, but it is larger than the point-particle P5N terms (which are currently unknown) by $\sim (R/M)^5 \sim 10^5$.

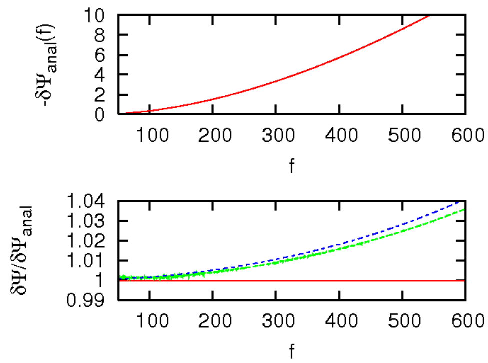

Figure 2.2: [Top] Analytic approximation (2.10) to the tidal perturbation to the gravitational wave phase for two identical $1.4M_{\odot}$ neutron stars of radius $R = 15\,\mathrm{km}$, modeled as $n = 1.0$ polytropes, as a function of gravitational wave frequency f. [Bottom] A comparison of different approximations to the tidal phase perturbation: the numerical solution (lower dashed, green curve) to the system (2.6), and the adiabatic analytic approximation (2.9) (upper dashed, blue), both in the limit (2.11) and divided by the leading order approximation (2.10).

2.5 Accuracy of Model

We will analyze the information contained in the portion of the signal before $f = 400\,\mathrm{Hz}$. This frequency was chosen to be at least 20% smaller than the frequency of the innermost stable circular orbit [58] for a conservatively large polytropic NS model with $n = 1.0$, $M = 1.4M_{\odot}$, and $R = 19\,\mathrm{km}$. We now argue that in this frequency band, the simple model (2.10) of the phase correction is sufficiently accurate for our purposes.

We consider six types of corrections to (2.10). For each correction, we estimate its numerical value at the frequency $f = 400$ Hz for a binary of two identical $m = 1.4M_{\odot}$, $R = 15$, $n = 1.0$ stars: (i) Corrections due to modes with $l \geq 3$ which are excited by higher order tidal tensors $\mathcal{E}_{ijk}, \ldots$. The $l = 3$ correction to $E(\omega)$, computed using the above methods in the low frequency limit, is smaller than the $l = 2$ contribution by a factor of $65k_3R^2/(45k_2r^2)$, where k_2, k_3 are apsidal constants. For Newtonian polytropes we have $k_2 = 0.26$, $k_3 = 0.106$ [46], and the ratio is $0.58(R/r)^2 = 0.04(R/15\,\mathrm{km})^2$. (ii) To assess the accuracy of the $\omega \ll \omega_n$ limit underlying (2.10) we simplify the model (2.5) by taking

$$\omega_n = \omega_0 \quad \text{for all } n, \tag{2.11}$$

so that $Q^n_{ij}/\lambda_{1,n}$ is independent of n. This simplification does not affect (2.10) and *increases* the size of the finite frequency corrections in (2.9) since $\omega_n \geq \omega_0$ [1]. This will yield an upper bound on the size of the corrections. (Also the $n \geq 1$ modes contribute typically less than $1-2\%$ of the Love number [47].) Figure 2 shows the phase correction $\delta\Psi$ computed numerically from Eqs. (2.6), and the approximations (2.9) and (2.10) in the limit (2.11). We see that the adiabatic approximation (2.9) is extremely accurate, to better than 1%, and so the dominant error is the difference between (2.9) and (2.10). The fractional correction to (2.10) is $\sim 0.7x^2 \sim 0.2(f/f_0)^2$, where $f_0 = \omega_0/(2\pi)$, neglecting unobservable terms of the form $\alpha + \beta f$. This ratio is $\lesssim 0.03$ for $f \leq 400$ Hz and for $f_0 \geq 1000$ Hz as is the case for f-mode frequencies for most NS models [15]. (iii) We have linearized in λ_1; the corresponding fractional corrections scale as $(R/r)^5 \sim 10^{-3}(R/15\,\mathrm{km})^5$ at 400 Hz. (iv) The leading nonlinear hydrodynamic corrections can be computed by adding a term $-\alpha Q^0_{ij}Q^0_{jk}Q^0_{ki}$ to the Lagrangian (2.5), where α is a constant. This corrects the phase shift (2.10) by a factor $1 - 285\alpha\lambda^2_{1,0}\omega^2/968 \sim 0.9995$, where we have used the models of Ref. [59] to estimate α. (v) Fractional corrections to the tidal signal due to spin scale as $\sim f^2_{\mathrm{spin}}/f^2_{\mathrm{max}}$, where f_{spin} is the spin frequency and f_{max} the maximum allowed spin frequency. These can be neglected as $f_{\mathrm{max}} \gtrsim 1000$ Hz for most models and f_{spin} is expected to be much smaller than this. (vi) Post-1-Newtonian corrections to the tidal signal (2.10) will be of order $\sim M/r \sim 0.05$.

[1]Buoyancy forces and associated g-modes for which $\omega_n \leq \omega_0$ have a negligible influence on the waveform's phase[50].

However these corrections will depend only on λ_1 when $\omega \ll \omega_n$, and can easily be computed and included in the data analysis method we suggest here.

Thus, systematic errors in the measured value of λ due to errors in the model should be $\lesssim 10\%$, which is small compared to the current uncertainty in λ (see Fig. 1).

2.6 Measuring the Love Number

The binary's parameters are extracted from the noisy GW signal by integrating the waveform $h(t)$ against theoretical inspiral templates $h(t, \theta^i)$, where θ^i are the parameters of the binary. The best-fit parameters $\hat{\theta}^i$ are those that maximize the overlap integral. The probability distribution for the signal parameters for strong signals and Gaussian detector noise is $p(\theta^i) = \mathcal{N} \exp\left(-1/2\, \Gamma_{ij} \Delta\theta^i \Delta\theta^j\right)$ [60], where $\Delta\theta^i = \theta^i - \hat{\theta}^i$, $\Gamma_{ij} = (\partial h/\partial\theta^i\,,\, \partial h/\partial\theta^j)$ is the Fisher information matrix, and the inner product is defined by Eq. (2.4) of Ref. [60]. The rms statistical measurement error in θ^i is then $\sqrt{(\Gamma^{-1})^{ii}}$.

Using the stationary phase approximation and neglecting corrections to the amplitude, the Fourier transform of the waveform for spinning point masses is given by $\tilde{h}(f) = \mathcal{A} f^{-7/6} \exp(i\Psi)$. Here the phase Ψ is

$$\begin{aligned}
\Psi(f) \;=\; & 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3M}{128\mu}(\pi M f)^{-5/3}\left[1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\frac{\mu}{M}\right) v^2\right. \\
& -4(4\pi - \beta)v^3 + 10\left(\frac{3058673}{1016064} + \frac{5429}{1008}\frac{\mu}{M} + \frac{617}{144}\frac{\mu^2}{M^2} - \sigma\right) v^4 \\
& + \left(\frac{38645\pi}{252} - \frac{65}{3}\frac{\mu}{M}\right)\ln v + \left(\frac{11583231236531}{4694215680} - \frac{640\pi^2}{3} - \frac{6848\gamma}{21}\right) v^6 \\
& + \frac{\mu}{M}\left(\frac{15335597827}{3048192} + \frac{2255\pi^2}{12} + \frac{47324}{63} - \frac{7948}{9}\right) v^6 \\
& + \left(\frac{76055}{1728}\frac{\mu^2}{M^2} - \frac{127825}{1296}\frac{\mu^3}{M^3} - \frac{6848}{21}\ln(4v)\right) v^6 \\
& \left. + \pi\left(\frac{77096675}{254016} + \frac{378515}{1512}\frac{\mu}{M} - \frac{74045}{756}\frac{\mu^2}{M^2}\right) v^7 \right], \qquad (2.12)
\end{aligned}$$

where $v = (\pi M f)^{1/3}$, β and σ are spin parameters, and γ is Euler's constant [61]. The tidal term (2.10) adds linearly to this, yielding a phase model with 7 parameters $(t_c, \phi_c, M, \mu, \beta, \sigma, \tilde{\lambda})$, where $\tilde{\lambda} = [(11m_2 + M)\lambda_1/m_1 + (11m_1 + M)\lambda_2/m_2]/26$ is a weighted average of λ_1 and λ_2. We incorporate the maximum spin constraint for the NSs by assuming a Gaussian prior for β and σ as in Ref. [60].

Figure 1 [bottom panel] shows the 90% confidence upper limit $\tilde{\lambda} \leqslant 20.1 \times 10^{36}\,\mathrm{g\,cm^2 s^2}$ we obtain for LIGO II (horizontal line) for two nonspinning $1.4M_\odot$ NSs at a distance of 50 Mpc (signal-to-noise of 95 in the frequency range 20 − 400Hz) with cutoff frequency $f_c = 400\,\mathrm{Hz}$, as well as the corresponding values of λ for relativistic polytropes with $n = 0.5$ (dashed curve) and $n = 1.0$ (solid line). The corresponding constraint on radius assuming identical $1.4M_\odot$ stars would be $R \leqslant 13.6\,\mathrm{km}$ $(15.3\,\mathrm{km})$ for $n = 0.5$ $(n = 1.0)$ polytropes. Current NS models span the range $10\,\mathrm{km} \lesssim R \lesssim 15\,\mathrm{km}$.

Our phasing model (2.12) is the most accurate available model, containing terms up to post-3.5-Newtonian (P3.5N) order. We have experimented with using lower order phase models (P2N, P2.5N, P3N), and we find that the resulting upper bound on $\tilde{\lambda}$ varies by factors of order ~ 2. Thus there is some associated systematic uncertainty in our result. To be conservative, we have adopted the most pessimistic (largest) upper bound on $\tilde{\lambda}$, which is that obtained from the P3.5N waveform.

In conclusion, even if the internal structure signal is too small to be seen, the analysis method suggested here could start to give interesting constraints on NS internal structure for nearby events.

This research was supported in part by NSF grants PHY-0140209 and PHY-0457200. We thank an anonymous referee for helpful comments and suggestions.

CHAPTER 3

TIDAL LOVE NUMBERS OF NEUTRON STARS

SUMMARY: For a variety of fully relativistic polytropic neutron star models we calculate the star's tidal Love number k_2. Most realistic equations of state for neutron stars can be approximated as a polytrope with an effective index $n \approx 0.5 - 1.0$. The equilibrium stellar model is obtained by numerical integration of the Tolman-Oppenheimer-Volkhov equations. We calculate the linear $l = 2$ static perturbations to the Schwarzschild spacetime following the method of Thorne and Campolattaro. Combining the perturbed Einstein equations into a single second-order differential equation for the perturbation to the metric coefficient and matching the exterior solution to the asymptotic expansion of the metric in the star's local asymptotic rest frame gives the Love number. Our results agree well with the Newtonian results in the weak field limit. The fully relativistic values differ from the Newtonian values by up to $\sim 24\%$. The Love number is potentially measurable in gravitational wave signals from inspiralling binary neutron stars.

Originally appeared in The Astrophysical Journal, **677***, 1216 (2008)*

3.1 Introduction and Motivation

A key challenge of current astrophysical research is to obtain information about the equation of state (EoS) of the ultra-dense nuclear matter making up neutron stars (NSs). The observational constraints on the internal structure of NSs are weak: the observed range of NS masses is $M \sim 1.1 - 2.2 M_{\odot}$ [10], and there is no current method to directly measure the radius. Some estimates using data from X-ray spectroscopy exist, but those are highly model-dependent (e. g. [11]). Different theoretical models for the NS internal structure predict, for a neutron star of mass $M \sim 1.4 M_{\odot}$, a central density in the range of $\rho_c \sim 2 - 8 \times 10^{14}\mathrm{gcm}^{-3}$ and a radius in the range of $R \sim 7 - 16$km [10]. Potential observations of pulsars rotating at frequencies above 1400Hz could be used to constrain the EoS if the pulsar's mass could also be measured (e. g. [62]).

Direct and model-independent constraints on the EoS of NSs could be obtained from gravitational wave observations. Coalescing binary neutron stars are one of the most important sources for ground-based gravitational wave detectors [63]. LIGO observations have established upper limits on the coalescence rate per comoving volume [64], and at design sensitivity LIGO II is expected to detect inspirals at a rate of ~ 2/day [43].

In the early, low frequency part of the inspiral ($f \leq 100$Hz, where f is the gravitational wave frequency), the waveform's phase evolution is dominated by the point-mass dynamics and finite-size effects are only a small correction. Toward the end of the inspiral the internal degrees of freedom of the bodies start to appreciably influence the signal, and there have been many investigations of how well the EoS can be constrained using the last several orbits and merger, including constraints from the gravitational wave energy spectrum [12] and from the NS tidal disruption signal for NS-black hole binaries [13]. Several numerical simulations of the hydrodynamics of NS-NS mergers have studied the dependence of the gravitational wave spectrum on the radius and EoS (see, e.g. [45] and references therein). However, trying to extract EoS information from this late time regime presents several difficulties: (i) the highly complex behavior requires solving the full nonlinear equations of general relativity together with relativistic hydrodynamics; (ii) the signal depends on unknown quantities such as the spins and angular momentum distribution inside the stars, and (iii) the signals from the hydrodynamic merger are outside of LIGO's most sensitive band.

During the early regime of the inspiral the signal is very clean and the influence of tidal effects is only a small correction to the waveform's phase. However, signal detection is based on matched filtering, i. e. integrating the measured waveform against theoretical templates, where the requirement on the templates is that the phasing remain accurate to ~ 1 cycle over the inspiral. If the accumulated phase shift due to the tidal corrections becomes of order unity or larger, it could corrupt the detection of NS-NS signals or alternatively, detecting a phase perturbation could give information about the NS structure. This has motivated several analytical and numerical investigations of tidal effects in NS binaries [65, 47, 49, 14, 46, 66, 67, 15, 68]. The influence of the internal structure on the gravitational wave phase in this early regime of the inspiral is characterized by a single parameter, namely the ratio λ of the induced quadrupole to the perturbing tidal field. This ratio λ is related to the star's tidal Love number k_2 by $k_2 = 3G\lambda R^{-5}/2$, where R is the star's radius. The authors of Ref. [69] have shown that for an inspiral of two non-spinning $1.4M_\odot$ NSs at a distance of 50 Mpc, LIGO II detectors will be able to constrain λ to $\lambda \leq 2.01 \times 10^{37}\,\mathrm{g\,cm^2 s^2}$ with 90% confidence. This number is an upper limit on λ in the case that no tidal phase shift is observed. The corresponding constraint on radius would be $R \leq 13.6\,\mathrm{km}$ $(15.3\,\mathrm{km})$ for a $n = 0.5$ $(n = 1.0)$ fully relativistic polytrope, for $1.4M_\odot$ NSs [69].

Because neutron stars are compact objects with strong internal gravity, their Love numbers could be very different from those for Newtonian stars that have been computed previously, e. g. in Ref. [54].

Knowledge of Love number values could also be useful for comparing different numerical simulations of NS binary inspiral by focusing on models with the same masses and values of λ.

In Ref. [69], the $l = 2$ tidal Love numbers for fully relativistic neutron star models with polytropic pressure-density relation $P = K\rho^{1+1/n}$, where K and n are constants, were computed for the first time. The present paper will give details of this computation. Using polytropes allows us to explore a wide range of stellar models, since most realistic models can be reasonably approximated as a polytrope with an effective index in the range $n \sim 0.5 - 1.0$ [10]. Our prescription for computing λ is valid for an arbitrary pressure-density relation and not restricted to polytropes. In Sec. 3.2, we start by defining λ in the fully relativistic context in terms of coefficients in an asymptotic expansion of the metric in the star's local asymptotic rest frame and discuss the extent to which it is uniquely defined. In Sec. 3.3, we discuss our method of calculating λ, which is based on static linearized perturbations of the equilibrium configuration in the Regge-Wheeler gauge as in Ref. [53]. Section 4.4.4 contains the results of the numerical computations together with a discussion. Unless otherwise specified, we use units in which $c = G = 1$.

3.2 Definition of the Love number

Consider a static, spherically symmetric star of mass M placed in a static external quadrupolar tidal field $\mathcal{E}_{ij}$. The star will develop in response a quadrupole moment Q_{ij} [1]. In the star's local asymptotic rest frame (asymptotically mass centered

[1]The induced quadrupolar deformation of the star can be described in terms of the star's $l = 2$ mode eigenfunctions of oscillation.

Cartesian coordinates) at large r the metric coefficient g_{tt} is given by [51]:

$$\begin{aligned}\frac{(1-g_{tt})}{2} &= -\frac{M}{r}-\frac{3Q_{ij}}{2r^3}\left(n^in^j-\frac{1}{3}\delta^{ij}\right)+O\left(\frac{1}{r^3}\right)\\ &\quad+\frac{1}{2}\mathcal{E}_{ij}x^ix^j+O\left(r^3\right),\end{aligned}\tag{3.1}$$

where $n^i = x^i/r$; this expansion defines $\mathcal{E}_{ij}$ [2] and Q_{ij}. In the Newtonian limit, Q_{ij} is related to the density perturbation $\delta\rho$ by

$$Q_{ij} = \int d^3x\delta\rho(\mathbf{x})\left(x_ix_j-\frac{1}{3}r^2\delta_{ij}\right),\tag{3.2}$$

and $\mathcal{E}_{ij}$ is given in terms of the external gravitational potential $\Phi_{\rm ext}$ as

$$\mathcal{E}_{ij} = \frac{\partial^2\Phi_{\rm ext}}{\partial x^i\partial x^j}.\tag{3.3}$$

We are interested in applications to fully relativistic stars, which requires going beyond Newtonian physics. In the strong field case, Eqs. (3.2) and (3.3) are no longer valid but the expansion of the metric (3.1) still holds in the asymptotically flat region and serves to define the moments Q_{ij} and $\mathcal{E}_{ij}$.

We briefly review here the extent to which these moments are uniquely defined since there are considerable coordinate ambiguities in performing asymptotic expansions of the metric. For an isolated body in a static situation these moments are uniquely defined: $\mathcal{E}_{ij}$ and Q_{ij} are the coordinate independent moments defined by Geroch [71] and Hansen [72] for stationary, asymptotically flat spacetimes in terms of certain combinations of the derivatives of the norm and twist of the timelike Killing vector at spatial infinity. In the case of an isolated object in a dynamical situation, there are ambiguities related to gravitational radiation, for example angular momentum is not uniquely defined [73]. For the application to the adiabatic part of a NS binary inspiral, we are interested in the case of a non-isolated object in a quasi-static situation. In this case there are still ambiguities

[2] The $l=2$ tidal moment can be related to a component of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ of the external pieces of the metric in Fermi normal coordinates at $r=0$ as $\mathcal{E}_{ij}=R_{0i0j}$ (see [70]).

(related to the choice of coordinates) but their magnitudes can be estimated [56] and are at a high post-Newtonian order and therefore can be neglected. We are also interested in (i) working to linear order in $\mathcal{E}_{ij}$ and (ii) in the limit where the source of $\mathcal{E}_{ij}$ is very far away. In this limit the ambiguities disappear.

To linear order in $\mathcal{E}_{ij}$, the induced quadrupole will be of the form

$$Q_{ij} = -\lambda \mathcal{E}_{ij}. \tag{3.4}$$

Here λ is a constant which is related to the $l = 2$ tidal Love number (apsidal constant) k_2 by [69]

$$k_2 = \frac{3}{2} G \lambda R^{-5}. \tag{3.5}$$

Note the difference in terminology: in Ref. [69], λ was called the Love number, whereas in this paper, we reserve that name for the dimensionless quantity k_2.

The tensor multipole moments Q_{ij} and $\mathcal{E}_{ij}$ can be decomposed as

$$\mathcal{E}_{ij} = \sum_{m=-2}^{2} \mathcal{E}_m \mathcal{Y}_{ij}^{2m}, \tag{3.6}$$

and

$$Q_{ij} = \sum_{m=-2}^{2} Q_m \mathcal{Y}_{ij}^{2m}, \tag{3.7}$$

where the symmetric traceless tensors $\mathcal{Y}_{ij}^{2m}$ are defined by [74]

$$Y_{2m}(\theta, \varphi) = \mathcal{Y}_{ij}^{2m} n^i n^j \tag{3.8}$$

with $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Thus, the relation (3.4) can be written as

$$Q_m = -\lambda \mathcal{E}_m. \tag{3.9}$$

Without loss of generality, we can assume that only one $\mathcal{E}_m$ is nonvanishing, this is sufficient to compute λ.

3.3 Calculation of the Love number

3.3.1 Equilibrium configuration

The geometry of spacetime of a spherical, static star can be described by the line element [70]

$$ds_0^2 = g^{(0)}_{\alpha\beta} dx^\alpha dx^\beta = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.10}$$

The star‘s stress-energy tensor is given by

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta + p g^{(0)}_{\alpha\beta}, \tag{3.11}$$

where $\vec{u} = e^{-\nu/2}\partial_t$ is the fluid’s four-velocity and ρ and p are the density and pressure. Numerical integration of the Tolman-Oppenheimer-Volkhov equations (see e.g. [70]) for neutron star models with a polytropic pressure-density relation

$$P = K\rho^{1+1/n}, \tag{3.12}$$

where K is a constant and n is the polytropic index, gives the equilibrium stellar model with radius R and total mass $M = m(R)$.

3.3.2 Static linearized perturbations due to an external tidal field

We examine the behavior of the equilibrium configuration under linearized perturbations due to an external quadrupolar tidal field following the method of Thorne and Campolattaro [53]. The full metric of the spacetime is given by

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h_{\alpha\beta}, \tag{3.13}$$

where $h_{\alpha\beta}$ is a linearized metric perturbation. We analyze the angular dependence of the components of $h_{\alpha\beta}$ into spherical harmonics as in Ref. [75]. We restrict our analysis to the $l = 2$, static, even-parity perturbations in the Regge-Wheeler gauge [75]. With these specializations, $h_{\alpha\beta}$ can be written as [75, 53]:

$$h_{\alpha\beta} = \text{diag}\left[e^{-\nu(r)}H_0(r),\ e^{\lambda(r)}H_2(r),\ r^2K(r),\ r^2\sin^2\theta K(r)\right]Y_{2m}(\theta,\varphi). \tag{3.14}$$

The nonvanishing components of the perturbations of the stress-energy tensor (3.11) are $\delta T_0^0 = -\delta\rho = -(dp/d\rho)^{-1}\delta p$ and $\delta T_i^i = \delta p$. We insert this and the metric metric perturbation (3.14) into the the linearized Einstein equation $\delta G_\alpha^\beta = 8\pi\delta T_\alpha^\beta$ and combine various components. From $\delta G_\theta^\theta - \delta G_\phi^\phi = 0$ it follows that that $H_2 = H_0 \equiv H$, then $\delta G_\theta^r = 0$ relates K' to H, and after using $\delta G_\theta^\theta + \delta G_\phi^\phi = 16\pi\delta p$ to eliminate δp, we finally subtract the $r - r$ component of the Einstein equation from the $t - t$ component to obtain the following differential equation for $H_0 \equiv H$ (for $l = 2$):

$$\begin{aligned} &H'' + H'\left[\frac{2}{r} + e^{\lambda}\left(\frac{2m(r)}{r^2} + 4\pi r\left(p - \rho\right)\right)\right] \\ &+H\left[-\frac{6e^{\lambda}}{r^2} + 4\pi e^{\lambda}\left(5\rho + 9p + \frac{\rho + p}{(dp/d\rho)}\right) - \nu'^2\right] = 0, \end{aligned} \tag{3.15}$$

where the prime denotes d/dr. The boundary conditions for Eq. (3.15) can be obtained as follows. Requiring regularity of H at $r = 0$ and solving for H near $r = 0$ yields

$$H(r) = a_0 r^2\left[1 - \frac{2\pi}{7}\left(5\rho(0) + 9p(0) + \frac{\rho(0) + p(0)}{(dp/d\rho)(0)}\right)r^2 + O(r^3)\right], \tag{3.16}$$

where a_0 is a constant. To single out a unique solution from this one-parameter family of solutions parameterized by a_0, we use the continuity of $H(r)$ and its derivative across $r = R$. Outside the star, Eq. (3.15) reduces to

$$H'' + \left(\frac{2}{r} - \lambda'\right)H' - \left(\frac{6e^{\lambda}}{r^2} + \lambda'^2\right)H = 0, \tag{3.17}$$

and changing variables to $x = (r/M - 1)$ as in Ref. [53] transforms Eq. (3.17) to a form of the associated Legendre equation with $l = m = 2$:

$$\left(x^2 - 1\right) H'' + 2xH' - \left(6 + \frac{4}{x^2 - 1}\right) H = 0. \tag{3.18}$$

The general solution to Eq. (3.18) in terms of the associated Legendre functions ${Q_2}^2(x)$ and ${P_2}^2(x)$ is given by

$$H = c_1 {Q_2}^2 \left(\frac{r}{M} - 1\right) + c_2 {P_2}^2 \left(\frac{r}{M} - 1\right), \tag{3.19}$$

where c_1 and c_2 are coefficients to be determined. Substituting the expressions for ${Q_2}^2(x)$ and ${P_2}^2(x)$ from Ref. [76] yields for the exterior solution

$$\begin{aligned} H \;=\; & c_1 \left(\frac{r}{M}\right)^2 \left(1 - \frac{2M}{r}\right) \left[-\frac{M(M-r)(2M^2 + 6Mr - 3r^2)}{r^2(2M-r)^2} + \frac{3}{2} \log\left(\frac{r}{r-2M}\right)\right] \\ & + c_2 \left(\frac{r}{M}\right)^2 \left(1 - \frac{2M}{r}\right). \end{aligned} \tag{3.20}$$

The asymptotic behavior of the solution (3.20) at large r is

$$H = \frac{8}{5}\left(\frac{M}{r}\right)^3 c_1 + O\left(\left(\frac{M}{r}\right)^4\right) + 3\left(\frac{r}{M}\right)^2 c_2 + O\left(\left(\frac{r}{M}\right)\right), \tag{3.21}$$

where the coefficients c_1 and c_2 are determined by matching the asymptotic solution (3.21) to the expansion (3.1) and using Eq. (3.9):

$$c_1 = \frac{15}{8} \frac{1}{M^3} \lambda \mathcal{E}, \qquad c_2 = \frac{1}{3} M^2 \mathcal{E}. \tag{3.22}$$

We now solve for λ in terms of H and its derivative at the star's surface $r = R$ using Eqs. (3.22) and (3.20), and use the relation (3.5) to obtain the expression:

$$\begin{aligned} k_2 = & \frac{8C^5}{5}\left(1 - 2C^2\right)\left[2 + 2C\left(y - 1\right) - y\right] \times \\ & \Big\{2C\left(6 - 3y + 3C(5y - 8)\right) + 4C^3\left[13 - 11y + C(3y - 2) + 2C^2(1 + y)\right] \\ & + 3(1 - 2C^2)\left[2 - y + 2C(y - 1)\right] \log\left(1 - 2C\right)\Big\}^{-1}, \end{aligned} \tag{3.23}$$

where we have defined the star's compactness parameter $C \equiv M/R$ and the quantity $y \equiv RH'(R)/H(R)$, which is obtained by integrating Eq. (3.15) outwards in the region $0 < r < R$.

3.3.3 Newtonian limit

The first term in the expansion of the expression (3.23) in M/R reproduces the Newtonian result:

$$k_2^N = \frac{1}{2}\left(\frac{2-y}{y+3}\right), \tag{3.24}$$

where the superscript N denotes "Newtonian". In the Newtonian limit, the differential equation (3.15) inside the star becomes

$$H'' + \frac{2}{r}H' + \left(\frac{4\pi\rho}{dp/d\rho} - \frac{6}{r^2}\right) H = 0. \tag{3.25}$$

For a polytropic index of $n = 1$, Eq. (3.25) can be transformed to a Bessel equation with the solution that is regular at $r = 0$ given by $H = A\sqrt{r/R}\, J_{5/2}(\pi r/R)$, where A is a constant. At $r = R$, we thus have $y = RH'/H = (\pi^2 - 9)/3$, and from Eq. (3.23) it follows that

$$k_2^N(n=1) = \left(-\frac{1}{2} + \frac{15}{2\pi^2}\right) \approx 0.25991, \tag{3.26}$$

which agrees with the known result [54].

3.4 Results and Discussion

The range of dimensionless Love numbers k_2 obtained by numerical integration of Eq. (3.23) is shown in Fig. 3.1 as a function of M/R and n for a variety of different neutron star models, and representative values are given in Table 3.1. These values can be approximated to an accuracy of $\sim 6\%$ in the range $0.5 \leq n \leq 1.0$ and $0.1 \leq (M/R) \leq 0.24$ by the fitting formula

$$k_2 \approx \frac{3}{2}\left(-0.41 + \frac{0.56}{n^{0.33}}\right)\left(\frac{M}{R}\right)^{-0.003}. \tag{3.27}$$

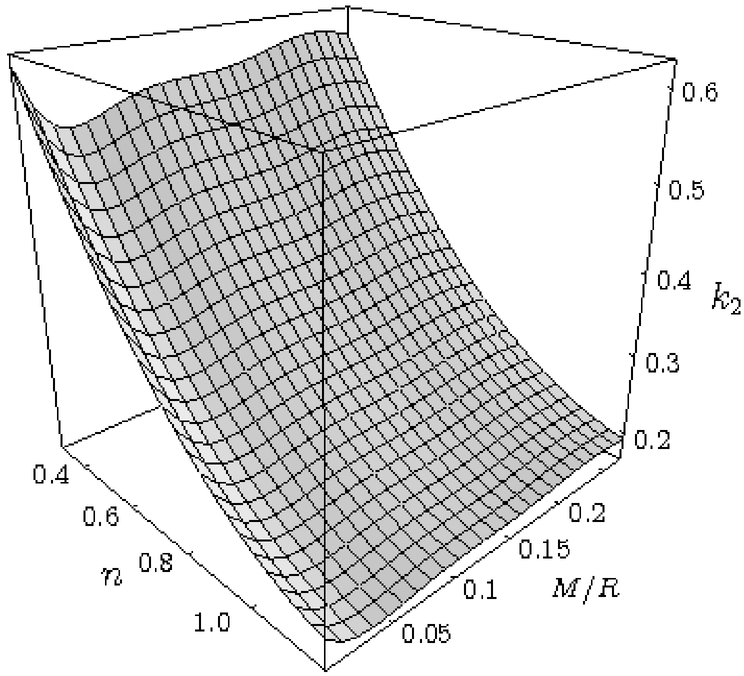

Figure 3.1: The relativistic Love numbers k_2.

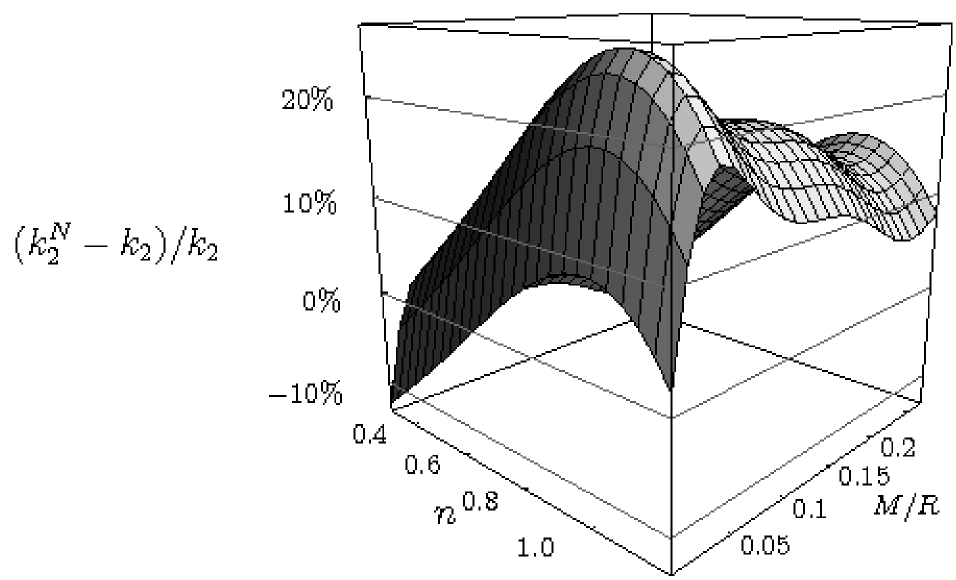

Figure 3.2: The difference in percent between the relativistic dimensionless Love numbers k_2 and the Newtonian values k_2^N.

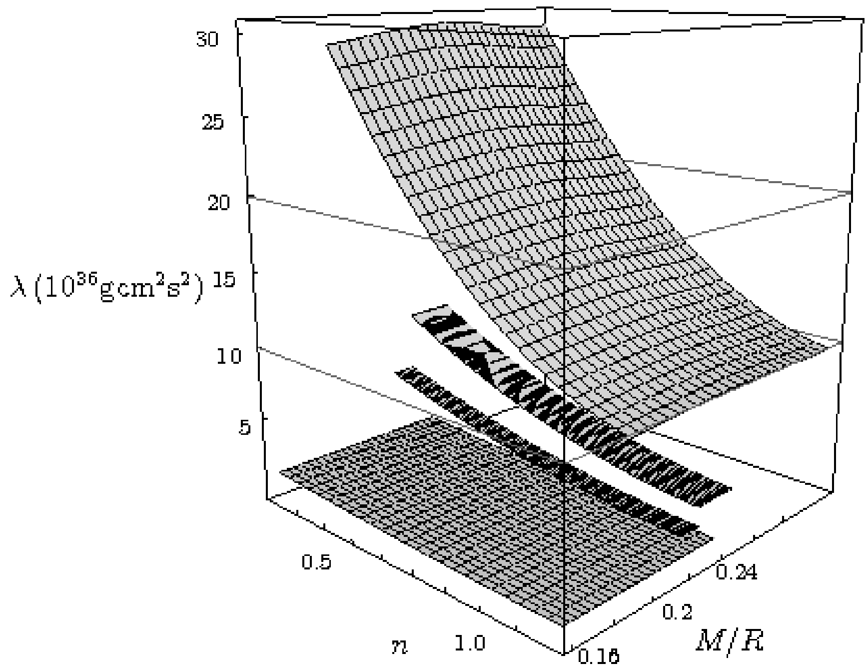

Figure 3.3: The range of Love numbers for the estimated NS parameters from X-ray observations. Top to bottom sheets: EXO0748-676, ωCen, M 13, NGC 2808. For an inspiral of two $1.4M_{\odot}$ NSs at a distance of 50 Mpc, LIGO II detectors will be able to constrain λ to $\lambda \leq 20.1 \times 10^{36}\mathrm{g\,cm^2 s^2}$ with 90% confidence.

Both Fig. 3.1 and Table 3.1 illustrate that the dimensionless Love numbers k_2 depend more strongly on the polytropic index n than on the compactness $C = M/R$. [3] This is expected since the weak field, Newtonian values k_2^N given by Eq. (3.24) just depend on n (through the dependence on y). The additional dependence on the compactness for the Love numbers k_2 in Eq. (3.23) is a relativistic correction to this. For $M/R \sim 10^{-5}$ our results for k_2 agree well with the Newtonian results of Ref. [54]. Figure 3.2 shows the percent difference $(k_2^N - k_2)/k_2$ between the relativistic and Newtonian dimensionless Love numbers. As can be seen from the figure, the relativistic values are lower than the Newtonian ones for higher values of n. This can be explained by the fact that the Love number encodes

[3] Note, however, that LIGO measurements will yield the combination $k_2 R^5$ and therefore will be more sensitive to the compactness than the polytropic index.

Table 3.1: Relativistic Love numbers k_2

n	M/R	k_2
0.3	10^{-5}	0.5511
0.3	0.1	0.5401
0.3	0.15	0.5691
0.3	0.2	0.6146
0.5	10^{-5}	0.4491
0.5	0.1	0.4260
0.5	0.15	0.4349
0.5	0.2	0.4489
0.5	0.25	0.4589
0.7	10^{-5}	0.3626
0.7	0.1	0.3373
0.7	0.15	0.3369
0.7	0.2	0.3363
0.7	0.25	0.3267
1.0	10^{-5}	0.2599
1.0	0.1	0.2405
1.0	0.15	0.2363
1.0	0.2	0.2282
1.0	0.25	0.2081
1.2	10^{-5}	0.2062
1.2	0.1	0.1936
1.2	0.15	0.1900
1.2	0.2	0.1811

Table 3.2: Estimated neutron star parameters from X-ray observations from Webb and Barrett and Ozel used to generate the figure.

Cluster / object	$M(M_\odot)$	R(km)	M/R
ω Cen [a]	1.61 ± 0.15	10.99 ± 0.71	0.18 ± 0.04
M 13 [a]	1.36 ± 0.04	9.89 ± 0.08	0.2
NGC 2808 [a]	0.84 ± 0.12	7.34 ± 0.96	0.22 ± 0.01
EXO 0748-676 [b]	$\geq 2.1 \pm 0.28$	$\geq 13.8 \pm 1.8$	0.2256

information about the degree of central condensation of the star. Stars with a higher the polytropic index n are more centrally condensed and therefore have a smaller response to a tidal field, resulting in a smaller Love number.

CHAPTER 4

TWO TIMESCALE ANALYSIS OF EXTREME MASS RATIO INSPIRALS IN KERR. I. ORBITAL MOTION

SUMMARY: Inspirals of stellar mass compact objects into massive black holes are an important source for future gravitational wave detectors such as Advanced LIGO and LISA. Detection of these sources and extracting information from the signal relies on accurate theoretical models of the binary dynamics. We cast the equations describing binary inspiral in the extreme mass ratio limit in terms of action angle variables, and derive properties of general solutions using a two-timescale expansion. This provides a rigorous derivation of the prescription for computing the leading order orbital motion. As shown by Mino, this leading order or adiabatic motion requires only knowledge of the orbit-averaged, dissipative piece of the self force. The two timescale method also gives a framework for calculating the post-adiabatic corrections. For circular and for equatorial orbits, the leading order corrections are suppressed by one power of the mass ratio, and give rise to phase errors of order unity over a complete inspiral through the relativistic regime. These post-1-adiabatic corrections are generated by the fluctuating piece of the dissipative, first order self force, by the conservative piece of the first order self force, and by the orbit-averaged, dissipative piece of the second order self force. We also sketch a two-timescale expansion of the Einstein equation, and deduce an analytic formula for the leading order, adiabatic gravitational waveforms generated by an inspiral.

To be published in Physical Review D **15**, *(2008), with É. Flanagan*

4.1 Introduction and Summary

4.1.1 Background and Motivation

Recent years have seen great progress in our understanding of the two body problem in general relativity. Binary systems of compact bodies undergo an inspiral driven by gravitational radiation reaction until they merge. As illustrated in Fig. 4.1, there are three different regimes in the dynamics of these systems, depending on the values of the total and reduced masses M and μ of the system and the orbital separation r : (i) The early, weak field regime at $r \gg M$, which can be accurately modeled using post-Newtonian theory, see, for example, the review [61]. (ii) The relativistic, equal mass regime $r \sim M$, $\mu \sim M$, which must be treated using numerical relativity. Over the last few years, numerical relativists have succeeded for the first time in simulating the merger of black hole binaries, see, for example, the review [77] and references therein. (iii) The relativistic, extreme mass ratio regime $r \sim M$, $\mu \ll M$. Over timescales short compared to the dephasing time $\sim M\sqrt{M/\mu}$, systems in this regime can be accurately modeled using black hole perturbation theory[78], with the mass ratio $\varepsilon \equiv \mu/M$ serving as the expansion parameter. The subject of this paper is the approximation methods that are necessary to treat such systems over the longer inspiral timescale $\sim M^2/\mu$ necessary for computation of complete inspirals.

This extreme mass ratio regime has direct observational relevance: Compact objects spiraling into much larger black holes are expected to be a key source for both LIGO and LISA. *Intermediate-mass-ratio inspirals* (IMRIs) are inspirals of black holes or neutron stars into intermediate mass ($50 \leq M \leq 1000 M_\odot$) black holes; these would be visible to Advanced LIGO out to distances of several hundred

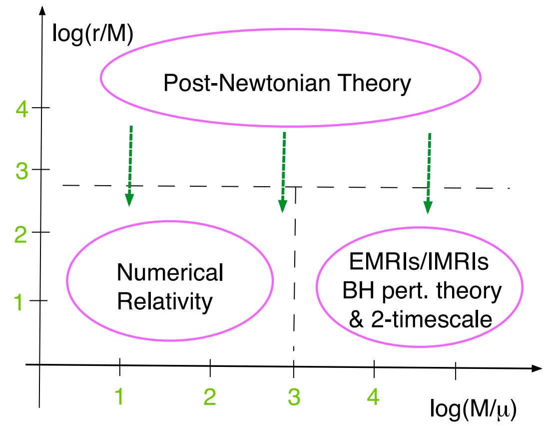

Figure 4.1: The parameter space of inspiralling compact binaries in general relativity, in terms of the inverse mass ratio $M/\mu = 1/\varepsilon$ and the orbital radius r, showing the different regimes and the computational techniques necessary in each regime. Individual binaries evolve downwards in the diagram (green dashed arrows).

Mpc [29], where the event rate could be about $3 - 30$ per year [29, 30]. *Extreme-mass-ratio inspirals* (EMRIs) are inspirals of stellar-mass compact objects (black holes, neutron stars, or possibly white dwarfs) into massive ($10^4 \leq M \leq 10^7 M_\odot$) black holes in galactic nuclei; these will be visible to LISA out to redshifts $z \approx 1$ [23, 24, 25]. It has been estimated [26, 27] that LISA should see about 50 such events per year, based on calculations of stellar dynamics in galaxies' central cusps[28]. Because of an IMRI's or EMRI's small mass ratio $\varepsilon = \mu/M$, the small body lingers in the large black hole‘s strong-curvature region for many wave cycles before merger: hundreds of cycles for LIGO's IMRIs; hundreds of thousands for LISA's EMRIs [23]. In this relativistic regime the post-Newtonian approximation has completely broken down, and full numerical relativity simulations become prohibitively difficult as ε is decreased. Modeling of these sources therefore requires a specialized approximation method.

Gravitational waves from these sources will be rich with information [24, 25]:

- The waves carry not only the details of the evolving orbit, but also a map of the large body's spacetime geometry, or equivalently the values of all its multipole moments, as well as details of the response of the horizon to tidal forces [79, 80]. Extracting the map (*bothrodesy*) is a high priority for LISA, which can achieve ultrahigh accuracy, and a moderate priority for LIGO, which will have a lower (but still interesting) accuracy [29]. Measurements of the black hole's quadrupole (fractional accuracy about 10^{-3} for LISA [81, 82], about 1 for Advanced LIGO [29]) will enable tests of the black hole's *no hair property*, that all of the mass and current multipole moments are uniquely determined in terms of the first two, the mass and spin. Potentially, these measurements could lead to the discovery of non-black-hole central objects such as boson stars [31, 32] or naked singularities.

- One can measure the mass and spin of the central black hole with fractional accuracies of order 10^{-4} for LISA [83, 84] and about 10^{-2}–10^{-1} for Advanced LIGO [29]. Observing many events will therefore provide a census of the masses and spins of the massive central black holes in non-active galactic nuclei like M31 and M32. The spin can provide useful information about the hole's growth history (mergers versus accretion) [33].

- For LISA, one can measure the inspiralling objects' masses with precision about 10^{-4}, teaching us about the stellar population in the central parsec of galactic nuclei.

- If the LISA event rate is large enough, one can measure the Hubble constant H_0 to about 1% [34], which would indirectly aid dark energy studies [35]. The idea is to combine the measured luminosity distance of cosmological ($z \sim 1/2$) EMRIs with a statistical analysis of the redshifts of candidate host galaxies located within the error box on the sky.

To realize the science goals for these sources requires accurate theoretical models of the waveforms for matched filtering. The accuracy requirement is roughly that the theoretical template's phase must remain accurate to ~ 1 cycle over the $\sim \varepsilon^{-1}$ cycles of waveform in the highly relativistic regime ($\sim 10^2$ cycles for LIGO, $\sim 10^5$ for LISA). For signal detection, the requirement is slightly less stringent than this, while for parameter extraction the requirement is slightly more stringent: The waveforms are characterized by 14 parameters, which makes a fully coherent search of the entire data train computationally impossible. Therefore, detection templates for LISA will use short segments of the signal and require phase coherence for $\sim 10^4$ cycles [27]. Once the presence of a signal has been established, the source parameters will be extracted using measurement templates that require a fractional phase accuracy of order the reciprocal of the signal to noise ratio [27], in order to keep systematic errors as small as the statistical errors.

4.1.2 Methods of computing orbital motion and waveforms

A variety of approaches to computing waveforms have been pursued in the community. We now review these approaches in order to place the present paper in context. The foundation for all approaches is the fact that, since $\varepsilon = \mu/M \ll 1$, the field of the compact object can be treated as a small perturbation to the large black hole's gravitational field. On short timescales $\sim M$, the compact object moves on a geodesic of the Kerr geometry, characterized by its conserved energy E, z-component of angular momentum L_z, and Carter constant Q. Over longer timescales $\sim M/\varepsilon$, radiation reaction causes the parameters E, L_z and Q to evolve adiabatically and the orbit to shrink. The effect of the internal structure of the object is negligible[1], so it can be treated as a point particle. At the end of the inspiral, the particle passes through an innermost stable orbit where adiabaticity breaks down, and it transitions onto a geodesic plunge orbit [89, 90, 91, 92]. In this paper we restrict attention to the adiabatic portion of the motion.

Numerical Relativity: Numerical relativity has not yet been applied to the extreme mass ratio regime. However, given the recent successful simulations in the equal mass regime $\varepsilon \sim 1$, one could contemplate trying to perform simulations with smaller mass ratios. There are a number of difficulties that arise as ε gets small: (i) The orbital timescale and the radiation reaction timescale are separated by the large factor $\sim 1/\varepsilon$. The huge number of wave cycles implies an impractically large computation time. (ii) There is a separation of lengthscales: the compact object is smaller than the central black hole by a factor ε. (iii) Most importantly, in the strong field region near the small object, the piece of the metric perturbation responsible for radiation reaction is of order ε, and since one requires errors in the radiation reaction to be of order ε, the accuracy requirement on the metric perturbation is of order ε^2. These difficulties imply that numerical simulations will likely not be possible in the extreme mass ratio regime in the foreseeable future, unless major new techniques are devised to speed up computations.

Use of post-Newtonian methods: Approximate waveforms which are qualitatively similar to real waveforms can be obtained using post-Newtonian methods or using hybrid schemes containing some post-Newtonian elements [93, 88, 94]. Although

[1] There are two exceptions, where corrections to the point-particle model can be important: (i) White dwarf EMRIs, where tidal interactions can play a role [85]. (ii) The effect due to the spin, if any, of the inspiralling object, whose importance has been emphasized by Burko [86, 87]. While this effect is at most marginally relevant for signal detection [88], it is likely quite important for information extraction. We neglect the spin effect in the present paper, since it can be computed and included in the waveforms relatively easily.

these waveforms are insufficiently accurate for the eventual detection and data analysis of real signals, they have been very useful for approximately scoping out the detectability of inspiral events and the accuracy of parameter measurement, both for LIGO [29] and LISA [27, 88]. They have the advantage that they can be computed relatively quickly.

Black hole perturbation theory – first order: There is a long history of using first order perturbation theory [78] to compute gravitational waveforms from particles in geodesic orbits around black holes [95, 96, 97, 98]. These computations have recently been extended to fully generic orbits [99, 100, 101]. However first order perturbation theory is limited to producing "snapshot" waveforms that neglect radiation reaction.[2] Such waveforms fall out of phase with true waveforms after a dephasing time $\sim M/\sqrt{\varepsilon}$, the geometric mean of the orbital and radiation reaction timescales, and so are of limited utility.[3]

Black hole perturbation theory – second order: One can in principle go to second order in perturbation theory [103, 104, 105]. At this order, the particle's geodesic motion must be corrected by self-force effects describing its interaction with its own spacetime distortion. This gravitational self force is analogous to the electromagnetic Abraham-Lorentz-Dirac force. Although a formal expression for the self force is known [106, 107], it has proved difficult to translate this expression into a practical computational scheme for Kerr black holes because of the mathematical complexity of the self-field regularization which is required. Research into this topic is ongoing; see, for example the review [108] and Refs.

[2]The source for the linearized Einstein equation must be a conserved stress energy tensor, which for a point particle requires a geodesic orbit.

[3]Drasco has argued that snapshot waveforms may still be useful for signal detections in certain limited parts of the IMRI/EMRI parameter space, since the phase coherence time is actually $\sim 100M/\sqrt{\varepsilon}$ [102].

[109, 110, 111, 112, 113, 114, 115, 105] for various approaches and recent progress.

When the self-force is finally successfully computed, second order perturbation theory will provide a self-consistent framework for computing the orbital motion and the waveform, but only over short timescales. The second order waveforms will fall out of phase with the true waveforms after only a dephasing time $\sim M/\sqrt{\varepsilon}$ [4] [116, 117]. Computing accurate waveforms describing a full inspiral therefore requires going beyond black hole perturbation theory.

Use of conservation laws: This well-explored method allows tracking an entire inspiral for certain special classes of orbits. Perturbation theory is used to compute the fluxes of E and L_z to infinity and down the horizon for geodesic orbits, and imposing global conservation laws, one infers the rates of change of the orbital energy and angular momentum. For circular orbits and equatorial orbits these determine the rate of change of the Carter constant Q, and thus the inspiralling trajectory. The computation can either be done in the frequency domain [95, 96, 97, 98], or in the time domain by numerically integrating the Teukolsky equation as a 2+1 PDE with a suitable numerical model of the point particle source [118, 119, 120, 121, 122, 123, 124, 125, 126, 127]. However, this method fails for generic orbits since there is no known global conservation law associated with the Carter constant Q.

Adiabatic approximation – leading order: Over the last few years, it has been discovered how to compute inspirals to leading order for generic orbits. The method

[4]The reason is as follows. Geodesic orbits and true orbits become out of phase by ~ 1 cycle after a dephasing time. Therefore, since the linear metric perturbation is sourced by a geodesic orbit, fractional errors in the linear metric perturbation must be of order unity. Therefore the second order metric perturbation must become comparable to the first order term after a dephasing time.

is based on the Mino's realization [128] that, in the adiabatic limit, one needs only the time averaged, dissipative piece of the first order self force, which can be straightforwardly computed from the half retarded minus half advanced prescription. This sidesteps the difficulties associated with regularization that impede computations of the full, first order self force. From the half advanced minus half retarded prescription, one can derive an explicit formula for a time-average of $\dot{Q}$ in terms of mode expansion [99, 40, 41, 129, 130]. Using this formula it will be straightforward to compute inspirals to the leading order.

We now recap and assess the status of these various approaches. All of the approaches described above have shortcomings and limitations [117]. Suppose that one computes the inspiral motion, either from conservation laws, or from the time-averaged dissipative piece of the first order self-force, or from the exact first order self-force when it becomes available. It is then necessary to compute the radiation generated by this inspiral. One might be tempted to use linearized perturbation theory for this purpose. However, two problems then arise:

- As noted above, the use of linearized perturbation theory with nongeodesic sources is mathematically inconsistent. This inconsistency has often been remarked upon, and various ad hoc methods of modifying the linearized theory to get around the difficulty have been suggested or implemented [107, 131, 132].
- A related problem is that the resulting waveforms will depend on the gauge chosen for the linearized metric perturbation, whereas the exact waveforms must be gauge invariant.

It has often been suggested that these problems can be resolved by going to second order in perturbation theory [108, 105]. However, as discussed above, a second order computation will be valid only for a dephasing time, and not for a full inspiral.

Of course, the above problems are not fatal, since the motion is locally very nearly geodesic, and so the inconsistencies and ambiguities are suppressed by a factor $\sim \varepsilon$ relative to the leading order waveforms.[5] Nevertheless, it is clearly desirable to have a well defined approximation method that gives a unique, consistent result for the leading order waveform. Also, for parameter extraction, it will be necessary to compute the phase of the waveform beyond the leading order. For this purpose it will clearly be necessary to have a more fundamental computational framework.

4.1.3 The two timescale expansion method

In this paper we describe an approximation scheme which addresses and resolves all of the theoretical difficulties discussed above. It is based on the fact that the systems evolve adiabatically: the radiation reaction timescale $\sim M/\varepsilon$ is much longer than the orbital timescale $\sim M$ [128]. It uses two-timescale expansions, which are a systematic method for studying the cumulative effect of a small disturbance on a dynamical system that is active over a long time [133].

The essence of the method is simply an ansatz for the dependence of the metric $g_{ab}(\varepsilon)$ on ε, and an ansatz for the dependence of the orbital motion on ε, that are justified a posteriori order by order via substitution into Einstein's equation. The ansatz for the metric is more complex than the Taylor series ansatz which

[5]This is true both for the instantaneous amplitude and for the accumulated phase of the waveform.

underlies standard perturbation theory. The two timescale method has roughly the same relationship to black hole perturbation theory as post-Newtonian theory has to perturbation theory of Minkowski spacetime. The method is consistent with standard black hole perturbation theory locally in time, at each instant, but extends the domain of validity to an entire inspiral. The method provides a systematic procedure for computing the leading order waveforms, which we call the adiabatic waveforms, as well as higher order corrections. We call the $O(\varepsilon)$ corrections the post-1-adiabatic corrections, the $O(\varepsilon^2)$ corrections post-2-adiabatic, etc., paralleling the standard terminology in post-Newtonian theory.

The use of two timescale expansions in the extreme mass ratio regime was first suggested in Refs. [116, 134]. The method has already been applied to some simplified model problems: a computation of the inspiral of a point particle in Schwarzschild subject to electromagnetic radiation reaction forces by Pound and Poisson [135], and a computation of the scalar radiation generated by a inspiralling particle coupled to a scalar field by Mino and Price [136]. We will extend and generalize these analyses, and develop a complete approximation scheme.

There are two, independent, parts to the the approximation scheme. The first is a two timescale analysis of the inspiralling orbital motion, which is the focus of the present paper. Our formalism enables us to give a rigorous derivation and clarification of the prescription for computing the leading order motion that is valid for all orbits, and resolves some controversies in the literature [135]. It also allows us to systematically calculate the higher order corrections. For these corrections, we restrict attention to inspirals in Schwarzschild, and to circular and equatorial inspirals in Kerr. Fully generic inspirals in Kerr involve a qualitatively new feature – the occurrence of transient resonances – which we will discuss in the forthcoming papers [137, 138].

The second part to the approximation scheme is the application of the two timescale method to the Einstein equation, and a meshing of that expansion to the analysis of the orbital motion. This allows computation of the observable gravitational waveforms, and is described in detail in the forthcoming paper [139]. We briefly sketch this formalism in Sec. 4.1.5 below, and give an analytic result for the leading order waveforms.

We note that alternative methods of attempting to overcome the problems with standard perturbation theory, and of going beyond adiabatic order, have been developed by Mino [131, 117, 140, 141, 142]. These methods have some overlap with the method discussed here, but differ in some crucial aspects. We do not believe that these methods provide a systematic framework for going to higher orders, unlike the two-timescale method.

4.1.4 Orbital Motion

We now turn to a description of our two timescale analysis of the orbital motion. The first step is to exploit the Hamiltonian structure of the unperturbed, geodesic motion to rewrite the governing equations in terms of generalized action angle variables. We start from the forced geodesic equation

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau} = \varepsilon a^{(1)\,\nu} + \varepsilon^2 a^{(2)\,\nu} + O(\varepsilon^3). \tag{4.1}$$

Here τ is proper time and $a^{(1)\,\nu}$ and $a^{(2)\,\nu}$ are the first order and second order self-accelerations. In Sec. 4.2 we augment these equations to describe the leading order backreaction of the inspiral on the mass M and spin a of the black hole, and show they can be rewritten as [cf. Eqs. (4.59) below]

$$\frac{dq_\alpha}{d\tau} = \omega_\alpha(J_\sigma) + \varepsilon g^{(1)}_\alpha(q_r, q_\theta, J_\sigma) + \varepsilon^2 g^{(2)}_\alpha(q_r, q_\theta, J_\sigma) + O(\varepsilon^3), \tag{4.2a}$$

$$\frac{dJ_\lambda}{d\tau} = \varepsilon G^{(1)}_\lambda(q_r, q_\theta, J_\sigma) + \varepsilon^2 G^{(2)}_\lambda(q_r, q_\theta, J_\sigma) + O(\varepsilon^3). \tag{4.2b}$$

Here the variables J_λ are the three conserved quantities of geodesic motion, with the dependence on the particle mass scaled out, together with the black hole mass and spin parameters:

$$J_\lambda = (E/\mu, L_z/\mu, Q/\mu^2, M, a). \tag{4.3}$$

The variables $q_\alpha = (q_r, q_\theta, q_\phi, q_t)$ are a set of generalized angle variables associated with the r, θ, ϕ and t motions in Boyer-Lindquist coordinates, and are defined more precisely in Sec. 4.2.4 below. The variables q_r, q_θ, and q_ϕ each increase by 2π after one cycle of motion of the corresponding variable r, θ or ϕ. The functions $\omega_\alpha(J_\sigma)$ are the fundamental frequencies of geodesic motion in the Kerr metric. The functions $g^{(1)}_\alpha$, $G^{(1)}_\lambda$ are currently not known explicitly, but their exact form will not be needed for the analysis of this paper. They are determined by the first order self acceleration [106, 107]. Similarly, the functions $g^{(2)}_\alpha$ and $G^{(2)}_\lambda$ are currently not known explicitly, and are determined in part by the second order self acceleration [143, 144, 145, 146, 147]; see Sec. 4.2.6 for more details.

In Secs. 4.4 – 4.5 below we analyze the differential equations (4.2) using two timescale expansions. In the non-resonant case, and up to post-1-adiabatic order, the results can be summarized as follows. Approximate solutions of the equations can be constructed via a series of steps:

- We define the slow time variable $\tilde{\tau} = \varepsilon\tau$.

- We construct a set of functions $\psi_\alpha^{(0)}(\tilde{\tau})$, $\mathcal{J}_\lambda^{(0)}(\tilde{\tau})$, $\psi_\alpha^{(1)}(\tilde{\tau})$ and $\mathcal{J}_\lambda^{(1)}(\tilde{\tau})$ of the slow time. These functions are defined by a set of differential equations that involve the functions ω_α, $g_\alpha^{(1)}$, $G_\lambda^{(1)}$, $g_\alpha^{(2)}$ and $G_\lambda^{(2)}$ and which are independent of ε [Eqs. (4.188), (4.193), (4.191), (4.201), (4.199) below].

- We define a set of auxiliary phase variables ψ_α by

$$\psi_\alpha(\tau,\varepsilon) = \frac{1}{\varepsilon}\psi_\alpha^{(0)}(\varepsilon\tau) + \psi_\alpha^{(1)}(\varepsilon\tau) + O(\varepsilon), \tag{4.4}$$

where the $O(\varepsilon)$ symbol refers to the limit $\varepsilon \to 0$ at fixed $\tilde{\tau} = \varepsilon\tau$.

- Finally, the solution to post-1-adiabatic order is given by

$$\begin{aligned} q_\alpha(\tau,\varepsilon) &= \psi_\alpha + O(\varepsilon), && (4.5\text{a}) \\ J_\lambda(\tau,\varepsilon) &= \mathcal{J}_\lambda^{(0)}(\varepsilon\tau) + \varepsilon\mathcal{J}^{(1)}(\varepsilon\tau) \\ &\quad + H_\lambda[\psi_r, \psi_\theta, \mathcal{J}_\sigma^{(0)}(\varepsilon\tau)] + O(\varepsilon^2), && (4.5\text{b}) \end{aligned}$$

where the $O(\varepsilon)$ and $O(\varepsilon^2)$ symbols refer to $\varepsilon \to 0$ at fixed $\tilde{\tau}$ and ψ_α. Here H_λ is a function which is periodic in its first two arguments and which can computed from the function $G_\lambda^{(1)}$ [Eq. (4.243) below].

We now turn to a discussion of the implications of the final result (4.5). First, we emphasize that the purpose of the analysis is *not* to give a convenient, practical scheme to integrate the orbital equations of motion. Such a scheme is not needed, since once the self-acceleration is known, it is straightforward to numerically integrate the forced geodesic equations (4.1). Rather, the main benefit of the analysis is to give an analytic understanding of the dependence of the motion on ε in the limit $\varepsilon \to 0$. This serves two purposes. First, it acts as a foundation for the two timescale expansion of the Einstein equation and the computation of waveforms (Sec. 4.1.5 below and Ref. [139]). Second, it clarifies the utility of different approximations to the self-force that have been proposed, by determining which pieces of the self-force contribute to the adiabatic order and post-1-adiabatic order motions [99, 40]. This second issue is discussed in detail in Sec. 4.7 below. Here we give a brief summary.

Consider first the motion to adiabatic order, given by the functions $\psi_\alpha^{(0)}$ and $\mathcal{J}_\lambda^{(0)}$. These functions are obtained from the differential equations [Eqs. (4.188), (4.193) and (4.191) below]

$$\begin{aligned}\frac{d\psi_\alpha^{(0)}}{d\tilde{\tau}}(\tilde{\tau}) &= \omega_\alpha[\mathcal{J}_\sigma^{(0)}(\tilde{\tau})], && (4.6a)\\ \frac{d\mathcal{J}_\lambda^{(0)}}{d\tilde{\tau}}(\tilde{\tau}) &= \left\langle G_\lambda^{(1)}\right\rangle[\mathcal{J}_\sigma^{(0)}(\tilde{\tau})], && (4.6b)\end{aligned}$$

where $\langle\ldots\rangle$ denotes the average[6] over the 2-torus

$$\left\langle G_\lambda^{(1)}\right\rangle(J_\sigma) \equiv \frac{1}{(2\pi)^2}\int_0^{2\pi} dq_r \int_0^{2\pi} dq_\theta\, G_\lambda^{(1)}(q_r, q_\theta, J_\sigma). \tag{4.7}$$

This zeroth order approximation describes the inspiralling motion of the particle. In Sec. 4.2.7 below we show that only the dissipative (ie half retarded minus half advanced) piece of the self force contributes to the torus average (4.7). Thus, the leading order motion depends only on the dissipative self-force, as argued by Mino [128]. Our result extends slightly that of Mino, since he advocated using an infinite time average on the right hand side of Eq. (4.6b), instead of the phase space or torus average. The two averaging methods are equivalent for generic geodesics, but not for geodesics for which the ratio of radial and azimuthal periods is a rational number. The time-average prescription is therefore correct for generic geodesics, while the result (4.6) is valid for all geodesics. The effect of the nongeneric geodesics is discussed in detail in Refs. [137, 138].

Consider next the subleading, post-1-adiabatic corrections to the inspiral given by the functions $\psi_\alpha^{(1)}$ and $\mathcal{J}_\lambda^{(1)}$. These corrections are important for assessing the

[6] This phase space average is uniquely determined by the dynamics of the system, and resolves concerns in the literature about inherent ambiguities in the choice of averaging [135].

accuracy of the adiabatic approximation, and will be needed for accurate data analysis of detected signals. The differential equations determining $\psi_\alpha^{(1)}$ and $\mathcal{J}_\lambda^{(1)}$ are Eqs. (4.201) and (4.199) below. These equations depend on (i) the oscillating (not averaged) piece of the dissipative, first order self force; (ii) the conservative piece of the first order self force, and (iii) the torus averaged, dissipative piece of the second order self force. Thus, all three of these quantities will be required to compute the inspiral to subleading order, confirming arguments made in Refs. [148, 99, 40, 149]. In particular, knowledge of the full first order self force will not enable computation of more accurate inspirals until the averaged, dissipative piece of the second order self force is known.[7]

4.1.5 Two timescale expansion of the Einstein equations and adiabatic waveforms

We now turn to a brief description of the two timescale expansion of the Einstein equations; more details will be given in the forthcoming paper [139]. We focus attention on a region $\mathcal{R}$ of spacetime defined by the conditions (i) The distance from the particle is large compared to its mass μ; (ii) The distance r from the large black hole is small compared to the inspiral time, $r \ll M^2/\mu$; and (iii) The extent of the region in time covers the entire inspiral in the relativistic regime. In this domain we make an ansatz for the form of the metric that is justified a posteriori order by order.

At distances $\sim \mu$ from the particle, one needs to use a different type of analysis (eg black hole perturbation theory for a small black hole), and to mesh that analysis

[7]This statement remains true when one takes into account resonances [138].

with the solution in the region $\mathcal{R}$ by matching in a domain of common validity. This procedure is very well understood and is the standard method for deriving the first order self force [106, 108]. It is valid for our metric ansatz (6.323) below since that ansatz reduces, locally in time at each instant, to standard black hole perturbation theory. Therefore we do not focus on this aspect of the problem here. Similarly, at large distances, one needs to match the solution within $\mathcal{R}$ onto an outgoing wave solution in order to read off the asymptotic waveforms.[8]

Within the region $\mathcal{R}$, our ansatz for the form of the metric in the non-resonant case is

$$\begin{aligned} g_{\alpha\beta}(\bar{t},\bar{x}^j;\varepsilon) &= g^{(0)}_{\alpha\beta}(\bar{x}^j) + \varepsilon g^{(1)}_{\alpha\beta}(q_r, q_\theta, q_\phi, \tilde{t}, \bar{x}^j) \\ &\quad + \varepsilon^2 g^{(1)}_{\alpha\beta}(q_r, q_\theta, q_\phi, \tilde{t}, \bar{x}^j) + O(\varepsilon^3). \end{aligned} \tag{4.8}$$

Here $g^{(0)}_{\alpha\beta}$ is the background, Kerr metric. The coordinates $(\bar{t},\bar{x}^j)$ can be any set of coordinates in Kerr, subject only to the restriction that $\partial/\partial\bar{t}$ is the timelike Killing vector. On the right hand side, $\tilde{t}$ is the slow time variable $\tilde{t} = \varepsilon\bar{t}$, and the quantities q_r, q_θ and q_ϕ are the values of the orbit's angle variables at the intersection of the inspiralling orbit with the hypersurface $\bar{t}$ = constant. These are functions of $\bar{t}$ and of ε, and can be obtained from the solutions (4.4) and (4.5a) of the inspiral problem by eliminating the proper time τ. The result is of the form

$$q_i(\bar{t},\varepsilon) = \frac{1}{\varepsilon} f^{(0)}_i(\tilde{t}) + f^{(1)}_i(\tilde{t}) + O(\varepsilon), \tag{4.9}$$

for some functions $f^{(0)}_i$, $f^{(1)}_i$. On the right hand side of Eq. (6.323), the $O(\varepsilon^3)$ refers to an asymptotic expansion associated with the limit $\varepsilon \to 0$ at fixed q_i, $\bar{x}^k$ and $\tilde{t}$. Finally the functions $g^{(1)}_{\alpha\beta}$ and $g^{(2)}_{\alpha\beta}$ are assumed to be multiply periodic in q_r, q_θ and q_ϕ with period 2π in each variable.

[8] This matching is not necessary at the leading, adiabatic order, for certain special choices of time coordinate in the background spacetime, as argued in Ref. [136]. It is needed to higher orders.

By inserting the ansatz (6.323) into Einstein's equations, one obtains a set of equations that determines the free functions, order by order. At leading order we obtain an equation of the form

$$\mathcal{D} g^{(0)}_{\alpha\beta} = 0, \tag{4.10}$$

where $\mathcal{D}$ is a linear differential operator on the six dimensional manifold with coordinates $(q_r, q_\theta, q_\phi, \bar{x}^j)$. In solving this equation, $\tilde{t}$ is treated as a constant. The solution that matches appropriately onto the worldline source can be written as

$$\begin{aligned} g^{(1)}_{\alpha\beta} \;\; = \;\; & \frac{\partial g^{(0)}_{\alpha\beta}}{\partial M} \delta M(\tilde{t}) + \frac{\partial g^{(0)}_{\alpha\beta}}{\partial a} \delta a(\tilde{t}) + \ldots \\ & + \mathcal{F}_{\alpha\beta}[q_r, q_\theta, q_\phi, \bar{x}^j, E(\tilde{t}), L_z(\tilde{t}), Q(\tilde{t})]. \end{aligned} \tag{4.11}$$

Here the terms on the first line are the secular pieces of the solution. They arise since the variable $\tilde{t}$ is treated as a constant, and so one can obtain a solution by taking the perturbation to the metric generated by allowing the parameters of the black hole (mass, spin, velocity, center of mass location) to vary as arbitrary functions of $\tilde{t}$. For example, the mass of the black hole can be written as $M(\tilde{t}) = M + \delta M(\tilde{t})$, where $M = M(0)$ is the initial mass. The functions $\delta M(\tilde{t})$, $\delta a(\tilde{t})$ etc. are freely specifiable at this order, and will be determined at the next (post-1-adiabatic) order.

The second line of Eq. (4.11) is the oscillatory piece of the solution. Here one obtains a solution by taking the function $\mathcal{F}_{\alpha\beta}$ to be the function

$$\mathcal{F}_{\alpha\beta}(q_r, q_\theta, q_\phi, \bar{x}^j, E, L_z, Q)$$

that one obtains from standard linear perturbation theory with a geodesic source. This function is known analytically in Boyer-Lindquist coordinates (t, r, θ, ϕ) in terms of a mode expansion.[9, 10]

The gauge freedom in this formalism consists of those one parameter families of diffeomorphisms which preserve the form (6.323) of the metric ansatz. To the leading order, these consist of (i) gauge transformations of the background coordinates that are independent of ε, which preserve the timelike Killing vector, and (ii) transformations of the form

$$x^\alpha \to x^\alpha + \varepsilon \xi^\alpha(q_r, q_\theta, q_\phi, \tilde{t}, x^j) + O(\varepsilon^2). \tag{4.12}$$

Note that this is *not* the standard gauge freedom of linear perturbation theory, since ξ^α depends on 4 "time variables" instead of one. This modified gauge group allows the two timescale method to evade the two problems discussed at the end of Sec. 4.1.2 above, since the gradual evolution is described entirely by the $\tilde{t}$ dependence, and, at each fixed $\tilde{t}$, the leading order dependence on the variables q_r, q_θ, q_ϕ, r, θ and ϕ is the same as in standard perturbation theory with the same gauge transformation properties.

[9]In coordinates $\bar{t} = t - r$, r, θ, ϕ, the explicit form of the asymptotic solution can be obtained by taking Eq. (3.1) of Ref. [102], eliminating the phases χ_{lmkn} using Eq. (8.29) of Ref. [40], and making the identifications $q_r = \Omega_r[t - r - t_0 + \hat{t}_r(-\lambda_{r0}) - \hat{t}_\theta(-\lambda_{\theta 0})] - \Upsilon_r \lambda_{r0}$, $q_\theta = \Omega_\theta[t - r - t_0 + \hat{t}_r(-\lambda_{r0}) - \hat{t}_\theta(-\lambda_{\theta 0})] - \Upsilon_\theta \lambda_{\theta 0}$, and $q_\phi = \Omega_\phi[t - r - t_0 + \hat{t}_r(-\lambda_{r0}) - \hat{t}_\theta(-\lambda_{\theta 0})] + \phi_0 - \hat{\phi}_r(-\lambda_{r0}) + \hat{\phi}_\theta(-\lambda_{\theta 0})$.

[10]The function $\mathcal{F}_{\alpha\beta}$ depends on q_ϕ and ϕ only through the combination $q_\phi - \phi$. This allows us to show that the two-timescale form (6.323) of the metric reduces to a standard Taylor series expansion, locally in time near almost every value $\tilde{t}_0$ of $\tilde{t}$. For equatorial orbits there is no dependence on q_θ, and the ε dependence of the metric has the standard form up to linear order, in coordinates (t', r', θ', ϕ') defined by $t' = (\tilde{t} - \tilde{t}_0)/\varepsilon + [f_r^{(0)}(\tilde{t}_0)/\varepsilon]/\omega_{r0}$, $\phi' = \phi + \omega_{\phi 0}[f_r^{(0)}(\tilde{t}_0)/\varepsilon]/\omega_{r0} - [f_\phi^{(0)}(\tilde{t}_0)/\varepsilon]$, $r' = r$, $\theta' = \theta$, where $\omega_{r0} = f_r^{(0)\prime}(\tilde{t}_0)$, $\omega_{\phi 0} = f_\phi^{(0)\prime}(\tilde{t}_0)$, and for any number x, $[x] \equiv x + 2\pi n$ where the integer n is chosen so that $0 \le [x] < 2\pi$. A similar construction works for circular orbits for which there is no dependence on q_r. For generic orbits a slightly more involved construction works, but only if $\omega_{r0}/\omega_{\phi 0}$ is irrational [139], which occurs for almost every value of $\tilde{t}_0$.

4.1.6 Organization of this Paper

The organization of this paper is as follows. In Sec. 4.2 we derive the fundamental equations describing the inspiral of a point particle into a Kerr black hole in terms of generalized action-angle variables. In Sec. 4.3 we define a class of general, weakly perturbed dynamical systems of which the inspiral motion in Kerr is a special case. We then study the solutions of this class of systems using two-timescale expansions, first for a single degree of freedom in Sec. 4.4, and then for the general case in Sec. 4.5. Section 4.6 gives an example of a numerical integration of a system of this kind, and Sec. 4.7 gives the final discussion and conclusions.

4.1.7 Notation and Conventions

Throughout this paper we use units with $G = c = 1$. Lower case Roman indices $a, b, c, \ldots$ denote abstract indices in the sense of Wald [73]. We use these indices both for tensors on spacetime and for tensors on the eight dimensional phase space. Lower case Greek indices $\nu, \lambda, \sigma, \tau, \ldots$ from the middle of the alphabet denote components of spacetime tensors on a particular coordinate system; they thus transform under spacetime coordinate transformations. They run over $0, 1, 2, 3$. Lower case Greek indices $\alpha, \beta, \gamma \ldots$ from the start of the alphabet label position or momentum coordinates on 8 dimensional phase space that are not associated with coordinates on spacetime. They run over $0, 1, 2, 3$ and do not transform under spacetime coordinate transformations. In Sec. 4.5, and just in that section, indices $\alpha, \beta, \gamma, \delta, \varepsilon, \ldots$ from the start of the Greek alphabet run over $1 \ldots N$, and indices $\lambda, \mu, \nu, \rho, \sigma, \ldots$ from the second half of the alphabet run over $1 \ldots M$. Bold face quantities generally denote vectors, as in $\mathbf{J} = (J_1, \ldots, J_M)$, although in Sec. 4.2

the bold faced notation is used for differential forms.

4.2 Extreme Mass Ratio Inspirals in Kerr formulated using action-angle variables

In this section we derive the form of the fundamental equations describing the inspiral of a point particle into a Kerr black hole, using action-angle type variables. Our final result is given in Eqs. (4.59) below, and the properties of the solutions of these equations are analyzed in detail in the remaining sections of this paper.

The description of geodesic motion in Kerr in terms of action angle variables was first given by Schmidt [150], and has been reviewed by Glampedakis and Babak [151]. We follow closely Schmidt's treatment, except that we work in an eight dimensional phase space instead of a six dimensional phase space, thus treating the time and spatial variables on an equal footing. We also clarify the extent to which the fundamental frequencies of geodesic motion are uniquely determined and gauge invariant, as claimed by Schmidt.

We start in subsection 4.2.1 by reviewing the geometric definition of action angle variables in Hamiltonian mechanics, which is based on the Liouville-Arnold theorem [152]. This definition does not apply to geodesic motion in Kerr, since the level surfaces defined by the conserved quantities in the eight dimensional phase space are non-compact. In subsection 4.2.2 we discuss how generalized action angle variables can be defined for non-compact level surfaces, and in subsection 4.2.3 we apply this to give a coordinate-independent construction of generalized action angle variables for generic bound geodesics in Kerr. Subsection 4.2.4 specializes to Boyer-Lindquist coordinates on phase space, and describes explicitly, following Schmidt [150], the explicit canonical transformation from those coordinates to the generalized action angle variables.

We then turn to using these variables to describe a radiation-reaction driven inspiral. In subsection 4.2.5 we derive the equations of motion in terms of the generalized action angle variables. These equations define a flow on the eight dimensional phase space, and do not explicitly exhibit the conservation of rest mass. In subsection 4.2.6 we therefore switch to a modified set of variables and equations in which the conservation of rest mass is explicit. We also augment the equations to describe the backreaction of gravitational radiation passing through the horizon of the black hole.

4.2.1 Review of action-angle variables in geometric Hamiltonian mechanics

We start by recalling the standard geometric framework for Hamiltonian mechanics [152]. A Hamiltonian system consists of a $2N$-dimensional differentiable manifold $\mathcal{M}$ on which there is defined a smooth function H (the Hamiltonian), and a non-degenerate 2-form Ω_{ab} which is closed, $\nabla_{[a}\Omega_{bc]} = 0$. Defining the tensor Ω^{ab} by $\Omega^{ab}\Omega_{bc} = \delta^a_c$, the Hamiltonian vector field is defined as

$$v^a = \Omega^{ab}\nabla_b H, \tag{4.13}$$

and the integral curves of this vector fields give the motion of the system. The two form Ω_{ab} is called the symplectic structure. Coordinates (q_α, p_α) with $1 \le \alpha \le N$ are called symplectic coordinates if the symplectic structure can be written as $\mathbf{\Omega} = dp_\alpha \wedge dq_\alpha$, i.e. $\Omega_{ab} = 2\nabla_{[a} p_\alpha \nabla_{b]} q_\alpha$.

We shall be interested in systems that possess $N-1$ first integrals of motion which, together with the Hamiltonian H, form a complete set of N independent first integrals. We denote these first integrals by P_α, $1 \le \alpha \le N$, where $P_1 = H$. These quantities are functions on $\mathcal{M}$ for which the Poisson brackets

$$\{P_\alpha, H\} \equiv \Omega^{ab}(\nabla_a P_\alpha)(\nabla_b H) \tag{4.14}$$

vanish for $1 \le \alpha \le N$. If the first integrals satisfy the stronger condition that all the Poisson brackets vanish,

$$\{P_\alpha, P_\beta\} = 0 \tag{4.15}$$

for $1 \le \alpha, \beta \le N$, then the first integrals are said to be in involution. If the 1-forms $\nabla_a P_\alpha$ for $1 \le \alpha \le N$ are linearly independent, then the first integrals are said to be independent. A system is said to be *completely integrable* in some open region $\mathcal{U}$ in $\mathcal{M}$ if there exist N first integrals which are independent and in involution at every point of $\mathcal{U}$.

For completely integrable systems, the phase space $\mathcal{M}$ is foliated by invariant level sets of the first integrals. For a given set of values $\mathbf{p} = (p_1, \ldots, p_N)$, we define the level set

$$\mathcal{M}_{\mathbf{p}} = \{x \in \mathcal{M} |\ P_\alpha(x) = p_\alpha, 1 \le \alpha \le N\}, \tag{4.16}$$

which is an N-dimensional submanifold of $\mathcal{M}$. The level sets are invariant under the Hamiltonian flow by Eq. (4.14). Also the pull back of the symplectic structure $\boldsymbol{\Omega}$ to $\mathcal{M}_{\mathbf{p}}$ vanishes, since the vector fields $\vec{v}_\alpha$ defined by

$$v_\alpha^a = \Omega^{ab}\nabla_b P_\alpha \tag{4.17}$$

for $1 \le \alpha \le N$ form a basis of the tangent space to $\mathcal{M}_{\mathbf{p}}$ at each point, and satisfy $\Omega_{ab} v_\alpha^a v_\beta^b = 0$ for $1 \le \alpha, \beta \le N$ by Eq. (4.15).

A classic theorem of mechanics, the Liouville-Arnold theorem [152], applies to systems which are completely integrable in a neighborhood of some level set $\mathcal{M}_{\mathbf{p}}$ that is connected and compact. The theorem says that

- The level set $\mathcal{M}_{\mathbf{p}}$ is diffeomorphic to an N-torus T^N. Moreover there is a neighborhood $\mathcal{V}$ of $\mathcal{M}_{\mathbf{p}}$ which is diffeomorphic to the product $T^N \times \mathcal{B}$ where $\mathcal{B}$ is an open ball, such that the level sets are the N-tori.
- There exist symplectic coordinates (q_α, J_α) for $1 \leq \alpha \leq N$ (action-angle variables) on $\mathcal{V}$ for which the angle variables q_α are periodic,

  $$q_\alpha + 2\pi \equiv q_\alpha,$$

  and for which the first integrals depend only on the action variables, $P_\alpha = P_\alpha(J_1, \ldots, J_N)$ for $1 \leq \alpha \leq N$.

An explicit and coordinate-invariant prescription for computing a set of action variables J_α is as follows [152]. A symplectic potential $\mathbf{\Theta}$ is a 1-form which satisfies $d\mathbf{\Theta} = \mathbf{\Omega}$. Since the 2-form $\mathbf{\Omega}$ is closed, such 1-forms always exist locally. For example, in any local symplectic coordinate system (q_α, p_α), the 1-form $\mathbf{\Theta} = p_\alpha dq_\alpha$ is a symplectic potential. It follows from the hypotheses of the Liouville-Arnold theorem that there exist symplectic potentials that are defined on a neighborhood of $\mathcal{M}_{\mathbf{p}}$ [153]. The first homotopy group $\Pi_1(\mathcal{M}_{\mathbf{p}})$ is defined to be the set of equivalence classes of loops on $\mathcal{M}_{\mathbf{p}}$, where two loops are equivalent if one can be continuously deformed into the other. Since $\mathcal{M}_{\mathbf{p}}$ is diffeomorphic to the N-torus, this group is isomorphic to $(\mathbf{Z}^N, +)$, the group of N-tuples of integers under addition. Pick a set of generators $\gamma_1, \ldots, \gamma_N$ of $\Pi_1(\mathcal{M}_{\mathbf{p}})$, and for each loop γ_α define

$$J_\alpha = \frac{1}{2\pi} \int_{\gamma_\alpha} \mathbf{\Theta}. \tag{4.18}$$

This integral is independent of the choice of symplectic potential $\mathbf{\Theta}$.[11] It is also independent of the choice of loop γ_α in the equivalence class of the generator of $\Pi_1(\mathcal{M}_{\mathbf{p}})$, since if γ_α and γ'_α are two equivalent loops, we have

$$\int_{\gamma_\alpha} \mathbf{\Theta} - \int_{\gamma'_\alpha} \mathbf{\Theta} = \int_{\partial\mathcal{R}} \mathbf{\Theta} = \int_{\mathcal{R}} d\mathbf{\Theta} = \int_{\mathcal{R}} \mathbf{\Omega} = 0. \tag{4.19}$$

Here $\mathcal{R}$ is a 2-dimensional surface in $\mathcal{M}_{\mathbf{p}}$ whose boundary is $\gamma_\alpha - \gamma'_\alpha$, we have used Stokes theorem, and in the last equality we have used the fact that the pull back of $\mathbf{\Omega}$ to the level set $\mathcal{M}_{\mathbf{p}}$ vanishes.

Action-angle variables for a given system are not unique [154]. There is a freedom to redefine the coordinates via

$$q_\alpha \to A_{\alpha\beta} q_\beta, \qquad J_\alpha \to B_{\alpha\beta} J_\beta, \tag{4.20}$$

where $A_{\alpha\beta}$ is a constant matrix of integers with determinant ± 1, and $A_{\alpha\beta} B_{\alpha\gamma} = \delta_{\beta\gamma}$. This is just the freedom present in choosing a set of generators of the group $\Pi_1(\mathcal{M}_{\mathbf{p}}) \sim (\mathbf{Z}^N, +)$. Fixing this freedom requires the specification of some additional information, such as a choice of coordinates on the torus; once the coordinates q_α are chosen, one can take the loops γ_α to be the curves $q_\beta =$ constant for $\beta \neq \alpha$. There is also a freedom to redefine the origin of the angle variables separately on each torus:

$$q_\alpha \to q_\alpha + \frac{\partial Z(J_\beta)}{\partial J_\alpha}, \qquad J_\alpha \to J_\alpha. \tag{4.21}$$

Here $Z(J_\beta)$ can be an arbitrary function of the action variables.

[11] The type of argument used in Ref. [153] can be used to show that the pullback to $\mathcal{M}_{\mathrm{p}}$ of the difference between two symplectic potentials is exact since it is closed.

4.2.2 Generalized action-angle variables for non-compact level sets

One of the crucial assumptions in the Liouville-Arnold theorem is that the level set $\mathcal{M}_{\mathbf{p}}$ is compact. Unfortunately, this assumption is not satisfied by the dynamical system of bound orbits in Kerr which we discuss in Sec. 4.2.3 below, because we will work in the 8 dimensional phase space and the motion is not bounded in the time direction. We shall therefore use instead a generalization of the Liouville-Arnold theorem to non-compact level sets, due to Fiorani, Giachetta and Sardanashvily [153].

Consider a Hamiltonian system which is completely integrable in a neighborhood $\mathcal{U}$ of a connected level set $\mathcal{M}_{\mathbf{p}}$, for which the N vector fields (4.17) are complete on $\mathcal{U}$, and for which the level sets $\mathcal{M}_{\mathbf{p}'}$ foliating $\mathcal{U}$ are all diffeomorphic to one another. For such systems Fiorani et. al. [153] prove that

- There is an integer k with $0 \leq k \leq N$ such that the level set $\mathcal{M}_{\mathbf{p}}$ is diffeomorphic to the product $T^k \times \mathbf{R}^{N-k}$, where $\mathbf{R}$ is the set of real numbers. Moreover there is a neighborhood $\mathcal{V}$ of $\mathcal{M}_{\mathbf{p}}$ which is diffeomorphic to the product $T^k \times \mathbf{R}^{N-k} \times \mathcal{B}$ where $\mathcal{B}$ is an open ball.
- There exist symplectic coordinates (q_α, J_α) for $1 \leq \alpha \leq N$ (generalized action-angle variables) on $\mathcal{V}$ for which the first k variables q_α are periodic,

  $$q_\alpha + 2\pi \equiv q_\alpha, \quad 1 \leq \alpha \leq k,$$

  and for which the first integrals depend only on the action variables, $P_\alpha = P_\alpha(J_1, \ldots, J_N)$ for $1 \leq \alpha \leq N$.

Thus, there are k compact dimensions in the level sets, and $N-k$ non-compact dimensions. In our application to Kerr below, the values of these parameters will be $k=3$ and $N-k=1$.

The freedom in choosing generalized action-angle variables is larger than the corresponding freedom for action-angle variables discussed above. The first k action variables can be computed in the same way as before, via the integral (4.18) evaluated on a set of generators $\gamma_1, \ldots, \gamma_k$ of $\Pi_1(\mathcal{M}_{\mathbf{p}})$, which in this case is isomorphic to $(\mathbf{Z}^k, +)$. This prescription is unique up to a group of redefinitions of the form [cf. Eq. (4.20) above]

$$q_\alpha \to \sum_{\beta=1}^{k} A_{\alpha\beta} q_\beta, \qquad J_\alpha \to \sum_{\beta=1}^{k} B_{\alpha\beta} J_\beta, \tag{4.22}$$

for $1 \le \alpha \le k$, where the $k \times k$ matrix $A_{\alpha\beta}$ is a constant matrix of integers with determinant ± 1, and $A_{\alpha\beta} B_{\alpha\gamma} = \delta_{\beta\gamma}$. There is additional freedom present in the choice of the rest of the action variables $J_{k+1}, \ldots, J_N$. As a consequence, the remaining freedom in choosing generalized action-angle variables consists of the transformations (4.21) discussed earlier, together with transformations of the form

$$q_\alpha \to A_{\alpha\beta} q_\beta, \qquad J_\alpha \to B_{\alpha\beta} J_\beta, \tag{4.23}$$

where $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are constant real $N \times N$ matrices with $A_{\alpha\beta} B_{\alpha\gamma} = \delta_{\beta\gamma}$ such that $J_1, \ldots, J_k$ are preserved.

In generalized action-angle variables, the equations of motion take the simple form

$$\dot{q}_\alpha = \frac{\partial H(\mathbf{J})}{\partial J_\alpha} \tag{4.24}$$

and

$$\dot{J}_\alpha = -\frac{\partial H(\mathbf{J})}{\partial q_\alpha} = 0. \tag{4.25}$$

We define the quantities

$$\Omega_\alpha(\mathbf{J}) \equiv \frac{\partial H(\mathbf{J})}{\partial J_\alpha}, \tag{4.26}$$

which are angular frequencies for $1 \le \alpha \le k$ but not for $k+1 \le \alpha \le N$. The solutions of the equations of motion are then

$$q_\alpha(t) = \Omega_\alpha(\mathbf{J}_0)t + q_{\alpha 0} \tag{4.27a}$$

$$J_\alpha(t) = J_{\alpha 0}, \tag{4.27b}$$

for some constants $\mathbf{J}_0$ and $\mathbf{q}_0$.

4.2.3 Application to bound geodesic motion in Kerr

We now apply the general theory discussed above to give a coordinate-invariant definition of action-angle variables for a particle on a bound orbit in the Kerr spacetime. We denote by $(\mathcal{M}_{\rm K}, g_{ab})$ the Kerr spacetime, and we denote by ξ^a and η^a the timelike and axial Killing vector fields. The cotangent bundle over $\mathcal{M}_{\rm K}$ forms an 8-dimensional phase space $\mathcal{M} = T^*\mathcal{M}_{\rm K}$. Given any coordinate system x^ν on the Kerr spacetime, we can define a coordinate system (x^ν, p_ν) on $\mathcal{M}$, such that the point (x^ν, p_ν) corresponds to the covector or one form $p_\nu dx^\nu$ at x^ν in $\mathcal{M}_{\rm K}$. The natural symplectic structure on $\mathcal{M}$ is then defined by demanding that all such coordinate systems (x^ν, p_ν) be symplectic [152]. The Killing vector fields ξ^a and η^a on $\mathcal{M}_{\rm K}$ have natural extensions to vector fields on phase space which Lie derive the symplectic structure.

Consider now a particle of mass μ on a bound geodesic orbit. A Hamiltonian on $\mathcal{M}$ that generates geodesic motion is given by

$$H(x^\nu, p_\nu) = \frac{1}{2} g^{\nu\sigma}(x^\rho) p_\nu p_\sigma; \tag{4.28}$$

this definition is independent of the choice of coordinate system x^ν. If we interpret p_ν to be the 4-momentum of the particle, then the conserved value of H is $-\mu^2/2$, and the evolution parameter is the affine parameter $\lambda = \tau/\mu$ where τ is proper time.

As is well known, geodesics on Kerr possess three first integrals, the energy $E = -\xi^a p_a$, the z-component of angular momentum $L_z = \eta^a p_a$, and Carter constant $Q = Q^{ab} p_a p_b$ where Q^{ab} is a Killing tensor [155]. Together with the Hamiltonian we therefore have four first integrals:

$$P_\alpha = (P_0, P_1, P_2, P_3) = (H, E, L_z, Q). \tag{4.29}$$

An explicit computation of the 4-form $dH \wedge dE \wedge dL_z \wedge dQ$ on $\mathcal{M}$ shows that it is non vanishing for bound orbits except for the degenerate cases of circular (i.e. constant Boyer-Lindquist radial coordinate) and equatorial orbits. Also the various Poisson brackets $\{P_\alpha, P_\beta\}$ vanish: $\{E, H\}$ and $\{L_z, H\}$ vanish since ξ^a and η^a are Killing fields, $\{E, L_z\}$ vanishes since these Killing fields commute, $\{Q, H\}$ vanishes since Q^{ab} is a Killing tensor, and finally $\{E, Q\}$ and $\{L_z, Q\}$ vanish since the Killing tensor is invariant under the flows generated by ξ^a and η^a. Therefore for generic orbits the theorem due to Fiorani et. al. discussed in the last subsection applies.[12] The relevant parameter values are $k = 3$ and $N = 4$, since the level sets $\mathcal{M}_{\mathbf{p}}$ are non-compact in the time direction only. Thus geodesic motion can be parameterized in terms of generalized action-angle variables.

We next discuss how to resolve in this context the non-uniqueness in the choice of generalized action angle variables discussed in the last subsection. Consider first the freedom (4.22) associated with the choice of generators of $\Pi_1(\mathcal{M}_{\mathbf{p}})$. One of these generators can be chosen to be an integral curve of the extension to $\mathcal{M}$ of the

[12] One can check that the two other assumptions in the theorem listed in the second paragraph of Sec. 4.2.2 are satisfied.

axial Killing field η^a. The other two can be chosen as follows. Let $\pi : \mathcal{M} \to \mathcal{M}_{\rm K}$ be the natural projection from phase space $\mathcal{M}$ to spacetime $\mathcal{M}_{\rm K}$ that takes (x^ν, p_ν) to x^ν. A loop $(x^\nu(\lambda), p_\nu(\lambda))$ in the level set $\mathcal{M}_{\rm p}$ then projects to the curve $x^\nu(\lambda)$ in $\pi(\mathcal{M}_{\rm p})$. Requiring that this curve intersect the boundary of $\pi(\mathcal{M}_{\rm p})$ only twice determines the two other generators of $\Pi_1(\mathcal{M}_{\mathbf{p}})$.[13] The resulting three generators coincide with the generators obtained from the motions in the r, θ and ϕ directions in Boyer-Lindquist coordinates [150]. We denote the resulting generalized action-angle variables by $(q_t, q_r, q_\theta, q_\phi, J_t, J_r, J_\theta, J_\phi)$.

The remaining ambiguity (4.23) is of the form

$$J_i \to J_i, \qquad J_t \to \gamma J_t + v^i J_i, \tag{4.30}$$

where i runs over r, θ and ϕ and the parameters γ and v^i are arbitrary. The corresponding transformation of the frequencies (4.26) is

$$\Omega_t \to \gamma^{-1}\Omega_t, \qquad \Omega_i \to \Omega_i - \gamma^{-1} v^i \Omega_t. \tag{4.31}$$

A portion of this ambiguity (the portion given by $\gamma = 1$, $v^r = v^\theta = 0$) is that associated with the choice of rotational frame, $\phi \to \phi + \Omega t$ where Ω is an angular velocity. It is not possible to eliminate this rotational-frame ambiguity using only the spacetime geometry in a neighborhood of the orbit. In this sense, the action angle variables are not uniquely determined by local geometric information. However, we can resolve the ambiguity using global geometric information, by choosing

$$J_t = \frac{1}{2\pi} \int_{\gamma_t} \mathbf{\Theta}, \tag{4.32}$$

where γ_t is an integral curve of length 2π of the extension to $\mathcal{M}$ of the timelike

[13] This excludes, for example, loops which wind around twice in the r direction and once in the θ direction.

Killing field ξ^a.[14] The definition (4.32) is independent of the choice of such a curve γ_t and of the choice of symplectic potential $\mathbf{\Theta}$.

To summarize, we have a given a coordinate-invariant definition of the generalized action-angle variables $(q_t, q_r, q_\theta, q_\phi, J_t, J_r, J_\theta, J_\phi)$ for generic bound orbits in Kerr. These variables are uniquely determined up to relabeling and up to the residual ambiguity (4.21). A similar construction has been given by Schmidt [150], except that Schmidt first projects out the time direction of the level sets, and then defines three action variables (J_r, J_θ, J_ϕ) and three angle variables (q_r, q_θ, q_ϕ).

4.2.4 Explicit expressions in terms of Boyer-Lindquist coordinates

In Boyer-Lindquist coordinates (t, r, θ, ϕ), the Kerr metric is

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 \\ &\quad + \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta\, d\phi^2 \\ &\quad - \frac{4Mar}{\Sigma}\sin^2\theta\, dt\, d\phi, \end{aligned} \tag{4.33}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2, \tag{4.34}$$

and M and a are the black hole mass and spin parameters. The timelike and axial Killing fields are $\vec{\xi} = \partial/\partial t$ and $\vec{\eta} = \partial/\partial\phi$, and so the energy and angular momentum are

$$E = -\vec{\xi}\cdot\vec{p} = -p_t \tag{4.35a}$$

[14] The Killing field ξ^a encodes global geometric information since it is defined to be timelike and of unit norm at spatial infinity.

and

$$L_z = \vec{\eta} \cdot \vec{p} = p_\phi. \tag{4.35b}$$

The Carter constant is given by [155]

$$Q = p_\theta^2 + a^2 \cos^2\theta \left(\mu^2 - p_t^2\right) + \cot^2\theta p_\phi^2, \tag{4.35c}$$

and the Hamiltonian (4.28) is

$$\begin{aligned} H &= \frac{\Delta}{2\Sigma} p_r^2 + \frac{1}{2\Sigma} p_\theta^2 + \frac{(p_\phi + a \sin^2\theta p_t)^2}{2\Sigma \sin^2\theta} \\ &\quad - \frac{\left[(r^2 + a^2)p_t + a p_\phi\right]^2}{2\Sigma\Delta}. \end{aligned} \tag{4.35d}$$

Following Schmidt [150], we can obtain an invertible transformation from the Boyer-Lindquist phase space coordinates (x^ν, p_ν) to the generalized action angle variables (q_α, J_α) as follows. Equations (4.35) can be inverted to express the momenta p_ν in terms of x^ν and the four first integrals

$$P_\alpha = (H, E, L_z, Q) = \left(-\frac{1}{2}\mu^2, E, L_z, Q\right) \tag{4.36}$$

up to some signs [155]:

$$p_t = -E, \quad p_\phi = L_z, \quad p_r = \pm\frac{\sqrt{V_r(r)}}{\Delta}, \quad p_\theta = \pm\sqrt{V_\theta(\theta)}. \tag{4.37}$$

Here the potentials $V_r(r)$ and $V_\theta(\theta)$ are defined by

$$\begin{aligned} V_r(r) &= \left[(r^2 + a^2)E - aL_z\right]^2 \\ &\quad - \Delta\left[\mu^2 r^2 + (L_z - aE)^2 + Q\right], \end{aligned} \tag{4.38a}$$

$$V_\theta(\theta) = Q - \left[(\mu^2 - E^2)a^2 + \frac{L_z^2}{\sin^2\theta}\right]\cos^2\theta. \tag{4.38b}$$

Using these formulae together with the symplectic potential $\mathbf{\Theta} = p_\nu dx^\nu$ in the definitions (4.18) and (4.32) gives

$$
\begin{aligned}
J_r &= \frac{1}{2\pi}\oint \frac{\sqrt{V_r}}{\Delta} dr && (4.39a)\\
J_\theta &= \frac{1}{2\pi}\oint \sqrt{V_\theta} d\theta && (4.39b)\\
J_\phi &= \frac{1}{2\pi}\oint p_\phi d\phi = L_z && (4.39c)\\
J_t &= \frac{1}{2\pi}\int_0^{2\pi} p_t dt = -E. && (4.39d)
\end{aligned}
$$

These expressions give the action variables as functions of the first integrals, $J_\alpha = J_\alpha(P_\beta)$. The theorem discussed in Sec. 4.2.2 above guarantees that these relations can be inverted to give

$$P_\alpha = P_\alpha(J_\beta). \tag{4.40}$$

Next, to obtain expressions for the corresponding generalized angle variables, we use the canonical transformation from the symplectic coordinates (x^ν, p_ν) to (q_α, J_α) associated with a general solution of the Hamilton Jacobi equation

$$H\left[x^\nu, \frac{\partial \mathcal{S}}{\partial x^\nu}\right] + \frac{\partial \mathcal{S}}{\partial \lambda} = 0. \tag{4.41}$$

As shown by Carter [155], this equation is separable and the general solution[15] can be written in terms of the first integrals P_α

$$\mathcal{S}(x^\nu, P_\alpha, \lambda) = -H\lambda + \mathcal{W}(x^\nu, P_\alpha) \tag{4.42}$$

where $H = -\mu^2/2$,

$$\mathcal{W}(x^\nu, P_\alpha) = -Et + L_z\phi \pm \mathcal{W}_r(r) \pm \mathcal{W}_\theta(\theta), \tag{4.43}$$

$$\mathcal{W}_r(r) = \int^r dr \frac{\sqrt{V_r}}{\Delta}, \tag{4.44}$$

[15]As indicated by the ± signs in Eq. (4.43), there are actually four different solutions, one on each of the four coordinate patches on which (x^ν, P_α) are good coordinates, namely $\mathrm{sgn}(p_r) = \pm 1$, $\mathrm{sgn}(p_\theta) = \pm 1$.

and

$$\mathcal{W}_\theta(\theta) = \int^\theta d\theta \sqrt{V_\theta}. \tag{4.45}$$

Using the relation (4.40) the function $\mathcal{W}$ can be expressed in terms of the Boyer-Lindquist coordinates x^ν and the action variables J_α:

$$\mathcal{W} = \mathcal{W}(x^\nu, J_\alpha). \tag{4.46}$$

This is a type II generating function that generates the required canonical transformation from (x^ν, p_ν) to (q_α, J_α):

$$p_\nu = \frac{\partial \mathcal{W}}{\partial x^\nu}(x^\nu, J_\beta) \tag{4.47a}$$

$$q_\alpha = \frac{\partial \mathcal{W}}{\partial J_\alpha}(x^\nu, J_\beta). \tag{4.47b}$$

Equation (4.47a) is already satisfied by virtue of the definition (4.43) of $\mathcal{W}$ together with Eqs. (4.37). Equation (4.47b) furnishes the required formulae for the generalized angle variables q_α.[16]

Although it is possible in principle to express the first integrals P_α in terms of the action variables J_α using Eqs. (4.39), it is not possible to obtain explicit analytic expressions for $P_\alpha(J_\beta)$. However, as pointed out by Schmidt [150], it is possible to obtain explicit expressions for the partial derivatives $\partial P_\alpha / \partial J_\beta$, and this is sufficient to compute the frequencies Ω_α. We review this in appendix 4.9.

[16]The freedom (4.21) to redefine the origin of the angle variables on each torus is just the freedom to add to $\mathcal{W}$ any function of P_α. We choose to resolve this freedom by demanding that $q_r = 0$ at the minimum value of r, and $q_\theta = 0$ at the minimum value of θ.

4.2.5 Application to slow inspiral motion in Kerr

The geodesic equations of motion in terms of the generalized action angle variables (q_α, J_α) are [cf. Eqs. (4.24) – (4.26) above]

$$\frac{dq_\alpha}{d\lambda} = \Omega_\alpha(J_\beta), \tag{4.48a}$$
$$\frac{dJ_\alpha}{d\lambda} = 0, \tag{4.48b}$$

for $0 \le \alpha \le 3$. Here $\lambda = \tau/\mu$ where τ is proper time and μ is the mass of the particle. In this section we derive the modifications to these equations required to describe the radiation-reaction driven inspiral of a particle in Kerr. Our result is of the form

$$\frac{dq_\alpha}{d\lambda} = \Omega_\alpha(J_\beta) + \mu^2 f_\alpha(q_\beta, J_\beta), \tag{4.49a}$$
$$\frac{dJ_\alpha}{d\lambda} = \mu^2 F_\alpha(q_\beta, J_\beta). \tag{4.49b}$$

We will derive explicit expressions for the forcing terms f_α and F_α in these equations.

The equation of motion for a particle subject to a self-acceleration a^ν is

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda} = \mu^2 a^\nu. \tag{4.50}$$

Rewriting this second order equation as two first order equations allows us to use the Jacobian of the coordinate transformation $\{x^\nu, p_\nu\} \to \{q_\alpha, J_\alpha\}$ to relate the forcing terms for the two sets of variables:

$$\frac{dx^\nu}{d\lambda} = g^{\nu\sigma}p_\sigma, \tag{4.51a}$$
$$\frac{dp_\nu}{d\lambda} = -\frac{1}{2}g^{\sigma\rho}{}_{,\nu}p_\sigma p_\rho + \mu^2 a_\nu. \tag{4.51b}$$

We start by deriving the equation of motion for the action variables J_α. Taking a derivative with respect to λ of the relation $J_\alpha = J_\alpha(x^\nu, p_\nu)$ and using Eqs. (4.51) gives

$$\begin{aligned} \frac{dJ_\alpha}{d\lambda} &= \frac{\partial J_\alpha}{\partial x^\nu} p^\nu + \frac{\partial J_\alpha}{\partial p_\nu}\frac{dp_\nu}{d\lambda} \\ &= \left[\frac{\partial J_\alpha}{\partial x^\nu} g^{\nu\sigma} p_\sigma - \frac{1}{2}\frac{\partial J_\alpha}{\partial p_\nu} g^{\sigma\rho}{}_{,\nu} p_\sigma p_\rho\right] \\ &\quad + \mu^2 \frac{\partial J_\alpha}{\partial p_\nu} a_\nu. \end{aligned} \tag{4.52}$$

The term in square brackets must vanish identically since J_α is conserved in the absence of any acceleration a_ν. Rewriting the second term using $J_\alpha = J_\alpha(P_\beta)$ and the chain rule gives an equation of motion of the form (4.49b), where the forcing terms F_α are

$$F_\alpha = \frac{\partial J_\alpha}{\partial P_\beta}\left(\frac{\partial P_\beta}{\partial p_\nu}\right)_x a_\nu. \tag{4.53}$$

Here the subscript x on the round brackets means that the derivative is to be taken holding x^ν fixed. When the sum over β is evaluated the contribution from $P_1 = H$ vanishes since $a_\nu p^\nu = 0$, and we obtain using Eqs. (4.29) and (4.39)

$$F_t = a_t, \tag{4.54a}$$

$$F_r = -\frac{\partial J_r}{\partial E} a_t + \frac{\partial J_r}{\partial Q} a_Q + \frac{\partial J_r}{\partial L_z} a_\phi, \tag{4.54b}$$

$$F_\theta = -\frac{\partial J_\theta}{\partial E} a_t + \frac{\partial J_\theta}{\partial Q} a_Q + \frac{\partial J_\theta}{\partial L_z} a_\phi, \tag{4.54c}$$

$$F_\phi = a_\phi. \tag{4.54d}$$

Here we have defined $a_Q = 2Q^{\nu\sigma} p_\nu a_\sigma$ and the various coefficients $\partial J_\alpha / \partial P_\beta$ are given explicitly as functions of P_α in Appendix 4.9.

We use a similar procedure to obtain the equation of motion (4.49a) for the generalized angle variables q_α. Differentiating the relation $q_\alpha = q_\alpha(x^\nu, p_\nu)$ with respect to λ and combining with the two first order equations of motion (4.51) gives

$$\begin{aligned}\frac{dq_\alpha}{d\lambda} &= \left[\frac{\partial q_\alpha}{\partial x^\nu} g^{\nu\sigma} p_\sigma - \frac{1}{2}\frac{\partial q_\alpha}{\partial p_\nu} g^{\sigma\rho}{}_{,\nu} p_\sigma p_\rho\right] \\ &+\mu^2 \frac{\partial q_\alpha}{\partial p_\nu} a_\nu. \end{aligned} \tag{4.55}$$

By comparing with Eq. (4.48a) in the case of vanishing acceleration we see that the term in square brackets is $\Omega_\alpha(J_\beta)$. This gives an equation of motion of the form (4.49a), where the where the forcing term f_α is

$$f_\alpha = \left(\frac{\partial q_\alpha}{\partial p_\nu}\right)_x a_\nu. \tag{4.56}$$

Using the expression (4.47b) for the angle variable q_α together with $J_\alpha = J_\alpha(P_\beta)$ gives

$$\begin{aligned}\left(\frac{\partial q_\alpha}{\partial p_\nu}\right)_x &= \left(\frac{\partial P_\gamma}{\partial p_\nu}\right)_x \left[\frac{\partial P_\beta}{\partial J_\alpha}\left(\frac{\partial^2 \mathcal{W}}{\partial P_\beta \partial P_\gamma}\right)_x \right. \\ &\left. + \left(\frac{\partial \mathcal{W}}{\partial P_\beta}\right)_x \frac{\partial}{\partial P_\gamma}\left(\frac{\partial P_\beta}{\partial J_\alpha}\right)\right]. \end{aligned} \tag{4.57}$$

This yields for the forcing term

$$\begin{aligned}f_\alpha &= a_\nu \left(\frac{\partial P_\gamma}{\partial p_\nu}\right)_x \frac{\partial P_\delta}{\partial J_\alpha}\left[\left(\frac{\partial^2 \mathcal{W}}{\partial P_\delta \partial P_\gamma}\right)_x \right. \\ &\left. - \left(\frac{\partial \mathcal{W}}{\partial P_\beta}\right)_x \frac{\partial P_\beta}{\partial J_\varepsilon}\frac{\partial^2 J_\varepsilon}{\partial P_\gamma \partial P_\delta}\right]. \end{aligned} \tag{4.58}$$

In this expression the first two factors are the same as the factors which appeared in the forcing term (4.53) for the action variables. The quantities $\partial P_\delta/\partial J_\alpha$, $\partial P_\beta/\partial J_\varepsilon$ and $\partial^2 J_\varepsilon/(\partial P_\gamma \partial P_\delta)$ can be evaluated explicitly as functions of P_α using the techniques discussed in Appendix 4.9. The remaining factors in Eq. (4.58) can be evaluated by differentiating the formula (4.43) for Hamilton's principal function $\mathcal{W}$ and using the formulae (4.38) for the potentials V_r and V_θ.

4.2.6 Rescaled variables and incorporation of backreaction on the black hole

We now augment the action-angle equations of motion (4.49) in order to describe the backreaction of the gravitational radiation on the black hole. We also modify the equations to simplify and make explicit the dependence on the mass μ of the particle. The resulting modified equations of motion, whose solutions we will analyze in the remainder of the paper, are

$$\begin{aligned}
\frac{dq_\alpha}{d\tau} &= \omega_\alpha(\tilde{P}_j, M_B) + \varepsilon g_\alpha^{(1)}(q_A, \tilde{P}_j, M_B) \\
&\quad + \varepsilon^2 g_\alpha^{(2)}(q_A, \tilde{P}_j, M_B) + O(\varepsilon^3), && \text{(4.59a)} \\
\frac{d\tilde{P}_i}{d\tau} &= \varepsilon G_i^{(1)}(q_A, \tilde{P}_j, M_A) + \varepsilon^2 G_i^{(2)}(q_A, \tilde{P}_j, M_B) \\
&\quad + O(\varepsilon^3), && \text{(4.59b)} \\
\frac{dM_A}{d\tau} &= \varepsilon^2 \hat{G}_A(q_A, \tilde{P}_j, M_B) + O(\varepsilon^3). && \text{(4.59c)}
\end{aligned}$$

Here α runs over $0, 1, 2, 3$, i, j run over $1, 2, 3$, A, B run over $1, 2$, $q_A = (q_r, q_\theta)$, $M_A = (M_1, M_2)$ and $\tilde{P}_i = (\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)$. Also all of the functions ω_α, $g_\alpha^{(1)}$, $g_\alpha^{(2)}$, $G_i^{(1)}$, $G_i^{(2)}$ and $\hat{G}_A$ that appear on the right hand sides are smooth functions of their arguments whose precise form will not be needed for this paper (and are currently unknown aside from ω_α).

Our final equations (4.59) are similar in structure to the original equations (4.49), but there are a number of differences:

- We have switched the independent variable in the differential equations from affine parameter λ to proper time $\tau = \mu\lambda$.
- We have introduced the ratio

$$\varepsilon = \frac{\mu}{M} \qquad (4.60)$$

of the particle mass μ and black hole mass M, and have expanded the forcing terms as a power series in ε.

- The forcing terms $g_\alpha^{(1)}$, $g_\alpha^{(2)}$, $G_i^{(1)}$, $G_i^{(2)}$, and $\hat{G}_A$ depend only on the two angle variables $q_A \equiv (q_r, q_\theta)$, and are independent of q_t and q_ϕ.

- Rather than evolving the action variables J_α, we evolve two different sets of variables, $\tilde{P}_i$ and M_A. The first of these sets consists of three of the first integrals of the motion, with the dependence on the mass μ of the particle scaled out:

$$\tilde{P}_i = (\tilde{P}_1, \tilde{P}_2, \tilde{P}_3) \equiv (E/\mu, L_z/\mu, Q/\mu^2). \tag{4.61}$$

  The second set consists of the mass and spin parameters of the black hole, which gradually evolve due to absorption of gravitational radiation by the black hole:

$$M_A = (M_1, M_2) = (M, a). \tag{4.62}$$

We now turn to a derivation of the modified equations of motion (4.59). The derivation consists of several steps. First, since the mapping (4.39) between the first integrals P_α and the action variables J_α is a bijection, we can use the P_α as dependent variables instead of J_α.[17] Equation (4.49a) is unmodified except that the right hand side is expressed as a function of P_α instead of J_α. Equation (4.49b) is replaced by

$$\frac{dP_\alpha}{d\lambda} = \mu^2 \left(\frac{\partial P_\alpha}{\partial p_\nu}\right)_x a_\nu. \tag{4.63a}$$

Second, we switch to using modified versions $\tilde{P}_\alpha$ of the first integrals P_α with the dependence on the mass μ scaled out. These rescaled first integrals are defined

[17]Note that since the variables J_α are adiabatic invariants, so are the variables P_α.

by

$$\begin{aligned}\tilde{P}_\alpha &= (\tilde{H}, \tilde{E}, \tilde{L}_z, \tilde{Q}) \\ &\equiv (H/\mu^2, E/\mu, L_z/\mu, Q/\mu^2). \end{aligned} \tag{4.64}$$

We also change the independent variable from affine parameter λ to proper time $\tau = \mu\lambda$. This gives from Eqs. (4.49) and (4.56) the system of equations

$$\frac{dq_\alpha}{d\tau} = \frac{1}{\mu}\Omega_\alpha(P_\beta) + \mu \left(\frac{\partial q_\alpha}{\partial p_\nu}\right)_x a_\nu, \tag{4.65a}$$

$$\frac{d\tilde{P}_\alpha}{d\tau} = \mu^{1-n_\alpha} \left(\frac{\partial P_\alpha}{\partial p_\nu}\right)_x a_\nu, \tag{4.65b}$$

where we have defined $n_\alpha = (2, 1, 1, 2)$.

Third, we analyze the dependence on the mass μ of the right hand sides of these equations. Under the transformation $(x^\nu, p_\nu) \to (x^\nu, sp_\nu)$ for $s > 0$, we obtain the following transformation laws for the first integrals (4.36), the action variables (4.39), and Hamilton's principal function (4.43):

$$P_\alpha \to s^{n_\alpha} P_\alpha \quad \text{with} \quad n_\alpha = (2, 1, 1, 2), \tag{4.66a}$$

$$J_\alpha \to sJ_\alpha, \tag{4.66b}$$

$$\mathcal{W} \to s\mathcal{W}. \tag{4.66c}$$

From the definitions (4.26) and (4.47b) of the angular frequencies Ω_α and the angle variables q_α we also deduce

$$\Omega_\alpha \to s\Omega_\alpha, \tag{4.67a}$$

$$q_\alpha \to q_\alpha. \tag{4.67b}$$

If we write the angular velocity Ω_α as a function $\omega_\alpha(P_\beta)$ of the first integrals P_β, then it follows from the scalings (4.66a) and (4.67a) that the first term on the right hand side of Eq. (4.65a) is

$$\frac{\Omega_\alpha}{\mu} = \frac{\omega_\alpha(P_\beta)}{\mu} = \frac{\omega_\alpha(\mu^{n_\beta}\tilde{P}_\beta)}{\mu} = \omega_\alpha(\tilde{P}_\beta). \tag{4.68}$$

This quantity is thus independent of μ at fixed $\tilde{P}_\beta$, as we would expect.

Similarly, if we write the angle variable q_α as a function $\bar{q}_\alpha(x^\nu, p_\nu)$ of x^ν and p_ν, then the scaling law (4.67b) implies that $\bar{q}_\alpha(x^\nu, sp_\nu) = \bar{q}_\alpha(x^\nu, p_\nu)$, and it follows that the coefficient of the 4-acceleration in Eq. (4.65a) is [18]

$$\mu \frac{\partial \bar{q}_\alpha}{\partial p_\nu}(x^\sigma, p_\sigma) = \mu \frac{\partial \bar{q}_\alpha}{\partial p_\nu}(x^\sigma, \mu u_\sigma) = \frac{\partial \bar{q}_\alpha}{\partial p_\nu}(x^\sigma, u_\sigma), \tag{4.69}$$

where u_σ is the 4-velocity. This quantity is also independent of μ at fixed $\tilde{P}_\beta$. We will denote this quantity by $f^\nu_\alpha(q_\beta, \tilde{P}_\beta)$. It can be obtained explicitly by evaluating the coefficient of a_ν in Eq. (4.58) at $P_\alpha = \tilde{P}_\alpha$, $p_\nu = u_\nu$. A similar analysis shows that the driving term on the right hand side of Eq. (4.65b) can be written in the form

$$F^\nu_\alpha(q_\beta, \tilde{P}_\beta) a_\nu \equiv (0, -a_t, a_\phi, 2Q^{\nu\sigma} u_\nu a_\sigma). \tag{4.70}$$

The resulting rescaled equations of motion are

$$\begin{aligned} \frac{dq_\alpha}{d\tau} &= \omega_\alpha(\tilde{P}_\beta) + f^\nu_\alpha(q_\beta, \tilde{P}_\beta) a_\nu, && (4.71a) \\ \frac{d\tilde{P}_\alpha}{d\tau} &= F^\nu_\alpha(q_\beta, \tilde{P}_\beta) a_\nu. && (4.71b) \end{aligned}$$

Note that this formulation of the equations is completely independent of the mass μ of the particle (except for the dependence on μ of the radiation reaction acceleration a_ν which we will discuss below).

Fourth, since $P_0 = H = -\mu^2/2$, the rescaled variable is $\tilde{P}_0 = -1/2$ from Eq. (4.64). Thus we can drop the evolution equation for $\tilde{P}_0$, and retain only the equations for the remaining rescaled first integrals

$$\tilde{P}_i = (\tilde{P}_1, \tilde{P}_2, \tilde{P}_3) = (\tilde{E}, \tilde{L}_z, \tilde{Q}). \tag{4.72}$$

[18] Note that $\mu\, \partial/\partial p_\nu$ cannot be simplified to $\partial/\partial u_\nu$ because we are working in the eight dimensional phase space $\mathcal{M}$ where μ is a coordinate and not a constant.

We can also omit the dependence on $\tilde{P}_0$ in the right hand sides of the evolution equations (4.71), since $\tilde{P}_0$ is a constant. This yields

$$\frac{dq_\alpha}{d\tau} = \omega_\alpha(\tilde{P}_j) + f_\alpha^\nu(q_\beta, \tilde{P}_j)a_\nu, \tag{4.73a}$$

$$\frac{d\tilde{P}_i}{d\tau} = F_i^\nu(q_\beta, \tilde{P}_j)a_\nu. \tag{4.73b}$$

Fifth, the self-acceleration of the particle can be expanded in powers of the mass ratio $\varepsilon = \mu/M$ as

$$a_\nu = \varepsilon a_\nu^{(1)} + \varepsilon^2 a_\nu^{(2)} + O(\varepsilon^3). \tag{4.74}$$

Here $a_\nu^{(1)}$ is the leading order self-acceleration derived by Mino, Sasaki and Tanaka [106] and by Quinn and Wald [107], discussed in the introduction. The subleading self-acceleration $a_\nu^{(2)}$ has been computed in Refs. [143, 144, 145, 146, 147]. The accelerations $a_\nu^{(1)}$ and $a_\nu^{(2)}$ are independent of μ and thus depend only on x^ν and u_ν, or, equivalently, on q_α and $\tilde{P}_i$. This yields the system of equations

$$\frac{dq_\alpha}{d\tau} = \omega_\alpha(\tilde{P}_j) + \varepsilon g_\alpha^{(1)}(q_\beta, \tilde{P}_j) + \varepsilon^2 g_\alpha^{(2)}(q_\beta, \tilde{P}_j) + O(\varepsilon^3), \tag{4.75a}$$

$$\frac{d\tilde{P}_i}{d\tau} = \varepsilon G_i^{(1)}(q_\beta, \tilde{P}_j) + \varepsilon^2 G_i^{(2)}(q_\beta, \tilde{P}_j) + O(\varepsilon^3). \tag{4.75b}$$

Here the forcing terms are given by

$$g_\alpha^{(s)} = f_\alpha^\nu a_\nu^{(s)}, \tag{4.76a}$$

$$G_i^{(s)} = F_i^\nu a_\nu^{(s)}, \tag{4.76b}$$

for $s = 1, 2$.

The formula (4.74) for the self-acceleration, with the explicit formula for $a_\nu^{(1)}$ from Refs. [106, 107], is valid when one chooses the Lorentz gauge for the metric perturbation. The form of Eq. (4.74) is also valid in a variety of other gauges; see Ref. [156] for a discussion of the gauge transformation properties of the self force. However, there exist gauge choices which are incompatible with Eq. (4.74), which can be obtained by making ε-dependent gauge transformations. We shall restrict attention to classes of gauges which are consistent with our ansatz (6.323) for the metric, as discussed in Sec. 4.1.5 above. This class of gauges has the properties that (i) the deviation of the metric from Kerr is $\lesssim \varepsilon$ over the entire inspiral, and (ii) the expansion (4.74) of the self-acceleration is valid. These restrictions exclude, for example, the gauge choice which makes $a_\nu^{(1)} \equiv 0$, since in that gauge the particle does not inspiral, and the metric perturbation must therefore become of order unity over an inspiral time. We note that alternative classes of gauges have been suggested and explored by Mino [131, 117, 142, 140].

Sixth, from the formula (4.47b) for the generalized angle variables q_α together with Eqs. (4.43) and (4.39d) it follows that q_t can be written as

$$q_t = t + f_t(r, \theta, P_\alpha) \tag{4.77}$$

for some function f_t. All of the other angle and action variables are independent of t. Therefore the vector field $\partial/\partial t$ on phase space is just $\partial/\partial q_t$; the symmetry $t \to t + \Delta t$ with x^i, p_μ fixed is the same as the symmetry $q_t \to q_t + \Delta t$ with q_r, q_θ, q_ϕ and J_α fixed. Since the self-acceleration as well as the background geodesic motion respect this symmetry, all of the terms on the right hand side of Eqs. (4.75) must be independent of q_t. A similar argument shows that they are independent of q_ϕ.

This gives

$$\begin{aligned}
\frac{dq_\alpha}{d\tau} &= \omega_\alpha(\tilde{P}_j) + \varepsilon g^{(1)}_\alpha(q_A, \tilde{P}_j) + \varepsilon^2 g^{(2)}_\alpha(q_A, \tilde{P}_j) \\
&\quad + O(\varepsilon^3), && (4.78a) \\
\frac{d\tilde{P}_i}{d\tau} &= \varepsilon G^{(1)}_i(q_A, \tilde{P}_j) + \varepsilon^2 G^{(2)}_i(q_A, \tilde{P}_j) \\
&\quad + O(\varepsilon^3), && (4.78b)
\end{aligned}$$

where $q_A \equiv (q_r, q_\theta)$.

Seventh, consider the evolution of the black hole background. So far in our analysis we have assumed that the particle moves in a fixed Kerr background, and is subject to a self-force $a_\nu = \varepsilon a^{(1)}_\nu + \varepsilon^2 a^{(2)}_\nu + O(\varepsilon^3)$. In reality, the center of mass, 4-momentum and spin angular momentum of the black hole will gradually evolve due to the gravitational radiation passing through the event horizon. The total change in the mass M of the black hole over the inspiral timescale $\sim M/\varepsilon$ is $\sim M\varepsilon$. It follows that the timescale for the black hole mass to change by a factor of order unity is $\sim M/\varepsilon^2$. The same timescale governs the evolution of the other black hole parameters.

This effect of evolution of the black hole background will alter the inspiral at the first subleading order (post-1-adiabatic order) in our two-timescale expansion. A complete calculation of the inspiral to this order requires solving simultaneously for the motion of the particle and the gradual evolution of the background. We introduce the extra variables

$$M_A = (M_1, M_2) = (M, a), \tag{4.79}$$

the mass and spin parameters of the black hole. We modify the equations of motion (4.78) by showing explicitly the dependence of the frequencies ω_α and the forcing functions $g^{(n)}_\alpha$ and $G^{(n)}_i$ on these parameters (the dependence has up to now been implicit). We also add to the system of equations the following evolution equations for the black hole parameters:

$$\frac{dM_A}{d\tau} = \varepsilon^2 \hat{G}_A(q_B, \tilde{P}_j, M_B) + O(\varepsilon^3), \tag{4.80}$$

where $A = 1, 2$. Here $\hat{G}_A$ are some functions describing the fluxes of energy and angular momentum down the horizon, whose explicit form will not be important for our analyses. They can in principle be computed using, for example, the techniques developed in Ref. [157].[19] The reason for the prefactor of ε^2 is that the evolution timescale for the black hole parameters is $\sim M/\varepsilon^2$, as discussed above. The functions $\hat{G}_A$ are independent of q_t and q_ϕ for the reason discussed near Eq. (4.78): the fluxes through the horizon respect the symmetries of the background spacetime. Finally, we have omitted in the set of new variables (4.79) the orientation of the total angular momentum, the location of the center of mass, and the total linear momentum of the system, since these parameters are not coupled to the inspiral motion at the leading order. However, it would be possible to enlarge the set of variables M_A to include these parameters without modifying in any way the analyses in the rest of this paper.

These modifications result in the final system of equations (4.59).

Finally we note that an additional effect arises due to the fact that the action-angle variables we use are defined, at each instant, to be the action-angle variables associated with the black hole background at that time. In other words the coordi-

[19] These techniques naturally furnish the derivatives of M_A with respect to Boyer Lindquist time t, not proper time τ as in Eq. (4.80). However this difference is unimportant; one can easily convert from one variable to the other by multiplying the functions $\hat{G}_A$ by the standard expression for $dt/d\tau$ [1],

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{\Sigma}\left(\frac{\varpi^4}{\Delta} - a^2 \sin^2\theta\right) + \frac{a\tilde{L}_z}{\Sigma}\left(1 - \frac{\varpi^2}{\Delta}\right),$$

where $\varpi = \sqrt{r^2 + a^2}$. This expression can be written in terms of of $q_A, \tilde{P}_i$ and M_A, and is valid for accelerated motion as well as geodesic motion by Eqs. (4.37) and (4.51a).

nate transformation on phase space from $(x^\nu, p_\nu) \to (q_\alpha, J_\alpha)$ acquires an additional dependence on time. Therefore the Jacobian of this transformation, which was used in deriving the evolution equations (4.49), has an extra term. However, the corresponding correction to the evolution equations can be absorbed into a redefinition of the forcing term $g^{(2)}_\alpha$.

4.2.7 Conservative and dissipative pieces of the forcing terms

In this subsection we define a splitting of the forcing terms g_α and G_i in the equations of motion (4.59) into conservative and dissipative pieces, and review some properties of this decomposition derived by Mino [128].

We start by defining some notation. Suppose that we have a particle at a point $\mathcal{P}$ with four velocity u^μ, and that we are given a linearized metric perturbation $h_{\mu\nu}$ which is a solution (not necessarily the retarded solution) of the linearized Einstein equation equation for which the source is a delta function on the geodesic determined by $\mathcal{P}$ and u^μ. The self-acceleration of the particle is then some functional of $\mathcal{P}$, u^μ, $h_{\mu\nu}$ and of the spacetime metric $g_{\mu\nu}$, which we write as

$$a^\mu \left[\mathcal{P}, u^\mu, g_{\mu\nu}, h_{\mu\nu}\right]. \tag{4.81}$$

Note that this functional does not depend on a choice of time orientation for the manifold, and also it is invariant under $u^\mu \to -u^\mu$. The retarded self-acceleration is defined as

$$a^\mu_{\rm ret} \left[\mathcal{P}, u^\mu, g_{\mu\nu}\right] = a^\mu \left[\mathcal{P}, u^\mu, g_{\mu\nu}, h^{\rm ret}_{\mu\nu}\right], \tag{4.82}$$

where $h^{\rm ret}_{\mu\nu}$ is the retarded solution to the linearized Einstein equation obtained using the time orientation that is determined by demanding that u^μ be future directed. This is the physical self-acceleration which is denoted by a^μ throughout the rest of this paper. Similarly, the advanced self-acceleration is

$$a^\mu_{\rm adv}\left[\mathcal{P}, u^\mu, g_{\mu\nu}\right] = a^\mu\left[\mathcal{P}, u^\mu, g_{\mu\nu}, h^{\rm adv}_{\mu\nu}\right], \tag{4.83}$$

where $h^{\rm adv}_{\mu\nu}$ is the advanced solution. It follows from these definitions that

$$a^\mu_{\rm ret}\left[\mathcal{P}, -u^\mu, g_{\mu\nu}\right] = a^\mu_{\rm adv}\left[\mathcal{P}, u^\mu, g_{\mu\nu}\right]. \tag{4.84}$$

We define the conservative and dissipative self-accelerations to be

$$a^\mu_{\rm cons} = \frac{1}{2}\left(a^\mu_{\rm ret} + a^\mu_{\rm adv}\right), \tag{4.85}$$

and

$$a^\mu_{\rm diss} = \frac{1}{2}\left(a^\mu_{\rm ret} - a^\mu_{\rm adv}\right). \tag{4.86}$$

The physical self-acceleration can then be decomposed as

$$a^\mu = a^\mu_{\rm ret} = a^\mu_{\rm cons} + a^\mu_{\rm diss}. \tag{4.87}$$

A similar decomposition applies to the forcing functions (4.76):

$$g^{(s)}_\alpha = g^{(s)}_{\alpha\,{\rm cons}} + g^{(s)}_{\alpha\,{\rm diss}}, \tag{4.88a}$$

$$G^{(s)}_i = G^{(s)}_{i\,{\rm cons}} + G^{(s)}_{i\,{\rm diss}}, \tag{4.88b}$$

for $s = 1, 2$.

Next, we note that if ψ is any diffeomorphism from the spacetime to itself, then the self acceleration satisfies the covariance relation

$$a^\nu_{\rm ret}[\psi(\mathcal{P}), \psi^* u^\nu, \psi^* g_{\mu\nu}] = \psi^* a^\nu_{\rm ret}[\mathcal{P}, u^\nu, g_{\mu\nu}]. \tag{4.89}$$

Taking the point $\mathcal{P}$ to be $(t_0, r_0, \theta_0, \phi_0)$ in Boyer-Lindquist coordinates, and choosing ψ to be $t \to 2t_0 - t$, $\phi \to 2\phi_0 - \phi$, then ψ is an isometry, $\psi^* g_{\mu\nu} = g_{\mu\nu}$. It follows that

$$a^{\nu}_{\rm ret}(-u_t, u_r, u_\theta, -u_\phi) = -\epsilon_\nu a^{\nu}_{\rm ret}(u_t, u_r, u_\theta, u_\phi), \tag{4.90}$$

where

$$\epsilon_\nu = (1, -1, -1, 1) \tag{4.91}$$

and there is no summation over ν on the right hand side. Combining this with the identity (4.84) gives

$$a^{\nu}_{\rm adv}(u_t, u_r, u_\theta, u_\phi) = -\epsilon_\nu a^{\nu}_{\rm ret}(u_t, -u_r, -u_\theta, u_\phi). \tag{4.92}$$

Now, under the transformation $p_r \to -p_r$, $p_\theta \to -p_\theta$ with other quantities fixed, the action variables and the quantities P_α are invariant, the angle variables q_r and q_θ transform as $q_r \to 2\pi - q_r$, $q_\theta \to 2\pi - q_\theta$, while $q_t - t$ and $q_\phi - \phi$ flip sign. This can be seen from the definitions (4.43) and (4.47b). Explicitly we have

$$\bar{q}_t(x^\gamma, \epsilon_\delta p_\delta) - t = -[\bar{q}_t(x^\gamma, p_\delta) - t], \tag{4.93a}$$

$$\bar{q}_\phi(x^\gamma, \epsilon_\delta p_\delta) - \phi = -[\bar{q}_\phi(x^\gamma, p_\delta) - \phi], \tag{4.93b}$$

$$\bar{q}_A(x^\gamma, \epsilon_\delta p_\delta) = 2\pi - \bar{q}_A(x^\gamma, p_\delta), \tag{4.93c}$$

$$P_i(x^\gamma, \epsilon_\delta p_\delta) = P_i(x^\gamma, p_\delta), \tag{4.93d}$$

where we use the values (4.91) of ϵ_α, the functions $\bar{q}_\alpha$ are defined before Eq. (4.69), and $q_A = (q_r, q_\theta)$. If we now differentiate with respect to p_α holding x^α fixed and use the definitions (4.69), (4.65b) and (4.71b) of the functions f^{ν}_{α} and F^{ν}_{i} we obtain

$$f^{\nu}_{\alpha}(x^\beta, \epsilon_\gamma u_\gamma) = -\epsilon_\nu f^{\nu}_{\alpha}(x^\beta, u_\gamma), \tag{4.94a}$$

$$F^{\nu}_{i}(x^\beta, \epsilon_\gamma u_\gamma) = \epsilon_\nu F^{\nu}_{i}(x^\beta, u_\gamma). \tag{4.94b}$$

We now compute the conservative and dissipative pieces of the forcing functions $g_\alpha^{(1)}$ and $G_i^{(1)}$, using the definitions (4.76) and (4.88). Using the results (4.92) and (4.94) we obtain

$$\begin{aligned} g^{(1)}_{\alpha\,\mathrm{adv}}(u_\gamma) &= f^\nu_\alpha(u_\gamma)\, a^{(1)}_{\nu\,\mathrm{adv}}(u_\gamma) \\ &= \left[-\epsilon_\nu f^\nu_\alpha(\epsilon_\gamma u_\gamma)\right] \left[-\epsilon_\nu a^{(1)}_{\nu\,\mathrm{ret}}(\epsilon_\gamma u_\gamma)\right] \\ &= g^{(1)}_{\alpha\,\mathrm{ret}}(\epsilon_\gamma u_\gamma). \end{aligned} \tag{4.95}$$

A similar computation gives

$$G^{(1)}_{i\,\mathrm{adv}}(u_\gamma) = -G^{(1)}_{i\,\mathrm{ret}}(\epsilon_\gamma u_\gamma), \tag{4.96}$$

and using that the mapping $x^\nu \to x^\nu$, $u_\mu \to \epsilon_\mu u_\mu$ corresponds to $\tilde{P}_j \to \tilde{P}_j$, $q_r \to 2\pi - q_r$, $q_\theta \to 2\pi - q_\theta$ finally yields the identities

$$g^{(1)}_{\alpha\,\mathrm{cons}}(q_A, \tilde{P}_j) = \left[g^{(1)}_\alpha(q_r, q_\theta, \tilde{P}_j) + g^{(1)}_\alpha(2\pi - q_r, 2\pi - q_\theta, \tilde{P}_j)\right]/2, \tag{4.97a}$$

$$g^{(1)}_{\alpha\,\mathrm{diss}}(q_A, \tilde{P}_j) = \left[g^{(1)}_\alpha(q_r, q_\theta, \tilde{P}_j) - g^{(1)}_\alpha(2\pi - q_r, 2\pi - q_\theta, \tilde{P}_j)\right]/2, \tag{4.97b}$$

and

$$G^{(1)}_{i\,\mathrm{cons}}(q_A, \tilde{P}_j) = \left[G^{(1)}_i(q_r, q_\theta, \tilde{P}_j) - G^{(1)}_i(2\pi - q_r, 2\pi - q_\theta, \tilde{P}_j)\right]/2, \tag{4.98a}$$

$$G^{(1)}_{i\,\mathrm{diss}}(q_A, \tilde{P}_j) = \left[G^{(1)}_i(q_r, q_\theta, \tilde{P}_j) + G^{(1)}_i(2\pi - q_r, 2\pi - q_\theta, \tilde{P}_j)\right]/2. \tag{4.98b}$$

Here we have used the fact that the forcing functions are independent of q_t and q_ϕ, as discussed in the last subsection. Similar equations apply with $g_\alpha^{(1)}$ and $G_i^{(1)}$ replaced by the higher order forcing terms $g_\alpha^{(s)}$ and $G_i^{(s)}$, $s \geq 2$.

It follows from the identity (4.98a) that, for the action-variable forcing functions $G_i^{(1)}$, the average over the 2-torus parameterized by q_r and q_θ of the conservative piece vanishes. For generic orbits (for which ω_r and ω_θ are incommensurate), the torus-average is equivalent to a time average, and so it follows that the time average vanishes, a result first derived by Mino [128]. Similarly from Eqs. (4.97) it follows that the torus-average of the dissipative pieces of $g_\alpha^{(1)}$ vanish.

4.3 A general weakly perturbed dynamical system

In the remainder of this paper we will study in detail the behavior of a one-parameter family of dynamical systems parameterized by a dimensionless parameter ε. We shall be interested in the limiting behavior of the systems as $\varepsilon \to 0$. The system contains $N + M$ dynamical variables

$$\mathbf{q}(t) = \big(q_1(t), q_2(t), \ldots, q_N(t)\big), \tag{4.99a}$$

$$\mathbf{J}(t) = \big(J_1(t), J_2(t), \ldots, J_M(t)\big), \tag{4.99b}$$

and is defined by the equations

$$\frac{dq_\alpha}{dt} = \omega_\alpha(\mathbf{J}, \tilde{t}) + \varepsilon g_\alpha(\mathbf{q}, \mathbf{J}, \tilde{t}, \varepsilon), \quad 1 \le \alpha \le N, \tag{4.100a}$$

$$\frac{dJ_\lambda}{dt} = \varepsilon G_\lambda(\mathbf{q}, \mathbf{J}, \tilde{t}, \varepsilon), \quad 1 \le \lambda \le M. \tag{4.100b}$$

Here the variable $\tilde{t}$ is the "slow time" variable defined by

$$\tilde{t} = \varepsilon t. \tag{4.101}$$

We assume that the functions g_α and G_λ can be expanded as

$$\begin{aligned} g_\alpha(\mathbf{q}, \mathbf{J}, \tilde{t}, \varepsilon) &= \sum_{s=1}^{\infty} g_\alpha^{(s)}(\mathbf{q}, \mathbf{J}, \tilde{t}) \varepsilon^{s-1} \\ &= g_\alpha^{(1)}(\mathbf{q}, \mathbf{J}, \tilde{t}) + g_\alpha^{(2)}(\mathbf{q}, \mathbf{J}, \tilde{t})\varepsilon + O(\varepsilon^2), \end{aligned} \tag{4.102}$$

and

$$\begin{aligned} G_\lambda(\mathbf{q}, \mathbf{J}, \tilde{t}, \varepsilon) &= \sum_{s=1}^{\infty} G_\lambda^{(s)}(\mathbf{q}, \mathbf{J}, \tilde{t}) \varepsilon^{s-1} \\ &= G_\lambda^{(1)}(\mathbf{q}, \mathbf{J}, \tilde{t}) + G_\lambda^{(2)}(\mathbf{q}, \mathbf{J}, \tilde{t})\varepsilon + O(\varepsilon^2). \end{aligned} \tag{4.103}$$

These series are assumed to be asymptotic series in ε as $\varepsilon \to 0$ that are uniform in $\tilde{t}$.[20] We assume that the functions ω_α, $g_\alpha^{(s)}$ and $G_\lambda^{(s)}$ are smooth functions of their arguments, and that the frequencies ω_α are nowhere vanishing. Finally the functions g_α and G_λ are assumed to be periodic in each variable q_α with period 2π:

$$\begin{aligned} g_\alpha(\mathbf{q}+2\pi\mathbf{k},\mathbf{J},\tilde{t}) &= g_\alpha(\mathbf{q},\mathbf{J},\tilde{t}), \quad 1 \le \alpha \le N, & (4.104a) \\ G_\lambda(\mathbf{q}+2\pi\mathbf{k},\mathbf{J},\tilde{t}) &= G_\lambda(\mathbf{q},\mathbf{J},\tilde{t}), \quad 1 \le \lambda \le M, & (4.104b) \end{aligned}$$

where $\mathbf{k} = (k_1, \ldots, k_N)$ is an arbitrary N-tuple of integers.

The equations (4.59) derived in the previous section describing the inspiral of a point particle into a Kerr black hole are a special case of the dynamical system (4.100). This can be seen using the identifications $t = \tau$, $\mathbf{q} = (q_t, q_r, q_\theta, q_\phi)$, $\mathbf{J} = (\tilde{P}_2, \tilde{P}_3, \tilde{P}_4, M_1, M_2)$, $G_\lambda^{(1)} = (G_2^{(1)}, G_3^{(1)}, G_4^{(1)}, 0, 0)$ and $G_\lambda^{(2)} = (G_2^{(2)}, G_3^{(2)}, G_4^{(2)}, \hat{G}_1, \hat{G}_2)$. The forcing functions $g_\alpha^{(s)}$ and $G_\lambda^{(s)}$ are periodic functions of q_α since they depend only on the variables $q_A = (q_r, q_\theta)$ which are angle variables; they do not depend on the variable q_t which is not an angle variable. Note that the system (4.100) allows the forcing functions $g_\alpha^{(s)}$, $G_\lambda^{(s)}$ and frequencies ω_α to depend in an arbitrary way on the slow time $\tilde{t}$, whereas no such dependence is seen in the Kerr inspiral system (4.59). The system studied here is thus slightly more general than is required for our specific application. We include the dependence on $\tilde{t}$ for greater generality and because it does not require any additional complexity in the analysis.

[20]In other words, there exists $\tilde{T} > 0$ such that for every $\mathbf{q}$, $\mathbf{J}$, every integer N, and every $\delta > 0$ there exists $\epsilon_1 = \epsilon_1(\mathbf{q}, \mathbf{J}, N, \delta)$ such that

$$\left| g_\alpha(\mathbf{q},\mathbf{J},\tilde{t},\varepsilon) - \sum_{s=1}^{N} g_\alpha^{(s)}(\mathbf{q},\mathbf{J},\tilde{t})\varepsilon^{s-1} \right| < \delta\varepsilon^{N-1}$$

for all $\tilde{t}$ with $0 < \tilde{t} < \tilde{T}$ and for all ε with $0 < \varepsilon < \epsilon_1$.

Another special case of the system (4.100) is when $N = M$ and when there exists a function $H(\mathbf{J}, \tilde{t})$ such that

$$\omega_\alpha(\mathbf{J}, \tilde{t}) = \frac{\partial H(\mathbf{J}, \tilde{t})}{\partial J_\alpha} \tag{4.105}$$

for $1 \leq \alpha \leq N$. In this case the system (4.100) represents a Hamiltonian system with slowly varying Hamiltonian $H(\mathbf{J}, \tilde{t})$, with action angle variables (q_α, J_α), and subject to arbitrary weak perturbing forces that vary slowly with time. The perturbed system is not necessarily Hamiltonian.

Because of the periodicity conditions (4.104), we can without loss of generality interpret the variables q_α to be coordinates on the N-torus T^N, and take the equations (4.100) to be defined on the product of this N-torus with an open set. This interpretation will useful below.

In the next several sections we will study in detail the behavior of solutions of the system (4.100) in the limit $\varepsilon \to 0$ using a two timescale expansion. We follow closely the exposition in the book by Kevorkian and Cole [133], except that we generalize their analysis and also correct some errors (see Appendix 4.10). For clarity we treat first, in Sec. 4.4, the simple case of a single degree of freedom, $N = M = 1$. Section 4.5 treats the case of general N and M, but with the restriction that the forcing functions g_α and G_λ contain no resonant pieces (this is defined in Sec. 4.5.3). The general case with resonances is treated in the forthcoming papers [137, 138]. Finally in Sec. 4.6 we present a numerical integration of a particular example of a dynamical system, in order to illustrate and validate the general theory of Secs. 4.4 and 4.5.

4.4 Systems with a single degree of freedom

4.4.1 Overview

For systems with a single degree of freedom the general equations of motion (4.100) discussed in Sec. 4.3 reduce to

$$\dot{q}(t) = \omega(J, \tilde{t}) + \varepsilon g(q, J, \tilde{t}, \varepsilon), \tag{4.106a}$$

$$\dot{J}(t) = \varepsilon G(q, J, \tilde{t}, \varepsilon), \tag{4.106b}$$

for some functions G and g, where $\tilde{t} = \varepsilon t$ is the slow time variable. The asymptotic expansions (4.102) and (4.103) of the forcing functions reduce to

$$\begin{aligned} g(q, J, \tilde{t}, \varepsilon) &= \sum_{s=1}^{\infty} g^{(s)}(q, J, \tilde{t})\varepsilon^{s-1} \\ &= g^{(1)}(q, J, \tilde{t}) + g^{(2)}(q, J, \tilde{t})\varepsilon + O(\varepsilon^2), \end{aligned} \tag{4.107}$$

and

$$\begin{aligned} G(q, J, \tilde{t}, \varepsilon) &= \sum_{s=1}^{\infty} G^{(s)}(q, J, \tilde{t})\varepsilon^{s-1} \\ &= G^{(1)}(q, J, \tilde{t}) + G^{(2)}(q, J, \tilde{t})\varepsilon + O(\varepsilon^2). \end{aligned} \tag{4.108}$$

Also the periodicity conditions (4.104) reduce to

$$g(q + 2\pi, J, \tilde{t}) = g(q, J, \tilde{t}), \tag{4.109a}$$

$$G(q + 2\pi, J, \tilde{t}) = G(q, J, \tilde{t}). \tag{4.109b}$$

In this section we apply two-timescale expansions to study classes of solutions of Eqs. (4.106) in the limit $\varepsilon \to 0$. We start in Sec. 4.4.2 by defining our conventions and notations for Fourier decompositions of the perturbing forces. The heart of the method is the ansatz we make for the form of the solutions, which is given in Sec. 4.4.3. Sec. 4.4.4 summarizes the results we obtain at each order in the expansion, and the derivations are given in Sec. 4.4.5. Although the results of this section are not directly applicable to the Kerr inspiral problem, the analysis of this section gives an introduction to the method of analysis, and is considerably simpler than the multivariable case treated in Sec. 4.5 below.

4.4.2 Fourier expansions of the perturbing forces

The periodicity conditions (4.109) apply at each order in the expansion in powers of ε:

$$g^{(s)}(q+2\pi, J, \tilde{t}) = g^{(s)}(q, J, \tilde{t}), \tag{4.110a}$$

$$G^{(s)}(q+2\pi, J, \tilde{t}) = G^{(s)}(q, J, \tilde{t}). \tag{4.110b}$$

It follows that these functions can be expanded as Fourier series:

$$g^{(s)}(q, J, \tilde{t}) = \sum_{k=-\infty}^{\infty} g_k^{(s)}(J, \tilde{t}) e^{ikq}, \tag{4.111a}$$

$$G^{(s)}(q, J, \tilde{t}) = \sum_{k=-\infty}^{\infty} G_k^{(s)}(J, \tilde{t}) e^{ikq}, \tag{4.111b}$$

where

$$g_k^{(s)}(J, \tilde{t}) = \frac{1}{2\pi} \int_0^{2\pi} dq \; e^{-ikq} \, g^{(s)}(q, J, \tilde{t}), \tag{4.112a}$$

$$G_k^{(s)}(J, \tilde{t}) = \frac{1}{2\pi} \int_0^{2\pi} dq \; e^{-ikq} \, G^{(s)}(q, J, \tilde{t}). \tag{4.112b}$$

For any periodic function $f = f(q)$, we introduce the notation

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(q) dq \tag{4.113}$$

for the average part of f, and

$$\hat{f}(q) = f(q) - \langle f \rangle \tag{4.114}$$

for the remaining part of f. It follows from these definitions that

$$\langle g^{(s)}(q, J, \tilde{t}) \rangle = g_0^{(s)}(J, \tilde{t}), \quad \langle G^{(s)}(q, J, \tilde{t}) \rangle = G_0^{(s)}(J, \tilde{t}), \tag{4.115}$$

and that

$$\begin{aligned} \hat{g}^{(s)}(q, J, \tilde{t}) &= \sum_{k \neq 0} g_k^{(s)}(J, \tilde{t}) e^{ikq}, && (4.116a) \\ \hat{G}^{(s)}(q, J, \tilde{t}) &= \sum_{k \neq 0} G_k^{(s)}(J, \tilde{t}) e^{ikq}. && (4.116b) \end{aligned}$$

We also have the identities

$$\begin{aligned} \langle f_{,q} \rangle &= \langle \hat{f} \rangle = 0 && (4.117a) \\ \langle fg \rangle &= \langle \hat{f} \hat{g} \rangle + \langle f \rangle \langle g \rangle && (4.117b) \end{aligned}$$

for any periodic functions $f(q)$, $g(q)$.

For any periodic function f, we also define a particular anti-derivative $\mathcal{I}\hat{f}$ of $\hat{f}$ by

$$(\mathcal{I}\hat{f})(q) \equiv \sum_{k \neq 0} \frac{f_k}{ik} e^{ikq}, \tag{4.118}$$

where $f_k = \int dq e^{-ikq} f(q)/(2\pi)$ are the Fourier coefficients of f. This operator satisfies the identities

$$\begin{aligned} (\mathcal{I}\hat{f})_{,q} &= \hat{f}, && (4.119a) \\ \langle (\mathcal{I}\hat{f}) \hat{g} \rangle &= -\langle \hat{f} (\mathcal{I}\hat{g}) \rangle, && (4.119b) \\ \langle \hat{f} (\mathcal{I}\hat{f}) \rangle &= 0. && (4.119c) \end{aligned}$$

4.4.3 Two timescale ansatz for the solution

We now discuss the *ansatz* we use for the form of the solutions of the equations of motion. This ansatz will be justified a posteriori order by order in ε. The method used here is sometimes called the "method of strained coordinates" [133].

We assume that q and J have asymptotic expansions in ε as functions of two different variables, the slow time parameter $\tilde{t} = \varepsilon t$, and a phase variable Ψ (also called a "fast-time parameter"), the dependence on which is periodic with period 2π. Thus we assume

$$\begin{aligned} q(t,\varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s q^{(s)}(\Psi,\tilde{t}) \\ &= q^{(0)}(\Psi,\tilde{t}) + \varepsilon q^{(1)}(\Psi,\tilde{t}) + O(\varepsilon^2), \qquad (4.120\text{a}) \\ J(t,\varepsilon) &= \sum_{s=0}^{\infty} \varepsilon^s J^{(s)}(\Psi,\tilde{t}) \\ &= J^{(0)}(\Psi,\tilde{t}) + \varepsilon J^{(1)}(\Psi,\tilde{t}) + O(\varepsilon^2). \qquad (4.120\text{b}) \end{aligned}$$

These asymptotic expansions are assumed to be uniform in $\tilde{t}$. The expansion coefficients $J^{(s)}$ are each periodic in the phase variable Ψ with period 2π:

$$J^{(s)}(\Psi + 2\pi, \tilde{t}) = J^{(s)}(\Psi, \tilde{t}). \qquad (4.121)$$

The phase variable Ψ is chosen so that angle variable q increases by 2π when Ψ increases by 2π; this implies that the expansion coefficients $q^{(s)}$ satisfy

$$\begin{aligned} q^{(0)}(\Psi + 2\pi, \tilde{t}) &= q^{(0)}(\Psi, \tilde{t}) + 2\pi, \qquad (4.122\text{a}) \\ q^{(s)}(\Psi + 2\pi, \tilde{t}) &= q^{(s)}(\Psi, \tilde{t}), \quad s \geq 1. \qquad (4.122\text{b}) \end{aligned}$$

The angular velocity $\Omega = d\Psi/dt$ associated with the phase Ψ is assumed to depend only on the slow time variable $\tilde{t}$ (so it can vary slowly with time), and on

ε. We assume that it has an asymptotic expansion in ε as $\varepsilon \to 0$ which is uniform in $\tilde{t}$:

$$\frac{d\Psi}{dt} = \Omega(\tilde{t}, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \Omega^{(s)}(\tilde{t}) \quad (4.123)$$

$$= \Omega^{(0)}(\tilde{t}) + \varepsilon \Omega^{(1)}(\tilde{t}) + O(\varepsilon^2). \quad (4.124)$$

Equation (4.124) serves to define the phase variable Ψ in terms the angular velocity variables $\Omega^{(s)}(\tilde{t})$, $s = 0, 1, 2 \ldots$, up to constants of integration. One constant of integration arises at each order in ε. Without loss of generality we choose these constants of integration so that

$$q^{(s)}(0, \tilde{t}) = 0 \quad (4.125)$$

for all s, $\tilde{t}$. Note that this does not restrict the final solutions $q(t, \varepsilon)$ and $J(t, \varepsilon)$, as we show explicitly below, because there are additional constants of integration that arise when solving for the functions $q^{(s)}(\Psi, \tilde{t})$ and $J^{(s)}(\Psi, \tilde{t})$.

Roughly speaking, the meaning of these assumptions is the following. The solution of the equations of motion consists of a mapping from (t, ε) to (q, J). That mapping contains dynamics on two different timescales, the dynamical timescale ~ 1 and the slow timescale $\sim 1/\varepsilon$. The mapping can be uniquely written the composition of two mappings

$$(t, \varepsilon) \quad \to \quad (\Psi, \tilde{t}, \varepsilon) \quad \to \quad (q, J), \quad (4.126)$$

such that the first mapping contains all the fast dynamics, and is characterized by the slowly evolving frequency $\Omega(\tilde{t}, \varepsilon)$, and the second mapping contains dynamics only on the slow timescale.

4.4.4 Results of the two-timescale analysis

By substituting the ansatz (4.120b) – (4.125) into the equations of motion (4.106) we find that all of the assumptions made in the ansatz can be satisfied, and that all of the expansion coefficients are uniquely determined, order by order in ε. This derivation is given in Sec. 4.4.5 below. Here we list the results obtained for the various expansion coefficients up to the leading and sub-leading orders.

Terminology for various orders of the approximation

We can combine the definitions just summarized to obtain an explicit expansion for the quantity of most interest, the angle variable q as a function of time. From the periodicity condition (4.122a) it follows that the function $q^{(0)}(\Psi, \tilde{t})$ can be written as $\Psi + \bar{q}^{(0)}(\Psi, \tilde{t})$ where $\bar{q}^{(0)}$ is a periodic function of Ψ. [We shall see that $\bar{q}^{(0)}$ in fact vanishes, cf. Eq. (4.132) below.] From the definitions (4.101) and (4.124), we can write the phase variable Ψ as

$$\Psi = \frac{1}{\varepsilon}\psi^{(0)}(\tilde{t}) + \psi^{(1)}(\tilde{t}) + \varepsilon\psi^{(2)}(\tilde{t}) + O(\varepsilon^2), \tag{4.127}$$

where the functions $\psi^{(s)}(\tilde{t})$ are defined by

$$\psi^{(s)}(\tilde{t}) = \int^{\tilde{t}} d\tilde{t}' \Omega^{(s)}(\tilde{t}'). \tag{4.128}$$

Inserting this into the expansion (4.120a) of q and using the above expression for $q^{(0)}$ gives

$$\begin{aligned} q(t, \varepsilon) \quad = \quad & \frac{1}{\varepsilon}\psi^{(0)}(\tilde{t}) + \left[\psi^{(1)}(\tilde{t}) + \bar{q}^{(0)}(\Psi, \tilde{t})\right] \\ & + \varepsilon\left[\psi^{(2)}(\tilde{t}) + q^{(1)}(\Psi, \tilde{t})\right] + O(\varepsilon^2). \end{aligned} \tag{4.129}$$

We will call the leading order, $O(1/\varepsilon)$ term in Eq. (4.129) the *adiabatic approximation*, the sub-leading $O(1)$ term the *post-1-adiabatic* term, the next $O(\varepsilon)$ term the *post-2-adiabatic* term, etc. This choice of terminology is motivated by the terminology used in post-Newtonian theory.

It is important to note that the expansion in powers of ε in Eq. (4.129) is *not* a straightforward power series expansion at fixed $\tilde{t}$, since the variable Ψ depends on ε. [The precise definition of the expansion of the solution which we are using is given by Eqs. (4.120a) – (4.125).] Nevertheless, the expansion (4.129) as written correctly captures the ε dependence of the secular pieces of the solution, since the functions $\bar{q}^{(0)}$ and $q^{(1)}$ are periodic functions of Ψ and so have no secular pieces.

Adiabatic Order

First, the zeroth order action variable is given by

$$J^{(0)}(\Psi, \tilde{t}) = \mathcal{J}^{(0)}(\tilde{t}), \tag{4.130}$$

where $\mathcal{J}^{(0)}$ satisfies the differential equation

$$\frac{d\mathcal{J}^{(0)}(\tilde{t})}{d\tilde{t}} = G_0^{(1)}[\mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]. \tag{4.131}$$

Here the right hand side denotes the average over q of the forcing term $G^{(1)}[q, \mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]$, cf. Eqs. (4.111) above. The zeroth order angle variable is given by

$$q^{(0)}(\Psi, \tilde{t}) = \Psi, \tag{4.132}$$

and the angular velocity Ω that defines the phase variable Ψ is given to zeroth order by

$$\Omega^{(0)}(\tilde{t}) = \omega[\mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]. \tag{4.133}$$

Note that this approximation is equivalent to the following simple prescription: (i) Truncate the equations of motion (4.106) to the leading order in ε:

$$\dot{q}(t) = \omega(J, \tilde{t}) + \varepsilon g^{(1)}(q, J, \tilde{t}), \tag{4.134a}$$

$$\dot{J}(t) = \varepsilon G^{(1)}(q, J, \tilde{t}); \tag{4.134b}$$

(ii) Omit the driving term $g^{(1)}$ in the equation for the angle variable; and (iii) Replace the driving term $G^{(1)}$ in the equation for the action variable with its average over q.

Post-1-adiabatic Order

Next, the first order action variable is given by

$$J^{(1)}(\Psi, \tilde{t}) = \frac{\mathcal{I}\hat{G}^{(1)}[\Psi, \mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]}{\Omega^{(0)}(\tilde{t})} + \mathcal{J}^{(1)}(\tilde{t}), \tag{4.135}$$

where the symbol $\mathcal{I}$ on the right hand side denotes the integration operator (6.407) with respect to Ψ. In Eq. (4.135) the quantity $\mathcal{J}^{(1)}(\tilde{t})$ satisfies the differential equation

$$\begin{aligned} &\frac{d\mathcal{J}^{(1)}(\tilde{t})}{d\tilde{t}} - \frac{\partial G_0^{(1)}}{\partial J}[\mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]\mathcal{J}^{(1)}(\tilde{t}) \\ = \; &\frac{\langle \frac{\partial \hat{G}^{(1)}}{\partial J}\mathcal{I}\hat{G}^{(1)}\rangle}{\Omega^{(0)}(\tilde{t})} - \frac{\langle \hat{G}^{(1)}\hat{g}^{(1)}\rangle}{\Omega^{(0)}(\tilde{t})} + G_0^{(2)}. \end{aligned} \tag{4.136}$$

Here it is understood that the quantities on the right hand side are evaluated at $q = q^{(0)} = \Psi$ and $J = \mathcal{J}^{(0)}(\tilde{t})$. The sub-leading correction to the phase variable Ψ is given by

$$\Omega^{(1)}(\tilde{t}) = \frac{\partial \omega}{\partial J}[\mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]\mathcal{J}^{(1)}(\tilde{t}) + g_0^{(1)}[\mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]. \tag{4.137}$$

Finally, the sub-leading term in the angle variable is

$$q^{(1)}(\Psi, \tilde{t}) = \hat{q}^{(1)}(\Psi, \tilde{t}) + \mathcal{Q}^{(1)}(\tilde{t}), \tag{4.138}$$

where

$$\begin{aligned}\hat{q}^{(1)}(\Psi,\tilde{t}) &= \frac{1}{\Omega^{(0)}(\tilde{t})^2}\frac{\partial\omega}{\partial J}[\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}]\,\mathcal{I}^2\hat{G}^{(1)}[\Psi,\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}] \\ &\quad+\frac{1}{\Omega^{(0)}(\tilde{t})}\mathcal{I}\hat{g}^{(1)}[\Psi,\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}] \end{aligned} \tag{4.139}$$

and

$$\mathcal{Q}^{(1)}(\tilde{t}) = -\hat{q}^{(1)}(0,\tilde{t}). \tag{4.140}$$

Discussion

One of the key results of the general analysis of this section is the identification of which pieces of the external forces are required to compute the adiabatic and post-1-adiabatic solutions. From Eqs. (4.131), (4.133) and (4.129), the adiabatic solution depends only on the averaged piece $G_0^{(1)}(J,\tilde{t}) = \langle G^{(1)}(q,J,\tilde{t})\rangle$ of the leading order external force $G^{(1)}$. This quantity is purely dissipative, as can be seen in the Kerr inspiral context from Eqs. (4.98) and (4.97). More generally, if the perturbing forces g and G arise from a perturbation $\varepsilon\Delta H = \sum_s \varepsilon^s \Delta H^{(s)}$ to the Hamiltonian, then the forcing function $G^{(s)}$ is

$$G^{(s)}(q,J,\tilde{t}) = -\frac{\partial \Delta H^{(s)}(q,J,\tilde{t})}{\partial q},$$

and it follows that the average over q of $G^{(s)}$ vanishes.

At the next order, the post-1-adiabatic term $\psi^{(1)}(\tilde{t})$ depends on the averaged piece $G_0^{(2)}(J,\tilde{t}) = \langle G^{(2)}(q,J,\tilde{t})\rangle$ of the sub-leading force $G^{(2)}$, again purely dissipative, as well as the remaining conservative and dissipative pieces of the leading order forces $G^{(1)}(q,J,\tilde{t})$ and $g^{(1)}(q,J,\tilde{t})$. This can be seen from Eqs. (4.136) and (4.137). These results have been previously discussed briefly in the EMRI context in Refs. [99, 40]. For circular, equatorial orbits, the fact that there is a post-1-

adiabatic order contribution from the second order self-force was first argued by Burko [148].

Initial conditions and the generality of our ansatz

We will show in the next subsection that our ansatz (4.120a) – (4.125) is compatible with the one parameter family of differential equations (4.106). However, it does not necessarily follow that our ansatz is compatible with the most general one parameter family $[q(t,\varepsilon), J(t,\varepsilon)]$ of solutions, because of the possibility of choosing arbitrary, ε-dependent initial conditions $q(0,\varepsilon)$ and $J(0,\varepsilon)$ at the initial time $t = 0$.[21] In general, the ε dependence of the solutions arises from both the ε dependence of the initial conditions and the ε dependence of the differential equations. It is possible to choose initial conditions which are incompatible with our ansatz.

To see this explicitly, we evaluate the expansions (4.129) and (4.135) at $t = \tilde{t} = 0$. This gives

$$q(0,\varepsilon) = \varepsilon^{-1}\psi^{(0)}(0) + \psi^{(1)}(0) + O(\varepsilon), \tag{4.141a}$$

$$\begin{aligned} J(0,\varepsilon) &= \mathcal{J}^{(0)}(0) + \varepsilon\mathcal{J}^{(1)}(0) \\ &\quad + \varepsilon\frac{\mathcal{I}\hat{G}^{(1)}[\varepsilon^{-1}\psi^{(0)}(0) + \psi^{(1)}(0), \mathcal{J}^{(0)}(0), 0]}{\omega[\mathcal{J}^{(0)}, 0]} \\ &\quad + O(\varepsilon^2). \end{aligned} \tag{4.141b}$$

Recalling that parameters $\psi^{(0)}(0)$, $\psi^{(1)}(0)$, $\mathcal{J}^{(0)}(0)$ and $\mathcal{J}^{(1)}(0)$ are assumed to be independent of ε, we see that the conditions (4.141) strongly constrain the allowed ε dependence of the initial conditions. We note, however, that the choice of constant (ε independent) initial conditions

$$q(0,\varepsilon) = q_0, \qquad J(0,\varepsilon) = J_0 \tag{4.142}$$

[21] More generally we could consider specifying initial conditions at some time $t = t_0$. In that case we would modify the definition of the rescaled time coordinate to $\tilde{t} = \varepsilon(t - t_0)$.

can be accommodated, which is sufficient for most applications of the formalism. To achieve this one chooses

$$\psi^{(0)}(0) = 0, \quad \psi^{(1)}(0) = q_0, \quad \mathcal{J}^{(0)}(0) = J_0, \tag{4.143}$$

and

$$\mathcal{J}^{(1)}(0) = -\frac{\mathcal{I}\hat{G}^{(1)}[q_0, J_0, 0]}{\omega[J_0, 0]}. \tag{4.144}$$

4.4.5 Derivation

In this subsection we give the derivation of the results (4.130) – (4.140) summarized above. At each order s we introduce the notation $\mathcal{J}^{(s)}(\tilde{t})$ for the average part of $J^{(s)}(\Psi, \tilde{t})$:

$$\mathcal{J}^{(s)}(\tilde{t}) \equiv \langle J^{(s)}(\Psi, \tilde{t})\rangle = \frac{1}{2\pi}\int_0^{2\pi} J^{(s)}(\Psi, \tilde{t})d\Psi. \tag{4.145}$$

We denote by $\hat{J}^{(s)}$ the remaining part of $J^{(s)}$, as in Eq. (4.114). This gives the decomposition

$$J^{(s)}(\Psi, \tilde{t}) = \mathcal{J}^{(s)}(\tilde{t}) + \hat{J}^{(s)}(\Psi, \tilde{t}) \tag{4.146}$$

for all $s \geq 0$. Similarly for the angle variable we have the decomposition

$$q^{(s)}(\Psi, \tilde{t}) = \mathcal{Q}^{(s)}(\tilde{t}) + \hat{q}^{(s)}(\Psi, \tilde{t}) \tag{4.147}$$

for all $s \geq 1$. [We do not use this notation for the $s = 0$ case for the angle variable, since $q^{(0)}$ is not a periodic function of Ψ, by Eq. (4.122a)].

Using the expansions (4.120a) and (4.120b) of q and J together with the expansion (4.124) of $d\Psi/dt$, we obtain

$$\begin{aligned}\frac{dq}{dt} &= \Omega^{(0)}q^{(0)}_{,\Psi} + \varepsilon\left[\Omega^{(1)}q^{(0)}_{,\Psi} + \Omega^{(0)}q^{(1)}_{,\Psi} + q^{(0)}_{,\tilde{t}}\right] \\ &\quad +\varepsilon^2\left[\Omega^{(2)}q^{(0)}_{,\Psi} + \Omega^{(0)}q^{(2)}_{,\Psi} + \Omega^{(1)}q^{(1)}_{,\Psi} + q^{(1)}_{,\tilde{t}}\right] \\ &\quad +O(\varepsilon^3).\end{aligned} \tag{4.148}$$

Here we use commas to denote partial derivatives. We now insert this expansion together with a similar expansion for dJ/dt into the equations of motion (4.106) and use the expansions (4.107) and (4.108) of the external forces g and G. Equating coefficients[22] of powers of ε then gives at zeroth order

$$\Omega^{(0)} q^{(0)}_{,\Psi} = \omega, \tag{4.149a}$$

$$\Omega^{(0)} J^{(0)}_{,\Psi} = 0, \tag{4.149b}$$

at first order

$$\Omega^{(0)} q^{(1)}_{,\Psi} - \omega_{,J} J^{(1)} = -\Omega^{(1)} q^{(0)}_{,\Psi} - q^{(0)}_{,\tilde{t}} + g^{(1)}, \tag{4.150a}$$

$$\Omega^{(1)} J^{(0)}_{,\Psi} + \Omega^{(0)} J^{(1)}_{,\Psi} = -J^{(0)}_{,\tilde{t}} + G^{(1)}, \tag{4.150b}$$

and at second order

$$\begin{aligned} \Omega^{(0)} q^{(2)}_{,\Psi} - \omega_{,J} J^{(2)} &= \frac{1}{2}\omega_{,JJ}(J^{(1)})^2 + g^{(1)}_{,q} q^{(1)} + g^{(1)}_{,J} J^{(1)} \\ &\quad + g^{(2)} - \Omega^{(2)} q^{(0)}_{,\Psi} - \Omega^{(1)} q^{(1)}_{,\Psi} \\ &\quad - q^{(1)}_{,\tilde{t}}, \end{aligned} \tag{4.151a}$$

$$\begin{aligned} \Omega^{(2)} J^{(0)}_{,\Psi} + \Omega^{(0)} J^{(2)}_{,\Psi} &= G^{(1)}_{,q} q^{(1)} + G^{(1)}_{,J} J^{(1)} - \Omega^{(1)} J^{(1)}_{,\Psi} \\ &\quad - J^{(1)}_{,\tilde{t}} + G^{(2)}. \end{aligned} \tag{4.151b}$$

Here it is understood that all functions of q and J are evaluated at $q^{(0)}$ and $J^{(0)}$.

Zeroth order analysis

The zeroth order equations (4.149) can be written more explicitly as

$$\Omega^{(0)}(\tilde{t}) q^{(0)}_{,\Psi}(\Psi, \tilde{t}) = \omega[J^{(0)}(\Psi, \tilde{t}), \tilde{t}], \tag{4.152a}$$

$$\Omega^{(0)}(\tilde{t}) J^{(0)}_{,\Psi}(\Psi, \tilde{t}) = 0. \tag{4.152b}$$

[22] As is well known, this procedure is valid for asymptotic series as well as normal power series.

The second of these equations implies that $J^{(0)}$ is independent of Ψ, so we obtain $J^{(0)}(\Psi, \tilde{t}) = \mathcal{J}^{(0)}(\tilde{t})$. The first equation then implies that $q^{(0)}_{,\Psi}$ is independent of Ψ, and integrating with respect to Ψ gives

$$q^{(0)}(\Psi, \tilde{t}) = \frac{\omega[\mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]}{\Omega^{(0)}(\tilde{t})}\Psi + \mathcal{Q}^{(0)}(\tilde{t}), \tag{4.153}$$

where $\mathcal{Q}^{(0)}$ is some function of $\tilde{t}$. The periodicity condition (4.122a) now implies that the coefficient of Ψ in Eq. (4.153) must be unity, which gives the formula (4.133) for the angular velocity $\Omega^{(0)}(\tilde{t})$. Finally, the assumption (4.125) forces $\mathcal{Q}^{(0)}(\tilde{t})$ to vanish, and we recover the formula (4.132) for $q^{(0)}(\Psi, \tilde{t})$.

First order analysis

The first order equation (4.150b) can be written more explicitly as

$$\begin{aligned} \Omega^{(0)}(\tilde{t}) J^{(1)}_{,\Psi}(\Psi, \tilde{t}) &= -\mathcal{J}^{(0)}_{,\tilde{t}}(\tilde{t}) \\ &\quad + G^{(1)}[\Psi, \mathcal{J}^{(0)}(\tilde{t}), \tilde{t}], \end{aligned} \tag{4.154}$$

where we have simplified using the zeroth order solutions (4.130) and (4.132). We now take the average with respect to Ψ of this equation. The left hand side vanishes since it is a total derivative, and we obtain using the definition (4.112) the differential equation (4.131) for $\mathcal{J}^{(0)}(\tilde{t})$. Next, we subtract from Eq. (4.154) its averaged part, and use the decomposition (4.146) of $J^{(1)}$. This gives

$$\Omega^{(0)}(\tilde{t}) \hat{J}^{(1)}_{,\Psi}(\Psi, \tilde{t}) = \hat{G}^{(1)}[\Psi, \mathcal{J}^{(0)}(\tilde{t}), \tilde{t}]. \tag{4.155}$$

We solve this equation using the Fourier decomposition (4.116b) of $\hat{G}^{(1)}$ to obtain

$$\hat{J}^{(1)}(\Psi, \tilde{t}) = \sum_{k \neq 0} \frac{G^{(1)}_k[\mathcal{J}^{(0)}(\tilde{t}), \tilde{t}] e^{ik\Psi}}{ik\Omega^{(0)}(\tilde{t})}. \tag{4.156}$$

This yields the first term in the result (4.135) for $J^{(1)}$ when we use the notation (6.407).

Next, we simplify the first order equation (4.150a) using the zeroth order solutions (4.130) and (4.132), to obtain

$$\Omega^{(0)}(\tilde{t})q^{(1)}_{,\Psi}(\Psi,\tilde{t}) - \omega_{,J}[\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}]J^{(1)}[\Psi,\tilde{t}]$$
$$= -\Omega^{(1)}(\tilde{t}) + g^{(1)}[\Psi,\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}]. \tag{4.157}$$

Averaging with respect to Ψ and using the decompositions (4.146) and (4.147) of $J^{(1)}$ and $q^{(1)}$ now gives the formula (4.137) for $\Omega^{(1)}(\tilde{t})$. Note however that the function $\mathcal{J}^{(1)}(\tilde{t})$ in that formula has not yet been determined; it will be necessary to go to one higher order to compute this function.

Finally, we subtract from Eq. (4.157) its average over Ψ using the decompositions (4.146) and (4.147) and then integrate with respect to Ψ using the notation (6.407). This gives

$$\begin{aligned}\hat{q}^{(1)}(\Psi,\tilde{t}) \;=\; & \frac{1}{\Omega^{(0)}(\tilde{t})}\Big\{\omega_{,J}[\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}]\,\mathcal{I}\hat{J}^{(1)}[\Psi,\tilde{t}] \\ & +\mathcal{I}\hat{g}^{(1)}[\Psi,\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}]\Big\}.\end{aligned} \tag{4.158}$$

Using the result for $\hat{J}^{(1)}$ given by the first term in Eq. (4.135) now yields the formula (4.139) for $\hat{q}^{(1)}(\Psi,\tilde{t})$, and the result (4.138) for $q^{(1)}$ then follows from the decomposition (4.147) together with the initial condition (4.125).

Second order analysis

We simplify the second order equation (4.151b) using the zeroth order solutions (4.130) and (4.132), average over Ψ, and simplify using the decompositions (4.146) and (4.147) and the identities (4.117). The result is

$$\begin{aligned}\mathcal{J}^{(1)}_{,\tilde{t}}(\tilde{t}) &= G^{(1)}_{0,J}[\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}]\mathcal{J}^{(1)}(\tilde{t}) + G^{(2)}_{0}[\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}] \\ &\quad + \left\langle \hat{q}^{(1)}(\Psi,\tilde{t})\,\hat{G}^{(1)}_{,q}[\Psi,\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}] \right\rangle \\ &\quad + \left\langle \hat{J}^{(1)}(\Psi,\tilde{t})\,\hat{G}^{(1)}_{,J}[\Psi,\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}] \right\rangle . \end{aligned} \tag{4.159}$$

Using the expressions (4.139) and (4.135) for $\hat{q}^{(1)}$ and $\hat{J}^{(1)}$ and simplifying using the identities (4.119) now gives the differential equation (4.136) for $\mathcal{J}^{(1)}$.

Extension to arbitrary order

In this subsection we prove by induction that solutions are uniquely determined at each order in ε. Our inductive hypothesis is that, given the equations up to order s, we can compute all of the expansion coefficients $q^{(u)}(\Psi,\tilde{t})$, $J^{(u)}(\Psi,\tilde{t})$ and $\Omega^{(u)}(\tilde{t})$ for $0 \le u \le s$, except for the averaged piece $\mathcal{J}^{(s)}(\tilde{t})$ of $J^{(s)}(\Psi,\tilde{t})$, and except for $\Omega^{(s)}(\tilde{t})$, which is assumed to be determined by $\mathcal{J}^{(s)}(\tilde{t})$. From the preceding subsections this hypothesis is true for $s = 0$ and for $s = 1$. We shall assume it is true at order $s - 1$ and prove it is true at order s.

The equations of motion at order s, when simplified using the zeroth zeroth order solutions (4.130) and (4.132), can be written as

$$\begin{aligned}
\Omega^{(0)} q^{(s)}_{,\Psi} + \Omega^{(s)} - \omega_{,J} J^{(s)} &= \omega_{,JJ} J^{(1)} J^{(s-1)} + g^{(1)}_{,q} q^{(s-1)} \\
&\quad + g^{(1)}_{,J} J^{(s-1)} - \Omega^{(1)} q^{(s-1)}_{,\Psi} \\
&\quad - \Omega^{(s-1)} q^{(1)}_{,\Psi} - q^{(s-1)}_{,\tilde{t}}, \\
&\quad + \mathcal{S} \qquad\qquad (4.160a)\\
\Omega^{(0)} J^{(s)}_{,\Psi} &= G^{(1)}_{,q} q^{(s-1)} + G^{(1)}_{,J} J^{(s-1)} \\
&\quad - \Omega^{(s-1)} J^{(1)}_{,\Psi} - \Omega^{(1)} J^{(s-1)}_{,\Psi} \\
&\quad - J^{(s-1)}_{,\tilde{t}} + \mathcal{T}. \qquad\qquad (4.160b)
\end{aligned}$$

Here $\mathcal{S} = \mathcal{S}(\Psi, \tilde{t})$ and $\mathcal{T} = \mathcal{T}(\Psi, \tilde{t})$ are expressions involving the forces $G^{(u)}$ and $g^{(u)}$ for $0 \leq u \leq s$ evaluated at $q = q^{(0)} = \Psi$ and $J = J^{(0)} = \mathcal{J}^{(0)}$, and involving the coefficients $q^{(u)}$, $J^{(u)}$ and $\Omega^{(u)}$ for $0 \leq u \leq s-2$ which by the inductive hypothesis are known. Therefore we can treat $\mathcal{S}$ and $\mathcal{T}$ as known functions.

Averaging Eq. (4.160b) over Ψ yields the differential equation

$$\begin{aligned}
\mathcal{J}^{(s-1)}_{,\tilde{t}} - G^{(1)}_{0,J} \mathcal{J}^{(s-1)} &= \langle \mathcal{T} \rangle + \langle \hat{G}^{(1)}_{,q} \hat{q}^{(s-1)} \rangle \\
&\quad + \langle \hat{G}^{(1)}_{,J} \hat{J}^{(s-1)} \rangle. \qquad\qquad (4.161)
\end{aligned}$$

By the inductive hypothesis all the terms on the right hand side are known, so we can solve this differential equation to determine $\mathcal{J}^{(s-1)}$.

Next, averaging Eq. (4.160a) yields

$$\begin{aligned}
\Omega^{(s)} - \omega_{,J} \mathcal{J}^{(s)} &= \omega_{,JJ} \langle \hat{J}^{(1)} \hat{J}^{(s-1)} \rangle + \omega_{,JJ} \mathcal{J}^{(1)} \mathcal{J}^{(s-1)} \\
&\quad + \langle \hat{g}^{(1)}_{,q} \hat{q}^{(s-1)} \rangle + \langle \hat{g}^{(1)}_{,J} \hat{J}^{(s-1)} \rangle \\
&\quad + g^{(1)}_{0,J} \mathcal{J}^{(s-1)} - \mathcal{Q}^{(s-1)}_{,\tilde{t}} + \langle \mathcal{S} \rangle. \qquad\qquad (4.162)
\end{aligned}$$

Since $\mathcal{J}^{(s-1)}$ has already been determined, the right hand side of this equation is known and therefore the equation can be used to solve for $\Omega^{(s)}$ once $\mathcal{J}^{(s)}$ is specified, in accord with the inductive hypothesis. Next, Eq. (4.160b) with the average part subtracted can be used to solve for $\hat{J}^{(s)}$, and once $\hat{J}^{(s)}$ is known Eq. (4.160a) with the average part subtracted can be used to solve for $\hat{q}^{(s)}$. Finally, the averaged piece $\mathcal{Q}^{(s)}(\tilde{t})$ of $q^{(s)}(\Psi, \tilde{t})$ can be computed from $\hat{q}^{(s)}$ using the initial condition (4.125) and the decomposition (4.147). Thus the inductive hypothesis is true at order s if it is true at order $s-1$.

4.5 Systems with several degrees of freedom subject to non-resonant forcing

4.5.1 Overview

In this section we generalize the analysis of the preceding section to the general system of equations (4.100) with several degrees of freedom. For convenience we reproduce those equations here:

$$\begin{aligned} \frac{dq_\alpha}{dt} &= \omega_\alpha(\mathbf{J}, \tilde{t}) + \varepsilon g_\alpha^{(1)}(\mathbf{q}, \mathbf{J}, \tilde{t}) + \varepsilon^2 g_\alpha^{(2)}(\mathbf{q}, \mathbf{J}, \tilde{t}) \\ &\quad + O(\varepsilon^3), \quad 1 \le \alpha \le N, \end{aligned} \tag{4.163a}$$

$$\begin{aligned} \frac{dJ_\lambda}{dt} &= \varepsilon G_\lambda^{(1)}(\mathbf{q}, \mathbf{J}, \tilde{t}) + \varepsilon^2 G_\lambda^{(2)}(\mathbf{q}, \mathbf{J}, \tilde{t}) \\ &\quad + O(\varepsilon^3), \quad 1 \le \lambda \le M. \end{aligned} \tag{4.163b}$$

For the remainder of this paper, unless otherwise specified, indices $\alpha, \beta, \gamma, \delta, \varepsilon, \ldots$ from the start of the Greek alphabet will run over $1 \ldots N$, and indices $\lambda, \mu, \nu, \rho, \sigma, \ldots$ from the second half of the alphabet will run over $1 \ldots M$.

The generalization from one to several variables is straightforward except for the treatment of resonances [133]. The key new feature in the N variable case is that the asymptotic expansions now have additional terms proportional to $\sqrt{\varepsilon}$, $\varepsilon^{3/2}, \ldots$ as well as the integer powers of ε. The coefficients of these half-integer powers of ε obey source-free differential equations, except at resonances. Therefore, in the absence of resonances, all of these coefficients can be set to zero without loss of generality. In this paper we develop the general theory with both types of terms present, but we specialize to the case where no resonances occur. Subsequent papers [137, 138] will extend the treatment to include resonances, and derive the form of the source terms for the half-integer power coefficients.

We start in Sec. 4.5.2 by defining our conventions and notations for Fourier decompositions of the perturbing forces. In Sec. 4.5.3 we discuss the assumptions we make that prevent the occurrence of resonances in the solutions. The heart of the method is the ansatz we make for the form of the solutions, which is given in Sec. 4.5.4. Section 4.5.5 summarizes the results we obtain at each order in the expansion, and the derivations are given in Sec. 4.5.6. The implications of the results are discussed in detail in Sec. 4.7 below.

4.5.2 Fourier expansions of perturbing forces

The periodicity condition (4.104) applies at each order in the expansion in powers of ε, so we obtain

$$g_\alpha^{(s)}(\mathbf{q}+2\pi\mathbf{k},\mathbf{J},\tilde{t}) = g_\alpha^{(s)}(\mathbf{q},\mathbf{J},\tilde{t}), \tag{4.164a}$$

$$G_\lambda^{(s)}(\mathbf{q}+2\pi\mathbf{k},\mathbf{J},\tilde{t}) = G_\lambda^{(s)}(\mathbf{q},\mathbf{J},\tilde{t}), \tag{4.164b}$$

for $s \geq 1$, $1 \leq \alpha \leq N$, and $1 \leq \lambda \leq M$. Here $\mathbf{k} = (k_1, \ldots, k_N)$ can be an arbitrary N-tuple of integers. It follows from Eqs. (4.164) that these functions can be expanded as multiple Fourier series:

$$g_\alpha^{(s)}(\mathbf{q},\mathbf{J},\tilde{t}) = \sum_{\mathbf{k}} g_{\alpha\,\mathbf{k}}^{(s)}(\mathbf{J},\tilde{t})e^{i\mathbf{k}\cdot\mathbf{q}}, \tag{4.165a}$$

$$G_\lambda^{(s)}(\mathbf{q},\mathbf{J},\tilde{t}) = \sum_{\mathbf{k}} G_{\lambda\,\mathbf{k}}^{(s)}(\mathbf{J},\tilde{t})e^{i\mathbf{k}\cdot\mathbf{q}}, \tag{4.165b}$$

where

$$g_{\alpha\,\mathbf{k}}^{(s)}(\mathbf{J},\tilde{t}) = \frac{1}{(2\pi)^N}\int d^Nq\; e^{-i\mathbf{k}\cdot\mathbf{q}}\, g_\alpha^{(s)}(\mathbf{q},\mathbf{J},\tilde{t}), \tag{4.166a}$$

$$G_{\lambda\,\mathbf{k}}^{(s)}(\mathbf{J},\tilde{t}) = \frac{1}{(2\pi)^N}\int d^Nq\; e^{-i\mathbf{k}\cdot\mathbf{q}}\, G_\lambda^{(s)}(\mathbf{q},\mathbf{J},\tilde{t}). \tag{4.166b}$$

Here we adopt the usual notations

$$\sum_{\mathbf{k}} \equiv \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}, \tag{4.167}$$

$$\int d^Nq \equiv \int_0^{2\pi} dq_1 \ldots \int_0^{2\pi} dq_N. \tag{4.168}$$

and

$$\mathbf{k}\cdot\mathbf{q} \equiv \sum_{\alpha=1}^{N} k_\alpha q_\alpha. \tag{4.169}$$

For any multiply periodic function $f = f(\mathbf{q})$, we introduce the notation

$$\langle f\rangle = \frac{1}{(2\pi)^N}\int d^Nq f(\mathbf{q}) \tag{4.170}$$

for the average part of f, and

$$\hat{f}(\mathbf{q}) = f(\mathbf{q}) - \langle f\rangle \tag{4.171}$$

for the remaining part of f. It follows from these definitions that

$$\langle g_\alpha^{(s)}(\mathbf{q},\mathbf{J},\tilde{t})\rangle = g_{\alpha\,\mathbf{0}}^{(s)}(\mathbf{J},\tilde{t}), \quad \langle G_\lambda^{(s)}(\mathbf{q},\mathbf{J},\tilde{t})\rangle = G_{\lambda\,\mathbf{0}}^{(s)}(\mathbf{J},\tilde{t}), \tag{4.172}$$

and that

$$\hat{g}_\alpha^{(s)}(\mathbf{q},\mathbf{J},\tilde{t}) = \sum_{\mathbf{k}\neq\mathbf{0}} g_{\alpha\,\mathbf{k}}^{(s)}(\mathbf{J},\tilde{t})e^{i\mathbf{k}\cdot\mathbf{q}}, \tag{4.173a}$$

$$\hat{G}_\lambda^{(s)}(\mathbf{q},\mathbf{J},\tilde{t}) = \sum_{\mathbf{k}\neq\mathbf{0}} G_{\lambda\,\mathbf{k}}^{(s)}(\mathbf{J},\tilde{t})e^{i\mathbf{k}\cdot\mathbf{q}}. \tag{4.173b}$$

We also have the identities

$$\left\langle \frac{\partial f}{\partial q_\alpha} \right\rangle = \langle \hat{f} \rangle = 0 \tag{4.174a}$$

$$\langle fg \rangle = \langle \hat{f}\hat{g} \rangle + \langle f \rangle \langle g \rangle \tag{4.174b}$$

for any multiply periodic functions $f(\mathbf{q})$, $g(\mathbf{q})$.

For any multiply periodic function f and for any vector $\mathbf{v} = (v_1, \ldots, v_N)$, we also define the quantity $\mathcal{I}_\mathbf{v} \hat{f}$ by

$$(\mathcal{I}_\mathbf{v} \hat{f})(\mathbf{q}) \equiv \sum_{\mathbf{k} \neq \mathbf{0}} \frac{f_\mathbf{k}}{i\mathbf{k} \cdot \mathbf{v}} e^{i\mathbf{k} \cdot \mathbf{q}}, \tag{4.175}$$

where $f_\mathbf{k} = \int d^N q e^{-i\mathbf{k} \cdot \mathbf{q}} f(\mathbf{q})/(2\pi)^N$ are the Fourier coefficients of f. The operator $\mathcal{I}_\mathbf{v}$ satisfies the identities

$$\mathcal{I}_\mathbf{v}(\mathbf{v} \cdot \boldsymbol{\nabla} \hat{f}) = \hat{f}, \tag{4.176a}$$

$$\langle (\mathcal{I}_\mathbf{v} \hat{f}) \hat{g} \rangle = -\langle \hat{f} (\mathcal{I}_\mathbf{v} \hat{g}) \rangle, \tag{4.176b}$$

$$\langle \hat{f} (\mathcal{I}_\mathbf{v} \hat{f}) \rangle = 0. \tag{4.176c}$$

4.5.3 The no-resonance assumption

For each set of action variables $\mathbf{J}$ and for each time $\tilde{t}$, we will say that an N-tuple of integers $\mathbf{k} \neq 0$ is a *resonant N-tuple* if

$$\mathbf{k} \cdot \boldsymbol{\omega}(\mathbf{J}, \tilde{t}) = 0. \tag{4.177}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N)$ are the frequencies that appear on the right hand side of the equation of motion (4.100a). This condition governs the occurrence of resonances in our perturbation expansion, as is well known in context of perturbations of multiply periodic Hamiltonian systems [152]. We will assume that for a given $\mathbf{k}$, the set of values of $\tilde{t}$ at which the quantity

$$\sigma_{\mathbf{k}}(\tilde{t}) \equiv \mathbf{k} \cdot \boldsymbol{\omega}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}] \tag{4.178}$$

vanishes (i.e. the resonant values) consists of isolated points. Here $\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t})$ is the leading order solution for $\mathbf{J}$ given by Eq. (4.191) below. This assumption excludes persistent resonances that last for a finite interval in $\tilde{t}$. Generically we expect this to be true because of the time dependence of $\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t})$.

Our no-resonance assumption is essentially that the Fourier components of the forcing terms vanish for resonant N-tuples. More precisely, for each fixed $\mathbf{k}$ and for each time $\tilde{t}_{\mathrm{r}}$ for which $\sigma_{\mathbf{k}}(\tilde{t}_{\mathrm{r}}) = 0$, we assume that

$$g^{(s)}_{\alpha\,\mathbf{k}}\left[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}\right] = 0, \tag{4.179a}$$

$$G^{(s)}_{\lambda\,\mathbf{k}}\left[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}\right] = 0, \tag{4.179b}$$

for $s \geq 1$ and for all $\tilde{t}$ in a neighborhood of $\tilde{t}_{\mathrm{r}}$. Our no-resonance assumption will be relaxed in the forthcoming papers [137, 138].

In our application to inspirals in Kerr black holes, the no-resonance condition will be automatically satisfied for two classes of orbits: circular and equatorial orbits. This is because for these orbits there is either no radial motion, or no motion in θ, and so the two-dimensional torus (q_r, q_θ) reduces to a one-dimensional circle. The resonance condition $k_r\omega_r + k_\theta\omega_\theta = 0$ reduces to $k_r\omega_r = 0$ for equatorial orbits, or $k_\theta\omega_\theta = 0$ for circular orbits, and these conditions can never be satisfied since the fundamental frequencies ω_r and ω_θ are positive.

4.5.4 Two timescale ansatz for the solution

We now discuss the two-timescale *ansatz* we use for the form of the solutions of the equations of motion. This ansatz will be justified a posteriori order by order in $\sqrt{\varepsilon}$. Our ansatz essentially consists of the assumption that the mapping from (t, ε) to $(\mathbf{q}, \mathbf{J})$ can be written as an asymptotic expansion in $\sqrt{\varepsilon}$, each term of which is the composition of two maps, the first from (t, ε) to an abstract N-torus with coordinates $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_N)$, and the second from $(\mathbf{\Psi}, \tilde{t}, \varepsilon)$ to $(\mathbf{q}, \mathbf{J})$. Here $\tilde{t} = \varepsilon t$ is the slow time parameter. All the fast timescale dynamics is encapsulated in the first mapping. More precisely, we assume

$$\begin{aligned} q_\alpha(t, \varepsilon) &= \sum_{n=0}^{\infty} \varepsilon^{n/2} q_\alpha^{(n/2)}(\mathbf{\Psi}, \tilde{t}) \\ &= q_\alpha^{(0)}(\mathbf{\Psi}, \tilde{t}) + \sqrt{\varepsilon} q_\alpha^{(1/2)}(\mathbf{\Psi}, \tilde{t}) + \varepsilon q_\alpha^{(1)}(\mathbf{\Psi}, \tilde{t}) \\ &\quad + \varepsilon^{3/2} q_\alpha^{(3/2)}(\mathbf{\Psi}, \tilde{t}) + O(\varepsilon^2), \end{aligned} \tag{4.180a}$$

$$\begin{aligned} J_\lambda(t, \varepsilon) &= \sum_{n=0}^{\infty} \varepsilon^{n/2} J_\lambda^{(n/2)}(\mathbf{\Psi}, \tilde{t}) \\ &= J_\lambda^{(0)}(\mathbf{\Psi}, \tilde{t}) + \sqrt{\varepsilon} J_\lambda^{(1/2)}(\mathbf{\Psi}, \tilde{t}) + \varepsilon J_\lambda^{(1)}(\mathbf{\Psi}, \tilde{t}) \\ &\quad + \varepsilon^{3/2} J_\lambda^{(3/2)}(\mathbf{\Psi}, \tilde{t}) + O(\varepsilon^2). \end{aligned} \tag{4.180b}$$

These asymptotic expansions are assumed to be uniform in $\tilde{t}$. The expansion coefficients $J_\lambda^{(s)}$, where $s = 0, 1/2, 1, \ldots$, are multiply periodic in the phase variables Ψ_α with period 2π in each variable:

$$J_\lambda^{(s)}(\mathbf{\Psi} + 2\pi \mathbf{k}, \tilde{t}) = J_\lambda^{(s)}(\mathbf{\Psi}, \tilde{t}). \tag{4.181}$$

Here $\mathbf{k} = (k_1, \ldots, k_N)$ is an arbitrary N-tuple of integers. The mapping of the abstract N-torus with coordinates $\mathbf{\Psi}$ into the torus in phase space parameterized by $\mathbf{q}$ is assumed to have a trivial wrapping, so that the angle variable q_α increases by 2π when Ψ_α increases by 2π; this implies that the expansion coefficients $q^{(s)}$

satisfy

$$q_\alpha^{(0)}(\mathbf{\Psi} + 2\pi\mathbf{k}, \tilde{t}) = q_\alpha^{(0)}(\mathbf{\Psi}, \tilde{t}) + 2\pi k_\alpha, \tag{4.182a}$$

$$q_\alpha^{(s)}(\mathbf{\Psi} + 2\pi\mathbf{k}, \tilde{t}) = q_\alpha^{(s)}(\mathbf{\Psi}, \tilde{t}), \quad s \geq 1/2, \tag{4.182b}$$

for arbitrary $\mathbf{k}$. The variables $\Psi_1, \ldots, \Psi_N$ are sometimes called "fast-time parameters".

The angular velocity

$$\Omega_\alpha = d\Psi_\alpha/dt \tag{4.183}$$

associated with the phase Ψ_α is assumed to depend only on the slow time variable $\tilde{t}$ (so it can vary slowly with time), and on ε. We assume that it has an asymptotic expansion in $\sqrt{\varepsilon}$ as $\varepsilon \to 0$ which is uniform in $\tilde{t}$:

$$\Omega_\alpha(\tilde{t}, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{n/2} \Omega_\alpha^{(n/2)}(\tilde{t}) \tag{4.184}$$

$$= \Omega_\alpha^{(0)}(\tilde{t}) + \sqrt{\varepsilon}\Omega_\alpha^{(1/2)}(\tilde{t}) + \varepsilon\Omega_\alpha^{(1)}(\tilde{t}) + \varepsilon^{3/2}\Omega_\alpha^{(3/2)}(\tilde{t}) + O(\varepsilon^2). \tag{4.185}$$

Equations (4.183) and (4.185) serve to define the phase variable Ψ_α in terms the angular velocity variables $\Omega_\alpha^{(s)}(\tilde{t})$, $s = 0, 1/2, 1 \ldots$, up to constants of integration. One constant of integration arises at each order in $\sqrt{\varepsilon}$, for each α. Without loss of generality we choose these constants of integration so that

$$q_\alpha^{(s)}(\mathbf{0}, \tilde{t}) = 0 \tag{4.186}$$

for all α, s and $\tilde{t}$. Note that this does not restrict the final solutions $q_\alpha(t, \varepsilon)$ and $J_\lambda(t, \varepsilon)$, as we show explicitly below, because there are additional constants of integration that arise when solving for the functions $q_\alpha^{(s)}(\mathbf{\Psi}, \tilde{t})$ and $J_\lambda^{(s)}(\mathbf{\Psi}, \tilde{t})$.

4.5.5 Results of the two-timescale analysis

By substituting the ansatz (4.180b) – (4.186) into the equations of motion (4.100) we find that all of the assumptions made in the ansatz can be satisfied, and that all of the expansion coefficients are uniquely determined, order by order in $\sqrt{\varepsilon}$. This derivation is given in Sec. 4.5.6 below. Here we list the results obtained for the various expansion coefficients up to the first three orders.

Terminology for various orders of the approximation

We can combine the definitions just summarized to obtain an explicit expansion for the quantity of most interest, the angle variables q_α as a function of time. From the periodicity condition (4.122a) it follows that the function $q_\alpha^{(0)}(\mathbf{\Psi}, \tilde{t})$ can be written as $\Psi_\alpha + \bar{q}_\alpha^{(0)}(\mathbf{\Psi}, \tilde{t})$ where $\bar{q}_\alpha^{(0)}$ is a multiply periodic function of $\mathbf{\Psi}$. From the definitions (4.101) and (4.185), we can write the phase variables Ψ_α as

$$\begin{aligned}\Psi_\alpha &= \frac{1}{\varepsilon}\psi_\alpha^{(0)}(\tilde{t}) + \frac{1}{\sqrt{\varepsilon}}\psi_\alpha^{(1/2)}(\tilde{t}) + \psi_\alpha^{(1)}(\tilde{t}) + \sqrt{\varepsilon}\psi_\alpha^{(3/2)}(\tilde{t}) \\ &\quad +\varepsilon\psi_\alpha^{(2)}(\tilde{t}) + O(\varepsilon^{3/2}),\end{aligned} \tag{4.187}$$

where the functions $\psi_\alpha^{(s)}(\tilde{t})$ are defined by

$$\psi_\alpha^{(s)}(\tilde{t}) = \int^{\tilde{t}} d\tilde{t}' \Omega_\alpha^{(s)}(\tilde{t}'). \tag{4.188}$$

Inserting this into the expansion (4.180a) of q_α gives

$$\begin{aligned}q_\alpha(t,\varepsilon) &= \frac{1}{\varepsilon}\psi_\alpha^{(0)}(\tilde{t}) + \frac{1}{\sqrt{\varepsilon}}\psi_\alpha^{(1/2)}(\tilde{t}) \\ &\quad + \left[\psi_\alpha^{(1)}(\tilde{t}) + \bar{q}_\alpha^{(0)}(\mathbf{\Psi}, \tilde{t})\right] \\ &\quad +\sqrt{\varepsilon}\left[\psi_\alpha^{(3/2)}(\tilde{t}) + q_\alpha^{(1/2)}(\mathbf{\Psi}, \tilde{t})\right] \\ &\quad +\varepsilon\left[\psi_\alpha^{(2)}(\tilde{t}) + q_\alpha^{(1)}(\mathbf{\Psi}, \tilde{t})\right] + O(\varepsilon^{3/2}).\end{aligned} \tag{4.189}$$

We will call the leading order, $O(1/\varepsilon)$ term in Eq. (4.189) the *adiabatic approximation*, the sub-leading $O(1/\sqrt{\varepsilon})$ term the *post-1/2-adiabatic* term, the next $O(1)$ term the *post-1-adiabatic* term, etc. Below we will see that the functions $\bar{q}_\alpha^{(0)}$ and $q_\alpha^{(1/2)}$ in fact vanish identically, and so the oscillatory, $\mathbf{\Psi}$-dependent terms in the expansion (4.189) arise only at post-2-adiabatic and higher orders.

As before we note that the expansion in powers of ε in Eq. (4.189) is *not* a straightforward power series expansion at fixed $\tilde{t}$, since the variable Ψ depends on ε. [The precise definition of the expansion of the solution which we are using is given by Eqs. (4.180a) – (4.186).] Nevertheless, the expansion (4.189) as written correctly captures the ε dependence of the secular pieces of the solution, since the functions $\bar{q}^{(0)}$, $q_\alpha^{(1/2)}$ and $q_\alpha^{(1)}$ are multiply periodic functions of Ψ and so have no secular pieces.

Adiabatic Order

The zeroth order action variables are given by

$$J_\lambda^{(0)}(\mathbf{\Psi}, \tilde{t}) = \mathcal{J}_\lambda^{(0)}(\tilde{t}), \tag{4.190}$$

where $\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}) = \left(\mathcal{J}_1^{(0)}(\tilde{t}), \ldots, \mathcal{J}_M^{(0)}(\tilde{t})\right)$ satisfies the set of coupled ordinary differential equations

$$\frac{d\mathcal{J}_\lambda^{(0)}(\tilde{t})}{d\tilde{t}} = G_{\lambda\mathbf{0}}^{(1)}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}]. \tag{4.191}$$

Here the right hand side denotes the average over $\mathbf{q}$ of the forcing term $G_\lambda^{(1)}[\mathbf{q}, \boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}]$, cf. Eqs. (4.166) above. The zeroth order angle variables are given by

$$q_\alpha^{(0)}(\mathbf{\Psi}, \tilde{t}) = \Psi_\alpha, \tag{4.192}$$

and the angular velocity Ω_α that defines the phase variable Ψ_α is given to zeroth order by

$$\Omega_\alpha^{(0)}(\tilde{t}) = \omega_\alpha[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}]. \tag{4.193}$$

Note that this approximation is equivalent to the following simple prescription: (i) Truncate the equations of motion (4.163) to the $O(\varepsilon)$; (ii) Omit the driving terms $g_\alpha^{(1)}$ in the equations for the angle variables; and (iii) Replace the driving terms $G_\lambda^{(1)}$ in the equations for the action variables with their averages over $\mathbf{q}$.

Post-1/2-adiabatic order

Next, the $O(\sqrt{\varepsilon})$ action variables are given by

$$J_\lambda^{(1/2)}(\boldsymbol{\Psi}, \tilde{t}) = \mathcal{J}_\lambda^{(1/2)}(\tilde{t}), \tag{4.194}$$

where $\boldsymbol{\mathcal{J}}^{(1/2)}(\tilde{t}) = \left(\mathcal{J}_1^{(1/2)}(\tilde{t}), \ldots, \mathcal{J}_M^{(1/2)}(\tilde{t})\right)$ satisfies the set of coupled, source-free ordinary differential equations

$$\frac{d\mathcal{J}_\lambda^{(1/2)}(\tilde{t})}{d\tilde{t}} - \frac{\partial G_{\lambda\mathbf{0}}^{(1)}}{\partial J_\mu}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}]\mathcal{J}_\mu^{(1/2)}(\tilde{t}) = 0. \tag{4.195}$$

Equation (4.195) will acquire a source term in Ref. [138] where we include the effects of resonances. The $O(\sqrt{\varepsilon})$ angle variables are given by

$$q_\alpha^{(1/2)}(\boldsymbol{\Psi}, \tilde{t}) = 0, \tag{4.196}$$

and the angular velocity Ω_α that defines the phase variable Ψ_α is given to $O(\sqrt{\varepsilon})$ by

$$\Omega_\alpha^{(1/2)}(\tilde{t}) = \frac{\partial \omega_\alpha}{\partial J_\lambda}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}]\mathcal{J}_\lambda^{(1/2)}(\tilde{t}). \tag{4.197}$$

Note that Eqs. (4.195) and (4.197) can be obtained simply by linearizing Eqs. (4.191) and (4.193) about the zeroth order solution. That is, $\boldsymbol{\mathcal{J}}^{(0)} + \sqrt{\varepsilon}\boldsymbol{\mathcal{J}}^{(1/2)}$ and $\boldsymbol{\Omega}^{(0)} + \sqrt{\varepsilon}\boldsymbol{\Omega}^{(1/2)}$ satisfy the zeroth order equations (4.191) and (4.193) to $O(\sqrt{\varepsilon})$.

This means that setting $\boldsymbol{\mathcal{J}}^{(1/2)}$ and $\boldsymbol{\Omega}^{(1/2)}$ to zero does not cause any loss of generality in the solutions (under the no-resonance assumption of this paper), as long as we allow initial conditions to have sufficiently general dependence on ε.

Post-1-adiabatic order

The first order action variable is given by

$$J_\lambda^{(1)}(\boldsymbol{\Psi},\tilde{t}) = \mathcal{I}_{\boldsymbol{\Omega}^{(0)}(\tilde{t})}\hat{G}_\lambda^{(1)}[\boldsymbol{\Psi},\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}] + \mathcal{J}_\lambda^{(1)}(\tilde{t}), \tag{4.198}$$

where the symbol $\mathcal{I}$ on the right hand side denotes the integration operator (4.175) with respect to $\boldsymbol{\Psi}$, $\hat{G}_\lambda^{(1)}$ is the non-constant piece of $G_\lambda^{(1)}$ as defined in Eq. (4.171), and $\boldsymbol{\Omega}^{(0)}$ is given by Eq. (4.193). In Eq. (4.198) the quantity $\boldsymbol{\mathcal{J}}^{(1)}(\tilde{t})$ satisfies the differential equation

$$\begin{aligned}
&\frac{d\mathcal{J}_\lambda^{(1)}(\tilde{t})}{d\tilde{t}} - \frac{\partial G_{\lambda\mathbf{0}}^{(1)}}{\partial J_\mu}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]\mathcal{J}_\mu^{(1)}(\tilde{t}) \\
= \;& G_{\lambda\mathbf{0}}^{(2)} + \frac{1}{2}\frac{\partial^2 G_{\lambda\mathbf{0}}^{(1)}}{\partial J_\mu \partial J_\sigma}\mathcal{J}_\mu^{(1/2)}\mathcal{J}_\sigma^{(1/2)} \\
&+ \left\langle \frac{\partial \hat{G}_\lambda^{(1)}}{\partial J_\mu}\mathcal{I}_{\boldsymbol{\Omega}^{(0)}}\hat{G}_\mu^{(1)} \right\rangle + \left\langle \frac{\partial \hat{G}_\lambda^{(1)}}{\partial q_\alpha}\mathcal{I}_{\boldsymbol{\Omega}^{(0)}}\hat{g}_\alpha^{(1)} \right\rangle \\
&+ \frac{\partial \omega_\alpha}{\partial J_\mu}\left\langle \frac{\partial \hat{G}_\lambda^{(1)}}{\partial q_\alpha}\mathcal{I}_{\boldsymbol{\Omega}^{(0)}}\mathcal{I}_{\boldsymbol{\Omega}^{(0)}}\hat{G}_\mu^{(1)} \right\rangle .
\end{aligned} \tag{4.199}$$

Here it is understood that the quantities on the right hand side are evaluated at $\mathbf{J} = \boldsymbol{\mathcal{J}}^{(0)}(\tilde{t})$ and $\mathbf{q} = \mathbf{q}^{(0)} = \boldsymbol{\Psi}$. The last three terms on the right hand side of Eq. (4.199) can be written more explicitly using the definition (4.175) of $\mathcal{I}$ and the definition (4.170) of the averaging $\langle \ldots \rangle$ as

$$\begin{aligned}
&\sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{\boldsymbol{\Omega}^{(0)}\cdot\mathbf{k}} \left\{ ik_\alpha \frac{\partial \omega_\alpha}{\partial J_\mu}\frac{G_{\lambda\mathbf{k}}^{(1)*}G_{\mu\mathbf{k}}^{(1)}}{\boldsymbol{\Omega}^{(0)}\cdot\mathbf{k}} - k_\alpha G_{\lambda\mathbf{k}}^{(1)*}g_{\alpha\mathbf{k}}^{(1)} \right. \\
&\left. -iG_{\mu\mathbf{k}}^{(1)}\frac{\partial G_{\lambda\mathbf{k}}^{(1)*}}{\partial J_\mu} \right\}.
\end{aligned} \tag{4.200}$$

The $O(\varepsilon)$ correction to the angular velocity Ω_α is given by

$$\begin{aligned}\Omega_\alpha^{(1)}(\tilde{t}) &= g_{\alpha\mathbf{0}}^{(1)}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}] + \frac{\partial\omega_\alpha}{\partial J_\lambda}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]\mathcal{J}_\lambda^{(1)}(\tilde{t}) \\ &+\frac{1}{2}\frac{\partial^2\omega_\alpha}{\partial J_\lambda \partial J_\mu}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]\mathcal{J}_\lambda^{(1/2)}(\tilde{t})\mathcal{J}_\mu^{(1/2)}(\tilde{t}).\end{aligned} \tag{4.201}$$

Finally, the sub-leading term in the angle variable is

$$q_\alpha^{(1)}(\boldsymbol{\Psi},\tilde{t}) = \hat{q}_\alpha^{(1)}(\boldsymbol{\Psi},\tilde{t}) + \mathcal{Q}_\alpha^{(1)}(\tilde{t}), \tag{4.202}$$

where

$$\begin{aligned}\hat{q}_\alpha^{(1)}(\boldsymbol{\Psi},\tilde{t}) &= \frac{\partial\omega_\alpha}{\partial J_\lambda}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}] \\ &\times\mathcal{I}_{\boldsymbol{\Omega}^{(0)}(\tilde{t})}\mathcal{I}_{\boldsymbol{\Omega}^{(0)}(\tilde{t})}\hat{G}_\lambda^{(1)}[\boldsymbol{\Psi},\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}] \\ &+\mathcal{I}_{\boldsymbol{\Omega}^{(0)}(\tilde{t})}\hat{g}_\alpha^{(1)}[\boldsymbol{\Psi},\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]\end{aligned} \tag{4.203}$$

and

$$\mathcal{Q}_\alpha^{(1)}(\tilde{t}) = -\hat{q}_\alpha^{(1)}(\mathbf{0},\tilde{t}). \tag{4.204}$$

Discussion

One of the key results of the general analysis of this section is the identification of which pieces of the external forces are required to compute the adiabatic, post-1/2-adiabatic and post-1-adiabatic solutions. From Eqs. (4.191), (4.193) and (4.189), the adiabatic solution depends only on the averaged piece $G_{\lambda\mathbf{0}}^{(1)}(\mathbf{J},\tilde{t}) = \langle G_\lambda^{(1)}(\mathbf{q},\mathbf{J},\tilde{t})\rangle$ of the leading order external force $G_\lambda^{(1)}$. Only the dissipative piece of the force $G_\lambda^{(1)}$ normally contributes to this average. For our application to inspirals in Kerr, this follows from the identity (4.98a) which shows that the average of the conservative piece of $G_\lambda^{(1)}$ vanishes. For a Hamiltonian system with $N = M$, if the perturbing forces g_α and G_β arise from a perturbation $\varepsilon\Delta H = \sum_s \varepsilon^s \Delta H^{(s)}$ to the Hamiltonian, then the forcing function $G_\beta^{(s)}$ is

$$G_\beta^{(s)}(\mathbf{q}, \mathbf{J}, \tilde{t}) = -\frac{\partial \Delta H^{(s)}(\mathbf{q}, \mathbf{J}, \tilde{t})}{\partial q_\beta},$$

and it follows that the average over $\mathbf{q}$ of $G_\beta^{(s)}$ vanishes.

At the next, post-1/2-adiabatic order, it follows from Eqs. (4.195) and (4.197) that the term $\psi_\alpha^{(1/2)}(\tilde{t})$ depends again only on the averaged, dissipative piece $G_{\lambda\mathbf{0}}^{(1)}$ of the leading order force. However, we shall see in the forthcoming paper [138] that when the effects of resonances are included, additional dependencies on the remaining (non-averaged) pieces of the first order self forces will arise.

At the next, post-1-adiabatic order, the term $\psi_\alpha^{(1)}(\tilde{t})$ in Eq. (4.189) depends on the averaged piece $G_{\lambda\mathbf{0}}^{(2)}(\mathbf{J}, \tilde{t}) = \langle G_\lambda^{(2)}(\mathbf{q}, \mathbf{J}, \tilde{t})\rangle$ of the sub-leading force $G_\lambda^{(2)}$, again normally purely dissipative, as well as the remaining conservative and dissipative pieces of the leading order forces $G_\lambda^{(1)}(\mathbf{q}, \mathbf{J}, \tilde{t})$ and $g_\alpha^{(1)}(\mathbf{q}, \mathbf{J}, \tilde{t})$. This can be seen from Eqs. (4.199) and (4.201). These results have been previously discussed briefly in the EMRI context in Refs. [99, 40]. For circular, equatorial orbits, the fact that there is a post-1-adiabatic order contribution from the second order self-force was first argued by Burko [148].

Finally, we consider the choice of initial conditions for the approximate differential equations we have derived. The discussion and conclusions here parallel those in the single variable case, given in Sec. 4.4.4 above, and the results are summarized in Sec. 4.7.3 below

4.5.6 Derivation

We will denote by $\mathcal{R}(\tilde{t})$ the set of resonant N-tuples $\mathbf{k}$ at time $\tilde{t}$, and by $\mathcal{R}^{\mathrm{c}}(\tilde{t})$ the remaining non-resonant nonzero N-tuples. The set of all N-tuples can therefore be written as the disjoint union

$$\mathbf{Z}^N = \{\mathbf{0}\} \,\dot{\cup}\, \mathcal{R}(\tilde{t}) \,\dot{\cup}\, \mathcal{R}^{\mathrm{c}}(\tilde{t}). \tag{4.205}$$

At each order s we introduce the notation $\mathcal{J}_\lambda^{(s)}(\tilde{t})$ for the average part of $J_\lambda^{(s)}(\mathbf{\Psi},\tilde{t})$:

$$\begin{aligned} \mathcal{J}_\lambda^{(s)}(\tilde{t}) &\equiv \langle J_\lambda^{(s)}(\mathbf{\Psi},\tilde{t})\rangle \qquad (4.206)\\ &= \frac{1}{(2\pi)^N}\int_0^{2\pi} d\Psi_1 \ldots \int_0^{2\pi} d\Psi_N J_\lambda^{(s)}(\mathbf{\Psi},\tilde{t}). \end{aligned}$$

We denote by $\hat{J}_\beta^{(s)}$ the remaining part of $J_\beta^{(s)}$, as in Eq. (4.171). This gives the decomposition

$$J_\lambda^{(s)}(\mathbf{\Psi},\tilde{t}) = \mathcal{J}_\lambda^{(s)}(\tilde{t}) + \hat{J}_\lambda^{(s)}(\mathbf{\Psi},\tilde{t}) \tag{4.207}$$

for all $s \geq 0$. Similarly for the angle variable we have the decomposition

$$q_\alpha^{(s)}(\mathbf{\Psi},\tilde{t}) = \mathcal{Q}_\alpha^{(s)}(\tilde{t}) + \hat{q}_\alpha^{(s)}(\mathbf{\Psi},\tilde{t}) \tag{4.208}$$

for all $s \geq 1/2$. For the case $s=0$ we use the fact that $q_\alpha^{(0)}(\mathbf{\Psi},\tilde{t}) - \Psi_\alpha$ is a multiply periodic function of $\mathbf{\Psi}$, by Eq. (4.182a), to obtain the decomposition

$$q_\alpha^{(0)}(\mathbf{\Psi},\tilde{t}) = \Psi_\alpha + \mathcal{Q}_\alpha^{(0)}(\tilde{t}) + \hat{q}_\alpha^{(0)}(\mathbf{\Psi},\tilde{t}), \tag{4.209}$$

where $\hat{q}_\alpha^{(0)}(\mathbf{\Psi},\tilde{t})$ is multiply periodic in $\mathbf{\Psi}$ with zero average.

Using the expansions (4.180a) and (4.180b) of q_α and J_β together with the expansion (4.185) of $d\Psi_\alpha/dt$, we obtain

$$\begin{aligned}
\frac{dq_\alpha}{dt} &= \Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(0)} + \sqrt{\varepsilon}\left[\Omega_\beta^{(1/2)} q_{\alpha,\Psi_\beta}^{(0)} + \Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(1/2)}\right] \\
&+\varepsilon\left[\Omega_\beta^{(1)} q_{\alpha,\Psi_\beta}^{(0)} + \Omega_\beta^{(1/2)} q_{\alpha,\Psi_\beta}^{(1/2)} + \Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(1)} + q_{\alpha,\tilde{t}}^{(0)}\right] \\
&+\varepsilon^{3/2}\left[\Omega_\beta^{(3/2)} q_{\alpha,\Psi_\beta}^{(0)} + \Omega_\beta^{(1)} q_{\alpha,\Psi_\beta}^{(1/2)} + \Omega_\beta^{(1/2)} q_{\alpha,\Psi_\beta}^{(1)}\right. \\
&\left.+\Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(3/2)} + q_{\alpha,\tilde{t}}^{(1/2)}\right] + \varepsilon^2\left[\Omega_\beta^{(2)} q_{\alpha,\Psi_\beta}^{(0)}\right. \\
&+\Omega_\beta^{(3/2)} q_{\alpha,\Psi_\beta}^{(1/2)} + \Omega_\beta^{(1)} q_{\alpha,\Psi_\beta}^{(1)} + \Omega_\beta^{(1/2)} q_{\alpha,\Psi_\beta}^{(3/2)} \\
&\left.+\Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(2)} + q_{\alpha,\tilde{t}}^{(1)}\right] + O(\varepsilon^{5/2}).
\end{aligned} \tag{4.210}$$

We now insert this expansion together with a similar expansion for dJ_λ/dt into the equations of motion (4.100) and use the expansions (4.102) and (4.103) of the external forces g_α and G_λ. Equating coefficients of powers[23] of $\sqrt{\varepsilon}$ then gives at zeroth order

$$\Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(0)} = \omega_\alpha, \tag{4.211a}$$

$$\Omega_\beta^{(0)} J_{\lambda,\Psi_\beta}^{(0)} = 0, \tag{4.211b}$$

at order $O(\sqrt{\varepsilon})$

$$\Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(1/2)} = -\Omega_\beta^{(1/2)} q_{\alpha,\Psi_\beta}^{(0)} + \omega_{\alpha,J_\lambda} J_\lambda^{(1/2)}, \tag{4.212a}$$

$$\Omega_\beta^{(0)} J_{\lambda,\Psi_\beta}^{(1/2)} = -\Omega_\beta^{(1/2)} J_{\lambda,\Psi_\beta}^{(0)}, \tag{4.212b}$$

at order $O(\varepsilon)$

$$\begin{aligned}
\Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(1)} &= -\Omega_\beta^{(1/2)} q_{\alpha,\Psi_\beta}^{(1/2)} - \Omega_\beta^{(1)} q_{\alpha,\Psi_\beta}^{(0)} - q_{\alpha,\tilde{t}}^{(0)} + g_\alpha^{(1)} \\
&+\omega_{\alpha,J_\lambda} J_\lambda^{(1)} + \frac{1}{2}\omega_{\alpha,J_\lambda J_\mu} J_\lambda^{(1/2)} J_\mu^{(1/2)},
\end{aligned} \tag{4.213a}$$

$$\begin{aligned}
\Omega_\beta^{(0)} J_{\lambda,\Psi_\beta}^{(1)} &= -\Omega_\beta^{(1/2)} J_{\lambda,\Psi_\beta}^{(1/2)} - \Omega_\beta^{(1)} J_{\lambda,\Psi_\beta}^{(0)} - J_{\lambda,\tilde{t}}^{(0)} \\
&+G_\lambda^{(1)},
\end{aligned} \tag{4.213b}$$

[23]This is justified since both sides are asymptotic expansions in powers of $\sqrt{\varepsilon}$ at fixed $\mathbf{\Psi}$, $\tilde{t}$.

at order $O(\varepsilon^{3/2})$

$$\begin{aligned}
\Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(3/2)} &= -\Omega_\beta^{(1/2)} q_{\alpha,\Psi_\beta}^{(1)} - \Omega_\beta^{(1)} q_{\alpha,\Psi_\beta}^{(1/2)} - \Omega_\beta^{(3/2)} q_{\alpha,\Psi_\beta}^{(0)} \\
&\quad - q_{\alpha,\tilde{t}}^{(1/2)} + g_{\alpha,q_\beta}^{(1)} q_\beta^{(1/2)} + g_{\alpha,J_\lambda}^{(1)} J_\lambda^{(1/2)} \\
&\quad + \omega_{\alpha,J_\lambda} J_\lambda^{(3/2)} + \omega_{\alpha,J_\lambda J_\mu} J_\lambda^{(1/2)} J_\mu^{(1)} \\
&\quad + \frac{1}{6} \omega_{\alpha,J_\lambda J_\mu J_\sigma} J_\lambda^{(1/2)} J_\mu^{(1/2)} J_\sigma^{(1/2)}, \qquad (4.214a) \\
\Omega_\beta^{(0)} J_{\lambda,\Psi_\beta}^{(3/2)} &= -\Omega_\beta^{(1/2)} J_{\lambda,\Psi_\beta}^{(1)} - \Omega_\beta^{(1)} J_{\lambda,\Psi_\beta}^{(1/2)} - \Omega_\beta^{(3/2)} J_{\lambda,\Psi_\beta}^{(0)} \\
&\quad - J_{\lambda,\tilde{t}}^{(1/2)} + G_{\lambda,q_\beta}^{(1)} q_\beta^{(1/2)} + G_{\lambda,J_\mu}^{(1)} J_\mu^{(1/2)}, \qquad (4.214b)
\end{aligned}$$

and at order $O(\varepsilon^2)$

$$\begin{aligned}
\Omega_\beta^{(0)} q_{\alpha,\Psi_\beta}^{(2)} &= -\Omega_\beta^{(1/2)} q_{\alpha,\Psi_\beta}^{(3/2)} - \Omega_\beta^{(1)} q_{\alpha,\Psi_\beta}^{(1)} - \Omega_\beta^{(3/2)} q_{\alpha,\Psi_\beta}^{(1/2)} \\
&\quad - \Omega_\beta^{(2)} q_{\alpha,\Psi_\beta}^{(0)} - q_{\alpha,\tilde{t}}^{(1)} + g_\alpha^{(2)} + g_{\alpha,q_\beta}^{(1)} q_\beta^{(1)} \\
&\quad + g_{\alpha,J_\lambda}^{(1)} J_\lambda^{(1)} + \frac{1}{2} g_{\alpha,q_\beta q_\gamma}^{(1)} q_\beta^{(1/2)} q_\gamma^{(1/2)} \\
&\quad + \frac{1}{2} g_{\alpha,J_\lambda J_\mu}^{(1)} J_\lambda^{(1/2)} J_\mu^{(1/2)} + g_{\alpha,q_\beta J_\lambda}^{(1)} q_\beta^{(1/2)} J_\lambda^{(1/2)} \\
&\quad + \omega_{\alpha,J_\lambda} J_\lambda^{(2)} + \frac{1}{2} \omega_{\alpha,J_\lambda J_\mu J_\sigma} J_\lambda^{(1)} J_\mu^{(1/2)} J_\sigma^{(1/2)} \\
&\quad + \frac{1}{2} \omega_{\alpha,J_\lambda J_\mu} J_\lambda^{(1)} J_\mu^{(1)} + \omega_{\alpha,J_\lambda J_\mu} J_\lambda^{(1/2)} J_\mu^{(3/2)} \\
&\quad + \frac{1}{24} \omega_{\alpha,J_\lambda J_\mu J_\sigma J_\tau} J_\lambda^{(1/2)} J_\mu^{(1/2)} J_\sigma^{(1/2)} J_\tau^{(1/2)}, \qquad (4.215a) \\
\Omega_\beta^{(0)} J_{\lambda,\Psi_\beta}^{(2)} &= -\Omega_\beta^{(1/2)} J_{\lambda,\Psi_\beta}^{(3/2)} - \Omega_\beta^{(1)} J_{\lambda,\Psi_\beta}^{(1)} - \Omega_\beta^{(3/2)} J_{\lambda,\Psi_\beta}^{(1/2)} \\
&\quad - \Omega_\beta^{(2)} J_{\lambda,\Psi_\beta}^{(0)} - J_{\lambda,\tilde{t}}^{(1)} + G_\lambda^{(2)} + G_{\lambda,q_\beta}^{(1)} q_\beta^{(1)} \\
&\quad + G_{\lambda,J_\mu}^{(1)} J_\mu^{(1)} + \frac{1}{2} G_{\lambda,q_\beta q_\gamma}^{(1)} q_\beta^{(1/2)} q_\gamma^{(1/2)} \\
&\quad + \frac{1}{2} G_{\lambda,J_\mu J_\sigma}^{(1)} J_\mu^{(1/2)} J_\sigma^{(1/2)} + G_{\lambda,q_\beta J_\mu}^{(1)} q_\beta^{(1/2)} J_\mu^{(1/2)}. \qquad (4.215b)
\end{aligned}$$

Here it is understood that all functions of $\mathbf{q}$ and $\mathbf{J}$ are evaluated at $\mathbf{q}^{(0)}$ and $\mathbf{J}^{(0)}$.

Zeroth order analysis

The zeroth order equations (4.211) can be written more explicitly as

$$\Omega_\beta^{(0)}(\tilde{t}) q^{(0)}_{\alpha,\Psi_\beta}(\boldsymbol{\Psi},\tilde{t}) \;=\; \omega_\alpha[\mathbf{J}^{(0)}(\boldsymbol{\Psi},\tilde{t}),\tilde{t}], \tag{4.216a}$$

$$\Omega_\beta^{(0)}(\tilde{t}) J^{(0)}_{\lambda,\Psi_\beta}(\boldsymbol{\Psi},\tilde{t}) \;=\; 0. \tag{4.216b}$$

Since $\mathbf{J}^{(0)}$ is a multiply periodic function of $\boldsymbol{\Psi}$ by Eq. (4.181), we can rewrite Eq. (4.216b) in terms of the Fourier components $J^{(0)}_{\lambda\,\mathbf{k}}(\tilde{t})$ of $J^{(0)}_\lambda$ as

$$\sum_{\mathbf{k}} \left[i\boldsymbol{\Omega}^{(0)}(\tilde{t})\cdot\mathbf{k}\right] J^{(0)}_{\lambda\,\mathbf{k}}(\tilde{t})\, e^{i\mathbf{k}\cdot\boldsymbol{\Psi}} = 0. \tag{4.217}$$

For non-resonant N-tuples $\mathbf{k}$ we have

$$\boldsymbol{\Omega}^{(0)}(\tilde{t})\cdot\mathbf{k} \neq 0 \tag{4.218}$$

by Eqs. (4.177) and (4.193) unless $\mathbf{k} = \mathbf{0}$. This implies that $J^{(0)}_{\lambda\,\mathbf{k}}(\tilde{t}) = 0$ for all nonzero non-resonant $\mathbf{k}$.

It follows that, for a given $\mathbf{k}$, $J^{(0)}_{\lambda\,\mathbf{k}}(\tilde{t})$ must vanish except at those values of $\tilde{t}$ at which $\mathbf{k}$ is resonant. Since we assume that $J^{(0)}_{\lambda\,\mathbf{k}}(\tilde{t})$ is a continuous function of $\tilde{t}$, and since the set of resonant values of $\tilde{t}$ for a given $\mathbf{k}$ consists of isolated points (cf. Sec. 4.5.3 above), it follows that $J^{(0)}_{\lambda\,\mathbf{k}}(\tilde{t})$ vanishes for all nonzero $\mathbf{k}$. The formula (4.190) now follows from the decomposition (4.207).

Next, substituting the formula (4.190) for $\mathbf{J}^{(0)}$ and the decomposition (4.209) of $\mathbf{q}^{(0)}$ into Eq. (4.216a) gives

$$\begin{aligned}\Omega_\alpha^{(0)}(\tilde{t}) \;&+\; \sum_{\mathbf{k}} \left[i\boldsymbol{\Omega}^{(0)}(\tilde{t})\cdot\mathbf{k}\right] \hat{q}^{(0)}_{\alpha\,\mathbf{k}}(\tilde{t})\, e^{i\mathbf{k}\cdot\boldsymbol{\Psi}} \\ &=\; \omega_\alpha[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}],\end{aligned} \tag{4.219}$$

where $\hat{q}^{(0)}_{\alpha\,\mathbf{k}}(\tilde{t})$ are the Fourier components of $\hat{q}^{(0)}_\alpha(\boldsymbol{\Psi},\tilde{t})$. The $\mathbf{k}=0$ Fourier component of this equation gives the formula (4.193) for the zeroth order angular velocity

$\mathbf{\Omega}^{(0)}$. The $\mathbf{k} \neq 0$ Fourier components imply, using an argument similar to that just given for Eq. (4.216b), that $\hat{q}^{(0)}_{\alpha\,\mathbf{k}}(\tilde{t}) = 0$ for all nonzero $\mathbf{k}$. The decomposition (4.209) then gives

$$q^{(0)}_\alpha(\mathbf{\Psi}, \tilde{t}) = \Psi_\alpha + \mathcal{Q}^{(0)}_\alpha(\tilde{t}). \tag{4.220}$$

Finally, the assumption (4.186) forces $\mathcal{Q}^{(0)}_\alpha(\tilde{t})$ to vanish, and we recover the formula (4.192) for $q^{(0)}_\alpha(\mathbf{\Psi}, \tilde{t})$.

Order $O(\sqrt{\varepsilon})$ analysis

The $O(\sqrt{\varepsilon})$ equation (4.212b) can be written more explicitly as

$$\Omega^{(0)}_\beta(\tilde{t}) J^{(1/2)}_{\lambda,\Psi_\beta}(\mathbf{\Psi}, \tilde{t}) = 0, \tag{4.221}$$

where we have simplified using the zeroth order solution (4.190). An argument similar to that given in Sec. 4.5.6 now forces the $\mathbf{\Psi}$ dependent piece of $\mathbf{J}^{(1/2)}$ to vanish, and so we obtain the formula (4.194).

Next, we simplify the order $O(\sqrt{\varepsilon})$ equation (4.212a) using the solutions (4.190), (4.192) and (4.194) to obtain

$$\begin{aligned} \Omega^{(0)}_\beta(\tilde{t}) q^{(1/2)}_{\alpha,\Psi_\beta}(\mathbf{\Psi}, \tilde{t}) &= \omega_{\alpha,J_\lambda}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}] \mathcal{J}^{(1/2)}_\lambda(\tilde{t}) \\ &\quad - \Omega^{(1/2)}_\alpha(\tilde{t}). \end{aligned} \tag{4.222}$$

After averaging with respect to $\mathbf{\Psi}$, the term on the left hand side vanishes since it is a total derivative, and we obtain the formula (4.197) for $\Omega^{(1/2)}(\tilde{t})$. Note however that the function $\boldsymbol{\mathcal{J}}^{(1/2)}(\tilde{t})$ in that formula has not yet been determined; it will be necessary to go to two higher orders in $\sqrt{\varepsilon}$ to compute this function.

Next, we subtract from Eq. (4.222) its averaged part and use the decomposition

(4.208) of $q_\alpha^{(1/2)}$ to obtain

$$\Omega_\beta^{(0)}(\tilde{t})\hat{q}^{(1/2)}_{\alpha,\Psi_\beta}(\mathbf{\Psi},\tilde{t}) = 0. \tag{4.223}$$

An argument similar to that given in Sec. 4.5.6 now shows that $\hat{\mathbf{q}}^{(1/2)} = 0$, and the result (4.196) then follows from the decomposition (4.208) together with the initial condition condition (4.186).

Order $O(\varepsilon)$ analysis

The first order equation (4.213b) can be written more explicitly as

$$\begin{aligned}\Omega_\beta^{(0)}(\tilde{t})J^{(1)}_{\lambda,\Psi_\beta}(\mathbf{\Psi},\tilde{t}) &= -\mathcal{J}^{(0)}_{\lambda,\tilde{t}}(\tilde{t}) \\ &\quad + G_\lambda^{(1)}[\mathbf{\Psi},\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}],\end{aligned} \tag{4.224}$$

where we have simplified using the zeroth order solutions (4.190) and (4.192) and the $O(\sqrt{\varepsilon})$ solution (4.194). We now take the average with respect to $\mathbf{\Psi}$ of this equation. The left hand side vanishes since it is a derivative, and we obtain using the definition (4.166) the differential equation (4.191) for $\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t})$. Next, we subtract from Eq. (4.224) its averaged part, and use the decomposition (4.207) of $\mathbf{J}^{(1)}$. This gives

$$\Omega_\beta^{(0)}(\tilde{t})\hat{J}^{(1)}_{\lambda,\Psi_\beta}(\mathbf{\Psi},\tilde{t}) = \hat{G}_\lambda^{(1)}[\mathbf{\Psi},\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]. \tag{4.225}$$

We solve this equation using the Fourier decomposition (4.173b) of $\hat{G}_\lambda^{(1)}$ to obtain

$$\begin{aligned}\hat{J}_\lambda^{(1)}(\mathbf{\Psi},\tilde{t}) &= \sum_{\mathbf{k}\in\mathcal{R}^c(\tilde{t})} \frac{G^{(1)}_{\lambda\,\mathbf{k}}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]}{i\mathbf{k}\cdot\mathbf{\Omega}^{(0)}(\tilde{t})} e^{i\mathbf{k}\cdot\mathbf{\Psi}} \\ &\quad + \sum_{\mathbf{k}\in\mathcal{R}(\tilde{t})} J^{(1)}_{\lambda\mathbf{k}}(\tilde{t})e^{i\mathbf{k}\cdot\mathbf{\Psi}}.\end{aligned} \tag{4.226}$$

Here the first term is a sum over non-resonant N-tuples, and the second term is a sum over resonant N-tuples, for which the coefficients are unconstrained by Eq.

(4.225). However for each fixed $\mathbf{k}$, the values of $\tilde{t}$ that correspond to resonances are isolated, and furthermore by the the no-resonance assumption (4.218) we have $G^{(1)}_{\beta\mathbf{k}}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}] = 0$ in the vicinity of those values of $\tilde{t}$. Therefore using the assumed continuity of $J^{(1)}_{\lambda\mathbf{k}}(\tilde{t})$ in $\tilde{t}$ we can simplify Eq. (4.226) to

$$\hat{J}^{(1)}_{\lambda}(\boldsymbol{\Psi},\tilde{t}) = \sum_{\mathbf{k}\neq\mathbf{0}} \frac{G^{(1)}_{\lambda\mathbf{k}}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]}{i\mathbf{k}\cdot\boldsymbol{\Omega}^{(0)}(\tilde{t})} e^{i\mathbf{k}\cdot\boldsymbol{\Psi}}, \tag{4.227}$$

where any terms of the form 0/0 that appear in the coefficients are interpreted to be 0. This yields the first term in the result (4.198) for $\mathbf{J}^{(1)}$ when we use the notation (4.175).

Next, we simplify the first order equation (4.213a) using the zeroth order solutions (4.190) and (4.192) and the $O(\sqrt{\varepsilon})$ solutions (4.194) and (4.196), to obtain

$$\begin{aligned}
\Omega^{(0)}_{\beta}(\tilde{t})q^{(1)}_{\alpha,\Psi_\beta}(\boldsymbol{\Psi},\tilde{t}) &= g^{(1)}_{\alpha}[\boldsymbol{\Psi},\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}] - \Omega^{(1)}_{\alpha}(\tilde{t}) \\
&+\omega_{\alpha,J_\lambda}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]J^{(1)}_{\lambda}[\boldsymbol{\Psi},\tilde{t}] \\
&+\frac{1}{2}\omega_{\alpha,J_\lambda J_\mu}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]\mathcal{J}^{(1/2)}_{\lambda}(\tilde{t})\mathcal{J}^{(1/2)}_{\mu}(\tilde{t}).
\end{aligned} \tag{4.228}$$

Averaging with respect to $\boldsymbol{\Psi}$ and using the decompositions (4.207) and (4.208) of $\mathbf{J}^{(1)}$ and $\mathbf{q}^{(1)}$ now gives the formula (4.201) for $\Omega^{(1)}(\tilde{t})$. Note however that the function $\boldsymbol{\mathcal{J}}^{(1)}(\tilde{t})$ in that formula has not yet been determined; it will be necessary to go to two higher orders in $\sqrt{\varepsilon}$ to compute this function.

Finally, we subtract from Eq. (4.228) its average over $\boldsymbol{\Psi}$ using the decompositions (4.207) and (4.208), and then solve the resulting partial differential equation using the notation (4.175) and the convention described after Eq. (4.227). This gives

$$\begin{aligned}
\hat{q}^{(1)}_{\alpha}(\boldsymbol{\Psi},\tilde{t}) &= \frac{\partial\omega_\alpha}{\partial J_\lambda}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]\,\mathcal{I}_{\boldsymbol{\Omega}^{(0)}(\tilde{t})}\hat{J}^{(1)}_{\lambda}[\boldsymbol{\Psi},\tilde{t}] \\
&\quad +\mathcal{I}_{\boldsymbol{\Omega}^{(0)}(\tilde{t})}\hat{g}^{(1)}_{\alpha}[\boldsymbol{\Psi},\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}].
\end{aligned} \tag{4.229}$$

Using the result for $\hat{J}_\beta^{(1)}$ given by the first term in Eq. (4.198) now yields the formula (4.203) for $\hat{q}_\alpha^{(1)}(\mathbf{\Psi}, \tilde{t})$, and the result (4.202) for $q_\alpha^{(1)}$ then follows from the decomposition (4.208) together with the initial condition (4.186).

Order $O(\varepsilon^{3/2})$ analysis

The $O(\varepsilon^{3/2})$ equation (4.214b) can be written more explicitly as

$$\begin{aligned}\Omega_\beta^{(0)}(\tilde{t}) J_{\lambda,\Psi_\beta}^{(3/2)}(\mathbf{\Psi}, \tilde{t}) &= -\Omega_\beta^{(1/2)}(\tilde{t}) J_{\lambda,\Psi_\beta}^{(1)}(\mathbf{\Psi}, \tilde{t}) - \mathcal{J}_{\lambda,\tilde{t}}^{(1/2)}(\tilde{t}) \\ &+ G_{\lambda,J_\mu}^{(1)}[\mathbf{\Psi}, \boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}] \mathcal{J}_\mu^{(1/2)}(\tilde{t}),\end{aligned} \tag{4.230}$$

where we have simplified using the lower order solutions (4.190), (4.192), (4.194) and (4.196). We now take the average with respect to $\mathbf{\Psi}$ of this equation. Two terms vanish since they are total derivatives, and we obtain using the definition (4.166) the differential equation (4.195) for $\boldsymbol{\mathcal{J}}^{(1/2)}(\tilde{t})$. The remaining non-zero Fourier components of Eq. (4.230) can be used to solve for $\hat{\mathbf{J}}^{(3/2)}$, which we will not need in what follows.

Next, we simplify the $O(\varepsilon^{3/2})$ equation (4.214a) using the lower order solutions (4.190), (4.192), (4.194) and (4.196) to obtain

$$\begin{aligned}\Omega_\beta^{(0)}(\tilde{t}) q_{\alpha,\Psi_\beta}^{(3/2)}(\mathbf{\Psi}, \tilde{t}) &= g_{\alpha,J_\lambda}^{(1)}[\mathbf{\Psi}, \boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}] \mathcal{J}_\lambda^{(1/2)}(\tilde{t}) \\ &- \Omega_\alpha^{(3/2)}(\tilde{t}) - \Omega_\beta^{(1/2)}(\tilde{t}) q_{\alpha,\Psi_\beta}^{(1)}(\mathbf{\Psi}, \tilde{t}) \\ &+ \omega_{\alpha,J_\lambda}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}] J_\lambda^{(3/2)}[\mathbf{\Psi}, \tilde{t}] \\ &+ \omega_{\alpha,J_\lambda J_\mu}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}] J_\lambda^{(1)}[\mathbf{\Psi}, \tilde{t}] \mathcal{J}_\mu^{(1/2)}(\tilde{t}) \\ &+ \frac{1}{2}\omega_{\alpha,J_\lambda J_\mu J_\sigma}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}), \tilde{t}] \mathcal{J}_\lambda^{(1/2)}(\tilde{t}) \mathcal{J}_\mu^{(1/2)}(\tilde{t}) \mathcal{J}_\sigma^{(1/2)}(\tilde{t}).\end{aligned} \tag{4.231}$$

The $\mathbf{k} = 0$ component of this equation yields a formula for $\mathbf{\Omega}^{(3/2)}(\tilde{t})$ in terms of

$\boldsymbol{\mathcal{J}}^{(1/2)}(\tilde{t})$ and $\boldsymbol{\mathcal{J}}^{(3/2)}(\tilde{t})$, and the Fourier components with $\mathbf{k} \neq \mathbf{0}$ yield a formula for $\hat{\mathbf{q}}^{(3/2)}$ which we shall not need.

Order $O(\varepsilon^2)$ analysis

We simplify the second order equation (4.215b) using the lower order solutions (4.190), (4.192), (4.194) and (4.196), average over $\mathbf{\Psi}$, and simplify using the decompositions (4.207) and (4.208) and the identities (4.174). The result is

$$\begin{aligned}
\frac{d}{d\tilde{t}}\mathcal{J}_\lambda^{(1)}(\tilde{t}) \quad = \quad & \frac{\partial G^{(1)}_{\lambda\,\mathbf{0}}}{\partial J_\mu}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]\mathcal{J}^{(1)}_\mu(\tilde{t}) + G^{(2)}_{\lambda\,\mathbf{0}}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}] \\
& \frac{1}{2}\frac{\partial^2 G^{(1)}_{\lambda\,\mathbf{0}}}{\partial J_\mu \partial J_\sigma}[\boldsymbol{\mathcal{J}}^{(0)}(\tilde{t}),\tilde{t}]\mathcal{J}^{(1/2)}_\mu(\tilde{t})\mathcal{J}^{(1/2)}_\sigma(\tilde{t}) \\
& + \left\langle \hat{q}^{(1)}_\alpha(\mathbf{\Psi},\tilde{t})\,\frac{\partial \hat{G}^{(1)}_\lambda}{\partial \Psi_\alpha}\left[\mathbf{\Psi},\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}\right]\right\rangle \\
& + \left\langle \hat{J}^{(1)}_\mu(\mathbf{\Psi},\tilde{t})\,\frac{\partial \hat{G}^{(1)}_\lambda}{\partial J_\mu}\left[\mathbf{\Psi},\mathcal{J}^{(0)}(\tilde{t}),\tilde{t}\right]\right\rangle . \qquad (4.232)
\end{aligned}$$

Using the expressions (4.203) and (4.198) for $\hat{q}^{(1)}_\alpha$ and $\hat{J}^{(1)}_\alpha$ now gives the differential equations (4.199) for $\boldsymbol{\mathcal{J}}^{(1)}$.[24]

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[24] We remark that a slight inconsistency arises in our solution ansatz (4.180) at this order, $O(\varepsilon^2)$. Consider the $\mathbf{k} \neq 0$ Fourier components of the second order equations (4.215). For resonant n-tuples $\mathbf{k}$, the left hand sides of these two equations vanish by definition, but the right hand sides are generically nonzero, due to the effects of subleading resonances. A similar inconsistency would arise in the $O(\varepsilon)$ equations (4.213), but for the fact that our no-resonance assumption (4.179) forces the right hand sides of those equations to vanish for resonant n-tuples. However, the no-resonance assumption (4.179) is insufficient to make the right hand sides of the $O(\varepsilon^2)$ equations (4.215) vanish, because of the occurrence of quadratic cross terms such as

$$g^{(1)}_{\alpha\,\mathbf{k}}\,g^{(1)}_{\beta\,\mathbf{k}'}\,e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{\Psi}}.$$

It can be shown, by an analysis similar to that given in Ref. [138], that the effect of these subleading resonances is to (i) restrict the domain of validity of the expansion (4.180) to exclude times $\tilde{t}$ at which subleading resonances occur, and (ii) to add source terms to the differential equation for $\boldsymbol{\mathcal{J}}^{(3/2)}$ which encode the effect of passing through a subleading resonance. These modifications do not affect any of the conclusions in the present paper.

4.6 Numerical Integration of an illustrative example

In this section we present a numerical integration of a particular example of a dynamical system, in order to illustrate and validate the general theory of Secs. 4.4 and 4.5.

Consider the system of equations

$$\dot{q} = \omega(J) + \varepsilon g^{(1)}(q, J) \tag{4.233a}$$

$$\dot{J} = \varepsilon G^{(1)}(q, J), \tag{4.233b}$$

where

$$\begin{aligned} \omega(J) &= 1 + J - J^2/4, \\ g^{(1)}(q, J) &= \sin(q)/J, \\ G^{(1)}(q, J) &= -J - J^2/4 - J\cos(q) - J^2 \sin(q), \end{aligned} \tag{4.234}$$

together with the initial conditions $q(0) = 1$, $J(0) = 1$, and with $\varepsilon = 10^{-3}$. The exact solution of this system is shown in Fig. 4.2.

Consider now the adiabatic approximation to this system. From Eqs. (4.128) – (4.133) the adiabatic approximation is given by the system

$$\frac{d\psi^{(0)}}{d\tilde{t}} = \omega(\mathcal{J}^{(0)}), \tag{4.235a}$$

$$\frac{d\mathcal{J}^{(0)}}{d\tilde{t}} = -\mathcal{J}^{(0)} - \mathcal{J}^{(0)\,2}/4, \tag{4.235b}$$

where $\tilde{t} = \varepsilon t$. The adiabatic solution $(q_{\rm ad}, J_{\rm ad})$ is given in terms of the functions $\psi^{(0)}(\tilde{t})$ and $\mathcal{J}^{(0)}(\tilde{t})$ by

$$q_{\rm ad}(t, \varepsilon) = \varepsilon^{-1}\psi^{(0)}(\varepsilon t), \qquad J_{\rm ad}(t, \varepsilon) = \mathcal{J}^{(0)}(\varepsilon t). \tag{4.236}$$

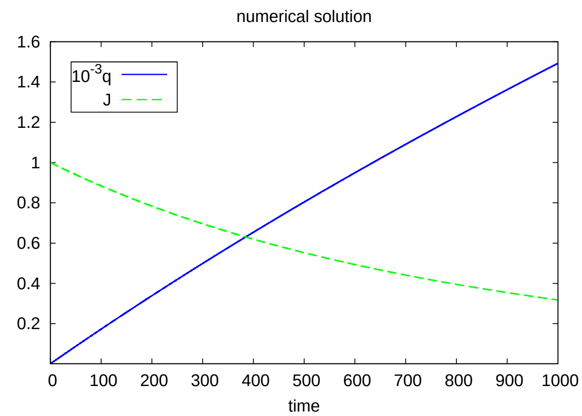

Figure 4.2: The exact numerical solution of the system of equations (4.233). After a time $\sim 1/\varepsilon$, the action variable J is $O(1)$, while the angle variable q is $O(1/\varepsilon)$.

To this order, the initial conditions on $(q_{\rm ad}, J_{\rm ad})$ are the same as those for (q, J), which gives $\psi^{(0)}(0) = \varepsilon$ [25] and $\mathcal{J}^{(0)}(0) = 1$. We expect to find that after a time $t \sim 1/\varepsilon$, the errors are of order ~ 1 for $q_{\rm ad}(t)$, and of order $\sim \varepsilon$ for $J_{\rm ad}(t)$. This is confirmed by the two upper panels in Fig. 4.3, which show the differences $q - q_{\rm ad}$ and $J - J_{\rm ad}$.

Consider next the post-1-adiabatic approximation. From Eqs. (4.136) and (4.137) this approximation is given by the system of equations

$$\frac{d\psi^{(1)}}{d\tilde{t}} = \omega_{,J}(\mathcal{J}^{(0)})\mathcal{J}^{(1)}, \tag{4.237a}$$

$$\frac{d\mathcal{J}^{(1)}}{d\tilde{t}} = -(1 + \mathcal{J}^{(0)}/2)\mathcal{J}^{(1)} + \frac{\mathcal{J}^{(0)}(\mathcal{J}^{(0)} + 1)}{2\omega(\mathcal{J}^{(0)})}, \tag{4.237b}$$

together with the adiabatic system (4.235). From Eqs. (4.129) and (4.135) the

[25] Strictly speaking, our derivations assumed that $\psi^{(0)}(\tilde{t})$ is independent of ε, and so it is inconsistent to use this initial condition for $\psi^{(0)}(0)$. Instead we should set $\psi^{(0)}(0) = 0$, and take account of the nonzero initial phase $q(0)$ at the next order, in the variable $\psi^{(1)}(0)$. However, moving a constant from $\psi^{(1)}(\tilde{t})$ to $\varepsilon^{-1}\psi^{(0)}(\tilde{t})$ does not affect the solution, and so we are free to choose the initial data as done here.

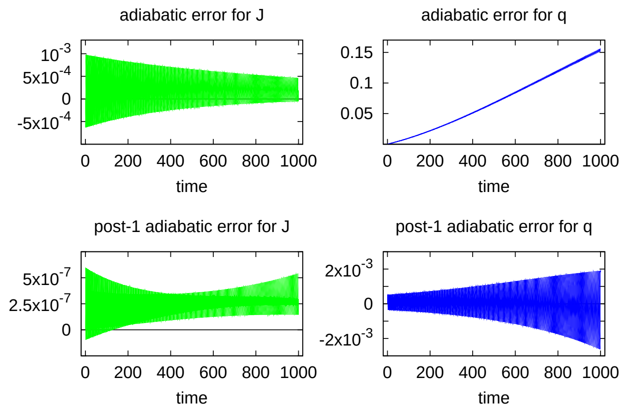

Figure 4.3: *Upper panels:* The difference between the solution of the exact dynamical system (4.233) and the adiabatic approximation given by Eqs. (4.235) and (4.236). For the action variable J, this difference is $O(\varepsilon)$, while for the angle variable q, this difference is $O(1)$, as expected. *Lower panels:* The difference between the exact solution and the post-1-adiabatic approximation given by Eqs. (4.235), (4.237) and (4.238). Again the magnitudes of these errors are as expected: $O(\varepsilon^2)$ for J and $O(\varepsilon)$ for q.

post-1-adiabatic solution $(q_{\rm p1a}, J_{\rm p1a})$ is given by

$$\begin{aligned} q_{\rm p1a}(t,\varepsilon) &= \varepsilon^{-1}\psi^{(0)}(\varepsilon t) + \psi^{(1)}(\varepsilon t), && (4.238a) \\ J_{\rm p1a}(t,\varepsilon) &= \mathcal{J}^{(0)}(\varepsilon t) + \varepsilon \mathcal{J}^{(1)}(\varepsilon t) \\ &\quad + \varepsilon H[\mathcal{J}^{(0)}(\varepsilon t), q_{\rm p1a}(t,\varepsilon)], && (4.238b) \end{aligned}$$

where the function H is given by

$$H(\mathcal{J}, q) = \frac{\mathcal{J}^2 \cos q - \mathcal{J} \sin q}{\omega(\mathcal{J})}. \tag{4.239}$$

Consider next the choice of initial conditions $\psi^{(0)}(0)$, $\psi^{(1)}(0)$, $\mathcal{J}^{(0)}(0)$ and $\mathcal{J}^{(1)}(0)$ for the system of equations (4.235) and (4.237). From Eqs. (4.238) these choices are constrained by, to $\mathrm{O}(\varepsilon^2)$,

$$\begin{aligned} q(0) &= \varepsilon^{-1}\psi^{(0)}(0) + \psi^{(1)}(0), && (4.240a) \\ J(0) &= \mathcal{J}^{(0)}(0) + \varepsilon \mathcal{J}^{(1)}(0) + \varepsilon H[J(0), q(0)]. && (4.240b) \end{aligned}$$

We solve these equations by taking $\psi^{(0)}(0) = 0$, $\psi^{(1)}(0) = q(0) = 1$, $\mathcal{J}^{(0)}(0) = J(0) = 1$, and $\mathcal{J}^{(1)}(0) = -H[J(0), q(0)]$. We expect to find that after a time $t \sim 1/\varepsilon$, the errors are of order $\sim \varepsilon$ for $q_{\rm p1a}(t)$, and of order $\sim \varepsilon^2$ for $J_{\rm p1a}(t)$. This is confirmed by the two lower panels in Fig. 4.3, which show the differences $q - q_{\rm p1a}$ and $J - J_{\rm p1a}$.

4.7 Discussion

In Sec. 4.2 above we derived the set of equations (4.59) describing the radiation-reaction driven inspiral of a particle into a spinning black hole, in terms of generalized action angle variables. Although those equations contain some functions which are currently unknown, it is possible to give a general analysis of the dependence of the solutions on the mass ratio $\varepsilon = \mu/M$ as $\varepsilon \to 0$, using two-timescale expansions. That analysis was presented in Secs. 4.3 – 4.6 above, for the general class of equation systems (4.100) of which the Kerr inspiral example (4.59) is a special case. In this final section we combine these various results and discuss the implications for our understanding of inspirals into black holes.

4.7.1 Consistency and uniqueness of approximation scheme

Our analysis has demonstrated that the adiabatic approximation method gives a simple and unique prescription for computing successive approximations to the exact solution, order by order, which is free of ambiguities. In this sense it is similar to the post-Newtonian approximation method.[26] This is shown explicitly in Sec. 4.4.5, which shows that the adiabatic method can be extended to all orders for the case of a single degree of freedom, and in Sec. 4.6, which shows how the method works in practice in a numerical example. In particular there is no ambiguity in the assignment of initial conditions when computing adiabatic or post-1-adiabatic approximations.

This conclusion appears to be at odds with a recent analysis of Pound and Poisson (PP) [135]. These authors conclude that "An adiabatic approximation to the exact differential equations and initial conditions, designed to capture the secular changes in the orbital elements and to discard the oscillations, would be very difficult to formulate without prior knowledge of the exact solution." The reason for the disagreement is in part a matter of terminology: PP's definition of "adiabatic approximation" is different to ours.[27] They take it to mean an approximation which (i) discards all the pieces of the true solutions that vary on the rapid timescale ~ 1, and retains the pieces that vary on the slow timescale $\sim 1/\varepsilon$; and (ii) is globally accurate to some specified order in ε over an inspiral time – throughout their paper they work to the first subleading order, i.e. post-1-adiabatic order. In our terminology, their approximation would consist of the adiabatic approximation, plus the secular piece of the post-1-adiabatic approximation [given by omitting the first term in Eq. (4.198)].

The difference in the terminology used here and in PP is not the only reason for the different conclusions. Our formalism shows that PP's "adiabatic approximation" is actually straightforward to formulate, and that prior knowledge of the exact solution is not required. The reason for the different conclusions is as fol-

[26]The analogy is closer when the two-timescale method is extended to include the field equations and wave generation as well as the inspiral motion [139].

[27]In a later version of their paper they call it instead a "secular approximation".

lows. By "exact solution" PP in fact meant any approximation which includes the rapidly oscillating pieces at post-1-adiabatic order. Their intended meaning was that, since the secular and rapidly oscillating pieces are coupled together at post-1-adiabatic order, any approximation which completely neglects the oscillations cannot be accurate to post-1-adiabatic order [158]. We agree with this conclusion.

On the other hand, we disagree with the overall pessimism of PP's conclusion, because we disagree with their premise. Since the qualitative arguments that were originally presented for the radiative approximation involved discarding oscillatory effects [128, 99], PP chose to examine general approximation schemes that neglect oscillatory effects[28] and correctly concluded that such schemes cannot be accurate beyond the leading order. However, our viewpoint is that there is no need to restrict attention to schemes that neglect all oscillatory effects. The two timescale scheme presented here yields leading order solutions which are not influenced by oscillatory effects, and higher order solutions whose secular pieces are. The development of a systematic approximation scheme that exploits the disparity in orbital and radiation reaction timescales need not be synonymous with neglecting all oscillatory effects.

4.7.2 Effects of conservative and dissipative pieces of the self force

As we have discussed in Secs. 4.4.4 and 4.5.5 above, our analysis shows rigorously that the dissipative piece of the self force contributes to the leading order, adiabatic motion, while the conservative piece does not, as argued in Refs. [128, 99]. It is

[28]In the strong sense of neglecting the influence of the oscillatory pieces of the solution on the secular pieces, as well as neglecting the oscillatory pieces themselves.

possible to understand this fundamental difference in a simple way as follows. We use units where the orbital timescale is ~ 1 and the inspiral timescale is $\sim 1/\varepsilon$. Then the total phase accumulated during the inspiral is $\sim 1/\varepsilon$, and this accumulated phase is driven by the dissipative piece of the self force.

Consider now the effect of the conservative piece of the self force. As a helpful thought experiment, imagine setting to zero the dissipative piece of the first order self force. What then is the effect of the conservative first order self-force on the dynamics? We believe that the perturbed motion is likely to still be integrable; arguments for this will be presented elsewhere [137, 138]. However, even if the perturbed motion is not integrable, the Kolmogorov-Arnold-Moser (KAM) theorem [152] implies that the perturbed motion will generically be confined to a torus in phase space for sufficiently small ε. The effect of the conservative self force is therefore roughly to give an $O(\varepsilon)$ distortion to this torus, and to give $O(\varepsilon)$ corrections to the fundamental frequencies.[29] If one now adds the effects of dissipation, we see that after the inspiral time $\sim 1/\varepsilon$, the corrections due to the conservative force will give a fractional phase correction of order $\sim \varepsilon$, corresponding to a total phase correction ~ 1. This correction therefore comes in at post-1-adiabatic order.

Another way of describing the difference is that the dissipative self-force produces secular changes in the orbital elements, while the conservative self-force does not at the leading order in ε. In Ref. [99] this difference was overstated: it was claimed that the conservative self-force does not produce any secular effects. However, once one goes beyond the leading order, adiabatic approximation, there are in fact conservative secular effects. At post-1-adiabatic order these are described by the first term on the right hand side of Eq. (4.201). This error was pointed out

[29]This corresponds to adding to the frequency ω_α in Eq. (4.163a) the average over $\mathbf{q}$ of the term $\varepsilon g_\alpha^{(1)}$.

by Pound and Poisson [159, 135].

4.7.3 The radiative approximation

So far in this paper we have treated the self force as fixed, and have focused on how to compute successive approximations to the inspiralling motion. However, as explained in the introduction, the first order self force is currently not yet known explicitly. The time-averaged, dissipative[30] piece is known from work of Mino and others [128, 99, 40, 41, 129]. The remaining, fluctuating piece of the dissipative first order self force has not been computed but will be straightforward to compute[31]. The conservative piece of the first order self force will be much more difficult to compute, and is the subject of much current research [108, 110, 111, 112, 113].

It is natural therefore to consider the *radiative approximation* obtained by using only the currently available, radiative piece of the first order self force, as suggested by Mino [128], and by integrating the orbital equations exactly (eg numerically). How well will this approximation perform?

From our analysis it follows that the motion as computed in this approximation will agree with the true motion to adiabatic order, and will differ at post-1-adiabatic order. At post-1-adiabatic order, it will omit effects due to the conservative first order force, and also effects due to the dissipative second order self force. It will include post-1-adiabatic effects due to the fluctuating pieces of the first order, dissipative self force, and so would be expected to be more accurate than the

[30]We use the terms radiative and dissipative interchangeably; both denote the time-odd piece of the self force, as defined by Eq. (4.86) above.

[31]For example, by evaluating $J_{\omega lmkn}$ from Eq. (8.21) of Ref. [40] at $\omega = \omega_{mk'n'}$ instead of $\omega = \omega_{mkn}$.

adiabatic approximation.[32] EMRI waveforms computed using this approximation will likely be the state of the art for quite some time.

Our conclusions about the radiative approximation appear to differ from those of PP [135], who argue that " The radiative approximation does not achieve the goals of an adiabatic approximation". Here, however, the different conclusions arise entirely from a difference in terminology, since PP define "adiabatic approximation" to include slowly varying pieces of the solution to at least post-1-adiabatic order. The radiative approximation does produce solutions that are accurate to adiabatic order, as we have defined it.

We now discuss in more detail the errors that arise in the radiative approximation. These errors occur at post-1-adiabatic order. For discussing these errors, we will neglect post-2-adiabatic effects, and so it is sufficient to use our post-1-adiabatic dynamical equations (4.199) and (4.201). These equations have the structure

$$\mathcal{D}\begin{bmatrix} \psi_\alpha^{(1)}(\tilde{t}) \\ \mathcal{J}_\lambda^{(1)}(\tilde{t}) \end{bmatrix} = \mathcal{S}, \tag{4.241}$$

where $\mathcal{D}$ is a linear differential operator and $\mathcal{S}$ is a source term. The appropriate initial conditions are [see Sec. 4.4.4]

$$\psi_\alpha^{(0)} = 0, \qquad \mathcal{J}_\lambda^{(0)}(0) = J_\lambda(0), \tag{4.242a}$$

$$\psi_\alpha^{(1)} = q_\alpha(0), \qquad \mathcal{J}_\lambda^{(1)}(0) = -H_\lambda[\mathbf{q}(0), \mathbf{J}(0)], \tag{4.242b}$$

where $\mathbf{q}(0)$ and $\mathbf{J}(0)$ are the exact initial conditions and the function H_λ is given by, from Eq. (4.198),

$$H_\lambda(\mathbf{q}, \mathbf{J}) = \mathcal{I}_{\mathbf{\Omega}^{(0)}{}_{(0)}} \hat{G}_\lambda^{(1)}[\mathbf{q}, \mathbf{J}, 0]. \tag{4.243}$$

[32] It is of course possible that, due to an accidental near-cancellation of different post-1-adiabatic terms, the adiabatic approximation may be closer to the true solution than the radiative approximation.

In terms of these quantities, the radiative approximation is equivalent to making the replacements

$$
\begin{aligned}
g_\alpha^{(1)}(\mathbf{q},\mathbf{J}) &\rightarrow g_{\alpha\,\mathrm{diss}}^{(1)}(\mathbf{q},\mathbf{J}), &\quad (4.244a)\\
G_i^{(1)}(\mathbf{q},\mathbf{J}) &\rightarrow G_{i\,\mathrm{diss}}^{(1)}(\mathbf{q},\mathbf{J}), &\quad (4.244b)\\
G_i^{(2)}(\mathbf{q},\mathbf{J}) &\rightarrow 0. &\quad (4.244c)
\end{aligned}
$$

These replacements have two effects: (i) they give rise to an error in the source term $\mathcal{S}$ in Eq. (4.241), and (ii) they give rise to an error in the function H_λ and hence in the initial conditions (4.242). There are thus two distinct types of errors that occur in the radiative approximation.[33]

The second type of error could in principle be removed by adjusting the initial conditions appropriately. For fixed initial conditions $\mathbf{q}(0)$ and $\mathbf{J}(0)$, such an adjustment would require knowledge of the conservative piece of the self force, and so is not currently feasible. However, in the context of searches for gravitational wave signals, matched filtering searches will automatically vary over a wide range of initial conditions. Therefore the second type of error will not be an impediment to detecting gravitational wave signals. It will, however, cause errors in parameter extraction.

This fact that the error in the radiative approximation can be reduced by adjusting the initial conditions was discovered by Pound and Poisson [160], who numerically integrated inspirals in Schwarzschild using post-Newtonian self-force expressions. Their "time-averaged" initial conditions, which they found to give the highest accuracy, correspond to removing the second type of error discussed above, that is, using the initial conditions (4.242) with the exact function H_λ rather than

[33] These two errors are both secular, varying on long timescales. There is in addition a rapidly oscillating error caused by the correction to the first term in the expression (4.198) for $J_\lambda^{(1)}$.

the radiative approximation to H_λ.

Finally, we note that given the radiative approximation to the self force, one can compute waveforms using the radiative approximation as described above, or compute waveforms in the adiabatic approximation by solving equations (4.188), (4.191) and (4.193) using the replacement (4.244b). This second option would be easier although somewhat less accurate.

4.7.4 Utility of adiabatic approximation for detection of gravitational wave signals

The key motivation for accurate computations of waveforms from inspiral events is of course their use for detecting and analyzing gravitational wave signals. How well will the adiabatic and radiative approximations perform in practice? In this section, we review the studies that have been made of this question. These studies are largely consistent with one another, despite differences in emphasis and interpretation that can be found in the literature. We restrict attention to inspirals in Schwarzschild, and to circular or equatorial inspirals in Kerr; fully general orbits present additional features that will be discussed elsewhere [137, 138].

First, we note that in this paper we have focused on how the post-1-adiabatic error in phase scales with the mass ratio $\varepsilon = \mu/M$. However, one can also ask how the error scales with the post-Newtonian expansion parameter $v/c \sim \sqrt{M/r}$. From Eq. (A10) of Ref. [40] it follows that the post-1-adiabatic phase errors scale as

$$\sim \left(\frac{\mu}{M}\right)^0 \left(\frac{v}{c}\right)^{-3};$$

this scaling is consistent with the more recent analysis of Ref. [160]. This scaling does imply that the error gets large in the weak field regime, as correctly argued in Ref. [160]. However, it does not necessarily imply large errors in the relativistic regime $v/c \sim 1$ relevant to LISA observations.

The first, order of magnitude estimates of the effects of the conservative piece of the self force were made by Burko in Refs. [161, 162]. Refs. [99, 40] computed the post-1-adiabatic phase error within the post-Newtonian approximation for circular orbits, minimized over some of the template parameters, and evaluated at frequencies relevant for LISA. The results indicated a total phase error of order one cycle, not enough to impede detection given that maximum coherent integration times are computationally limited to ~ 3 weeks [27]. This result was extended to eccentric orbits with eccentricities $\lesssim 0.4$ in Refs. [163, 164], with similar results. Similar computations were performed by Burko in Refs. [87, 165], although without minimization over template parameters.

These analyses all focused on extreme mass ratio inspirals for LISA. For intermediate mass ratio inspirals, potential sources for LIGO, the post-1-adiabatic corrections were studied within the post-Newtonian approximation in Refs. [29, 166]. Ref. [29] computed fitting factors in addition to phase errors, found that the associated loss of signal to noise ratio would be less than 10% in all but the most rapidly spinning cases, and concluded that it would be "worthwhile but not essential" to go beyond adiabatic order for detection templates.

The most definitive study to date of post-adiabatic errors for LISA in the Schwarzschild case was performed by Pound and Poisson (PP1) [160]. PP1 numerically integrated the geodesic equations with post-Newtonian expressions for the self force, with and without conservative terms. PP1 found large phase errors,

$\delta\phi \gtrsim 100$, in the weak field regime. However, the regime relevant to LISA observations is $p \lesssim 30$ [23][34], where p is the dimensionless semilatus rectum parameter defined by PP1, and PP1's results are focused mostly on values of p larger than this[35]. It is therefore difficult to compare the results of PP1 with earlier estimates or to use them directly to make inferences about signal detection with LISA. PP1's results do show clearly that the errors increase rapidly with increasing eccentricity.

We have repeated PP1's calculations, reproducing the results of their Fig. 6, and extended their calculations to more relativistic systems at lower values of p. More specifically, we performed the following computation: (i) Select values of the mass parameters M and μ, and the initial eccentricity e; (ii) Choose the initial value of semilatus rectum p to correspond to one year before the last stable orbit, which occurs on the separatrix $p = 6 + 2e$; (iii) Choose the radiative evolution and the exact evolution to line up at some matching time t_{m} during the last year of inspiral; (iv) Start the radiative and exact evolutions with slightly different initial conditions in order that the secular pieces of the evolutions initially coincide – this is the "time-averaged" initial data prescription of PP1; (v) Compute the maximum of the absolute value of the phase error $\delta\phi$ incurred during the last year; (vi) Minimize over the matching time t_{m}; and (vii) Repeat for different values of M, μ and e. As an example, for $M = 10^6 M_\odot$ and $\mu = 10 M_\odot$, an inspiral starting at $(p, e) = (10.77, 0.300)$ ends up at $(6.31, 0.153)$ after one year. We match the two evolutions at 0.2427 years before plunge, with the exact evolution starting at $(p, e) = (8.81933, 0.210700)$ and the radiative evolution starting at

[34]It is true that there will be some binaries visible to LISA at higher values of p, that do not merge within the LISA mission lifetime. However post-Newtonian templates should be sufficient for the detection of these systems.

[35]The second panel of their Fig. 6 does show phase shifts for smaller values of p, but these are all for a mass ratio of $\varepsilon = 0.1$, too large to be a good model of LISA observations; although the phase shift becomes independent of ε as $\varepsilon \to 0$, their Fig. 6 shows that it can vary by factors of up to ~ 10 as ε varies between 0.1 and 0.001.

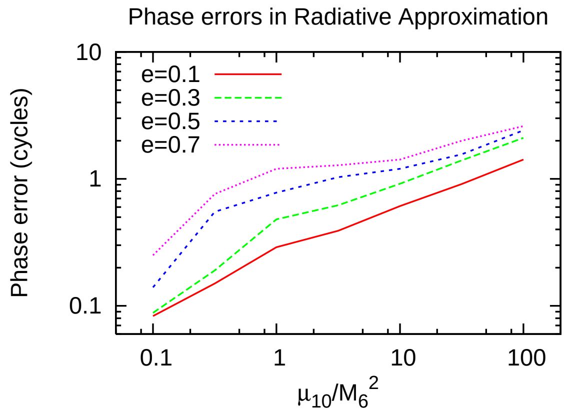

Figure 4.4: The maximum orbital phase error in cycles, $\delta N = \delta\phi/(2\pi)$, incurred in the radiative approximation during the last year of inspiral, as a function of the mass M_6 of the central black hole in units of $10^6 M_\odot$, the mass μ_{10} of the small object in units of $10\,M_\odot$, and the eccentricity e of the system at the start of the final year of inspiral. The exact and radiative inspirals are chosen to line up at some time $t_{\rm m}$ during the final year, and the value of $t_{\rm m}$ is chosen to minimize the phase error. The initial data at time $t_{\rm m}$ for the radiative evolution is slightly different to that used for the exact evolution in order that the secular pieces of the two evolutions initially coincide; this is the "time-averaged" initial data prescription of Pound and Poisson. All evolutions are computed using the hybrid equations of motion of Kidder, Will and Wiseman in the osculating-element form given by Pound and Poisson.

$(p, e) = (8.81928, 0.210681)$. The maximum phase error incurred in the last year is then 0.91 cycles.

The phase error incurred during an inspiral from some initial values of e and p to the plunge is independent of the masses M and μ in the small mass ratio limit. However the phase error incurred during the last year of inspiral is not, since the initial value of p depends on the inspiral timescale $\sim M^2/\mu$. The result is that the phase error depends only on the combination of masses M^2/μ to a good approximation.

Our results are shown in Fig. 4.4. This figure shows, firstly, that the computational method of PP1 gives results for low eccentricity systems that are roughly consistent with earlier, cruder, estimates, with total phase errors of less than one cycle over most of the parameter space. It also shows that for large eccentricity systems the total phase error can be as large as two or three cycles.

How much will the phase errors shown in Fig. 4.4 impede the use of the radiative approximation to detect signals? There are two factors which will help. First, Fig. 4.4 shows the maximum phase error during the last year of inspiral, while for detection phase coherence is needed only for periods of ~ 3 weeks [27]. Second, the matched filtering search process will automatically select parameter values to maximize the overlap between the template and true signal, and parameter mismatches will therefore be likely to reduce the effect of the phase error[36]. On the other hand, for large eccentricities, the phase error $\delta\phi(t)$ is typically a rapidly oscillating function, rather than a smooth function, which may counteract the helpful effects of smaller time windows or parameter mismatches. Also we note that a sign flip will occur in the integrand of an overlap integral once the gravitational wave phase error $2\delta\phi$ exceeds π, corresponding to the number of cycles plotted in Fig. 4.4 exceeding 1/4. This occurs in a large part of the parameter space.

Thus, there is a considerable amount of uncertainty as to whether the radiative approximation will be sufficiently accurate for signal detection. A detailed study would require computation of fitting factors and optimizing over all template parameters, and modeling the hierarchical detection algorithm discussed in Ref. [27]. Such a study is beyond the scope of this paper. Based on the results shown in

[36]We note that there are already two minimizations over parameters included in the phase errors shown in Fig. 4.4: a minimization over $t_{\rm m}$ as discussed above, and the replacement $m_1 \to m_1 + m_2$ used by PP1 in the derivation of their self-force expressions in order to eliminate the leading order piece of the self-force.

Fig. 4.4, we agree with the conclusions of PP1 that the early estimates based on circular orbits [99, 40] were too optimistic, and that it is not clear that the radiative approximation is sufficiently accurate. (Moreover parameter extraction will clearly require going beyond the radiative approximation.)

For gravitational wave searches, it might therefore be advisable to use *hybrid* waveforms, computed using the fully relativistic dissipative piece of the self force, and using post-Newtonian expressions for the conservative piece. Although the post-Newtonian expressions are not expected to be very accurate in the relativistic regime, improved versions have been obtained recently based on comparisons between post-Newtonian and fully numerical waveforms from binary black hole mergers; see, for example, the effective one body approximation of Refs. [167, 168]. It seems likely that hybrid EMRI waveforms incorporating such improved post-Newtonian expressions for the conservative self force will be more accurate than radiative waveforms. Hybrid waveforms may be the best that can be done until the fully relativistic conservative self-force is computed.

4.8 Conclusions

In this paper we have developed a systematic two-timescale approximation method for computing the inspirals of particles into spinning black holes. Future papers in this series will deal with the effects of transient resonances [137, 138], and will give more details of the two-timescale expansion of the Einstein equations [139] that meshes consistently with the approximation method for orbital motion discussed here.

4.8.1 Acknowledgements

We thank Steve Drasco, Marc Favata, John Friedman, Scott Hughes, Yasushi Mino, Eric Poisson, Adam Pound and Eran Rosenthal for helpful conversations. This research was supported in part by NSF grant PHY-0457200 and NASA grant NAGW-12906. TH was supported in part by the John and David Boochever Prize Fellowship in Theoretical Physics at Cornell.

4.9 Appendix: Explicit expressions for the coefficients in the action-angle equations of motion

From the formulae (4.39) for the action variables together with the definitions (4.38) of the potentials V_r and V_θ we can compute the partial derivatives $\partial J_\alpha/\partial P_\beta$. The non-trivial derivatives are

$$\frac{\partial J_r}{\partial H} = \frac{Y}{\pi}, \tag{4.245a}$$

$$\frac{\partial J_r}{\partial E} = \frac{W}{\pi}, \tag{4.245b}$$

$$\frac{\partial J_r}{\partial L_z} = -\frac{Z}{\pi}, \tag{4.245c}$$

$$\frac{\partial J_r}{\partial Q} = -\frac{X}{2\pi}, \tag{4.245d}$$

$$\frac{\partial J_\theta}{\partial H} = \frac{2\sqrt{z_+}a^2}{\pi\beta}\left[K(k) - E(k)\right], \tag{4.245e}$$

$$\frac{\partial J_\theta}{\partial E} = \frac{2\sqrt{z_+}Ea^2}{\pi\beta}\left[K(k) - E(k)\right], \tag{4.245f}$$

$$\frac{\partial J_\theta}{\partial L_z} = \frac{2L_z}{\pi\beta\sqrt{z_+}}\left[K(k) - \Pi(\pi/2, z_-, k)\right], \tag{4.245g}$$

$$\frac{\partial J_\theta}{\partial Q} = \frac{1}{\pi\beta\sqrt{z_+}}K(k). \tag{4.245h}$$

Here the quantities W, X, Y and Z are the radial integrals defined by Schmidt [37] as [150]

$$\begin{aligned}
W &= \int_{r_1}^{r_2} \frac{r^2 E(r^2+a^2) - 2Mra(L_z - aE)}{\Delta\sqrt{V_r}} dr, && (4.246a)\\
X &= \int_{r_1}^{r_2} \frac{dr}{\sqrt{V_r}}, && (4.246b)\\
Y &= \int_{r_1}^{r_2} \frac{r^2}{\sqrt{V_r}} dr, && (4.246c)\\
Z &= \int_{r_1}^{r_2} \frac{r\left[L_z r - 2M(L_z - aE)\right]}{\Delta\sqrt{V_r}} dr, && (4.246d)
\end{aligned}$$

where r_1 and r_2 are the turning points of the radial motion, i.e. the two largest roots of $V_r(r) = 0$. In these equations $K(k)$ is the complete elliptic integral of the first kind, $E(k)$ is the complete elliptic integral of the second kind, and $\Pi(\phi, n, k)$ is the Legendre elliptic integral of the third kind [169]:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \tag{4.247}$$

$$E(k) = \int_0^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2\theta}, \tag{4.248}$$

$$\Pi(\phi, n, k) = \int_0^{\phi} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}. \tag{4.249}$$

Also we have defined $\beta^2 = a^2(\mu^2 - E^2)$ and $k = \sqrt{z_-/z_+}$, where $z = \cos^2\theta$ [38] and z_- and z_+ are the two roots of $V_\theta(z) = 0$ with $0 < z_- < 1 < z_+$.

Combining the derivatives (4.245) with the chain rule in the form

$$\frac{\partial P_\alpha}{\partial J_\beta}\frac{\partial J_\beta}{\partial P_\gamma} = \delta^\alpha_\gamma \tag{4.250}$$

[37] There is a typo in the definition of W given in Eq. (44) of Schmidt [150].

[38] Here we follow Drasco and Hughes [100] rather than Schmidt who defines $z = \cos\theta$.

yields the following expression for the frequencies (4.26) as functions of P_α:

$$\Omega_t = \frac{K(k)W + a^2 z_+ E\left[K(k) - E(k)\right] X}{K(k)Y + a^2 z_+\left[K(k) - E(k)\right] X}, \tag{4.251a}$$

$$\Omega_r = \frac{\pi K(k)}{K(k)Y + a^2 z_+\left[K(k) - E(k)\right] X}, \tag{4.251b}$$

$$\Omega_\theta = \frac{\pi\beta\sqrt{z_+}X/2}{K(k)Y + a^2 z_+\left[K(k) - E(k)\right] X}, \tag{4.251c}$$

$$\Omega_\phi = \frac{K(k)Z + L_z[\Pi(\pi/2, z_-, k) - K(k)]X}{K(k)Y + a^2 z_+[K(k) - E(k)]X}. \tag{4.251d}$$

4.10 Appendix: Comparison with treatment of Kevorkian and Cole

As explained in Sec. 4.3 above, our two-timescale analysis of the general system of equations (4.100) follows closely that of the textbook [133] by Kevorkian and Cole (KC), which is a standard reference on asymptotic methods. In this appendix we explain the minor ways in which our treatment of Secs. 4.4 and 4.5 extends and corrects that of KC. Section 4.4 of KC covers the one variable case. We simplify this treatment by using action angle variables, and also extend it by showing that the method works to all orders in ε. Our general system of equations (4.100) is studied by KC in their section 4.5. We generalize this analysis by including the half-integer powers of ε, which are required for the treatment of resonances. A minor correction is that their solution (4.5.54a) is not generally valid, since it requires Ω_i and τ_i to be collinear, which will not always be the case. However it is easy to repair this error by replacing the expression with one constructed using Fourier methods, cf. Eq. (4.227) above. Finally, our treatment of resonances [137, 138] will closely follow KC's section 5.4, except that our analysis will apply to the general system (4.100), generalizing KC's treatment of special cases.

CHAPTER 5

EVOLUTION OF THE CARTER CONSTANT FOR INSPIRALS INTO A BLACK HOLE: EFFECT OF THE BLACK HOLE QUADRUPOLE

SUMMARY: We analyze the effect of gravitational radiation reaction on generic orbits around a body with an axisymmetric mass quadrupole moment Q to linear order in Q, to the leading post-Newtonian order, and to linear order in the mass ratio. This system admits three constants of the motion in absence of radiation reaction: energy, angular momentum, and a third constant analogous to the Carter constant. We compute instantaneous and time-averaged rates of change of these three constants. For a point particle orbiting a black hole, Ryan has computed the leading order evolution of the orbit's Carter constant, which is linear in the spin. Our result, when combined with an interaction quadratic in the spin (the coupling of the black hole's spin to its own radiation reaction field), gives the next to leading order evolution. The effect of the quadrupole, like that of the linear spin term, is to circularize eccentric orbits and to drive the orbital plane towards antialignment with the symmetry axis. In addition we consider a system of two point masses where one body has a single mass multipole or current multipole. To linear order in the mass ratio, to linear order in the multipole, and to the leading post-Newtonian order, we show that there does not exist an analog of the Carter constant for such a system (except for the cases of spin and mass quadrupole). With mild additional assumptions, this result falsifies the conjecture that all vacuum, axisymmetric spacetimes possess a third constant of geodesic motion.

Originally appeared in Phys. Rev. D **75** *124007, (2007), with É. Flanagan.*

5.1 Introduction and summary

The inspiral of stellar mass compact objects with masses μ in the range $\mu \sim 1 - 100 M_\odot$ into massive black holes with masses $M \sim 10^5 - 10^7 M_\odot$ is one of the most important sources for the future space-based gravitational wave detector LISA. Observing such events will provide a variety of information: (i) the masses and spins of black holes can be measured to high accuracy ($\sim 10^{-4}$); which can constrain the black hole's growth history [88]; (ii) the observations will give a precise test of general relativity in the strong field regime and unambiguously identify whether the central object is a black hole [151]; and (iii) the measured event rate will give insight into the complex stellar dynamics in galactic nuclei [88]. Analogous inspirals may also be interesting for the advanced stages of ground-based detectors: it has been estimated that advanced LIGO could detect up to $\sim 10 - 30$ inspirals per year of stellar mass compact objects into intermediate mass black holes with masses $M \sim 10^2 - 10^4 M_\odot$ in globular clusters [29]. Detecting these inspirals and extracting information from the datastream will require accurate models of the gravitational waveform as templates for matched filtering. For computing templates, we therefore need a detailed understanding of the how radiation reaction influences the evolution of bound orbits around Kerr black holes [151, 108, 170, 171].

There are three dimensionless parameters characterizing inspirals of bodies into black holes:

- the dimensionless spin parameter $a = |\mathbf{S}|/M^2$ of the black hole, where $\mathbf{S}$ is the spin.
- the strength of the interaction potential $\epsilon^2 = GM/rc^2$, i.e. the expansion parameter used in post-Newtonian (PN) theory.
- the mass ratio μ/M.

For LISA data analysis we will need waveforms that are accurate to all orders in a and ϵ^2, and to leading order in μ/M. However, it is useful to have analytic results in the regimes $a \ll 1$ and/or $\epsilon^2 \ll 1$. Such approximate results can be useful as a check of numerical schemes that compute more accurate waveforms, for scoping out LISA's data analysis requirements [172, 88], and for assessing the accuracy of the leading order in μ/M or adiabatic approximation [40, 163, 137, 138]. There is substantial literature on such approximate analytic results, and in this paper we will extend some of these results to higher order.

A long standing difficulty in computing the evolution of generic orbits has been the evolution of the orbit's "Carter constant", a constant of motion which governs the orbital shape and inclination. A theoretical prescription now exists for computing Carter constant evolution to all orders in ϵ and a in the adiabatic limit $\mu \ll M$ [128, 41, 37, 40], but it has not yet been implemented numerically. In this paper we focus on computing analytically the evolution of the Carter constant in the regime $a \ll 1$, $\epsilon \ll 1$, $\mu/M \ll 1$, extending earlier results by Ryan [173, 174].

We next review existing analytical work on the effects of multipole moments on inspiral waveforms. For non-spinning point masses, the phase of the $l = 2$ piece of the waveform is known to $O(\epsilon^7)$ beyond leading order [175], while spin corrections are not known to such high order. To study the leading order effects of the central body's multipole moments on the inspiral waveform, in the test mass limit $\mu \ll M$, one has to correct both the conservative and dissipative pieces of the forces on the bodies. For the conservative pieces, it suffices to use the Newtonian action for a binary with an additional multipole interaction potential. For the dissipative pieces, the multipole corrections to the fluxes at infinity of the conserved quantities can simply be added to the known PN point mass results. The lowest order spin-orbit coupling effects on the gravitational radiation were first derived by Kidder [176], then extended by Ryan [173, 174], Gergely [177], and Will [178]. Recently, the corrections of $O(\epsilon^2)$ beyond the leading order to the spin-orbit effects on the fluxes were derived [179, 180]. Corrections to the waveform due to the quadrupole - mass monopole interaction were first considered by Poisson [181], who derived the effect on the time averaged energy flux for circular equatorial orbits. Gergely [182] extended this work to generic orbits and computed the radiative instantaneous and time averaged rates of change of energy E, magnitude of angular momentum $|\mathbf{L}|$, and the angle $\kappa = \cos^{-1}(\mathbf{S} \cdot \mathbf{L})$ between the spin $\mathbf{S}$ and orbital angular momentum $\mathbf{L}$. Instead of the Carter constant, Gergely identified the angular average of the magnitude of the orbital angular momentum, $\bar{L}$, as a constant of motion. The fact that to post-2-Newtonian (2PN) order there is no time averaged secular evolution of the spin allowed Gergely to obtain expressions for $\dot{L}$ and $\dot{\kappa}$ from the quadrupole formula for the evolution of the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. In a different paper, Gergely [177] showed that in addition to the quadrupole, self-interaction spin effects also contribute at 2PN order, which was seen previously in the black hole perturbation calculations of Shibata et al. [183]. Gergely calculated the effect of this interaction on the instantaneous and time-averaged fluxes of E and $|\mathbf{L}|$ but did not derive the evolution of the third constant of motion.

In this paper, we will re-examine the effects of the quadrupole moment of the black hole and of the leading order spin self interaction. For a black hole, our analysis will thus contain all effects that are quadratic in spin to the leading order in ϵ^2 and in μ/M. Our work will extend earlier work by

- Considering generic orbits.
- Using a natural generalization of the Carter-type constant that can be defined for two point particles when one of them has a quadrupole. This facilitates applying our analysis to Kerr inspirals.
- Computing instantaneous as well as time-averaged fluxes for all three constants of motion: energy E, z-component of angular momentum L_z, and Carter-type constant K. For most purposes, only time-averaged fluxes are needed as only they are gauge invariant and physically relevant. However, there is one effect for which the time-averaged fluxes are insufficient, namely transient resonances that occur during an inspiral in Kerr in the vicinity of geodesics for which the radial and azimuthal frequencies are commensurate [137, 138]. The instantaneous fluxes derived in this paper will be used in [138] for studying the effect of these resonances on the gravitational wave phasing.

We will analyze the effect of gravitational radiation reaction on orbits around a body with an axisymmetric mass quadrupole moment Q to leading order in Q, to the leading post-Newtonian order, and to leading order in the mass ratio. With these approximations the adiabatic approximation holds: gravitational radiation reaction takes place over a time scale much longer than the orbital period, so the orbit looks geodesic on short time scales. We follow Ryan's method of computation [173]: First, we calculate the orbital motion in the absence of radiation reaction and the associated constants of motion. Next, we use the leading order radiation reaction accelerations that act on the particle (given by the Burke-Thorne formula

[1] augmented by the relevant spin corrections [173]) to compute the evolution of the constants of motion. In the adiabatic limit, the time-averaged rates of change of the constants of motion can be used to infer the secular orbital evolution. Our results show that a mass quadrupole has the same qualitative effect on the evolution as spin: it tends to circularize eccentric orbits and drive the orbital plane towards antialignment with the symmetry axis of the quadrupole.

The relevance of our result to point particles inspiralling into black holes is as follows. The vacuum spacetime geometry around any stationary body is completely characterized by the body's mass multipole moments $I_L = I_{a_1,a_2\ldots a_l}$ and current multipole moments $S_L = S_{a_1,a_2\ldots a_l}$ [184]. These moments are defined as coefficients in a power series expansion of the metric in the body's local asymptotic rest frame [185]. For nearly Newtonian sources, they are given by integrals over the source as

$$I_L \equiv I_{a_1,\ldots a_l} = \int \rho x_{<a_1} \ldots x_{a_l>} d^3x, \tag{5.1}$$

$$S_L \equiv S_{a_1,\ldots a_l} = \int \rho x_p v_q \epsilon_{pq<a_1} x_{a_2} \ldots x_{a_l>} d^3x. \tag{5.2}$$

Here ρ is the mass density and v_q is the velocity, and "$< \cdots >$" means "symmetrize and remove all traces". For axisymmetric situations, the tensor multipole moments I_L (S_L) contain only a single independent component, conventionally denoted by I_l (S_l) [184]. For a Kerr black hole of mass M and spin $\mathbf{S}$, these moments are given by [184]

$$I_l + iS_l = M^{l+1}(ia)^l, \tag{5.3}$$

where a is the dimensionless spin parameter defined by $a = |\mathbf{S}|/M^2$. Note that $S_l = 0$ for even l and $I_l = 0$ for odd l.

Consider now inspirals into an axisymmetric body which has some arbitrary mass and current multipoles I_l and S_l. Then we can consider effects that are linear in I_l and S_l for each l, effects that are quadratic in the multipoles proportional to $I_l I_{l'}$, $I_l S_{l'}$, $S_l S_{l'}$, effects that are cubic, etc. For a general body, all these effects can be separated using their scalings, but for a black hole, $I_l \propto a^l$ for even l and $S_l \propto a^l$ for odd l [see Eq.(5.3)], so the effects cannot be separated. For example, a physical effect that scales as $O(a^2)$ could be an effect that is quadratic in the spin or linear in the quadrupole; an analysis in Kerr cannot distinguish these two possibilities. For this reason, it is useful to analyze spacetimes that are more general than Kerr, characterized by arbitrary I_l and S_l, as we do in this paper. For recent work on computing exact metrics characterized by sets of moments I_l and S_l, see Refs. [186, 187] and references therein.

The leading order effect of the black hole's multipoles on the inspiral is the $O(a)$ effect computed by Ryan [174]. This $O(a)$ effect depends linearly on the spin S_1 and is independent of the higher multipoles S_l and I_l since these all scale as $O(a^2)$ or smaller. In this paper we compute the $O(a^2)$ effect on the inspiral, which includes the leading order linear effect of the black hole's quadrupole (linear in $I_2 \equiv Q$) and the leading order spin self-interaction (quadratic in S_1).

We next discuss how these $O(a^2)$ effects scale with the post-Newtonian expansion parameter ϵ. Consider first the conservative orbital dynamics. Here it is easy to see that fractional corrections that are linear in I_2 scale as $O(a^2\epsilon^4)$, while those quadratic in S_1 scale as $O(a^2\epsilon^6)$. Thus, the two types of terms cleanly separate. We compute only the leading order, $O(a^2\epsilon^4)$, term. For the dissipative contributions to the orbital motion, however, the scalings are different. There are corrections to the radiation reaction acceleration whose fractional magnitudes are $O(a^2\epsilon^4)$ from both types of effects linear in I_2 and quadratic in S_1. The effects quadratic in S_1 are due to the backscattering of the radiation off the piece of spacetime curvature due to the black hole's spin. This effect was first pointed out by Shibata et al. [183], who computed the time-averaged energy flux for circular orbits and small inclination angles based on a PN expansion of black hole perturbations. Later, Gergely [177] analyzed this effect on the instantaneous and time-averaged fluxes of energy and magnitude of orbital angular momentum within the PN framework.

The organization of this paper is as follows. In Sec. 5.2, we study the conservative orbital dynamics of two point particles when one particle is endowed with an axisymmetric quadrupole, in the weak field regime, and to leading order in the mass ratio. In Sec. 5.3, we compute the radiation reaction accelerations and the instantaneous and time-averaged fluxes. In order to have all the contributions at $O(a^2\epsilon^4)$ for a black hole, we include in our computations of radiation reaction acceleration the interaction that is quadratic in the spin S_1. The application to black holes in Sec. 5.4 briefly discusses the qualitative predictions of our results and also compares with previous results.

The methods used in this paper can be applied only to the black hole spin (as analyzed by Ryan [173]) and the black hole quadrupole (as analyzed here). We show in Sec. 5.5 that for the higher order mass and current multipole moments taken individually, an analog of the Carter constant cannot be defined to the order of our approximations. We then show that under mild assumptions, this non-existence result can be extended to exact spacetimes, thus falsifying the conjecture that all vacuum axisymmetric spacetimes possess a third constant of geodesic motion.

5.2 Effect of an axisymmetric mass quadrupole on the conservative orbital dynamics

Consider two point particles m_1 and m_2 interacting in Newtonian gravity, where $m_2 \ll m_1$ and where the mass m_1 has a quadrupole moment Q_{ij} which is axisymmetric:

$$\begin{aligned} Q_{ij} &= \int d^3x \rho(\mathbf{r}) \left[x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right] && (5.4) \\ &= Q \left(n_i n_j - \frac{1}{3} \delta_{ij} \right). && (5.5) \end{aligned}$$

For a Kerr black hole of mass M and dimensionless spin parameter a with spin axis along $\mathbf{n}$, the quadrupole scalar is $Q = -M^3 a^2$.

The action describing this system, to leading order in m_2/m_1, is

$$S = \int dt \left[\frac{1}{2} \mu \mathbf{v}^2 - \mu \Phi(\mathbf{r}) \right], \tag{5.6}$$

where $\mathbf{v} = \dot{\mathbf{r}}$ is the velocity, the potential is

$$\Phi(\mathbf{r}) = -\frac{M}{r} - \frac{3}{2r^5} x^i x^j Q_{ij}, \tag{5.7}$$

μ is the reduced mass and M the total mass of the binary, and we are using units with $G = c = 1$. We work to linear order in Q, to linear order in m_2/m_1, and to leading order in M/r. In this regime, the action (5.6) also describes the conservative effect of the black hole's mass quadrupole on bound test particles in Kerr, as discussed in the introduction. We shall assume that the quadrupole Q_{ij} is constant in time. In reality, the quadrupole will evolve due to torques that act to change the orientation of the central body. An estimate based on treating m_1 as a rigid body in the Newtonian field of m_2 gives the scaling of the time scale for the quadrupole to evolve compared to the radiation reaction time as (see Appendix A for details)

$$\frac{T_{\rm evol}}{T_{\rm rr}} \sim \left(\frac{m_1}{m_2}\right)\left(\frac{M}{r}\right)\left(\frac{\bar{S}}{\bar{Q}}\right) \sim \left(\frac{M}{\mu}\right)\left(\frac{M}{r}\right)\left(\frac{1}{a}\right). \tag{5.8}$$

Here, we have denoted the dimensionless spin and quadrupole of the body by $\bar{S}$ and $\bar{Q}$ respectively, and the last relation applies for a Kerr black hole. Since $\mu/M \ll 1$, the first factor in Eq. (5.8) will be large, and since $1/a \geq 1$ and for the relativistic regime $M/r \sim 1$, the evolution time is long compared to the radiation reaction time. Therefore we can neglect the evolution of the quadrupole at leading order.

This system admits three conserved quantities, the energy

$$E = \frac{1}{2}\mu \mathbf{v}^2 + \mu\Phi(\mathbf{r}), \tag{5.9}$$

the z-component of angular momentum

$$L_z = \mathbf{e}_z \cdot (\mu \mathbf{r} \times \mathbf{v}), \tag{5.10}$$

and the Carter-type constant

$$K = \mu^2(\mathbf{r}\times\mathbf{v})^2 - \frac{2Q\mu^2}{r^3}(\mathbf{n}\cdot\mathbf{r})^2 + \frac{Q\mu^2}{M}\left[(\mathbf{n}\cdot\mathbf{v})^2 - \frac{1}{2}\mathbf{v}^2 + \frac{M}{r}\right]. \tag{5.11}$$

(See below for a derivation of this expression for K).

5.2.1 Conservative orbital dynamics in a Boyer-Lindquist-like coordinate system

We next specialize to units where $M = 1$. We also define the rescaled conserved quantities by $\tilde{E} = E/\mu$, $\tilde{L}_z = L_z/\mu$, $\tilde{K} = K/\mu^2$, and drop the tildes. These specializations and definitions have the effect of eliminating all factors of μ and M

from the analysis. In spherical polar coordinates (r, θ, φ) the constants of motion E and L_z become

$$
\begin{aligned}
E &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2) - \frac{1}{r} + \frac{Q}{2r^3}(1 - 3\cos^2\theta), && (5.12)\\
L_z &= r^2 \sin^2\theta\dot{\varphi}. && (5.13)
\end{aligned}
$$

In these coordinates, the Hamilton-Jacobi equation is not separable, so a separation constant K cannot readily be derived. For this reason we switch to a different coordinate system $(\tilde{r}, \tilde{\theta}, \varphi)$ defined by

$$
\begin{aligned}
r\cos\theta &= \tilde{r}\cos\tilde{\theta}\left(1 + \frac{Q}{4\tilde{r}^2}\right),\\
r\sin\theta &= \tilde{r}\sin\tilde{\theta}\left(1 - \frac{Q}{4\tilde{r}^2}\right). && (5.14)
\end{aligned}
$$

We also define a new time variable $\tilde{t}$ by

$$
dt = \left[1 - \frac{Q}{2\tilde{r}^2}\cos(2\tilde{\theta})\right] d\tilde{t}. \tag{5.15}
$$

The action (5.6) in terms of the new variables to linear order in Q is

$$
\begin{aligned}
S = &\int d\tilde{t}\left\{\frac{1}{2}\left(\frac{d\tilde{r}}{d\tilde{t}}\right)^2 + \frac{1}{2}\tilde{r}^2\left(\frac{d\tilde{\theta}}{d\tilde{t}}\right)^2 + \frac{1}{2}\tilde{r}^2\sin^2\tilde{\theta}\left(\frac{d\varphi}{d\tilde{t}}\right)^2\left[1 - \frac{Q}{\tilde{r}^2}\sin^2\tilde{\theta}\right]\right.\\
&\left. + \frac{1}{\tilde{r}} + \frac{Q}{4\tilde{r}^3}\right\}. && (5.16)
\end{aligned}
$$

However, a difficulty is that the action (5.16) does not give the same dynamics as the original action (5.6). The reason is that for solutions of the equations of motion for the action (5.6), the variation of the action vanishes for paths with fixed endpoints for which the time interval Δt is fixed. Similarly, for solutions of the equations of motion for the action (5.16), the variation of the action vanishes for paths with fixed endpoints for which the time interval $\Delta\tilde{t}$ is fixed. The two sets of varied paths are not the same, since $\Delta t \neq \Delta\tilde{t}$ in general. Therefore, solutions of the Euler-Lagrange equations for the action (5.6) do not correspond to solutions of the Euler-Lagrange equations for the action (5.16). However, in the special case of zero-energy motions, the extra terms in the variation of the action vanish. Thus, a way around this difficulty is to modify the original action to be

$$\hat{S} = \int dt \left[\frac{1}{2}\mu \mathbf{v}^2 - \mu\Phi(\mathbf{r}) + E \right]. \tag{5.17}$$

This action has the same extrema as the action (5.6), and for motion with physical energy E, the energy computed with this action is zero. Transforming to the new variables yields, to linear order in Q:

$$\begin{aligned} \hat{S} &= \int d\tilde{t} \left\{ \frac{1}{2}\left(\frac{d\tilde{r}}{d\tilde{t}}\right)^2 + \frac{1}{2}\tilde{r}^2 \left(\frac{d\tilde{\theta}}{d\tilde{t}}\right)^2 \right. \\ & \quad + \frac{1}{2}\tilde{r}^2 \sin^2\tilde{\theta} \left(\frac{d\varphi}{d\tilde{t}}\right)^2 \left[1 - \frac{Q}{\tilde{r}^2}\sin^2\tilde{\theta}\right] \\ & \quad \left. + \frac{1}{\tilde{r}} + \frac{Q}{4\tilde{r}^3} + E - \frac{QE}{2\tilde{r}^2}\cos(2\tilde{\theta}) \right\}. \end{aligned} \tag{5.18}$$

The zero-energy motions for this action coincide with the zero energy motions for the action (5.17). We use this action (5.18) as the foundation for the remainder of our analysis in this section.

The z-component of angular momentum in terms of the new variables $(\tilde{r}, \tilde{\theta}, \varphi, \tilde{t})$ is

$$L_z = \tilde{r}^2 \sin^2\tilde{\theta} \left(\frac{d\varphi}{d\tilde{t}}\right) \left[1 - \frac{Q}{\tilde{r}^2}\sin^2\tilde{\theta}\right]. \tag{5.19}$$

We now transform to the Hamiltonian:

$$\begin{aligned} \hat{H} &= \frac{1}{2}p_{\tilde{r}}^2 - \frac{1}{\tilde{r}} - E - \frac{Q}{4\tilde{r}^3} + \frac{QL_z^2}{2\tilde{r}^4} \\ & \quad + \frac{1}{2\tilde{r}^2}\left[p_{\tilde{\theta}}^2 + \frac{L_z^2}{\sin^2\tilde{\theta}} + QE\cos(2\tilde{\theta})\right] \end{aligned} \tag{5.20}$$

and solve the Hamiltonian Jacobi equation. Denoting the separation constant by K we obtain the following two equations for the $\tilde{r}$ and $\tilde{\theta}$ motions:

$$\left(\frac{d\tilde{r}}{d\tilde{t}}\right)^2 = 2E + \frac{2}{\tilde{r}} - \frac{K}{\tilde{r}^2} + \frac{Q}{2}\left[\frac{1}{\tilde{r}^3} - \frac{2L_z^2}{\tilde{r}^4}\right], \tag{5.21}$$

and

$$\tilde{r}^4\left(\frac{d\tilde{\theta}}{d\tilde{t}}\right)^2 = K - \frac{L_z^2}{\sin^2\tilde{\theta}} - QE\cos(2\tilde{\theta}). \tag{5.22}$$

Note that the equations of motion (5.21) and (5.22) have the same structure as the equations of motion for Kerr geodesic motion. Using Eqs. (5.19), (5.21) and (5.22) together with the inverse of the transformation (5.14) to linear order in Q, we obtain the expression for K in spherical polar coordinates:

$$\begin{aligned} K &= r^4(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + Q(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + \frac{Q}{r} \\ &- \frac{Q}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) - \frac{2Q}{r}\cos^2\theta. \end{aligned} \tag{5.23}$$

This is equivalent to the formula (5.11) quoted earlier.

5.2.2 Effects linear in spin on the conservative orbital dynamics

To include the linear in spin effects, we repeat Ryan's analysis [173, 174] (he only gives the final, time averaged fluxes; we will also give the instantaneous fluxes). We can simply add these linear in spin terms to our results because any terms of order $O(SQ)$ will be higher than the order a^2 to which we are working. The correction to the action (5.6) due to spin-orbit coupling is

$$S^{\mathrm{spin-orbit}} = \int dt\left[-\frac{2\mu S n^i \epsilon_{ijk} x_j \dot{x}_k}{r^3}\right]. \tag{5.24}$$

We will restrict our analysis to the case when the unit vectors n_i corresponding to the axisymmetric quadrupole Q_{ij} and to the spin S_i coincide, as they do in Kerr.

Including the spin-orbit term in the action (5.6) results in the following modified expressions for L_z and K:

$$L_z = \mathbf{n}\cdot(\mu\mathbf{r}\times\mathbf{v}) - \frac{2S}{r^3}[\mathbf{r}^2 - (\mathbf{n}\cdot\mathbf{r})^2], \tag{5.25}$$

and

$$
\begin{aligned}
K &= (\mathbf{r}\times\mathbf{v})^2 - \frac{4S}{r}\mathbf{n}\cdot(\mathbf{r}\times\mathbf{v}) - \frac{2Q}{r^3}(\mathbf{n}\cdot\mathbf{r})^2 \\
&\quad + Q\left[(\mathbf{n}\cdot\mathbf{v})^2 - \frac{1}{2}\mathbf{v}^2 + \frac{1}{r}\right]. \qquad (5.26)
\end{aligned}
$$

In terms of the Boyer-Lindquist like coordinates, the conserved quantities with the linear in spin terms included are:

$$
L_z = \tilde{r}^2 \sin^2\tilde{\theta}\left(\frac{d\varphi}{d\tilde{t}}\right) - \frac{2S}{r}\sin^2\tilde{\theta} - Q\sin^4\tilde{\theta}, \qquad (5.27)
$$

$$
\begin{aligned}
K &= r^4(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - 4Sr\sin^2\theta\dot{\varphi} \\
&\quad - \frac{2Q}{r}\cos^2\theta + Q(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + \frac{QM}{r} \\
&\quad - \frac{Q}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2). \qquad (5.28)
\end{aligned}
$$

The equations of motion are

$$
\left(\frac{d\tilde{r}}{d\tilde{t}}\right)^2 = 2E + \frac{2}{\tilde{r}} - \frac{K}{\tilde{r}^2} - \frac{4SL_z}{\tilde{r}^3} + \frac{Q}{2}\left[\frac{1}{\tilde{r}^3} - \frac{2L_z^2}{\tilde{r}^4}\right], \qquad (5.29)
$$

and

$$
\tilde{r}^4\left(\frac{d\tilde{\theta}}{d\tilde{t}}\right)^2 = K - \frac{L_z^2}{\sin^2\tilde{\theta}} - QE\cos(2\tilde{\theta}). \qquad (5.30)
$$

5.3 Effects linear in quadrupole and quadratic in spin on the evolution of the constants of motion

5.3.1 Evaluation of the radiation reaction force

The relative acceleration of the two bodies can be written as

$$
\mathbf{a} = -\nabla\Phi(\mathbf{r}) + \mathbf{a}_{\mathrm{rr}}, \qquad (5.31)
$$

where $\mathbf{a}_{\rm rr}$ is the radiation-reaction acceleration. Combining this with Eqs. (5.9), (5.25) and (5.26) for E, L_z and K gives the following formulae for the time derivatives of the conserved quantities:

$$
\begin{aligned}
\dot{E} &= \mathbf{v} \cdot \mathbf{a}_{\rm rr}, && (5.32)\\
\dot{L}_z &= \mathbf{n} \cdot (\mathbf{r} \times \mathbf{a}_{\rm rr}), && (5.33)\\
\dot{K} &= 2(\mathbf{r} \times \mathbf{v}) \cdot (\mathbf{r} \times \mathbf{a}_{\rm rr}) - \frac{4S}{r} \mathbf{n} \cdot (\mathbf{r} \times \mathbf{a}_{\rm rr}) \\
& \quad +2Q(\mathbf{n} \cdot \mathbf{v}) \, (\mathbf{n} \cdot \mathbf{a}_{\rm rr}) - Q \mathbf{v} \cdot \mathbf{a}_{\rm rr}. && (5.34)
\end{aligned}
$$

The standard expression for the leading order radiation reaction acceleration acting on one of the bodies is [188]:

$$
\begin{aligned}
a^j_{\rm rr} &= -\frac{2}{5} I^{(5)}_{jk} x_k + \frac{16}{45} \epsilon_{jpq} S^{(6)}_{pk} x_k x_q + \frac{32}{45} \epsilon_{jpq} S^{(5)}_{pk} x_k v_q \\
& \quad + \frac{32}{45} \epsilon_{pq[j} S^{(5)}_{k]p} x_q v_k. && (5.35)
\end{aligned}
$$

Here the superscripts in parentheses indicate the number of time derivatives and square brackets on the indices denote antisymmetrization.

The multipole moments $I_{jk}(t)$ and $S_{jk}(t)$ in Eq. (5.35) are the total multipole moments of the spacetime, i.e. approximately those of the black hole plus those due to the orbital motion. The expression (5.35) is formulated in asymptotically Cartesian mass centered (ACMC) coordinates of the system, which are displaced from the coordinates used in Sec. 5.2 by an amount [185]

$$
\delta \mathbf{r}(t) = -\frac{\mu}{M} \, \mathbf{r}(t). \qquad (5.36)
$$

This displacement contributes to the radiation reaction acceleration in the following ways:

1. The black hole multipole moments I_l and S_l, which are time-independent

in the coordinates used in Sec. 5.2, will be displaced by $\delta\mathbf{r}$ and thus will contribute to the $(l+1)$th ACMC radiative multipole [185].

2. The constants of motion are defined in terms of the black hole centered coordinates used in Sec. 5.2, so the acceleration $\mathbf{a}_{\rm rr}$ we need in Eqs. (5.32) – (5.34) is the relative acceleration. This requires calculating the acceleration of both the black hole and the point mass in the ACMC coordinates using (5.35), and then subtracting to find $\mathbf{a}_{\rm rr} = \mathbf{a}^{\mu}_{\rm rr} - \mathbf{a}^{M}_{\rm rr}$ [173]. To leading order in μ, the only effect of the acceleration of the black hole is via a backreaction of the radiation field: the lth black hole moments couple to the $(l{+}1)$th radiative moments, thus producing an additional contribution to the acceleration.

For our calculations at $O(S_1\epsilon^3)$, $O(I_2\epsilon^4)$, $O(S_1^2\epsilon^4)$, we can make the following simplifications:

- *quadrupole corrections*: The fractional corrections linear in $I_2 = Q$ that scale as $O(a^2\epsilon^4)$ require only the effect of I_2 on the conservative orbital dynamics as computed in Sec. 5.2A and the Burke-Thorne formula for the radiation reaction acceleration [given by the first term in Eq. (5.35)].

- *spin-spin corrections*: As discussed in the introduction, the fractional corrections quadratic in S_1 to the conservative dynamics scale as $O(a^2\epsilon^6)$ and are subleading order effects which we neglect. At $O(a^2\epsilon^4)$, the only effect quadratic in S_1 is the backscattering of the radiation off the spacetime curvature due to the spin. As discussed in item 1. above, the black hole's current dipole $S_i = S_1\delta_{i3}$ (taking the z-axis to be the symmetry axis) will contribute to the radiative current quadrupole an amount

$$S^{\rm spin}_{ij} = -\frac{3}{2}\frac{\mu}{M}S_1 x_i \delta_{j3}. \tag{5.37}$$

The black hole's current dipole S_i will couple to the gravitomagnetic radiation field due to S_{ij} as discussed in item 2. above, and contribute to the relative acceleration as [173]:

$$a_{\rm rr}^{j\ {\rm spin}} = \frac{8}{15} S_1 \delta_{i3} S_{ij}^{(5)}. \tag{5.38}$$

For our purposes of computing terms quadratic in the spin, we substitute $S_{ij}^{\rm spin}$ for S_{ij} in Eq. (5.38). Evaluating these quadratic in spin terms requires only the Newtonian conservative dynamics, i.e. the results of Sec. 5.2 and Eqs. (5.32) – (5.34) with the quadrupole set to zero.

- *linear in spin corrections*: Contributions to these effects are from Eq. (5.35) with the current quadrupole replaced by just the spin contribution (5.37), and from Eq. (5.38) evaluated using only the orbital current quadrupole.

With these simplifications, we replace the expression (5.35) for the radiation reaction acceleration with

$$\begin{aligned} a_{\rm rr}^{j} \;=\; & -\frac{2}{5} I_{jk}^{(5)} x_k + \frac{16}{45} \epsilon_{jpq} S_{pk}^{(6)\ {\rm spin}} x_k x_q \\ & + \frac{32}{45} \epsilon_{jpq} S_{pk}^{(5)\ {\rm spin}} x_k v_q + \frac{32}{45} \epsilon_{pq[j} S_{k]p}^{(5)\ {\rm spin}} x_q v_k \\ & + \frac{8}{15} S_1 \delta_{i3} \left[S_{ij}^{(5)\ {\rm orbit}} + S_{ij}^{(5)\ {\rm spin}} \right]. \end{aligned} \tag{5.39}$$

To justify these approximations, consider the scaling of the contribution of black hole's acceleration to the orbital dynamics. The mass and current multipoles of the black hole contribute terms to the Hamiltonian that scale with ϵ as

$$\Delta H \sim S_l \epsilon^{2l+3} \ \& \ I_l \epsilon^{2l+2}. \tag{5.40}$$

Since the Newtonian energy scales as ϵ^2, the fractional correction to the orbital dynamics scale as

$$\Delta H/E \sim S_l \epsilon^{2l+1} \ \& \ I_l \epsilon^{2l}. \tag{5.41}$$

To $O(\epsilon^4)$, the only radiative multipole moments that contribute to the acceleration (5.35) are the mass quadrupole I_2, the mass octupole I_3, and the current quadrupole S_2 (cf. [176]). Since we are focusing only on the leading order terms quadratic in spin (these can simply be added to the known 2PN point particle and 1.5PN linear in spin results), the only terms in Eq. (5.35) relevant for our purposes are those given in Eq. (5.39). The results from a computation of the fully relativistic metric perturbation for black hole inspirals [183] show that quadratic in spin corrections to the $l = 2$ piece compared to the flat space Burke-Thorne formula first appear at $O(a^2\epsilon^4)$, which is consistent with the above arguments.

5.3.2 Instantaneous fluxes

We evaluate the radiation reaction force as follows. The total mass and current quadrupole moment of the system are

$$Q_{ij}^{\mathrm{T}} = Q_{ij} + \mu x_i x_j, \tag{5.42}$$

$$S_{ij}^{\mathrm{T}} = S_{ij}^{\mathrm{spin}} + x_i \epsilon_{jkm} x_k \dot{x}_m, \tag{5.43}$$

where from Eq. (5.14)

$$\begin{aligned} x_i = & \left[\tilde{r}\sin\tilde{\theta}\left(1 - \frac{Q}{4\tilde{r}^2}\right)\cos\varphi,\ \tilde{r}\sin\tilde{\theta}\left(1 - \frac{Q}{4\tilde{r}^2}\right)\sin\varphi,\right. \\ & \left.\tilde{r}\cos\tilde{\theta}\left(1 + \frac{Q}{4\tilde{r}^2}\right)\right]. \end{aligned} \tag{5.44}$$

Only the second term in Eq. (5.42) contributes to the time derivative of the quadrupole. We differentiate five times by using

$$\frac{d}{dt} = \left[1 + \frac{Q}{2\tilde{r}^2}\cos(2\tilde{\theta})\right]\frac{d}{d\tilde{t}}, \tag{5.45}$$

to the order we are working as discussed above. After each differentiation, we eliminate any occurrences of $d\varphi/d\tilde{t}$ using Eq. (5.27), and we eliminate any occurrences of the second order time derivatives $d^2\tilde{r}/d\tilde{t}^2$ and $d^2\tilde{\theta}/d\tilde{t}^2$ in favor of first order time derivatives using (the time derivatives of) Eqs. (5.29) and (5.30). For computing the terms linear and quadratic in S_1, we set the quadrupole Q to zero in all the formulae. We insert the resulting expression into the formula (5.39) for the self-acceleration, and then into Eqs. (5.32) – (5.34). We eliminate $(d\tilde{r}/d\tilde{t})^2$, $(d\tilde{\theta}/d\tilde{t})^2$, and $(d\varphi/d\tilde{t})$ in favor of E, L_z, and K using Eqs. (5.27) – (5.30). In the final expressions for the instantaneous fluxes, we keep only terms that are of $O(S)$, $O(Q)$ and $O(S^2)$ and obtain the following results:

$$
\begin{aligned}
\dot{E} &= \frac{160K}{3r^6} + \frac{64}{3r^5} + \frac{512E}{15r^4} - \frac{40K^2}{r^7} + \frac{272KE}{5r^5} + \frac{64E^2}{5r^3} \\
&+ \frac{SL_z}{r^9}\left(196K^2 + \frac{952}{3}r^2 - \frac{3668}{5}Kr - 352KEr^2 + \frac{1024}{3}Er^3 + \frac{128}{5}E^2r^4\right) \\
&+ \frac{2Q}{r^9}\left[-49K^2 - 169KL_z^2 + r\left(\frac{532}{5}K + \frac{3307}{15}L_z^2\right)\right] \\
&+ \frac{4Q}{r^7}\left(-\frac{20}{3} + 47KE + \frac{548}{5}L_z^2E\right) - \frac{160Q}{r^5}E^2\cos(2\theta) \\
&+ \frac{Q}{r^9}\left[\left(-562K^2 + \frac{2998}{3}Kr - \frac{320}{3}r^2 + \frac{5072}{5}KEr^2 - \frac{4048}{15}r^3E\right)\cos(2\theta)\right] \\
&+ \frac{Q}{r^6}\sin(2\theta)\left(439K - \frac{926}{3}r - \frac{1528}{5}r^2E\right)\dot{\theta}\dot{r} - \frac{2Q}{r^9}\left(\frac{152}{5}r^3E - 16r^4E^2\right) \\
&+ \frac{S^2}{r^9}\left(-K^2 + \frac{22}{3}Kr - \frac{28}{3}r^2 + \frac{32}{5}KEr^2 - \frac{236}{15}r^3E - \frac{32}{5}r^4E^2\right)\cos(2\theta) \\
&- \frac{S^2}{r^6}\sin(2\theta)\left(K + \frac{2}{3}r + \frac{8}{5}r^2E\right)\dot{\theta}\dot{r} - \frac{S^2}{r^5}\frac{224}{5}E^2 - \frac{S^2}{r^6}\frac{1652}{15}E \\
&+ \frac{S^2}{r^9}\left[-49K^2 + 6KL_z^2 + 2r\left(63K - \frac{16}{3}L_z^2 - \frac{98}{3}\right)\right] \\
&+ \frac{S^2}{r^7}\left(112KE - \frac{48}{5}L_z^2E\right),
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
\dot{L}_z &= \frac{32L_z}{r^4} + \frac{144L_zE}{5r^3} - \frac{24KL_z}{r^5} \\
&+ \frac{S}{r^7}\left[-50K^2 + 240KL_z^2 + \frac{62}{5}Kr - \frac{7376}{15}L_z^2 r + \frac{316}{3}r^2 + 56KEr^2\right] \\
&+ \frac{S}{r^5}\left[\frac{624}{5}Er - \frac{1824}{5}EL_z^2 + \frac{128}{5}E^2r^2\right] \\
&+ \frac{S}{r^7}\left(50K^2 - \frac{62}{5}Kr - \frac{316}{3}r^2 - 56KEr^2 - \frac{624}{5}Er^3 - \frac{128}{5}E^2r^4\right)\cos(2\theta) \\
&+ \frac{S}{r^4}\left(-104K + 64r + 64Er^2\right)\sin(2\theta)\dot{r}\dot{\theta} \\
&+ \frac{QL_z}{5r^7}\left[660Er^2 + 753r - 360L_z^2 - 435K\right] \\
&+ \frac{QL_z}{5r^7}\left(1601r + 1512r^2E - 1185K\right)\cos 2\theta \\
&+ \frac{174QL_z}{r^4}\sin(2\theta)\dot{r}\dot{\theta} + \frac{2S^2L_z}{r^7}\left[\frac{72}{5}Er^2 + 16r - 9K\right], \qquad (5.47)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{K} &= \frac{16K}{5r^5}\left(20r + 18r^2E - 15K\right) \\
&+ \frac{SL_z}{r^7}\left(280K^2 - \frac{14008}{15}Kr + \frac{1264}{3}r^2 + \frac{2496}{5}Er^3 - \frac{2528}{5}KEr^2\right) \\
&+ \frac{512SL_z}{5r^3}E^2 \\
&+ \frac{2Q}{15r^7}\left[2\left(-555K^2 - 1035KL_z^2 + 956Kr + 747L_z^2r + 80r^2 + 834KEr^2\right)\right] \\
&+ \frac{4Q}{15r^5}\left(360L_z^2E + 128Er + 48E^2r^2\right) - \frac{4Q}{15r^3}\cos(2\theta)168E^2 \\
&+ \frac{4Q}{15r^7}\cos(2\theta)\left(-2175K^2 + 2975Kr + 80r^2 + 3012KEr^2 - 112Er^3\right) \\
&+ \frac{2Q}{15r^4}\left(3075K - 20r - 192Er^2\right)\sin(2\theta)\dot{\theta}\dot{r} \\
&+ \frac{2S^2}{r^7}\left[\left(7K - 2L_z^2\right)\left(-3K + \frac{16}{3}r + \frac{24}{5}Er^2\right)\right] \\
&+ \frac{2S^2}{r^7}\left[K\cos(2\theta)\left(3K - \frac{16}{3}r - \frac{24}{5}Er^2\right)\right] \\
&+ \frac{2S^2}{r^4}\sin(2\theta)\left(-4K + \frac{14}{3}r + \frac{16}{5}Er^2\right)\dot{\theta}\dot{r}. \qquad (5.48)
\end{aligned}
$$

5.3.3 Alternative set of constants of the motion

A body in a generic bound orbit in Kerr traces an open ellipse precessing about the hole's spin axis. For stable orbits the motion is confined to a toroidal region whose shape is determined by E, L_z, K. The motion can equivalently be characterized by the set of constants inclination angle ι, eccentricity e, and semi-latus rectum p defined by Hughes [189]. The constants ι, p and e are defined by $\cos\iota = L_z/\sqrt{K}$, and by $\tilde{r}_\pm = p/(1 \pm e)$, where $\tilde{r}_\pm$ are the turning points of the radial motion, and $\tilde{r}$ is the Boyer-Lindquist radial coordinate. This parameterization has a simple physical interpretation: in the Newtonian limit of large p, the orbit of the particle is an ellipse of eccentricity e and semilatus rectum p on a plane whose inclination angle to the hole's equatorial plane is ι. In the relativistic regime $p \sim M$, this interpretation of the constants e, p, and ι is no longer valid because the orbit is not an ellipse and ι is not the angle at which the object crosses the equatorial plane (see Ryan [173] for a discussion).

We adopt here analogous definitions of constants of motion ι, e and p, namely

$$\cos(\iota) = L_z/\sqrt{K}, \tag{5.49}$$

$$\frac{p}{1 \pm e} = \tilde{r}_\pm. \tag{5.50}$$

Here K is the conserved quantity (5.26) or (5.28), and $\tilde{r}_\pm$ are the turning points of the radial motion using the $\tilde{r}$ coordinate defined by Eq. (5.14), given by the vanishing of the right-hand side of Eq. (5.29).

We now rewrite our results in terms of the new constants of the motion e, p and ι. We can use Eq. (5.29) together with Eqs. (5.49) and (5.50) to write E, L_z

and K as functions of p, e and ι. To leading order in Q and S we obtain

$$\begin{aligned}
K &= p\left[1-\frac{2S\cos\iota}{p^{3/2}}\left(3+e^2\right)-\left(1+e^2\right)\frac{2Q\cos^2\iota}{p^2}\right.\\
&\qquad\left.+\left(3+e^2\right)\frac{Q}{4p^2}\right], && (5.51)\\
E &= -\frac{(1-e^2)}{2p}\left[1+\frac{2S\cos\iota}{p^{3/2}}\left(1-e^2\right)\right.\\
&\qquad\qquad\left.+\left(1-e^2\right)\frac{Q}{p^2}\left(\cos^2\iota-\frac{1}{4}\right)\right], && (5.52)\\
L_z &= \sqrt{p}\cos\iota\left[1-\frac{S\cos\iota}{p^{3/2}}(3+e^2)-\left(1+e^2\right)\frac{Q\cos^2\iota}{p^2}\right.\\
&\qquad\qquad\left.+\left(3+e^2\right)\frac{Q}{8p^2}\right]. && (5.53)
\end{aligned}$$

As discussed in the introduction, the effects quadratic in S on the conservative dynamics scale as $O(a^2\epsilon^6)$ and thus are not included in this analysis to $O(a^2\epsilon^4)$.

Inserting these relations into the expressions (5.46)–(5.48) gives, dropping terms of $O(QS)$, $O(Q^2)$ and $O(QS^2)$:

$$
\begin{aligned}
\dot{E} &= -\frac{8}{15p^2r^7}\left[75p^4 - 100p^3r + p^2r^2\left(11-51e^2\right) + 32pr^3\left(1-e^2\right)\right] \\
&+ \frac{48}{15p^2r^3}\left(1-e^2\right) \\
&+ \frac{4S\cos\iota}{15p^{7/2}r^9}\left[735p^6 - 2751p^5r + 10p^4r^2(365-6e^2) - 128pr^5(1-e^2)^2\right] \\
&+ \frac{64S\cos\iota}{15p^{3/2}r^6}\left[5p(-23+3e^2) - 3r(-9+e^2+8e^4)\right] - \frac{64S\cos\iota}{5p^{7/2}r^3}(e^2-1)^3 \\
&- \frac{Q}{15p^4r^9}\left[4005p^6 - 6499p^5r + 2p^4r^2\left(1577-1977e^2\right) - 24r^6\left(1-e^2\right)^3\right] \\
&- \frac{Q}{15p^4r^9}\left[-32p^3r^3\left(8-33e^2\right) + 64pr^5\left(1-2e^2+e^4\right)\right] \\
&- \frac{Q}{15p^4r^9}\left[24p^2r^4\left(5-27e^2+22e^4\right)\right] \\
&+ \frac{Q}{15p^3r^6}\sin(2\theta)\left(6585p^2 - 4630pr + 2292r^2(1-e^2)\right)\dot{\theta}\dot{r} \\
&- \frac{Q}{15p^4r^9}\left[2p^2\cos(2\theta)\left(4215p^4 - 7495p^3r + 4p^2r^2(1151-951e^2)\right)\right] \\
&- \frac{2Q}{15p^2r^6}\cos(2\theta)\left[300r(1-2e^2+e^4) - 1012p(1-e^2)\right] \\
&- \frac{Q}{15p^4r^9}\cos(2\iota)\left[2535p^6 - 3307p^5r + 12p^4r^2(37-237e^2) - 48r^6(1-e^2)^3\right] \\
&- \frac{Q}{15p^4r^9}\cos(2\iota)\left[800p^3r^3(1+e^2) + 128pr^5(1-2e^2+e^4)\right] \\
&+ \frac{204Q}{15p^2r^5}\cos(2\iota)\left(1+2e^2-3e^4\right) - \frac{4S^2}{15r^7}(446-201e^2) \\
&- \frac{2S^2}{15p^2r^9}\left[84r^4(1-e^2)^2(1+e^2)^2 + 345p^4 - 905p^3r - 413pr^3(1-e^2)\right] \\
&- \frac{S^2}{15p^2r^9}\cos(2\theta)\left[15p^4 - 110p^3r + 4p^2r^2(47-12e^2) - 118pr^3(1-e^2)\right] \\
&- \frac{24S^2}{15p^2r^5}\cos(2\theta)(1-e^2)^2(1+e^2)^2 \\
&+ \frac{S^2}{15r^9}\cos(2\iota)\left[45p^2 - 80pr + 36r^2(1-e^2)\right] \\
&+ \frac{S^2}{15pr^6}\sin(2\theta)\dot{r}\dot{\theta}\left[15p^2 + 10pr - 12r^2(1-e^2)\right],
\end{aligned}
\tag{5.54}
$$

$$
\begin{aligned}
\dot{L}_z &= -\frac{8\cos\iota}{5\sqrt{p}r^5}\left[15p^2-20pr+9r^2(1-e^2)\right] \\
&+ \frac{2S}{15p^2r^7}\left[525p^4-1751p^3r+34p^2r^2(61-6e^2)+12pr^3(-69+29e^2)\right] \\
&+ \frac{2S}{15p^2r^7}\left[6r^4(17+2e^2-19e^4)\right]-\frac{96S}{15p^2r^3}(1-2e^2+e^4)\cos(2\theta) \\
&+ \frac{2S}{15p^2r^7}\left[375p^4-93p^3r+468pr^3(1-e^2)-10p^2r^2(58+21e^2)\right]\cos(2\theta) \\
&+ \frac{4S}{15p^2r^7}\left[450p^4-922p^3r-60pr^3(3+e^2)-9p^2r^2(-83+23e^2)\right)\cos(2\iota) \\
&+ \frac{4S}{15p^2r^3}27(1+2e^2-3e^4)\cos(2\iota) \\
&- \frac{8S}{pr^4}\left[13p^2-8pr+4r^2(1-e^2)\right]\sin(2\theta)\dot{r}\dot{\theta} \\
&- \frac{Q\cos\iota}{5p^{5/2}r^7}\left[615p^4-753p^3r+15p^2r^2\left(19-31e^2\right)+20pr^3\left(1+3e^2\right)\right] \\
&- \frac{Q\cos\iota}{5p^{1/2}r^7}\cos(2\theta)\left(1185p^2-1601pr+756r^2(1-e^2)\right) \\
&- \frac{2Q\cos\iota}{5p^{5/2}r^7}\left[2\cos(2\iota)\left(45p^4-18r^4e^2(1-e^2)-45p^2r^2(1+e^2)\right)\right] \\
&- \frac{40Q\cos\iota}{5p^{5/2}r^4}p(1+e^2)2\cos(2\iota) \\
&- \frac{9Q\cos\iota}{5p^{5/2}r^3}\left(1-6e^2+5e^4\right)+\frac{2Q\cos\iota}{5p^{5/2}r^4}435p^3\sin(2\theta)\dot{\theta}\dot{r} \\
&- \frac{2S^2\cos\iota}{p^{1/2}r^7}\left[9p^2-16pr+\frac{36}{5}r^2(1-e^2)\right], \qquad (5.55)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{K} &= \frac{16}{5r^5}\left[20pr - 15p^2 - 9r^2(1-e^2)\right] \\
&+ \frac{8S\cos\iota}{15p^{3/2}r^7}\left[525p^4 - 1751p^3r + 2p^2r^2(1172-57e^2) + 12pr^3(-99+19e^2)\right] \\
&- \frac{8S\cos\iota}{15p^{3/2}r^3}24(-11+4e^2+7e^4) + \frac{48Q}{15p^2r^3}(1+7e^2-8e^4) \\
&+ \frac{2Q}{15p^2r^7}\left[-2145p^4 + 2659p^3r - 8pr^3(31+29e^2) - 2p^2r^2(427-867e^2)\right] \\
&+ \frac{2Q}{15p^2r^7}\left[2\cos(2\theta)\left(2175p^4 - 2975p^3r - 56pr^3(1-e^2)\right)\right] \\
&+ \frac{2Q}{15p^2r^3}\left[2\cos(2\theta)42(1-2e^2+e^4)\right] + \frac{2Q}{15p^2r^3}\left[3\cos(2\iota)36(1+2e^2-3e^4)\right] \\
&+ \frac{8Q}{15r^5}\cos(2\theta)(713-753e^2) \\
&+ \frac{2Q}{15p^2r^7}\left[3\cos(2\iota)\left(-345p^4 + 249p^3r - 160pr^3(1+e^2) + 120p^2r^2(1+3e^2)\right)\right] \\
&+ \frac{2Q}{15pr^4}\sin(2\theta)\left(3075p^2 - 20pr + 96r^2(1-e^2)\right)\dot{r}\dot{\theta} \\
&+ \frac{4S^2}{r^7}\left[-9p^2 + 16pr - \frac{36}{5}r^2(1-e^2)\right] \\
+ &+ \frac{2S^2}{r^7}\left(\cos(2\theta)+\cos(2\iota)\right)\left[3p^2 - \frac{16}{3}pr + \frac{12}{5}r^2(1-e^2)\right] \\
&+ \frac{4S^2}{pr^4}\sin(2\theta)\dot{r}\dot{\theta}\left[-2p^2 + \frac{7}{3}pr - \frac{4}{5}r^2(1-e^2)\right]. \qquad (5.56)
\end{aligned}
$$

5.3.4 Time averaged fluxes

In this section we will compute the infinite time-averages $\langle \dot{E} \rangle$, $\langle \dot{L}_z \rangle$ and $\langle \dot{K} \rangle$ of the fluxes. These averages are defined by

$$
\langle \dot{E} \rangle \equiv \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \dot{E}(t)dt. \qquad (5.57)
$$

These time-averaged fluxes are sufficient to evolve orbits in the adiabatic regime (except for the effect of resonances) [128, 137]. In Appendix B, we present two different ways of computing the time averages. The first approach is based on decoupling the $\tilde{r}$ and $\tilde{\theta}$ motion using the analog of the Mino time parameter for geodesic motion in Kerr [128]. The second approach uses the explicit Newtonian parameterization of the orbital motion. Both averaging methods give the following results:

$$
\begin{aligned}
\langle \dot{E} \rangle &= -\frac{32}{5}\frac{(1-e^2)^{3/2}}{p^5}\left[1+\frac{73}{24}e^2+\frac{37}{96}e^4\right. \\
&\quad -\frac{S}{p^{3/2}}\left(\frac{73}{12}+\frac{823}{24}e^2+\frac{949}{32}e^4+\frac{491}{192}e^6\right)\cos(\iota) \\
&\quad -\frac{Q}{p^2}\left(\frac{1}{2}+\frac{85}{32}e^2+\frac{349}{128}e^4+\frac{107}{384}e^6\right) \\
&\quad -\frac{Q}{p^2}\left(\frac{11}{4}+\frac{273}{16}e^2+\frac{847}{64}e^4+\frac{179}{192}e^6\right)\cos(2\iota) \\
&\quad -\frac{S^2}{p^2}\left(\frac{13}{192}+\frac{247}{384}e^2+\frac{299}{512}e^4+\frac{39}{1024}e^6\right) \\
&\quad \left.+\frac{S^2}{p^2}\left(\frac{1}{192}+\frac{19}{384}e^2+\frac{23}{512}e^4+\frac{3}{1024}e^6\right)\cos(2\iota)\right],
\end{aligned}
\tag{5.58}
$$

$$
\begin{aligned}
\langle \dot{L}_z \rangle &= -\frac{32}{5}\frac{(1-e^2)^{3/2}}{p^{7/2}}\cos\iota\left[1+\frac{7}{8}e^2\right. \\
&\quad -\frac{S}{2p^{3/2}\cos\iota}\left\{\frac{61}{24}+7e^2+\frac{271}{64}e^4+\left(\frac{61}{8}+\frac{91}{4}e^2+\frac{461}{64}e^4\right)\cos(2\iota)\right\} \\
&\quad -\frac{Q}{16p^2}\left\{-3-\frac{45}{4}e^2+\frac{19}{8}e^4+\left(45+148e^2+\frac{331}{8}e^4\right)\cos(2\iota)\right\} \\
&\quad \left.+\frac{S^2}{16p^2}\left\{1+3e^2+\frac{3}{8}e^4\right\}\right]
\end{aligned}
\tag{5.59}
$$

$$
\begin{aligned}
\langle \dot{K} \rangle &= -\frac{64}{5}\frac{(1-e^2)^{3/2}}{p^3}\left[1+\frac{7}{8}e^2-\frac{S}{2p^{3/2}}\left(\frac{97}{6}+37e^2+\frac{211}{16}e^4\right)\cos(\iota)\right. \\
&\quad -\frac{Q}{p^2}\left\{\frac{1}{2}+\frac{55}{48}e^2+\frac{139}{192}e^4+\left(\frac{13}{4}+\frac{841}{96}e^2+\frac{449}{192}e^4\right)\cos(2\iota)\right\} \\
&\quad \left.+\frac{S^2}{p^2}\left\{\frac{13}{192}+\frac{13}{64}e^2+\frac{13}{512}e^4-\left(\frac{1}{192}+\frac{1}{64}e^2+\frac{1}{512}e^4\right)\cos(2\iota)\right\}\right]
\end{aligned}
\tag{5.60}
$$

Using Eqs. (5.51) and (5.53), we obtain from (5.58) – (5.61) the following time averaged rates of change of the orbital elements e, p, ι:

$$\begin{aligned}
\langle\dot{p}\rangle &= -\frac{64}{5}\frac{(1-e^2)^{3/2}}{p^3}\left[1+\frac{7}{8}e^2-\frac{S}{2p^{3/2}}\left(\frac{97}{6}+37e^2+\frac{211}{16}e^4\right)\cos(\iota)\right. \\
&\quad -\frac{Q}{p^2}\left\{\frac{1}{2}+\frac{55}{48}e^2+\frac{139}{192}e^4+\left(\frac{13}{4}+\frac{841}{96}e^2+\frac{449}{192}e^4\right)\cos(2\iota)\right\} \\
&\quad \left.+\frac{S^2}{p^2}\left\{\frac{13}{192}+\frac{13}{64}e^2+\frac{13}{512}e^4-\left(\frac{1}{192}+\frac{1}{64}e^2+\frac{1}{512}e^4\right)\cos(2\iota)\right\}\right]
\end{aligned} \tag{5.61}$$

$$\begin{aligned}
\langle\dot{e}\rangle &= -\frac{304}{15}\frac{(1-e^2)^{3/2}}{p^4 e}\left[e^2\left(1+\frac{121}{304}e^2\right)\right. \\
&\quad -\frac{S}{p^{3/2}}\left(-\frac{12}{19}+\frac{573}{76}e^2+\frac{105}{8}e^4+\frac{1757}{608}e^6\right)\cos(\iota) \\
&\quad -\frac{Q}{p^2}\left(\frac{193}{304}+\frac{1209}{1216}e^2+\frac{385}{1216}e^4\right) \\
&\quad -\frac{Q}{p^2}\left(-\frac{3}{19}+\frac{1109}{304}e^2+\frac{1887}{304}e^4+\frac{157}{152}e^6\right)\cos(2\iota) \\
&\quad \left.+\frac{S^2}{p^2}\frac{15e^2}{9728}\left(8+12e^2+e^4\right)\left(13-\cos(2\iota)\right)\right]
\end{aligned} \tag{5.62}$$

$$\begin{aligned}
\langle\dot{\iota}\rangle &= \frac{(1-e^2)^{3/2}}{5p^{11/2}}S\csc(\iota)\left[\frac{266}{3}+184e^2+\frac{151}{4}e^4+\left(\frac{22}{3}-62e^2-\frac{39}{4}e^4\right)\cos(2\iota)\right] \\
&\quad +\frac{22(1-e^2)^{3/2}}{5p^6}Q\cot(\iota)\left[1+\frac{355}{132}e^2+\frac{221}{264}e^4\right] \\
&\quad +\frac{22(1-e^2)^{3/2}}{5p^6}Q\cot(\iota)\left[\frac{7}{11}-\frac{47}{66}e^2-\frac{95}{264}e^4\right]\cos(2\iota) \\
&\quad -\frac{(1-e^2)^{3/2}}{240p^6}S^2e^2\sin(2\iota)\left[8+3e^2(8+e^2)\right]
\end{aligned} \tag{5.63}$$

5.4 Application to black holes

5.4.1 Qualitative discussion of results

The above results for the fluxes, Eqs. (5.62), (5.62) and (5.63) show that the correction terms at $O(a^2\epsilon^4)$ due to the quadrupole have the same type of effect on the evolution as the linear spin correction computed by Ryan: they tend to circularize eccentric orbits and change the angle ι such as to become antialigned with the symmetry axis of the quadrupole.

The effects of the terms quadratic in spin are qualitatively different. In the expression (5.58) for $\langle \dot{E} \rangle$, the coefficient of $\cos(2\iota)$ due to the spin self-interaction has the opposite sign to the quadrupole term, while the terms not involving ι have the same sign. The terms involving $\cos(2\iota)$ in Eq. (5.61) for $\langle \dot{K} \rangle$ of $O(Q)$ and $O(S^2)$ have the same sign, while the terms not involving ι have the opposite sign. The fractional spin-spin correction to $\langle \dot{L}_z \rangle$, Eq. (5.59), has no ι-dependence, and in expression (5.63) for $\langle i \rangle$, the dependence on ι of the two effects $O(Q)$ and $O(S^2)$ is different, too. This is not surprising as the $O(Q)$ effects included here are corrections to the conservative orbital dynamics, while the effects of $O(S^2)$ that we included are due to radiation reaction.

5.4.2 Comparison with previous results

The terms linear in the spin in our results for the time averaged fluxes, Eqs. (5.58) – (5.63), agree with those computed by Ryan, Eqs. (14a) – (15c) of [190], and with those given in Eqs. (2.5) – (2.7) of Ref. [191], when we use the transformations to the variables used by Ryan given in Eqs. (2.3) – (2.4) in [191].

Equation (5.58) for the time averaged energy flux agrees with Eq. (3.10) of Gergely [182] and Eq. (4.15) of [177] when we use the following transformations:

$$
\begin{aligned}
K &= \bar{L}^2\left[1-\frac{Q}{2\bar{L}^4}\left(\bar{A}^2\sin^2\kappa\cos\delta-(1-\bar{A}^2)\cos^2\kappa\right)\right] \\
&= \bar{L}^2\left[1-\frac{Q}{\bar{L}^4}\;E\;\cos^2\kappa\right. \\
&\qquad\left.-\frac{Q}{2\bar{L}^4}(1+2\bar{L}^2)\sin^2\kappa\cos\delta\right], && (5.64)\\
\cos\iota &= \cos\kappa\left[1+\frac{Q}{2\bar{L}^4}\;E\;\cos^2\kappa\right. \\
&\qquad\left.+\frac{Q}{2\bar{L}^4}(1+2\bar{L}^2)\sin^2\kappa\cos\delta\right], && (5.65)\\
\xi_0 &= \frac{1}{2}(\delta+\kappa), && (5.66)\\
\xi_0 &= (\psi_0-\psi_i)+\frac{\pi}{2}, && (5.67)
\end{aligned}
$$

where $\bar{A}$, $\bar{L}$, κ, δ, ψ_0 and ψ_i are the quantities used by Gergely. The first relation here is obtained from the turning points of the radial motion as follows. We compute $\tilde{r}_\pm$ in terms of E and K and map these expressions back to r using Eqs. (5.14). The result can then be compared with the turning points in Gergely's variables, Eq. (2.19) of [182], using the fact that E is the same in both cases. Instead of the evolution of the constants of motion K and L_z, Gergely computes the rates of change of the magnitude L of the orbital angular momentum and of the angle κ defined by $\cos\kappa = (\mathbf{L}\cdot\mathbf{S})/L$. Using the transformations (5.64) – (5.67) and the definition of κ we verify that our Eq. (5.59) agrees with the $\langle \dot{L}_z\rangle$ computed using Gergely's Eqs. (3.23) and (3.35) in [182] and Eq. (4.30) of [177].

In the limit of the circular equatorial orbits analyzed by Poisson [181], our Eq. (5.58) agrees with Poisson's Eq. (22) when we use the transformations and specializations:

$$p = \frac{1}{v^2}\left[1 - \frac{Q}{4}v^4\right], \tag{5.68}$$

$$\iota = 0, \tag{5.69}$$

$$e^2 = 0, \tag{5.70}$$

$$\cos\alpha_A = 1, \tag{5.71}$$

where v and α_A are the variables used by Poisson and the relation (5.68) is obtained by comparing the expressions for the constants of motion in the two sets of variables.

The main improvement of our analysis over Gergely's is that we express the results in terms of the Carter-type constant K, which facilitates comparing our results with other analyses of black hole inspirals. Our computations also include the spin curvature scattering effects for all three constants of motion; Gergely [177] only considers these effects for two of them: the energy and magnitude of angular momentum, not for the third conserved quantity.

When we expand Eq. (5.58) for small inclination angles and specialize to circular orbits, then after converting p to the parameter v using Eq. (5.68), we obtain

$$\begin{aligned}\langle \dot{E} \rangle &= -\frac{32}{5p^5}\left[1 - \frac{1}{p^2}\left(2Q + \frac{S^2}{16}\right) + \frac{\iota^2}{2p^2}\left(11Q - \frac{S^2}{48}\right)\right] \\ &= -\frac{32}{5p^5}\left[1 - \frac{a^2v^4}{16}\left\{33 - \frac{527}{6}\iota^2\right\}\right].\end{aligned} \tag{5.72}$$

This result agrees with the terms at $O(a^2v^4)$ of Eq. (3.13) of Shibata et al. [183], whose calculations were based on the fully relativistic expressions. This agreement is a check that we have taken into account all the contributions at $O(a^2\epsilon^4)$. The analysis in Ref. [183] could not distinguish between effects due to the quadrupole and those due curvature scattering, but we can see from Eq. (5.72) that those two interactions have the opposite dependence on ι. Comparing (5.72) with Eq. (3.7) of [183] (which gives the fluxes into the different modes ($l = 2, m, n$), where m and n are the multiples of the φ and θ frequencies), we see that the terms in the $(2, \pm 2, 0)$ and the $(2, \pm 1, \pm 1)$ modes are entirely due to the quadrupole, while the spin-spin interaction effects are fully contained in the $(2, \pm 1, 0)$ and $(2, 0, \pm 1)$ modes.

5.5 Non-existence of a Carter-type constant for higher multipoles

In this section, we show that for a single axisymmetric multipole interaction, it is not possible to find an analog of the Carter constant (a conserved quantity which does not correspond to a symmetry of the Lagrangian), except for the cases of spin (treated by Ryan [174]) and mass quadrupole moment (treated in this paper). Our proof is valid only in the approximations in which we work – expanding to linear order in the mass ratio, to the leading post-Newtonian order, and to linear order in the multipole. However we will show below that with very mild additional smoothness assumptions, our non-existence result extends to exact geodesic motion in exact vacuum spacetimes.

We start in Sec. 5.5.1 by showing that there is no coordinate system in which the Hamilton-Jacobi equation is separable. Now separability of the Hamilton-Jacobi equation is a sufficient but not a necessary condition for the existence of a additional conserved quantity. Hence, this result does not yield information about the existence or non-existence of an additional constant. Nevertheless we find it to be a suggestive result. Our actual derivation of the non-existence is based on Poisson bracket computations, and is given in Sec. 5.5.2.

5.5.1 Separability analysis

Consider a binary of two point masses m_1 and m_2, where the mass m_1 is endowed with a single axisymmetric current multipole moment S_l or axisymmetric mass multipole moment I_l. In this section, we show that the Hamilton-Jacobi equation for this motion, to linear order in the multipoles, to linear order in the mass ratio and to the leading post-Newtonian order, is separable only for the cases S_1 and I_2.

We choose the symmetry axis to be the z-axis and write the action for a general multipole as

$$S = \int dt \quad \Big[\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) + \frac{1}{r} \\ + f(r,\theta) + g(r,\theta)\dot{\varphi} + E\Big]\,. \tag{5.73}$$

For mass moments, $g(r,\theta) = 0$, while for current moments $f(r,\theta) = 0$. For an axisymmetric multipole of order l, the functions f and g will be of the form

$$f(r,\theta) = \frac{c_l I_l P_l(\cos\theta)}{r^{l+1}}, \quad g(r,\theta) = \frac{d_l S_l \sin\theta\partial_\theta P_l(\cos\theta)}{r^l}, \tag{5.74}$$

where $P_l(\cos\theta)$ are the Legendre polynomials and c_l and d_l are constants. We will work to linear order in f and g. In Eq. (5.73), we have added the energy term needed when doing a change of time variables, cf. the discussion before Eq. (5.17) in Sec. 5.3. Since φ is a cyclic coordinate, $p_\varphi = L_z$ is a constant of motion and the system has effectively only two degrees of freedom. Note that in the case of a current moment, there will be correction term in L_z:

$$L_z = r^2\sin^2\theta\dot{\varphi} + g(r,\theta). \tag{5.75}$$

Next, we switch to a different coordinate system $(\tilde{r}, \tilde{\theta}, \varphi)$ defined by

$$\begin{aligned} r &= \tilde{r} + \alpha(\tilde{r}, \tilde{\theta}, L_z), & (5.76) \\ \theta &= \tilde{\theta} + \beta(\tilde{r}, \tilde{\theta}, L_z), & (5.77) \end{aligned}$$

where the functions α and β are yet undetermined. We also define a new time variable $\tilde{t}$ by

$$dt = \left[1 + \gamma(\tilde{r}, \tilde{\theta}, L_z)\right] d\tilde{t}. \quad (5.78)$$

Since we work to linear order in f and g, we can work to linear order in α, β, and γ. We then compute the action in the new coordinates and drop the tildes. The Hamiltonian is given by

$$\begin{aligned} H = & \frac{1}{2} p_r^2 (1 + \gamma - 2\alpha_{,r}) + \frac{p_\theta^2}{2r^2} (1 - \frac{2\alpha}{r} - 2\beta_{,\theta} + \gamma) \\ & + \frac{p_r p_\theta}{r^2} (-\alpha_{,\theta} - r^2 \beta_{,r}) - E(1 + \gamma) \\ & + \frac{L_z^2}{2r^2 \sin^2\theta} (1 + \gamma - \frac{2\alpha}{r} - 2\beta \cot\theta) \\ & - \frac{1}{r} (1 - \frac{\alpha}{r} + \gamma) - f - \frac{g L_z}{r^2 \sin^2\theta} \end{aligned} \quad (5.79)$$

and the corresponding Hamilton-Jacobi equation is

$$\begin{aligned} 0 = & \left(\frac{\partial W}{\partial r}\right)^2 \hat{C}_1 + \left(\frac{\partial W}{\partial \theta}\right)^2 \frac{\hat{C}_2}{r^2} \\ & + 2 \left(\frac{\partial W}{\partial r}\right) \left(\frac{\partial W}{\partial \theta}\right) \frac{\hat{C}_3}{r^2} + 2\hat{V}, \end{aligned} \quad (5.80)$$

where we have denoted

$$\begin{aligned}
\hat{C}_1 &= J(r,\theta)\left[1+\gamma-2\alpha_{,r}\right] = 1+\gamma-2\alpha_{,r}+j, && (5.81)\\
\hat{C}_2 &= J(r,\theta)\left[1-\frac{2\alpha}{r}-2\beta_{,\theta}+\gamma\right]\\
&= 1-\frac{2\alpha}{r}-2\beta_{,\theta}+\gamma+j, && (5.82)\\
\hat{C}_3 &= J(r,\theta)\left[-\alpha_{,\theta}-r^2\beta_{,r}\right] = -\alpha_{,\theta}-r^2\beta_{,r}, && (5.83)\\
\hat{V} &= J(r,\theta)\left[\frac{L_z^2}{2r^2\sin^2\theta}(1+\gamma-\frac{2\alpha}{r}-2\beta\cot\theta)\right.\\
&\qquad -\frac{1}{r}(1-\frac{\alpha}{r}+\gamma)-E(1+\gamma)\\
&\qquad \left.-f-\frac{gL_z}{r^2\sin^2\theta}\right]\\
&= \frac{L_z^2}{2r^2\sin^2\theta}(1+\gamma-\frac{2\alpha}{r}-2\beta\cot\theta+j)\\
&\quad -E(1+\gamma+j)-\frac{1}{r}(1-\frac{\alpha}{r}+\gamma+j)\\
&\quad -f-\frac{gL_z}{r^2\sin^2\theta}. && (5.84)
\end{aligned}$$

The unperturbed problem is separable, so to make the perturbed problem separable, we have multiplied the Hamilton-Jacobi equation by an arbitrary function $J(r,\theta)$, which can be expanded as $J(r,\theta) = 1 + j(r,\theta)$, where $j(r,\theta)$ is a small perturbation.

To find a solution of the form $W = W_r(r) + W_\theta(\theta)$, we first specialize to the case where $\hat{C}_3 = 0$:

$$-\hat{C}_3 = \beta_{,r}r^2 + \alpha_{,\theta} = 0. \tag{5.85}$$

We differentiate Eq. (5.80) with respect to θ, using Eq. (5.80) to write $(dW_r/dr)^2$ in terms of $(dW_\theta/d\theta)^2$ and then differentiate the result with respect to r to obtain

$$\begin{aligned}
0 &= \left(\frac{dW_\theta}{d\theta}\right)^2 \partial_r\left[\frac{\partial_\theta\hat{C}_2}{\hat{C}_2}-\frac{\partial_\theta\hat{C}_1}{\hat{C}_1}\right]\\
&\quad +2\partial_r\left[r^2\frac{\partial_\theta\hat{V}}{\hat{C}_2}-\frac{r^2\hat{V}\partial_\theta\hat{C}_1}{\hat{C}_1\hat{C}_2}\right]. && (5.86)
\end{aligned}$$

Expanding Eq. (5.86) to linear order in the small quantities then yields the two conditions for the kinetic and the potential part of the Hamiltonian to be separable:

$$
\begin{aligned}
0 &= \partial_r \partial_\theta \left(2\alpha_{,r} - \frac{2\alpha}{r} - 2\beta_{,\theta} \right), && (5.87) \\
0 &= \frac{L_z^2}{\sin^2\theta} \left(2\beta_{,r} \cot^2\theta - 3\beta_{,r\theta} \cot\theta + \beta_{,r} \csc^2\theta \right) \\
&\quad + \frac{L_z^2}{\sin^2\theta} \partial_r \left[-\frac{\alpha_{,\theta}}{r} + \alpha_{,r\theta} \right] \\
&\quad - \partial_r \partial_\theta \left[\frac{c_l I_l}{r^{l-1}} P_l(\cos\theta) + \frac{d_l S_l L_z}{r^l \sin\theta} \partial_\theta P_l(\cos\theta) \right] \\
&\quad - \partial_r \left[r \left(2\alpha_{,r\theta} - \frac{\alpha_{,\theta}}{r} \right) + 2Er^2 \alpha_{,r\theta} \right], && (5.88)
\end{aligned}
$$

where we have used Eq. (5.74) for f and g. Therefore, the following conditions must be satisfied:

$$
\begin{aligned}
M_4(\theta) - N(r) &= \frac{\alpha}{r} + \beta_{,\theta} - 2\alpha_{,r}, && (5.89) \\
M_1(\theta) &= 2\beta \cot^2\theta + \beta \csc^2\theta + \beta_{,\theta\theta} \\
&\quad - 3\beta_{,\theta} \cot\theta, && (5.90) \\
M_2(\theta) &= r^2 \partial_r (r^2 \beta_{,r}), && (5.91) \\
M_3(\theta) &= 2r\alpha_{,r\theta} - \alpha_{,\theta} + \frac{I_l}{r^{l-1}} \partial_\theta P_l(\cos\theta) \\
&\quad - \frac{S_l L_z}{r^l} \partial_\theta (\csc\theta \, \partial_\theta P_l(\cos\theta)). && (5.92)
\end{aligned}
$$

Here, the functions M and N are arbitrary integration constants.

Solving the condition for the kinetic term to be separable, Eq. (5.89), together with Eq. (5.85) gives the general solution that goes to zero at large r as

$$
\begin{aligned}
\alpha &= \frac{A}{r^{n-1}} \cos(n\theta + \nu), && (5.93) \\
\beta &= -\frac{A}{r^n} \sin(n\theta + \nu), && (5.94)
\end{aligned}
$$

where A and ν are arbitrary and n is an integer. These functions must satisfy the conditions (5.90) – (5.92) in order for the potential term to be separable as well.

To see when this will be the case, we start by considering Eq. (5.92). Substituting the general ansatz $\alpha = a_1(r)a_2(\theta)$ shows that $a_2' = P_l'$ or $a_2' = (csc\theta\, P_l')'$ depending on whether a mass or a current multipole is present. The function $a_1(r)$ is then determined from

$$0 = 2ra_1' - a_1 + \begin{cases} c_l I_l / r^{(l-1)} \\ \\ d_l S_l L_z / r^l \end{cases} \tag{5.95}$$

Hence,

$$a_1 = \begin{cases} \left[c_l I_l/(2l)\right] r^{(1-l)} \\ \\ \left[d_l S_l L_z/(2l+1)\right] r^{-l} \end{cases} \tag{5.96}$$

so that we obtain for mass moments

$$\alpha = \frac{c_l I_l}{2l} \frac{P_l(\cos\theta)}{r^{l-1}}, \qquad \beta = \frac{c_l I_l}{2l^2} \frac{P_l'(\cos\theta)}{r^l} \tag{5.97}$$

and for current moments

$$\alpha = \frac{d_l S_l L_z}{2l+1} \frac{\csc\theta P_l'(\cos\theta)}{r^l}, \tag{5.98}$$

$$\beta = \frac{d_l S_l L_z}{(2l+1)(l+1)} \frac{(\csc\theta\, P_l'(\cos\theta))'}{r^{l+1}}, \tag{5.99}$$

where we have used the condition (5.85) to solve for β.

Substituting this in Eq. (5.91) determines that $l = 2$ for mass moments and $l + 1 = 2$ for current moments. For an $l = 2$ mass moment, conditions (5.89) and (5.90) are satisfied as well, with $n = 2$ and $\nu = 0$. For the case of an $l = 1$ current moment, the extra term in H is independent of θ anyway. But for any other multipole interaction, the Hamilton-Jacobi equation will not be separable. For example, for the current octupole S_{ijk}, the last term in Eq. (5.79) is proportional to $S_3 L_z (5\cos^2\theta - 1)/r^5$ and is therefore not separable. From Eq. (5.74) one can see that, for a general multipole, the functions f or g contain different powers of $\cos\theta$ appearing with the same power of r since the Legendre polynomials can be expanded as [169]:

$$P_l(\cos\theta) = \sum_{n=0}^{N} \frac{(-1)^n (2l-2n)!}{2^l n!(l-n)!(l-2n)!} (\cos\theta)^{l-2n}, \tag{5.100}$$

where $N = l/2$ for even l and $N = (l+1)/2$ for odd l. It will not be possible to cancel all of these terms with (5.93) – (5.94) for $l > 2$.

The case when $\hat{C}_3$ is non-vanishing will only be separable if all the coefficients are functions of r or of θ only, and if in addition, the potential also depends only on r or on θ. Achieving this for our problem will not be possible because the potential cannot be transformed to the form required for separability.

5.5.2 Derivation of non-existence of additional constants of the motion

In this subsection, we show using Poisson brackets that for a single axisymmetric multipole interaction, to linear order in the multipole and the mass ratio, a first integral analogous to the Carter constant does not exist, except for the cases of mass quadrupole and spin.

Suppose that such a constant does exist. We write the Hamiltonian corresponding to the action (5.73) as $H = H_0 + \delta H$ and the Carter-type constant as $K = K_0 + \delta K(p_r, p_\theta, L_z, r, \theta)$, where

$$\begin{aligned}
H_0 &= \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + \frac{L_z^2}{2r^2 \sin^2\theta} - \frac{1}{r}, && (5.101)\\
\delta H &= -\frac{c_l I_l}{r^{l+1}} P_l(\cos\theta) - \frac{d_l S_l L_z}{r^{l+2}\sin\theta} \partial_\theta P_l(\cos\theta), && (5.102)\\
K_0 &= p_\theta^2 + \frac{L_z^2}{\sin^2\theta}. && (5.103)
\end{aligned}$$

Computing the Poisson bracket gives, to linear order in the perturbations

$$0 = \{H_0, \delta K\} + \{\delta H, K_0\} \tag{5.104a}$$
$$= \frac{d}{dt}\delta K + \{\delta H, K_0\}, \tag{5.104b}$$

where we have used that $\{H_0, K_0\} = 0$ and the fact that $\{H_0, \delta K\} = d(\delta K)/dt$. Here, d/dt denotes the total time derivative along an orbit $(r(t), \theta(t), p_r(t), p_\theta(t))$ of H_0 in phase space. The partial differential equation (5.104a) for δK thus reduces to a set of ordinary differential equations that can be integrated along the individual orbits in phase space.

The unperturbed motion for a bound orbit is in a plane, so we can switch from spherical to plane polar coordinates (r, ψ). In terms of these coordinates, we have $H_0 = p_r^2/2 + p_\psi^2/2r^2$, $K_0 = p_\psi^2$, and $\cos\theta = \sin\iota \sin(\psi + \psi_0)$, with $\cos\iota = L_z/\sqrt{K}$ and the constant ψ_0 denoting the angle between the direction of the periastron and the intersection between the orbital and equatorial plane. Then Eq. (5.104) becomes

$$\frac{d}{dt}\delta K = \eta(t), \tag{5.105}$$
$$\eta(t) = -\frac{2p_\psi \ d_l S_l L_z}{\sin\iota \ r^{l+2}(t)}\partial_\psi\left(\frac{\partial_\psi P_l(\sin\iota\sin(\psi(t)+\psi_0))}{\cos(\psi(t)+\psi_0)}\right) + \frac{2p_\psi \ c_l I_l}{r^{l+1}(t)}\partial_\psi P_l(\sin\iota\sin(\psi(t)+\psi_0)). \tag{5.106}$$

For unbound orbits, one can always integrate Eq. (5.105) to determine δK. However, for bound periodic orbits there is a possible obstruction: the solution for the conserved quantity $K_0 + \delta K$ will be single valued if and only if the integral of the source over the closed orbit vanishes,

$$\oint_0^{T_{\rm orb}} \eta(t) dt = 0. \tag{5.107}$$

Here, $T_{\rm orb}$ is the orbital period. In other words, the partial differential equation (5.104) has a solution δK if and only if the condition (5.107) is satisfied. This is the same condition as obtained by the Poincare-Mel'nikov-Arnold method, a technique for showing the non-integrability and existence of chaos in certain classes of perturbed dynamical systems [192].

Thus, it suffices to show that the condition (5.107) is violated for all multipoles other than the spin and mass quadrupole. To perform the integral in Eq. (5.107), we use the parameterization for the unperturbed motion, $r = K/(1 + e\cos\psi)$ and $dt/d\psi = K^{3/2}/(1 + e\cos\psi)^2$, so that the condition for the existence of a conserved quantity $K_0 + \delta K$ becomes

$$0 = \int_0^{2\pi} d\psi \Big[c_l I_l (1 + e\cos\psi)^{l-1} \partial_\psi P_l(\sin\iota \sin(\psi + \psi_0)) \\ - \frac{d_l S_l L_z}{K\sin\iota} (1 + e\cos\psi)^l \partial_\psi \left(\frac{\partial_\psi P_l(\sin\iota\sin(\psi+\psi_0))}{\cos(\psi + \psi_0)} \right) \Big]. \quad (5.108)$$

In terms of the variable $\chi = \psi + \psi_0 - \pi/2$, Eq. (5.108) can be written as

$$\begin{aligned} 0 &= \int_0^{2\pi} d\chi c_l I_l \left[1 + e(\sin\psi_0\cos\chi - \cos\psi_0 \sin\chi)\right]^{l-1} \frac{d}{d\chi} P_l(\sin\iota\cos\chi) \\ &+ \int_0^{2\pi} d\chi \frac{d_l S_l L_z}{\sin\iota} \left[1 + e(\sin\psi_0\cos\chi - \cos\psi_0\sin\chi)\right]^l \\ &\qquad \frac{d}{d\chi}\left(\frac{1}{\sin\chi}\frac{d}{d\chi} P_l(\sin\iota\cos\chi)\right). \end{aligned} \quad (5.109)$$

Inserting the expansion (5.100) for $P_l(\cos\chi)$, taking the derivatives, and using the binomial expansion for the first term in Eq. (5.109), we get

$$\begin{aligned} 0 &= c_l I_l \sum_{n=0}^{N} \sum_{j=0}^{l-1} \sum_{k=0}^{j} A_{lnjk}\, e^j (\sin\iota)^{l-2n} (\sin\psi_0)^k (\cos\psi_0)^{j-k} \\ &\qquad \int_0^{2\pi} d\chi\, (\sin\chi)^{j-k+1} (\cos\chi)^{k+l-2n-1} \\ &+ \frac{d_l S_l L_z}{K} \sum_{n=0}^{N} \sum_{j=0}^{l} \sum_{k=0}^{j} B_{lnjk}\, e^j (\sin\iota)^{l-2n-1} (\sin\psi_0)^k (\cos\psi_0)^{j-k} \\ &\qquad \int_0^{2\pi} d\chi\, (\sin\chi)^{j-k+1} (\cos\chi)^{k+l-2n-2}. \end{aligned} \quad (5.110)$$

The coefficients A_{lnkj} and B_{lnkj} are

$$
\begin{aligned}
A_{lnkj} &= \frac{(-1)^{n+k+1}(l-1)!(2l-2n)!}{2^l n!(l-1-j)!k!(j-k)!(l-n)!(l-2n-1)!}, && (5.111)\\
B_{lnkj} &= \frac{(-1)^{n+k} l!(2l-2n)!}{2^l n!(l-j)!k!(j-k)!(l-n)!(l-2n-2)!}. && (5.112)
\end{aligned}
$$

The only non-vanishing contribution to the integrals in Eq. (5.110) will come from terms with even powers of both $\cos\chi$ and $\sin\chi$. These can be evaluated as multiples of the beta function:

$$
\begin{aligned}
0 &= c_l I_l \sum_{n=0}^{N}\sum_{j=0}^{l-1}\sum_{k=0}^{j} C_{lnjk}\, e^j (\sin\iota)^{l-2n} (\sin\psi_0)^k (\cos\psi_0)^{j-k}\ \delta_{(j-k+1),\mathrm{even}}\, \delta_{(l+k-1),\mathrm{even}}\\
&+ \frac{d_l S_l L_z}{K} \sum_{n=0}^{N}\sum_{j=0}^{l}\sum_{k=0}^{j} D_{lnjk}\, e^j (\sin\iota)^{l-2n-1} (\sin\psi_0)^k (\cos\psi_0)^{j-k}\\
&\qquad \delta_{(j-k+1),\mathrm{even}}\, \delta_{(l+k),\mathrm{even}}. && (5.113)
\end{aligned}
$$

Here, the coefficients are

$$
\begin{aligned}
C_{lnjk} &= \frac{2\Gamma(\frac{j}{2}-\frac{k}{2}+1)\Gamma(\frac{k}{2}+\frac{l}{2}-n)}{\Gamma(\frac{j}{2}+\frac{l}{2}-n+1)} A_{lnkj}, && (5.114)\\
D_{lnjk} &= \frac{2\Gamma(\frac{j}{2}-\frac{k}{2}+1)\Gamma(\frac{k}{2}+\frac{l}{2}-n-\frac{1}{2})}{\Gamma(\frac{j}{2}+\frac{l}{2}-n+\frac{3}{2})} B_{lnkj} && (5.115)
\end{aligned}
$$

Eq. (5.113) shows that for even l, terms with j =even (odd) and k =odd (even) give a non-vanishing contribution for the case of a mass (current) multipole, and hence $K_0 + \delta K$ is not a conserved quantity for the perturbed motion. Note that terms with j =even and k =odd for even l occur only for $l > 3$, so for $l = 2$ the mass quadrupole term in Eq. (5.113) vanishes and therefore there exists an analog of the Carter constant, which is consistent with our results of Sec. 5.2 and our separability analysis. For odd l, terms with j =odd (even) and k =even (odd) are finite for I_l (S_l). Note that for the case $l = 1$ of the spin, the derivatives with respect to χ in Eq. (5.109) evaluate to zero, so in this case there also exists a Carter-type constant. These results show that for a general multipole other than I_2 and S_1, there will not be a Carter-type constant for such a system.

Exact vacuum spacetimes

Our result on the non-existence of a Carter-type constant can be extended, with mild smoothness assumptions, to falsify the conjecture that all exact, axisymmetric vacuum spacetimes possess a third constant of the motion for geodesic motion. Specifically, we fix a multipole of order l, and we assume:

- There exists a one parameter family

  $$(M, g_{ab}(\lambda))$$

  of spacetimes, which is smooth in the parameter λ, such that $\lambda = 0$ is Schwarzschild, and each spacetime $g_{ab}(\lambda)$ is stationary and axisymmetric with commuting Killing fields $\partial/\partial t$ and $\partial/\partial\phi$, and such that all the mass and current multipole moments of the spacetime vanish except for the one of order l. On physical grounds, one expects a one parameter family of metrics with these properties to exist.

- We denote by $H(\lambda)$ the Hamiltonian on the tangent bundle over M for geodesic motion in the metric $g_{ab}(\lambda)$. By hypothesis, there exists for each λ a conserved quantity $M(\lambda)$ which is functionally independent of the conserved energy and angular momentum. Our second assumption is that $M(\lambda)$ is differentiable in λ at $\lambda = 0$. One would expect this to be true on physical grounds.

- We assume that the conserved quantity $M(\lambda)$ is invariant under the symmetries of the system:

  $$\mathcal{L}_{\vec{\xi}} M(\lambda) = \mathcal{L}_{\vec{\eta}} M(\lambda) = 0,$$

  where $\vec{\xi}$ and $\vec{\eta}$ are the natural extensions to the 8 dimensional phase space of the Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$. This is a very natural assumption.

These assumptions, when combined with our result of the previous section, lead to a contradiction, showing that the conjecture is false under our assumptions.

To prove this, we start by noting that $M(0)$ is a conserved quantity for geodesic motion in Schwarzschild, so it must be possible to express it as some function f of the three independent conserved quantities:

$$M(0) = f(E, L_z, K_0). \tag{5.116}$$

Here E is the energy, L_z is the angular momentum, and K_0 is the Carter constant. Differentiating the exact relation $\{H(\lambda), M(\lambda)\} = 0$ and evaluating at $\lambda = 0$ gives

$$\{H_0, M_1\} = \frac{\partial f}{\partial E}\{E, H_1\} + \frac{\partial f}{\partial L_z}\{L_z, H_1\} + \frac{\partial f}{\partial K_0}\{K_0, H_1\}, \tag{5.117}$$

where $H_0 = H(0)$, $H_1 = H'(0)$, and $M_1 = M'(0)$. As before, we can regard this is a partial differential equation that determines M_1, and a necessary condition for solutions to exist and be single valued is that the integral of the right hand side over any closed orbit must vanish:

$$\oint \left[\frac{\partial f}{\partial E}\{E, H_1\} + \frac{\partial f}{\partial L_z}\{L_z, H_1\} + \frac{\partial f}{\partial K_0}\{K_0, H_1\}\right] = 0. \tag{5.118}$$

Now strictly speaking, there are no closed orbits in the eight dimensional phase space. However, the argument of the previous section applies to orbits which are closed in the four dimensional space with coordinates $(r, \theta, p_r, p_\theta)$, since by the third assumption above everything is independent of t and ϕ, and p_t and p_ϕ are conserved. Here (t, r, θ, ϕ) are Schwarzschild coordinates and $(p_t, p_r, p_\theta, p_\phi)$ are the corresponding conjugate momenta.

Next, we can pull the partial derivatives $\partial f/\partial E$ etc. outside of the integral. It is then easy to see that the first two terms vanish, since there do exist a conserved energy and a conserved z-component of angular momentum for the perturbed system. Thus, Eq. (5.118) reduces to

$$\frac{\partial f}{\partial K_0} \oint \{K_0, H_1\} = 0. \tag{5.119}$$

Since $M(0)$ is functionally independent of E and L_z, the prefactor $\partial f / \partial K_0$ must be nonzero, so we obtain

$$\oint \{K_0, H_1\} = 0. \tag{5.120}$$

The result (5.120) applies to fully relativistic orbits in Schwarzschild. We need to take the Newtonian limit of this result in order to use the result we derived in the previous section. However, the Newtonian limit is a little subtle since Newtonian orbits are closed and generic relativistic orbits are not closed. We now discuss how the limit is taken.

The integral (5.120) is taken over any closed orbit in the four dimensional phase space $(r, \theta, p_r, p_\theta)$ which corresponds to a geodesic in Schwarzschild. Such orbits are non generic; they are the orbits for which the ratio between the radial and angular frequencies ω_r and ω_θ is a rational number. We denote by q_r and q_θ the angle variables corresponding to the r and θ motions [150]. These variables evolve with proper time τ according to

$$q_r = q_{r,0} + \omega_r \tau, \tag{5.121a}$$

$$q_\theta = q_{\theta,0} + \omega_\theta \tau, \tag{5.121b}$$

where $q_{r,0}$ and $q_{\theta,0}$ are the initial values. We denote the integrand in Eq. (5.120) by

$$\mathcal{I}(q_r, q_\theta, p, e, \iota),$$

where $\mathcal{I}$ is some function, and p, e and ι are the parameters of the geodesic defined by Hughes [189] (functions of E, L_z and K_0). The result (5.120) can be written as

$$\frac{1}{T} \int_{-T/2}^{T/2} d\tau \, \mathcal{I}[q_r(\tau), q_\theta(\tau), p, e, \iota] = 0, \tag{5.122}$$

where $T = T(p, e, \iota)$ is the period of the r, θ motion.

Since the variables q_r and q_θ are periodic with period 2π, we can express the function $\mathcal{I}$ as a Fourier series

$$\mathcal{I}(q_r, q_\theta, p, e, \iota) = \sum_{n,m=-\infty}^{\infty} \mathcal{I}_{nm}(p, e, \iota) e^{inq_r + imq_\theta}. \tag{5.123}$$

Now combining Eqs. (5.121), (5.122) and (5.123) gives

$$\begin{aligned} 0 &= \sum_{n,m=-\infty}^{\infty} \mathcal{I}_{nm}(p, e, \iota) e^{inq_{r,0} + imq_{\theta,0}} \\ &\quad \times \mathrm{Si}\left[(n\omega_r + m\omega_\theta)T/2\right], \end{aligned} \tag{5.124}$$

where $\mathrm{Si}(x) = \sin(x)/x$. Since the initial conditions $q_{r,0}$ and $q_{\theta,0}$ are arbitrary, it follows that

$$\mathcal{I}_{nm}(p, e, \iota)\mathrm{Si}\left[(n\omega_r + m\omega_\theta)T/2\right] = 0 \tag{5.125}$$

for all n, m.

Next, for closed orbits the ratio of the frequencies must be a rational number, so

$$\frac{\omega_r}{\omega_\theta} = \frac{j}{q}, \tag{5.126}$$

where j and q are integers with no factor in common. These integers depend on p, e and ι. The period T is given by $2\pi/T = q\omega_r = j\omega_\theta$. The second factor in Eq. (5.125) now simplifies to

$$\mathrm{Si}\left[\frac{(nj + mq)\pi}{jq}\right], \tag{5.127}$$

which vanishes if and only if

$$n = \bar{n}q, \quad m = \bar{m}j, \quad \bar{n} + \bar{m} \neq 0, \tag{5.128}$$

for integers $\bar{n}$, $\bar{m}$. It follows that

$$\mathcal{I}_{nm}(p, e, \iota) = 0 \tag{5.129}$$

for all n, m except for values of n, m which satisfy the condition (5.128).

Consider now the Newtonian limit, which is the limit $p \to \infty$ while keeping fixed e and ι and the mass of the black hole. We denote by $\mathcal{I}_{\rm N}(q_r, q_\theta, p, e, \iota)$ the Newtonian limit of the function $\mathcal{I}(q_r, q_\theta, p, e, \iota)$. The integral (5.122) in the Newtonian limit is given by the above computation with $j = q = 1$, since $\omega_r = \omega_\theta$ in this limit. This gives

$$\frac{1}{T}\oint d\tau \mathcal{I}_{\rm N} = \sum_{n=-\infty}^{\infty} \mathcal{I}_{{\rm N}\, n,-n}(p, e, \iota)\, e^{in(q_{r,0}-q_{\theta,0})}, \tag{5.130}$$

where $\mathcal{I}_{{\rm N}\, nm}$ are the Fourier components of $\mathcal{I}_{\rm N}$. In the previous subsection, we showed that this function is non-zero, which implies that there exists a value k of n for which $\mathcal{I}_{{\rm N}\, k,-k} \neq 0$.

Now as $p \to \infty$, we have $\omega_r/\omega_\theta \to 1$, and hence from Eq. (5.126) there exists a critical value p_c of p such that the values of j and q exceed k for all closed orbits with $p > p_c$. (We are keeping fixed the values of e and ι). It follows from Eqs. (5.128) and (5.129) that

$$\frac{\mathcal{I}_{k,-k}(p, e, \iota)}{\mathcal{I}_{{\rm N}\, k,-k}(p, e, \iota)} = 0 \tag{5.131}$$

for all such values of p. However this contradicts the fact that

$$\frac{\mathcal{I}_{k,-k}(p, e, \iota)}{\mathcal{I}_{{\rm N}\, k,-k}(p, e, \iota)} \to 1 \tag{5.132}$$

as $p \to \infty$. This completes the proof.

Hence, if the three assumptions listed at the start of this subsection are satisfied, then the conjecture that all vacuum, axisymmetric spacetimes possess a third constant of the motion is false.

Finally, it is sometimes claimed in the classical dynamics literature that perturbation theory is not a sufficiently powerful tool to assess whether the integrability of a system is preserved under deformations. An example that is often quoted is the Toda lattice Hamiltonian [193, 194]. This system is integrable and admits a full set of constants of motion in involution. However, if one approximates the Hamiltonian by Taylor expanding the potential about the origin to third order, one obtains a system which is not integrable. This would seem to indicate that perturbation theory can indicate a non-integrability, while the exact system is still integrable.

In fact, the Toda lattice example does not invalidate the method of proof we use here. If we write the Toda lattice Hamiltonian as $H(\mathbf{q}, \mathbf{p})$, then the situation is that $H(\lambda\mathbf{q}, \mathbf{p})$ is integrable for $\lambda = 1$, but it is not integrable for $0 < \lambda < 1$. Expanding $H(\lambda\mathbf{q}, \mathbf{p})$ to third order in λ gives a non-integrable Hamiltonian. Thus, the perturbative result is not in disagreement with the exact result for $0 < \lambda < 1$, it only disagrees with the exact result for $\lambda = 1$. In other words, the example shows that perturbation theory can fail to yield the correct result for finite values of λ, but there is no indication that it fails in arbitrarily small neighborhoods of $\lambda = 0$. Our application is qualitatively different from the Toda lattice example since we have a one parameter family of Hamiltonians $H(\lambda)$ which by assumption are integrable for all values of λ.

5.6 Conclusion

We have examined the effect of an axisymmetric quadrupole moment Q of a central body on test particle inspirals, to linear order in Q, to the leading post-Newtonian order, and to linear order in the mass ratio. Our analysis shows that a natural generalization of the Carter constant can be defined for the quadrupole interaction.

We have also analyzed the leading order spin self-interaction effect due to the scattering of the radiation off the spacetime curvature due to the spin. Combining the effects of the quadrupole and the leading order effects linear and quadratic in the spin, we have obtained expressions for the instantaneous as well as time-averaged evolution of the constants of motion for generic orbits under gravitational radiation reaction, complete at $O(a^2\epsilon^4)$. We have also shown that for a single multipole interaction other than Q or spin, in our approximations, a Carter-type constant does not exist. With mild additional assumptions, this result can be extended to exact spacetimes and falsifies the conjecture that all axisymmetric vacuum spacetimes possess a third constant of motion for geodesic motion.

5.7 Acknowledgments

This research was partially supported by NSF grant PHY-0457200. We thank Jeandrew Brink for useful correspondence.

5.8 Appendix: Time variation of quadrupole: order of magnitude estimates

In this appendix, we give an estimate of the timescale $T_{\rm evol}$ for the quadrupole to change. The analysis in the body of this paper is valid only when $T_{\rm evol} \gg T_{\rm rr}$, where T_{rr} is the radiation reaction time, since we have neglected the time evolution of the quadrupole. We distinguish between two cases: (i) when the central body is exactly nonspinning but has a quadrupole, and (ii) when the central body has finite spin in addition to the quadrupole.

5.8.1 Estimate of the scaling for the nonspinning case

For the purpose of a crude estimate, the relevant interaction is the tidal interaction with energy

$$Q_{ij}\mathcal{E}_{ij} \sim -\frac{m_2}{r^3}\bar{Q}I\cos^2\theta, \tag{5.133}$$

where $\mathcal{E}_{ij}$ is the tidal field, θ is the angle between the symmetry axis and the normal to the orbital plane of m_2, and we have written the quadrupole as $Q \sim \bar{Q}I$, where $\bar{Q}$ is dimensionless and I is the moment of inertia. For small deviations from equilibrium, the relevant piece of the Lagrangian is schematically

$$L \sim I\dot{\psi}^2 + \bar{Q}I\frac{m_2}{r^3}\psi^2. \tag{5.134}$$

We define the evolution timescale $T_{\rm evol}$ to be the time it takes for the angle to change by an amount of order unity, and since the amplitude of the oscillation scales roughly as $\sim m_2/m_1$, the evolution time scales as

$$T_{\rm evol}^{-2} \sim \frac{m_2^2}{m_1^2}\bar{Q}\left(\frac{m_2}{M}\right)\omega_{\rm orbit}^2, \tag{5.135}$$

where $\omega_{\rm orbit}^2 = M/r^3$. Thus, the ratio of the evolution timescale compared to the radiation reaction timescale scales as

$$T_{\rm evol}/T_{\rm rr} \sim \left(1/\sqrt{\bar{Q}}\right)\frac{m_1}{m_2}\left(\frac{\mu}{M}\right)^{1/2}\left(\frac{M}{r}\right)^{5/2}. \tag{5.136}$$

5.8.2 Estimate of the scaling for the spinning case

When the body is spinning the effect of the tidal coupling is to cause a precession. For the purpose of this estimate, we calculate the torque on m_1 due to the companion's Newtonian field. The torque $\mathbf{N}$ scales as

$$N_i \sim \epsilon_{imj}Q_{mk}\mathcal{E}_{jk}. \tag{5.137}$$

We assume that the precession is slow, i.e.

$$\omega_{\rm prec} \ll \bar{S}/m_1 \left(\frac{m_2}{M}\right), \tag{5.138}$$

where $\omega_{\rm prec}$ is the precession frequency and $\bar{S} = S/m_1^2$ is the dimensionless spin. This gives the approximate scaling of the precession timescale as (cf. [195])

$$T_{\rm prec}/T_{\rm rr} \sim \frac{\bar{S}}{\bar{Q}} \left(\frac{M}{r}\right). \tag{5.139}$$

and the evolution timescale is thus

$$T_{\rm evol}/T_{\rm rr} \sim \frac{m_1}{m_2} \frac{\bar{S}}{\bar{Q}} \left(\frac{M}{r}\right). \tag{5.140}$$

Because of our assumption (5.138) that the precession is slow, equation (5.140) is valid only when

$$1 \gg \left(\frac{\mu}{M}\right) \frac{\bar{S}^2}{\bar{Q}} \left(\frac{r}{M}\right)^3. \tag{5.141}$$

When $\bar{S}$ is sufficiently small that the condition (5.141) is violated, the relevant timescale is instead given by Eq. (5.135).

5.8.3 Application to Kerr inspirals

For Kerr inspirals,

$$\bar{S} \sim a, \quad \bar{Q} \sim a^2, \quad \mu/M \ll 1 \text{ and } r \sim M. \tag{5.142}$$

Therefore, the condition (5.141) is satisfied, and the precession time is longer than the radiation reaction time by

$$T_{\rm prec}/T_{\rm rr} \sim \frac{1}{a} \left(\frac{M}{r}\right). \tag{5.143}$$

Note that for Kerr inspirals, since $r \sim M$ both formulas (5.135) and (5.139) give the same scaling.

Moreover, for Kerr inspirals, the amplitude of the precession will be small, of order the mass ratio μ/M. This is because of angular momentum conservation: in the relativistic regime, the orbital angular momentum is a factor of μ/M smaller than the angular momentum of the black hole and can therefore not cause a large precession amplitude. Even if the orbital angular momentum at infinity is large, most of it will be radiated away as outgoing gravitational waves during the earlier phase of the inspiral. This factor of μ/M is taken into account when we consider the evolution timescale, which for Kerr inspirals reduces to

$$T_{\rm evol}/T_{\rm rr} \sim \left(\frac{M}{\mu}\right)\left(\frac{1}{a}\right)\left(\frac{M}{r}\right). \tag{5.144}$$

Since $1/a \geq 1$, $M/r \sim 1$ and $M/\mu \gg 1$, the evolution time is long compared to the radiation reaction time and we can neglect the time variation of the quadrupole at leading order.

5.9 Appendix: Computation of time averaged fluxes

5.9.1 Averaging method that parallels fully relativistic averaging

We start by noting that the differential equations (5.29) and (5.30) governing the $\tilde{r}$ and $\tilde{\theta}$ motions decouple if we define a new time parameter $\hat{t}$ by

$$d\hat{t} = \frac{1}{\tilde{r}^2} d\tilde{t}. \tag{5.145}$$

This is the analog of the Mino time parameter for geodesic motion in Kerr [128]. The equations of motion (5.29)–(5.27) then become

$$
\begin{aligned}
\left(\frac{d\tilde{r}}{d\hat{t}}\right)^2 &= \hat{V}_{\tilde{r}}(\tilde{r}), && (5.146)\\
\hat{V}_{\tilde{r}}(\tilde{r}) &= 2E\tilde{r}^4 + 2\tilde{r}^3 - K\tilde{r}^2 + \frac{Q}{2}\left(\tilde{r} - 2L_z^2\right), && (5.147)\\
\left(\frac{d\tilde{\theta}}{d\hat{t}}\right)^2 &= \hat{V}_{\tilde{\theta}}(\tilde{\theta}), && (5.148)\\
\hat{V}_{\tilde{\theta}}(\tilde{\theta}) &= K - \frac{L_z^2}{\sin^2\tilde{\theta}} - QE\cos 2\tilde{\theta}, && (5.149)\\
\left(\frac{d\varphi}{d\hat{t}}\right) &= \hat{V}_{\varphi\tilde{r}}(\tilde{r}) + \hat{V}_{\varphi\tilde{\theta}}(\tilde{\theta}), && (5.150)\\
\hat{V}_{\varphi\tilde{r}}(\tilde{r}) &= \frac{QL_z}{\tilde{r}^2}, \qquad \hat{V}_{\varphi\tilde{\theta}}(\tilde{\theta}) = \frac{L_z}{\sin^2\tilde{\theta}}. && (5.151)
\end{aligned}
$$

The parameters t and $\hat{t}$ are related by:

$$
\begin{aligned}
\frac{dt}{d\hat{t}} &= \hat{V}_{t\tilde{r}}(\tilde{r}) + \hat{V}_{t\tilde{\theta}}(\tilde{\theta}) && (5.152)\\
\hat{V}_{t\tilde{r}}(\tilde{r}) &= \tilde{r}^2, \qquad \hat{V}_{t\tilde{\theta}}(\tilde{\theta}) = \frac{Q}{2}\cos 2\tilde{\theta}. && (5.153)
\end{aligned}
$$

It follows from Eqs. (6.213) and (6.214) that the functions $\tilde{r}(\hat{t})$ and $\tilde{\theta}(\hat{t})$ are periodic; and we denote their periods by $\Lambda_{\tilde{r}}$ and $\Lambda_{\tilde{\theta}}$. We define the fiducial motion associated with the constants of motion E, L_z and K to be the motion with the initial conditions $\tilde{r}(0) = \tilde{r}_{\rm min}$ and $\tilde{\theta}(0) = \tilde{\theta}_{\rm min}$, where $\tilde{r}_{\rm min}$ and $\tilde{\theta}_{\rm min}$ are given by the vanishing of the right-hand sides of Eqs. (6.213) and (6.214) respectively. The functions $\hat{r}(\hat{t})$ and $\hat{\theta}(\hat{t})$ associated with this fiducial motion are given by

$$
\begin{aligned}
\int_{\tilde{r}_{\rm min}}^{\hat{r}(\hat{t})} \frac{d\tilde{r}}{\pm\sqrt{\hat{V}_{\tilde{r}}(\tilde{r})}} &= \hat{t}, && (5.154)\\
\int_{\tilde{\theta}_{\rm min}}^{\hat{\theta}(\hat{t})} \frac{d\tilde{\theta}}{\pm\sqrt{\hat{V}_{\tilde{\theta}}(\tilde{\theta})}} &= \hat{t}. && (5.155)
\end{aligned}
$$

From Eq. (6.216) it follows that

$$
t(\hat{t}) = t_0 + \int_0^{\hat{t}} dt' \left(\hat{V}_{t\tilde{r}}[\tilde{r}(t')] + \hat{V}_{t\tilde{\theta}}[\tilde{\theta}(t')]\right), \qquad (5.156)
$$

where $t_0 = t(0)$. Next, we define the constant Γ to be the following average value:

$$\Gamma = \frac{1}{\Lambda_{\tilde{r}}} \int_0^{\Lambda_{\tilde{r}}} dt' \hat{V}_{t\tilde{r}}[\hat{r}(t')] + \frac{1}{\Lambda_{\tilde{\theta}}} \int_0^{\Lambda_{\tilde{\theta}}} dt' \hat{V}_{t\tilde{\theta}}[\hat{\theta}(t')]. \tag{5.157}$$

Then we can write $t(\hat{t})$ as a sum of a linear term and terms that are periodic:

$$t(t) = t_0 + \Gamma \hat{t} + \delta t(\hat{t}), \tag{5.158}$$

where $\delta t(\hat{t})$ denotes the oscillatory terms in Eq. (6.219).

To average a function over the time parameter $\hat{t}$, it is convenient to parameterize $\tilde{r}$ and $\tilde{\theta}$ in terms of angular variables as follows. For the average over $\tilde{\theta}$ we introduce the parameter χ by

$$\cos^2 \hat{\theta}(\hat{t}) = z_- \cos^2 \chi, \tag{5.159}$$

where $z_- = \cos^2 \tilde{\theta}_-$ with z_- being the smaller root of Eq. (6.214):

$$z_{\pm} = \frac{1}{2\beta} \left[K + 3QE \pm \sqrt{(K - QE)^2 + 4QEL_z^2} \right] \tag{5.160}$$

and where $\beta = 2QE$. Then from the definition (6.218) of $\hat{\theta}$ together with Eq. (6.214) and the requirement that χ increases monotonically with $\hat{t}$ we obtain

$$\frac{d\chi}{d\hat{t}} = \sqrt{\beta \left(z_+ - z_- \cos^2 \chi \right)}. \tag{5.161}$$

Then we can write the average over $\hat{t}$ of a function $F_{\tilde{\theta}}(\hat{t})$ which is periodic with period $\Lambda_{\tilde{\theta}}$ in terms of χ as

$$\begin{aligned} \langle F_{\tilde{\theta}} \rangle_{\hat{t}} &= \frac{1}{\Lambda_{\tilde{\theta}}} \int_0^{\Lambda_{\tilde{\theta}}} d\hat{t} F_{\tilde{\theta}}(\hat{t}) \\ &= \frac{1}{\Lambda_{\tilde{\theta}}} \int_0^{2\pi} d\chi \frac{F_{\tilde{\theta}}[\hat{t}(\chi)]}{\sqrt{\beta \left(z_+ - z_- \cos^2 \chi \right)}}, \end{aligned} \tag{5.162}$$

where

$$\Lambda_{\tilde{\theta}} = \int_0^{2\pi} d\chi \frac{1}{\sqrt{\beta \left(z_+ - z_- \cos^2 \chi \right)}}. \tag{5.163}$$

Similarly, to average a function $F_{\tilde{r}}(\hat{t})$ that is periodic with period $\Lambda_{\tilde{r}}$, we introduce a parameter ξ via

$$\tilde{r} = \frac{p}{1 + e\cos\xi}, \tag{5.164}$$

where the parameter ξ varies from 0 to 2π as $\tilde{r}$ goes through a complete cycle. Then,

$$\begin{aligned} \frac{d\xi}{d\hat{t}} &= P(\xi), & (5.165) \\ P(\xi) &\equiv \left(\hat{V}_{\tilde{r}}[\tilde{r}(\xi)]\right)^{1/2} \left[\frac{pe \mid \sin\xi \mid}{(1+e\cos\xi)^2}\right]^{-1} & (5.166) \end{aligned}$$

The average over $\hat{t}$ of $F_{\tilde{r}}(\hat{t})$ can then be computed from

$$\langle F_{\tilde{r}} \rangle_{\hat{t}} = \frac{\int_0^{2\pi} d\xi\, F_{\tilde{r}}/P(\xi)}{\int_0^{2\pi} d\xi/P(\xi)}. \tag{5.167}$$

Now, a generic function $F_{\tilde{r},\tilde{\theta}}[\tilde{r}(\hat{t}), \tilde{\theta}(\hat{t})]$ will be biperiodic in $\hat{t}$: $F_{\tilde{r},\tilde{\theta}}[\tilde{r}(\hat{t}+\Lambda_{\tilde{r}}), \tilde{\theta}(\hat{t}+\Lambda_{\tilde{\theta}})] = F_{\tilde{r},\tilde{\theta}}[\tilde{r}(\hat{t}), \tilde{\theta}(\hat{t})]$. Combining the results (6.224) and (6.229) we can write its average as a double integral over χ and ξ as

$$\langle F_{\tilde{r},\tilde{\theta}} \rangle_{\hat{t}} = \frac{1}{\Lambda_{\tilde{\theta}}\Lambda_{\tilde{r}}} \int_0^{2\pi} d\chi \int_0^{2\pi} d\xi \frac{F_{\tilde{r},\tilde{\theta}}[\tilde{r}(\xi), \tilde{\theta}(\chi)]}{\sqrt{\beta\,(z_+ - z_- \cos^2\chi)}P(\xi)}. \tag{5.168}$$

To compute the time average of $\dot{E}$, $\dot{L}_z$, and $\dot{K}$, we need to convert the average of a function over $\hat{t}$ calculated from (6.230) to the average over t. As explained in detail in [40], in the adiabatic limit we can choose a time interval Δt which is long compared to the orbital timescale but short compared to the radiation reaction time. From Eq. (6.219) we have $\Delta t = \Gamma\hat{t} + \text{osc.terms}$. The oscillatory terms will be bounded and will therefore be negligible in the adiabatic limit, so we have to a good approximation

$$\langle \dot{E} \rangle_t = \frac{1}{\Gamma} \langle \dot{E}\,\hat{V}_t \rangle_{\hat{t}}, \tag{5.169}$$

where $\hat{V}_t \equiv \hat{V}_{t\tilde{r}} + \hat{V}_{t\tilde{\theta}}$, cf. Eq. (6.216), and similarly for $\dot{L}_z$ and $\dot{K}$.

The explicit results we obtain using this method are given in section 5.3, Eqs. (5.58), (5.59), and (5.61).

5.9.2 Averaging method using the explicit parameterization of Newtonian orbits

To perform the time-averaging using this method, we define a parameter ξ via

$$\tilde{r} = \frac{p}{1 + e\cos\xi}, \tag{5.170}$$

where the parameter ξ varies from 0 to 2π as $\tilde{r}$ goes through a complete cycle. Note that θ appears in Eqs. (5.46) – (5.48) only in terms that are linear in Q, so we can write θ in terms of ξ using the Newtonian relation

$$x_3 = r\cos\theta = r\sin\iota\sin(\xi + \xi_0). \tag{5.171}$$

Here, ξ_0 is the angle between the direction of the perihelion and the intersection of the orbital and equatorial plane. Similarly, for the $\dot{r}\dot{\theta}$ terms in Eqs. (5.47) and (5.56) we can use the Newtonian relations $\dot{r} = e/\sqrt{p}\sin\xi$ and $\dot{\xi} = \sqrt{p}/r^2$. From Eqs. (5.30) and (6.226) it follows that

$$\begin{aligned}
\frac{d\tilde{t}}{d\xi} &= \frac{p^{3/2}}{(1+e\cos\xi)^2}\left\{1 - \frac{Q}{8p^2}\left[-3 + e^2 - 2e\cos\xi + 2\cos^2\iota(8 - e^2 + 8e\cos\xi)\right]\right. \\
&\qquad \left. - \frac{Q}{4p^2}e^2\cos^2\iota\cos 2\xi\right\},
\end{aligned} \tag{5.172}$$

and from Eq. (5.15)

$$\frac{dt}{d\tilde{t}} = \left\{1 + \frac{Q}{2p^2}\left(1 + e\cos\xi\right)\left[2\sin^2\iota\sin^2(\xi+\xi_0) - 1\right]\right\}. \tag{5.173}$$

Using these expressions, we compute the time-averaged fluxes from

$$\langle\dot{E}\rangle = \frac{\int_0^{2\pi} d\xi\;\dot{E}\,(dt/d\tilde{t})\,(d\tilde{t}/d\xi)}{\int_0^{2\pi} d\xi\,(dt/d\tilde{t})\,(d\tilde{t}/d\xi)} \tag{5.174}$$

and obtain:

$$
\begin{aligned}
\langle \dot{E} \rangle &= -\frac{32}{5}\frac{(1-e^2)^{3/2}}{p^5}\left[1+\frac{73}{24}e^2+\frac{37}{96}e^4\right.\\
&-\frac{S}{p^{3/2}}\left(\frac{73}{12}+\frac{823}{24}e^2+\frac{949}{32}e^4+\frac{491}{192}e^6\right)\cos(\iota)\\
&-\frac{Q}{p^2}\left\{\frac{1}{2}+\frac{85}{32}e^2+\frac{349}{128}e^4+\frac{107}{384}e^6\right\}\\
&-\frac{Q}{p^2}\left\{\left(\frac{11}{4}+\frac{273}{16}e^2+\frac{847}{64}e^4+\frac{179}{192}e^6\right)\cos(2\iota)\right\}\\
&-\frac{S^2}{p^2}\left\{\frac{13}{192}+\frac{247}{384}e^2+\frac{299}{512}e^4+\frac{39}{1024}e^6\right\}\\
&+\frac{S^2}{p^2}\left\{\left(\frac{1}{192}+\frac{19}{384}e^2+\frac{23}{512}e^4+\frac{3}{1024}e^6\right)\cos(2\iota)\right\}\\
&-\frac{Q}{p^2}e^2\left(\frac{869}{48}+\frac{1595}{96}e^2+\frac{121}{128}e^4\right)\cos(2\xi_0)\sin^2\iota\\
&\left.+\frac{S^2}{p^2}e^2\left(\frac{1}{384}+\frac{5}{384}e^2+\frac{3}{2084}e^4\right)\cos(2\xi_0)\sin^2\iota\right], \qquad (5.175)
\end{aligned}
$$

$$
\begin{aligned}
\langle \dot{L}_z \rangle &= -\frac{32}{5}\frac{(1-e^2)^{3/2}}{p^{7/2}}\cos\iota\left[1+\frac{7}{8}e^2-\frac{S}{2p^{3/2}\cos\iota}\left\{\frac{61}{24}+7e^2+\frac{271}{64}e^4\right\}\right.\\
&-\frac{S}{2p^{3/2}\cos\iota}\left\{\left(\frac{61}{8}+\frac{91}{4}e^2+\frac{461}{64}e^4\right)\cos(2\iota)\right\}\\
&-\frac{Q}{16p^2}\left\{-3-\frac{45}{4}e^2+\frac{19}{8}e^4+\left(45+148e^2+\frac{331}{8}e^4\right)\cos(2\iota)\right\}\\
&+\frac{S^2}{16p^2}\left\{1+3e^2+\frac{3}{8}e^4\right\}\\
&\left.-\frac{Q}{p^2}e^2\cos(2\xi_0)\sin^2\iota\left(\frac{201}{32}+\frac{51}{32}e^2\right)\right], \qquad (5.176)
\end{aligned}
$$

$$
\begin{aligned}
\langle \dot{K} \rangle &= -\frac{64}{5}\frac{(1-e^2)^{3/2}}{p^3}\left[1+\frac{7}{8}e^2-\frac{S}{2p^{3/2}}\left(\frac{97}{6}+37e^2+\frac{211}{16}e^4\right)\cos(\iota)\right.\\
&-\frac{Q}{p^2}\left\{\frac{1}{2}+\frac{55}{48}e^2+\frac{139}{192}e^4+\left(\frac{13}{4}+\frac{841}{96}e^2+\frac{449}{192}e^4\right)\cos(2\iota)\right\}\\
&+\frac{S^2}{p^2}\left\{\frac{13}{192}+\frac{13}{64}e^2+\frac{13}{512}e^4-\left(\frac{1}{192}+\frac{1}{64}e^2+\frac{1}{512}e^4\right)\cos(2\iota)\right\}\\
&\left.-\frac{Q}{p^2}\left(\frac{391}{48}+\frac{37}{24}e^2\right)e^2\cos(2\xi_0)\sin^2\iota\right]. \qquad (5.177)
\end{aligned}
$$

In the adiabatic limit, the terms involving $\cos(2\xi_0)$ can be omitted because they average to zero. As explained by Ryan [174], the radiation reaction timescale for terms involving ξ_0 is much longer than the precession timescale for most orbits, so the terms involving ξ_0 will average away. This is consistent with our results for the adiabatic infinite time-averaged fluxes using the Mino time parameter. The Mino-time averaging method was based on the assumption that the fundamental frequencies are incommensurate and the motion fills up the whole torus, which is equivalent to averaging over ξ_0.

CHAPTER 6

CARTER CONSTANT EVOLUTION IN THE ADIABATIC REGIME

SUMMARY: A key source for LISA will be the inspiral of compact objects into massive black holes. Recently Mino has shown that in the adiabatic limit, gravitational waveforms for these sources can be computed by using for the radiation reaction force the gradient of one half the difference between the retarded and advanced metric perturbations. We describe an explicit computational procedure for obtaining waveforms based on Minos result and derive an explicit expression for the time-averaged time derivative of the Carter constant. The result is not new, but the intent is to give self-contained treatment in a unified notation and more details on the derivation than previously available, starting with the Kerr metric, and ending with formulae for the time evolution of all three constants of the motion that are sufficiently explicit to be used immediately in a numerical code. We have added some new material based on the two-timescale formalism. The derivation uses detailed properties of mode expansions, Greens functions and bound geodesic orbits in the Kerr spacetime, which we review in detail. This paper follows closely a previous treatment of scalar radiation reaction but extended to the tensor case.

6.1 Introduction

The inspiral of compact objects into massive black holes will be an important source for LISA. Observing these inspirals requires accurate templates for matched filtering. There are several approaches for generating the model waveforms, all of which are based on treating the small object as a linear perturbation to the Kerr spacetime of the large black hole. On short timescales, the compact object moves on a bound geodesic orbit, characterized by its energy E, z-component of angular momentum L_z and Carter constant K. Over longer time scales, these parameters evolve due to self-force effects. A formal expression for the gravitational self-acceleration in terms of the retarded metric perturbation now exists [196], [39]; however, the practical implementation is difficult because of regularization problems.

An approximation that bypasses the challenge of regularization calculations is to compute the time-average rates of change of the constants of motion due to radiation reaction, and use those to evolve the orbit as a flow through successive geodesics as suggested by Mino [128]. Mino showed that in the adiabatic limit (when the radiation reaction timescale is much longer than the orbital timescale) an approximate radiation reaction force constructed from the half-retarded minus half-advanced field gives the same time averages $\langle dE/dt\rangle, \langle dL_z/dt\rangle$ and $\langle dK/dt\rangle$ as the full self-force [128]. This half retarded minus half advanced prescription is the standard prescription for scalar and electromagnetic radiation reaction in flat spacetime, and was previously conjectured by Gal'tsov [197] to apply to gravitational waves in Kerr. The fact that the adiabatic limit requires only the radiative self field, which is a solution to a homogeneous wave equation, allows us to avoid the reconstruction of the full metric perturbation from the Teukolsky functions.

The rates of change of E and L_z can be computed by imposing conservation of energy and angular momentum to infer the amounts lost by the particle from the fluxes at infinity and down the black hole horizon. These fluxes can be computed directly from a mode expansion. Evolving generic orbits also requires evolving the third constant K, which presents a difficulty since it is not directly related to asymptotic gravitational waves, and there is no known conservation law associated with K.

Mino [128] showed that $\langle dK/dt \rangle$ could be computed from the radiative self field, which served as a basis for further developments: Recently, the authors of [40] used a scalar charge model to derive an explicit formula for the adiabatic evolution of K in terms of a mode expansion that can immediately be used in a numerical code. Sago et al generalized this formula to the tensor case in Ref. [41] and obtained an apparently different result. However, Drasco and Sago [198] then showed that the two results are fully equivalent in the scalar case.

The key property of the final expressions for the evolution of E, L_z and K in the adiabatic limit is that, unlike for local self-force computations, they avoid the problem of reconstructing the metric perturbation from the curvature perturbations. They fail to include the properties of the perturbed spacetime associated with the nonradiating $l = 0$ and $l = 1$ degrees of freedom. However, these modes (which correspond to properties such as shifts in mass and angular momentum due to the perturbation) contribute only to the conservative components of the self-acceleration [199], and can be neglected in the adiabatic limit [37, 40, 200].

In this chapter, we rederive an explicit expression for the time-averaged rate of change of the Carter constant in the tensor case that can be used for numerically computing adiabatic waveforms. This paper contains no new results but more details on the derivation than previously available and gives a self-contained treatment in a unified notation. Our derivation and review closely follows that of the scalar case [40], from which we have taken over several paragraphs verbatim, as well as Gal'tsov [197] and Chrzanowski [201], and is based on the radiative self-force and the mode expansion of the radiative Greens function.

Our final result for the evolution of the Carter constant in the adiabatic limit, Eq. (6.305) below is formulated in terms of two different amplitudes. We give explicit expressions for these amplitudes in terms of sums over the three fundamental frequency components of geodesic motion and an integral over the torus in phase space, Eqs. (6.250) and (6.306), with the various quantities defined in (6.314), (6.275), (6.302), and (6.304). Drasco [198] and Sago [41] have shown that that the new amplitude can be written fully in terms of the same amplitudes that appear in the expressions for $\langle dE/dt \rangle$ and $\langle dL_z/dt \rangle$ for the scalar model. We extend this derivation to the tensor case, which leads to the expression in Eq. (6.315) below, together with an average over a geodesic given in Eq. (6.314).

6.2 The Kerr spacetime

6.2.1 Teukolsky perturbation formalism

This section reviews the Teukolsky formalism for treating linearized perturbations of Kerr, which is based on the Newman-Penrose tetrad formalism. These formalisms are valid for general spin weight $s = -2, -1, 0, 1, 2$, but in this chapter, we will specialize to the tensor case $s = \pm 2$.

The Newman-Penrose formalism is based on a null tetrad $(\vec{l}, \vec{n}, \vec{m}, \vec{m}^*)$ consisting of two real null vectors $\vec{l}$, $\vec{n}$ and a complex spacelike vector $\vec{m}$, which obey the orthonormality relations $\vec{l} \cdot \vec{n} = -1$ and $\vec{m} \cdot \vec{m}^* = 1$, with all other products vanishing. The metric can be written in terms of the corresponding one-forms as

$$g_{ab} = -2 l_{(a} n_{b)} + 2 m_{(a} m^*_{b)}. \tag{6.1}$$

The asterisk in Eq. (6.1) means complex conjugation.

The 10 independent tetrad components of the Weyl tensor C_{abcd} of the full spacetime can be written as 5 complex scalars $\psi_0 \ldots, \psi_4$ by contracting C_{abcd} with the basis vectors in all possible nontrivial ways:

$$
\begin{aligned}
\psi_0 &= -C_{abcd} l^a m^b l^c m^d, \qquad \psi_1 = -C_{abcd} l^a n^b l^c m^d, \\
\psi_2 &= -\frac{1}{2} C_{abcd} \left(l^a n^b l^c n^d + l^a n^b m^c m^{*d} \right), \\
\psi_3 &= -C_{abcd} l^a n^b m^{*c} n^d, \qquad \psi_4 = -C_{abcd} n^a m^{*b} n^c m^{*d}.
\end{aligned} \tag{6.2}
$$

The full metric of the spacetime is

$$
g_{ab}^{\text{entire}} = g_{ab} + h_{ab}, \tag{6.3}
$$

where g_{ab} is the background Kerr metric given in Eq. (6.1) and h_{ab} is a perturbation. We will consider only linearized perturbations here. We choose the background tetrad so that $\vec{l}$ and $\vec{n}$ are along the repeated principal null directions of the Weyl tensor. There is then only one non-vanishing unperturbed Weyl tensor component in the background:

$$
\psi_0^{(0)} = \psi_1^{(0)} = \psi_3^{(0)} = \psi_4^{(0)} = 0, \qquad \psi_2^{(0)} \neq 0, \tag{6.4}
$$

where the superscript (0) denotes the unperturbed Weyl scalars.

Teukolsky showed that with the choice of tetrad of Eq. (6.4), the linearized perturbation equations governing ψ_0 and ψ_4 can be decoupled and that the perturbations ψ_0 and ψ_4 are invariant under infinitesimal gauge and tetrad transformations [202]. In his derivation, Teukolsky then used part of the remaining freedom in the choice of background tetrad to make a null rotation so that the spin coefficient ϵ vanishes, and he defined the master variables

$$
{}_s\Psi = {}_sM^{ab} h_{ab} = \begin{cases} \psi_0 = -C_{abcd} l^a m^b l^c m^d, & s = 2, \\ (\psi_2^{(0)})^{-4/3}\ \psi_4 = -(\psi_2^{(0)})^{-4/3} C_{abcd} n^a m^{*b} n^c m^{*d}, & s = -2. \end{cases} \tag{6.5}
$$

This equation defines the second order differential operators ${}_2M^{ab}$ and ${}_{-2}M^{ab}$ that project the Teukolsky scalars from the metric perturbation. The uncoupled differential equation for ${}_s\Psi$ is called the master perturbation equation or Teukolsky equation and can be written as:

$$ {}_s\mathcal{O}\ {}_s\Psi = 4\pi\ {}_s\tau_{ab}T^{ab}. \tag{6.6}$$

This equation serves to define, up to a multiplicative function, the two second order differential operators ${}_s\mathcal{O}$ and ${}_s\tau_{ab}$ for $s = \pm 2$, which project the linearized Einstein operator and the source term T^{ab} from the linearized Einstein equation to the Teukolsky equation. The full definition of these operators will be given in Sec. IB. The presence of the factor of $(\psi_2^{(0)})^{-4/3}$ in front of ψ_4 in Eq. (6.5) is related to the background null rotation used to set the spin coefficient $\epsilon = 0$ to later achieve separability of the decoupled equations. (A different choice would lead to a different factor while leaving the separable master perturbation equation for ${}_s\Psi$ invariant).

Relation of the metric perturbation to solutions of the vacuum Teukolsky equations

Wald has shown, based on earlier results by Cohen and Kegeles [203] and Chrzanowski [201], that for linearized vacuum perturbations of Kerr, and for each $s = 2,\ s = -2$, the metric perturbation h_{ab} can be constructed by applying a second order differential operator to a scalar potential ${}_s\Phi$ that is a solution to the adjoint of the vacuum Teukolsky equation for ${}_s\Psi$ [1] [2]. Wald's derivation shows that the existence of a scalar which is both gauge invariant and tetrad-gauge invariant and leads to decoupled equations is sufficient to guarantee that the two

[1]The potential ${}_s\Phi$ is often called a Hertz or Debye potential.

degrees of freedom of the metric perturbation h_{ab} are explicitly determined by the information in a single complex scalar ${}_s\Phi$, except for the non-radiative multipoles $l = 0, 1$ [2] and up to the remaining gauge freedom.

Wald [2] give the following definition for the adjoint of an operator. If a linear differential tensor operator M acts on an $n-$ index tensor ψ, taking it to a $k-$index field $M\psi$, we define its adjoint $M^\dagger$ in such a way that $M^\dagger$ is also a linear operator and

$$(M_1 M_2)^\dagger = M_2^\dagger M_1^\dagger \tag{6.7}$$

for any pair of operators M_1 and M_2 whose composition is well-defined. The adjoint operator thus acts on $k-$index tensors ϕ, taking them to $n-$index tensors $M^\dagger\phi$. If we require that for all ψ and ϕ,

$$\phi^*_{ab...k} \left(M\psi\right)^{ab...k} - \left(M^\dagger\phi\right)^*_{ab...n} \psi^{ab...n} = \nabla_c t^c, \tag{6.8}$$

where the right hand side is a total divergence term, then property (6.7) holds, and we can take Eq. (6.8) as the definition of the adjoint operator.

Wald's result is that the metric perturbation for vacuum solutions can be obtained from the potential ${}_s\Phi$ via

$$\tilde{h}_{ab} = \ {}_s\tau^\dagger_{ab}\ {}_s\Phi - \nabla_{(a}\xi_{b)}, \tag{6.9}$$

where ${}_s\tau^\dagger_{ab}$ is the adjoint of the operator defined by Eq. (6.6) and ξ_b are arbitrary functions. Note that h_{ab} in Eq. (6.9) has two physical degrees of freedom but we omit the explicit decomposition. The master variables are related to the potential

[2]Wald [204] showed that the two perturbations associated with variations of the black hole mass and spin parameters M and a are the only ones of reals frequency for which the master variables ${}_s\Psi$ vanish. This implies that all solutions, except a two-dimensional subspace, can be constructed from ${}_s\Phi$ and also that none of the constructed (real frequency) $\tilde{h}_{ab}$ are pure gauge

by:

$$
\begin{aligned}
{}_s\Psi &= {}_sM^{ab}\ {}_s\tau^{\dagger}_{ab}\ {}_s\Phi &(6.10)\\
{}_{-s}\Psi &= {}_{-s}M^{ab}\ {}_s\tau^{\dagger}_{ab}\ {}_s\Phi. &(6.11)
\end{aligned}
$$

We now briefly review Wald's derivation of these results [3]. The metric perturbation h_{ab} satisfies the source-free differential equation

$$E^{abcd}h_{cd} = 0, \tag{6.12}$$

where E^{abcd} denotes one-half the linearized Einstein operator. By introducing the new variables ${}_s\Psi$ made of linear combinations of components of h_{ab} and their derivatives and combining Eqs. (6.12) and their derivatives, Teukolsky found decoupled equations of the form

$${}_s\mathcal{O}\ {}_s\Psi = 0. \tag{6.13}$$

This implies that there exists a linear operator ${}_sM^{ab}$ such that ${}_s\Psi = {}_sM^{ab}h_{ab}$. Since it is possible to obtain the decoupled scalar equation (6.13) from linear manipulations of Eqs. (6.12), this also implies that there exists another linear operator ${}_s\tau_{ab}$ which represents these manipulations necessary to derive Eq. (6.13) from Eqs. (6.12) and with the property that the following operator identity holds [4]:

$${}_s\tau_{ab}E^{abcd} = {}_s\mathcal{O}\ {}_sM^{cd}. \tag{6.14}$$

This identity means that when both sides of Eq. (6.14) act on a solution h_{ab} of Eq. (6.12) the result is Eq. (6.13). The operators ${}_s\tau_{ab}$ can most easily be read off

[3] Wald's notation for the operators which we denote by E^{abcd}, ${}_s\tau_{ab}$, ${}_s\mathcal{O}$ and ${}_sM^{ab}$ is $\mathcal{E}_G$, $\mathcal{S}_G$, $\mathcal{O}_G$ and $\sim\mathcal{T}_G$ respectively.

[4] This identity is not applicable if (i) the derivation of the decoupled equations is based on introducing a potential by using integrability conditions from Eq. (6.12), or (ii) the decoupled variable is gauge dependent and the derivation of Eq. (6.13) relies on a gauge choice. Neither of these caveats applies for the case considered here, so the identity holds.

from the source term of the inhomogeneous version of Eq. (6.13), since the source term encodes the manipulations necessary for the decoupling of the equations.

One can obtain a solution to the original Einstein equation from a solution of the vacuum Teukolsky equation as follows. Taking the adjoint of the identity (6.14) and using the fact that the Einstein operator E^{abcd} is self-adjoint implies that

$$E^{abcd}{}_s\tau^{\dagger}_{ab} = \left({}_sM^{cd}\right)^{\dagger}\ {}_s\mathcal{O}^{\dagger}, \tag{6.15}$$

where we have taken into account the property in Eq. (6.7). Therefore, a function ${}_s\Phi$ that solves the adjoint of the vacuum Teukolsky equation for spin s,

$${}_s\mathcal{O}^{\dagger}\ {}_s\Phi = 0, \tag{6.16}$$

will also be a solution to

$$E^{abcd}\ {}_s\tau^{\dagger}_{ab}\ {}_s\Phi = 0. \tag{6.17}$$

Comparing this to Eq. (6.12) establishes the result (6.9). Acting with ${}_sM^{ab}$ and ${}_{-s}M^{ab}$ respectively on the metric perturbation (6.9) and using the definition (6.5) leads to the expressions (6.10) and (6.11). Therefore, the operator $({}_sM^{ab}{}_s\tau^{\dagger}_{ab})$ maps solutions ${}_s\Phi$ of the adjoint equation (6.16) into solutions ${}_s\Psi$ to the vacuum Teukolsky equation (6.6) and the operator $({}_{-s}M^{ab}{}_s\tau^{\dagger}_{ab})$ maps solutions ${}_s\Phi$ into vacuum solutions ${}_{-s}\Psi$.

The metric perturbation $({}_s\tau^{\dagger}_{ab}\ {}_s\Phi)$ obtained from the operators ${}_s\tau^{\dagger}_{ab}$ is in a particular gauge determined by the gauge choice for the operator ${}_s\tau_{ab}$. By the Bianchi identity, one can add a term $\eta_a\nabla_b$ to ${}_s\tau_{ab}$, where η_a is an arbitrary vector field, which results in adding the term $\nabla_{(b}\ {}_s\Phi\ \eta_{a)} = -\nabla_{(a}\xi_{b)}$ to the solution $\tilde{h}_{ab}$, where $\xi_a = \ {}_s\Phi\eta_a$. To date, it has only been possible to reconstruct the vacuum metric perturbation in Kerr from a potential ${}_s\Phi$ in the class of radiation gauges, in which the decoupled Teukolsky equation is derived. The choice of $s = 2$ or $s = -2$ for the operators ${}_s\tau_{ab}$ determines which of the radiation gauges: in a gauge where $h_{ab}\, l^a = 0$, we use $s = 2$, whereas in the gauge with the ingoing and outgoing directions reversed, with $h_{ab}\, n^a = 0$, we use $s = -2$. [5]

6.2.2 Boyer-Lindquist coordinates

To proceed further with the formal expressions given in the previous subsection, we need to specialize to a particular coordinate system. We will work in Boyer-Lindquist coordinates (t, r, θ, φ), where the Kerr metric is

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma} dt d\varphi + \left(\varpi^4 - \Delta a^2 \sin^2\theta\right) \frac{\sin^2\theta}{\Sigma} d\varphi^2 \\ &\quad + \Sigma d\theta^2 + \frac{\Sigma}{\Delta} dr^2. \end{aligned} \tag{6.18}$$

Here

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{6.19}$$

$$\Delta = r^2 + a^2 - 2Mr, \tag{6.20}$$

$$\varpi = \sqrt{r^2 + a^2}, \tag{6.21}$$

and M, a are the black hole mass and spin parameter. The square root of the determinant of the metric is

$$\sqrt{-g} = \Sigma \sin\theta \tag{6.22}$$

and the background Weyl scalar is

$$\psi_2^{(0)} = -M\rho^3, \tag{6.23}$$

[5] In addition to these conditions, the metric perturbation is trace-free, so the components of h_{ab} are overdetermined. Therefore, one cannot find a radiation gauge for a generic metric and source. However, for the case of the kinds of radiative perturbations of Kerr of interest in this chapter, such gauges exist [205].

where [6]

$$\rho = (r - ia\cos\theta)^{-1}. \tag{6.24}$$

Note that $\Sigma = (\rho\rho^*)^{-1}$.

In the Boyer-Lindquist coordinate basis, the Kinnersley tetrad is given by

$$\begin{aligned} \vec{l} &= \frac{\varpi^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi, \qquad \vec{n} = \frac{\varpi^2}{2\Sigma}\partial_t - \frac{\Delta}{2\Sigma}\partial_r + \frac{a}{2\Sigma}\partial_\phi, \\ \vec{m} &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta\partial_t + \partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right). \end{aligned} \tag{6.25}$$

This tetrad has $\vec{l}$ along the outgoing direction and is well-behaved on the past event horizon but singular on the future event horizon, where the Boyer-Lindquist coordinates become singular [1] (one manifestation of this singularity is that as infalling particles or photons approach the horizon, the coordinate time diverges). The corresponding one-forms are

$$\begin{aligned} \mathbf{l} &= -dt + a\sin^2\theta d\varphi + \frac{\Sigma}{\Delta}dr, \qquad \mathbf{n} = -\frac{\Delta}{2\Sigma}dt + \frac{a\Delta\sin^2\theta}{2\Sigma}d\varphi - \frac{1}{2}dr, \\ \mathbf{m} &= \frac{\rho^*}{\sqrt{2}}\left(-ia\sin\theta dt + \Sigma d\theta + i\varpi^2\sin\theta d\varphi\right). \end{aligned} \tag{6.26}$$

A tetrad that is regular on the future horizon can be obtained from the tetrad (6.25) by the transformation $(t, \varphi) \to (-t, -\varphi)$, which is an isometry of the Kerr metric. This transformation, which we will denote by a bar, acts on the basis vectors via a pullback and results in the interchange $(\vec{l}, \vec{n}) \to (\vec{n}, \vec{l})$ and $(\vec{m}, \vec{m}^*) \to (\vec{m}^*, \vec{m})$ together with the appropriate renormalization [7]:

$$\bar{l}^a = -\frac{2\Sigma}{\Delta}n^a, \qquad \bar{n}^a = -\frac{\Delta}{2\Sigma}l^a, \qquad \bar{m}^a = \frac{\rho^*}{\rho}m^{*a}. \tag{6.27}$$

[6]Our notation for ρ is related to the variable z used by Gal'tsov [197] by $z = 1/\rho^*$.

[7]Note that in the literature, this bar transformation is often denoted by a + or a †, and bars denote complex conjugation.

The master variable corresponding to the barred tetrad (6.27) is given by projecting the Weyl tensor along the barred tetrad in analog to Eq. (6.5):

$$ {}_s\bar{\Psi} = \begin{cases} -C_{abcd}\bar{l}^a\bar{m}^b\bar{l}^c\bar{m}^d, & s = 2, \\ -\bar{\rho}^{-4}C_{abcd}\bar{n}^a\bar{m}^{*b}\bar{n}^c\bar{m}^{*d}, & s = -2. \end{cases} \tag{6.28} $$

Teukolsky [202] has shown that the variable defined in Eq. (6.28) is related to that in the unbarred tetrad by

$$ {}_s\bar{\Psi} = \left(\frac{2}{\Delta}\right)^s {}_{-s}\Psi, \tag{6.29} $$

which can be seen as follows. We first note that the expression (6.24) is invariant under the transformation $(t, \varphi) \to (-t, -\varphi)$, so $\bar{\rho} = \rho$. Using Eq. (6.27) in Eq. (6.28) leads to the expressions

$$ {}_2\bar{\Psi} = -\left(\frac{2}{\Delta}\right)^2 \rho^{-4}C_{abcd}n^a m^{*b}n^c m^{*d} = \left(\frac{2}{\Delta}\right)^2 {}_{-2}\Psi \tag{6.30} $$

$$ {}_{-2}\bar{\Psi} = -\rho^{-4}\left(\frac{\Delta}{2\Sigma}\right)^2\left(\frac{\rho}{\rho^*}\right)^2 C_{abcd}l^a m^b l^c m^d = \left(\frac{2}{\Delta}\right)^{-2} {}_2\Psi. \tag{6.31} $$

Combining the results of Eqs. (6.30) and (6.31) gives Eq. (6.28).

In Boyer-Lindquist coordinates, the differential operator ${}_s\mathcal{O}$ defined in Eq. (6.6) can be written as

$$ {}_s\mathcal{O} = \Sigma^{-1}\ {}_s\Box, \tag{6.32} $$

where the operator ${}_s\Box$ is given by

$$ \begin{aligned} {}_s\Box = {} & \left[\frac{\varpi^4}{\Delta} - a^2\sin^2\theta\right]\partial_t^2 - \frac{4Mar}{\Delta}\partial_t\partial_\varphi + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\partial_\varphi^2 + \frac{1}{\Delta^s}\partial_r\left(\Delta^{s+1}\partial_r\right) \\ & + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\partial_\varphi \\ & + 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\partial_t + \left(s^2\cot^2\theta - s\right). \end{aligned} \tag{6.33} $$

For notational convenience, we will include the factor of Σ in Eq. (6.32) with the source term and write the decoupled master equation (6.6) as

$$ {}_s\Box\ {}_s\Psi = {}_s\mathcal{T}, \tag{6.34} $$

where ${}_s\mathcal{T}$ is given by

$$ {}_s\mathcal{T} = 4\pi\Sigma\ {}_s\tau_{ab}T^{ab}. \tag{6.35}$$

We define the angular and radial differential operators $\mathcal{L}_s$ and $\mathcal{D}_n$, for the integers s and n, in terms of directional derivatives along the tetrad:

$$\begin{aligned} \mathcal{L}_s &= \frac{\sqrt{2}}{\rho}m^{*a}\partial_a + s\cot\theta, &(6.36)\\ \mathcal{D}_n &= l^a\partial_a + n\,\frac{2(r-M)}{\Delta}. &(6.37)\end{aligned}$$

The operators corresponding to the tetrad (6.27) are given by

$$\begin{aligned} \bar{\mathcal{D}}_n &= \bar{l}^a\partial_a + \frac{2n(r-M)}{\Delta} = -\frac{2\Sigma}{\Delta}n^a\partial_a + \frac{2n(r-M)}{\Delta}\\ \bar{\mathcal{L}}_s &= \frac{\sqrt{2}}{\rho^*}m^a\partial_a + s\cot\theta. &(6.38)\end{aligned}$$

We can read off the operators ${}_s\tau_{ab}$ from the decoupled equations derived in Ref. [202], Eqs. (2.13) and (2.15) with the specialization $\epsilon = 0 = \epsilon^*$ and the changes in notation

$$\begin{aligned} D^{\rm Teuk} &\to\ l^a\partial_a = \mathcal{D}_0, \quad \Delta^{\rm Teuk} \to n^a\partial_a = \frac{\Delta}{2\Sigma}\bar{\mathcal{D}}_0, \quad \delta^{\rm Teuk} \to m^a\partial_a = \frac{\rho^*}{\sqrt{2}}\bar{\mathcal{L}}_0,\\ \delta^{*\ \rm Teuk} &\to\ m^{*a}\partial_a = \frac{\rho}{\sqrt{2}}\mathcal{L}_0. &(6.39)\end{aligned}$$

This gives the following expressions [8]:

$$\begin{aligned} {}_2\tau_{ab} &= \rho^4\rho^*\left[\sqrt{2}\left(\bar{\mathcal{L}}_{-1}\frac{(\rho^*)^2}{\rho^4}\mathcal{D}_0 + \mathcal{D}_0\frac{(\rho^*)^2}{\rho^4}\bar{\mathcal{L}}_{-1}\right)\frac{1}{(\rho^*)^2}l_{(a}m_{b)} - \bar{\mathcal{L}}_{-1}\frac{1}{\rho^4}\bar{\mathcal{L}}_0\rho^*(l_al_b)\right.\\ &\quad\left.-2\mathcal{D}_0\frac{1}{\rho^4}\mathcal{D}_0\frac{1}{\rho^*}(m_am_b)\right], &(6.40)\\ {}_{-2}\tau_{ab} &= -\rho^4\rho^*\left[\frac{\Delta}{\sqrt{2}}\left(\bar{\mathcal{D}}_{-1}\frac{(\rho^*)^2}{\rho^4}\mathcal{L}_{-1} + \mathcal{L}_{-1}\frac{(\rho^*)^2}{\rho^4}\bar{\mathcal{D}}_{-1}\right)\Sigma^2\ n_{(a}m^*_{b)}\right.\\ &\quad\left.+\mathcal{L}_{-1}\frac{1}{\rho^4}\mathcal{L}_0\frac{\Sigma}{\rho}(n_an_b) + \frac{\Delta^2}{2}\bar{\mathcal{D}}_0\frac{1}{\rho^4}\bar{\mathcal{D}}_0\frac{\rho^*}{\rho^2}(m^*_am^*_b)\right], &(6.41)\end{aligned}$$

[8]Note that there are two typos in the corresponding Eq. (2.3) in Gal'tsov:
(1) in his expression for ${}_2\tau$, the last term should be $m\otimes m$ instead of $n\otimes m$
(2) in the last term in his expression for ${}_{-2}\tau$, his operator $\mathcal{D}_{-1}$ in the prefactor of $\bar{m}\otimes\bar{m}$ should be replaced by $\mathcal{D}_0$.

where the notation $v^{(a}w^{b)}$ means symmetrization on these indices.

To obtain the expressions for the operators ${}_sM^{ab}$ in Eq. (6.5), one can use the vacuum case and write the perturbations of the Riemann tensor δR_{abcd} in terms of the metric perturbation:

$$\delta R_{abcd} = \frac{1}{2}\left(\nabla_b\nabla_c h_{ad} + \nabla_a\nabla_d h_{bc} - \nabla_a\nabla_c h_{bd} - \nabla_b\nabla_d h_{ac}\right) - R^{(0)}_{ab[c}{}^{e} h_{d]e}. \tag{6.42}$$

Projecting this result along the tetrad legs as in Eq. (6.5) gives following expression:

$${}_2\Psi = -\frac{1}{2}\left(l^a m^b m^c l^d + m^a l^b l^c m^d - m^a m^b l^c l^d - l^a l^b m^c m^d\right)\nabla_c\nabla_d h_{ab} = {}_2M^{ab}h_{ab}, \tag{6.43}$$

and similarly for ψ_4. Next, we expand h_{ab} in terms of the tetrad vectors:

$$\begin{aligned} h_{ab} &= h_{ll}n_a n_b + h_{nn}l_a l_b + h_{mm}m^*_a m^*_b + h_{m^*m^*}m_a m_b - h_{lm}n_a m^*_b - h_{nm^*}l_a m_b \\ &\quad - h_{nm}l_a m^*_b - h_{lm^*}n_a m_b. \end{aligned} \tag{6.44}$$

Using Eq. (6.44) in Eq. (6.43) and rewriting the result in terms of the operators $\mathcal{L}_s$ and $\mathcal{D}_n$ defined in Eqs. (6.36) and (6.37) gives the following expressions:

$$\begin{aligned} {}_2M^{ab} &= -\frac{\rho^*}{2}\left[\frac{1}{2}\bar{\mathcal{L}}_{-1}\bar{\mathcal{L}}_0\rho^*(l^a l^b) + \mathcal{D}_0^2\frac{1}{\rho^*}(m^a m^b)\right] \\ &\quad + \frac{\rho^*}{2\sqrt{2}}\left(\mathcal{D}_0(\rho^*)^2\bar{\mathcal{L}}_{-1}\frac{1}{(\rho^*)^2} + \bar{\mathcal{L}}_{-1}(\rho^*)^2\mathcal{D}_0\frac{1}{(\rho^*)^2}\right)l^{(a}m^{b)}, \end{aligned} \tag{6.45}$$

$$\begin{aligned} {}_{-2}M^{ab} &= -\frac{\rho^*}{2}\left[\frac{1}{2}\mathcal{L}_{-1}\mathcal{L}_0\frac{\Sigma}{\rho}(n^a n^b) + \frac{\Delta^2}{4}\bar{\mathcal{D}}_0^2\frac{\rho^*}{\rho^2}(m^{*a}m^{*b})\right] \\ &\quad - \frac{\rho^*\Delta^2}{4\sqrt{2}}\left(\bar{\mathcal{D}}_0\frac{(\rho^*)^2}{\Delta}\mathcal{L}_{-1}\Sigma^2 + \mathcal{L}_{-1}(\rho^*)^2\bar{\mathcal{D}}_0\frac{\Sigma^2}{\Delta}\right)n^{(a}m^{*b)}. \end{aligned} \tag{6.46}$$

These are the same expressions as in Ref. [201] with the translations

$$\begin{aligned} \mathcal{L}_s &= \frac{\sqrt{2}}{\rho}\left(\delta^* + 2s\beta^*\right), \quad \bar{\mathcal{L}}_s = \frac{\sqrt{2}}{\rho^*}\left(\delta + 2s\beta\right), \\ \mathcal{D}_n &= D + 4n(\rho\rho^*)^{-1}\left(\gamma - \mu\right), \quad \bar{\mathcal{D}}_n = \frac{\rho^*}{\mu^*}\left[\Delta + 2n\left(\mu^* - \gamma^*\right)\right]. \end{aligned} \tag{6.47}$$

One can compute the adjoint operators ${}_s\tau^\dagger_{ab}$ from Eqs.(6.40) and (6.41) together with the following identities that one can check using Eq. (6.8):

$$\mathcal{L}^\dagger_s = -\Sigma^{-1}\bar{\mathcal{L}}_{1-s}\,\Sigma, \qquad \mathcal{D}^\dagger_n = -\Sigma^{-1}\mathcal{D}_{-n}\,\Sigma. \tag{6.48}$$

This gives the following expressions:

$$\begin{aligned}
{}_2\tau^\dagger_{ab} &= \left[\sqrt{2}\; l_{(a}m^*_{b)}\frac{\rho^*}{\rho}\left(\mathcal{L}_2\frac{\rho^2}{(\rho^*)^4}\mathcal{D}_0 + \mathcal{D}_0\frac{\rho^2}{(\rho^*)^4}\mathcal{L}_2\right) - (l_a l_b)\rho^2\rho^*\mathcal{L}_1\frac{1}{(\rho^*)^4}\mathcal{L}_2\right](\rho^*)^3 \\
&\quad - \left[2(m^*_a m^*_b)\rho^*\mathcal{D}_0\frac{1}{(\rho^*)^4}D_0\right](\rho^*)^3, \qquad (6.49)\\
{}_{-2}\tau^\dagger_{ab} &= \frac{1}{\sqrt{2}}n_{(a}m_{b)}\Sigma\left(\bar{\mathcal{D}}_1\frac{\rho^2}{(\rho^*)^4}\bar{\mathcal{L}}_2 + \bar{\mathcal{L}}_2\frac{\rho^2}{(\rho^*)^4}\bar{\mathcal{D}}_1\right)\Delta(\rho^*)^3 \\
&\quad - \left[(n_a n_b)\frac{1}{\rho^*}\bar{\mathcal{L}}_1\frac{1}{(\rho^*)^4}\bar{\mathcal{L}}_2\frac{1}{2}m_a m_b\frac{\rho^2}{\rho^*}\bar{\mathcal{D}}_0\frac{1}{(\rho^*)^4}\bar{\mathcal{D}}_0\Delta^2\right](\rho^*)^3. \qquad (6.50)
\end{aligned}$$

Using Eqs. (6.45) and (6.46) results in the following expressions for the operators ${}_sM^{ab}{}_s\tau^\dagger_{ab}$:

$$\begin{aligned}
{}_2M^{ab}{}_2\tau^\dagger_{ab} &= \mathcal{D}^4_0, \qquad (6.51)\\
{}_{-2}M^{ab}{}_{-2}\tau^\dagger_{ab} &= \frac{1}{16}\Delta^2\bar{\mathcal{D}}^4_0\Delta^2, \qquad (6.52)\\
{}_2M^{ab}({}_2\tau^\dagger_{ab})^* = \;\; &0 \;\; = {}_{-2}M^{ab}({}_{-2}\tau^\dagger_{ab})^*. \qquad (6.53)
\end{aligned}$$

We will not need the expressions for ${}_{-s}M^{ab}{}_s\tau^\dagger_{ab}$, which give two differential relations involving $\mathcal{L}_s$ and $\bar{\mathcal{L}}_s$ instead of $\mathcal{D}_0$ and $\bar{\mathcal{D}}_0$, because both ${}_2\Psi$ and ${}_{-2}\Psi$ encode the same information, so it suffices to compute one of them. The operators ${}_sM^{ab}{}_s\tau^\dagger_{ab}$, in addition to the ${}_s\tau_{ab}$ necessary to compute the source term, will be all we need to construct the radiative Green's function for the metric perturbation from the Green's function for the Teukolsky equation in later sections of the chapter. As discussed in the introduction, in the adiabatic limit we do not need to reconstruct the metric perturbation, and therefore we will not discuss the challenges associated with this task, such as the presence of sources, gauge issues, low multipoles, etc.

6.3 Vacuum equations

6.3.1 Separation of variables

We now review the separation of variables first carried out by Teukolsky [202]. In this section, we specialize to the homogeneous version of the Teukolsky equation (6.34). The Teukolsky operator separates into a radial and an angular part as [9]

$$
\begin{aligned}
{}_s\Box &= {}_s\Box^{(r)} + {}_s\Box^{(\theta)}, &(6.55)\\
{}_s\Box^{(r)} &= \frac{1}{\Delta^s}\partial_r\left(\Delta^{s+1}\partial_r\right) + \frac{1}{\Delta}\left[-\varpi^4\partial_t^2 + 2a\varpi^2\partial_t\partial_\varphi - a^2\partial_\varphi^2\right] + s + |s| \\
&\quad -\frac{2s(r-M)}{\Delta}(-\varpi^2\partial_t + a\partial_\varphi) - 4sr\partial_t + a^2\partial_t^2 - 2a\partial_t\partial_\varphi &(6.56)\\
{}_s\Box^{(\theta)} &= \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) - a^2\cos^2\theta\partial_t^2 + \csc^2\theta\partial_\varphi^2 - 2ias\cos\theta\partial_t \\
&\quad +\frac{2is\cos\theta}{\sin^2\theta}\partial_\varphi - s^2\cot^2\theta - |s|. &(6.57)
\end{aligned}
$$

From Eqs. (6.25), (6.36), and (6.37), the expressions for the operators $\mathcal{L}_s$ and $\mathcal{D}_n$ are

$$
\begin{aligned}
\mathcal{L}_s &= -ia\sin\theta\partial_t + \partial_\theta - \frac{i}{\sin\theta}\partial_\varphi + s\cot\theta, &(6.58)\\
\mathcal{D}_n &= \frac{\varpi^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\varphi + \frac{2n(r-M)}{\Delta}. &(6.59)
\end{aligned}
$$

Note that the radial operators ${}_s\Box^{(r)}$ and $\mathcal{D}_n$ are real, while the angular operators ${}_s\Box^{(\theta)}$ and $\mathcal{L}_s$ are complex.

To obtain separable solutions, we make the ansatz

$$
{}_s\Psi = {}_sR(r)_s\Theta(\theta)e^{im\varphi}e^{-i\omega t}. \tag{6.60}
$$

[9] Separability of the equations can be achieved in any coordinates $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\varphi})$ related to Boyer-Lindquist by

$$
\tilde{t} = t + f_1(r) + f_2(\theta), \qquad \tilde{r} = g(r), \qquad \tilde{\theta} = h(\theta), \qquad \tilde{\varphi} = \varphi + j_1(r) + j_2(\theta), \tag{6.54}
$$

for arbitrary functions f_1, f_2, g, h, j_1, j_2.

Substituting the ansatz (6.60) into the homogeneous version of Eq. (6.34) results in the two equations:

$$\begin{aligned}
0 &= \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\,{}_s\Theta}{d\theta}\right) + \left[-s^2\cot^2\theta + \lambda - |s|\right]{}_s\Theta \\
&\quad + \left[a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta}\right]{}_s\Theta, \qquad (6.61)\\
0 &= \frac{1}{\Delta^s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d\,{}_sR}{dr}\right) + \left[\frac{K_{m\omega}^2 - 2is(r-M)K_{m\omega}}{\Delta} + 4is\omega r - \lambda\right]{}_sR \\
&\quad + \left[-a^2\omega^2 + 2am\omega + s + |s|\right]{}_sR = 0. \qquad (6.62)
\end{aligned}$$

Here, λ is the separation constant and we have defined

$$K_{m\omega} = \omega\varpi^2 - am. \qquad (6.63)$$

The separation constant λ is related to the constant A used by Teukolsky [206] by $\lambda = A + s + |s|$. We denote the eigenvalues of the angular equation (6.61) by $\lambda_{s\omega lm}$, where the integer l labels the successive eigenvalues with $l \geq |s|$ and $|m| \leq l$. In the special case $a\omega = 0$ we have $\lambda_{slm} = l(l+1) - s^2 + |s|$[207]. The angular equations for $s = 2$ and $s = -2$ have the same set of eigenvalues λ [207] but not of A. The solutions to Eq. (6.61) are the real functions ${}_s\Theta_{\omega lm}(\theta)$ that are regular on $[0, \pi]$. These quantities also depend on $a\omega$, i.e. ${}_s\Theta_{lm}(a\omega, \theta)$ and $\lambda_{slm}(a\omega)$, but we do not show this dependence explicitly here. The angular differential equation (6.61) is invariant under the transformation $(s, \omega, m) \to (-s, -\omega, -m)$ holding λ fixed, so we can choose the relative normalization to be:

$${}_s\Theta_{\omega lm}(\theta) = {}_{-s}\Theta_{(-\omega)l(-m)}(\theta). \qquad (6.64)$$

The functions

$${}_sS_{\omega lm}(\theta, \varphi) = e^{im\varphi}\,{}_s\Theta_{\omega lm}(\theta) \qquad (6.65)$$

are the spin-weighted spheroidal harmonics, and we can choose them to be orthonormal:

$$\int d^2\Omega\ {}_sS^*_{\omega lm}(\theta, \varphi)\ {}_sS_{\omega l'm'}(\theta, \varphi) = \delta_{ll'}\delta_{mm'}. \qquad (6.66)$$

We can choose the phases of the spheroidal harmonics to satisfy (cf. Galt'sov):

$$(P_s S_{\omega lm})(\theta,\varphi) \equiv {}_sS_{\omega lm}(\pi-\theta,\pi+\varphi) = (-1)^l {}_{-s}S_{\omega lm}(\theta,\varphi), \tag{6.67}$$

where P is the parity operator that maps $(\theta,\varphi) \to (\pi-\theta,\pi+\varphi)$.

For a single Fourier mode $\propto e^{im\varphi}e^{-i\omega t}$, the differential operators $\mathcal{L}_s$ and $\mathcal{D}_n$ reduce to

$$\begin{aligned} \mathcal{L}_{sm\omega} &= -a\omega\sin\theta + \partial_\theta + \frac{m}{\sin\theta} + s\cot\theta, & (6.68)\\ \mathcal{D}_{nm\omega} &= -i\omega\frac{\varpi^2}{\Delta} + \partial_r + \frac{iam}{\Delta} + \frac{2n(r-M)}{\Delta}. & (6.69) \end{aligned}$$

The transformation $(t,\varphi) \to (-t,-\varphi)$ reduces to $(\omega,m) \to (-\omega,-m)$ in this context. We denote this reduced transformation $(\omega,m) \to (-\omega,-m)$ by a "+": [10]

$$\mathcal{L}^+_{sm\omega} = \mathcal{L}_{s(-m)(-\omega)}, \qquad \mathcal{D}^+_{nm\omega} = \mathcal{D}_{n(-m)(-\omega)}. \tag{6.70}$$

Note that the specialization to the ansatz (6.60) has changed the complexity of the operators: now the angular differential equation (6.61) is real, while the radial equation (6.62) is complex. In term of these operators (6.68) and (6.69), the angular and radial differential equations can be written more compactly as:

$$\begin{aligned} \left(\mathcal{L}_{-1m\omega}\mathcal{L}^+_{2m\omega} + 6a\omega\cos\theta\right)\ {}_{-2}\Theta_{\omega lm} &= -\lambda_{slm}\ {}_{-2}\Theta_{\omega lm}, & (6.71)\\ \left(\mathcal{L}^+_{-1m\omega}\mathcal{L}_{2m\omega} - 6a\omega\cos\theta\right)\ {}_{2}\Theta_{\omega lm} &= -\lambda_{slm}\ {}_{2}\Theta_{\omega lm}, & (6.72)\\ \left(\Delta\mathcal{D}_{-1m\omega}\mathcal{D}^+_{0m\omega} + 6i\omega r\right)\Delta^2\ {}_{2}R_{\omega lm} &= \lambda_{slm}\Delta^2\ {}_{2}R_{\omega lm}, & (6.73)\\ \left(\Delta\mathcal{D}^+_{-1m\omega}\mathcal{D}_{0m\omega} - 6i\omega r\right)\ {}_{-2}R_{\omega lm} &= \lambda_{slm}\ {}_{-2}R_{\omega lm}. & (6.74) \end{aligned}$$

The radial equation (6.62) can be simplified by defining the tortoise coordinate r^* by

$$dr^*/dr = \varpi^2/\Delta. \tag{6.75}$$

[10]Often, a † is used to denote this transformation

We can express r^* as

$$r^* = r + \frac{2r_+}{r_+ - r_-} \ln \frac{r - r_+}{2} - \frac{2r_-}{r_+ - r_-} \ln \frac{r - r_-}{2}, \tag{6.76}$$

where

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{6.77}$$

are the two roots of $\Delta(r) = 0$. The radial equation (6.62) then becomes

$$\left[\frac{d^2}{dr^{*2}} + 2G \frac{d}{dr^*} + \frac{K_{m\omega}^2 - 2is(r - M)K_{m\omega} + \Delta(4irs - \lambda)}{\varpi^4} \right] {}_sR = 0, \tag{6.78}$$

where $G = s(r - M)/\varpi^2 + r\Delta/\varpi^4$. This can be written as an effective potential equation for the variable ${}_su(r)$ defined by

$${}_sR(r) = e^{-\int G dr^*} {}_su(r) = \Delta^{-s/2} \varpi^{-1} {}_su(r). \tag{6.79}$$

The resulting simplified homogeneous radial equation is

$$0 = \frac{d^2 \ {}_su}{dr^{*2}} + {}_sV_{\omega lm} \ {}_su(r^*). \tag{6.80}$$

The effective potential ${}_sV_{\omega lm}$ is complex (it is real for $s = 0$) and given by

$$\begin{aligned} {}_sV_{\omega lm} \quad = \quad & \omega^2 + \frac{1}{\varpi^4} \left\{ -4aMrm\omega + a^2m^2 - 2is(r - M)K + \right\} \\ & + \frac{\Delta}{\varpi^4} \left(4ir\omega s - \lambda_{\omega lm} + |s| - a^2\omega^2 \right) - \frac{s^2(r - M)^2}{\varpi^4} \\ & + \frac{\Delta}{\varpi^6} \left(4Mr - 3r^2 - a^2 \right) + \frac{3r^2\Delta^2}{\varpi^8}. \end{aligned} \tag{6.81}$$

6.3.2 Basis of modes

We review here the definition of the basis of modes found in Refs. [197, 201]. This basis is characterized by positive and negative exponents of r^* as $r^* \to \pm\infty$.

Consider first the limit $r^* \to -\infty$ ($r \to r_+$), the past and future event horizons. In this limit, $\Delta \to 0$ and $\varpi^2 \to 2Mr_+$. We find that the radial potential becomes:

$$
\begin{aligned}
{}_sV_{\omega lm} &\to \omega^2 - \frac{\omega}{Mr_+}\left[am + is(r_+ - M)\right] + \frac{a^2m^2}{4M^2r_+^2} + \frac{isam(r_+ - M)}{2M^2r_+^2} \\
&\quad - \frac{s^2(r_+ - M)^2}{(2Mr_+)^2} && (6.82) \\
&= p_{m\omega}^2 - \frac{2is(r_+ - M)p_{m\omega}}{2Mr_+} - \frac{s^2(r_+ - M)^2}{(2Mr_+)^2} && (6.83) \\
&= p_{m\omega}^2\kappa_{sm\omega}^2, && (6.84)
\end{aligned}
$$

where we have defined the quantities $p_{m\omega}$ and $\kappa_{sm\omega}$ by

$$
\begin{aligned}
p_{m\omega} &= \omega - \frac{am}{2Mr_+}, && (6.85) \\
\kappa_{sm\omega} &= 1 - \frac{is(r_+ - r_-)}{4Mr_+p_{m\omega}}. && (6.86)
\end{aligned}
$$

The last term in Eq.(6.85) is the angular velocity of the horizon $\omega_+ = am/(2Mr_+)$. From Eqs. (6.80) and (6.84), the solutions of the radial equation near the horizon are of the form

$$
{}_su(r) \propto e^{\pm ip_{m\omega}\kappa_{sm\omega}r^*} = \Delta^{\pm s/2}e^{\pm ip_{m\omega}r^*}\left[1 + O\left(\frac{1}{r^*}\right)\right]. \tag{6.87}
$$

The last equality in (6.87) follows from the leading order form of $\Delta = (r - r_+)(r - r_-)$ at the event horizon:

$$
\Delta \to (r - r_+)(r_+ - r_-). \tag{6.88}
$$

In the limit of $r^* \to \infty$ ($r \to \infty$), past and future null infinity, the potential has the asymptotic behavior

$$
V = \omega^2 + \frac{2is\omega}{r} + O\left(\frac{1}{r^2}\right), \tag{6.89}
$$

so the radial solutions are of the form

$$
{}_su(r) \propto r^{\mp s}e^{\pm i\omega r^*}. \tag{6.90}
$$

6.3.3 "in", "up", "out", and "down" modes

The general solution to the second order ordinary differential equation (6.80) can be spanned by any pair of independent solutions. The most convenient bases are those characterized by the asymptotic positive and negative exponential dependence on r^*. We define, following Galt'sov, the solution

$$
{}_s u^{\rm in}_{\omega lm} = \alpha_{s\omega lm} \begin{cases} \tau_{s\omega lm} \mid p_{m\omega} \mid^{-1/2} \Delta^{-s/2} e^{-ip_{m\omega} r^*}, & r^* \to -\infty, \\ \mid \omega \mid^{-1/2} \left[r^s e^{-i\omega r^*} + \sigma_{s\omega lm} r^{-s} e^{i\omega r^*} \right], & r^* \to \infty. \end{cases} \tag{6.91}
$$

This equation defines the mode as well as the complex transmission and reflection coefficients $\tau_{s\omega lm}$ and $\sigma_{s\omega lm}$. The coefficient $\alpha_{s\omega lm}$ is a normalization constant. The "in" mode (6.91) is a mixture of outgoing and ingoing components at past and future null infinity, since the mode function is multiplied by $e^{-i\omega t}$. At the past and future event horizon, the mode is purely ingoing when the sign of $p_{m\omega}$ is the same as the sign of ω. However, from the definition (6.85) of $p_{m\omega}$ we see that $\omega p_{m\omega}$ can be negative; this occurs for superradiant modes. Thus, at the future event horizon the "in" modes can be either ingoing or outgoing.

The important feature of the "in" modes is that they vanish on the past event horizon. This feature will be used later in constructing the various Green's functions. A more precise statement of the result is that a solution ${}_s\Psi$ of the Teukolsky equation which is a linear combination of "in" modes with coefficients $c_{\omega lm}$, such that the coefficients depend smoothly on ω (a reasonable requirement), must vanish at the past event horizon. To see this, note from Eqs. (6.60) and (6.79) that the solution can be written as

$$
{}_s\Psi(t, r, \theta, \phi) = \int_{-\infty}^{\infty} d\omega \sum_{l=2}^{\infty} \sum_{m=-l}^{l} e^{-i\omega t} c_{\omega lm}\ {}_s S_{\omega lm}(\theta, \phi) \frac{{}_s u^{\rm in}_{\omega lm}(r^*)}{\varpi(r^*) \Delta^{s/2}(r^*)}. \tag{6.92}
$$

We now insert the asymptotic form (6.91) of the mode function near the horizon,

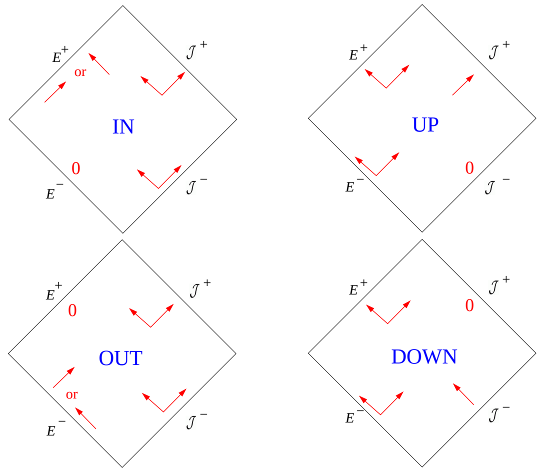

Figure 6.1: An illustration of the various types of modes in black hole spacetimes. Here $\mathcal{J}^-$ denotes past null infinity, $\mathcal{J}^+$ future null infinity, E^- the past event horizon, and E^+ the future event horizon. The four panels give the behavior of the four different modes "in", "out", "up" or "down" as indicated. A zero indicates the mode vanishes at the indicated boundary. Two arrows indicates that the mode consists of a mixture of ingoing and outgoing radiation at that boundary. Two arrows with an "OR" means that the mode is either purely ingoing or purely outgoing at that boundary, depending on the relative sign of $p_{m\omega}$ and ω. The "in" modes vanish on the past event horizon, and the "up" modes vanish on past null infinity. Thus the "in" and "up" modes together form a complete basis of modes. Similarly the "down" and "out" modes together form a complete basis of modes. From Drasco, Flanagan and Hughes, 2005.

and we use the definition (6.85) of $p_{m\omega}$. This gives

$$\begin{aligned} {}_s\Psi(t,r,\theta,\phi) &= \frac{1}{\varpi\Delta^s}\int_{-\infty}^{\infty} d\omega \sum_{lm} e^{-i\omega(t+r^*)} c_{\omega lm}\ {}_sS_{\omega lm}(\theta,\phi) \\ &\qquad \times \alpha_{s\omega lm}\tau_{s\omega lm}|p_{m\omega}|^{-1/2} e^{im\omega_+ r^*} \qquad (6.93) \\ &\equiv \frac{1}{\varpi\Delta^s}\sum_{lm}\ {}_sG_{lm}(t+r^*;\theta,\phi)e^{im\omega_+ r^*}. \qquad (6.94) \end{aligned}$$

Now all of the quantities that depend on ω in the integrand are smooth functions of ω. Since Fourier transforms of smooth functions go to zero at infinity, it follows that the function ${}_sG_{lm}(v;\theta,\phi)$ defined by Eq. (6.94) satisfies $G_{lm} \to 0$ as $v \to -\infty$, where $v = t + r^*$. Thus, ${}_s\Psi$ will vanish as $v \to -\infty$, on the past event horizon.

We next define the "up" modes:

$$
{}_s u^{\mathrm{up}}_{\omega lm} = \beta_{s\omega lm} \begin{cases} \mid p_{m\omega} \mid^{-1/2} \frac{\omega p_{m\omega}}{|\omega p_{m\omega}|} \left[\mu_{s\omega lm} \Delta^{s/2} e^{i p_{m\omega} r^*} \right. & \\ \qquad \left. + \ \nu_{s\omega lm} \Delta^{-s/2} e^{-i p_{m\omega} r^*} \right], & r^* \to -\infty, \\ \mid \omega \mid^{-1/2} r^{-s} e^{i\omega r^*}, & r^* \to \infty. \end{cases} \tag{6.95}
$$

This defines the mode as well as the complex coefficients $\mu_{s\omega lm}$ and $\nu_{s\omega lm}$. The coefficient $\beta_{s\omega lm}$ is a normalization constant. The "up" modes are a mixture of ingoing and outgoing components at the past and future event horizons. At future null infinity, the mode is purely outgoing. A similar argument as above for the "in" modes shows that the "up" modes vanish at past null infinity, so they are orthogonal to the "in" modes and both sets of modes together form a basis.

From (6.79), (6.91) and (6.95) we find the asymptotic forms of the radial function:

$$
{}_s R^{\mathrm{in}}_{\omega lm} = \alpha_{s\omega lm} \begin{cases} \tau_{s\omega lm} |p_{m\omega}|^{-1/2} (2Mr_+)^{-1/2} \Delta^{-s} e^{-i p_{m\omega} r^*} & , r^* \to -\infty, \\ |\omega|^{-1/2} \left[r^{-1} e^{-i\omega r^*} + \sigma_{s\omega lm} r^{-2s-1} e^{i\omega r^*} \right] & , r^* \to \infty, \end{cases} \tag{6.96}
$$

$$
{}_s R^{\mathrm{up}}_{\omega lm} = \beta_{s\omega lm} \begin{cases} |p_{m\omega}|^{-1/2} (2Mr_+)^{-1/2} \frac{\omega p}{|\omega p|} \left[\mu_{s\omega lm} e^{i p_{m\omega} r^*} \right. & \\ \qquad \left. + \ \nu_{s\omega lm} \Delta^{-s} e^{-i p_{m\omega} r^*} \right] & , r^* \to -\infty, \\ |\omega|^{-1/2} r^{-2s-1} e^{i\omega r^*} & , r^* \to \infty. \end{cases} \tag{6.97}
$$

Here we have used that $\varpi \to \sqrt{2Mr_+}$ near the horizon and $\Delta^{s/2}\varpi \to r^{s+1}$ near infinity.

Next, we note that the effective potential ${}_sV_{\omega lm}$ of Eq. (6.81) has the symmetry ${}_{-s}V^*_{\omega lm} = {}_sV_{\omega lm}$. It follows that ${}_{-s}u^{\mathrm{in}\,*}$ is also a solution to the radial differential equation (6.80). We can therefore define another basis: the "out" and "down" modes

$$
\begin{aligned}
{}_s u^{\text{out}}_{\omega lm} &= {}_{-s}u^{\text{in}\ *}_{\omega lm}, & (6.98)\\
{}_s u^{\text{down}}_{\omega lm} &= {}_{-s}u^{\text{up}\ *}_{\omega lm}. & (6.99)
\end{aligned}
$$

The asymptotic forms of the "out" modes are:

$$
{}_s u^{\text{out}}_{\omega lm} = \alpha^*_{-s\omega lm} \begin{cases} \tau^*_{-s\omega lm} \mid p_{m\omega} \mid^{-1/2} \Delta^{s/2} e^{ip_{m\omega}r^*}, & r^* \to -\infty \\ \mid \omega \mid^{-1/2} \left[r^{-s} e^{i\omega r^*} + \sigma^*_{-s\omega lm} r^s e^{-i\omega r^*} \right], & r^* \to \infty \end{cases} \tag{6.100}
$$

These modes vanish on the future horizon. The asymptotic forms of the "down" modes are:

$$
{}_s u^{\text{down}}_{\omega lm} = \beta^*_{-s\omega lm} \begin{cases} \mid p_{m\omega} \mid^{-1/2} \frac{\omega p_{m\omega}}{|\omega p_{m\omega}|} \left[\mu^*_{-s\omega lm} \Delta^{-s/2} e^{-ip_{m\omega}r^*} \right. & \\ \qquad \left. + \ \nu^*_{-s\omega lm} \Delta^{s/2} e^{ip_{m\omega}r^*} \right], & r^* \to -\infty \\ \mid \omega \mid^{-1/2} r^s e^{-i\omega r^*}, & r^* \to \infty \end{cases} \tag{6.101}
$$

These modes vanish on future null infinity and thus the "out" and "down" modes together form a complete basis.

We now define the following complete Teukolsky mode functions:

$$
\begin{aligned}
{}_s\Psi^{\text{in}}_{\omega lm}(t,r,\theta,\varphi) &= e^{-i\omega t}\ {}_sR^{\text{in}}_{\omega lm}(r)\ {}_sS_{\omega lm}(\theta,\varphi), & (6.102)\\
{}_s\Psi^{\text{up}}_{\omega lm}(t,r,\theta,\varphi) &= e^{-i\omega t}\ {}_sR^{\text{up}}_{\omega lm}(r)\ {}_sS_{\omega lm}(\theta,\varphi), & (6.103)\\
{}_s\Psi^{\text{out}}_{\omega lm}(t,r,\theta,\varphi) &= e^{-i\omega t}\ {}_sR^{\text{out}}_{\omega lm}(r)\ {}_sS_{\omega lm}(\theta,\varphi), & (6.104)\\
{}_s\Psi^{\text{down}}_{\omega lm}(t,r,\theta,\varphi) &= e^{-i\omega t}\ {}_sR^{\text{down}}_{\omega lm}(r)\ {}_sS_{\omega lm}(\theta,\varphi). & (6.105)
\end{aligned}
$$

6.3.4 Relations between the scattering and transmission coefficients

Wronskian Relations

In what follows, we will use the shorthand notation $\Lambda = \{\omega l m\}$. Relations between the coefficients $\sigma_{s\Lambda}$,$\tau_{s\Lambda}$, $\mu_{s\Lambda}$ and $\nu_{s\Lambda}$ can be derived by using the fact that the Wronskian

$$W(u_1, u_2) = u_1 \frac{du_2}{dr^*} - \frac{du_1}{dr^*} u_2 \tag{6.106}$$

is conserved for any two solutions u_1 and u_2 of the homogeneous radial equation (6.80). Throughout this subsection, we will specialize to fixed values of ω, l and m. Evaluating $W({}_s u^{\,\rm up},\ {}_s u^{\,\rm in})$ at $r^* = \pm\infty$ using the asymptotic relations (6.95) and (6.91) and equating the results we obtain:

$$1 = \mu_{s\Lambda}\ \tau_{s\Lambda} \left[1 - \frac{is(r_+ - M)}{2Mr_+ p_{m\omega}}\right] = \mu_{s\Lambda}\ \tau_{s\Lambda}\ \kappa_{sm\omega}. \tag{6.107}$$

where we have used the definition (6.86) of $\kappa_{sm\omega}$. A similar calculation with the modes ${}_s u^{\rm up}$, ${}_s u^{\rm out}$ yields

$$\tau^*_{-s\Lambda}\ \nu_{s\Lambda}\ \kappa_{sm\omega} = -\sigma^*_{-s\Lambda}, \tag{6.108}$$

and using ${}_s u^{\rm in}$ and ${}_s u^{\rm out}$ gives the "unitarity condition":

$$\frac{\omega p_{m\omega}}{\mid \omega p_{m\omega} \mid} \tau_{s\Lambda}\ \tau^*_{-s\Lambda}\ \kappa_{sm\omega} + \sigma_{s\Lambda}\ \sigma^*_{-s\Lambda} = 1. \tag{6.109}$$

Since the "in" and "up" modes form a basis of modes, we can express the "down" and "out" modes as linear combinations of the the "in" and "up" modes. Using the asymptotic forms (6.91) and (6.95) of the modes at $r^* \to \infty$ together with the definition (6.101) and the asymptotic forms at $r^* \to -\infty$ with the definition

(6.98) allows us to identify the coefficients for ${}_s u_\Lambda^{\rm down}$ and ${}_s u_\Lambda^{\rm out}$, giving

$$
\begin{aligned}
{}_s u_\Lambda^{\rm down} &= \frac{\beta^*_{-s\Lambda}}{\alpha_{s\Lambda}}\, {}_s u_\Lambda^{\rm in} - \frac{\beta^*_{-s\Lambda}\ \sigma_{s\Lambda}}{\beta_{s\Lambda}}\, {}_s u_\Lambda^{\rm up}, && (6.110)\\
{}_s u_\Lambda^{\rm out} &= \frac{\omega p_{m\omega}}{\mid \omega p_{m\omega} \mid} \frac{\alpha^*_{-s\Lambda}\ \tau^*_{-s\Lambda}}{\beta_{s\Lambda}\ \mu_{s\Lambda}}\, {}_s u_\Lambda^{\rm up} - \frac{\alpha^*_{-s\Lambda}\ \tau^*_{-s\Lambda}\ \nu_{s\Lambda}}{\alpha_{s\Lambda}\ \tau_{s\Lambda}\ \mu_{s\Lambda}}\, {}_s u_\Lambda^{\rm in}. && (6.111)
\end{aligned}
$$

The second expression (6.111) can be simplified using Eqs. (6.107), (6.108), and (6.109) to yield:

$$
{}_s u_\Lambda^{\rm out} = \frac{\alpha^*_{-s\Lambda}}{\beta_{s\Lambda}}(1 - \sigma_{s\Lambda}\ \sigma^*_{-s\Lambda})\, {}_s u_\Lambda^{\rm up} + \frac{\alpha^*_{-s\Lambda}\ \sigma^*_{-s\Lambda}}{\alpha_{s\Lambda}}\, {}_s u_\Lambda^{\rm in}. \tag{6.112}
$$

Spin-inversion Relations

We have already discussed the fact that either of the two sets of functions ${}_{\pm s}\Psi$ and ${}_{\pm s}\bar{\Psi}$ contains complete information, and shown how they are related in Eq. (6.29). In this subsection we review how one can compute the local value of all the variables from knowing the local values of one of them by obtaining their transformation properties under spin weight inversion $s \to -s$. Teukolsky and Starobinsky have shown that there exist relations between quantities of positive and negative spin weight, the "Starobinsky identities". These identities link a given solution of the radial equation (6.62), and a solution to the angular equation (6.61) to the unique corresponding solution with negative spin weight and are given by [208]:

$$
\begin{aligned}
\mathcal{L}_{-1m\omega}\ \mathcal{L}_{0m\omega}\ \mathcal{L}_{1m\omega}\ \mathcal{L}_{2m\omega}\ {}_2\Theta_{\omega lm}(\theta) &= F_{\omega lm}\ {}_{-2}\Theta_{\omega lm}(\theta), && (6.113)\\
\mathcal{L}^+_{-1m\omega}\ \mathcal{L}^+_{0m\omega}\ \mathcal{L}^+_{1m\omega}\ \mathcal{L}^+_{2m\omega}\ {}_{-2}\Theta_{\omega lm}(\theta) &= F_{\omega lm}\ {}_2\Theta_{\omega lm}(\theta), && (6.114)\\
\Delta^2 \left(\mathcal{D}^+_{0m\omega}\right)^4 \Delta^2\ {}_2R_{\omega lm} &= BC^*_{\omega lm}\ {}_{-2}R_{\omega lm}, && (6.115)\\
\mathcal{D}^4_{0m\omega}\ {}_{-2}R_{\omega lm} &= \frac{C_{\omega lm}}{B}\ {}_2R_{\omega lm}. && (6.116)
\end{aligned}
$$

Here, $C_{\omega lm}$ is the Starobinsky constant given by [208]:

$$
\begin{aligned}
|C_{\omega lm}|^2 &= \lambda_{\omega lm}^2 (\lambda_{\omega lm}+2)^2 + 8\lambda_{\omega lm}(5\lambda_{\omega lm}+6)\left(am\omega - a^2\omega^2\right) + 144M^2\omega^2 \\
&\quad + 96a^2\omega^2\lambda_{\omega lm} + 144a^2\omega^2 (m-a\omega)^2, \qquad (6.117) \\
C_{\omega lm} &= F_{\omega lm} + 12iM\omega, \quad F_{\omega lm} = \Re(C_{\omega lm}), \quad C^*_{\omega lm} = C_{-\omega l - m}. \qquad (6.118)
\end{aligned}
$$

The constant B in Eqs. (6.119) and (6.120) is a numerical factor that depends on the choice of relative normalization of the radial functions with opposite spin weight. With our choice for the normalization of the angular functions in Eq. (6.64), we obtain from Eq. (6.29) that $B = 4$. In the remainder of this chapter, we will therefore specialize to the case $B = 4$. [11]

The relations (6.115) and (6.116) were recently corrected by Bardeen [209], who showed that the left and right hand side of these equations contain different linear combinations of the "even-parity-like" and "odd-parity-like" parts of ${}_sR_{\omega lm}$ [this decomposition is defined in Eq. (6.429) below]. Therefore, to get the correct form of these identities, one needs to split each function ${}_sR_{\omega lm}$ into an "even-parity-like" part ${}_sR^{\rm E}_{\omega lm}$ and "odd-parity-like" part ${}_sR^{\rm O}_{\omega lm}$. We will summarize his results here and give a sketch of the derivation in the Appendix. The correct radial identities are:

$$
\begin{aligned}
\Delta^2 \left(\mathcal{D}^{+}_{0m\omega}\right)^4 \Delta^2 \left[{}_2R^{\rm in\ E}_{\omega lm} + {}_2R^{\rm in\ O}_{\omega lm} \right] &= B\left[C^*_{\omega lm}\ {}_{-2}R^{\rm in\ E}_{\omega lm} + C_{\omega lm}\ {}_{-2}R^{\rm in\ O}_{\omega lm}\right] \qquad (6.119) \\
\mathcal{D}^4_{0m\omega}\left[{}_{-2}R^{\rm in\ E}_{\omega lm} + {}_{-2}R^{\rm in\ O}_{\omega lm} \right] &= \frac{1}{B}\left[C_{\omega lm}\ {}_2R^{\rm in\ E}_{\omega lm} + C^*_{\omega lm}\ {}_2R^{\rm in\ O}_{\omega lm}\right] \qquad (6.120)
\end{aligned}
$$

Here,

$$
{}_sR^{\rm in\ E}_{\omega lm} \equiv {}_sR^{\rm in}_{\omega lm} + {}_sR^{\rm in\ *}_{(-\omega)l(-m)}, \qquad {}_sR^{\rm in\ O}_{\omega lm} \equiv {}_sR^{\rm in}_{\omega lm} - {}_sR^{\rm in\ *}_{(-\omega)l(-m)}. \qquad (6.121)
$$

Similar results hold for the "out", "down" and "up" modes.

---

[11] This is also Gal'tsov's [197] and Bardeen's [209] convention, Sago [129] chooses $B = 1$.

We can now obtain further relations between the coefficients of the scattering states by using the in the relations (6.119) and (6.120) the asymptotic forms of the radial modes of Eqs. (6.96) and (6.97).

Near the event horizon, we have that

$$\mathcal{D}_{0m\omega} \to \partial_r - \frac{2Mr_+}{\Delta}\, ip_{m\omega}, \qquad \mathcal{D}^{+}_{0m\omega} \to \partial_r + \frac{2Mr_+}{\Delta}\, ip_{m\omega}, \tag{6.122}$$

where we have used that $r_+^2 + a^2 = 2Mr_+$. For a function of the form $f(r)e^{\pm ip_{m\omega}r^*}$ we compute the following leading order behavior near the horizon:

$$\begin{aligned} \mathcal{D}_{0m\omega}\ f(r)e^{ip_{m\omega}r^*} &\to \frac{df}{dr}e^{ip_{m\omega}r^*} \\ \mathcal{D}_{0m\omega}\ f(r)e^{-ip_{m\omega}r^*} &\to \left(\frac{df}{dr} - \frac{4Mr_+}{\Delta}\, ip_{m\omega}\right) e^{-ip_{m\omega}r^*}, \end{aligned} \tag{6.123}$$

where we have used the asymptotic form of the definition (6.75) of r^*. The corresponding expressions for $\mathcal{D}_0^+$ can be obtained by the "+" transformation $(\omega, m) \to (-\omega, -m)$. In the following paragraph, we will omit the subscripts ωm on $\mathcal{D}_{0m\omega}$, $p_{m\omega}$, and $\kappa_{sm\omega}$. We can compute

$$\mathcal{D}_0^4\ \Delta^s\ e^{-ipr^*} = (4Mr_+p)^4 \kappa_{-s}\ \kappa_{-s+1}\ \kappa_{-s+2}\ \kappa_{-s+3}\ \Delta^{s-4}\ e^{-ipr^*}. \tag{6.124}$$

Here, we have rewritten the derivative of Eq. (6.88) in terms of κ_s defined in Eq. (6.86). To obtain the corresponding expression with $(\mathcal{D}_0^+)^4$, just use the + transformation on this result.

For the leading terms of ${}_{-2}R^{\rm in}_{\omega lm}$ and ${}_{-2}R^{\rm up}$ in Eqs. (6.96) and (6.97) we thus obtain near the horizon:

$$\mathcal{D}_0^4\ \Delta^2\ e^{-ipr^*} = (4Mr_+p)^4 \kappa_{-2}\ \kappa_{-1}\ \kappa_1\ \Delta^{-2}\ e^{-ipr^*} \tag{6.125}$$

$$\begin{aligned} \Delta^2 (\mathcal{D}_0^+)^4\ \Delta^2\ e^{ipr^*} &= \Delta^2 \left[\mathcal{D}_0^4 \left(\Delta^2 e^{-ipr^*}\right)\right]^+ \\ &= (4Mr_+p)^4\ \kappa_2\ \kappa_1\ \kappa_{-1}\ \Delta^{-2}\ e^{-ipr^*}, \end{aligned} \tag{6.126}$$

where we have used that from the definition (6.86), it follows that $\kappa_{s-m-\omega} = \kappa_{-sm\omega}$.

Substituting the asymptotic form of ${}_sR^{\rm in}_\Lambda$ for $r_* \to -\infty$ from Eq. (6.96) on the right hand side of Eq. (6.149) and similarly for ${}_{-s}R^{\rm up}_\Lambda$ from Eq. (6.97), where as before $\Lambda = \{\omega l m\}$, we obtain the relations

$$(\alpha_{-s\Lambda}\ \tau_{-s\Lambda})^{\rm E} = 2^{-s-|s|}C^{s/2}(2Mr_+p)^{-2s}(\kappa_{-2}\kappa_{-1}\kappa_1)^{-s/2}\ (\alpha_{s\Lambda}\ \tau_{s\Lambda})^{\rm E}, \quad (6.127)$$

$$(\beta_{-s\Lambda}\ \nu_{-s\Lambda})^{\rm E} = 2^{-s-|s|}C^{s/2}(2Mr_+p)^{-2s}(\kappa_{-2}\kappa_{-1}\kappa_1)^{-s/2}\ (\beta_{s\Lambda}\ \nu_{s\Lambda})^{\rm E}, \quad (6.128)$$

$$(\beta_{-s\Lambda}\ \mu_{-s\Lambda})^{\rm E} = 2^{s}(C^*)^{-s/2}(2Mr_+p)^{2s}(\kappa_{2}\kappa_{1}\kappa_{-1})^{s/2}\ (\beta_{s\Lambda}\ \mu_{s\Lambda})^{\rm E}. \quad (6.129)$$

The relations for the "O"-parts can be obtained from these relations by interchanging $C \leftrightarrow C^*$.

Next, we use that for $r \to \infty$ the operators become $\mathcal{D}_0 \to \partial_r - i\omega$ and $\Delta \to r^2$ to compute the leading order behavior

$$\mathcal{D}_0\ f(r)e^{-i\omega r^*} \to -2i\omega f(r)e^{-i\omega r^*}, \qquad \mathcal{D}_0\ f(r)e^{i\omega r^*} \to 0. \quad (6.130)$$

As before, the corresponding expressions for $\mathcal{D}_0^+$ can be obtained by the "+" relabeling. A similar computation as for the horizon behavior leads to the following relations:

$$(\beta_{-s\Lambda})^{\rm E} = 2^s\omega^{2s}(C^*)^{-s/2}\ (\beta_{s\Lambda})^{\rm E}, \quad (6.131)$$

$$(\alpha_{-s\Lambda})^{\rm E} = 2^{-s}(2\omega)^{-2s}C^{s/2}\ (\alpha_{s\Lambda})^{\rm E}, \quad (6.132)$$

$$(\alpha_{-s\Lambda}\ \sigma_{-s\Lambda})^{\rm E} = 2^s\omega^{2s}(C^*)^{-s/2}\ (\alpha_{s\Lambda}\ \sigma_{s\Lambda})^{\rm E}, \quad (6.133)$$

and as before, the the "O"-parts can be obtained from these relations by interchanging $C \leftrightarrow C^*$.

6.3.5 Mode expansion of the potential for the metric perturbation

In this section, we give the explicit form of the potential ${}_s\Phi$ for the metric perturbation. We use the two requirements on ${}_s\Phi$ discussed in Sec.(IIA):

1. The potential ${}_s\Phi$ satisfies the adjoint of the homogeneous Teukolsky equation for ${}_s\Psi$,
2. the Teukolsky functions ${}_s\Psi$ are related to ${}_s\Phi$ by Eqs. (6.10) and (6.11).

We therefore make the ansatz to decompose ${}_s\Phi$ into normal modes as we did for ${}_s\Psi$:

$$ {}_s\Phi_{\omega lm} = A_{s\omega lm}\ {}_sB_{\omega lm}(r)\ {}_sG_{\omega lm}(\theta)e^{im\varphi - i\omega t}, \tag{6.134}$$

where the functions ${}_sB$ and ${}_sG$ are to be determined by finding the adjoint Teukolsky operator for the parameter s. We can compute the adjoint operators from the definition (6.8). It is convenient to rewrite this in terms of the scalar product of two tensor fields ϕ and ψ of equal rank on spacetime, which we define to be

$$\langle \phi, \psi \rangle = \int d^4x\sqrt{-g}\ \phi^*_{ab\ldots}\ \psi^{ab\ldots}. \tag{6.135}$$

The adjoint of an operator can then be computed by requiring that

$$\langle \phi, M\psi \rangle = \langle M^\dagger \phi, \psi \rangle. \tag{6.136}$$

To compute the adjoint of the Teukolsky operator for parameter s, it is easiest to use Eq. (6.48) in Eq. (6.136), together with the angular and radial equations in the form given in Eqs. (6.71) – (6.74). Defining the operators

$$\begin{aligned} {}_{-2}\mathcal{O}_{m\omega} &= {}_{-2}\mathcal{O}^{(\theta)}_{m\omega} + {}_{-2}\mathcal{O}^{(r)}_{m\omega} \qquad (6.137) \\ &= \Sigma^{-1}\left(\mathcal{L}_{-1m\omega}\mathcal{L}^{+}_{2m\omega} + 6a\omega\cos\theta\right) + \Sigma^{-1}\left(\Delta\mathcal{D}^{+}_{-1m\omega}\mathcal{D}_{0m\omega} - 6i\omega r\right) \\ {}_{2}\mathcal{O}_{m\omega} &= {}_{2}\mathcal{O}^{(\theta)}_{m\omega} + {}_{2}\mathcal{O}^{(r)}_{m\omega} \qquad (6.138) \\ &= \Sigma^{-1}\left(\mathcal{L}^{+}_{-1m\omega}\mathcal{L}_{2\ m\omega} - 6a\omega\cos\theta\right) + \Sigma^{-1}\left(\Delta\mathcal{D}_{1m\omega}\mathcal{D}^{+}_{2m\omega} + 6i\omega r\right), \end{aligned}$$

where in the last expression we have used that $\Delta\mathcal{D}_{n+1} = \mathcal{D}_n\Delta$, we can easily compute the adjoint to be:

$$ {}_{s}\mathcal{O}^{\dagger}_{m\omega} = {}_{s}\mathcal{O}^{(\theta)}_{m\omega} + {}_{-s}\mathcal{O}^{(r)}_{m\omega}. \qquad (6.139)$$

Since the angular operator is self-adjoint, the function ${}_sG$ satisfies the equation:

$$ {}_{s}\Box^{(\theta)}_{m\omega}\ {}_{s}G_{\omega lm}(\theta) = 0. \qquad (6.140)$$

Therefore, we can choose

$$ {}_{s}G_{\omega lm} = {}_{s}\Theta_{\omega lm}. \qquad (6.141)$$

For the radial function, the adjoint of the radial operator for parameter s is the radial operator with parameter $-s$, so that ${}_sB$ satisfies the differential equation:

$$ {}_{-s}\Box^{(r)}_{m\omega}\ {}_{s}B_{\omega lm}(r) = 0. \qquad (6.142)$$

It follows that we can choose

$$ {}_{s}B_{\omega lm}(r) = {}_{-s}R_{\omega lm}(r). \qquad (6.143)$$

As we did in Eqs. (6.102) – (6.105), we define the complete mode functions:

$$\begin{aligned} {}_{s}\Phi^{\text{in}}_{\omega lm}(t,r,\theta,\varphi) &= A_{s\omega lm}\ {}_{-s}R^{\text{in}}_{\omega lm}(r)\ {}_{s}S_{\omega lm}(\theta,\varphi)e^{-i\omega t}, \qquad (6.144) \\ {}_{s}\Phi^{\text{up}}_{\omega lm}(t,r,\theta,\varphi) &= A_{s\omega lm}\ {}_{-s}R^{\text{up}}_{\omega lm}(r)\ {}_{s}S_{\omega lm}(\theta,\varphi)e^{-i\omega t}, \qquad (6.145) \\ {}_{s}\Phi^{\text{out}}_{\omega lm}(t,r,\theta,\varphi) &= A_{s\omega lm}\ {}_{-s}R^{\text{out}}_{\omega lm}(r)\ {}_{s}S_{\omega lm}(\theta,\varphi)e^{-i\omega t}, \qquad (6.146) \\ {}_{s}\Phi^{\text{down}}_{\omega lm}(t,r,\theta,\varphi) &= A_{s\omega lm}\ {}_{-s}R^{\text{down}}_{\omega lm}(r)\ {}_{s}S_{\omega lm}(\theta,\varphi)e^{-i\omega t}, \qquad (6.147) \end{aligned}$$

where

$$
{}_{-s}R^{\text{in,up,out,down}}_{\omega lm} = \frac{\Delta^{s/2}}{\varpi}\ {}_{-s}u^{\text{in,up,out,down}}_{\omega lm}. \tag{6.148}
$$

The constant $A_{s\omega lm}$ will be determined from the relation of ${}_s\Phi$ to the Teukolsky functions ${}_s\Psi$. We define, for any solution ${}_sR$ of the homogeneous radial equation, the spin-inversion operators by

$$
{}_sU^{(r)}_{m\omega}\left[\ {}_sR^{\mathrm{E}}_{\omega lm} + \ {}_sR^{\mathrm{O}}_{\omega lm}\right] = \gamma_{s\omega lm}\left[{}_{-s}R^{\mathrm{E}}_{\omega lm} + \delta_{s\omega lm}\ {}_{-s}R^{\mathrm{O}}_{\omega lm}\right], \tag{6.149}
$$

where, from Eqs. (6.119) and (6.120),

$$
\begin{aligned}
{}_2U^{(r)} &= \Delta^2(\mathcal{D}_0^+)^4\Delta^2, \qquad {}_{-2}U^{(r)} = \mathcal{D}_0^4 && (6.150)\\
\gamma_s &= 2^s\ C^{-s/4+1/2}\ (C^*)^{s/4+1/2}, \qquad \delta_s = \left(\frac{C}{C^*}\right)^{s/2}, && (6.151)
\end{aligned}
$$

i.e. $\gamma_2 = 4C^*$, $\gamma_{-2} = C/4$. The operators ${}_sM^{cd}\ {}_s\tau^\dagger_{cd}$ given in Eqs. (6.51) – (6.52) can be expressed in terms of the radial spin-inversion operators defined in Eq. (6.149) as:

$$
\begin{aligned}
{}_sM^{ab}\ {}_s\tau^\dagger_{ab} &= 2^{s-2}\ {}_{-s}U^{(r)} && (6.152)\\
{}_sM^{ab}({}_s\tau^\dagger_{ab})^* &= 0. && (6.153)
\end{aligned}
$$

From these expressions, combined with Eq. (6.10) we can determine the constant A_s by requiring that we recover the properly normalized Teukolsky functions ${}_s\Psi$ when acting on ${}_s\Phi$ with the operator ${}_sM^{ab}\ {}_s\tau^\dagger_{ab}$. We find that

$$
\begin{aligned}
A^{\mathrm{E}}_s &= 2^{-s+2}\gamma_s^{-1}, && (6.154)\\
A^{\mathrm{O}}_s &= \delta_s^{-1}\ A^{\mathrm{E}}_s, && (6.155)
\end{aligned}
$$

i.e. $A^{\mathrm{E}}_2 = 4/C$ and $A^{\mathrm{E}}_{-2} = 4/C^*$.

6.4 Construction of the Green's functions for the Teukolsky variables

6.4.1 Formula for the retarded Green's function

The retarded Green's function ${}_sG_{\rm ret}(x, x')$ is defined such that if ${}_s\Psi$ obeys the Teukolsky equation (6.34) with source ${}_s\mathcal{T}$

$$
{}_s\Box\Psi = {}_s\mathcal{T}, \tag{6.156}
$$

then the retarded solution is

$$
{}_s\Psi_{\rm ret}(x) = \int d^4x' \sqrt{-g(x')}\ {}_sG_{\rm ret}(x, x')\ {}_s\mathcal{T}(x'). \tag{6.157}
$$

The expression for the retarded Green's function in terms of the complete mode functions defined in previous sections is

$$
\begin{aligned}
{}_sG_{\rm ret}(x, x') \quad &= \quad \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\omega \sum_{l=2}^{\infty}\sum_{m=-l}^{l} \frac{1}{\alpha_{s\omega lm}\beta_{s\omega lm}} \frac{1}{A^*_{s\omega lm}} \frac{\omega}{|\omega|} \\
&\left[{}_s\Psi^{\rm up}_{\omega lm}(x)\ {}_s\Phi^{\rm out}_{\omega lm}{}^*(x')\theta(r - r') + {}_s\Psi^{\rm in}_{\omega lm}(x)\ {}_s\Phi^{\rm down}_{\omega lm}{}^*(x')\theta(r' - r)\right].
\end{aligned} \tag{6.158}
$$

Here $\theta(x)$ is the step function, defined to be $+1$ for $x \geq 0$ and 0 otherwise.

Expression (6.158) can be expanded into more explicit form by using the definitions (6.146) and (6.147) of the mode functions ${}_s\Phi_{\omega lm}$ in terms of the radial mode functions, together with the definitions (6.100) and (6.101) of the "out" and "down" modes. This gives

$$
\begin{aligned}
{}_sG_{\rm ret}(x, x') \quad &= \quad \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\omega \sum_{l=2}^{\infty}\sum_{m=-l}^{l} \frac{1}{\alpha_{s\omega lm}\beta_{s\omega lm}} \frac{\omega}{|\omega|} e^{-i\omega(t-t')} \\
&{}_sS_{\omega lm}(\theta, \phi)\ {}_sS^*_{\omega lm}(\theta', \phi') \frac{1}{\varpi\varpi'}(\Delta\Delta')^{-s/2} \\
&\left[{}_su^{\rm up}_{\omega lm}(r)\ {}_su^{\rm in}_{\omega lm}(r')\theta(r - r') {}_su^{\rm in}_{\omega lm}(r)\ {}_su^{\rm up}_{\omega lm}(r')\theta(r' - r)\right].
\end{aligned} \tag{6.159}
$$

Note that the expression (6.159) is independent of the values chosen for the normalization constants $\alpha_{s\omega lm}$ and $\beta_{s\omega lm}$, since the factor of $1/\alpha$ cancels a factor of α present in the definition (6.91) of the "in" modes, and similarly for β and the "up" modes.

6.4.2 Derivation

We now discuss the derivation of the formula (6.159). Suppose that the source $\mathcal{T}(x)$ is non-zero only in the finite range of values of r

$$r_{\min} \leq r \leq r_{\max}. \tag{6.160}$$

Then, the retarded solution ${}_s\Psi_{\rm ret}(x)$ will be a solution of the homogeneous equation in the regions $r < r_{\min}$ and $r > r_{\max}$. Now, the retarded solution is determined uniquely by the condition that it vanish on the past event horizon E^- and on past null infinity $\mathcal{J}^-$. This property will be guaranteed if we impose the following boundary conditions:

1. When we expand ${}_s\Psi_{\rm ret}$ in the region $r < r_{\min}$ on the basis of solutions $e^{-i\omega t}{}_sS_{\omega lm}(\theta,\varphi)\ {}_sR^{\rm in}_{\omega lm}(x)$ and $e^{-i\omega t}{}_sS_{\omega lm}(\theta,\varphi)\ {}_sR^{\rm up}_{\omega lm}(x)$ of the homogeneous equation, only the "in" modes contribute. Then, since the "in" modes vanish on the past event horizon, ${}_s\Psi_{\rm ret}$ must also vanish on the past event horizon.
2. When we expand ${}_s\Psi_{\rm ret}$ in the region $r > r_{\max}$ on the basis of solutions $e^{-i\omega t}{}_sS_{\omega lm}(\theta,\varphi)\ {}_sR^{\rm in}_{\omega lm}(x)$ and $e^{-i\omega t}{}_sS_{\omega lm}(\theta,\varphi)\ {}_sR^{\rm up}_{\omega lm}(x)$, only the "up" modes contribute. Then, since the "up" modes vanish on past null infinity, ${}_s\Psi_{\rm ret}$ must also vanish on past null infinity.

We define the Fourier transformed quantities

$$ {}_s\tilde{\mathcal{T}}(\omega, r, \theta, \varphi) = \int_{-\infty}^{\infty} dt e^{i\omega t} {}_s\mathcal{T}(t, r, \theta, \varphi) \tag{6.161} $$

and

$$ {}_s\tilde{\Psi}(\omega, r, \theta, \varphi) = \int_{-\infty}^{\infty} dt e^{i\omega t} {}_s\Psi(t, r, \theta, \varphi). \tag{6.162} $$

For the remainder of this section we omit the subscript "ret" on ${}_s\Psi$. We make the following ansatz for the Green's function:

$$ {}_sG_{\rm ret}(x, x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} {}_s\tilde{G}_{\rm ret}(r, \theta, \varphi; r', \theta', \varphi'; \omega). \tag{6.163} $$

Inserting these definitions into the defining relation (6.157) and using $\sqrt{-g} = \Sigma dr dt d^2\Omega$ gives

$$ {}_s\tilde{\Psi}(\omega, r, \theta, \varphi) = \int_0^{\infty} dr' \int d^2\Omega' \; \Sigma(r', \theta') \; {}_s\tilde{G}_{\rm ret}(r, \theta, \varphi; r', \theta', \varphi'; \omega) \; {}_s\tilde{\mathcal{T}}(\omega, r', \theta', \varphi'). \tag{6.164} $$

Next, we decompose the quantities ${}_s\tilde{\Psi}$ and $\Sigma {}_s\tilde{\mathcal{T}}$ on the basis of spin-weighted spheroidal harmonics:

$$ {}_s\tilde{\Psi}(\omega, r, \theta, \varphi) = \sum_{lm} {}_sS_{\omega lm}(\theta, \varphi) \; {}_sR_{\omega lm}(r) \tag{6.165} $$

and

$$ \Sigma \; {}_s\tilde{\mathcal{T}}(\omega, r, \theta, \varphi) = r^2 \sum_{lm} {}_sS_{\omega lm}(\theta, \varphi) \; {}_s\tilde{\mathcal{T}}_{\omega lm}(r). \tag{6.166} $$

The factor of r^2 is included so that the coefficients ${}_s\tilde{\mathcal{T}}_{\omega lm}$ reduce to the conventional spin weighted spherical harmonic coefficients for $a = 0$. From the orthogonality relation (6.66), the inverse transformations are

$$ {}_sR_{\omega lm}(r) = \int d^2\Omega \, {}_sS^*_{\omega lm}(\theta, \varphi) \, {}_s\tilde{\Psi}(\omega, r, \theta, \varphi) \tag{6.167} $$

and

$$ r^2 \, {}_s\tilde{\mathcal{T}}_{\omega lm}(r) = \int d^2\Omega \, {}_sS^*_{\omega lm}(\theta, \varphi) \, \Sigma(r, \theta) {}_s\tilde{\mathcal{T}}(\omega, r, \theta, \varphi). \tag{6.168} $$

Next, we insert these decompositions into the Fourier transform of the differential equation (6.80) and include the source term

$$ {}_s s_{\omega lm} = \varpi^{-3}\Delta^{1+s/2}\ r^2\ {}_s\tilde{\mathcal{T}}_{\omega lm}. \tag{6.169}$$

This gives the inhomogeneous equation

$$ \frac{d^2\ {}_s u_{\omega lm}}{dr^{*2}} + {}_s V_{\omega lm}\ {}_s u_{\omega lm}(r^*) = {}_s s_{\omega lm}, \tag{6.170}$$

where

$$ {}_s u_{\omega lm}(r) = \Delta(r)^{s/2}\varpi\ {}_s R_{\omega lm}(r), \tag{6.171}$$

where the potential ${}_s V_{\omega lm}$ is given by Eq. (6.81). We denote by ${}_s G_{\omega lm}(r^*, r^{*\prime})$ the Green's function for the differential equation (6.170):

$$ {}_s u_{\omega lm}(r^*) = \int_{-\infty}^{\infty} dr^{*\prime}\ {}_s G_{\omega lm}(r^*, r^{*\prime})\ {}_s s_{\omega lm}(r^{*\prime}). \tag{6.172}$$

We note that we can express the Fourier-transformed retarded Green's function $\tilde{G}_{\rm ret}(r,\theta,\varphi;r',\theta',\varphi';\omega)$ in terms of $G_{\omega lm}$ as:

$$ {}_s\tilde{G}_{\rm ret}(x,x';\omega) = \sum_{lm} {}_s S_{\omega lm}(\theta,\varphi)\ {}_s S^*_{\omega lm}(\theta',\varphi')\frac{{}_s G^{\rm ret}_{\omega lm}(r^*, r^{*\prime})}{\Delta^{s/2}\Delta'^{s/2}\varpi\varpi'}. \tag{6.173}$$

We verify this by direct substitution of the ansatz (6.173) into the relation (6.164) and simplifying using Eqs. (6.66), (6.169), (6.172), and (6.171):

$$\begin{aligned}
{}_s\tilde{\Psi}(\omega,r,\theta,\varphi) &= \int_0^\infty dr' \int d^2\Omega' r'^2 \sum_{lm} {}_s S_{\omega lm}(\theta',\varphi')\ {}_s\tilde{\mathcal{T}}_{\omega lm}(r') \\
&\qquad \sum_{l'm'} {}_s S_{\omega l'm'}(\theta,\varphi)\ {}_s S^*_{\omega l'm'}(\theta',\varphi')\frac{{}_s G_{\omega l'm'}(r^*, r^{*\prime})}{\Delta^{s/2}\Delta'^{s/2}\varpi\varpi'} \\
&= \sum_{lm}\int_0^\infty r'^2 dr'\ {}_s S_{\omega lm}(\theta,\varphi)\ {}_s\tilde{\mathcal{T}}_{\omega lm}(r')\frac{{}_s G_{\omega l'm'}(r^*, r^{*\prime})}{\Delta^{s/2}\Delta'^{s/2}\varpi\varpi'} \\
&= \sum_{lm}\int_0^\infty r'^2\frac{\Delta'}{\varpi'^2}dr^{*\prime}\ {}_s S_{\omega lm}(\theta,\varphi)\frac{\varpi'^3}{r'^2}\Delta'^{-(1+s/2)}\ {}_s s_{\omega lm}\frac{{}_s G_{\omega lm}(r^*, r^{*\prime})}{\Delta^{s/2}\Delta'^{s/2}\varpi\varpi'} \\
&= \sum_{lm} {}_s S_{\omega lm}(\theta,\varphi)\frac{{}_s u_{\omega lm}(r^*)}{\Delta^{s/2}\varpi} \\
&= \sum_{lm} {}_s S_{\omega lm}(\theta,\varphi)\ {}_s R_{\omega lm}.
\end{aligned}$$

Comparing the result in the last line with the definition (6.165) shows that the ansatz (6.173) is correct.

We now derive the formula for the retarded Green's function ${}_sG_{\omega lm}(r^*, r^{*\prime})$. From the discussion at the beginning of this section, the relevant boundary conditions to impose are that

$$ {}_sG^{\rm ret}_{\omega lm}(r^*, r^{*\prime}) \propto {}_su^{\rm in}_{\omega lm}(r^*), \quad r^* \to -\infty \tag{6.174} $$

and

$$ {}_sG^{\rm ret}_{\omega lm}(r^*, r^{*\prime}) \propto {}_su^{\rm up}_{\omega lm}(r^*), \quad r^* \to \infty. \tag{6.175} $$

Consider now the expression

$$ \begin{aligned} {}_sG^{\rm ret}_{\omega lm}(r^*, r^{*\prime}) \quad &= \quad \frac{1}{W({}_su^{\rm in}_{\omega lm},\ {}_su^{\rm up}_{\omega lm})}\Big[{}_su^{\rm up}_{\omega lm}(r)\ {}_su^{\rm in}_{\omega lm}(r')\theta(r-r') \\ &\qquad + {}_su^{\rm in}_{\omega lm}(r)\ {}_su^{\rm up}_{\omega lm}(r')\theta(r-r')\Big], \end{aligned} \tag{6.176} $$

where W is the conserved Wronskian (6.106). This expression satisfies the boundary conditions (6.174) and (6.175) as well as the differential equation (6.170) with the source replaced by $\delta(r^* - r^{*\prime})$, using the fact that the "in" and "up" modes satisfy the homogeneous version of the differential equation. This establishes the formula (6.176).

Next, we compute the Wronskian $W({}_su^{\rm in}_{\omega lm}, {}_su^{\rm up}_{\omega lm})$ using the asymptotic expressions (6.91) and (6.95) for the mode functions for $r^* \to \infty$. This gives

$$ W({}_su^{\rm in}_{\omega lm},\ {}_su^{\rm up}_{\omega lm}) = 2i\alpha_{s\omega lm}\ \beta_{s\omega lm}\frac{\omega}{|\omega|}. \tag{6.177} $$

Then, the retarded Green's function for the differential equation (6.170) becomes:

$$ \begin{aligned} {}_sG_{\omega lm}(r^*, r^{*\prime}) \quad &= \quad \frac{1}{4\pi i}\frac{1}{\alpha_{s\omega lm}\ \beta_{s\omega lm}}\left[{}_su^{\rm up}_{\omega lm}(r)\ {}_su^{\rm in}_{\omega lm}(r')\theta(r-r')\right. \\ &\qquad \left. + {}_su^{\rm in}_{\omega lm}(r)\ {}_su^{\rm up}_{\omega lm}(r')\theta(r'-r)\right]. \end{aligned} \tag{6.178} $$

Inserting this into Eq. (6.176) and then into Eqs. (6.173) and (6.163) finally yields the formula (6.159).

Advanced Green's function

The definition of the advanced Green's function ${}_sG_{\rm adv}(x, x')$ is the analog of Eq. (6.157). The expression for the advanced Green's function is

$$\begin{aligned} {}_sG_{\rm adv}(x, x') &= \frac{-1}{4\pi i}\int_{-\infty}^{\infty} d\omega \sum_{l=2}^{\infty}\sum_{m=-l}^{l} \frac{1}{\alpha^*_{-s\omega lm}\ \beta^*_{-s\omega lm}} \frac{\omega}{|\omega|} e^{-i\omega(t-t')} \qquad (6.179) \\ & \qquad {}_sS_{\omega lm}(\theta,\varphi)\ {}_sS^*_{\omega lm}(\theta',\varphi') \frac{1}{\varpi\varpi'}(\Delta\Delta')^{-s/2} \\ & \qquad \left[{}_s u^{\rm down}_{\omega lm}(r')\ {}_s u^{\rm out}_{\omega lm}(r)\theta(r'-r) + {}_s u^{\rm out}_{\omega lm}(r)\ {}_s u^{\rm down}_{\omega lm}(r')\theta(r-r')\right]. \end{aligned}$$

In terms of the complete mode functions, this can be written as

$$\begin{aligned} {}_sG_{\rm adv}(x, x') = \frac{-1}{4\pi i}\int_{-\infty}^{\infty} d\omega \sum_{lm} \frac{\omega}{|\omega|} \frac{1}{\alpha^*_{-s\omega lm}\beta^*_{-s\omega lm}} \frac{1}{A^*_{-s\omega lm}} \qquad (6.180) \\ \left[\ {}_s\Psi^{\rm down}_{\omega lm}(x)\, {}_s\Phi^{\rm in\ *}_{\omega lm}(x')\theta(r-r') + {}_s\Psi^{\rm out}_{\omega lm}(x)\ {}_s\Phi^{\rm up\ *}_{\omega lm}(x')\theta(x'-x)\right]. \end{aligned}$$

Derivation

The advanced solution is determined uniquely by the condition that it vanish on the future horizon and on future null infinity. From Fig. (6.1), we see that the relevant basis of solutions is the "(out, down)" basis. We need to impose the following boundary conditions:

1. When we expand ${}_s\Psi_{\rm adv}$ in the region $r < r_{\rm min}$ on the basis of solutions $e^{-i\omega t}{}_sS_{\omega lm}(\theta,\varphi)\ {}_sR^{\rm out}_{\omega lm}(x)$ and $e^{-i\omega t}{}_sS_{\omega lm}(\theta,\varphi)\ {}_sR^{\rm down}_{\omega lm}(x)$ of the homogeneous equation, only the "out" modes contribute. Then, since the "out" modes

vanish on the future event horizon, ${}_s\Psi_{\rm adv}$ must also vanish on the future event horizon.

2. When we expand ${}_s\Psi_{\rm adv}$ in the region $r > r_{\rm max}$ on the basis of solutions $e^{-i\omega t}{}_sS_{\omega lm}(\theta,\varphi)\ {}_sR^{\rm out}_{\omega lm}(x)$ and $e^{-i\omega t}{}_sS_{\omega lm}(\theta,\varphi)\ {}_sR^{\rm down}_{\omega lm}(x)$, only the "down" modes contribute. Then, since the "down" modes vanish on future null infinity, ${}_s\Psi_{\rm adv}$ must also vanish on future null infinity.

Therefore, the advanced Green's function has to satisfy:

$$ {}_sG^{\rm adv}_{\omega lm}(r^*,r^{*\prime}) \propto {}_su^{\rm out}_{\omega lm}(r^*), \quad r^* \to -\infty \tag{6.181} $$

and

$$ {}_sG^{\rm adv}_{\omega lm}(r^*,r^{*\prime}) \propto {}_su^{\rm down}_{\omega lm}(r^*), \quad r^* \to \infty. \tag{6.182} $$

Consider now the expression

$$ \begin{aligned} {}_sG^{\rm adv}_{\omega lm}(r^*,r^{*\prime}) &= \frac{1}{W({}_su^{\rm out}_{\omega lm},\ {}_su^{\rm down}_{\omega lm})} \\ &\left[{}_su^{\rm down}_{\omega lm}(r)\ {}_su^{\rm out}_{\omega lm}(r')\theta(r-r') + {}_su^{\rm out}_{\omega lm}(r)\ {}_su^{\rm down}_{\omega lm}(r')\theta(r-r')\right]. \end{aligned} \tag{6.183} $$

This expression satisfies the boundary conditions (6.181) and (6.182) as well as the differential equation (6.170) with the source replaced by $\delta(r^* - r^{*\prime})$, using the fact that the "out" and "down" modes satisfy the homogeneous version of the differential equation. This establishes the formula (6.183).

A similar computation as for the "in" and "up" modes gives for the Wronskian

$$ W({}_su^{\rm out}_{\omega lm}, {}_su^{\rm down}_{\omega lm}) = -2i\alpha^*_{-s\omega lm}\ \beta^*_{-s\omega lm}\frac{\omega}{|\omega|}. \tag{6.184} $$

Using this is Eq. (6.183) yields the final result in Eq. (6.392).

Note that the advanced Green's function is simply obtained by applying the "bar" transformation to the retarded Green's function and taking the complex conjugate.

6.4.3 Construction of the radiative Green's function for the Teukolsky variables

Formula for the radiative Green's function

Using the retarded and advanced Green's function ${}_sG_{\rm ret}(x,x')$ and ${}_sG_{\rm adv}(x,x')$ discussed in the last sections we can construct the retarded and advanced solutions ${}_s\Psi_{\rm ret}(x)$ and ${}_s\Psi_{\rm adv}(x)$ of the Teukolsky equation (6.34). One half the retarded solution minus one half the advanced solution gives the radiative solution:

$$ {}_s\Psi^{\rm rad}(x) = \frac{1}{2}\left[{}_s\Psi^{\rm ret}(x) - {}_s\Psi^{\rm adv}(x)\right]. \tag{6.185}$$

Clearly the radiative solution is given in terms of a radiative Green's function

$$ {}_s\Psi_{\rm rad}(x) = \int d^4x' \sqrt{-g(x')}\ {}_sG_{\rm rad}(x,x')\ {}_s\mathcal{T}(x'), \tag{6.186}$$

where

$$ {}_sG_{\rm rad}(x,x') = \frac{1}{2}\left[{}_sG_{\rm ret}(x,x') - {}_sG_{\rm adv}(x,x')\right]. \tag{6.187}$$

The expression for the radiative Green's function in terms of the modes defined in Sec. III is [197]

$$\begin{aligned} {}_sG_{\rm rad}(x,x') = {} & \frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{\omega}{\mid \omega \mid} e^{-i\omega t}\frac{1}{A^*_{s\omega lm}} \\ & \left[\frac{1}{\alpha^*_{-s\omega lm}\alpha_{s\omega lm}}\,{}_s\Psi^{\rm out}_{\omega lm}(x)\ {}_s\Phi^{\rm out}_{\omega lm}{}^{*}(x')\right. \\ & \left. + \frac{1}{\beta_{s\omega lm}\beta^*_{-s\omega lm}}\frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\kappa_{s\omega m}\tau_{s\omega lm}\tau^*_{-s\omega lm}\ {}_s\Psi^{\rm down}_{\omega lm}(x)\ {}_s\Phi^{\rm down}_{\omega lm}{}^{*}(x')\right]. \end{aligned} \tag{6.188}$$

This expression can be expanded into more explicit form as

$$
\begin{aligned}
{}_sG_{\rm rad}(x,x') = \frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid} e^{-i\omega(t-t')} \sum_{lm} {}_sS_{\omega lm}(\theta,\varphi)\ {}_sS^*_{\omega lm}(\theta',\varphi') \\
\frac{(\Delta/\Delta')^{-s/2}}{\varpi\varpi'}\left[\frac{1}{\alpha^*_{-s\omega lm}\alpha_{s\omega lm}} {}_su^{\rm out}_{\omega lm}(r)\ {}_{-s}u^{\rm out\ *}_{\omega lm}(r')\right. \qquad (6.189)\\
\left. + \frac{1}{\beta_{s\omega lm}\beta^*_{-s\omega lm}} \frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\kappa_{s\omega m}\tau_{s\omega lm}\tau^*_{-s\omega lm}\ {}_su^{\rm down}_{\omega lm}(r)\ {}_{-s}u^{\rm down\ *}_{\omega lm}(r')\right].
\end{aligned}
$$

Note that this expression is actually independent of the values chosen for the normalization constants $\alpha_{s\omega lm}$ and $\beta_{s\omega lm}$, since the factor of $1/(\alpha_s\alpha^*_{-s})$ cancels factors of α_s present in the definition (6.100) of the "out" modes, and similarly for β_s and the "down" modes.

Derivation

In this subsection, we will again use the notation $\Lambda = \{\omega lm\}$ for convenience. Inserting the expressions (6.159), and (6.392) into Eq. (6.201) gives

$$
\begin{aligned}
&{}_sG^{\rm rad}(x,x') = \frac{1}{2}\left[{}_sG^{\rm ret}(x,x') - {}_sG^{\rm adv}(x,x')\right] \qquad (6.190)\\
&= \frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid} e^{-i\omega(t-t')} \sum_{lm} \frac{1}{\alpha_{s\Lambda}\beta_{s\Lambda}} {}_sS_\Lambda(\theta,\varphi)\ {}_sS^*_\Lambda(\theta',\varphi')\frac{(\Delta\Delta')^{-s/2}}{\varpi\varpi'}\\
&\times\left[{}_su^{\rm up}_\Lambda(r)\ {}_su^{\rm in}_\Lambda(r')\theta(r-r') + {}_su^{\rm in}_\Lambda(r)\ {}_su^{\rm up}_\Lambda(r')\theta(r'-r)\right]\\
&+\frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid} e^{-i\omega(t-t')} \sum_{lm} \frac{1}{\alpha^*_{-s\Lambda}\beta^*_{-s\Lambda}} {}_sS_\Lambda(\theta,\varphi)\ {}_sS^*_\Lambda(\theta',\varphi')\frac{(\Delta\Delta')^{s/2}}{\varpi\varpi'}\\
&\times\left[{}_{-s}u^{\rm up\ *}_\Lambda(r)\ {}_{-s}u^{\rm in\ *}_\Lambda(r')\theta(r-r') + {}_{-s}u^{\rm up\ *}_\Lambda(r')\ {}_{-s}u^{\rm in\ *}_\Lambda(r)\theta(r'-r)\right].\\
&= \frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid} e^{-i\omega(t-t')} \sum_{lm} {}_sS_\Lambda(\theta,\varphi)\ {}_sS^*_\Lambda(\theta',\varphi')\frac{(\Delta\Delta')^{-s/2}}{\varpi\varpi'} \qquad (6.191)\\
&\quad\left\{\left[\frac{1}{\alpha_{s\Lambda}\beta_{s\Lambda}} {}_su^{\rm up}_\Lambda(r)\ {}_su^{\rm in}_\Lambda(r') + \frac{1}{\alpha^*_{-s\Lambda}\beta^*_{-s\Lambda}} {}_{-s}u^{\rm up\ *}_\Lambda(r)\ {}_{-s}u^{\rm in\ *}_\Lambda(r')\right]\theta(r-r')\right.\\
&\quad\left. + \left[\frac{1}{\alpha_{s\Lambda}\beta_{s\Lambda}} {}_su^{\rm in}_\Lambda(r)\ {}_su^{\rm up}_\Lambda(r') + \frac{1}{\alpha^*_{-s\Lambda}\beta^*_{-s\Lambda}} {}_{-s}u^{\rm up\ *}_\Lambda(r')\ {}_{-s}u^{\rm in\ *}_\Lambda(r)\right]\theta(r'-r)\right\}.
\end{aligned}
$$

Consider now the coefficient of $\theta(r-r')$ inside the curly brackets in Eq. (6.191). We denote this quantity by $H_s(r,r')$:

$$H_s(r,r') = \frac{1}{\alpha_{s\Lambda}\beta_{s\Lambda}}{}_s u_\Lambda^{\rm up}(r)\ {}_s u_\Lambda^{\rm in}(r') + \frac{1}{\alpha^*_{-s\Lambda}\beta^*_{-s\Lambda}}{}_{-s}u_\Lambda^{\rm up\ *}(r)\ {}_{-s}u_\Lambda^{\rm in\ *}(r'). \qquad (6.192)$$

We solve the expression (6.111) for the "out" modes in terms of the "(in, up)" basis for ${}_s u_\Lambda^{\rm up}$:

$${}_s u_\Lambda^{\rm up} = \frac{\beta_{s\Lambda}}{\alpha^*_{-s\Lambda}}\frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\frac{1}{\kappa_{sm\omega}\ \tau_{s\Lambda}\ \tau^*_{-s\Lambda}}\ {}_{-s}u_\Lambda^{\rm in\ *} - \frac{\beta_{s\Lambda}\ \sigma^*{}_{-s\Lambda}}{\alpha_{s\Lambda}}\ {}_s u_\Lambda^{\rm in}. \qquad (6.193)$$

Substituting this into Eq. (6.192) gives

$$\begin{aligned} H_s(r,r') \ &= \ \frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\frac{1}{\kappa_{sm\omega}\ \tau_{s\Lambda}\ \tau^*_{-s\Lambda}}\Bigg[\frac{1}{\alpha_{s\Lambda}\alpha^*_{-s\Lambda}}{}_{-s}u_\Lambda^{\rm in*}(r)\ {}_s u_\Lambda^{\rm in}(r') \\ &\quad -\frac{\sigma^*_{-s\Lambda}}{\alpha^2_{s\Lambda}}\ {}_s u_\Lambda^{\rm in}(r)\ {}_s u_\Lambda^{\rm in}(r') + \frac{1}{\alpha_{s\Lambda}\alpha^*_{-s\Lambda}}{}_s u_\Lambda^{\rm in}(r)\ {}_{-s}u_\Lambda^{\rm in\ *}(r') \\ &\quad - \frac{\sigma_{s\Lambda}}{(\alpha^*_{-s\Lambda})^2}\ {}_{-s}u_\Lambda^{\rm in\ *}(r)\ {}_{-s}u_\Lambda^{\rm in\ *}(r')\Bigg]. \end{aligned} \qquad (6.194)$$

Next, we use Eq. (6.193) to evaluate

$$\begin{aligned} {}_{-s}u_\Lambda^{\rm up\ *}(r)\ {}_s u_\Lambda^{\rm up}(r') &= \frac{\beta_{s\Lambda}\beta^*_{-s\Lambda}}{(\kappa_{sm\omega}\ \tau_{s\Lambda}\ \tau^*_{-s\Lambda})^2}\left(\frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\right)^2 \\ &\Bigg[\frac{1}{\alpha_{s\Lambda}\alpha^*_{-s\Lambda}}{}_s u_\Lambda^{\rm in}(r)\ {}_{-s}u_\Lambda^{\rm in\ *}(r') + \frac{1}{\alpha_{s\Lambda}\alpha^*_{-s\Lambda}}\sigma_{s\Lambda}\ \sigma^*_{-s\Lambda}\ {}_{-s}u_\Lambda^{\rm in\ *}(r)\ {}_s u_\Lambda^{\rm in}(r') \\ &\quad -\frac{\sigma^*_{-s\Lambda}}{(\alpha_{s\Lambda})^2}{}_s u_\Lambda^{\rm in}(r)\ {}_s u_\Lambda^{\rm in}(r') - \frac{\sigma_{s\Lambda}}{(\alpha^*_{-s\Lambda})^2}{}_{-s}u_\Lambda^{\rm in\ *}(r)\ {}_{-s}u_\Lambda^{\rm in\ *}(r')\Bigg]. \end{aligned}$$

Using the unitarity condition in Eq. (6.109), $H_s(r,r')$ can be written as

$$\begin{aligned} H_s(r,r') \ &= \ \frac{1}{\alpha_{s\Lambda}\alpha^*_{-s\Lambda}}{}_{-s}u_\Lambda^{\rm in\ *}(r)\ {}_s u_\Lambda^{\rm in}(r') \\ &\quad +\frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\frac{\kappa_{sm\omega}\ \tau_{s\Lambda}\ \tau^*_{-s\Lambda}}{\beta_{s\Lambda}\beta^*_{-s\Lambda}}{}_{-s}u_\Lambda^{\rm up\ *}(r)\ {}_s u_\Lambda^{\rm up}(r') \qquad (6.195) \\ &= \ \frac{1}{\alpha_{s\Lambda}\alpha^*_{-s\Lambda}}{}_s u_\Lambda^{\rm out}(r)\ {}_{-s}u_\Lambda^{\rm out\ *}(r') \\ &\quad +\frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\frac{\kappa_{sm\omega}\ \tau_{s\Lambda}\ \tau^*_{-s\Lambda}}{\beta_{s\Lambda}\beta^*_{-s\Lambda}}{}_s u^{\rm down}(r)\ {}_{-s}u^{\rm down\ *}(r'). \qquad (6.196) \end{aligned}$$

Now the right hand side of Eq. (6.195) is explicitly invariant under the combined transformations of interchanging r and r' and taking the complex conjugate together with interchanging $s \to -s$. However, from the definition (6.192) of $H_s(r,r')$, the left hand side is invariant under combined complex conjugation and spin weight inversion. It follows that both sides of Eq. (6.195) are symmetric under interchange of r and r':

$$H(r,r') = H(r',r) \tag{6.197}$$

and also satisfy

$$H_s(r,r') = H_{-s}(r,r')^*. \tag{6.198}$$

Next, the quantity inside the curly brackets in the expression (6.191) for $G_{\rm rad}$ is

$$H_s(r,r')\theta(r-r') + H_s(r',r)\theta(r'-r). \tag{6.199}$$

Using the symmetry property (6.197) together with $\theta(r-r') + \theta(r'-r) = 1$, this can be written simply as $H_s(r,r')$. Therefore we can replace the expression in curly brackets in (6.191) with the expression (6.192) for $H_s(r,r')$. This gives the final expression for the Green's function that contains no step function:

$$\begin{aligned}
{}_sG^{\rm rad}(x,x') = \frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid \omega \mid} e^{-i\omega(t-t')} \sum_{lm} {}_sS_{\omega lm}(\theta,\varphi)\ {}_sS_{\omega lm}(\theta',\varphi')^* \\
\frac{(\Delta/\Delta')^{-s/2}}{\varpi\varpi'} \left[\frac{1}{\alpha^*_{-s\omega lm}\alpha_{s\omega lm}} {}_su^{\rm out}_{\omega lm}(r)\ {}_{-s}u^{\rm out}_{\omega lm}{}^*(r') \right. \\
\left. + \frac{1}{\beta_{s\omega lm}\beta^*_{-s\omega lm}} \frac{\omega p_{m\omega}}{|\omega p_{m\omega}|} \kappa_{sm\omega}\ \tau_{s\omega lm}\ \tau^*_{-s\omega lm}\ {}_su^{\rm down}_{\omega lm}(r)\ {}_{-s}u^{\rm down}_{\omega lm}{}^*(r') \right]
\end{aligned} \tag{6.200}$$

We depart from Gal'tsov [197] in the derivation as follows. He considers the Green's function for the metric perturbation, while the result of this section is for the Teukolsky function (we will obtain from this the tensor Green's function in the following section). We directly substitute the form of the advanced Green's function (6.392) determined in the last section, while Gal'tsov uses the reciprocity relation $G_{\rm adv}(x,x') = G_{\rm ret}(x',x)$. To simplify the quantity that is analogous to our coefficient $H(r,r')$, he uses the "denormalized" radial mode functions, so that his manipulations on this quantity do not involve the constants α_s and β_s. Both approaches lead to the same final result.

6.4.4 The inhomogeneous potentials

The retarded and radiative fields ${}_s\Psi$

Using the expression (6.159) for the retarded Green's function together with the integral expression (6.157), we can compute the retarded field ${}_s\Psi_{\rm ret}(x)$ generated by the source ${}_s\mathcal{T}(x)$. For the case we are interested in, ${}_s\mathcal{T}(x)$ will be nonzero only in a finite range of values of r of the form

$$r_{\rm min} \le r \le r_{\rm max}. \tag{6.201}$$

For $r > r_{\rm max}$ only the first term in the square brackets in Eq. (6.159) will contribute, and the function $\theta(r-r')$ will always be 1. This gives, using the definitions (6.157) and (6.135):

$$\begin{aligned}
{}_s\Psi_{\rm ret}(x) &= \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid} e^{-i\omega t}\sum_{lm}\frac{1}{\alpha_{s\omega lm}\beta_{s\omega lm}}\,{}_sR^{\rm up}_{\omega lm}(r)\ {}_sS_{lm\omega}(\theta,\varphi)e^{-i\omega t} \\
&\quad \frac{1}{A^*_{s\omega lm}}\int d^4x'\sqrt{-g}\ {}_s\Phi^{\rm out}_{\omega lm}{}^*(x')\ {}_s\tau_{ab}(x')T^{ab}(x'), \\
&= \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid}\sum_{lm}\frac{1}{\alpha_{s\omega lm}\beta_{s\omega lm}}\,{}_sR^{\rm up}_{\omega lm}(r)\ {}_sS_{lm\omega}(\theta,\varphi)e^{-i\omega t} \\
&\quad \frac{1}{A^*_{s\omega lm}}\langle {}_s\tau^\dagger_{ab}(x')\ {}_s\Phi^{\rm out}_{\omega lm}(x')\ T^{ab}\rangle \\
&= \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid}\,{}_sZ^{\rm out}_{\omega lm}\ {}_s\Psi^{\rm up}_{\omega lm}(x), \qquad r > r_{\rm max},
\end{aligned} \tag{6.202}$$

where the amplitude ${}_sZ^{\rm out}_{\omega lm}$ is given by the the following inner product:

$$ {}_sZ^{\rm out}_{\omega lm} = \frac{1}{A^*_{s\omega lm}} \langle {}_s\tau^\dagger_{ab}\ {}_s\Phi^{\rm out}_{\omega lm},\ T^{ab}\rangle. \tag{6.203}$$

Similarly for $r < r_{\rm min}$ we obtain

$$ {}_s\Psi_{\rm ret}(x) = \frac{1}{4\pi i}\frac{1}{\alpha_s\beta_s}\sum\int d\omega \frac{\omega}{|\omega|}{}_sZ^{\rm down}_{\omega lm}\,{}_s\Psi^{\rm in}(x), \qquad r < r_{\rm min}, \tag{6.204}$$

where

$$ {}_sZ^{\rm down}_{\omega lm} = \frac{1}{A^*_{s\omega lm}} \langle {}_s\tau^\dagger_{ab}\ {}_s\Phi^{\rm down}_{\omega lm},\ T^{ab}\rangle. \tag{6.205}$$

Then, from the expression (6.201) for the radiative Green's function, together with Eq. (6.185), we obtain the radiative field:

$$\begin{aligned} {}_s\Psi_{\rm rad}(x) \quad = \quad & \frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid}\sum_{l=2}^{\infty}\sum_{m=l}^{l}\Bigg[\frac{1}{\alpha^*_{-s\omega lm}\ \alpha_{s\omega lm}}\ {}_sZ^{\rm out}_{\omega lm}\ {}_s\Psi^{\rm out}_{\omega lm}(x) \qquad (6.206)\\ & + \frac{1}{\beta_{s\omega lm}\ \beta^*_{-s\omega lm}}\frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\kappa_{s\omega m}\ \tau_{s\omega lm}\ \tau^*_{-s\omega lm}\ \ {}_sZ^{\rm down}_{\omega lm}\ {}_s\Psi^{\rm down}_{\omega lm}(x)\Bigg]. \end{aligned}$$

All of these expressions depend on the amplitudes ${}_sZ^{\rm out}_{\omega lm}$ and ${}_sZ^{\rm down}_{\omega lm}$.

Radiative metric perturbation

The important property of the radiative metric perturbation $h^{\rm rad}_{ab}$ is that it is a solution to the linearized Einstein equation in vacuum. We can therefore use the results of Secs. (II) and (III E) to construct $h^{\rm rad}_{ab}$ from the Teukolsky functions. From Eq. (6.206), we can write down the formula for the radiative potential ${}_s\Phi_{\rm rad}$:

$$\begin{aligned} {}_s\Phi_{\rm rad}(x) \quad = \quad & \frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid}\sum_{l=2}^{\infty}\sum_{m=l}^{l}\Bigg[\frac{1}{\alpha^*_{-s\omega lm}\ \alpha_{s\omega lm}}\ {}_sZ^{\rm out}_{\omega lm}\ {}_s\Phi^{\rm out}_{\omega lm}(x) \qquad (6.207)\\ & + \frac{1}{\beta_{s\omega lm}\ \beta^*_{-s\omega lm}}\frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\kappa_{s\omega m}\ \tau_{s\omega lm}\ \tau^*_{-s\omega lm}\ \ {}_sZ^{\rm down}_{\omega lm}\ {}_s\Phi^{\rm down}_{\omega lm}(x)\Bigg]. \end{aligned}$$

Acting on Eq. (6.207) with the operator ${}_s\tau^\dagger_{ab}(x)$ gives the radiative metric perturbation:

$$h^{\rm rad}_{ab}(x) = \frac{1}{8\pi i}\int_{-\infty}^{\infty} d\omega \frac{\omega}{\mid\omega\mid}\sum_{l=2}^{\infty}\sum_{m=l}^{l}\left[\frac{1}{\alpha^*_{-s\omega lm}\ \alpha_{s\omega lm}}\ {}_sZ^{\rm out}_{\omega lm}\ h^{\rm out}_{ab\ \omega lm}(x)\right.$$
$$\left. + \frac{1}{\beta_{s\omega lm}\ \beta^*_{-s\omega lm}}\frac{\omega p_{m\omega}}{|\omega p_{m\omega}|}\kappa_{s\omega m}\ \tau_{s\omega lm}\ \tau^*_{-s\omega lm}\ \ {}_sZ^{\rm down}_{\omega lm}\ h^{\rm down}_{ab\ \omega lm}(x)\right]. \quad (6.208)$$

Here, we have defined the mode functions

$$h^{\rm out}_{ab\ \omega lm}(x) = {}_s\tau^\dagger_{ab}(x)\ {}_s\Phi^{\rm out}_{\omega lm}(x), \quad (6.209)$$

$$h^{\rm down}_{ab\ \omega lm}(x) = {}_s\tau^\dagger_{ab}(x)\ {}_s\Phi^{\rm down}_{\omega lm}(x). \quad (6.210)$$

One can verify that acting on Eq. (6.208) with ${}_sM^{ab}(x)$ reproduces Eq. (6.206). As discussed at the end of Sec. II A, the subscript s that is present in Eq. (6.208) serves to indicate which gauge we are using to compute h_{ab}. The final result for the Carter constant evolution will be expressed in terms of inner products and will be gauge independent.

Note here that using Eqs. (6.209) and (6.210), we can write the amplitudes ${}_sZ^{\rm out,down}_{\omega lm}$ as

$${}_sZ^{\rm out,down}_{\omega lm} = \frac{1}{A^*_{s\omega lm}}\langle h^{\rm out,down}_{ab\ \omega lm},\ T^{ab}\rangle. \quad (6.211)$$

6.4.5 Harmonic decomposition of the amplitudes

Geodesic Motion

The equations of geodesic motion in Kerr decouple if we use the Mino time parameter λ, which is related to proper time τ by

$$d\lambda = \frac{1}{\Sigma}d\tau. \quad (6.212)$$

The equations of motion are

$$\left(\frac{dr}{d\lambda}\right)^2 = V_r(r) = \left[E\varpi^2 - aL_z\right]^2 - \Delta\left(r^2 + K\right), \tag{6.213}$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = V_\theta(\theta) = K - \frac{L_z^2}{\sin^2\theta} - a^2 E\cos 2\theta, \tag{6.214}$$

$$\left(\frac{d\varphi}{d\lambda}\right) = V_{\varphi r}(r) + V_{\varphi\theta}(\theta) = -\frac{a^2 L_z}{\Delta} + aE\left(\frac{\varpi^2}{\Delta} - 1\right) + \frac{L_z}{\sin^2\theta}. \tag{6.215}$$

The parameters t and λ are related by:

$$\frac{dt}{d\lambda} = V_{tr}(r) + V_{t\theta}(\theta) = E\left[\frac{\varpi^4}{\Delta} - a^2\sin^2\theta\right] + aL_z\left(1 - \frac{\varpi^2}{\Delta}\right) \tag{6.216}$$

It follows from Eqs. (6.213) and (6.214) that the functions $r(\lambda)$ and $\theta(\lambda)$ are periodic; and we denote their periods by Λ_r and Λ_θ. We define the fiducial motion associated with the constants of motion E, L_z and K to be the motion with the initial conditions $r(0) = r_{\min}$ and $\theta(0) = \theta_{\min}$, where $r_{\min}$ and $\theta_{\min}$ are given by the minimum values of r and θ for which the right-hand sides of Eqs. (6.213) and (6.214) vanish. The functions $\hat{r}(\lambda)$ and $\hat{\theta}(\lambda)$ associated with this fiducial motion are given by

$$\int_{r_{\min}}^{\hat{r}(\lambda)} \frac{dr}{\pm\sqrt{V_r(r)}} = \lambda, \tag{6.217}$$

$$\int_{\theta_{\min}}^{\hat{\theta}(\lambda)} \frac{d\theta}{\pm\sqrt{V_\theta(\theta)}} = \lambda. \tag{6.218}$$

From Eq. (6.216) it follows that

$$t(\lambda) = t_0 + \int_0^\lambda dt'\left(V_{tr}[r(t')] + V_{t\theta}[\theta(t')]\right), \tag{6.219}$$

where $t_0 = t(0)$. Next, we define the constant Γ to be the following average value:

$$\Gamma = \frac{1}{\Lambda_r}\int_0^{\Lambda_r} dt' V_{tr}[\hat{r}(t')] + \frac{1}{\Lambda_\theta}\int_0^{\Lambda_\theta} dt' V_{t\theta}[\hat{\theta}(t')]. \tag{6.220}$$

Then we can write $t(\lambda)$ as a sum of a linear term and terms that are periodic:

$$t(\lambda) = t_0 + \Gamma\lambda + \delta t(\lambda), \tag{6.221}$$

where $\delta t(\lambda)$ denotes the oscillatory terms in Eq. (6.219).

To average a function over the time parameter λ, it is convenient to parameterize r and θ in terms of angular variables as follows (this parametrization was first introduced by Hughes [189]). For the average over θ we introduce the parameter $\chi = \chi(\lambda)$ by

$$\cos^2 \hat{\theta}(\lambda) = z_- \cos^2 \chi, \tag{6.222}$$

where $z_- = \cos^2 \theta_-$ with z_- being the smaller root of Eq. (6.214) and where $\beta = a^2(1 - E^2)$. Then from the definition (6.218) of $\hat{\theta}$ together with Eq. (6.214) and the requirement that χ increases monotonically with λ we obtain

$$\frac{d\chi}{d\lambda} = \sqrt{\beta \left(z_+ - z_- \cos^2 \chi\right)}. \tag{6.223}$$

Then we can write the average over λ of a function $F_\theta(\lambda)$ which is periodic with period Λ_θ in terms of χ as

$$\begin{aligned} \langle F_\theta \rangle_\lambda &= \frac{1}{\Lambda_\theta} \int_0^{\Lambda_\theta} d\lambda F_\theta(\lambda) \\ &= \frac{1}{\Lambda_\theta} \int_0^{2\pi} d\chi \frac{F_\theta[\lambda(\chi)]}{\sqrt{\beta \left(z_+ - z_- \cos^2 \chi\right)}}, \end{aligned} \tag{6.224}$$

where

$$\Lambda_\theta = \int_0^{2\pi} d\chi \frac{1}{\sqrt{\beta \left(z_+ - z_- \cos^2 \chi\right)}}. \tag{6.225}$$

Similarly, to average a function $F_r(\lambda)$ that is periodic with period Λ_r, we introduce a parameter ξ via

$$r = \frac{p}{1 + e \cos \xi}, \tag{6.226}$$

where the parameter ξ varies from 0 to 2π as r goes through a complete cycle. Then,

$$\frac{d\xi}{d\lambda} = P(\xi), \tag{6.227}$$

$$P(\xi) \equiv \left(V_r[r(\xi)]\right)^{1/2} \left[\frac{pe \sin \xi}{\left(1 + e \cos \xi\right)^2}\right]^{-1} \tag{6.228}$$

The average over λ of $F_r(\hat{t})$ can then be computed from

$$\langle F_r \rangle_\lambda = \frac{\int_0^{2\pi} d\xi\, F_r/P(\xi)}{\int_0^{2\pi} d\xi/P(\xi)}. \tag{6.229}$$

Now, a generic function $F_{r,\theta}[r(\lambda), \theta(\lambda)]$ will be biperiodic in λ: $F_{r,\theta}[r(\lambda+\Lambda_r), \theta(\lambda+\Lambda_\theta)] = F_{r,\theta}[\tilde{r}(\lambda), \tilde{\theta}(\lambda)]$. Combining the results (6.224) and (6.229) we can write its average as a double integral over χ and ξ as

$$\langle F_{r,\theta} \rangle_\lambda = \frac{1}{\Lambda_\theta \Lambda_r} \int_0^{2\pi} d\chi \int_0^{2\pi} d\xi \frac{F_{r,\theta}[r(\xi), \theta(\chi)]}{\sqrt{\beta\,(z_+ - z_- \cos^2\chi)}P(\xi)}. \tag{6.230}$$

We will use these results for the averages below to compute the time derivatives of the constants of motion in the adiabatic limit.

Amplitudes

For the case considered here where the source is a point particle on a bound geodesic orbit $z(\tau)$, the energy-momentum tensor is

$$T^{ab}(x) = \mu \int d\tau \frac{u^a u^b}{\sqrt{-g}} \delta^{(4)}(x - z(\tau)). \tag{6.231}$$

For bound geodesics, the amplitudes ${}_sZ^{\rm out}_{\omega lm}$ and ${}_sZ^{\rm down}_{\omega lm}$ can be expressed as discrete sums over delta functions:

$${}_sZ^{\rm out/down}_{\omega lm} = \sum_{k,n=-\infty}^{\infty} {}_sZ^{\rm out/down}_{lmkn} \delta(\omega - \omega_{mkn}). \tag{6.232}$$

Here,

$$\omega_{mkn} = m\Omega_\varphi + k\Omega_\theta + n\Omega_r; \qquad \Omega_\varphi = \frac{\langle V_\varphi \rangle_\lambda}{\Gamma}, \quad \Omega_\theta = \frac{2\pi}{\Lambda_\theta \Gamma}, \quad \Omega_r = \frac{2\pi}{\Lambda_r \Gamma}. \tag{6.233}$$

The formula for the coefficients ${}_sZ_{lmkn}$ is

$$\begin{aligned} {}_sZ^{\rm out}_{lmkn} \;\; = \;\; & \frac{2\pi\mu}{\Gamma \Lambda_r \Lambda_\theta} e^{-im\varphi_0} e^{i\omega_{mkn} t_0} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta \; e^{i\Gamma(k\Omega_\theta \lambda_\theta + n\Omega_r)} e^{-im\Delta\varphi_r(\lambda_r)} \\ & \times e^{-im\Delta\varphi_\theta(\lambda_\theta)} e^{i\omega_{mkn}\Delta t_r(\lambda_r)} e^{i\omega_{mkn}\Delta t_\theta(\lambda_\theta)} \Sigma[r(\lambda_r), \theta(\lambda_\theta)] \\ & \times {}_sR^{\rm in}_{\omega lm}[r(\lambda_r)] \; {}_s\Theta^*_{\omega lm}[\theta(\lambda_\theta)] \; {}_s\tau_{ab} u^a u^b [r(\lambda_r), \theta(\lambda_\theta)]. \end{aligned} \tag{6.234}$$

6.4.6 Derivation

We start by inserting the expression (6.231) of the source ${}_s\mathcal{T}(x)$ into the definition of ${}_sZ^{\rm out}_{\omega lm}$ given by Eqs. (6.203) and (6.135). This gives an expression consisting of an integral along the geodesic of the mode function:

$$
{}_sZ^{\rm out}_{\omega lm} = \mu \int d\tau \; {}_sR^{\rm in}_{\omega lm}[r(\tau)] \; {}_sS^{*}_{\omega lm}[\theta(\tau),\varphi(\tau)] e^{i\omega t(\tau)} \; {}_s\tau_{ab} u^a u^b [t(\tau), r(\tau), \theta(\tau), \varphi(\tau)]. \tag{6.235}
$$

We next change the variable of integration from proper time τ to Mino time λ and use that

$$
\begin{aligned}
t(\lambda) &= t_0 + \Gamma\lambda + \Delta t(\lambda) && (6.236)\\
\varphi(\lambda) &= \varphi_0 + \langle V_\varphi \rangle \lambda + \Delta\varphi(\lambda) && (6.237)
\end{aligned}
$$

Then,

$$
\begin{aligned}
{}_sZ^{\rm out}_{lm\omega} = \mu e^{-im\varphi_0} e^{i\omega t_0} \int_{-\infty}^{\infty} d\lambda e^{i\lambda(\omega\Gamma - m\langle V_\varphi\rangle)} \Sigma[r(\lambda),\theta(\lambda)] e^{-im\Delta\varphi_r} e^{-im\Delta\varphi_\theta} \\
e^{i\omega\Delta t_r} e^{i\omega\Delta t_\theta} \; {}_{-s}R^{\rm out}_{\omega lm}{}^{*}[r(\lambda)] \; {}_s\Theta^{*}_{\omega lm}[\theta(\lambda)] \; {}_s\tau_{ab} u^a u^b [r(\lambda),\theta(\lambda)].
\end{aligned} \tag{6.238}
$$

We now define the function of two variables

$$
\begin{aligned}
{}_sJ_{\omega lm}(\lambda_r,\lambda_\theta) = \mu e^{-im\varphi_0} e^{i\omega t_0} e^{-im\Delta\varphi_r(\lambda_r)} e^{-im\Delta\varphi_\theta(\lambda_\theta)} e^{i\omega\Delta t_r(\lambda_r)} e^{i\omega\Delta t_\theta(\lambda_\theta)} \\
\Sigma[r(\lambda_r),\theta(\lambda_\theta)] \; {}_{-s}R^{\rm out}_{\omega lm}{}^{*}[r(\lambda)] \; {}_s\Theta^{*}_{\omega lm}[\theta(\lambda)] \; {}_s\tau_{ab} u^a u^b [r(\lambda),\theta(\lambda)].
\end{aligned} \tag{6.239}
$$

When this function is evaluated at $\lambda_r = \lambda_\theta = \lambda$, the "out" amplitude is given by

$$
{}_sZ^{\rm out}_{\omega lm} = \int d\lambda {}_sJ_{\omega lm}(\lambda,\lambda) e^{i(\Gamma\omega - m\langle V_\varphi\rangle)}. \tag{6.240}
$$

Note that the function ${}_sJ_{\omega lm}(\lambda_r,\lambda_\theta)$ is biperiodic in r and θ:

$$
{}_sJ_{\omega lm}(\lambda_r + \Lambda_r, \lambda_\theta) = {}_sJ_{\omega lm}(\lambda_r,\lambda_\theta), \qquad {}_sJ_{\omega lm}(\lambda_r, \lambda_\theta + \Lambda_\theta) = {}_sJ_{\omega lm}(\lambda_r,\lambda_\theta) \tag{6.241}
$$

Therefore, the function ${}_sJ_{\omega lm}$ can be expanded in a double Fourier series:

$$ {}_sJ_{\omega lm}(\lambda_r, \lambda_\theta) = \sum_{k,n=-\infty}^{\infty} {}_sJ_{\omega lmkn} e^{-i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r\lambda_r)}, \tag{6.242} $$

where the coefficients ${}_sJ_{\omega lmkn}$ are given by

$$ {}_sJ_{\omega lmkn} = \frac{1}{\Lambda_r\Lambda_\theta} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta e^{i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r\lambda_r)} {}_sJ_{\omega lm}(\lambda_r\lambda_\theta). \tag{6.243} $$

Inserting the Fourier series (6.242) evaluated at $\lambda_r = \lambda_\theta = \lambda$ in the definition of ${}_sZ^{\text{out}}$ gives

$$ {}_sZ^{\text{out}}_{\omega lm} = \sum_{kn} \int d\lambda e^{i\Gamma(\omega - m\Omega_\varphi - k\Omega_\theta - n\Omega_r)} {}_sJ_{\omega lmkn} \tag{6.244} $$

$$ = \sum_{kn} \frac{2\pi}{\Gamma} \delta(\omega - \omega_{mkn}) {}_sJ_{\omega lmkn}. \tag{6.245} $$

Note that it follows from the harmonic decomposition (6.232) that for geodesic sources, the continuous frequency ω and the discrete indices l, m are replaced with the four discrete indices k, n, l, and m. In this context the operation

$$ \omega \to -\omega, \qquad m \to -m, \qquad l \to l \tag{6.246} $$

associated with the symmetries of the functions ${}_sR_{\omega lm}$ and ${}_sS_{\omega lm}$ is replaced by the operation

$$ k \to -k, \qquad n \to -n, \qquad m \to -m, \qquad l \to l. \tag{6.247} $$

Dependence of the amplitudes on parameters of the geodesic

This dependence is derived and explained in detail in [40], we cite the result here. The parameters characterizing the geodesic are E, L_z, Q, t_0, φ_0, λ_{r0}, $\lambda_{\theta 0}$. We write the dependence of the amplitude on these parameters as

$$ Z^{\text{out}}_{lmkn} = Z^{\text{out}}_{lmkn}(E, L_z, Q, t_0, \varphi_0, \lambda_{r0}, \lambda_{\theta 0}). \tag{6.248} $$

For the fiducial geodesic associated with $\mathcal{E} = E,\ L_z,$ or Q, we have $t_0 = \varphi_0 = \lambda_{r0} = \lambda_{\theta 0} = 0$. For this case we can simplify the formula (6.234) by setting t_0 and φ_0 to zero, by replacing the motions $r(\lambda)$ and $\theta(\lambda)$ with the fiducial motions $\hat{r}(\lambda)$ and $\hat{\theta}(\lambda)$, and by replacing the functions Δt_r, Δt_θ, $\Delta\varphi_r$ and $\Delta\varphi_\theta$ with the functions $\hat{t}_r$, $\hat{t}_\theta$, $\hat{\varphi}_r$, and $\hat{\varphi}_\theta$. This yields

$$\begin{aligned}
{}_sZ^{\rm out}_{lmkn}(E, L_z, Q, 0, 0, 0, 0) &= \frac{2\pi\mu}{\Gamma\Lambda_r\Lambda_\theta}\int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta\ e^{i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r)} e^{-im\hat{\varphi}_r(\lambda_r)} \\
& e^{-im\hat{\varphi}_\theta(\lambda_\theta)} e^{i\omega_{mkn}\hat{t}_r(\lambda_r)} e^{i\omega_{mkn}\hat{t}_\theta(\lambda_\theta)} \Sigma[\hat{r}(\lambda_r), \hat{\theta}(\lambda_\theta)]\ {}_s\Theta_{\omega_{mkn}lm}[\hat{\theta}(\lambda_\theta)]^* \\
& {}_{-s}R^{\rm out *}_{\omega_{mkn}lm}[\hat{r}(\lambda_r)]\ {}_s\tau_{ab}u^a u^b[\hat{r}(\lambda_r), \hat{\theta}(\lambda_\theta)]. \qquad (6.249)
\end{aligned}$$

The term ${}_s\tau_{ab}u^a u^b$ can be expanded into more explicit form by using the expressions (6.40) or (6.41) for the operator ${}_s\tau_{ab}$ and writing the four-velocity in terms of the components on the tetrad. The results are known explicitly, and can be found, e.g. in Drasco and Hughes.

For more general geodesics, the amplitude $Z^{\rm out}_{lmkn}$ depends on the parameters t_0, φ_0, λ_{r0} and $\lambda_{\theta 0}$ only through an overall phase. We have

$$\begin{aligned}
{}_sZ^{\rm out}_{lmkn}(E, L_z, Q, 0, 0, 0, 0) &= \frac{2\pi\mu}{\Gamma\Lambda_r\Lambda_\theta}\int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta\ e^{i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r)} e^{-im\hat{\varphi}_r(\lambda_r)} \\
& \times\ e^{-im\hat{\varphi}_\theta(\lambda_\theta)} e^{i\omega_{mkn}\hat{t}_r(\lambda_r)} e^{i\omega_{mkn}\hat{t}_\theta(\lambda_\theta)} \Sigma[\hat{r}(\lambda_r), \hat{\theta}(\lambda_\theta)]\ {}_s\Theta_{\omega_{mkn}lm}[\hat{\theta}(\lambda_\theta)]^* \\
& \times\ {}_{-s}R^{\rm out *}_{\omega_{mkn}lm}[\hat{r}(\lambda_r)]\ {}_s\tau_{ab}u^a u^b[\hat{r}(\lambda_r), \hat{\theta}(\lambda_\theta)]. \qquad (6.250)
\end{aligned}$$

and thus

$${}_sZ^{\rm out}_{lmkn}(E, L_z, Q, t_0, \varphi_0, \lambda_{r0}, \lambda_{\theta 0}) = e^{i\chi_{lmkn}(t_0,\varphi_0,\lambda_{r0},\lambda_{\theta 0})} Z^{\rm out}_{lmkn}(E, L_z, Q, 0, 0, 0, 0), \qquad (6.251)$$

where

$$\begin{aligned}
\chi_{lmkn}(t_0, \varphi_0, \lambda_{r0}, \lambda_{\theta 0}) &= \Gamma\left[k\Omega_\theta\lambda_{\theta 0} + n\Omega_r\lambda_{r0} + m\left(\hat{\varphi}_r(-\lambda_{r0}) + \hat{\varphi}_\theta(-\lambda_{\theta 0}) - \varphi_0\right)\right] \\
& -\omega_{mkn}\left[\hat{t}_r(-\lambda_{r0}) + \hat{t}_\theta(-\lambda_{\theta 0}) - t_0\right]. \qquad (6.252)
\end{aligned}$$

This formula can be derived by substituting the expressions given in Sec. (IVB) for the functions Δt_r, Δt_θ, $\Delta\varphi_r$ and $\Delta\varphi_\theta$ into Eq. (6.232), making the changes of variables in the integral

$$\lambda_r \to \tilde{\lambda}_r = \lambda_r - \lambda_{r0}, \qquad \lambda_\theta \to \tilde{\lambda}_\theta = \lambda_\theta - \lambda_{\theta 0}, \tag{6.253}$$

and comparing with Eq. (6.249). Finally we note that the phase (6.252) and amplitude (6.251) are invariant under the transformations

$$\lambda_{r0} \quad \to \quad \tilde{\lambda}_{r0} = \lambda_{r0} + \Delta\lambda \tag{6.254}$$

$$\lambda_{\theta 0} \quad \to \quad \tilde{\lambda}_{\theta 0} = \lambda_{\theta 0} + \Delta\lambda \tag{6.255}$$

that correspond to the re-parameterization $\lambda \to \lambda + \Delta\lambda$. This invariance serves as a consistency check of the formulae, since we expect the invariance on physical grounds.

6.4.7 Expressions for the time derivatives of the constants of motion

Time averages

Let $\mathcal{E}$ be one of the three conserved quantities of geodesic motion, E, L_z or Q. For the purpose of evolving the orbit we would like to compute the quantity

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_t, \tag{6.256}$$

that is, the average with respect to the Boyer-Lindquist time coordinate t of the derivative of $\mathcal{E}$ with respect to t. However, the quantity that is most naturally computed is the derivative with respect to proper time τ, and the type of average that is most easily computed is the average with respect to Mino time λ. In this section we therefore rewrite the quantity (6.256) in terms of a Mino-time average of $d\mathcal{E}/d\tau$.

In the adiabatic limit, we can choose a time interval Δt which is long compared to the orbital timescales but short compared to the radiation reaction time[12]. Then, to a good approximation we have

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_t = \frac{\Delta\mathcal{E}}{\Delta t}, \tag{6.257}$$

where $\Delta\mathcal{E}$ is the change in $\mathcal{E}$ over this interval. Now let $\Delta\lambda$ be the change in Mino time over the interval. From Eq. (6.221) we have

$$\Delta t = \Gamma \Delta\lambda + \text{oscillatory terms}. \tag{6.258}$$

Now the oscillatory terms will be bounded as Δt is taken larger and larger, and therefore in the adiabatic limit they will give a negligible fractional correction to Δt. Hence we get

$$\begin{aligned} \left\langle \frac{d\mathcal{E}}{dt} \right\rangle_t &= \frac{1}{\Gamma}\frac{\Delta\mathcal{E}}{\Delta\lambda} \\ &= \frac{1}{\Gamma}\left\langle \frac{d\mathcal{E}}{d\lambda} \right\rangle_\lambda, \end{aligned} \tag{6.259}$$

where the λ subscript on the angular brackets means an average with respect to λ. Note that using the definition (6.220) of Γ we can rewrite this formula as

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_t = \frac{\langle d\mathcal{E}/d\lambda\rangle_\lambda}{\langle dt/d\lambda\rangle_\lambda}. \tag{6.260}$$

Finally we can use Eq. (6.212) to rewrite the Mino-time derivative in Eq. (6.259) in terms of a proper time derivative. This gives the final formula which we will

[12]A natural choice for Δt is the geometric mean of the orbital time and the radiation reaction time; this is the time it takes for the phase difference between the geodesic orbit and the true orbit to become of order unity.

use:

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle_t = \frac{1}{\Gamma} \left\langle \Sigma \frac{d\mathcal{E}}{d\tau} \right\rangle_\lambda . \tag{6.261}$$

Formulas for the energy and angular momentum fluxes

In the following, we will use the shorthand notation $V = \{knlm\}$, and

$$\sum_V = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} . \tag{6.262}$$

For the case $\mathcal{E} = E,\ L_z$ we find:

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = \frac{1}{8\pi^2} \sum_V \left\{ \begin{matrix} \omega_{mkn} \\ \\ m \end{matrix} \right\} \frac{\omega_{mkn}}{|\omega_{mkn}|} \left[B_{sV}^{\text{out}} \mid {}_sZ_V^{\text{out}} \mid^2 \right.$$

$$\left. + \frac{\omega_{mkn} p_{mkn}}{|\omega_{mkn} p_{mkn}|} B_{sV}^{\text{down}} \mid {}_sZ_V^{\text{down}} \mid^2 \right] , \tag{6.263}$$

where the coefficients B_{sV} are given by

$$B_{sV}^{\text{out}} = \frac{A_{sV}}{\alpha_{sV}\ \alpha_{-sV}^*}, \qquad B_{sV}^{\text{down}} = \frac{A_{sV}\ \tau_{sV}\ \tau_{-sV}^*\ \kappa_{sm\omega}}{\beta_{sV}\beta_{-sV}^*}, \tag{6.264}$$

and

$$p_{mkn} = \omega_{mkn} - \frac{am}{2(1+\sqrt{1-a^2})} . \tag{6.265}$$

Using the relations for the spin-inverted coefficients, we can rewrite the coefficients as

$$B_{sV}^{\text{out}} = 2^{s+2} (2\omega)^{2s} C^{-(3s/4+1/2)} (C^*)^{-s/4+1/2} \left[(\alpha_{sV}^{\text{E}})^2 + \frac{C^*}{C} (\alpha_{sV}^{\text{O}})^2 \right]^{-1} . \tag{6.266}$$

Derivation

The energy and angular momentum can be written as the inner product of a Killing vector and the 4-velocity:

$$\mathcal{E} = \xi_\alpha u^\alpha \tag{6.267}$$

where $\vec{\xi} = -\partial_t$ for E and $\vec{\xi} = \partial_\varphi$ for $\mathcal{E} = L_z$. Taking a time derivative and using that for a Killing vector, $\nabla_{(\alpha}\xi_{\beta)} = 0$ gives:

$$\frac{d\mathcal{E}}{d\tau} = u^\beta \nabla_\beta \left(\xi_\alpha u^\alpha\right) = u^\alpha u^\beta \nabla_\beta \xi_\alpha + \xi_\alpha u^\beta \nabla_\beta u^\alpha = \xi_\alpha a^\alpha \tag{6.268}$$

where a^α is the 4-acceleration. In the adiabatic regime, the self-acceleration is given in terms of the radiative field as

$$a^\alpha = -\frac{1}{2}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\left(2\nabla_\delta h^{\rm rad}_{\beta\gamma} - \nabla_\beta h^{\rm rad}_{\gamma\delta}\right) u^\gamma u^\delta \tag{6.269}$$

Inserting this in Eq. (6.268) gives

$$\begin{aligned}
\frac{d\mathcal{E}}{d\tau} &= -\frac{1}{2}\left(\xi^\beta + \mathcal{E} u^\beta\right)\left(2u^\gamma u^\delta \nabla_\delta h^{\rm rad}_{\beta\gamma} - u^\gamma u^\delta \nabla_\beta h^{\rm rad}_{\gamma\delta}\right) &(6.270)\\
&= -\left(\xi^\beta + \frac{\mathcal{E}}{2} u^\beta\right) u^\gamma \frac{d}{d\tau} h^{\rm rad}_{\beta\gamma} + \frac{1}{2} u^\gamma u^\delta \xi^\beta \nabla_\beta h^{\rm rad}_{\gamma\delta} &(6.271)\\
&= -\xi^\beta \frac{d}{d\tau}\left(u^\gamma h^{\rm rad}_{\beta\gamma} + \frac{1}{2} u_\beta u^\gamma u^\delta h^{\rm rad}_{\beta\gamma}\right) + \left(\xi^\beta + \mathcal{E} u^\beta\right) h^{\rm rad}_{\beta\gamma} a^\gamma \\
&\quad + \xi_\alpha a^\alpha h^{\rm rad}_{\beta\gamma} u_\beta u^\gamma + \frac{1}{2} u^\gamma u^\delta \xi^\beta \nabla_\beta h^{\rm rad}_{\gamma\delta} &(6.272)
\end{aligned}$$

To leading order in μ, all the terms except the last term in (6.272) can be neglected because they are either a total time derivative (and so the change in $\mathcal{E}$ over an interval from τ_1 to τ_2 associated with these terms will oscillate but will not grow secularly with time and thus will be smaller than the contribution of the last term by $T_{\rm orb}/T_{\rm inspiral}$) or they are proportional to a^α and hence higher order in μ. Dropping all these terms and substituting (6.261) gives

$$\langle \frac{d\mathcal{E}}{dt} \rangle_t = \frac{1}{2\Gamma} \langle \Sigma \xi^\beta u^\gamma u^\delta \nabla_\beta h^{\rm rad}_{\gamma\delta} \rangle_\lambda \tag{6.273}$$

The radiative field can be written as

$$\begin{aligned}
h^{\rm rad}_{ab}(x) &= \frac{1}{8\pi i} \sum_\Lambda \sum_{p=\pm 1} \frac{\omega_{mkn}}{|\omega_{mkn}|}\left(1 + pP\right)\left[\frac{1}{\alpha_{s\Lambda}\alpha^*_{-s\Lambda}}\, {}_s Z^{\rm out}_V \; h^{\rm out}_{ab\;V}(x)\right. \\
&\quad \left. + \frac{\omega_{mkn} p_{mkn}}{|\omega_{mkn} p_{mkn}|} \frac{\tau_{sV} \tau^*_{-sV} \kappa_{sm\omega}}{\beta_{sV} \beta^*_{-sV}}\, {}_s Z^{\rm down}_V \; h^{\rm down}_{ab\;V}(x)\right]. &(6.274)
\end{aligned}$$

Using that the operator $\xi^\alpha \nabla_\alpha$ gives a factor of $i\omega$ or im when acting on ${}_s\pi^{\text{out/down}}_{\Lambda\, ab}$ then gives

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = \frac{1}{4\pi\Gamma} \sum_V \left\{ \begin{array}{c} \omega_{mkn} \\ m \end{array} \right\} \frac{\omega_{mkn}}{|\omega_{mkn}|} \left[\frac{B^{\text{out}}_{sV}}{A_{sV}} {}_s Z^{\text{out}}_{\Lambda} \langle \Sigma u^a u^b \; h^{\text{out}}_{ab\;V}(x) \rangle_\lambda \right.$$
$$\left. + \frac{\omega_{mkn} p_{mkn}}{|\omega_{mkn} p_{mkn}|} \frac{B^{\text{down}}_{sV}}{A_{sV}} {}_s Z^{\text{down}}_{\Lambda} \langle \Sigma u^a u^b \; h^{\text{down}}_{ab,\;V}(x) \rangle \right] \tag{6.275}$$

From the decompositions of the amplitudes, (6.239) and (6.240), it follows that we can write

$$\frac{1}{A_{sV}} (\Sigma u^a u^b \; {}_s\tau^\dagger_{ab} \; {}_s\Phi^{\text{down}}_V)[z^\alpha(\lambda)] = J^{\text{out}\,*}_{\omega_{mkn} lm}(\lambda, \lambda) e^{-i\lambda(\Gamma\omega_{mkn} - m\langle V_\varphi \rangle)}$$
$$= \sum_{k',n'} J^{\text{out}\,*}_{\omega_{mkn} l m k' n'} e^{-i\lambda\Gamma(\omega_{mkn} - \omega_{mk'n'})} \tag{6.276}$$

and averaging will result in collapsing the sum to $\delta_{kk'}\delta_{nn'}$, so that

$$\frac{1}{A_{sV}} \langle \Sigma u^a u^b \; {}_s\tau^\dagger_{ab} \; {}_s\Phi^{\text{down}}_V)[z^\alpha(\lambda)] \rangle_\lambda = J^{\text{out}\,*}_{\omega_{mkn} lmkn} = \frac{\Gamma}{2\pi} Z^{\text{out}\,*}_{\Lambda} \tag{6.277}$$

and we obtain the final expression

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = \frac{1}{8\pi^2} \sum_V \left\{ \begin{array}{c} \omega_{mkn} \\ m \end{array} \right\} \frac{\omega_{mkn}}{|\omega_{mkn}|} \left[B^{\text{out}}_{sV} \mid {}_s Z^{\text{out}}_V \mid^2 + \frac{\omega_{mkn} p_{mkn}}{|\omega_{mkn} p_{mkn}|} B^{\text{down}}_{sV} \mid {}_s Z^{\text{down}}_V \mid^2 \right] \tag{6.278}$$

Time derivative of the Carter constant

The final result for the time derivative of the Carter constant is

$$\langle \frac{dK}{dt} \rangle_t = \frac{1}{4\pi^2} \sum_V \frac{\omega_{mkn}}{|\omega_{mkn}|} \left[B^{\text{out}}_{sV} \; {}_s Z^{\text{out}}_V \; {}_s \tilde{Z}^{\text{out}\,*}_V + \frac{\omega_{mkn} p_{mkn}}{|\omega_{mkn} p_{mkn}|} B^{\text{down}}_{sV} \; {}_s Z^{\text{down}}_V \; {}_s \tilde{Z}^{\text{down}\,*}_V \right], \tag{6.279}$$

where the new amplitude ${}_s\tilde{Z}_V^{\rm down}$ is given by

$$\begin{aligned}
{}_s\tilde{Z}_V^{\rm out} \quad &= \quad \frac{2\pi\mu}{\Gamma\Lambda_r\Lambda_\theta} e^{-im\varphi_0} e^{i\omega_{mkn}t_0} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta \\
&\times \left\{ g_{mkn}[\lambda_r, \lambda_\theta] + G(\lambda_r, \lambda_\theta)\partial_r \right\} \Sigma[r(\lambda_r), \theta(\lambda_\theta)]\ {}_sR^{\rm in}_{\omega lm}[r(\lambda_r)] \\
&\times\ {}_s\Theta^*_{\omega lm}[\theta(\lambda_\theta)]_s\tau_{ab}u^a u^b[r(\lambda_r), \theta(\lambda_\theta)] e^{i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r)} \\
&\times\ e^{-im\Delta\varphi_r(\lambda_r)} e^{-im\Delta\varphi_\theta(\lambda_\theta)} e^{i\omega_{mkn}\Delta t_r(\lambda_r)} e^{i\omega_{mkn}\Delta t_\theta(\lambda_\theta)}, \qquad (6.280)
\end{aligned}$$

with

$$g_{mkn}(\lambda, \lambda) \quad = \quad \frac{1}{\Delta} \left(-\varpi^2 E + aL_z\right) \left(\varpi^2 \omega_{mkn} - am\right) \qquad (6.281)$$

$$G(\lambda, \lambda) \quad = \quad i\Delta u_r. \qquad (6.282)$$

Following Drasco and Sago [**?**], the result (6.280) can be written in terms of just the untilded amplitudes as

$$\begin{aligned}
\left\langle \frac{dK}{dt} \right\rangle_t \quad &= \quad \frac{1}{4\pi^2} \sum_\Lambda \frac{\omega_{mkn}}{|\omega_{mkn}|} \left[B^{\rm out}_{sV}\ H_{mkn} \mid {}_sZ_V^{\rm out} \mid^2 \right. \\
&\left. + \frac{\omega_{mkn}p_{mkn}}{|\omega_{mkn}p_{mkn}|} B^{\rm down}_{sV}\ H_{mkn} \mid {}_sZ_V^{\rm down} \mid^2 \right], \qquad (6.283)
\end{aligned}$$

where

$$H_{mkn} = -\langle \frac{1}{\Delta} \left(\varpi^2 E - aL_z\right) \left(\varpi^2 \omega_{mkn} - am\right) \rangle + n\Gamma\Omega_r. \qquad (6.284)$$

The expressions for the time derivative of the Carter constant have a similar structure as those for E and L_z and are independent of the parameters t_0, φ_0, λ_{r0} and $\lambda_{\theta 0}$.

Derivation

The Carter constant $K = Q + (L_z - aE)^2$ (where Q is the separation constant for the r and θ motions in Kerr) can be written in terms of the Killing tensor and the 4-velocity as

$$K = K^{\alpha\beta} u_\alpha u_\beta \qquad (6.285)$$

where

$$K^{\alpha\beta} = 2\Sigma l^{(\alpha} n^{\beta)} + r^2 g^{\alpha\beta}. \tag{6.286}$$

Taking a time derivative of Eq. (6.285) and using the Killing tensor equation $\nabla_{(\gamma} K_{\alpha\beta)} = 0$ gives

$$\frac{dK}{d\tau} = u^\gamma \nabla_\gamma \left(K_{\alpha\beta} u^\alpha u^\beta \right) = u^\alpha u^\beta u^\gamma \nabla_{(\gamma} K_{\alpha\beta)} + 2K_{\alpha\beta} u^\alpha a^\beta = 2K_{\alpha\beta} u^\alpha a^\beta \tag{6.287}$$

Substituting the formula for the self-acceleration in the adiabatic limit gives

$$\begin{aligned}
\frac{dK}{d\tau} &= -\left(K^{\alpha\beta} u_\alpha + K u^\beta \right) \left(2u^\gamma u^\alpha \nabla_\alpha h^{\rm rad}_{\beta\gamma} - u^\gamma u^\alpha \nabla_\beta h^{\rm rad}_{\gamma\alpha} \right) \\
&= -\left(2K^{\alpha\beta} u_\alpha + K u^\beta \right) u^\gamma \frac{dh_{\beta\gamma}}{d\tau} + K^{\alpha\beta} u_\alpha u^\gamma u^\delta \nabla_\beta h^{\rm rad}_{\gamma\delta} \qquad (6.288) \\
&\sim K^{\alpha\beta} u_\alpha \nabla_\beta \left(h^{\rm rad}_{\gamma\delta} u^\gamma u^\delta \right) + h^{\alpha\beta} u_\alpha \left(u^\gamma u^\delta \nabla_\delta K_{\beta\gamma} - K^{\gamma\delta} u_\gamma \nabla_\delta u_\beta \right), \qquad (6.289)
\end{aligned}$$

where in the last line we have integrated by parts and neglected all terms that are total derivatives with respect to τ and all those that involve the acceleration a^α. The second term here can also be neglected, which can be seen as follows. The important property we need is that for Kerr geodesics, $u_r = u_r(r)$, $u_\theta = u_\theta(\theta)$ and u_t and u_φ are constant. Then, $\nabla_\delta u_\beta = \nabla_\beta u_\delta$ and we can rewrite the second term as

$$\begin{aligned}
& h^{\alpha\beta} u_\alpha \left(u^\gamma u^\delta \nabla_\delta K_{\beta\gamma} - K^{\gamma\delta} u_\gamma \nabla_\delta u_\beta \right) \\
&\qquad = h^{\alpha\beta} u_\alpha \left(u^\gamma u^\delta \nabla_\delta K_{\beta\gamma} - \nabla_\beta K + K^{\gamma\delta} u_\delta \nabla_\beta u_\gamma + u^\delta u^\gamma \nabla_\beta K_{\gamma\delta} \right) \\
&\qquad = h^{\alpha\beta} u_\alpha \left(-u^\gamma u^\delta \nabla_\gamma K_{\beta\delta} + K^{\gamma\delta} u_\delta \nabla_\beta u_\gamma \right) \qquad (6.290)
\end{aligned}$$

where in the second line we have used the Killing equation $\nabla_{(\delta} K_{\beta\gamma)} = 0$. Comparing the left and right hand sides, it follows that they must be zero.

Thus,

$$\left\langle \frac{dK}{dt} \right\rangle_t = \frac{1}{\Gamma}\langle \Sigma K^{\alpha\beta} u_\alpha \nabla_\beta h^{\rm rad}_{\gamma\delta} u^\gamma u^\delta \rangle_\lambda \tag{6.291}$$

$$= \frac{1}{2\pi i \Gamma}\sum_V \frac{\omega_{mkn}}{|\omega_{mkn}|}\left[B^{\rm out}_{sV}\, {}_sZ^{\rm out}_V \langle \Sigma K^{\alpha\beta} u_\alpha \nabla_\beta\; u^a u^b h^{\rm out}_{ab\;V}\rangle_\lambda \right.$$

$$\left. + \frac{\omega_{mkn} p_{mkn}}{|\omega_{mkn} p_{mkn}|} B^{\rm down}_{sV}\, {}_sZ^{\rm down}_V \langle \Sigma K^{\alpha\beta} u_\alpha \nabla_\beta\; u^a u^b h^{\rm down}_{ab\;V}\rangle_\lambda \right]. \tag{6.292}$$

To evaluate the amplitudes $\langle \Sigma K^{ab} u_a \nabla_b u^c u^d h^{\rm out/down}_{ab\;V}\rangle_\lambda$, we start by simplifying the operator $K^{\mu\alpha} u_\mu \nabla_\alpha$. Using expression (6.286) and the definitions of $\vec{l}$ and $\vec{n}$ in Eq. (6.25) gives

$$K^{\mu\alpha} u_\mu \nabla_\alpha = \Sigma l^\alpha u_\alpha n^\beta \nabla_\beta + \Sigma n^\alpha u_\alpha l^\beta \nabla_\beta + r^2 \frac{d}{d\tau} \tag{6.293}$$

$$= \frac{1}{2}\left(-\frac{\varpi^2}{\Delta}E + u_r + \frac{a}{\Delta}L_z\right)\left(\varpi^2 \partial_t - \Delta\partial_r + a\partial_\phi\right)$$

$$+ \frac{1}{2}\left(-\varpi^2 E - \Delta u_r + aL_z\right)\left(\frac{\varpi^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi\right) + r^2\frac{d}{d\tau}$$

$$= \frac{1}{\Delta}\left(-\varpi^2 E + aL_z\right)\left(\varpi^2\partial_t + a\partial_\phi\right) - \Delta u_r \partial_r + r^2 \frac{d}{d\tau} \tag{6.294}$$

We now define a new amplitude $\tilde{Z}$ by

$${}_s\tilde{Z}^{\rm out}_{lmkn} = \frac{2\pi}{i\Gamma}\frac{1}{A^*_{sV}}\langle \Sigma K^{\alpha\beta} u_\alpha \nabla_\beta u^c u^d \left(h^{\rm out}_{cd\;V}\right)^* \rangle_\lambda \tag{6.295}$$

Substituting Eq. (6.294) gives:

$${}_s\tilde{Z}^{\rm out}_{lmkn} = \frac{2\pi}{i\Gamma}\frac{1}{A^*_{sV}}\left\langle \Sigma\left[\frac{i}{\Delta}\left(-\varpi^2 E + aL_z\right)\left(\varpi^2\omega_{mkn} - am\right)\right]\right.$$

$$\left. -\Delta u_r\partial_r + r^2\frac{d}{d\tau}\right] u^c u^d\; h^{\rm out\;*}_{cd\;V}\Big\rangle_\lambda \tag{6.296}$$

Consider the contribution of the term involving $r^2 d/d\tau$:

$$\frac{2\pi}{i\Gamma}\langle r^2 \Sigma \frac{d}{d\tau} u^c u^d\; h^{\rm out\;*}_{cd\;V}\rangle_\lambda = \frac{2\pi}{i\Gamma}\langle \frac{d}{d\lambda}\left(r^2 u^c u^d\; h^{\rm out\;*}_{cd\;V}\right) - 2r\Delta u_r u^c u^d\; h^{\rm out\;*}_{cd\;V}\rangle_\lambda, \tag{6.297}$$

where we have integrated by parts with respect to λ and used that $dr/d\lambda = \Delta u_r$. Neglecting all terms that are not leading order in μ this gives

$$\frac{2\pi i}{\Gamma}\langle u^c u^d\; h^{\rm out\;*}_{cd\;V} 2r\frac{dr}{d\lambda}\rangle_\lambda = \frac{4\pi i}{\Gamma}\langle u^c u^d\; h^{\rm out\;*}_{cd\;V}\; r\Delta u_r\rangle_\lambda. \tag{6.298}$$

The amplitude then becomes

$$
\begin{aligned}
{}_s\tilde{Z}^{\rm out}_{lmkn} &= \frac{2\pi}{\Gamma}\frac{1}{A^*_{sV}}\left\langle\Sigma\left[\frac{1}{\Delta}\left(-\varpi^2 E + aL_z\right)\left(\varpi^2\omega_{mkn} - am\right)\right.\right. \\
&\qquad\left.\left. + i\Delta u_r\partial_r + \frac{2ir\Delta}{\Sigma}u_r\right] u^c u^d\ h^{\rm out}_{cd\ V}{}^*\right\rangle_\lambda \qquad (6.299) \\
&= \frac{2\pi}{\Gamma}\frac{1}{A^*_{sV}}\left\langle\Sigma G_{mkn}(\lambda,\lambda)u^c u^d\ h^{\rm out}_{cd\ V}{}^* + \Sigma G(\lambda,\lambda)\partial_r u^c u^d\ h^{\rm out}_{cd\ V}{}^*\right\rangle_\lambda, \qquad (6.300)
\end{aligned}
$$

where we have defined the quantities

$$
\begin{aligned}
G_{mkn}(\lambda_r,\lambda_\theta) &= \frac{1}{\Delta}\left(-\varpi^2 E + aL_z\right)\left(\varpi^2\omega_{mkn} - am\right) + \frac{1}{\Sigma}2ir\Delta u_r, \qquad (6.301) \\
G(\lambda_r,\lambda_\theta) &= i\Delta u_r. \qquad (6.302)
\end{aligned}
$$

Following Drasco and Sago [**?**], we can further rewrite this expression by noting that $2r = \partial_r\Sigma$, so that if we combine the term $2ir\Delta u_r$ in $G_{\omega mkn}$ with the derivative term, we can move the factor of Σ through and obtain

$$
\begin{aligned}
{}_s\tilde{Z}^{\rm out}_{lmkn} &= \frac{2\pi}{\Gamma A^*_{sV}}\left\langle g_{mkn}(\lambda,\lambda)\Sigma u^c u^d h^{\rm out}_{cd\ V}{}^*\right. \\
&\qquad\left. + G(\lambda,\lambda)\partial_r\left[\Sigma u^c u^d\ h^{\rm out}_{cd\ V}{}^*\right]\right\rangle_\lambda, \qquad (6.303) \\
g_{mkn}(\lambda,\lambda) &= \frac{1}{\Delta}\left(-\varpi^2 E + aL_z\right)\left(\varpi^2\omega_{mkn} - am\right). \qquad (6.304)
\end{aligned}
$$

Using the definition of the amplitudes in Eq. (6.211), the expression for the time derivative of the Carter constant can then be written as

$$
\langle\frac{dK}{dt}\rangle_t = \frac{1}{4\pi^2}\sum_V \frac{\omega_{mkn}}{|\omega_{mkn}|}\left[B^{\rm out}_{sV}\,{}_sZ^{\rm out}_V\ {}_s\tilde{Z}^{\rm out\,*}_V + \frac{\omega_{mkn}p_{mkn}}{|\omega_{mkn}p_{mkn}|}B^{\rm down}_{sV}\ {}_sZ^{\rm down}_V\ {}_s\tilde{Z}^{\rm down\,*}_V\right] \qquad (6.305)
$$

The dependence of the amplitudes ${}_s\tilde{Z}^{\rm down}_\Lambda$ on the parameters of the geodesic is

$$
\begin{aligned}
{}_s\tilde{Z}^{\rm down}_V &= \frac{2\pi}{\Gamma\Lambda_r\Lambda_\theta}e^{-im\varphi_0}e^{i\omega_{mkn}t_0}\int_0^{\Lambda_r}d\lambda_r\int_0^{\Lambda_\theta}d\lambda_\theta \\
&\left\{g_{mkn}[\lambda_r,\lambda_\theta] + G(\lambda_r,\lambda_\theta)\partial_r\right\}\left\{\Sigma\ \left(u^c u^d\ {}_s\tau^\dagger_{cd}\ {}_{-s}R^{\rm out}_V\ {}_s\Theta_V\right)^*\right\}[r(\lambda_r),\theta(\lambda_\theta)] \\
&e^{i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r)}e^{-im\Delta\varphi_r(\lambda_r)}e^{-im\Delta\varphi_\theta(\lambda_\theta)}e^{i\omega_{mkn}\Delta t_r(\lambda_r)}e^{i\omega_{mkn}\Delta t_\theta(\lambda_\theta)}. \qquad (6.306)
\end{aligned}
$$

Note in particular that the dependence of ${}_s\tilde{Z}_V$ on the parameters t_0, φ_0, λ_{r0} and $\lambda_{\theta 0}$ via an overall phase is the same as that of the amplitudes ${}_sZ_\Lambda$, so that as expected in the adiabatic limit, the time derivative of the Carter constant is independent of these parameters since they cancel out.

We can simplify the expression (6.306) to look like that given in Ref. [41] as follows. Consider first the result of differentiating with respect to λ_r:

$$\begin{aligned}
& i\frac{d}{d\lambda_r}\Big\{ \left[\Sigma u^c u^d h^{\mathrm{out}\ *}_{cd\ V}\right] [r(\lambda_r),\theta(\lambda_\theta)] e^{i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r)} e^{-im\Delta\varphi_r(\lambda_r)} e^{-im\Delta\varphi_\theta(\lambda_\theta)} \\
& \qquad e^{i\omega_{mkn}\Delta t_r(\lambda_r)} e^{i\omega_{mkn}\Delta t_\theta(\lambda_\theta)} \Big\} \\
& = \Big[i\Delta u_r \partial_r - \Gamma n \Omega_r + m\left(V_{\varphi r} - \langle V_{\varphi r}\rangle\right) - \omega_{mkn}\left(V_{tr} - \langle V_{tr}\rangle\right) \Big] \Sigma u^c u^d h^{\mathrm{out}\ *}_{cd\ V} \\
& \times e^{i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r)} e^{-im\Delta\phi_r} e^{-im\Delta\phi_\theta} e^{i\omega_{mkn}\Delta t_r} e^{i\omega_{mkn}\Delta t_\theta}. \qquad (6.307)
\end{aligned}$$

Here, we have used the following expressions for various derivatives:

$$\frac{dr}{d\lambda_r} = \Delta u_r, \qquad (6.308)$$

$$\frac{d\Delta\varphi_r}{d\lambda_r} = V_{\varphi r} - \langle V_{\varphi r}\rangle = \frac{a}{\Delta}\left(\varpi^2 E - aL_z\right) - \langle \frac{a}{\Delta}\left(\varpi^2 E - aL_z\right)\rangle, \qquad (6.309)$$

$$\frac{d\Delta t_r}{d\lambda_r} = V_{tr} - \langle V_{tr}\rangle = \frac{\varpi^2}{\Delta}\left(\varpi^2 E - aL_z\right) - \langle \frac{\varpi^2}{\Delta}\left(\varpi^2 E - aL_z\right)\rangle. \qquad (6.310)$$

Now, the left hand side of (6.307) will vanish when we integrate over a radial period, and we can use (6.307) to substitute for the $r-$ derivative in (6.306) and combine terms to obtain an expression without any derivatives:

$$\begin{aligned}
{}_s\tilde{Z}^{\mathrm{out}}_\Lambda &= \frac{2\pi}{\Gamma\Lambda_r\Lambda_\theta} e^{-im\phi_0} e^{i\omega_{mkn}t_0} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta \\
& \qquad H_{mkn} \Sigma u^c u^d \frac{1}{A^*_{s\Lambda}} h^{\mathrm{out}\ *}_{cd\ V}[r(\lambda_r),\theta(\lambda_\theta)] \\
& \quad \times e^{i\Gamma(k\Omega_\theta\lambda_\theta + n\Omega_r)} e^{-im\Delta\phi_r(\lambda_r)} e^{-im\Delta\phi_\theta(\lambda_\theta)} e^{i\omega_{mkn}\Delta t_r(\lambda_r)} e^{i\omega_{mkn}\Delta t_\theta(\lambda_\theta)} \qquad (6.311) \\
&= \frac{2\pi}{\Gamma A^*_{sV}} H_{mkn} \langle \Sigma u^c u^d h^{\mathrm{out}\ *}_{cd\ V} \rangle \qquad (6.312) \\
&= H_{mkn}\ {}_sZ^{\mathrm{out}\ *}_V, \qquad (6.313)
\end{aligned}$$

where

$$H_{mkn} = -\langle \frac{1}{\Delta} \left(\varpi^2 E - aL_z\right) \left(\varpi^2 \omega_{mkn} - am\right) \rangle + n\Gamma\Omega_r. \tag{6.314}$$

Using Eq. (6.313), we can rewrite the time derivative of the Carter constant in terms of the same amplitudes as for E and L_z and the average (6.314) over the geodesic as

$$\langle \frac{dK}{dt} \rangle_t = \frac{1}{4\pi^2} \sum_V \frac{\omega_{mkn}}{|\omega_{mkn}|} \left[B^{\text{out}}_{sV} H_{mkn} \mid {}_sZ^{\text{out}}_V \mid^2 + \frac{\omega_{mkn} p_{mkn}}{|\omega_{mkn} p_{mkn}|} B^{\text{down}}_{sV} H_{mkn} \mid {}_sZ^{\text{down}}_V \mid^2 \right] \tag{6.315}$$

6.5 Comparison of the notation to other conventions

The various coefficients defined in this chapter are related to those defined by Hughes [189], in which $s = -2$ throughout, as follows. The variable κ_s we define is related to Hughes' ε by

$$\kappa_s = 1 - \frac{2is\sqrt{M^2 - a^2}}{4Mr_+ p_{m\omega}} = 1 - \frac{2is}{p_{m\omega}} \varepsilon^{\text{Hughes}}. \tag{6.316}$$

The various amplitudes B and D defined by Hughes correspond to the following combinations of our variables:

$$B^{\text{hole}} = \frac{\alpha_s \tau_s}{\sqrt{2Mr_+ |p_{m\omega}|}} \tag{6.317}$$

$$B^{\text{out}} = \frac{\alpha_s \sigma_s}{|\omega|^{1/2}} \tag{6.318}$$

$$B^{\text{in}} = \frac{\alpha_s}{|\omega|^{1/2}} \tag{6.319}$$

$$D^{\text{out}} = \frac{\beta_s \mu_s}{\sqrt{2Mr_+ |p_{m\omega}|}} \frac{\omega p}{|\omega p|} \tag{6.320}$$

$$D^{\text{in}} = \frac{\beta_s \nu_s}{\sqrt{2Mr_+ |p_{m\omega}|}} \frac{\omega p}{|\omega p|} \tag{6.321}$$

$$D^{\infty} = \frac{\alpha_s}{|\omega|^{1/2}} \tag{6.322}$$

6.6 Two-timescale method

6.6.1 Analysis of the $O(\varepsilon)$ Einstein equation

In this section, we use the methods of [200] [also Chapter 5] to give an explicit prescription for computing the leading order waveform.

We restrict the analysis to a region whose extent in time covers the entire inspiral time $\tau_{\text{inspiral}} \sim M/\varepsilon$ and whose spatial extent is $\mu \ll r \ll M/\varepsilon$. A global, consistent solution is obtained by matching in a common domain of validity to solutions obtained by different types of analysis outside of this regime (e.g., black hole perturbation theory for a small black hole for $r \sim \mu$ and, at large r, matching on to an outgoing wave solution. We will show below that the matching to an outgoing wave solution is not necessary at leading order). Because we restrict the domain to $r \ll \tau_{\text{inspiral}}$, we can take the foliation to be a constant-time hypersurface that intersects the worldline. We exclude the case when the source exhibits resonances.

We make the following ansatz for the metric:

$$g_{\alpha\beta}(t, x^i; \varepsilon) = g^{(0)}_{\alpha\beta}(x^i, \tilde{t}) + \varepsilon h^{(1)}_{\alpha\beta}(q^i, \tilde{t}, x^i) + \varepsilon^2 h^{(2)}_{\alpha\beta}(q^i, \tilde{t}, x^i). \tag{6.323}$$

Here, $\tilde{t} \equiv \varepsilon t$, and the dependence of $g^{(0)}_{ab}$ on $\tilde{t}$ is an *implicit* dependence that arises because the parameters of the black hole $P_B(\tilde{t}) = [M(\tilde{t}), a(\tilde{t})]$ (its mass and spin) are assumed to be slowly evolving due to the absorption of gravitational radiation (since we restrict the discussion here to the leading order, it is sufficient to assume that P_B depend on $\tilde{t}$ only, see Ref. [200]). As discussed in Ref. [139], the leading order Einstein equation reduces to the standard equation for Kerr at fixed $\tilde{t}$, so that the $\tilde{t}-$dependence of $g^{(0)}$ is unspecified at that order but will be determined at the next to leading order. There is no *explicit* dependence on $\tilde{t}$ because we assume $\partial/\partial t$ to be a timelike Killing field.

The functions q_i, for $i = 1, 2, 3 = r, \theta, \varphi$, are coordinates on the three-torus given by the following asymptotic expansion at fixed $\tilde{t}$:

$$q_i = \frac{1}{\varepsilon} f_i^{(0)}(\tilde{t}) + f_i^{(1)}(\tilde{t}) + O(\varepsilon), \tag{6.324}$$

they are the angle variables obtained from the analysis of the orbital motion after eliminating proper time τ in favor of t.

The mathematical meaning of Eq. (6.323) is that it is an asymptotic expansion as $\varepsilon \to 0$ holding $\tilde{t}$, f_i and x^i fixed. The dependence of the metric on the q^i is assumed to be 2π-periodic, and this periodicity is what leads to unique solutions at each order in ε.

The differential equations we obtain below that determine the leading order gravitational waveform are similar to those obtained from usual black hole perturbation theory, except that they are equations at fixed $\tilde{t}$ on a six-dimensional manifold with coordinates $(x^i, q_r(t), q_\theta(t), q_\varphi(t))$, where $q_i(t)$ are coordinates on the three-torus.

We use the Newman-Penrose null tetrad to write the background metric as

$$g_{ab}^{(0)} = -2 l_{(a}^{(0)} n_{b)}^{(0)} + 2 m_{(a}^{(0)} m_{b)}^{*(0)}, \tag{6.325}$$

where the superscript (0) denotes the unperturbed quantities. As discussed in Ref. [139], if the covariant derivative acts on a function of q_i, $\tilde{t}$ and x^i, it can be expanded in a double expansion on the six-dimensional manifold as

$$\nabla_a = \nabla_a^{(0,0)} + \varepsilon \left[\nabla_a^{(0,1)} + \nabla_{[h^{(1)}]\,a}^{(1,0)} \right] + O(\varepsilon^2). \tag{6.326}$$

The type of double-expansion we are using here is such that a quantity with a superscript (n, m) will contain n factors of $h^{(1)}$ and m derivatives with respect to $\tilde{t}$, as well as derivatives with respect to f_i that involve the angular frequency $df_i^{(m)}/dt = \Omega_i^{(m)}(\tilde{t})$.

We can use the expansion in Eq. (6.326) to obtain a similar expansion for the Riemann tensor:

$$R_{abcd} = R_{abcd}^{(0,0)} + \varepsilon \left(R_{abcd}^{(1,0)}[h^{(1)}] + R_{abcd}^{(0,1)}[g^{(0)}] \right) + O(\varepsilon^2). \tag{6.327}$$

The first term here is just the Riemann tensor of the Kerr background at fixed $\tilde{t}$ on the larger manifold. The second term in Eq. (6.327) is given explicitly by:

$$\begin{aligned} R_{abcd}^{(1,0)} &= \frac{1}{2} \left(\nabla_b^{(0,0)} \nabla_c^{(0,0)} h_{ad}^{(1)} + \nabla_a^{(0,0)} \nabla_d^{(0,0)} h_{bc}^{(1)} - \nabla_a^{(0,0)} \nabla_c^{(0,0)} h_{bd}^{(1)} - \nabla_b^{(0,0)} \nabla_d^{(0,0)} h_{ac}^{(1)} \right) \\ &\quad - R_{ab[c}^{(0,0)\,e} h_{d]e}^{(1)}. \end{aligned} \tag{6.328}$$

We will analyze the piece $R_{abcd}^{(0,1)}$ separately in Ref. [139]; it corresponds to non-radiative degrees of freedom and schematically, it involves derivatives of the form $R_{abcd}^{(0,1)} \sim$ ($\delta_{d0}\ \partial_c$ & $\Gamma_{cd}^{(0)}$) $(\partial g_{ab}/\partial P_B)$ $(dP_B/d\tilde{t})$.

The ten independent tetrad components of the Weyl tensor C_{abcd} can be written as five complex scalars $\psi_0 \ldots, \psi_4$ by contracting C_{abcd} with the basis vectors in all possible nontrivial ways:

$$\begin{aligned} \psi_0 &= -C_{abcd} l^a m^b l^c m^d, \qquad \psi_1 = -C_{abcd} l^a n^b l^c m^d, \\ \psi_2 &= -\frac{1}{2} C_{abcd} \left(l^a n^b l^c n^d + l^a n^b m^c m^{*d} \right), \\ \psi_3 &= -C_{abcd} l^a n^b m^{*c} n^d, \qquad \psi_4 = -C_{abcd} n^a m^{*b} n^c m^{*d}. \end{aligned} \tag{6.329}$$

We choose the background tetrad so that $\vec{l}^{(0)}$ and $\vec{n}^{(0)}$ are along the repeated principal null directions of the Weyl tensor. There is then only one non-vanishing unperturbed Weyl tensor component in the background:

$$\psi_0^{(0)} = \psi_1^{(0)} = \psi_3^{(0)} = \psi_4^{(0)} = 0, \qquad \psi_2^{(0)} \neq 0. \tag{6.330}$$

We define the variables

$$
\begin{aligned}
{}_s\Psi^{(1)}(q_i,\tilde{t},x^i) &= {}_sM^{ab(0)}h^{(1)}_{ab}(q_i,\tilde{t},x^i) \qquad (6.331)\\
&= \begin{cases} \psi_0^{(1)} = -C^{(1,0)}_{abcd}l^{a(0)}m^{b(0)}l^{c(0)}m^{d(0)}, & s=2,\\ (\psi_2^{(0)})^{-4/3}\ \psi_4^{(1)} = -(\psi_2^{(0)})^{-4/3}C^{(1,0)}_{abcd}n^{a(0)}m^{*b(0)}n^{c(0)}m^{*d(0)}, & s=-2. \end{cases}
\end{aligned}
$$

The operators ${}_sM^{ab(0)}$ can be read off by projecting Eq. (6.328) along the tetrad as in Eq. (6.331) and using the expansion of $h^{(1)}_{ab}$ in terms of the tetrad vectors:

$$
\begin{aligned}
h^{(1)}_{ab} &= h^{(1)}_{ll}n_an_b + h^{(1)}_{nn}l_al_b + h^{(1)}_{mm}m^*_am^*_b + h^{(1)}_{m^*m^*}m_am_b - h^{(1)}_{lm}n_am^*_b\\
&\quad -h^{(1)}_{nm^*}l_am_b - h^{(1)}_{nm}l_am^*_b - h^{(1)}_{lm^*}n_am_b. \qquad (6.332)
\end{aligned}
$$

Here, we have omitted the superscripts (0) on the tetrad legs.

The master variables ${}_s\Psi^{(1)}(q_i,\tilde{t},x^i)$ satisfy the Teukolsky equation

$$
{}_s\mathcal{O}^{(0)}\ {}_s\Psi^{(1)}(q_i,\tilde{t},x^i) = 4\pi\ {}_s\tau^{(0)}_{ab}T^{ab(1)}(q_i,\tilde{t},x^i), \qquad (6.333)
$$

where the operators ${}_s\tau^{(0)}_{ab}$ and ${}_s\mathcal{O}^{(0)}$ satisfy the schematic identity

$$
{}_s\tau^{(0)}G^{(1,0)}[h^{(1)}] = {}_s\mathcal{O}^{(0)}{}_sM^{(0)}. \qquad (6.334)
$$

We use the Kinnersley tetrad in Boyer-Lindquist coordinates given explicitly in Eq. (6.25) and define the angular and radial differential operators $\mathcal{L}^{(0)}_s$ and $\mathcal{D}^{(0)}_n$, for the integers s and n as

$$
\begin{aligned}
\mathcal{L}_s &= -ia\sin\theta\Omega^{(0)}_i\partial_{f_i} + \partial_\theta - \frac{i}{\sin\theta}\partial_\varphi + s\cot\theta, \qquad (6.335)\\
\mathcal{D}_n &= \frac{\varpi^2}{\Delta}\Omega^{(0)}_i\partial_{f_i} + \partial_r + \frac{a}{\Delta}\partial_\varphi + \frac{2n(r-M)}{\Delta}. \qquad (6.336)
\end{aligned}
$$

In terms of these operators, the differential operators that project the source term are given by Eqs. (6.40) and (6.41).

The differential operator ${}_s\mathcal{O}^{(0)}$ when acting on a function $f = f(q_i, \tilde{t}, x^i)$ can be written as

$$
{}_s\mathcal{O}^{(0)} = \Sigma^{-1}\ {}_s\Box^{(0)}, \tag{6.337}
$$

where the operator ${}_s\Box^{(0)}$ is given by

$$
\begin{aligned}
{}_s\Box^{(0)} \quad = \quad & \left[\frac{\varpi^4}{\Delta} - a^2 \sin^2\theta\right]\left(\Omega_i^{(0)}\partial_{f_i}\right)^2 - \frac{4Mar}{\Delta}\Omega_i^{(0)}\partial_{f_i}\partial_\varphi \\
& + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\partial_\varphi^2 + \frac{1}{\Delta^s}\partial_r\left(\Delta^{s+1}\partial_r\right) + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) \\
& + 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\partial_\varphi + \left(s^2\cot^2\theta - s\right) \\
& + 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\Omega_i^{(0)}\partial_{f_i},
\end{aligned} \tag{6.338}
$$

where, $\Omega_i^{(0)} = df_i^{(0)}/dt$. As discussed above, this differs from the usual Teukolsky operator in that it is a differential operator on the larger, 6-dimensional manifold at fixed $\tilde{t}$. For notational convenience, we will include the factor of Σ in Eq. (6.337) with the source term and write the decoupled master equation (6.333) as

$$
{}_s\Box^{(0)}\ {}_s\Psi^{(1)}(q_i, \tilde{t}, x^i) = {}_s\mathcal{T}^{(1)}(q_i, \tilde{t}, x^i), \tag{6.339}
$$

where ${}_s\mathcal{T}^{(1)}$ is given by

$$
{}_s\mathcal{T}^{(1)} = 4\pi\Sigma\ {}_s\tau_{ab}^{(0)} T^{ab\ (1)}. \tag{6.340}
$$

Separation of variables

We now specialize to the homogeneous version of the Teukolsky equation (6.339). The Teukolsky operator in Eq. (6.338) separates into a radial and an angular part as follows:

$$
\begin{aligned}
{}_s\Box^{(0)} &= {}_s\Box^{(r)\ (0)} + {}_s\Box^{(\theta)\ (0)}, && (6.341)\\
{}_s\Box^{(r)\ (0)} &= \frac{1}{\Delta^s}\partial_r\left(\Delta^{s+1}\partial_r\right) + \frac{1}{\Delta}\left[-\varpi^4\left(\Omega_i^{(0)}\partial_{f_i}\right)^2 + 2a\varpi^2\Omega_i^{(0)}\partial_{f_i}\partial_\varphi - a^2\partial_\varphi^2\right]\\
&\quad -\frac{2s(r-M)}{\Delta}(-\varpi^2\Omega_i^{(0)}\partial_{f_i} + a\partial_\varphi)\\
&\quad -4sr\Omega_i^{(0)}\partial_{f_i} + a^2\left(\Omega_i^{(0)}\partial_{f_i}\right)^2 - 2a\Omega_i^{(0)}\partial_{f_i}\partial_\varphi + s + |s| && (6.342)\\
{}_s\Box^{(\theta)\ (0)} &= \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) - a^2\cos^2\theta\left(\Omega_i^{(0)}\partial_{f_i}\right)^2 + \csc^2\theta\partial_\varphi^2\\
&\quad -2ias\cos\theta\Omega_i^{(0)}\partial_{f_i} + \frac{2is\cos\theta}{\sin^2\theta}\partial_\varphi - s^2\cot^2\theta - |s|. && (6.343)
\end{aligned}
$$

To obtain separable solutions, we make the ansatz

$$
{}_s\Psi^{(1)}_{\mathbf{k}lm} = {}_sR(r){}_s\Theta(\theta)e^{im\varphi}e^{-ik_jq_j}. \qquad (6.344)
$$

Explicitly, $k_jq_j = k_rq_r + k_\theta q_\theta + k_\varphi q_\varphi$ with $q_i = f_i^{(0)}(\tilde{t})/\varepsilon + f_i^{(1)}(\tilde{t}) + O(\varepsilon)$. Substituting the ansatz (6.344) into the homogeneous version of Eq. (6.339) and keeping only the leading order term $\Omega_i^{(0)}$ in the expansion of df_i/dt results in the two equations:

$$
\begin{aligned}
0 &= \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\ {}_s\Theta}{d\theta}\right) + \left[a^2\left(k_j\Omega_j^{(0)}\right)^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2ak_j\Omega_j^{(0)}s\cos\theta\right.\\
&\quad \left. -\frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + \lambda - |s|\right]{}_s\Theta, && (6.345)\\
0 &= \frac{1}{\Delta^s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d\ {}_sR}{dr}\right) + \left[\frac{K_{m\mathbf{k}}^{(0)\,2} - 2is(r-M)K_{m\mathbf{k}}^{(0)}}{\Delta} + 4isk_j\Omega_j^{(0)}r - \lambda\right.\\
&\quad \left. -a^2\left(k_j\Omega_j^{(0)}\right)^2 + 2amk_j\Omega_j^{(0)} + s + |s|\right]{}_sR. && (6.346)
\end{aligned}
$$

Here, $\lambda_{slm}(ak_j\Omega_j)$ is the separation constant and we have defined

$$
K_{m\mathbf{k}}^{(0)} = k_j\Omega_j^{(0)}\varpi^2 - am. \qquad (6.347)
$$

The solutions to Eq. (6.345) are the real functions ${}_s\Theta_{lm}(ak_j\Omega_j^{(0)},\theta)$ that are regular on $[0,\pi]$. In what follows, we do not show the dependence of ${}_s\Theta_{\mathbf{k}lm}(\theta)$ on $ak_j\Omega_j^{(0)}$

explicitly. The angular differential equation (6.345) is invariant under the transformation $(s, k_j\Omega_j^{(0)}, m) \to (-s, -k_j\Omega_j^{(0)}, -m)$ holding λ fixed, so we can choose the relative normalization to be:

$$ {}_s\Theta_{\mathbf{k}lm}(\theta) = {}_{-s}\Theta_{(-\mathbf{k})l(-m)}(\theta). \tag{6.348} $$

The functions

$$ {}_sS_{\mathbf{k}lm}(\theta, \varphi) = e^{im\varphi}{}_s\Theta_{\mathbf{k}lm}(\theta) \tag{6.349} $$

are the spin-weighted spheroidal harmonics, and we can choose them to be orthonormal:

$$ \int d^2\Omega \; {}_sS^*_{\mathbf{k}lm}(\theta, \varphi) \; {}_sS_{\mathbf{k}l'm'}(\theta, \varphi) = \delta_{ll'}\delta_{mm'}. \tag{6.350} $$

Following Galt'sov [197], we make the phase choice:

$$ {}_sS_{\mathbf{k}lm}(\pi - \theta, \pi + \varphi) = (-1)^l{}_{-s}S_{\mathbf{k}lm}(\theta, \varphi). \tag{6.351} $$

Basis of modes

The radial equation (6.346) can be simplified by defining the tortoise coordinate r^* by

$$ dr^*/dr = \varpi^2/\Delta. \tag{6.352} $$

We can express r^* as

$$ r^* = r + \frac{2r_+}{r_+ - r_-} \ln \frac{r - r_+}{2} - \frac{2r_-}{r_+ - r_-} \ln \frac{r - r_-}{2}, \tag{6.353} $$

where

$$ r_\pm = M \pm \sqrt{M^2 - a^2} \tag{6.354} $$

are the two roots of $\Delta(r) = 0$. Introducing the variable ${}_su(r)$ defined by

$$ {}_sR(r) = \Delta^{-s/2}\varpi^{-1}{}_su(r), \tag{6.355} $$

the homogeneous radial equation can be written as an effective potential equation

$$0 = \frac{d^2\ {}_s u}{dr^{*2}} + {}_s V_{\mathbf{k}lm}\ {}_s u(r^*). \tag{6.356}$$

The effective potential ${}_s V_{\mathbf{k}lm}$ is complex (it is real for $s = 0$) and given by

$$\begin{aligned} {}_s V_{\mathbf{k}lm} \ &= \ \left(k_j \Omega_j^{(0)}\right)^2 + \frac{1}{\varpi^4}\left[-4aMrmk_j\Omega_j^{(0)} + a^2m^2 - 2is(r-M)K^{(0)}\right] \\ &\quad + \frac{\Delta}{\varpi^4}\left[4irk_j\Omega_j^{(0)}s - \lambda_{\mathbf{k}lm} + |s| - a^2\left(k_j\Omega_j^{(0)}\right)^2\right] - s^2\frac{(r-M)^2}{\varpi^4} \\ &\quad + \frac{\Delta}{\varpi^6}\left(4Mr - 3r^2 - a^2\right) + \frac{3r^2\Delta^2}{\varpi^8}. \end{aligned} \tag{6.357}$$

In the limit $r^* \to -\infty$ $(r \to r_+)$, the past and future event horizons the radial potential becomes:

$${}_s V_{\mathbf{k}lm} = p_{m\mathbf{k}}^2 \kappa_{sm\mathbf{k}}^2, \qquad r^* \to -\infty, \tag{6.358}$$

where we have defined the quantities $p_{m\mathbf{k}}$ and $\kappa_{sm\mathbf{k}}$ by

$$p_{m\mathbf{k}} \ = \ k_j\Omega_j^{(0)} - \frac{am}{2Mr_+}, \tag{6.359}$$

$$\kappa_{sm\mathbf{k}} \ = \ 1 - \frac{is(r_+ - r_-)}{4Mr_+ p_{m\mathbf{k}}}. \tag{6.360}$$

The solutions of the radial equation near the horizon are of the form

$${}_s u(r) \propto e^{\pm ip_{m\mathbf{k}}\kappa_{sm\mathbf{k}}r^*} = \Delta^{\pm s/2} e^{\pm ip_{m\mathbf{k}}r^*}\left[1 + O\left(\frac{1}{r^*}\right)\right]. \tag{6.361}$$

In the limit of $r^* \to \infty$ $(r \to \infty)$, past and future null infinity, the potential has the asymptotic behavior

$$V = \left(k_j\Omega_j^{(0)}\right)^2 + \frac{2isk_j\Omega_j^{(0)}}{r} + O\left(\frac{1}{r^2}\right), \tag{6.362}$$

so the radial solutions are of the form

$${}_s u(r) \propto r^{\mp s} e^{\pm ik_j\Omega_j^{(0)}r^*}. \tag{6.363}$$

We define the ("in", "up") basis of modes to be those with the following asymptotic behavior:

$$
{}_s u^{\rm in}_{\mathbf{k}lm} = \alpha_{s\mathbf{k}lm}(\tilde t) \begin{cases} \tau_{s\mathbf{k}lm}(\tilde t) \mid p_{m\mathbf{k}}(\tilde t) \mid^{-1/2} \Delta^{-s/2} e^{-ip_{m\mathbf{k}} r^*}, & r^* \to -\infty, \\ \mid k_j \Omega_j^{(0)}(\tilde t) \mid^{-1/2} \Big[r^s e^{-ik_j\Omega_j^{(0)} r^*} & \\ \qquad + \sigma_{s\mathbf{k}lm}(\tilde t) r^{-s} e^{ik_j \Omega_j^{(0)} r^*} \Big], & r^* \to \infty \end{cases} \tag{6.364}
$$

and

$$
{}_s u^{\rm up}_{\mathbf{k}lm} = \beta_{s\mathbf{k}lm}(\tilde t) \begin{cases} \mid p_{m\mathbf{k}}(\tilde t) \mid^{-1/2} \frac{k_j\Omega_j^{(0)}(\tilde t) p_{m\mathbf{k}}(\tilde t)}{|k_j\Omega_j^{(0)}(\tilde t) p_{m\mathbf{k}}(\tilde t)|} \Big[\mu_{s\mathbf{k}lm}(\tilde t) \Delta^{s/2} e^{ip_{m\mathbf{k}} r^*} & \\ \qquad + \nu_{s\mathbf{k}lm}(\tilde t) \Delta^{-s/2} e^{-ip_{m\mathbf{k}} r^*} \Big], & r^* \to -\infty, \\ \mid k_j \Omega_j^{(0)}(\tilde t) \mid^{-1/2} r^{-s} e^{ik_j\Omega_j^{(0)} r^*}, & r^* \to \infty. \end{cases} \tag{6.365}
$$

The modes (6.364) and (6.365) are similar to those defined in standard black hole perturbation theory except for the following properties:

1. The scattering, transmission and normalization coefficients depend on the slow variable $\tilde t$, i. e. they are constant only at fixed $\tilde t$.
2. They depend on the frequencies $k_j\Omega_j^{(0)}(\tilde t)$ rather than ω.

Noting that the effective potential ${}_sV_{\mathbf{k}lm}$ of Eq. (6.357) has the symmetry ${}_{-s}V^*_{\mathbf{k}lm} = {}_sV_{\mathbf{k}lm}$, we can define another basis: the "out" and "down" modes

$$
{}_s u^{\rm out}_{\mathbf{k}lm} = {}_{-s} u^{\rm in\ *}_{\mathbf{k}lm}, \tag{6.366}
$$

$$
{}_s u^{\rm down}_{\mathbf{k}lm} = {}_{-s} u^{\rm up\ *}_{\mathbf{k}lm}, \tag{6.367}
$$

with the following asymptotic behavior:

$$
{}_s u^{\rm out}_{\mathbf{k}lm} = \alpha^*_{-s\mathbf{k}lm}(\tilde t) \begin{cases} \tau^*_{-s\mathbf{k}lm}(\tilde t) \mid p_{m\mathbf{k}}(\tilde t) \mid^{-1/2} \Delta^{s/2} e^{ip_{m\mathbf{k}} r^*}, & r^* \to -\infty \\ \mid k_j \Omega_j^{(0)}(\tilde t) \mid^{-1/2} \Big[r^{-s} e^{ik_j\Omega_j^{(0)}(\tilde t) r^*} & \\ \qquad + \sigma^*_{-s\mathbf{k}lm}(\tilde t) r^{s} e^{-ik_j \Omega_j^{(0)}(\tilde t) r^*} \Big], & r^* \to \infty \end{cases} \tag{6.368}
$$

and

$$
{}_s u^{\rm down}_{\mathbf{k}lm} = \beta^*_{-s\mathbf{k}lm}(\tilde t)\begin{cases} \mid p_{m\mathbf{k}}(\tilde t)\mid^{-1/2} \frac{k_j\Omega^{(0)}_j(\tilde t)p_{m\mathbf{k}}(\tilde t)}{|k_j\Omega^{(0)}_j(\tilde t)p_{m\mathbf{k}}(\tilde t)|}\left[\mu^*_{-s\mathbf{k}lm}\Delta^{-s/2}e^{-ip_{m\mathbf{k}}r^*}\right. & \\ \qquad\qquad \left. + \ \nu^*_{-s\mathbf{k}lm}(\tilde t)\Delta^{s/2}e^{ip_{m\mathbf{k}}r^*}\right], & r^*\to-\infty \\ \mid \ k_j\Omega^{(0)}_j(\tilde t)\mid^{-1/2} r^s e^{-ik_j\Omega^{(0)}_j(\tilde t)r^*}, & r^*\to\infty \end{cases}
\tag{6.369}
$$

See Fig. (6.1) for an illustration of the asymptotic properties of the two bases of modes.

We now define the following complete Teukolsky mode functions:

$$
\begin{aligned}
{}_s\Psi^{\rm in}_{\mathbf{k}lm}(q_i,\tilde t,r,\theta,\varphi) &= e^{-ik_jf_j(\tilde t)}\ \Delta^{-s/2}\varpi^{-1}\ {}_s u^{\rm in}_{\mathbf{k}lm}(r)\ {}_sS_{\mathbf{k}lm}(\theta,\varphi), & (6.370)\\
{}_s\Psi^{\rm up}_{\mathbf{k}lm}(q_i,\tilde t,r,\theta,\varphi) &= e^{-ik_jf_j(\tilde t)}\ \Delta^{-s/2}\varpi^{-1}\ {}_s u^{\rm up}_{\mathbf{k}lm}(r)\ {}_sS_{\mathbf{k}lm}(\theta,\varphi), & (6.371)\\
{}_s\Psi^{\rm out}_{\mathbf{k}lm}(q_i,\tilde t,r,\theta,\varphi) &= e^{-ik_jf_j(\tilde t)}\Delta^{-s/2}\varpi^{-1}\ {}_s u^{\rm out}_{\mathbf{k}lm}(r)\ {}_sS_{\mathbf{k}lm}(\theta,\varphi), & (6.372)\\
{}_s\Psi^{\rm down}_{\mathbf{k}lm}(q_i,\tilde t,r,\theta,\varphi) &= e^{-ik_jf_j(\tilde t)}\ \Delta^{-s/2}\varpi^{-1}\ {}_s u^{\rm down}_{\mathbf{k}lm}(r)\ {}_sS_{\mathbf{k}lm}(\theta,\varphi). & (6.373)
\end{aligned}
$$

Retarded Green's function

The Green's function ${}_sG(x,x')$ is defined such that if ${}_s\Psi^{(1)}$ obeys the Teukolsky equation (6.339) with source ${}_s\mathcal{T}^{(1)}$

$$
{}_s\Box^{(0)}{}_s\Psi^{(1)}(q_i,\tilde t,x^i) = {}_s\mathcal{T}^{(1)}(q_i,\tilde t,x^i), \tag{6.374}
$$

then the solution is

$$
{}_s\Psi^{(1)}(q_i,\tilde t,x^i) = \int d^3q'_i\int d^3x'_i\sqrt{-g(x')}\ {}_sG(q_i,\tilde t,x^i,q'_i,\tilde t',x^{i\prime})\ {}_s\mathcal{T}^{(1)}(q'_i,\tilde t',x^{i\prime}). \tag{6.375}
$$

Since the variables f_i are periodic with period 2π, we can expand the various functions in Fourier series:

$$ {}_s\mathcal{T}^{(1)}(q_i,\tilde{t},x^i) = \sum_{\mathbf{k}} {}_s\tilde{\mathcal{T}}^{(1)}_{\mathbf{k}}(\tilde{t},x^i) e^{-ik_j q_j}, \tag{6.376} $$

where $\mathbf{k} = (k_r, k_\theta, k_\varphi)$ and the Fourier coefficients are given by

$$ {}_s\tilde{\mathcal{T}}^{(1)}_{\mathbf{k}}(\tilde{t},r,\theta,\varphi) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d^3q \; e^{ik_j q_j} {}_s\mathcal{T}^{(1)}(q_i,\tilde{t},r,\theta,\varphi) \tag{6.377} $$

and

$$ {}_s\tilde{\Psi}^{(1)}_{\mathbf{k}}(\tilde{t},r,\theta,\varphi) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d^3q \; e^{ik_j q_j} {}_s\Psi^{(1)}(q_i,\tilde{t},r,\theta,\varphi). \tag{6.378} $$

Here, we have used that $q_i = f_i + O(\varepsilon)$.

We make the following ansatz for the Green's function:

$$ {}_sG(q_i,\tilde{t},x^i,q_i',\tilde{t}',x^{i\prime}) = \sum_{\mathbf{k}} e^{-ik_j(q_j-q_j')} {}_s\tilde{G}_{\mathbf{k}}(r,\theta,\varphi;r',\theta',\varphi';\tilde{t}). \tag{6.379} $$

Here, we have used that $\tilde{t} = \tilde{t}'$ since we specialize to a t =const. foliation.

Inserting these definitions into the defining relation (6.375) and using $\sqrt{-g} = \Sigma\sin\theta$ gives

$$ {}_s\tilde{\Psi}^{(1)}_{\mathbf{k}}(\tilde{t},r,\theta,\varphi) = \int_0^\infty dr' \int d^2\Omega' \; \Sigma(r',\theta') \; {}_s\tilde{G}_{\mathrm{ret}\,\mathbf{k}}(r,\theta,\varphi;r',\theta',\varphi';\tilde{t}) \; {}_s\tilde{\mathcal{T}}^{(1)}_{\mathbf{k}}(\tilde{t},r',\theta',\varphi'). \tag{6.380} $$

We will omit the superscript (1) on Ψ and $\mathcal{T}$ for the remainder of this discussion. Next, we decompose the quantities ${}_s\tilde{\Psi}_{\mathbf{k}}$ and $\Sigma_s\tilde{\mathcal{T}}_{\mathbf{k}}$ on the basis of spin-weighted spheroidal harmonics:

$$ {}_s\tilde{\Psi}_{\mathbf{k}}(\tilde{t},r,\theta,\varphi) = \sum_{lm} {}_sS_{\mathbf{k}lm}(\theta,\varphi) \; {}_sR_{\mathbf{k}lm}(r) \tag{6.381} $$

and

$$ \Sigma \; {}_s\tilde{\mathcal{T}}_{\mathbf{k}}(\tilde{t},r,\theta,\varphi) = r^2 \sum_{lm} {}_sS_{\mathbf{k}lm}(\theta,\varphi) \; {}_s\tilde{\mathcal{T}}_{\mathbf{k}lm}(r), \tag{6.382} $$

and we insert these decompositions into the Fourier transform of the differential equation (6.80). This gives

$$-\frac{d^2\ {}_s u_{\mathbf{k}lm}}{dr^{*2}} + {}_sV_{\mathbf{k}lm}\ {}_s u_{\mathbf{k}lm}(r^*) = {}_s s_{\mathbf{k}lm}, \tag{6.383}$$

where

$${}_s u_{\mathbf{k}lm}(r) = \Delta(r)^{s/2} \varpi\ {}_s R_{\mathbf{k}lm}(r), \tag{6.384}$$

the potential ${}_sV_{\mathbf{k}lm}$ is given by Eq. (6.357), and the source term is

$${}_s s_{\mathbf{k}lm} = \varpi^{-3} \Delta^{1+s/2}\ r^2\ {}_s\tilde{\mathcal{T}}_{\mathbf{k}lm}. \tag{6.385}$$

We denote by ${}_sG_{\mathbf{k}lm}(r^*, r^{*\prime})$ the Green's function for the differential equation (6.383):

$${}_s u_{\mathbf{k}lm}(r^*) = \int_{-\infty}^{\infty} dr^{*\prime}\ {}_sG_{\mathbf{k}lm}(r^*, r^{*\prime})\ {}_s s_{\mathbf{k}lm}(r^{*\prime}). \tag{6.386}$$

We note that we can express the Fourier-transformed Green's function $\tilde{G}_{\mathbf{k}}(r, \theta, \varphi; r', \theta', \varphi'; \tilde{t})$ in terms of $G_{\mathbf{k}lm}$ as:

$${}_s\tilde{G}_{\text{ret }\mathbf{k}}(x^i, x_i'; \tilde{t}) = \sum_{lm} {}_sS_{\mathbf{k}lm}(\theta, \varphi)\ {}_sS^*_{\mathbf{k}lm}(\theta', \varphi') \frac{{}_sG^{\text{ret}}_{\mathbf{k}lm}(r^*, r^{*\prime})}{\Delta^{s/2} \Delta'^{s/2} \varpi \varpi'}. \tag{6.387}$$

We now derive the formula for the retarded Green's function ${}_sG_{\mathbf{k}lm}(r^*, r^{*\prime})$. Suppose that the source $\mathcal{T}(x)$ is non-zero only in the finite range of values of r

$$r_{\min} \le r \le r_{\max}. \tag{6.388}$$

Then, the retarded solution ${}_s\Psi_{\text{ret}}(x)$ will be a solution of the homogeneous equation in the regions $r < r_{\min}$ and $r > r_{\max}$. Now, the retarded solution is determined uniquely by the condition that it vanish on the past event horizon E^- and on past null infinity $\mathcal{J}^-$. This property will be guaranteed if we impose the following boundary conditions:

1. When we expand ${}_s\Psi_{\rm ret}$ in the region $r < r_{\rm min}$ on the basis of solutions ${}_s\Psi^{\rm in}_{\mathbf{k}lm}(q_i, x^i, \tilde{t})$ and ${}_s\Psi^{\rm up}_{\mathbf{k}lm}(q_i, x^i, \tilde{t})$ of the homogeneous equation, only the "in" modes contribute. Then, since the "in" modes vanish on the past event horizon, ${}_s\Psi_{\rm ret}$ must also vanish on the past event horizon.

2. When we expand ${}_s\Psi_{\rm ret}$ in the region $r > r_{\rm max}$ on the basis of solutions ${}_s\Psi^{\rm in}_{\mathbf{k}lm}(q_i, x^i, \tilde{t})$ and ${}_s\Psi^{\rm in}_{\mathbf{k}lm}(q_i, x^i, \tilde{t})$, only the "up" modes contribute. Then, since the "up" modes vanish on past null infinity, ${}_s\Psi_{\rm ret}$ must also vanish on past null infinity.

Consider now the expression

$$
\begin{aligned}
{}_sG^{\rm ret}_{\mathbf{k}lm}(r^*, r^{*\prime}) \quad = \quad & \frac{1}{W({}_su^{\rm in}_{\mathbf{k}lm},\ {}_su^{\rm up}_{\mathbf{k}lm})}\Big[{}_su^{\rm up}_{\mathbf{k}lm}(r)\ {}_su^{\rm in}_{\mathbf{k}lm}(r')\theta(r-r') \\
& + {}_su^{\rm in}_{\mathbf{k}lm}(r)\ {}_su^{\rm up}_{\mathbf{k}lm}(r')\theta(r-r')\Big],
\end{aligned}
\tag{6.389}
$$

where $W(\tilde{t})$ is the conserved Wronskian. This expression satisfies the boundary conditions listed above as well as the differential equation (6.383) with the source replaced by $\delta(r^* - r^{*\prime})$, using the fact that the "in" and "up" modes satisfy the homogeneous version of the differential equation. This establishes the formula (6.389).

Next, we compute the Wronskian $W({}_su^{\rm in}_{\omega lm}, {}_su^{\rm up}_{\omega lm})$ using the asymptotic expressions (6.91) and (6.95) for the mode functions for $r^* \to \infty$. This gives

$$
W({}_su^{\rm in}_{\mathbf{k}lm},\ {}_su^{\rm up}_{\mathbf{k}lm}) = 2i\alpha_{s\mathbf{k}lm}(\tilde{t})\ \beta_{s\mathbf{k}lm}(\tilde{t})\frac{k_j\Omega_j(\tilde{t})}{|k_j\Omega_j(\tilde{t})|}. \tag{6.390}
$$

Inserting this into Eq. (6.389) and then into Eqs. (6.387) and (6.379) finally yields the formula

$$
\begin{aligned}
{}_sG_{\mathrm{ret}}(q_i, x^i, q_i', x^{i\prime}; \tilde{t}) = \frac{1}{2i} \sum_{\mathbf{k}} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} & \frac{1}{\alpha_{s\mathbf{k}lm}(\tilde{t})\beta_{s\mathbf{k}lm}(\tilde{t})} \frac{k_j \Omega_j(\tilde{t})}{|k_j \Omega_j(\tilde{t})|} e^{-ik_j(q_j - q_j')} \\
& {}_sS_{\mathbf{k}lm}(\theta, \varphi)\ {}_sS^*_{\mathbf{k}lm}(\theta', \varphi') \frac{1}{\varpi\varpi'} (\Delta\Delta')^{-s/2} \\
& \left[{}_su^{\mathrm{up}}_{\mathbf{k}lm}(r)\ {}_su^{\mathrm{in}}_{\mathbf{k}lm}(r')\theta(r - r') + {}_su^{\mathrm{in}}_{\mathbf{k}lm}(r)\ {}_su^{\mathrm{up}}_{\mathbf{k}lm}(r')\theta(r' - r) \right].
\end{aligned} \tag{6.391}
$$

Note that the expression (6.391) is independent of the values chosen for the normalization constants $\alpha_{s\mathbf{k}lm}(\tilde{t})$ and $\beta_{s\mathbf{k}lm}(\tilde{t})$, since the factor of $1/\alpha$ cancels a factor of α present in the definition (6.91) of the "in" modes, and similarly for β and the "up" modes.

The expression for the advanced Green's function is

$$
\begin{aligned}
{}_sG_{\mathrm{adv}}(q_i, x^i, q_i', x^{i\prime}; \tilde{t}) = \frac{-1}{2i} \sum_{\mathbf{k}} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} & \frac{1}{\alpha^*_{-s\mathbf{k}lm}(\tilde{t})\beta^*_{-s\mathbf{k}lm}(\tilde{t})} \frac{k_j \Omega_j(\tilde{t})}{|k_j \Omega_j(\tilde{t})|} \\
& e^{-ik_j(q_j - q_j')}\ {}_sS_{\mathbf{k}lm}(\theta, \varphi)\ {}_sS^*_{\mathbf{k}lm}(\theta', \varphi') \frac{1}{\varpi\varpi'} (\Delta\Delta')^{-s/2} \\
& \left[{}_su^{\mathrm{down}}_{\mathbf{k}lm}(r')\ {}_su^{\mathrm{out}}_{\mathbf{k}lm}(r)\theta(r' - r) + {}_su^{\mathrm{out}}_{\mathbf{k}lm}(r)\ {}_su^{\mathrm{down}}_{\mathbf{k}lm}(r')\theta(r - r') \right].
\end{aligned} \tag{6.392}
$$

Using the retarded and advanced Green's function ${}_sG_{\mathrm{ret}}(q_i, x^i, q_i', x^{i\prime}; \tilde{t})$ and ${}_sG_{\mathrm{adv}}(q_i, x^i, q_i', x^{i\prime}; \tilde{t})$ discussed above, we can construct the retarded and advanced solutions ${}_s\Psi^{(1)}_{\mathrm{ret}}(q_i, x^i, \tilde{t})$ and ${}_s\Psi^{(1)}_{\mathrm{adv}}(q_i, x^i, \tilde{t})$ of the Teukolsky equation (6.339). One half the retarded solution minus one half the advanced solution gives the radiative solution:

$$
{}_s\Psi^{\mathrm{rad}}(x^i, q_i, \tilde{t}) = \frac{1}{2} \left[{}_s\Psi^{\mathrm{ret}}(x^i, q_i, \tilde{t}) - {}_s\Psi^{\mathrm{adv}}(x^i, q_i, \tilde{t}) \right]. \tag{6.393}
$$

The radiative solution is given in terms of a radiative Green's function

$$
{}_s\Psi_{\mathrm{rad}}(x^i, q_i, \tilde{t}) = \int d^3q' \int d^3x' \sqrt{-g(x')}\ {}_sG_{\mathrm{rad}}(x^i, q_i, x^{i\prime}, q_i'; \tilde{t})\ {}_s\mathcal{T}(x^{i\prime}, q_i'; \tilde{t}), \tag{6.394}
$$

where

$$
{}_sG_{\mathrm{rad}}(q_i, x^i, q_i', x^{i\prime}; \tilde{t}) = \frac{1}{2} \left[{}_sG_{\mathrm{ret}}(q_i, x^i, q_i', x^{i\prime}; \tilde{t}) - {}_sG_{\mathrm{adv}}(q_i, x^i, q_i', x^{i\prime}; \tilde{t}) \right]. \tag{6.395}
$$

The expression for the radiative Green's function is

$$
\begin{aligned}
{}_sG_{\rm rad}(q_i, x^i, q_i', x^{i\prime}; \tilde t) = \frac{1}{4i} \sum_{\mathbf{k}} \frac{k_j \Omega_j(\tilde t)}{|k_j\Omega_j(\tilde t)|} e^{-ik_j(q_j - q_j')} \sum_{lm} {}_sS_{\mathbf{k}lm}(\theta,\varphi)\ {}_sS^*_{\mathbf{k}lm}(\theta',\varphi') \\
\frac{(\Delta/\Delta')^{-s/2}}{\varpi\varpi'} \left[\frac{1}{\alpha^*_{-s\mathbf{k}lm}(\tilde t)\alpha_{s\mathbf{k}lm}(\tilde t)} {}_su^{\rm out}_{\mathbf{k}lm}(r)\ {}_{-s}u^{{\rm out}\ *}_{\mathbf{k}lm}(r') \right. \qquad (6.396) \\
\left. + \frac{\kappa_{s\mathbf{k}m}\tau_{s\mathbf{k}lm}(\tilde t)\tau^*_{-s\mathbf{k}lm}(\tilde t)}{\beta_{s\mathbf{k}lm}(\tilde t)\beta^*_{-s\mathbf{k}lm}(\tilde t)} \frac{k_j\Omega_j p_{m\mathbf{k}}}{|k_j\Omega_j p_{m\mathbf{k}}|}\ {}_su^{\rm down}_{\mathbf{k}lm}(r)\ {}_{-s}u^{{\rm down}\ *}_{\mathbf{k}lm}(r') \right].
\end{aligned}
$$

Using the expression (6.391) for the retarded Green's function together with the integral expression (6.375), we can compute the retarded field ${}_s\Psi_{\rm ret}(q_i, x^i, \tilde t)$ generated by the source ${}_s\mathcal{T}(q_i, x^i, \tilde t)$. For the case of interest here, ${}_s\mathcal{T}(q_i, x^i, \tilde t)$ will be nonzero only in a finite range of values of r of the form

$$
r_{\rm min} \le r \le r_{\rm max}. \qquad (6.397)
$$

For $r > r_{\rm max}$, only the first term in the square brackets in Eq. (6.391) will contribute, and the function $\theta(r - r')$ will always be 1. This gives, using the definition (6.375):

$$
\begin{aligned}
{}_s\Psi_{\rm ret}(x^i, q_i, \tilde t) &= \frac{1}{2i} \sum_{\mathbf{k}} \sum_{lm} \frac{k_j\Omega_j(\tilde t)}{|\ k_j\Omega_j(\tilde t)\ |} e^{-ik_jq_j} \frac{{}_sR^{\rm up}_{\mathbf{k}lm}(r)\ {}_sS_{lm\mathbf{k}}(\theta,\varphi)}{\alpha_{s\mathbf{k}lm}(\tilde t)\beta_{s\mathbf{k}lm}(\tilde t)} \\
&\quad \frac{1}{(2\pi)^3} \int d^3q' \int d^3x' \sqrt{-g}\ {}_{-s}R^{{\rm out}\ *}_{\mathbf{k}lm}(r')\ {}_sS^*_{\mathbf{k}lm}(\theta'\varphi') e^{ik_jq_j'}\ {}_s\tau_{ab}(x') T^{ab}(q_i', \tilde t, x^{i\prime}) \\
&= \frac{1}{2i} \sum_{\mathbf{k}} \frac{1}{\alpha_{s\mathbf{k}lm}(\tilde t)\beta_{s\mathbf{k}lm}(\tilde t)} \frac{k_j\Omega_j(\tilde t)}{|\ k_j\Omega_j(\tilde t)\ |} {}_sZ^{\rm out}_{\mathbf{k}lm}(\tilde t)\ {}_s\Psi^{\rm up}_{\mathbf{k}lm}(q_i, \tilde t, x^i), \qquad (6.398)
\end{aligned}
$$

where the amplitude ${}_sZ^{\rm out}_{\mathbf{k}lm}(\tilde t)$ is given by the the following inner product:

$$
{}_sZ^{\rm out}_{\mathbf{k}lm}(\tilde t) = \langle {}_s\tau^\dagger_{ab}\ {}_{-s}R^{\rm out}_{\mathbf{k}lm}\ {}_sS_{\mathbf{k}lm} e^{-ik_jq_j},\ T^{ab} \rangle, \qquad (6.399)
$$

where the angular brackets denote the scalar product on the 6-dimensional manifold. For two tensor fields $\phi(x^i, q_i, \tilde t)$ and $\psi(x^i, q_i, \tilde t)$ of equal rank it is given by:

$$
\langle \phi(x^i, q_i, \tilde t), \psi(x^i, q_i, \tilde t) \rangle = \frac{1}{(2\pi)^3} \int d^3x \int_0^{2\pi} d^3q\ \sqrt{-g}\ \phi^*_{ab\ldots}(x^i, q_i, \tilde t)\ \psi^{ab\ldots}(x^i, q_i, \tilde t). \qquad (6.400)
$$

Similarly for $r < r_{\rm min}$ we obtain for the retarded field

$$ {}_s\Psi_{\rm ret}(q_i,\tilde{t},x^i) = \frac{1}{2i}\sum_{\mathbf{k}}\sum_{lm}\frac{1}{\alpha_{s\mathbf{k}lm}(\tilde{t})\beta_{s\mathbf{k}lm}(\tilde{t})}\frac{k_j\Omega_j(\tilde{t})}{\mid k_j\Omega_j(\tilde{t})\mid}{}_sZ^{\rm down}_{\mathbf{k}lm}(\tilde{t}){}_s\Psi^{\rm in}_{\mathbf{k}lm}(x^i,q_i,\tilde{t}), $$

where

$$ {}_sZ^{\rm down}_{\mathbf{k}lm}(\tilde{t}) = \langle {}_s\tau^{\dagger}_{ab}\ {}_{-s}R^{\rm down}_{\mathbf{k}lm}\ {}_sS_{\mathbf{k}lm}(\theta'\varphi')e^{-ik_jq_j},\ T^{ab}\rangle. \tag{6.401} $$

Similarly, from the expression (6.395) for the radiative Green's function, together with Eq. (6.393), we obtain the radiative field:

$$ \begin{aligned} {}_s\Psi_{\rm rad}(q_i,\tilde{t},x^i) = \frac{1}{4i}\sum_{\mathbf{k}}\sum_{l=2}^{\infty}\sum_{m=l}^{l}\frac{k_j\Omega_j(\tilde{t})}{\mid k_j\Omega_j(\tilde{t})\mid}\Bigg[\frac{1}{\alpha^*_{-s\mathbf{k}lm}\ \alpha_{s\mathbf{k}lm}}\ {}_sZ^{\rm out}_{\mathbf{k}lm}(\tilde{t})\ {}_s\Psi^{\rm out}_{\mathbf{k}lm}(x) \\ +\frac{1}{\beta_{s\mathbf{k}lm}\ \beta^*_{-s\mathbf{k}lm}}\frac{k_j\Omega_jp_{m\mathbf{k}}}{|k_j\Omega_jp_{m\mathbf{k}}|}\kappa_{s\mathbf{k}m}\ \tau_{s\mathbf{k}lm}\ \tau^*_{-s\mathbf{k}lm}\ \ {}_sZ^{\rm down}_{\mathbf{k}lm}(\tilde{t})\ {}_s\Psi^{\rm down}_{\mathbf{k}lm}(x)\Bigg]. \end{aligned} $$

All of these expressions depend on the amplitudes ${}_sZ^{\rm out}_{\mathbf{k}lm}(\tilde{t})$ and ${}_sZ^{\rm down}_{\mathbf{k}lm}(\tilde{t})$.

Amplitudes

In this subsection, we show that the amplitudes ${}_sZ_{\mathbf{k}lm} = {}_sZ_{k_rk_\theta k_\varphi lm}$ contain a term $\delta_{k_\varphi,m}$, i. e. that there are only four independent indices k_r, k_θ, l, m just as in the standard formalism. From the treatment of the orbital motion in Ref. [200] [and Chapter 5], it follows that the orbital phase $\varphi(t)$ can be written as

$$ \varphi(t) = q_\varphi(t) + \sum_{k_A}\Phi_{k_A}(J_\lambda,\tilde{t})e^{ik_Aq_A} \equiv q_\varphi(t) + \delta\phi(q_A,\tilde{t}), \tag{6.402} $$

where we use the notation of Ref. [200] to denote $k_A = (k_r, k_\theta)$ and $q_A = (q_r, q_\theta)$. The particle's stress-energy tensor is given by

$$ T^{(1)}_{ab} = \mu\frac{u_au_b}{\sqrt{-g}}\left(\frac{dt}{d\tau}\right)^{-1}\delta\left(r - r(q_A,\tilde{t})\right)\delta\left(\theta - \theta(q_A,\tilde{t})\right)\delta\left(\varphi - \varphi(q_\varphi,q_A,\tilde{t})\right). \tag{6.403} $$

Here, $u_a = [-E^{(0)}(\tilde{t}), u_r^{(0)}[q_r, \tilde{t}], u_\theta^{(0)}[q_\theta, \tilde{t}], L_z^{(0)}(\tilde{t})]$ and $(dt/d\tau) = \Omega_t^{(0)} + \sum ik_A \Omega_A^{(0)} T_{k_A}(J_\lambda^{(0)}) \exp[ik_A q_A] + O(\varepsilon)$. Substituting Eq. (6.403), together with Eq. (6.402) into the expression for the amplitude in Eq. (6.399) and using the definition of the inner product in Eq. (6.400) yields

$$
\begin{aligned}
{}_sZ^{\text{out}}_{\mathbf{k}lm}(\tilde{t}) &= \frac{\mu}{(2\pi)^3} \int d^2q_A \int dq_\varphi \int d^3x \; {}_{-s}R^{\text{up}\,*}_{\mathbf{k}lm} \left[r(q_A, \tilde{t})\right] \; {}_s\Theta_{\mathbf{k}lm} \left[\theta(q_A, \tilde{t})\right] \\
&\qquad e^{im\varphi(q_\varphi, q_A, \tilde{t})} \; e^{-ik_\varphi q_\varphi - ik_A q_A} \mathcal{S}(q_A, \tilde{t}) \qquad (6.404) \\
&= \frac{\mu}{(2\pi)^2} \int d^2q_A \int d^3x \; {}_{-s}R^{\text{up}\,*}_{\mathbf{k}lm} \left[r(q_A, \tilde{t})\right] \; {}_s\Theta_{\mathbf{k}lm} \left[\theta(q_A, \tilde{t})\right] \; e^{im\delta\phi(q_A, \tilde{t})} \\
&\qquad e^{-ik_A q_A} \mathcal{S}(q_A, \tilde{t}) \; \delta_{k_\varphi \; m}. \qquad (6.405)
\end{aligned}
$$

Here, we denote $\mathcal{S}(q_A, \tilde{t}) = {}_s\tau_{ab} u^a u^b$.

Waveforms

For $r \to \infty$, the quantity $\rho^4{}_{-2}\Psi^{(1)} = \psi_4^{(1)}$ is related to $h_{ab}^{(1)}$ by

$$
\psi_4^{(1)} = \frac{1}{2} \left(\Omega_i^{(0)} \partial_{f_i} \right)^2 \left(h_+^{(1)} - ih_\times^{(1)} \right). \qquad (6.406)
$$

For any multiply periodic function f and for any vector $\mathbf{v} = (v_1, \ldots, v_N)$, we define the quantity $\mathcal{I}_\mathbf{v}\hat{f}$ by

$$
(\mathcal{I}_\mathbf{v}\hat{f})(\mathbf{q}) \equiv \sum_{\mathbf{k} \neq \mathbf{0}} \frac{f_\mathbf{k}}{i\mathbf{k} \cdot \mathbf{v}} e^{i\mathbf{k} \cdot \mathbf{q}}, \qquad (6.407)
$$

where $f_\mathbf{k} = \int d^N q e^{-i\mathbf{k} \cdot \mathbf{q}} f(\mathbf{q})/(2\pi)^N$ are the Fourier coefficients of f.

Using Eq. (6.407) in Eq. (6.406) gives for the waveform

$$
h_+^{(1)} - ih_\times^{(1)} = 2\mathcal{I}_{\mathbf{\Omega^{(0)}}} \mathcal{I}_{\mathbf{\Omega^{(0)}}} \rho^4{}_{-2}\Psi^{(1)}, \qquad (6.408)
$$

and substituting the expression (6.402) with $s = -2$ we obtain the explicit formula for the radiative fields

$$[h_+^{(1)\ \mathrm{rad}} - i h_\times^{(1)\ \mathrm{rad}}](q_i, \tilde{t}, x^i) = \frac{1}{2} \sum_{k_A} \sum_{l=2}^{\infty} \sum_{m=l}^{l} \frac{1}{\left(k_A \Omega_A^{(0)}(\tilde{t}) + m\Omega_\varphi^{(0)}(\tilde{t})\right)^2}$$

$$\left[\frac{\gamma_{k_A lm}^{\mathrm{out}}(\tilde{t})}{\alpha^*_{2k_A lm}(\tilde{t})}\ \rho^4\ {}_{-2}\Psi_{k_A lm}^{\mathrm{out}}(q_i, \tilde{t}, x^i) + \frac{\tilde{\gamma}_{k_A lm}^{\mathrm{down}}(\tilde{t})}{\beta^*_{2k_A lm}(\tilde{t})}\ \rho^4\ {}_{-2}\Psi_{k_A lm}^{\mathrm{down}}(q_i, \tilde{t}, x^i) \right].$$

Here, we have defined the following coefficients

$$\gamma_{k_A lm}^{\mathrm{out}}(\tilde{t}) = \frac{k_A \Omega_A(\tilde{t}) + m\Omega_\varphi(\tilde{t})}{\mid k_A \Omega_A(\tilde{t}) + m\Omega_\varphi(\tilde{t}) \mid} \frac{1}{i\alpha_{-2k_A lm}(\tilde{t})}\ {}_{-2}Z_{k_A lm}^{\mathrm{out}}(\tilde{t}), \tag{6.409}$$

$$\gamma_{k_A lm}^{\mathrm{down}}(\tilde{t}) = \frac{k_A \Omega_A(\tilde{t}) + m\Omega_\varphi(\tilde{t})}{\mid k_A \Omega_A(\tilde{t}) + m\Omega_\varphi(\tilde{t}) \mid} \frac{1}{i\beta_{-2k_A lm}(\tilde{t})}\ {}_{-2}Z_{k_A lm}^{\mathrm{down}}(\tilde{t}) \tag{6.410}$$

$$\tilde{\gamma}_{k_A lm}^{\mathrm{down}}(\tilde{t}) = \frac{[k_A \Omega_A(\tilde{t}) + m\Omega_\varphi(\tilde{t})]p_{mk_A}(\tilde{t})}{|[k_A \Omega_A(\tilde{t}) + m\Omega_\varphi(\tilde{t})]p_{mk_A}(\tilde{t})|} \kappa_{-2k_A m}(\tilde{t})\ \tau_{-2k_A lm}(\tilde{t})\ \tau^*_{2k_A lm}(\tilde{t})\ \gamma_{k_A lm}^{\mathrm{down}}(\tilde{t}).$$

The retarded fields are given by a similar expression, namely

$$[h_+^{(1)\ \mathrm{ret}} - i h_\times^{(1)\ \mathrm{ret}}](q_i, \tilde{t}, x^i) = \sum_{k_A=-\infty}^{\infty} \sum_{l=2}^{\infty} \sum_{m=l}^{l} \frac{1}{\left(k_A \Omega_A^{(0)}(\tilde{t}) + m\Omega_\varphi^{(0)}(\tilde{t})\right)^2}$$

$$\left[\frac{\gamma_{k_A lm}^{\mathrm{out}}(\tilde{t})}{\beta_{-2k_A lm}(\tilde{t})}\ \rho^4\ {}_{-2}\Psi_{k_A lm}^{\mathrm{up}}(q_i, \tilde{t}, x^i) + \frac{\gamma_{k_A lm}^{\mathrm{down}}(\tilde{t})}{\alpha_{-2k_A lm}(\tilde{t})}\ \rho^4\ {}_{-2}\Psi_{k_A lm}^{\mathrm{in}}(q_i, \tilde{t}, x^i) \right].$$

Note that, as discussed below Eq. (6.391), this expression is actually independent of the normalization functions α and β.

In the limit $r \to \infty$, $\rho^4 \to r^{-4}$, and using Eq. (6.365), the leading order behavior of the radial function ${}_{-2}R^{\mathrm{up}}$ is

$${}_{-2}R^{\mathrm{up}} \to \beta_{-2k_A lm} \mid k_A \Omega_A^{(0)}(\tilde{t}) + m\Omega_\varphi(\tilde{t}) \mid^{-1/2}\ r^3\ e^{i\left[k_A \Omega_A^{(0)} + m\Omega_\varphi^{(0)}\right] r^*}. \tag{6.411}$$

The leading order retarded waveform at $r \to \infty$ then has the behavior

$$h_+^{(1)\ \infty} - i h_\times^{(1)\ \infty} = \frac{1}{r} \sum_{k_A=-\infty}^{\infty} \sum_{l=2}^{\infty} \sum_{m=l}^{l} \frac{1}{\left(k_A \Omega_A^{(0)}(\tilde{t}) + m\Omega_\varphi^{(0)}(\tilde{t})\right)^2}$$

$$\frac{\gamma_{k_A lm}^{\mathrm{out}}(\tilde{t})}{|k_A \Omega_A^{(0)}(\tilde{t}) + m\Omega_\varphi^{(0)}(\tilde{t})|^{1/2}}\ {}_{-2}S_{k_A lm}(\theta, \varphi) e^{-ik_j\left[f_j(\tilde{t}) - \Omega_j^{(0)} r^\star\right]}. \tag{6.412}$$

This shows that at this order, no matching at large r is required o read off the asymptotic waveform.

6.7 Appendix: Sketch of the derivation of the Teukolsky-Starobinsky identities

In his derivation of these identities, Bardeen follows Teukolsky and Press [208] and considers the asymptotic behavior at infinity for an ingoing solution

$$ {}_2\Psi^{\rm in} = \int d\omega \sum_{lm} {}_2R^{\rm in}_{\omega lm}\ {}_2\Theta_{\omega lm}(\theta) e^{-i\omega v} e^{im\varphi}. \tag{6.413} $$

Here $v = t + r^*$ is the advanced time coordinate we already used in the discussion of the asymptotic behavior of the "in" modes. The asymptotic forms of the relevant Newman-Penrose quantities in the limit $r \to \infty$, $v =$ fixed, are given in Appendix B of [208]. Working to leading order in $1/r$, one can combine the perturbed Newman-Penrose equations to obtain (see [208, 209])

$$ \mathcal{L}_{-1}\ \mathcal{L}_0\ \mathcal{L}_1 \mathcal{L}_2\ {}_2\Psi = 64\ \partial_v^4\ {}_{-2}\Psi + 24\sqrt{2}\ M\partial_v^2\ \mathcal{L}_{-1}\ \pi^{\rm pert.}, \tag{6.414} $$

where $\pi^{\rm pert.}$ means the linearized perturbation to the spin coefficient π. Next, taking the complex conjugate of Eq. (B2) in [208] and using the result in their Eq. (B5) gives that

$$ \partial_v^2\ \mathcal{L}_{-1}\ \pi^{\rm pert.} = \frac{1}{2\sqrt{2}} \partial_v\ {}_2\Psi^*. \tag{6.415} $$

Combining Eqs. (6.414) and (6.415) gives the final result

$$ \mathcal{L}_{-1}\ \mathcal{L}_0\ \mathcal{L}_1\ \mathcal{L}_2\ {}_2\Psi - 12M\partial_v\ {}_2\Psi^* = 64\ \partial_v^4\ {}_{-2}\Psi. \tag{6.416} $$

The mode expansion of ${}_2\Psi^*$ is the complex conjugate of Eq. (6.413):

$$ {}_2\Psi^{\rm in\ *} = \int d\omega \sum_{lm} {}_2R^{\rm in\ *}_{\omega lm}\ {}_2\Theta^*_{\omega lm}(\theta) e^{i\omega v} e^{-im\varphi}. \tag{6.417} $$

In order for all the functions in Eq. (6.416) to have the same phase factor $e^{-i\omega t + im\varphi}$, we reverse the signs of ω and m in Eq. (6.417) and rewrite it as:

$$ {}_2\Psi^{\rm in\ *} = \int d\omega \sum_{lm} {}_2R^{\rm in\ *}_{(-\omega)l(-m)}\ {}_{-2}\Theta_{\omega lm}(\theta) e^{-i\omega v} e^{im\varphi}, \tag{6.418} $$

where we have used the fact that the angular function is real and satisfies Eq. (6.348). The function ${}_2R^{\text{in}\,*}_{-\omega l-m}$ satisfies the same differential equation as the function ${}_2R^{\text{in}}_{\omega lm}$ [as can be seen from Eq. (6.56) or (6.81)], but the key result of Bardeen is that these functions are not equal. The relation (6.416) then becomes:

$$
\begin{aligned}
&\int d\omega \sum_{lm} e^{-i\omega v+im\varphi} \big\{\mathcal{L}_{-1m\omega}\ \mathcal{L}_{0m\omega}\ \mathcal{L}_{1m\omega}\ \mathcal{L}_{2m\omega}\ {}_2\Theta_{\omega lm}\ {}_2R^{\text{in}}_{\omega lm} \\
&\qquad +12Mi\omega\ {}_{-2}\Theta_{\omega lm}\ {}_2R^{\text{in}*}_{(-\omega)l(-m)}\big\} \\
&= \sum_{lm} e^{-i\omega v+im\varphi}(64\ \omega^4)\ {}_{-2}\Theta_{\omega lm}\ {}_{-2}R^{\text{in}}_{\omega lm}. \qquad (6.419)
\end{aligned}
$$

Next, we use the fact that Teukolsky [206] has shown that the angular functions satisfy the relations (6.113) and (6.114). One can verify these as follows. Equation (6.113) can be reformulated with the aid of Eqs. (6.71) and (6.72) to be:

$$
\begin{aligned}
&\mathcal{L}_{-1m\omega}\ \mathcal{L}_{0m\omega}\ \mathcal{L}_{1m\omega}\ \mathcal{L}_{2m\omega}\left(\mathcal{L}_{-1m\omega}\ \mathcal{L}^+_{2m\omega} - 6a\omega\cos\theta\right)\ {}_2\Theta_{\omega lm} \\
&= \left(\mathcal{L}_{-1m\omega}\ \mathcal{L}^+_{2m\omega} - 6a\omega\cos\theta\right)\mathcal{L}_{-1m\omega}\ \mathcal{L}_{0m\omega}\ \mathcal{L}_{1m\omega}\ \mathcal{L}_{2m\omega}\ {}_2\Theta_{\omega lm}. \quad (6.420)
\end{aligned}
$$

This expression, and the corresponding relation obtained from the "+" transformation $(\omega, m) \to (-\omega, -m)$ are equivalent to

$$
\begin{aligned}
\mathcal{L}_{-1m\omega}\ \mathcal{L}_{0m\omega}\ \mathcal{L}_{1m\omega}\ \mathcal{L}_{2m\omega}\ {}_2\Theta_{\omega lm} &= F_{-2\omega lm}\ {}_{-2}\Theta_{\omega lm} \\
\mathcal{L}^+_{-1m\omega}\ \mathcal{L}^+_{0m\omega}\ \mathcal{L}^+_{1m\omega}\ \mathcal{L}^+_{2m\omega}\ {}_{-2}\Theta_{\omega lm} &= F_{2\omega lm}\ {}_2\Theta_{\omega lm}. \qquad (6.421)
\end{aligned}
$$

The (real) coefficients F_2 and F_{-2} can be determined using the normalization integral for the functions ${}_s\Theta_{\omega lm}$ [we chose both ${}_2\Theta_{\omega lm}$ and ${}_{-2}\Theta_{\omega lm}$ to be normalized to unity in Eq. (6.66)]:

$$
\begin{aligned}
F^2_{-2\omega lm} &= F^2_{-2\omega lm}\int_0^\pi \, {}_{-2}\Theta^2_{\omega lm}\sin\theta d\theta \\
&= \int_0^\pi (\mathcal{L}_{-1m\omega}\ \mathcal{L}_{0m\omega}\ \mathcal{L}_{1m\omega}\ \mathcal{L}_{2m\omega}\ {}_2\Theta_{\omega lm})^2\sin\theta d\theta \qquad (6.422)\\
&= \int_0^\pi \, {}_2\Theta_{\omega lm}\ \mathcal{L}^+_{-1m\omega}\ \mathcal{L}^+_{0m\omega}\ \mathcal{L}^+_{1m\omega}\ \mathcal{L}^+_{2m\omega}\ \mathcal{L}_{-1m\omega}\ \mathcal{L}_{0m\omega}\ \mathcal{L}_{1m\omega}\ \mathcal{L}_{2m\omega}\ {}_2\Theta_{\omega lm}\sin\theta d\theta \\
&= F_{-2\omega lm}F_{2\omega lm}\int_0^\pi \, {}_2\Theta^2_{\omega lm}\sin\theta d\theta \\
&= F_{-2\omega lm}F_{2\omega lm}, \qquad (6.423)
\end{aligned}
$$

where in the second line we have used integration by parts [which is equivalent to using Eq. (6.48)]. This establishes that

$$F_{-2\omega lm} = F_{2\omega lm} \equiv F_{\omega lm}. \qquad (6.424)$$

Working out the algebra for the operator in Eq. (6.422) yields [208]: $F^2_{\omega lm} = |C_{\omega lm}|^2 - (12M\omega)^2 = (\Re C_{\omega lm})^2$, where $C_{\omega lm}$ is given by Eqs. (6.117) and (6.118).

We now use the relation (6.113) in Eq. (6.419), which leads to the radial relation

$$
\begin{aligned}
\int d\omega \sum_{lm} e^{-i\omega v + im\varphi}\ {}_{-2}\Theta_{\omega lm} \quad & \Big\{ F_{\omega lm}\ {}_2R^{\rm in}_{\omega lm} + 12iM\omega\ {}_2R^{{\rm in}\ *}_{(-\omega)l(-m)} \\
& = (64\ \omega^4)\ {}_{-2}R^{\rm in}_{\omega lm}\Big\}. \qquad (6.425)
\end{aligned}
$$

Noting that asymptotically, $\mathcal{D}_{0m\omega}\ {}_{-2}\Psi^{\rm in}_{\omega lm} \sim 2\omega\ {}_{-2}\Psi^{\rm in}_{\omega lm}$, and using Eqs. (6.117) and (6.118), Eq. (6.425) is equivalent to

$$
\begin{aligned}
\int d\omega \sum_{lm} e^{-i\omega v + im\varphi}\ {}_{-2}\Theta_{\omega lm} \quad & \Big\{ 4\ \mathcal{D}^4_{0m\omega}\ {}_{-2}R^{\rm in}_{\omega lm} \qquad (6.426)\\
& = \Re(C_{\omega lm})\ {}_2R^{\rm in}_{\omega lm} + i\ \Im(C_{\omega lm})\ {}_2R^{{\rm in}\ *}_{(-\omega)l(-m)}\Big\}.
\end{aligned}
$$

Taking the complex conjugate of Eq. (6.419), and relabeling $(\omega, m) \to (-\omega, -m)$ in order to have the same phase factor, using the expressions (6.68) and (6.69), and finally using the angular relation (6.114) gives:

$$\int d\omega \sum_{lm} e^{-i\omega v+im\varphi}\ {}_2\Theta_{\omega lm} \left\{4\ \mathcal{D}^4_{0m\omega}\ {}_{-2}R^{\mathrm{in}\ *}_{(-\omega)l(-m)} \right.$$
$$= \Re(C_{\omega lm})\ {}_2R^{\mathrm{in}\ *}_{(-\omega)l(-m)} + i\ \Im(C_{\omega lm})\ {}_2R^{\mathrm{in}}_{\omega lm} \Big\}\,. \qquad (6.427)$$

We can rewrite this using the parity operator and Eq. (6.67):

$$\int d\omega \sum_{lm}\ pP\ e^{-i\omega v+im\varphi}\ {}_{-2}\Theta_{\omega lm} \left\{4\ \mathcal{D}^4_{0m\omega}\ {}_{-2}R^{\mathrm{in}\ *}_{(-\omega)l(-m)} \right.$$
$$= \Re(C_{\omega lm})\ {}_2R^{\mathrm{in}\ *}_{(-\omega)l(-m)} + i\ \Im(C_{\omega lm})\ {}_2R^{\mathrm{in}}_{\omega lm} \Big\}\,. \qquad (6.428)$$

We will ultimately be interested in the sum over $p = \pm 1$. We now define

$${}_sR^{\mathrm{in\ E}}_{\omega lm} = {}_sR^{\mathrm{in}}_{\omega lm} + {}_sR^{\mathrm{in}\ *}_{(-\omega)l(-m)}, \qquad {}_sR^{\mathrm{in\ O}}_{\omega lm} = {}_sR^{\mathrm{in}}_{\omega lm} - {}_sR^{\mathrm{in}\ *}_{(-\omega)l(-m)}. \qquad (6.429)$$

Then we can combine Eqs. (6.427) and (6.428) to obtain:

$$4\ \mathcal{D}^4_{0m\omega}\ {}_{-2}R^{\mathrm{in\ E}}_{\omega lm} = C_{\omega lm}\ {}_2R^{\mathrm{in\ E}}_{\omega lm}, \qquad p = +1, \qquad (6.430)$$

$$4\ \mathcal{D}^4_{0m\omega}\ {}_{-2}R^{\mathrm{in\ O}}_{\omega lm} = C^*_{\omega lm}\ {}_2R^{\mathrm{in\ O}}_{\omega lm}, \qquad p = -1, \qquad (6.431)$$

or

$$4\ \mathcal{D}^4_{0m\omega}\left[{}_{-2}R^{\mathrm{in\ E}}_{\omega lm} +\ {}_{-2}R^{\mathrm{in\ O}}_{\omega lm}\right] = \left[C_{\omega lm}\ {}_2R^{\mathrm{in\ E}}_{\omega lm} + C^*_{\omega lm}\ {}_2R^{\mathrm{in\ O}}_{\omega lm}\right]. \qquad (6.432)$$

The above derivation can be repeated for any other radial solution and will lead to the same result. Since the solution in the asymptotic limit at past null infinity uniquely defines the solution everywhere, these relations are valid everywhere, not just asymptotically.

The other pair of equations (6.120) and (6.114) can be obtained from Eqs. (6.119) and (6.113) via the transformation $(\omega, m) \to (-\omega, -m)$. The radial equations (6.73) and (6.74) show that ${}_sR_{-\omega l-m}$ satisfies the same differential equation as $\Delta^{-s}\ {}_{-s}R_{\omega lm}$. Using this fact and the symmetry (6.348) establishes the result.

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