GRE PHYSICS STUDY GUIDE

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1 Introduction

If you are considering applying to graduate school in physics, you are going to have to take the GRE Physics subject exam. You will find that this exam is different than any other physics test you have taken in a number of ways. Some of these ways will be obvious: the test covers everything you have learned in your undergraduate physics education, and yet is multiple choice and does not require lengthy integrals or other complicated mathematics. Other differences will be more subtle.

The purpose of this document is to help you better understand the exam, so that you can wisely spend your study and preparation time. Chapter 2 will familiarize you with the format of the exam, set a performance goal, and describe test-taking strategies you will need to meet that goal. Chapter 3 (the bulk of this document) will give some fundamental physical principles that nearly always appear on the exam, as well as example problems for each subject. Chapter 4 will include a mini-exam that concentrates on the physics covered in Chapter 3.

2 The GRE Physics Exam

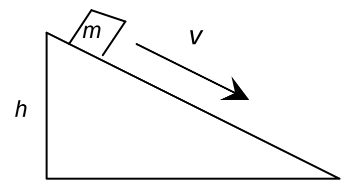
2.1 Structure of the Exam

The GRE Physics exam consists of 100 questions with a time limit of 170 minutes (2 hours 50 minutes), or just under two minutes per question. Each question is multiple-choice with five answer options given. According to the Educational Testing Service, the typical exam will cover the following subject matter (the percentages indicate on average how many questions address each subject):

- 20% - **Classical Mechanics** (such as kinematics, Newton's laws, work and energy, oscillatory motion, rotational motion about a fixed axis, dynamics of systems of particles, central forces and celestial mechanics, three-dimensional particle dynamics, Lagrangian and Hamiltonian formalism, noninertial reference frames, elementary topics in fluid dynamics)
- 18% - **Electromagnetism** (such as electrostatics, currents and DC circuits, magnetic fields in free space, Lorentz force, induction, Maxwell's equations and their applications, electromagnetic waves, AC circuits, magnetic and electric fields in matter)
- 9% - **Optics and Wave Phenomena** (such as wave properties, superposition, interference, diffraction, geometrical optics, polarization, Doppler effect)
- 10% - **Thermodynamics and Statistical Mechanics** (such as the laws of thermodynamics, thermodynamic processes, equations of state, ideal gases, kinetic theory, ensembles, statistical concepts and calculation of thermodynamic quantities, thermal expansion and heat transfer)
- 12% - **Quantum Mechanics** (such as fundamental concepts, solutions of the Schrödinger equation (including square wells, harmonic oscillators, and hydrogenic atoms), spin, angular momentum, wave function symmetry, elementary perturbation theory)
- 10% - **Atomic Physics** (such as properties of electrons, Bohr model, energy quantization, atomic structure, atomic spectra, selection rules, black-body radiation, x-rays, atoms in electric and magnetic fields)
- 6% - **Special Relativity** (such as introductory concepts, time dilation, length contraction, simultaneity, energy and momentum, four-vectors and Lorentz transformation, velocity addition)
- 6% - **Laboratory Methods** (such as data and error analysis, electronics, instrumentation, radiation detection, counting statistics, interaction of charged particles with matter, lasers and optical interferometers, dimensional analysis, fundamental applications of probability and statistics)

- 9% - **Specialized Topics** (Nuclear and Particle physics (e.g., nuclear properties, radioactive decay, fission and fusion, reactions, fundamental properties of elementary particles), Condensed Matter (e.g., crystal structure, x-ray diffraction, thermal properties, electron theory of metals, semiconductors, superconductors), Miscellaneous (e.g., astrophysics, mathematical methods, computer applications)

The exam is cleverly written. Many of the questions require more than just recalling which equation is applicable and plugging in numbers, but instead test your physical intuition. My favorite example is the following:



1. A block of mass m slides with constant velocity v down an inclined plane with height h and coefficient of friction μ. By the time the block reaches the bottom of the plane, how much energy has been dissipated as heat?

a. $mv^2/2$ b. $\mu mv^2/2$
c. mgh d. μmgh
e. $\mu mgh/2$

At first glance, your instinct may be to recall the formulae you usually apply when given a coefficient of friction, $\boldsymbol{F} = \mu \boldsymbol{N} = \mu mg \cos \theta$...but you are given neither an angle nor the length of the ramp, so you have reached a dead end. Instead, think of the problem this way – you are being asked for an energy, so let us first apply conservation of energy. Since the block is sliding with a constant velocity, the kinetic energy is not changing from the top to the bottom of the ramp. Only the potential energy is changing, and since it is not being transformed into kinetic energy, it must all be dissipated as heat. Thus the amount of energy released as heat equals the change in gravitational potential energy mgh, so the answer is (c).

Scoring of the exam is done in a couple of steps. First, you are credited one point for each correct answer, zero points for each omitted question (no answer chosen), and minus one-quarter point for each incorrect answer. This is called your raw score. So for example, if you answered 64 of the 100 questions, and answered 48 correctly and 16 incorrectly, your raw score would be 48 – (16/4) = 44.

Next, the raw score is converted to a scaled score, much like the SAT exam, though in this case the scores range from a high of 990 to a low of about 400. The conversion is applied to cancel out the difficulty level of a particular exam, by requiring that a certain number of examinees score a 990, a few more get 980, and so on. This allows scores to be comparable from one exam to the next. Table 1 (again taken from the ETS website) shows which scaled scores corresponded to which raw scores for each of three different exams.

Scaled Score	Raw Scores		
	Test A	Test B	Test C
900	73	68-69	64
800	58-59	54-55	50
700	44	41	38
600	30	27	27

Table 1: Raw scores needed to achieve certain scaled scores on three GRE Physics exams

When you receive your scores, you will see your raw score, your scaled score, and a percentile. This is the percentage of examinees whose scores you exceeded. Table 2 shows the raw and scaled scores associated with several percentiles for a practice exam on the ETS website.

The most important lesson to draw from these tables is that the GRE Physics exam is **tough**! Only a small number of examinees will exceed even 60% correct. So you should not prepare for the exam by trying to relearn everything, with the expectation of answering most of the questions correctly. We'll talk more about how this will affect your study plan later in this chapter.

Percentile	Scaled Score	Raw Score
99	990	85
90	900	73
80	820	61
70	760	53
60	710	45
50	660	38
40	620	33
30	580	27
20	530	20
10	480	13

Table 2: Raw and scaled scores for certain percentiles on sample GRE Physics exam

2.2 How the Exam is Used

It would be impossible to make a blanket statement about how every graduate school uses the GRE subject exam score in their evaluation of applications. Different schools will weight the exam differently in their decision-making process, and will have a different definition of "high" and "low" scores.

Having said that, the following is true for many graduate schools. Admissions boards look primarily at undergraduate grades (particularly in upper-level physics courses), recommendations, and any research experience (always a big plus). The GRE score is typically used as a check, a verification that the student has learned the material that the undergraduate transcript says (s)he has been exposed to. It has the advantage of being calibrated nationwide, unlike grades, which may represent a widely different level of accomplishment from one school to the next. An average or better GRE score raises no eyebrows. A low score, however, is a red flag to admissions officials that the transcript may not be telling the whole story.

2.3 Performance Goal

Given what you have learned about the subject exam, what are you trying to accomplish? Your goal should be to score well enough to avoid drawing attention away from your strong transcript and research background. So what score should you be trying to achieve?

Notice in Table 2 that the scores are most crowded together around the 30^{th}-50^{th} percentile (i.e., the difference in scores from one line to the next is the smallest). This corresponds to scaled scores in the low 600's and raw scores in the low 30's. So let us say that **your performance goal is a scaled score of 600**. This will put you ahead of at least one-third of examinees, and lumped in with a large crowd of similar scores. Thus it will be unlikely that your score stands out, which is what we want. Note from Table 1 that a scaled score of 600 is equivalent to a raw score of about 30; this will affect our test-taking strategies in the next section.

2.4 Preparation and Test-Taking Strategies

There are three things you can do to maximize your chance of scoring at least 600, both before and during the test:

- Study a finite set of physics topics thoroughly, rather than everything in brief
- Take a lot of practice tests
- Always guess if you are sure you can eliminate at least two answer choices

First, let us talk about studying. When preparing for a test about everything, it can be hard to know where to begin. The temptation is to try to review notes from every physics class you have taken. You usually end up covering too much too quickly, and when test time rolls comes you have not relearned any of the material well enough to apply it under pressure.

But remember, our goal is to get at least 30% of the questions correct, not 80-90%, so there is no need to study everything. If you can isolate a finite amount of material that constitutes at least a third of every GRE physics exam, and learn it thoroughly enough that you are confident that you can answer any question on those subjects, then you are well on your way to meeting your goal. This study guide is a collection of topics that nearly always show up on the exam, so for each concept that you master, you have effectively added a point to your expected raw score. Concentrate on learning these topics as if you were taking a final exam in this “course”. Only when you can get a perfect score on the sample exam in Chapter 4 should you expand your studying into other areas.

Second, you should be taking a lot of practice tests. This will familiarize you with the time limit and the types of questions usually asked on the exams, as well as point out to you which topics require further review. You should practice under real testing conditions (no books or equation sheets, adhering strictly to the time limit), and you should use practice exams as close to the real thing as possible. We recommend getting *GRE: Practicing to Take the Physics Test* from the Educational Testing Service, since it includes actual exams; it is out-of-print as of this writing, but you can find used copies at Amazon or Barnes & Noble. The ETS website (http://www.gre.org) also has a sample physics exam in .pdf format.

Third, in order to maximize your score you need to have the courage to guess. Note that if you had no idea what the answer to a question was, guessing would be an even gamble. You’d have a 20% chance of gaining one point, versus an 80% chance of losing one-quarter of a point, for a net expectation value of zero ((0.2 x 1) – (0.8 x 0.25) = 0). So if you can eliminate even one of the answer choices, it is to your advantage to guess.

We recommend that you not guess if you can eliminate only one answer choice. The reason is that on most questions, two choices can usually be eliminated for the same straightforward reason, so if you can only eliminate one you are probably doing something wrong. But **if you can eliminate two answer choices, you MUST guess**. To not do so is to give away several points off of your final raw score.

The first time you take a practice test, make a note of the questions on which you guess, and when you score the test, count what percent of these questions you answered incorrectly. It may be discouraging to find that you got 50-60% of your guesses wrong. But as long as you missed less than 80%, **you benefited** from making those guesses. Have courage!

So here is our test-taking strategy. Starting as early as August, start reviewing the material in this document, until you can answer every question on the sample test in Chapter 4. Get the ETS booklet and take a practice exam regularly, perhaps once a weekend, under real testing conditions. Get used to pacing yourself – if a question is on a topic you have studied well, take the time to get it right. If not, take a minute to see if you can eliminate two or more answer choices. If you can, guess. Either way, move on to the next question, so that you can see them all in the 170 minutes available. By the time you get to the real exam, these strategies should be ingrained, and you will do fine.

3 Subject Guide

Review material in each of the subject categories listed in section 2.1 is included below. An example problem accompanies each topic. Attempt each example problem **on your own** before looking at the solution.

3.1 Classical Mechanics

3.1.1 The Basics

Position, velocity, and acceleration are related by $x = x_0 + v_0 t + \frac{1}{2}at^2$, $v = v_0 + at$

Momentum = $p = mv$ Force = $F = ma = m\frac{dv}{dt} = \frac{dp}{dt}$

Kinetic energy = $\frac{1}{2}mv^2$ Gravitational potential energy = mgh Spring potential energy = $\frac{1}{2}kx^2$

Remember that both momentum and energy are conserved quantities. Also recall that the x, y, and z components of momentum are all independently conserved.

Example 1: A particle of mass m is moving along the x-axis with speed v when it collides with a particle of mass $2m$ initially at rest. After the collision, the first particle has come to rest, and the second particle has split into two equal-mass pieces that move at equal angles $\theta > 0$ with the x-axis, as shown in the figure below. Which of the following statements correctly describes the speed of the two pieces?

a. Each piece moves with speed v.
b. One of the pieces moves with speed v, the other moves with speed less than v.
c. Each piece moves with speed $v/2$.
d. One of the pieces moves with speed $v/2$, the other moves with speed greater than $v/2$.
e. Each piece moves with speed greater than $v/2$.

Answer: Note that we can rule out choices (b) and (d) by symmetry, so we cannot skip this question. The initial momentum in the x-direction is mv, so the final momentum of each particle in the x-direction is $mv/2 = m(v/2)$, which means that each particle must have a speed of at least $v/2$, even if all excess energy is converted to heat. If no heat is released (elastic collision), $\frac{1}{2}mv^2 = 2\left(\frac{1}{2}mv_f^2\right)$, so $v_f = \frac{v}{\sqrt{2}} \approx 0.7v$, which is less than v but greater than $v/2$. Thus the correct answer must be (e).

3.1.2 Collisions

The formulae for collisions can be derived from conservation of momentum and energy, but you may find that you can work more quickly if you have them memorized. In both cases, we assume a mass m_1 is moving at speed v toward a mass m_2 at rest.

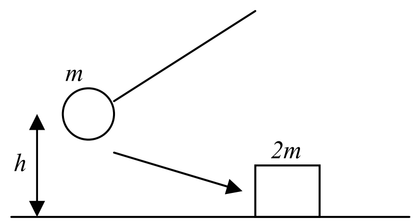
If the collision is elastic (the two particles do not stick together, no energy lost as heat), then

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v \qquad\qquad v_{2f} = \frac{2m_1}{m_1 + m_2} v$$

If the collision is inelastic (the two masses stick together, energy is lost as heat), then

$$v_f = \frac{m_1}{m_1 + m_2} v$$

Example 2: A pendulum bob with mass m is released from a height h as shown in the figure at right. When it reaches the bottom of its swing, it elastically strikes a mass $2m$ sitting on a frictionless surface. To what height does the pendulum bob swing back?



Answer: The initial gravitational potential energy of the pendulum bob is mgh. At the bottom of the swing, all of this potential energy has been converted to kinetic energy, so $\frac{1}{2}mv^2 = mgh$ or $v = \sqrt{2gh}$.

Since this is an elastic collision, $v_f = \frac{m - 2m}{m + 2m}\sqrt{2gh} = -\frac{\sqrt{2gh}}{3}$. Then as the pendulum swings back up, all of this kinetic energy is turned back into gravitational potential energy, so $mgh_f = \frac{1}{2}mv_f^2 = \frac{1}{2}m\left(-\frac{\sqrt{2gh}}{3}\right)^2$, which gives $h_f = h/9$.

3.1.3 Circular Motion

For circular motion, $a = v^2/r$. The circumference of one orbit is $2\pi r$, so the time required for one orbit (the period) will be T = distance/rate = $2\pi r/v$, and the frequency of the orbit will be $f = 1/T = v/2\pi r$.

This will most often come up in relation to planetary motion, so it is useful to mention the related gravitational formulae here: gravitational force = $\frac{GMm}{r^2}$, gravitational potential energy = $-\frac{GMm}{r}$ (with zero potential energy at infinity).

Finally, it may also help to remember two of Kepler's Laws:

2nd law: The line joining the Sun and a planet sweeps out equal areas in equal times.
3rd law: The square of a planet's orbital period is proportional to the cube of the semimajor axis of the orbit.

Example 3: A satellite of mass m orbits a planet of mass M in a circular orbit of radius R. The time required for one revolution is:

a. independent of M b. proportional to $\sqrt{m}$ c. linear in R
d. proportional to $R^{3/2}$ e. proportional to R^2

Answer: We need to find $T = 2\pi R / v$, which means we need to find v as follows:
$F = \frac{GMm}{R^2} = ma = m\frac{v^2}{R}$, so $v = \sqrt{\frac{GM}{R}}$ Thus $T = \frac{2\pi R}{\sqrt{GM/R}} = \frac{2\pi}{\sqrt{GM}}R^{3/2}$, and the correct answer is (d).

3.1.4 Rotational Motion

θ = angular position (in radians) ω = angular velocity α = angular acceleration
Note that if an object is moving at speed v through a circular path of radius r, it will travel through 2π radians when it travels a distance $2\pi r$, so $v=\omega r$.

Moment of inertia = $I = \int \rho r^2 dr$, where ρ is the density of an object and r is the distance of a given piece of the object from its center of mass. Some typical values for moment of inertia:

Rod (about axis perpendicular to the center of the rod) = $\frac{1}{12}Ml^2$, where l is the length of the rod.

Disc (about axis perpendicular to the center of the disk) = $\frac{1}{2}MR^2$

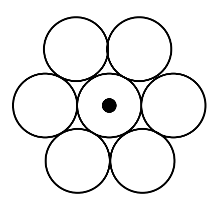
Sphere (about axis through the center) = $\frac{2}{5}MR^2$

A very useful rule is the *parallel-axis theorem*: The moment of inertia about an axis parallel to the axis through the center of mass is given by $I = I_{cm} + Mh^2$, where M is the mass of the object and h is the distance between the new axis and the center of mass. So for example, the moment of inertial about the end of a rod of length l is given by $I = I_{cm} + Mh^2 = \frac{1}{12}Ml^2 + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2$

Instead of force, in rotational motion we use torque = $\tau = rF\sin\theta = I\alpha$

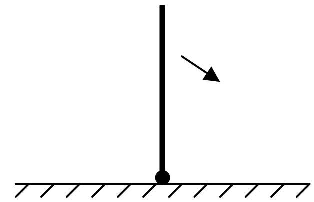
Angular momentum = $L = I\omega$ Rotational kinetic energy = $\frac{1}{2}I\omega^2$

Example 4: Seven pennies are arranged in a hexagonal, planar pattern so as to touch each neighbor, as shown in the figure at right. Each penny is a uniform disk of mass m and radius r. What is the moment of inertia of the system of seven pennies about an axis that passes through the center of the central penny and is normal to the plane of the pennies?



Answer: The moment of inertia through the center of a disk is $\frac{1}{2}Mr^2$. But for six of our seven disks, the axis of rotation is shifted a distance $2r$ from the center of mass; by the parallel-axis theorem, $I = \frac{1}{2}Mr^2 + M(2r)^2 = \frac{9}{2}Mr^2$. Thus the total moment of inertia is $\frac{1}{2}Mr^2 + 6\left(\frac{9}{2}\right)Mr^2 = \frac{55}{2}Mr^2$.

Example 5: A thin uniform rod of mass *M* and length *L* is positioned vertically above an anchored frictionless pivot point, as shown at right, and then allowed to fall to the ground. With what speed does the free end of the rod strike the ground?

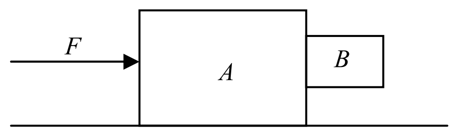


Answer: The initial potential energy of the rod is given by $mgh_{cm} = mgl/2$. This will all be converted to kinetic energy, so $\frac{mgl}{2} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\omega^2$, using the moment of inertia for the end of a rod that we found on the last page. Solving this equation yields $\omega = \sqrt{3g/l}$. Finally, since the end of the rod is moving through a circular path with radius *l*, $v = \omega l = \sqrt{3gl}$.

3.1.5 Friction

The force due to friction is given by $F = \mu N \sin\theta$, where μ is the coefficient of friction, *N* is the normal force on the object (*N =ma)*, and θ is the angle between the force and normal vectors.

Example 6: For the system consisting of the two blocks shown at right, the minimum horizontal force *F* is applied so that the block *B* does not fall under the influence of gravity. The masses of *A* and *B* are 16.0 kg and 4.00 kg, respectively. The horizontal surface is frictionless and the coefficient of friction between the two blocks is 0.50. What is the magnitude of *F*?



Answer: Gravity is trying to accelerate block *B* downward with force $F_g = m_B g$. Friction with block *A* is pushing block *B* in the opposite direction with force $F_f = \mu N = \mu m_B a$, where *a* is the acceleration of block *B* due to the force *F*. In order for the block *B* to not fall, the frictional force must equal the gravitational force, so $a = g/\mu = g/(0.5) = 2g$. Thus $F = m_{total}a = (16kg + 4kg)2(9.8m/s) = 392N$.

3.1.6 The Lagrangian

The Lagrangian is defined as (Kinetic energy – Potential energy), or symbolically, *L = T – V.* You can express this quantity using a position coordinate *q* and a velocity coordinate $\dot{q}$ (the time-derivative of *q*). For a given classical system, this gives you a convenient way to find the system's equation of motion. First find the Lagrangian, and then apply the following equation:

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

Example 7: Find the equation of motion for a mass *m* at a height *x* under the influence of gravity.

Answer: The kinetic energy of the mass is given by $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$, and the potential energy by $V = mgx$, so $L = \frac{1}{2}m\dot{x}^2 - mgx$. Taking the derivatives shown above gives $-mg = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$. Thus the equation of motion is $F = m\ddot{x} = ma = -mg$, where the negative sign means that gravity is pulling the mass downward with acceleration *g*, just as we would expect.

3.2 Electromagnetism

3.2.1 Electric Forces and Fields

The force from an electric charge is given by Coulomb's law, $F = \frac{kq_1q_2}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}$. Like charges repel, opposite charges attract.

A force from a spherically symmetric shell of charge can be determined very simply. If the point of interest is inside the shell, the charge shell has no effect (the individual force vectors from each part of the shell cancel out). If the point of interest is outside of the shell, all of the charge can be assumed to be at a point at the center of the shell.

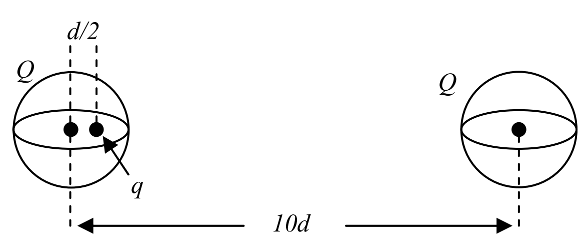
The electric field at a point is an expression of what force a test charge *q* would feel if placed at that point: *F=qE.* The electric field will always point in the direction that positive charges want to move – it can be expressed visually by drawing arrows away from positive charges and towards negative charges.

Recalling that work equals force times distance, we define the electric potential *V* such that $\Delta V = -\int E \cdot dl$; in other words, the energy required to move a charge from point *A* to point *B* equals $V_B - V_A$. Since this is a conservative force, it doesn't matter how you get from point *A* to point *B*, the change in energy is the same. *V* at a point is therefore just a scalar, a number without a direction. For an arrangement of discrete charges, $V = \Sigma \frac{kq}{r}$, with zero potential defined at a point infinitely far from the charge distribution.

Example 8: Five positive charges of magnitude *q* are arranged symmetrically around the circumference of a circle of radius *r*. What is the magnitude of the electric field at the center of the circle? What is the electric potential?

Answer: By symmetry, the electric force at the center of the circle must be zero; otherwise if you rotated the circle 1/5 of a revolution, the charge distribution would look the same, but the force would be pointing in a different direction. Since *F=qE*, the electric field magnitude must also be zero. The potential, however, is $V = \sum \frac{kq}{r} = \frac{5kq}{r}$.

Example 9: Two spherical, nonconducting, and very thin shells of uniformly distributed positive charge Q and radius *d* are located a distance 10*d* from each other. A positive point charge *q* is placed inside one of the shells at a distance *d/2* from the center, on the line connecting the centers of the two shells, as shown in the figure above. What is the net force on the charge *q*?



a. $\dfrac{qQ}{361\pi\varepsilon_0 d^2}$ to the left b. $\dfrac{qQ}{361\pi\varepsilon_0 d^2}$ to the right c. $\dfrac{qQ}{441\pi\varepsilon_0 d^2}$ to the left

d. $\dfrac{qQ}{441\pi\varepsilon_0 d^2}$ to the right e. $\dfrac{360qQ}{361\pi\varepsilon_0 d^2}$ to the left

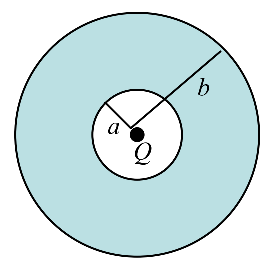
Answer: Since the charge is inside the left shell, this shell will have no effect on the charge. Thus only the right shell matters, and since it is the same charge as our test charge, the two will repel, causing a force to the left, which eliminates answers (b) and (d). The right shell can be treated as if all the charge Q were at the center of the shell, which is (10 - 1/2)d = (19/2)d from our test charge. By Coulomb's law, $F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qQ}{(19d/2)^2} = \dfrac{qQ}{361\pi\varepsilon_0 d^2}$, so the answer is (a).

3.2.2 Conductors

A conductor is a material in which charge is free to move about. One special property of conductors is that its charge will always redistribute itself in order to make the electric field inside the conductor equal to zero; if the field were nonzero, after all, then the charges in the conductor would feel a force due to $F=qE$ and continue moving.

How the charges redistribute is best understood through Gauss's Law: the sum of all the electric field lines passing through a closed surface is linearly related to the amount of charge enclosed in that surface. So for example:

Example 10: A neutral, conducting, hollow spherical shell with inner radius a and outer radius b has a charge Q enclosed at its center. What is the surface charge density on the inner surface of the spherical shell? On the outer surface?



Answer: The electric field must be zero everywhere inside the conducting material. So if I drew a concentric spherical surface with radius r, where $b > r > a$, the electric field must be zero everywhere on this surface, which in turn means that there can be no net charge enclosed in this surface. To cancel out the charge Q in the center, the inner surface of the conductor must have a charge $-Q$ spread across it, and since the surface area would be $4\pi a^2$, the surface charge density would be $-Q/4\pi a^2$. Since the sphere is neutral, any charge on the inner surface must be cancelled by an equal and opposite charge on the outer surface, so the outer surface must have a total charge Q, or a surface charge density of $Q/4\pi b^2$.

Example 11: A positive charge Q is located at a distance L above an infinite grounded conducting plane. What is the total charge induced on the plane?

Answer: The fact that the plane is grounded means that the electric potential must be zero everywhere on the plane. Since $V = \Sigma \dfrac{kq}{r}$, the simplest way to make $V = 0$ everywhere is to imagine another charge of magnitude $-Q$ an equal distance L below the plane; then for any point on the plane, $V = \dfrac{kQ}{r} + \dfrac{k(-Q)}{r} = 0$. We call this the **image charge**, with the conductor acting very much like a mirror. Thus a total charge of $-Q$ must be induced on the conducting plane.

3.2.3 Magnetic Forces and Fields

Magnetic force is expressed by $\vec{F} = q\vec{v} \times \vec{B} = qvB\sin\theta$. The direction of the force can be determined by the right-hand rule. Point the fingers of your right hand in the direction that the charge is moving, and then curl your fingers to point in the direction of the magnetic field. The direction that your thumb is pointing is the direction of the force applied to the moving charge. Note that if the charge is negative, the sign of the equation flips, so you must remember to take the opposite direction of what you get from the right-hand rule; don't forget!

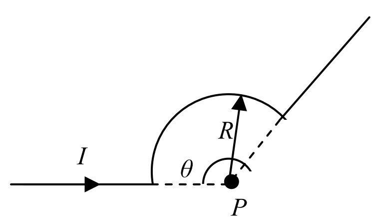
Magnetic fields are generated by the motion of electrical charge, as expressed by the Biot-Savart law, $B = \int \frac{\mu_0}{4\pi} \frac{\vec{I} dl \times \vec{r}}{r^2}$. For a given bit of electrical current, the direction of the magnetic field can again be found by a right-hand rule. Point the thumb of your right hand in the direction of the current (remember, current is the direction in which <u>positive</u> charge flows, even though in reality it is usually negative charges doing the moving). Then curl your fingers; this is the direction of the magnetic field.

In certain symmetrical situations, Ampere's law, $\oint B \cdot dl = \mu_0 I$, where I is the current encircled by a closed loop, can make finding magnetic fields a lot easier. For example, imagine a point a distance d from a wire carrying current I. Draw a circle around the wire with radius d; by symmetry the field must be the same at every point on this circle. Since the circumference of the circle is $2\pi d$, then by Ampere's law, $\oint B \cdot dl = B(2\pi d) = \mu_0 I$, so $B = \frac{\mu_0 I}{2\pi r}$.

Example 12: A proton moves in the $+z$-direction after being accelerated from rest through a potential difference V. The proton then passes through a region with a uniform electric field E in the $+x$-direction and a uniform magnetic field B in the +y-direction, but the proton's trajectory is not affected. If the experiment were repeated using a potential difference of $2V$, what would happen to the proton?

Answer: The electric field points in the direction that positive charges want to move, so the E field will deflect the proton in the $+x$-direction. Since the proton was not deflected, the B field must be giving it an equal deflection in the $-x$-direction, which is exactly what we get from $\vec{F} = q\vec{v} \times \vec{B}$ (try it with the right-hand rule and see). Now if the potential difference is increased, the energy and thus the velocity of the proton also increases. This doesn't affect the electric force, but by $\vec{F} = q\vec{v} \times \vec{B}$, the magnetic force will also increase. Thus the proton will now be deflected in the $-x$-direction.

Example 13: A segment of wire is bent into an arc of radius R and subtended angle θ, as shown in the figure at right. Point P is at the center of the circular segment. The wire carries current I. What is the magnitude of the magnetic field at P?



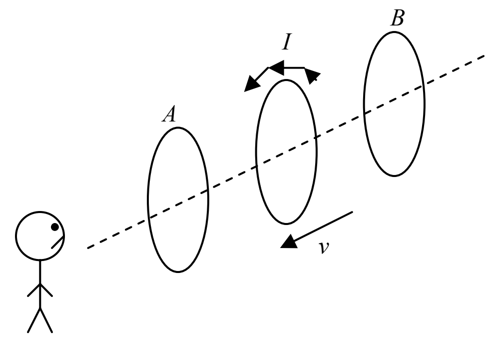
Answer: First, notice that the straight portions of the wire will not contribute to the magnetic field; since point P is at the very center of the wire, all of the current flow is outside of this point, so if we used Ampere's law, no current would be encircled and the field would be zero. So only the arc contributes to the field. The length is a fraction of the circumference, $2\pi R\left(\frac{\theta}{2\pi}\right) = \theta R$. Thus by the Biot-Savart law, $B = \int_0^{\theta R} \frac{\mu_0}{4\pi} \frac{Idl \times \hat{r}}{r^2} = \int_0^{\theta R} \frac{\mu_0}{4\pi} \frac{I}{R^2} dl = \frac{\mu_0 I}{4\pi R^2} \theta R = \frac{\mu_0 I \theta}{4\pi R}$.

3.2.4 Induction

Magnetic flux ϕ_B is defined as the sum of all of the magnetic field lines protruding through a surface. Conductors do not like changes in magnetic flux. Lenz's Law states that current will be induced to flow to cancel out any changes in the magnetic flux through a surface. Thus if the flux through a surface increases to the left, current will flow to generate a magnetic field pointing to the right. Mathematically this law is usually represented in terms of the induced electromotive force (or voltage), which causes current to flow: $\varepsilon = -\frac{d\phi_B}{dt}$.

The most important thing to remember with this rule is that the induced current flow cancels out the **change** in flux. So if the field through a surface is pointing to the right, but decreasing, the change is an increase in flux to the left, and thus current will flow to generate a field pointing right to cancel this change.

Example 14: Three wire loops and an observer are positioned as shown in the figure at right. From the observer's point of view, a current *I* flows counterclockwise in the middle loop, which is moving towards the observer with a velocity *v*. Loops *A* and *B* are stationary. In what direction does the observer note current flowing in loops *A* and *B*?



Answer: First use the right-hand rule to see that the magnetic field from the current *I* points toward the observer, and remember that the field will get weaker the farther you are from the loop. As the current loop moves, it gets closer to loop *A*, which means the flux through this loop toward the observer increases. Loop *A* will cancel this change by generating a magnetic field away from the observer, and using the right-hand rule, you should see that this means current will flow clockwise. The current loop is moving farther from loop *B*, however, so the flux toward the observer through loop *B* is decreasing. Loop *B* will cancel this change by generating a magnetic field toward the observer, which requires current to flow counterclockwise.

3.2.5 Maxwell's Equations

Maxwell's equations are just a summary of what we have discussed in the previous four sections. Below they are presented in integral form, differential form, and (most importantly) in plain English.

I. $\oint E \cdot dA = \frac{q}{\varepsilon_0}$ or $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ Gauss's Law, which states that the sum of all of the electric field lines through a closed surface is linearly related to the amount of charge enclosed in that surface.

II. $\oint B \cdot dA = 0$ or $\nabla \cdot B = 0$ Gauss's Law for magnetism. Because any field lines that emanate from a north pole must return to the associated south pole, any magnetic field line through a closed surface must eventually come back through that surface, so the net sum of all these field lines is zero.

III. $\oint E \cdot dl = -\frac{d\phi_B}{dt}$ or $\nabla \times E = -\frac{\partial B}{\partial t}$ Faraday's Law of Induction, stating that an electric field can be generated by a changing magnetic field.

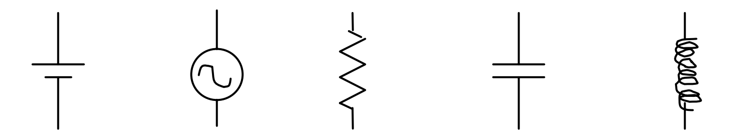
IV. $\oint B \cdot dl = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$ or $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ Similar to the law of induction, this states that a magnetic field can be generated by moving electrical charge or by a changing electric field.

Example 15: If magnetic charge (monopoles) existed and were conserved, which of these equations would have to be changed?

Answer: Equation I would be unchanged because magnetic poles do not generate electric field lines. Equation II would change, however, because if you had monopoles, magnetic field lines could pass through a surface and not have to return, but terminate at another monopole elsewhere. Equation III would also have to change, because an electric field could now be generated by moving magnetic monopoles in the same way that a magnetic field can be generated by moving electric charge. Equation IV would be unaffected, since this equation shows only how electric charge and fields generate magnetic fields. Thus only equations II and III would change – notice that after making this change, Maxwell's equations would be symmetric! The presence of electric charge but not magnetic charge is what breaks this symmetry.

3.2.6 RLC Circuits

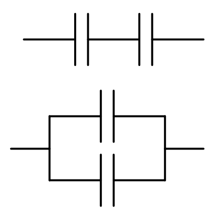
I can't stress enough how useful it will be to remember the few rules in this section. A typical GRE exam will have at least four straightforward questions about circuits. Let's start by reminding ourselves how to draw each circuit element:



Multiple resistors and capacitors are combined according to the following rules:

Series: $R = R_1 + R_2 \quad C = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \frac{C_1 C_2}{C_1 + C_2}$

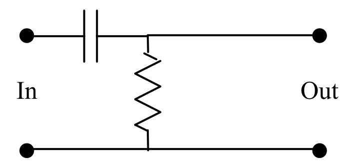
Parallel: $R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \frac{R_1 R_2}{R_1 + R_2} \quad C = C_1 + C_2$



The simplest relationship in circuits is given by Ohm's Law: $V=IR$. More voltage means more current. More resistance means less current. The capacitor and inductor also obey a form of Ohm's Law, but are a little different because (a) their "resistance" is frequency-dependent, and (b) they store energy rather than dissipate it as heat. Thus we use the term "impedance" rather than "resistance" for them and write $V=IZ$, where $Z_C = \frac{1}{\omega C}$ and $Z_L = \omega L$ where $\omega = 2\pi f$ is the angular frequency of the AC voltage source.

	Capacitor (C)	Inductor (L)
Low Frequency	High impedance (open circuit)	Low impedance (short circuit)
High Frequency	Low impedance (short circuit)	High impedance (open circuit)

The table opposite shows how the capacitor and inductor act as opposites. At low frequencies, the capacitor has time to fully charge and then block any further current flow, so it has high impedance. At high frequencies, the capacitor does not have time to charge up, and thus does not affect the circuit – it might as well be a wire. The inductor, on the other hand, works to prevent changes in current flow. At low frequencies, change is very gradual, so the inductor has little effect, and acts as a short circuit. At high frequencies, change is abrupt, so the inductor has high impedance as it tries to prevent these changes.



Filter circuits take advantage of this behavior. Look at the circuit at right. At low frequency, the impedance of the capacitor is very large, so by Ohm's Law, all of the voltage coming into the circuit will drop across the capacitor. Since there is none left to drop across the resistor, there is no output voltage. At high frequency, the capacitor is a short, so all the voltage drops across the resistor and is output by the circuit. Thus we have just designed a high-pass filter, since only high-frequency voltage signals are passed through.

A circuit with both inductors and capacitors will have a minimum total impedance when the two contribute equally. This will happen at the frequency $\omega_0 = \frac{1}{\sqrt{LC}}$, called the resonant frequency of the circuit, and is the frequency at which current flow will be a maximum and energy loss a minimum.

When an RC circuit is first connected, the current will be at a maximum, but will drop exponentially as the capacitor charges. Similarly, the voltage across the capacitor will start at zero, and increase over time. If the capacitor is fully charged and the circuit is broken, then the current and voltage across the capacitor will start at a maximum value, and both will decrease exponentially as the capacitor discharges. The time constant for all of these changes is *RC*. For example, as a capacitor discharges, after a time *RC* its voltage will drop to $1/e \approx 0.37$ of its original value.

When an RL circuit is first connected, the current will remain zero while the voltage across the inductor is a maximum, as the inductor resists changes in current flow. Over time the inductor voltage will exponentially decay to zero while the current reaches a maximum. When the circuit is broken, the current will stay at its current value while the inductor voltage again leaps to a maximum, after which both will exponentially decay to zero. The time constant for all of these changes is *L/R*.

The power dissipated by a resistor is given by $P = VI = IR^2 = V^2/R$. The energy stored by a capacitor or inductor is given by $\frac{1}{2}CV^2$ and $\frac{1}{2}LI^2$, respectively.

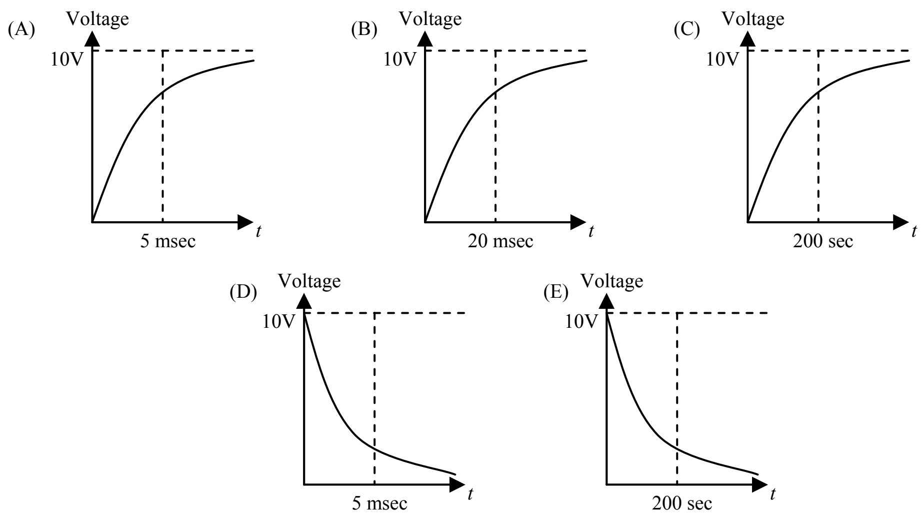
Example 16: An AC circuit consists of the following elements: R = 10,000 ohms, L = 25 millihenries, and an adjustable capacitance C. The AC voltage generator supplies a signal with an amplitude of 40 volts and angular frequency of 1,000 radians per second. For what value of C is the amplitude of the current maximized?

Answer: $\omega_0 = \frac{1}{\sqrt{LC}}$, so $C = \frac{1}{L\omega_0^2} = \frac{1}{(0.025)(1000)^2} = 4\times10^{-5}F = 40\mu F$.

Example 17: A 3-microfarad capacitor is connected in series with a 6-microfarad capacitor. When a 300-volt potential difference is applied across this combination, what is the total energy stored?

Answer: $C = \left(\frac{1}{3\mu F} + \frac{1}{6\mu F} \right)^{-1} = 2\mu F$, so $U = \frac{1}{2}CV^2 = \frac{1}{2}\left(2\times10^{-6}F\right)\left(3\times10^2V\right)^2 = 9\times10^{-2}J = 0.09J$.

Example 18: A switch is closed, completing a circuit that includes a 10V power source, a 2 ohm resistor, and a 10 millihenry inductor. Which of the following best represents the voltage across the inductor, as seen on an oscilloscope?



Answer: When the switch is first closed, the inductor will do everything it can to resist a change in current flow, so the voltage across the inductor will jump to a maximum. The voltage will then decay to zero with time constant $L/R = 0.01mH/2\Omega = 0.005\,\text{sec} = 5\text{ msec}$. Thus (D) is the correct graph.

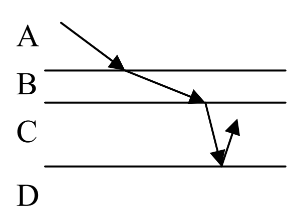
3.3 Optics and Wave Phenomena

3.3.1 Reflection/Refraction

For light approaching a reflective surface, the angle of incidence equals the angle of reflection. The angle is always measured between the light ray and the normal (perpendicular) to the surface, not the surface itself.

For substances with different indices of refraction, the light path is determined by Snell's Law: $n_i \sin\theta_i = n_f \sin\theta_f$. Index of refraction must always be ≥ 1, and represents how much light is slowed in a material – the speed of light in a substance is given by *c/n*. When light passes from high index of refraction to low, there may sometimes be no solution to Snell's Law, in which case the light experiences total internal reflection and bounces back through the high-index material.

Example 19: A light ray passes through four materials as shown at right. Which material has the highest index of refraction? Which has the lowest?



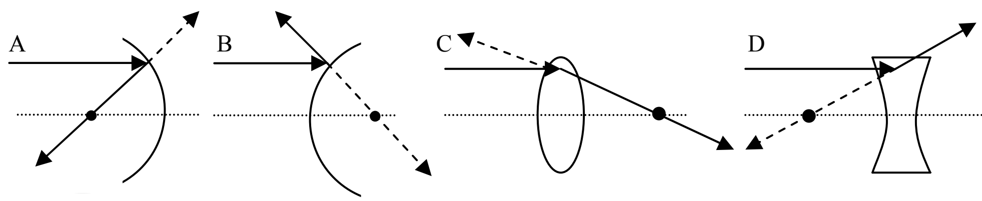
Answer: By Snell's Law, the higher the index of refraction, the smaller the angle to the normal. Thus material C must have the highest index of refraction. The lowest index belongs to material D – it is so low that no angle can satisfy Snell's Law, so total internal reflection occurs.

3.3.2 Ray Tracing

Given an object and a mirror or lens, the object's image may be located by finding the intersection of three rays:

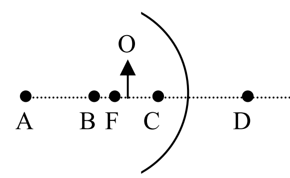
1. Coming in parallel to the optical axis, and leaving through the focal point.
2. Coming in through the focal point, and leaving parallel to the optical axis.
3. Approaching the center of the optic, and either passing straight through (lens) or reflecting back at the same angle (mirror).

Where you sight the focal point depends on the type of optic. Below is an example of the first type of ray for each of four optics:

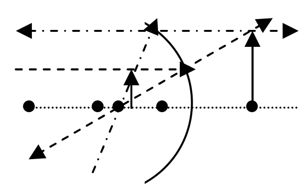


A: Convex mirror – focal length is positive, and focal point is on same side as object
B: Concave mirror – focal length is negative, and focal point is on opposite side as object
C: Converging lens – focal length is positive, and focal point is on opposite side as object
D: Diverging lens – focal length is negative, and focal point is on same side as object

Example 20: A spherical, concave mirror is shown in the figure at right. The focal point *F* and the object *O* are indicated. At what point will the image be located?



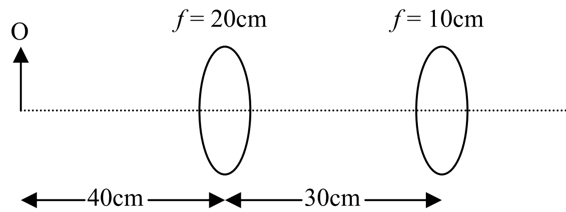
Answer: The diagram below the original problem shows the concave mirror with ray types 1 and 2 added. They intersect above point D, so that is where we expect the image to be. Note that in this case the image is virtual, since the rays intersect on the opposite side of the mirror.



3.3.3 Mirror/Lens Equations

Both mirrors and lenses obey the equation $\frac{1}{l_o}+\frac{1}{l_i}=\frac{1}{f}$, where l_o is the distance from the object to the optic, l_i is the image distance, and f is the focal length. Care must be taken with the signs in this equation. f will be negative for concave mirrors or diverging lenses. The sign of l_i will tell you on which side of the optic the image can be found. And in systems of multiple optics, the image from one optic may end up farther away than the second optic, resulting in a negative l_o.

Example 21: An object is located 40cm from the first of two thin converging lenses of focal lengths 20cm and 10cm, respectively, as shown on the next page. The lenses are separated by 30cm. Where is the final image located?



Answer: For the first lens, $\frac{1}{40cm}+\frac{1}{l_i}=\frac{1}{20cm}$, so l_i = 40cm. This is 10cm past the second lens, making the object distance -10cm. So for the second lens, $\frac{1}{-10cm}+\frac{1}{l_i}=\frac{1}{10cm}$, so l_i = 5cm. Thus the final image is 5cm past the second lens.

3.3.4 Polarization

Light is an electromagnetic wave. The axis along which the electric field is oscillating is called the polarization of the wave. Typically light from the sun or a lamp is made up of waves all polarized in different directions, so we treat the sum of all these waves as unpolarized. Light from some sources such as lasers, however, may be polarized in a specific direction.

Polarizers admit light in one polarization direction and block light polarized in the perpendicular direction. The intensity of light transmitted is given by Malus's Law, $I = I_0 \cos^2\theta$, where I_0 is the initial intensity of the light and θ is the angle between the light polarization and the preferred axis of the polarizer. Thus if $\theta = 0$, all of the light is transmitted, while if $\theta = \pi/2$, none of the light is transmitted. One-half of unpolarized light will always be transmitted regardless of the polarizer's orientation. The transmitted light will always have the same polarization as the preferred axis of the polarizer.

Example 22: Unpolarized light of intensity I_0 is incident on a series of three polarizing filters. The axis of the second filter is oriented at 45 degrees to that of the first filter, while the axis of the third filter is oriented at 90 degrees to that of the first filter. What is the intensity of the light transmitted through the third filter?

Answer: The intensity is reduced from I_0 to $I_0/2$ by the first filter. The second filter is oriented at 45 degrees to the first, so $I=\left(\frac{I_0}{2}\right)\cos^2\left(45^\circ\right)=\left(\frac{I_0}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2=\frac{I_0}{4}$. The light is now polarized in the direction of the second filter, which is at 45 degrees to the third, so the final intensity is $I=\left(\frac{I_0}{4}\right)\cos^2\left(45^\circ\right)=\frac{I_0}{8}$.

3.3.5 Interference

There are several formulae you can memorize related to interference – single-slit diffraction, double-slit diffraction, many-slit diffraction, Bragg diffraction, thin film interference, etc. However, I prefer to think of all of these problems in terms of their underlying principle, from which any of these formulae can be quickly derived: given two beams of coherent (in-phase) light, if one travels an integer number of wavelengths farther than the other, they will remain in phase and interfere constructively. If one travels (n + 1/2) wavelengths farther, where n is an integer, the two beams will be exactly out of phase and interfere destructively (cancel out).

One situation to be careful of is thin film interference, when light is reflecting off of materials with different indices of refraction. When reflecting off of a material with a higher index of refraction, a light wave will experience a phase flip, which is equivalent to traveling an extra half-wavelength. When reflecting off of a material with a lower index of refraction, there is no phase flip. Also, when in a material of index of refraction *n*, the wavelength of light is given by λ/n.

Example 23: Blue light of wavelength 480 nanometers is most strongly reflected off a thin film of oil on a glass slide when viewed near normal incidence. Assuming that the index of refraction of the oil is 1.2 and that of glass is 1.6, what is the minimum thickness of the oil film (other than zero)?

Answer: Since we are looking for when blue is "most strongly reflected", we are looking for *constructive* interference between two beams, one of which reflects off of the surface of the oil, the other of which passes through the oil and reflects off of the microscope slide. In both cases the light is reflecting off of a material with higher index of refraction (1.2 for oil is greater than 1 for air, and 1.6 for glass is greater than 1.2 for oil), so both beams travel an extra half-wavelength. Thus the only path length difference between the two is due to the thickness of the oil. Since one beam travels down and back through the oil, then twice the thickness of the oil must equal one wavelength. Thus $2t = \lambda / n$, or $t = (480nm)/2(1.2) = 200nm$.

3.3.6 Resolution

The maximum resolution of an optical system is expressed in terms of the smallest angular separation necessary to tell two objects apart, as follows: $\theta_{\min} \approx \lambda / a$, where *a* is the aperture size of the optical instrument. If the aperture is round (such as a typical telescope) with diameter *D*, the expression is exactly $\theta_{\min} = 1.22\lambda / D$.

Example 24: Two stars are separated by an angle of 3×10^{-5} radians. What is the diameter of the smallest telescope that can resolve the two stars using visible light (λ = 600 nm)?

Answer: $D = \dfrac{1.22\lambda}{\theta_{\min}} = \dfrac{1.22\left(6\times10^{-7}m\right)}{3\times10^{-5}rad} = 2.44\times10^{-2}m \approx 2.4cm$.

3.4 Thermodynamics and Statistical Mechanics

3.4.1 Compressions and Expansions

In thermodynamics we typically work with four idealized ways to compress or expand a gas:

1. Adiabatic – a rapid change that does not allow heat exchange with the outside world
2. Isothermal – a slow change while in contact with a reservoir at constant temperature
3. Isobaric – constant pressure is maintained while volume is changed
4. Isovolume – constant volume is maintained while pressure is changed

There are also several useful rules associated with these:

1. The First Law of Thermodynamics: $dU = dQ + dW = TdS - PdV$, where *dU* is the change in the gas's energy, *dQ* is change in heat, *dW* is work done on the gas, *T* is temperature, *dS* is change in entropy, *P* is pressure, and *dV* is change in volume.

2. The Ideal Gas Law, $PV = NkT$, where N is number of molecules and k is Boltzmann's constant.
3. $PV = (\gamma - 1)U$, where γ is a property of the gas (=5/3 for monatomic, =7/5 for rigid diatomic, =9/7 for nonrigid diatomic).
4. For an isothermal process, $PV = C$, a constant (this is a consequence of the Ideal Gas Law).
5. For an adiabatic process, $PV^\gamma = C$.

From these rules, we can construct the following table:

Compression/Expansion	*dQ*	*dW*	*dU*
Isovolume	$\dfrac{V(P_f - P_i)}{\gamma - 1}$	0	$\dfrac{V(P_f - P_i)}{\gamma - 1}$
Isobaric	$\dfrac{\gamma}{\gamma - 1} P\left(V_f - V_i\right)$	$-P\left(V_f - V_i\right)$	$\dfrac{P\left(V_f - V_i\right)}{\gamma - 1}$
Isothermal	$NkT \ln \dfrac{V_f}{V_i}$	$-NkT \ln \dfrac{V_f}{V_i}$	0
Adiabatic	0	$\dfrac{P_f V_f - P_i V_i}{\gamma - 1}$	$\dfrac{P_f V_f - P_i V_i}{\gamma - 1}$

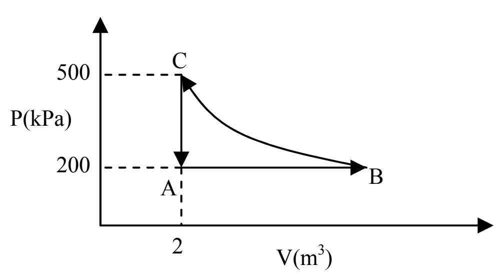
Remember that a positive dQ means heat has been added to the gas, and a positive dW means that work has been done on the gas by the environment. If you can't remember this entire table, concentrate on (a) the dW column and (b) the bottom two rows, as these are always asked about.

Example 25: Consider the quasi-static adiabatic expansion of an ideal gas from an initial state i to a final state f. Which of the following statements is NOT true?

a. No heat flows into or out of the gas.
b. The entropy of state i equals the entropy of state f.
c. The change of internal energy of the gas is $-\int PdV$.
d. The mechanical work done by the gas is $\int PdV$.
e. The temperature of the gas remains constant.

Answer: (a) is the definition of an adiabatic expansion. Since $dQ = TdS = 0$ for an adiabatic change, and the temperature is not zero, then $dS = 0$, so the entropy is constant and (b) is also true. The First Law then becomes $dU = -PdV$, so (c) is true. $dW = -PdV$ represents work done on the gas, so work done by the gas must be the opposite, so (d) is also true. Finally, (e) is the definition of an <u>isothermal</u> change, which is different from an adiabatic change, so the answer is (e).

Example 26: A constant amount of an ideal gas undergoes the cyclic process ABCA in the PV diagram shown below. The path BC is isothermal. What is the work done by the gas during one complete cycle?



Answer: First, we are not given the volume at point B, but we can find it from the fact that $PV = C$ for the isothermal step BC. Thus $V_B/2 = 500/200$, so $V_B = 5$ cubic meters. Now we can find the work done in each step. For A to B, $dW = -PdV = -(200kPa)\left(3m^3\right) = -600kJ$. For B to C,

$dW = -NkT\ln\frac{V_f}{V_i} = -PV\ln\frac{V_f}{V_i} = -(200kPa)\left(5m^3\right)\ln\frac{2}{5} \approx 916kJ$. (Note that I could substitute PV for NkT because of the Ideal Gas Law, and use PV from any point in the process because $PV = C$ for an isothermal expansion). For C to A, since $dV = 0$, no work is done. So the total is (-600+916)=316kJ. But this is the work done <u>on</u> the gas, so the work done <u>by</u> the gas is -316 kJ.

3.4.2 Boltzmann Factor

The probability of a particle being in a state with energy E is given by $Ce^{-E/kT}$, where C is a constant chosen to make all of the probabilities add up to 1. The quantity $e^{-E/kT}$ is called the Boltzmann factor; it is fundamental to all of statistical mechanics, and critical to remember for the exam. It may also be useful to remember that at room temperature, kT is roughly 1/40 of an electron volt.

The sum of the Boltzmann factors for all of the particles in a system is called the partition function for that system.

Example 27: An ensemble of systems is in thermal equilibrium with a reservoir at room temperature. State A has an energy that is 0.1eV above that of state B. If it is assumed that the systems obey Maxwell-Boltzmann statistics and that the degeneracies of the two states are the same, then what is the ratio of the number of systems in state A to the number in state B?

Answer: $\frac{\#A}{\#B} = \frac{prob(A)}{prob(B)} = \frac{e^{-(E+0.1eV)/kT}}{e^{-E/kT}} = e^{-0.1eV/(0.025eV)} = e^{-4} \approx 0.02$.

Example 28: In a Maxwell-Boltzmann system with two states of energies E and $2E$, respectively, and a degeneracy of 2 for each state, what is the partition function?

Answer: A degeneracy of 2 means there are two particles with energy E and two with energy $2E$. Adding up all of their Boltzmann factors gives $2\left[e^{-E/kT} + e^{-2E/kT}\right]$.

3.4.3 Other Useful Equations

Equipartition of Energy: Classically, a system will average $(1/2)kT$ of thermal energy for each degree of freedom of motion it has. Thus an atom will average $(3/2)kT$ (motion in x, y, and z), while a rigid diatomic molecule will average $(5/2)kT$ (three directions of motion plus two distinct axes of rotation).

Stefan-Boltzmann Law: The energy radiated by a blackbody is proportional to temperature to the fourth power.

Example 29: If the absolute temperature of a blackbody is increased by a factor of 3, the energy radiated per second per unit area does which of the following?

Answer: The energy scales as T^4, so if $T \rightarrow 3T$, $U \rightarrow (3T)^4 = 81T^4 = 81U$, so the energy radiated increases by a factor of 81.

3.5 Quantum Mechanics

3.5.1 The Wavefunction

Quantum mechanics is based on the idea that any system's behavior can be described by a wavefunction. A "well-behaved" wavefunction Ψ must (a) be single-valued everywhere, (b) have continuous and single-valued derivates, and (c) must approach zero at its extremes, such that $\int |\Psi|^2$ is finite.

You will often see a wavefunction written as $|\Psi\rangle$. A wavefunction can also be written as the linear combination of other wavefunctions – for example, $|\Psi\rangle = 4|1\rangle + 2|2\rangle - |3\rangle$ is the linear combination of quantum states 1, 2, and 3. Typically the states that make up these linear combinations are required to be orthonormal. Orthogonal means that the product of two states is zero unless they are the same state. Orthonormal is the same as orthogonal with the added requirement that the product be 1 if the states are the same; in other words, $|i\rangle \times |j\rangle = \langle i \mid j\rangle = \delta_{ij}$, the Dirac delta function.

Example 30: Which of the following functions could represent the radial wave function for an electron in an atom? (*r* is the distance of the electron from the nucleus; *A* and *b* are constants).

I. Ae^{-br} II. $A\sin(br)$ III. A/r

Answer: Function II does not approach zero as *r* approaches infinity, and function III explodes as *r* approaches zero, so neither will be integrable over all space. Thus only function I is viable.

Example 31: $|\psi_1\rangle = 5|1\rangle - 3|2\rangle + 2|3\rangle$ $|\psi_2\rangle = |1\rangle - 5|2\rangle + x|3\rangle$

States $|1\rangle$, $|2\rangle$, and $|3\rangle$ are orthonormal. For what values of *x* are the states ψ_1 and ψ_2 orthogonal?

Answer: Asking ψ_1 and ψ_2 to be orthogonal is the same as asking for their product to be zero. Since states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are orthonormal, all cross terms will cancel out, so the product is $|\psi_1\rangle \times |\psi_2\rangle = (5|1\rangle)(|1\rangle) + (-3|2\rangle)(-5|2\rangle) + (2|3\rangle)(x|3\rangle) = 5\langle 1 \mid 1\rangle + 15\langle 2 \mid 2\rangle + 2x\langle 3 \mid 3\rangle = 20 + 2x = 0$, and $x = -10$.

3.5.2 Probability and Normalization

The fundamental statement of quantum mechanics is that the probability of a particle being in a given place is equal to $|\Psi|^2$ evaluated at that point in space. But for this to work, we must *normalize* the function to make the probabilities over all space add up to 1. This is done by multiplying the function by the constant $A = \dfrac{1}{\sqrt{\int |\Psi|^2}} = \dfrac{1}{\sqrt{\langle \psi \mid \psi\rangle}}$.

If a wavefunction is written as a linear combination of states, then once it is normalized, the probability of the particle acting like it is in a certain state is equal to the square of the coefficient of that state.

Example 32: $|\psi\rangle = A\left[5|1\rangle - 3|2\rangle + 2|3\rangle\right]$

Find *A* such that ψ is normalized. What is the probability that the particle is in state 2?

Answer: $A = \dfrac{1}{\sqrt{\langle\psi\,|\,\psi\rangle}} = \dfrac{1}{\sqrt{\left(5|1\rangle\right)^2 + \left(-3|2\rangle\right)^2 + \left(2|3\rangle\right)^2}} = \dfrac{1}{\sqrt{25+9+4}} = \dfrac{1}{\sqrt{38}}$. So

$|\psi\rangle = \dfrac{5}{\sqrt{38}}|1\rangle - \dfrac{3}{\sqrt{38}}|2\rangle + \dfrac{2}{\sqrt{38}}|3\rangle$. The probability of behaving as state 2 is then $\left(-\dfrac{3}{\sqrt{38}}\right)^2 = \dfrac{9}{38}$.

3.5.3 Operators and Expectation Value

An operator is just what it sounds like – a mathematical expression that operates on a wavefunction. For example, the momentum operator is given by $\hat{p} = \dfrac{\hbar}{i}\dfrac{d}{dx}$. When the result of applying an operator to a wavefunction is a constant times the original wavefunction $\left(\hat{p}\Psi = C\Psi\right)$, then the function is called an eigenfunction or eigenvector of the operator, and the constant is called an eigenvalue.

Some operators are called *observables*, which means that they correspond to measurable quantities (such as momentum or energy). The constant generated when such an operator is applied is the value of that observable quantity for that state. Since the value is measurable, the eigenvalue must be real; operators that produce only real eigenvalues are also called Hermitian operators.

The expectation value for an operator is the average result expected when the operator is applied. It is found by taking the sum of all possible eigenvalues times the probability of each eigenvalue occurring. For example, the expectation value of the roll of a die is $\frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6) = 3.5$

Example 33: The state $\psi = \dfrac{1}{\sqrt{6}}\psi_{-1} + \dfrac{1}{\sqrt{2}}\psi_1 + \dfrac{1}{\sqrt{3}}\psi_2$ is a linear combination of three orthonormal eigenstates of the operator $\hat{O}$ corresponding to eigenvalues -1, 1, and 2. What is the expectation value of $\hat{O}$ in this state?

Answer: Note that the wavefunction is already normalized, since $\left(\dfrac{1}{\sqrt{6}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{3}}\right)^2 = 1$. Thus the expectation value $= \frac{1}{6}(-1) + \frac{1}{2}(1) + \frac{1}{3}(2) = 1$.

Example 34: The energy eigenstates for a particle of mass *m* in a box of length *L* have wavefunctions $\phi_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ and energies $E_n = n^2\pi^2\hbar^2/2mL^2$, where *n* = 1, 2, 3, …. At time *t* = 0, the particle is in a state described as $\psi(t=0) = \dfrac{1}{\sqrt{14}}\left[\phi_1 + 2\phi_2 + 3\phi_3\right]$. Which of the following is a possible result of a measurement of energy for the state ψ?

a. $2E_1$ b. $5E_1$ c. $7E_1$ d. $9E_1$ e. $14E_1$

Answer: Note that $E_n = n^2 E_1$, so $E_2 = 4E_1$ and $E_3 = 9E_1$. Since the particle is in a combination of only states 1, 2, and 3, a measurement of its energy can only give the energy of one of those three states (namely, E_1, $4E_1$, or $9E_1$). The only answer that corresponds to this list is (d).

Example 35: Let $|n\rangle$ represent the nth energy eigenstate of the one-dimensional harmonic oscillator, $H|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$. If $|\psi\rangle$ is a normalized ensemble state that can be expanded as a linear combination $|\psi\rangle = \frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle$ of the eigenstates, what is the expectation value of the energy operator in this ensemble state?

Answer: First note that $H_1 = \frac{3}{2}\hbar\omega, H_2 = \frac{5}{2}\hbar\omega,$ and $H_3 = \frac{7}{2}\hbar\omega$. Then the expectation value is given by

$$\left(\frac{1}{\sqrt{14}}\right)^2\left(\frac{3}{2}\hbar\omega\right)+\left(-\frac{2}{\sqrt{14}}\right)^2\left(\frac{5}{2}\hbar\omega\right)+\left(\frac{3}{\sqrt{14}}\right)^2\left(\frac{7}{2}\hbar\omega\right)=\hbar\omega\left(\frac{1}{14}\frac{3}{2}+\frac{4}{14}\frac{5}{2}+\frac{9}{14}\frac{7}{2}\right)=\frac{43}{14}\hbar\omega.$$

3.5.4 Commutators

A commutator is the mathematical expression $[A,B] = AB - BA$, which for certain combinations of operators will be nonzero. A couple of useful identities related to commutators are $[A,B] = -[B,A]$ and $[AB,C] = A[B,C]+[A,C]B$.

Example 36: The components of the orbital angular momentum operator $L = (L_x, L_y, L_z)$ satisfy the following commutation relations:

$$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

What is the value of the commutator $[L_x L_y, L_z]$?

Answer: $[L_x L_y, L_z] = L_x[L_y, L_z]+[L_x, L_z]L_y = L_x(i\hbar L_x)+(-i\hbar L_y)L_y = i\hbar\left(L_x^2 - L_y^2\right)$.

3.6 Atomic Physics

3.6.1 Electron Configuration

We describe electrons in an atom as inhabiting a series of energy levels about the nucleus. The state of any given electron is described primarily by two quantum numbers: the principal number n, which tells us what level the electron is in, and l, which tells us how much angular momentum the particular state has. n takes on integer values 1, 2, 3…, while l ranges from 0 to $n-1$. We also associate a letter with each value of l: 0 corresponds to the *s* orbital, 1 to the *p* orbital, 2 to the *d* orbital, 3 to the *f* orbital, and so on.

The chart at right shows the order in which the orbitals are filled, and how many electrons go into each orbital. It is generated by writing all of the $n = 1$ orbitals in a column, then stepping down a row and writing the $n = 2$ orbitals in the next column, then stepping down another row and writing the $n = 3$ orbitals, etc. Up to 2 electrons can fit in an *s* orbital, 6 in a *p*, 10 in a *d*, and 14 in an *f* orbital. Given a certain number of electrons, you start filling the orbitals from left to right, top to bottom, until you run out of electrons.

$1s^2$						
	$2s^2$					
	$2p^6$	$3s^2$				
		$3p^6$	$4s^2$			
		$3d^{10}$	$4p^6$	$5s^2$		
			$4d^{10}$	$5p^6$	$6s^2$	
			$4f^{14}$	$5d^{10}$	$6p^6$	$7s^2$

Example 37: Write the ground state electron configuration for iron, which has 26 electrons.

Answer: Following the chart, we get $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^5$**.** Things start getting non-intuitive when you fill the 4*s* orbital before the 3*d* orbital; even though the 4*s* has a higher principal quantum number, it has a lower energy. Follow the chart!

3.6.2 The Bohr Atom

Bohr's description of the hydrogen atom gave the energy of the *n*th energy level as $E_n = \frac{E_1}{n^2} = \frac{-13.6eV}{n^2}$. Note that the energy is negative because the electron is bound to the nucleus; if the energy of the electron were increased to zero or more, it would escape from the atom into free space.

When an electron moves from an excited energy level to a lower one, a photon is emitted. The energy of the photon is related to its wavelength by $E = h\nu = hc/\lambda$, where h is Planck's constant. Combining this with the energy level formula gives $\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$, where n_i is the initial energy level, $n_f < n_i$ is the final energy level, and $R \approx 1.1\times10^7 m^{-1}$ is the Rydberg constant. Transitions to $n_f = 1$ are called the Lyman series and are in ultraviolet wavelengths. Transitions to $n_f = 2$ are called the Balmer series and result in visible wavelengths.

If there are Z protons in the nucleus, the Bohr energy formula becomes $E_n = \frac{Z^2 E_1}{n^2}$. This only really works for one electron, however. With more than one, the closer electrons partially screen the nuclear charge from the farther ones, and the effective Z is less.

Example 38: In the hydrogen spectrum, what is the ratio of the wavelengths for Lyman-α radiation ($n = 2$ to $n = 1$) to Balmer-α radiation ($n = 3$ to $n = 2$)?

Answer: $\frac{1}{\lambda_L} = R\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = \frac{3}{4}R$ $\qquad$ $\frac{1}{\lambda_B} = R\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{5}{36}R$ $\qquad$ $\frac{\lambda_L}{\lambda_B} = \frac{1/\lambda_B}{1/\lambda_L} = \frac{5R/36}{3R/4} = \frac{5}{27}$

Example 39: The energy required to remove both electrons from the helium atom in its ground state is 79.0 eV. How much energy is required to ionize helium (i.e., to remove one electron)?

Answer: The energy to remove the last electron from helium is given by $Z^2 E_1 = 4E_1 = 54.4eV$. Since it takes 79.0 eV to remove both, the energy required to remove the first must be (79.0 – 54.4) = 24.6 eV.

3.7 Special Relativity

3.7.1 Length Contraction and Time Dilation

In an inertial (non-accelerating) reference frame, an object moving at a speed v with respect to an observer will appear to be shortened in its direction of motion, and its clock will appear to be running more slowly. Both effects are proportional to the quantity $\gamma = \frac{1}{\sqrt{1-\left(v^2/c^2\right)}}$. A useful number to remember is that $\gamma = 2$ at $\frac{\sqrt{3}}{2}c \approx 0.87c$.

Example 40: If a charged pion that decays in 10^{-8} seconds in its own rest frame is to travel 30 meters in the laboratory before decaying, how fast must the pion be moving?

Answer: Let's assume that the pion is moving close to the speed of light, and check that assumption later. In the laboratory reference frame, it should take the pion $(30m)/(3\times10^8 m/s) = 10^{-7}$ seconds to travel 30 meters. Since it will decay in 10^{-8} seconds, its clock must be running $10^{-7}/10^{-8} = 10$ times more slowly. So $\gamma = \frac{1}{\sqrt{1-\left(v^2/c^2\right)}} = 10$, or $v = 0.995c = 2.98 \times 10^8$ m/s. Our initial assumption looks OK.

3.7.2 Kinematics

Relativistic momentum = γmv.

Relativistic energy = $\gamma mc^2 = (\gamma - 1)mc^2 + mc^2$. The first term represents relativistic kinetic energy, and goes to zero when $v = 0$. The second term is called the *rest mass energy*, and is present even when the object is not moving.

Useful identity: $E^2 = (pc)^2 + \left(mc^2\right)^2$. Note that for light, which is massless, this implies that $E=pc$.

Example 41: If the total energy of a particle of mass m is equal to twice its rest energy, then what is the magnitude of the particle's relativistic momentum?

Answer: Energy = γmc^2=γ × rest energy, so γ = 2, and in turn $v = \frac{\sqrt{3}}{2}c$. Thus

$$p = \gamma mv = 2m\left(\frac{\sqrt{3}}{2}c\right) = \sqrt{3}mc .$$

Example 42: A particle leaving a cyclotron has a total relativistic energy of 10 GeV and a relativistic momentum of 8 GeV/c. What is the rest mass of this particle?

Answer: $(10GeV)^2 = (8GeV)^2 + \left(mc^2\right)^2$, so $m = 6\ GeV/c^2$.

3.8 Miscellaneous Topics

3.8.1 Watch Your Units!

Occasionally you can eliminate one or more answer choices simply because their units cannot possibly match those of the answer. And usually once per test, there will be a question solvable with units alone!

Example 43: Einstein's formula for the molar heat capacity, C, of solids is given by $C = 3kN_A\left(\frac{\hbar\nu}{kT}\right)^2 \frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}$. At high temperatures, C approaches which of the following?

a. 0 b. $3kN_A\left(\frac{h\nu}{kT}\right)$ c. $3kN_A h\nu$ d. $3kN_A$ e. $N_A h\nu$

Answer: Even if you forget that $e^x \approx 1 + x$ for small x, you should still recognize that since $h\nu$ and kT are both energies, their units cancel out, so the units of C must always match those of kN_A. Thus you can eliminate answers (c) and (e), since both contain an energy term. With only three answers remaining, it is in your favor to guess. Using the expansion above, by the way, the correct answer is (d).

3.8.2 Particle Physics

Nearly every GRE test has one question asking if a particular particle decay is legal. For a decay to occur, several quantities must be conserved:

1. Charge. The total charge must match on each side of the decay.
2. Baryon number conservation – protons and neutrons have a baryon number of +1. Antiprotons and antineutrons have a baryon number of -1. Mesons such as the pion (π) and kaon (K) have a baryon number of zero. The total baryon number on each side of the decay must match.
3. Lepton number – electrons and electron neutrinos (ν_e) have an electron lepton number of +1. Antielectrons and electron antineutrinos ($\overline{\nu_e}$) have an electron number of -1. Similarly, muons and muon neutrinos $\left(\mu \text{ and } \nu_\mu\right)$ have a muon lepton number of +1, and taus and tau neutrinos $\left(\tau \text{ and } \nu_\tau\right)$ have a tau lepton number of +1. Each type of lepton number must be balanced separately.
4. Parity.
5. Strangeness. These last two may show up as possible answers, but are almost never the correct one, since you would have to have the parity and strangeness of many different particles memorized to catch this.

Example 44: Is this mechanism for "electron capture", $p^+ + e^- \to n$, legal?

Answer: Total charge is zero on both sides, so that is conserved. There is one +1 baryon on either side, so that is conserved. But there is an electron number of +1 on the left and none on the right. An electron neutrino must also be emitted for this decay to be legal: $p^+ + e^- \to n + \nu_e$.

3.8.3 Nuclear Radiation

A given atomic isotope can be written as ${}^A_Z X$, where X is the atomic symbol, A is the number of nucleons (protons + neutrons), and Z is the number of protons. Such an isotope can decay in several ways:

1. Emission of an alpha particle, a He^{++} nucleus with two protons and two neutrons. This reduces Z by two and A by four.
2. Emission of a beta particle (an electron): $n \to p^+ + e^- + \overline{\nu}_e$. This increases Z by one while A is unchanged.

3. Emission of a positron (or antielectron): $p^+ \to n + e^+ + \nu_e$. This decreases Z by one while A is unchanged.
4. Electron capture: $p^+ + e^- \to n + \nu_e$. This is the same as positron emission, except that the charge of the atom also remains constant.

Example 45: ${}^{12}_{5}B$ decays into ${}^{12}_{6}C$ through what process?

Answer: Since Z increased by one while A did not change, the mechanism must be beta decay.

4 Practice Test

4.1 The Test

Starting on the next page is a 20-question miniature version of the GRE Physics exam. The proportion of questions from each subject area is correct, but the difficulty level is lower; you should be able to answer every question from reviewing just the material in this document. Give yourself 40 minutes to take this exam, in as much of a real test-taking environment as possible; try to minimize distractions and interruptions, don't use any reference books, etc. You will be provided the same list of physical constants that would be given on the real exam, but nothing more. Finally, make sure to note which answers are guesses, so you can see your success rate afterward.

Good luck!

TABLE OF INFORMATION

Rest mass of the electron	$m_e = 9.11 \times 10^{-31}$ kilogram $= 9.11 \times 10^{-28}$ gram
Magnitude of the electron charge	$e = 1.60 \times 10^{-19}$ coulomb $= 4.80 \times 10^{-10}$ statcoulomb (esu)
Avogadro's number	$N_A = 6.02 \times 10^{23}$ per mole
Universal gas constant	$R = 8.31$ joules/(mole · K)
Boltzmann's constant	$k = 1.38 \times 10^{-23}$ joule/K $= 1.38 \times 10^{-16}$ erg/K
Speed of light	$c = 3.00 \times 10^{8}$ m/s $= 3.00 \times 10^{10}$ cm/s
Planck's constant	$h = 6.63 \times 10^{-34}$ joule · second $= 4.14 \times 10^{-15}$ eV · second
	$\hbar = h/2\pi$
Vacuum permittivity	$\epsilon_0 = 8.85 \times 10^{-12}$ coulomb^2/(newton · meter^2)
Vacuum permeability	$\mu_0 = 4\pi \times 10^{-7}$ weber/(ampere · meter)
Universal gravitational constant	$G = 6.67 \times 10^{-11}$ meter^3/(kilogram · second^2)
Acceleration due to gravity	$g = 9.80$ m/s^2 $= 980$ cm/s^2
1 atmosphere pressure	1 atm $= 1.0 \times 10^5$ newtons/meter^2 $= 1.0 \times 10^5$ pascals (Pa)
1 angstrom	1Å $= 1 \times 10^{-10}$ meter
	1 weber/m^2 $=$ 1 tesla $= 10^4$ gauss

Moments of inertia about center of mass

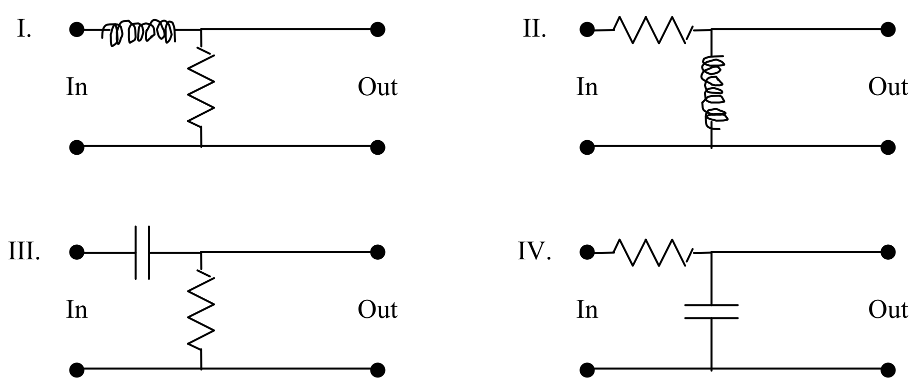
Rod	$\frac{1}{12} M\ell^2$
Disc	$\frac{1}{2} MR^2$
Sphere	$\frac{2}{5} MR^2$

Sample GRE Physics Test
20 Questions – 40 Minutes

1. A solid cylinder starts at rest, then rolls down an incline without slipping. What fraction of the cylinder's kinetic energy comes from rotational motion?

   (A) 2/3
   (B) 1/2
   (C) 1/3
   (D) 1/4
   (E) 1/6

2. Which of the following circuits are low-pass filters?



   (A) I and II
   (B) II and III
   (C) I and IV
   (D) II and III
   (E) II and IV

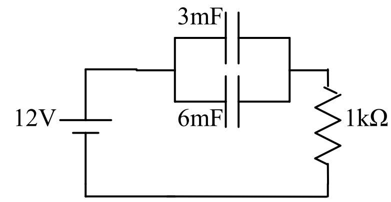
3. A muon has a lifetime of 2.2×10^{-6} seconds in its own rest frame. If an observer at rest measures a muon to travel 2 km before decaying, the muon's speed must be most nearly:

   (A) 0.33c
   (B) 0.87c
   (C) 0.90c
   (D) 0.95c
   (E) 0.99c

4. A hockey puck with a mass of 50g and an initial velocity of 2 m/s is sliding across an ice rink with coefficient of friction 0.03. How far will it travel before stopping?

   (A) 6.8 meters
   (B) 13.5 meters
   (C) 20 meters
   (D) 27 meters
   (E) 135 meters

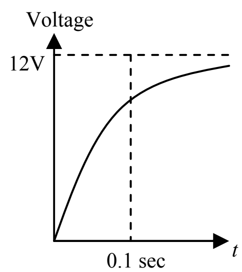
Questions 5 and 6 refer to the circuit shown at right. The 12V power can be switched on and off. The problem begins with the power having been switched on for a long time.

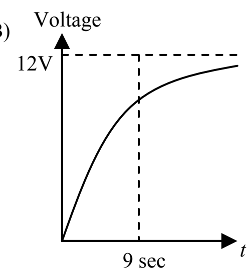


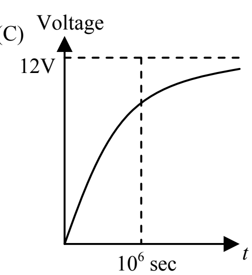
5. What is the energy stored in the capacitor?

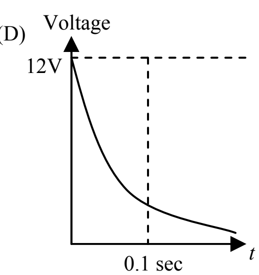
   (A) 0.14 J
   (B) 0.29 J
   (C) 0.58 J
   (D) 0.65 J
   (E) 1.30 J

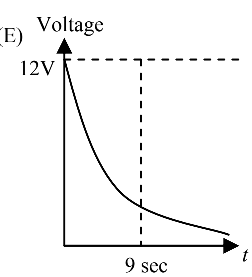
6. Once the 12V is switched off, which of the following best describes the voltage across the resistor as a function of time?

(A)


(B)


(C)


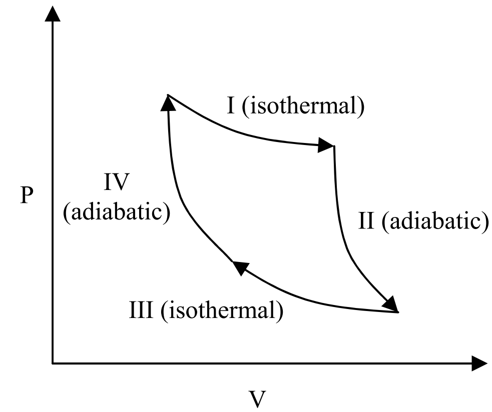
(D)


(E)


7. Unpolarized light of intensity I_0 is incident upon a series of three polarizing filters. The axis of the second filter is oriented at 30^0 to that of the first filter, while the axis of the third filter is oriented at 60^0 to that of the first filter and 30^0 to the second. What is the intensity of the light transmitted through the third filter?

   (A) (9/32) I_0
   (B) (9/16) I_0
   (C) (3/4) I_0
   (D) I_0
   (E) 0

8. In the Carnot cycle shown at right, during which step(s) is heat added to the system?



(A) I
(B) III
(C) I and II
(D) I and III
(E) III and IV

9. Given $\left[x, p\right] = i\hbar$, what is $[x, T]$, where $T = \dfrac{p^2}{2m}$ is the kinetic energy operator?

(A) 0
(B) $\dfrac{i\hbar}{m}$
(C) $\dfrac{ip}{m}$
(D) $\dfrac{-i\hbar p}{m}$
(E) $\dfrac{i\hbar p}{m}$

10. When ${}^{7}_{4}$Be transforms into ${}^{7}_{3}$Li, it does so by

(A) emitting an alpha particle only
(B) emitting an electron only
(C) emitting a neutron only
(D) emitting a positron only
(E) electron capture by the nucleus with emission of a neutrino

11. The Lagrangian of a system is given by $L = \frac{1}{4}\dot{x}^2 + 3x^2 + 4$. What is the equation of motion?

(A) $\ddot{x} = 2\sqrt{3x^2 + 4}$
(B) $\ddot{x} = 12x$
(C) $\ddot{x} = 12\dot{x}$
(D) $\ddot{x} = -12x$
(E) $\ddot{x} = 12x + 8$

Questions 12 and 13 refer to the following:

The eigenfunctions of the one-dimensional quantum harmonic oscillator are given by $|n\rangle$, where the associated energy eigenvalue is given by $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$. Let $|\psi\rangle$ be a normalized state of the oscillator, given by the linear combination

$$|\psi\rangle = \frac{2}{\sqrt{21}}|1\rangle - \frac{1}{\sqrt{21}}|2\rangle + \frac{4}{\sqrt{21}}|3\rangle$$

12. Which of the following is a possible result of a measurement of the energy of ψ?

    (A) $\frac{2}{\sqrt{21}}\hbar\omega$
    (B) $\frac{2}{21}\hbar\omega$
    (C) $\frac{4}{21}\hbar\omega$
    (D) $\frac{1}{2}\hbar\omega$
    (E) $\frac{3}{2}\hbar\omega$

13. What is the expectation value of the energy operator for ψ?

    (A) $\frac{43}{14}\hbar\omega$
    (B) $\frac{119}{42}\hbar\omega$
    (C) $\frac{13}{42}\hbar\omega$
    (D) $\frac{17}{21}\hbar\omega$
    (E) $\frac{29}{42}\hbar\omega$

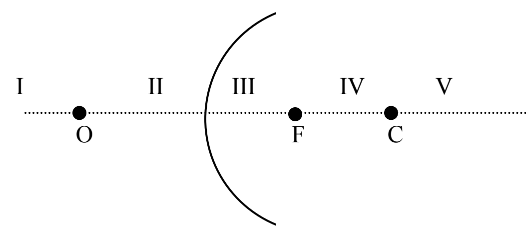
14. In this description of neutron decay: $n \rightarrow p + e^- + \nu_e$
    what rule is being violated?

    (A) Charge conservation
    (B) Parity conservation
    (C) Baryon number conservation
    (D) Lepton number conservation
    (E) No rules are being violated

15. Which of the following is an expression for the Planck time?

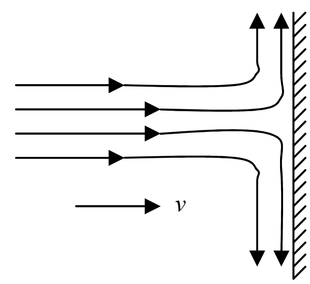
(A) $\hbar Gc$
(B) $\sqrt{hGc}$
(C) $\sqrt{\hbar c^2 / G}$
(D) $\sqrt{\hbar G / c^5}$
(E) $\sqrt{c^6 / G\hbar^2}$

16. A spherical, convex mirror with focal point F and center of curvature C is shown at right. If an object is placed at O, in what region will the image be located?



(A) I
(B) II
(C) III
(D) IV
(E) V

17. A stream of water of density ρ, cross-sectional area A, and speed v strikes a wall that is perpendicular to the direction of the stream, as shown at right. The water then flows sideways across the wall. The force exerted by the stream on the wall is



(A) $\rho v^2 A$
(B) $\rho v A / 2$
(C) $\rho g h A$
(D) $v^2 A / \rho$
(E) $4\rho v^2 A$

18. An object with mass M is moving linearly with nonrelativistic speed v. It explodes into two pieces of equal mass, one of which is produced at rest. What is the minimum amount of energy released by the explosion?

(A) 0
(B) $\frac{1}{8}Mv^2$
(C) $\frac{1}{4}Mv^2$
(D) $\frac{1}{2}Mv^2$
(E) Mv^2

19. The valence band gap in undoped silicon is approximately 1.0 eV. At what temperature would approximately 5% of electrons be found in the valence band?

(A) 300 K
(B) 3,000 K
(C) 4,000 K
(D) 6,000 K
(E) 12,000 K

20. A negatively-charged particle is moving in the positive *x*-direction past a magnetic field line pointing in the positive *z*-direction. Which of the following statements is true?

(A) The particle will feel a force in the negative-*y* direction.
(B) When looking from above (from $z > 0$ down to the origin), the particle will appear to be traveling in a clockwise path.
(C) The work done on the particle will depend linearly on the initial velocity of the particle.
(D) All of the above statements are true.
(E) None of the above statements are true.

STOP

If you finish before time is called, you may check your work on this test.

Solutions begin on page 38

4.2 Solutions

1. **(C)**
   The cross-section of a solid cylinder is a disc, so that is the moment of inertia we will use.
   $KE = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{4}M\left(\omega R\right)^2 = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 = \frac{3}{4}Mv^2$
   and thus the fraction from rotational motion is (1/4) / (3/4) = 1/3.

2. **(C)**
   At low frequencies, the impedance of an inductor is small, so in an RL circuit all of the voltage drop will be across the resistor. Since we want the voltage drop to be at the output, we want the RL circuit where the resistor is in parallel to the output, which is I. The capacitor, however, has high impedance at low frequencies, so all of the voltage drop will be across it. Thus we want the capacitor in parallel with the output for an RC circuit, so IV is also a low-pass filter.

3. **(D)**
   Let us start with the assumption that the muon is moving at close to the speed of light. Thus it will travel 2 km in $\left(2\times10^3 m\right)/\left(3\times10^8 m/s\right) = 6.7\times10^{-6} s$, or three times longer than the muon's lifetime.
   Thus the muon's clock must be slowed by a factor of three, so $\gamma = \frac{1}{\sqrt{1-\left(v^2/c^2\right)}} = 3$. Solving for v gives 0.942c, which is closest to 0.95c.

4. **(A)**
   The puck will decelerate due to friction at a rate given by $F = ma = \mu N = \mu mg$. Thus
   $a = \mu g = (0.03)\left(9.8m/s^2\right) = 0.294m/s^2$. Next, $v = v_0 + at$ implies that the puck will come to a stop when $0 = 2m/s - \left(0.294m/s^2\right)t$, which gives t = 6.8 seconds. Finally,
   $x = x_0 + v_0 t + \frac{1}{2}at^2 = 0 + (2m/s)(6.8s) - \frac{1}{2}\left(0.294m/s^2\right)(6.8s)^2 = 6.8m$.
   You could also have been really clever and noticed that all of the puck's kinetic energy would be used up as work overcoming the force of friction, so $\frac{1}{2}mv^2 = F\cdot d = \mu N\cdot d = \mu mgd$. Thus
   $d = \frac{mv^2}{2\mu mg} = \frac{v^2}{2\mu g} = \frac{(2m/s)^2}{2(0.03)(9.8m/s^2)} = 6.8m$.

5. **(D)**
   Since the capacitors are in parallel, the total capacitance is just 3mF + 6mF = 9mF. The energy stored is then $U = \frac{1}{2}CV^2 = \frac{1}{2}(0.009F)(12)^2 = 0.65J$.

6. **(E)**
   When the switch is opened, the capacitor will begin to discharge from, and all of the voltage drop will occur across the resistor. Thus the voltage across the resistor will begin at 12V and exponentially decrease. The time constant for this decrease is RC = (1000 Ω)(0.009 F) = 9 sec.

7. **(A)**
   Since the initial light is unpolarized, half of the light intensity will be lost at the first filter. For each

of the next two filters, the angle between the polarizer axis and the light polarization is 30 degrees, so $I = I_0 \cos^2 30^\circ = I_0\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}I_0$. The effect of all three filters is $\left(\frac{1}{2}\times\frac{3}{4}\times\frac{3}{4}\right)I_0 = \frac{9}{32}I_0$.

8. **(A)**
By definition, there is no heat flow in an adiabatic expansion, so steps II and IV are out. For an isothermal change, $dQ = NkT \ln\frac{V_f}{V_i}$, which will be positive when $V_f > V_i$. In other words, when heat is added to the gas, the gas expands, and when heat is taken away from the gas, the gas contracts. The only step in which the gas isothermally expands is step I.

9. **(E)**
$$\left[x, \frac{p^2}{2m}\right] = \frac{1}{2m}\left[x, p^2\right] = -\frac{1}{2m}\left[p^2, x\right] = -\frac{1}{2m}\left\{p[p,x]+[p,x]p\right\} = \frac{1}{2m}\left\{p[x,p]+[x,p]p\right\} = \frac{1}{2m}\left\{2i\hbar p\right\} = \frac{i\hbar p}{m}.$$

10. **(E)**
We start with 4 protons, 3 neutrons (7-4), and 4 electrons (since the atom is neutral), and finish with 3 protons, 4 neutrons, and 3 electrons. Thus we lose a proton and electron, and gain a neutron: $p^+ + e^- \to n$, which is the definition of electron capture. The neutrino emission balances lepton number in the decay (this is why answer D must be wrong – the emission of a positron <u>only</u> violates lepton number conservation.

11. **(B)**
$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$, so $6x = \frac{d}{dt}\left(\frac{1}{2}\dot{x}\right) = \frac{1}{2}\ddot{x}$. Thus the equation of motion is $\ddot{x} = 12x$.

12. **(E)**
For the energy formula given, $E_1 = \frac{3}{2}\hbar\omega, E_2 = \frac{5}{2}\hbar\omega,$ and $E_3 = \frac{7}{2}\hbar\omega$. These will be the only possible energy eigenvalues, and only choice (E) appears in this list.

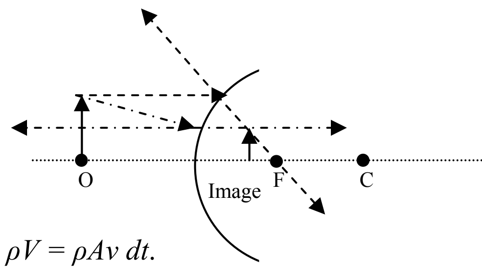
13. **(A)**
Using the energy eigenvalues found in the previous question, the expectation value is
$$\left(\frac{2}{\sqrt{21}}\right)^2\left(\frac{3}{2}\hbar\omega\right)+\left(-\frac{1}{\sqrt{21}}\right)^2\left(\frac{5}{2}\hbar\omega\right)+\left(\frac{4}{\sqrt{21}}\right)^2\left(\frac{7}{2}\hbar\omega\right)=\left[\frac{(4)(3)+(1)(5)+(16)(7)}{42}\right]\hbar\omega = \frac{129}{42}\hbar\omega = \frac{43}{14}\hbar\omega.$$

14. **(D)**
Even though I very rudely left off the positive sign on the proton, you should see that charge is conserved (I saw this on a real exam once). Electron lepton number, however, is zero on the left and +2 on the right. The neutrino should instead be an antineutrino to balance the decay.

15. **(D)**
The units of *c* are m/s, of *G* are m^3/(kg s^2), and of $\hbar$ are $J\cdot s = \frac{kg\ m^2}{s^2}s = \frac{kg\ m^2}{s}$. To cancel the kilogram unit, we must combine $\hbar\cdot G = \frac{kg\ m^2}{s}\frac{m^3}{kg\ s^2} = \frac{m^5}{s^3}$. To cancel the meter, we must divide by

c^5. The only choice that contains these quantities in these proportions is (D).

16. **(C)**
The rays incident parallel to the optical axis and through the opposite focal point are shown in the figure at right. They converge in region III, creating a virtual image.



17. **(A)**
The volume of water that will strike the wall in a time *dt* is $V = Av\,dt$. The mass of this volume of water is given by $M = \rho V = \rho Av\,dt$. The horizontal momentum of the water is $p = Mv = \rho Av^2\,dt$. All of this momentum will be lost upon impact with the wall. Thus $F = \dfrac{dp}{dt} = \rho Av^2$.

18. **(D)**
The initial mass has momentum Mv and kinetic energy $(1/2)Mv^2$. The two subsequent masses have mass $(1/2)M$. Since one is produced at rest, in order to conserve momentum, the other must have velocity $2v$, since $(1/2)M \times 2v = Mv$. But then this mass will have kinetic energy $\frac{1}{2}\left(\frac{1}{2}M\right)\left(2v^2\right) = Mv^2$. Thus at least $Mv^2 - \frac{1}{2}Mv^2 = \frac{1}{2}Mv^2$ must have been released by the explosion; more may have been contributed as heat.

19. **(C)**
$e^{-E/kT} = e^{-(1.0eV)/kT} = 0.05$, so $(1.0eV)/kT = -\ln(0.05) = 3$. Thus $kT = (1/3)$ eV. Now at room temperature (300K), $kT = (1/40)$ eV, so the temperature here is (40/3)(300K) = 4000K.

20. **(E)**
Using the right-hand rule, point your fingers in the positive *x*-direction and curl them in the positive *z*-direction. Your thumb should be pointing in the negative *y*-direction. But this is a *negatively-charged* particle, so the force will be in the positive *y*-direction, which will result in a counterclockwise path. Also, answer (C) is wrong because a magnetic field cannot do work; it does not change the kinetic energy of a particle, but only deflects its path.

5 You're Not Done!

So have you mastered all of the material in this guide? Great! But you're not done yet! This guide is only meant to give you a basis from which you can be confident that you will achieve a decent score. Go take some real practice tests, find out what you need more work on, and review that material as well. For example, if you've taken Quantum 2, you could also be looking at (a) spherical harmonics and how they relate to quantum numbers and angular momentum, and (b) first-order, time-independent perturbation theory, both of which often appear on the exam.

Good luck on the exam!