Group Velocity and Phase Velocity

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Physics 158
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Class Outline

- Meanings of wave velocity
    - Group Velocity
    - Phase Velocity
- Fourier Analysis
    - Spectral density
    - Power Spectrum
- Spectral measurements

Phase Velocity

For a sinusoidal wave, or a waveform comprised of many sinusoidal components that all propagate at the same velocity, the waveform will move at the **phase velocity** of the sinusoidal components

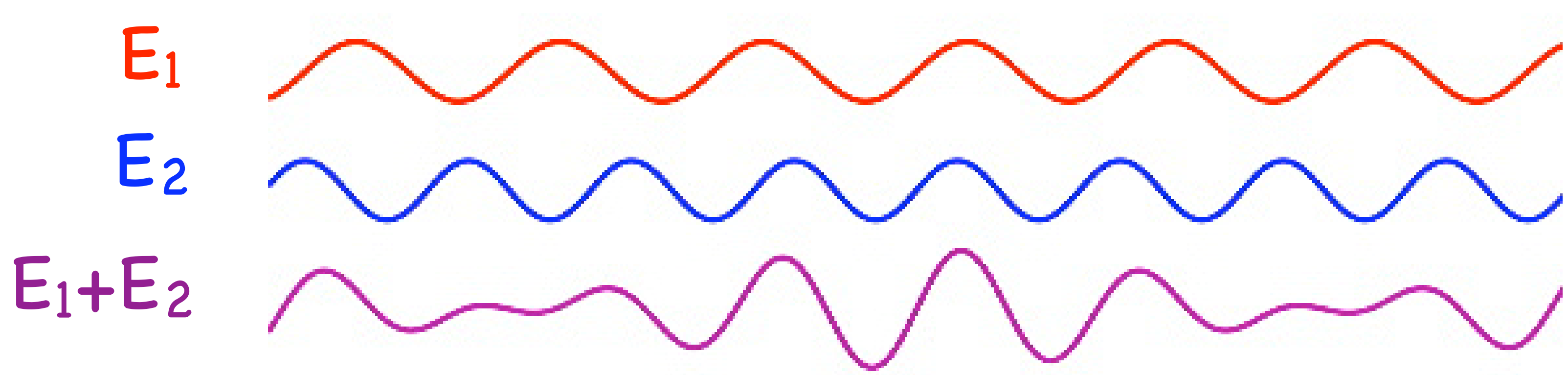
We've seen already that the phase velocity is

$$v_p = \omega / k$$

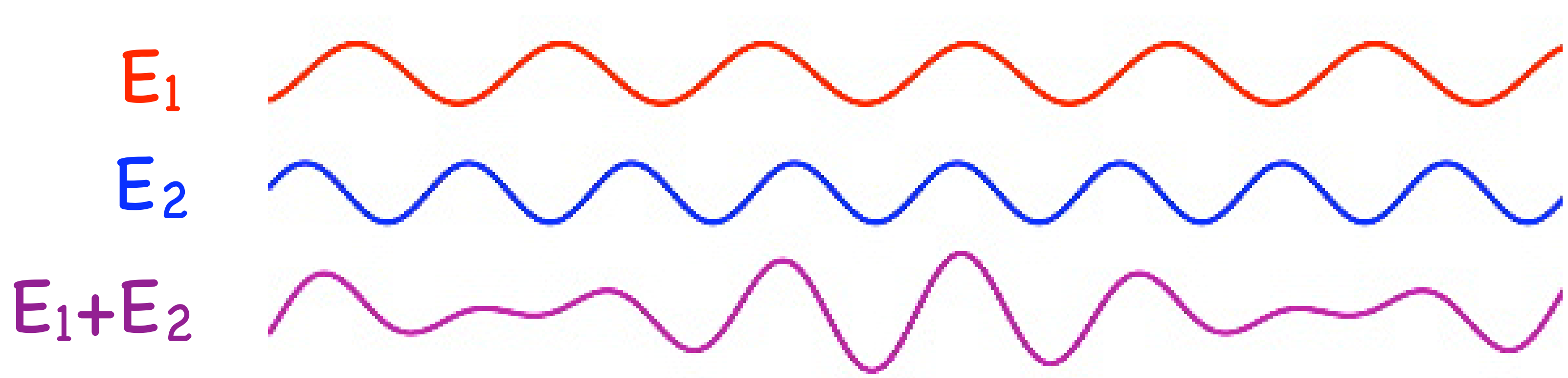
What happens if the different components of the wave have different phase velocities (i.e. because of dispersion)?

Phase and Group Velocity

No dispersion ($v_p = v_g$)



Dispersion ($v_p \neq v_g$)



Group Velocity

When the various frequency components of a waveform have different phase velocities, the phase velocity of the waveform is an average of these velocities (the phase velocity of the **carrier wave**), but the waveform itself moves at a different speed than the underlying carrier wave called the **group velocity**.

Group vs Phase velocity

An analogy that may be useful for understanding the difference comes from velodrome cycling:

Riders race as a team and take turns as leader with the old leader peeling away and going to the back of the pack

As riders make their way from the rear of the pack to the front they are moving faster than the group that they are in

Group Velocity

The phase velocity of a wave is

$$v = \frac{\omega}{k}$$

and comes from the change in the position of the wavefronts as a function of time

The waveform moves at a rate that depends on the relative position of the component wavefronts as a function of time. This is the group velocity and is

$$v_g = \frac{d\omega}{dk}$$

which can be found if you have

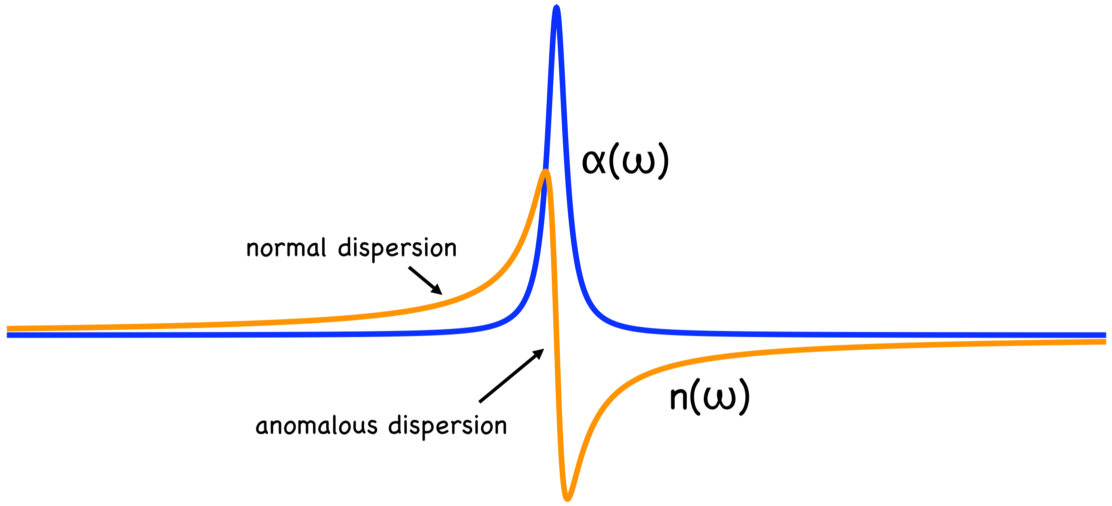
$$\omega = vk = \frac{c}{n(k)} k$$ giving $$v_g = v\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

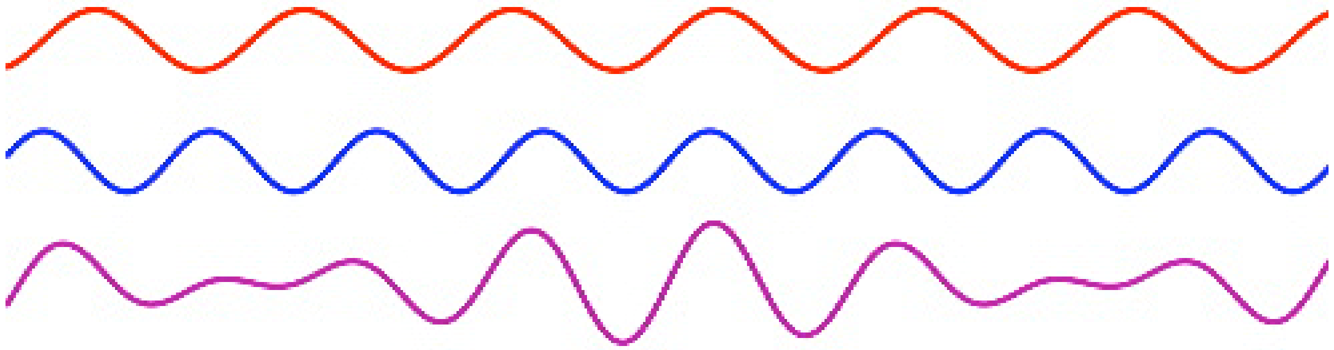
Slow Light

How slow can light be made to go?

In a Bose-Einstein Condensate light tuned to the atomic resonance tremendous dispersion and has been slowed to a speed of...

See Hau, et al. "Light speed reduction to 17 metres per second in an ultracold atomic gas", Nature 397, 594 - 598 (18 February 1999)





Example

Given the dispersion equation

$$n^2(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right)$$

where f_j is the fraction of electrons that have a resonant frequency of ω_{0j}, find the phase velocity and group velocity of high frequency electromagnetic waves ($\omega \gg \omega_{0j}$)

Example

$$n^2(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right)$$

The phase velocity is v=c/n so

$$v = \frac{c}{\sqrt{1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right)}} \approx c \left(1 + \frac{Ne^2}{2\epsilon_0 m_e \omega^2} \right)$$

The group velocity can be found from

$$v_g = \frac{d\omega}{dk}$$

Example

$$n^2(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m_e} \sum_j \left(\frac{f_j}{\omega_{0j}^2 - \omega^2} \right) \qquad v_g = \frac{d\omega}{dk}$$

using $\sum_j f_j = 1$ and $k = \frac{n\omega}{c}$

$$k = \frac{n\omega}{c} \approx \frac{\omega}{c}\left(1 - \frac{Ne^2}{2\epsilon_0 m_e \omega^2}\right)$$

$$\frac{dk}{d\omega} = \frac{n\omega}{c} \approx \left(\frac{1}{c} + \frac{Ne^2}{2\epsilon_0 m_e \omega^2}\right)$$

$$c = \frac{d\omega}{dk} = \frac{c}{1 + Ne^2/2m_e\epsilon_0\omega^2}$$

Modulated Light

The difference between group velocity and phase velocity is most relevant for light that is **modulated** at high frequencies or for pulses of light.

Light that is modulated is by definition non-sinusoidal, however it can be thought of as the sum of many sinusoidal components consider, consider the Fourier transform…

Fourier Transforms

When adding up the fields from an infinite number of monochromatic waves we can describe the field as an amplitude as a function of time (or space), or we can describe it by the amplitude and frequency (or wave-vector) of the waves that were added to produce it.

$$E(t) = \int_{-\infty}^{\infty} E(\omega) e^{i\omega t} d\omega$$

The function E(ω) is called the **Fourier Transform** of E(t)

Fourier Transforms

E(ω)can be found from E(t) as follows:

$$E(t) = \int_{-\infty}^{\infty} \widetilde{E}(\omega) e^{i\omega t} d\omega$$

multiply both sides by $e^{-i\omega t}$ and integrate over all time

$$\int_{-\infty}^{\infty} e^{-i\omega' t} E(t) dt = \int_{-\infty}^{\infty} e^{-i\omega' t} \int_{-\infty}^{\infty} \widetilde{E}(\omega) e^{i\omega t} d\omega dt$$

simplify the right hand side

$$\int_{-\infty}^{\infty} e^{-i\omega' t} E(t) dt = \int\int_{-\infty}^{\infty} \widetilde{E}(\omega) e^{i(\omega-\omega')t} d\omega dt$$

The oscillating function $e^{i(\omega-\omega')t}$ integrates to zero unless ω=ω′ so the integral is a delta function δ(ω-ω′)

$$\int_{-\infty}^{\infty} e^{-i\omega' t} E(t) dt = \delta(\omega - \omega') \int_{-\infty}^{\infty} \widetilde{E}(\omega) e^{i(\omega-\omega')t} d\omega$$

$$\int_{-\infty}^{\infty} e^{-i\omega' t} E(t) dt = \widetilde{E}(\omega') \quad \Rightarrow \quad \widetilde{E}(\omega) = \int_{-\infty}^{\infty} E(t) e^{-i\omega t} dt$$

Spectral Density

In mathematics E(ω) is called the **Fourier transform** of the function E(t)

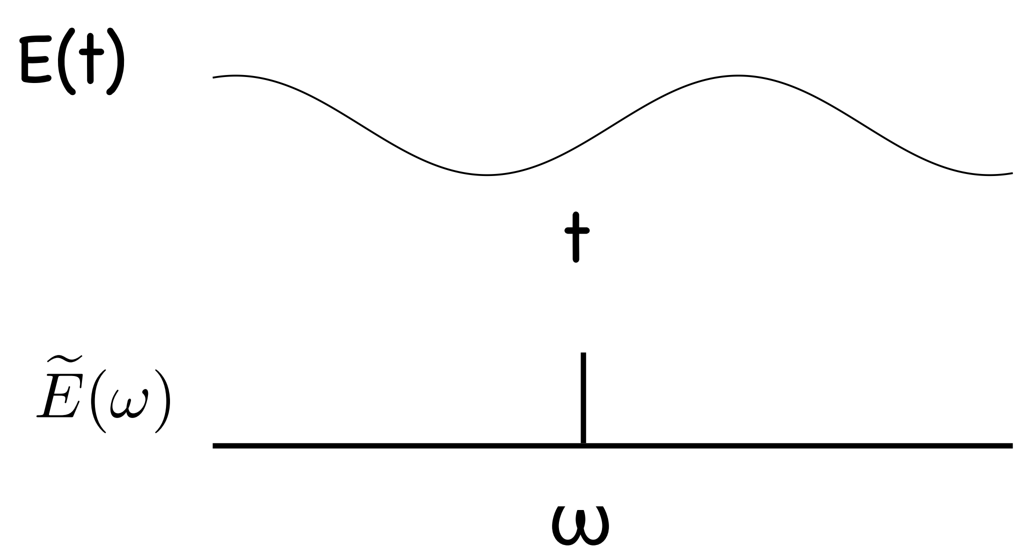
$$\widetilde{E}(\omega) = \int_{-\infty}^{\infty} E(t) e^{-i\omega t} dt$$

A more physical interpretation comes from calling it the **spectral density** of the electric field

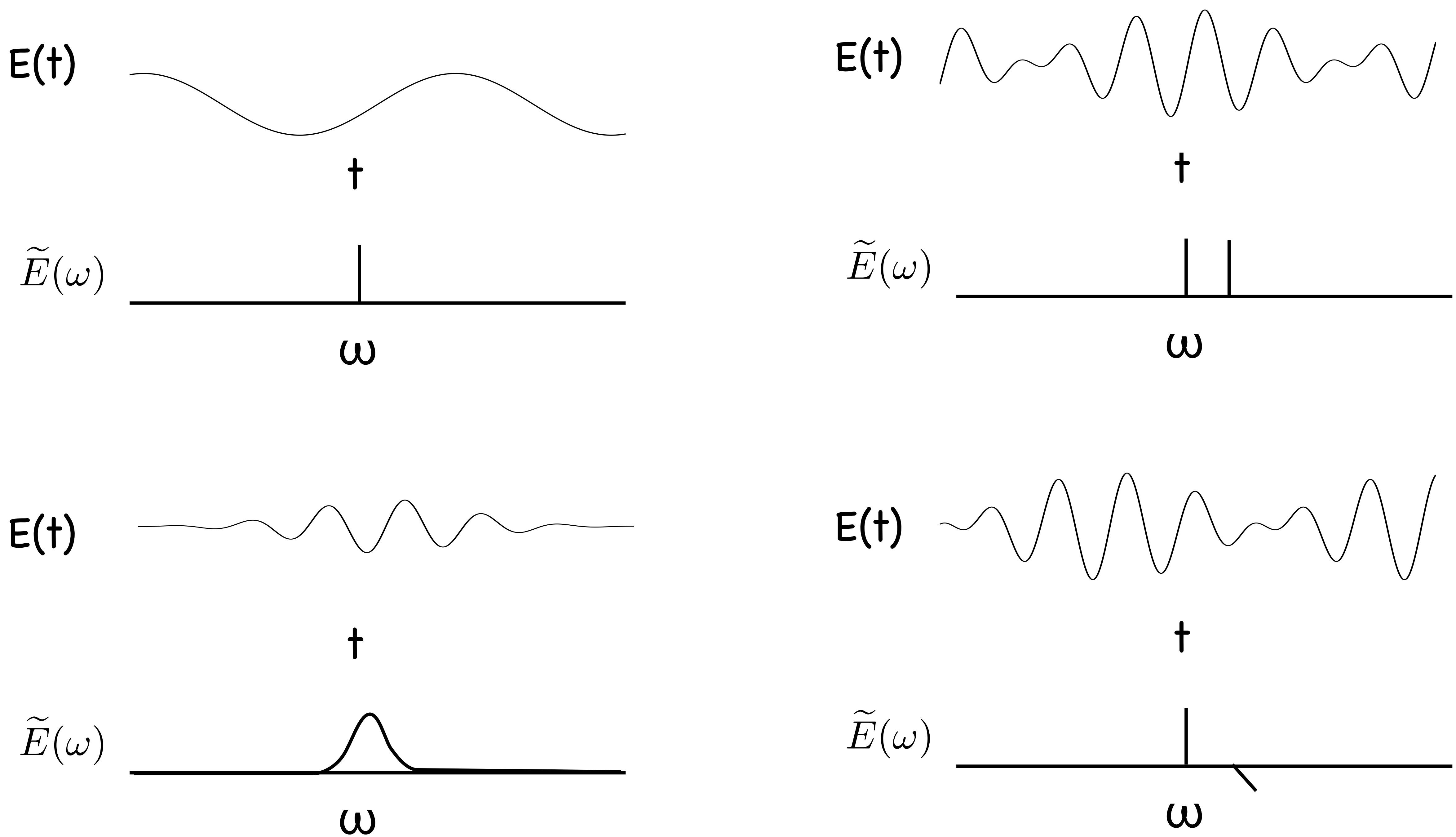
Consider the units

E(t) → V/m

E(ω) → (V/m) s → (V/m)/Hz



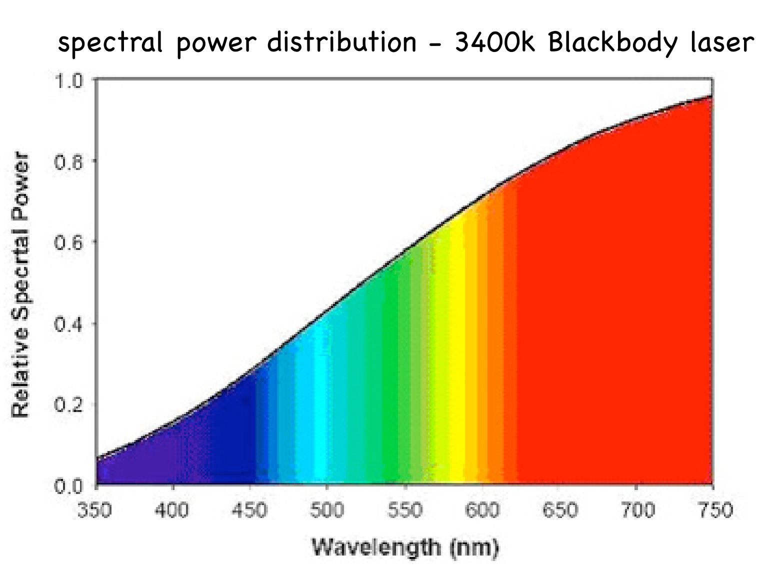
Spectral Density

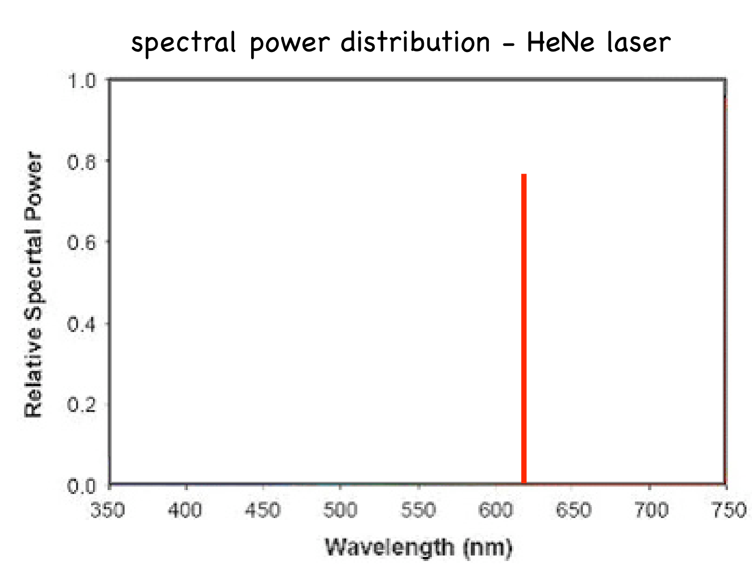


Power Spectrum

A plot of the spectral density shows the amplitude of the electric field as a function of frequency. It is often useful to consider the irradiance of a wave as a function of frequency, this is given by the **power spectral density**, aka the power spectrum of a light source.

$$\widetilde{P}(\omega) = \left|\widetilde{E}(\omega)\right|^2$$



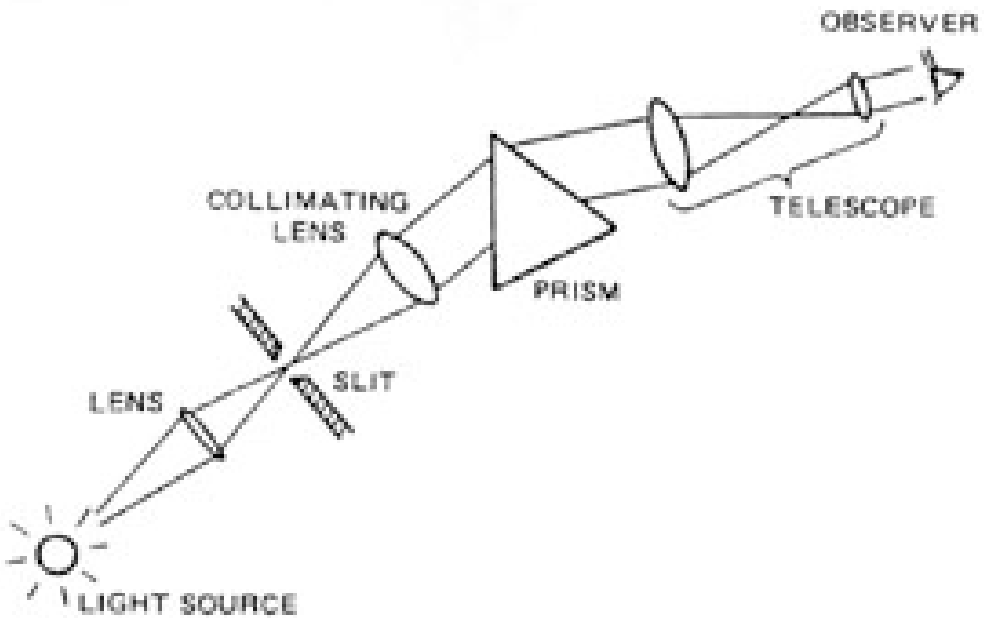


Prism Spectrometer

Principle: incident light is refracted by a dispersive prism. The angle of refraction is a function of the wavelength and can me measured.

Basic Components:

- Input slit to constrain the spatial width of the image
- Lenses to image the slit at the output and fully illuminate the prism
- Dispersive prism



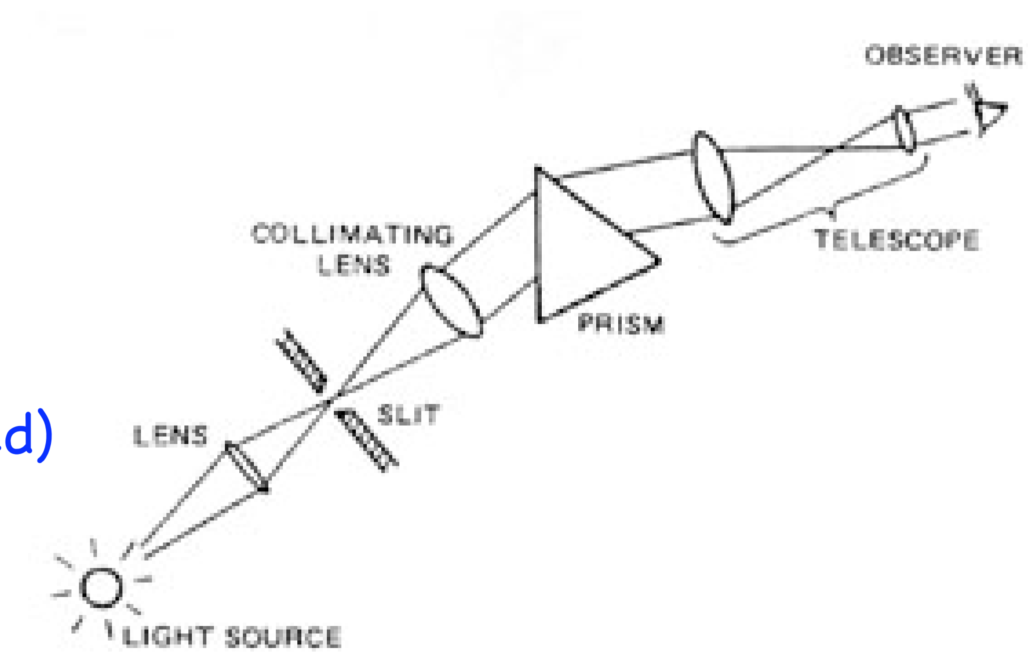
Instrumental Properties:

- Speed
- Spectral Transmission
- Resolving Power

Spectral Resolving Power

Spectral Resolving Power - the ability to resolve spectral features (it is the inverse of the spectral resolution)

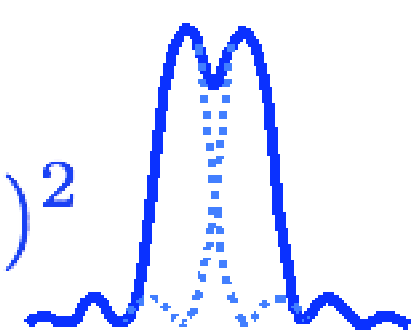
$$R = |\lambda_0/\Delta\lambda|$$



Using the "Rayleigh Criterion" two (diffraction limited) lines are just resolvable if the peak of one line coincides with the first minimum of another line.

The diffraction pattern of a uniformly illuminated aperture of width a is

$$I(\theta) = I_0 \left[\frac{\sin(a\pi\sin(\theta)/\lambda)}{a\pi\sin(\theta)/\lambda}\right]^2 \approx I_0 \mathrm{sinc}(a\pi\theta/\lambda)^2$$



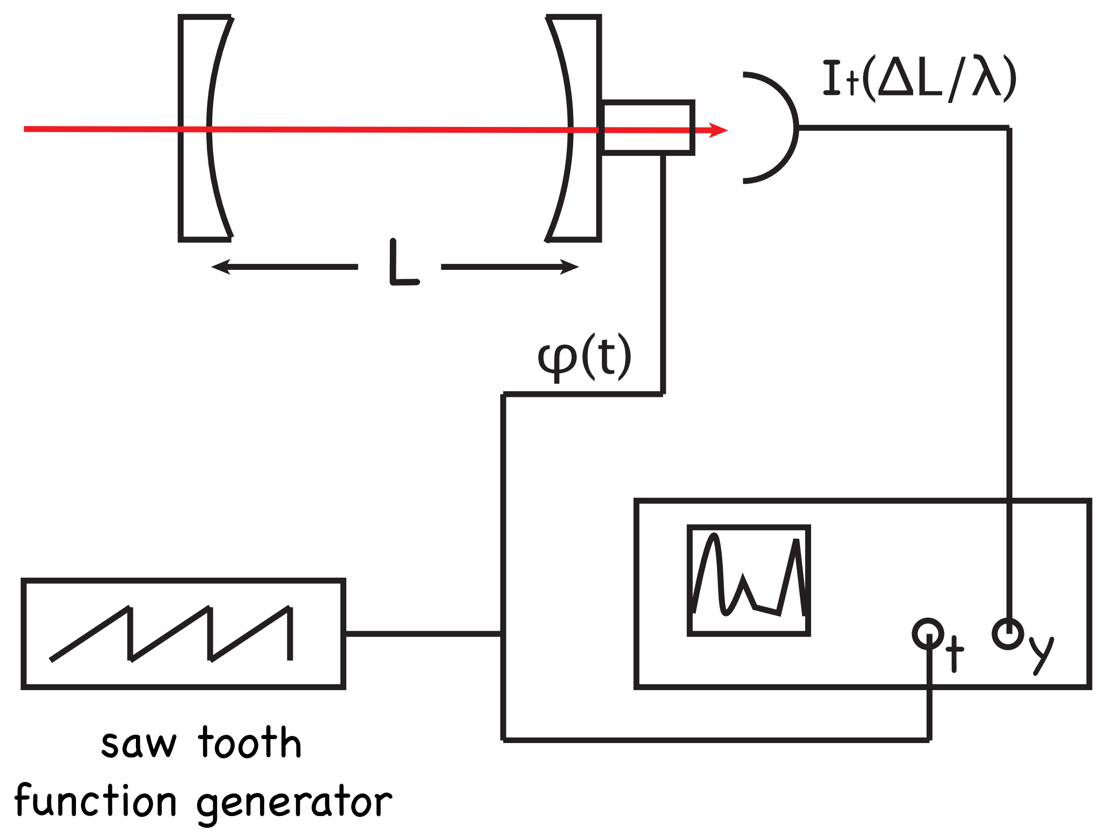
which after being imaged by a lens of focal length f_2 has a width from the central max to the first minimum of

$$\Delta x_2 = f_2\lambda/a$$

Scanning Confocal Fabry-Perot

- A Fabry-Perot is used as a wavelength selective filter
- Mirrors of the cavity are "confocal" $R_1=R_1=L$ so that transmitted intensity I(t) is only a function of $\Delta L(t)/\lambda$ (and does not depend on transverse modes in cavity produced by poor alignment)
- One mirror is scanned back and forth to dither the length by up to a full free spectral range
- Transmission of cavity is plotted versus cavity length (on an oscilloscope) to see the spectrum of the input light
- Most useful for CW sources when all of the power in the spectrum is within one free spectral range of the center wavelength

$\Delta L = 2L + \delta L \sin(\varphi(t))$



Calibrated by relating the time to scan through one free spectral range, to the known free spectral range of the instrument based on its length

Summary

- The wavefronts of a wave propagate at the phase velocity $v=\omega/k$
- The waveform of a wave propagates at the group velocity $v_g=d\omega/dk$
- In dispersive media $v \neq v_g$
- Waveforms can be deconstructed into sinusoidal components through Fourier analysis to obtain the spectral density and power spectrum