

Hamiltonian Systems and Chaos Overview

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Abstract: In this paper we will give an overview of Hamiltonian systems with specific examples, including the classical pendulum example. Upon addressing the properties of Hamiltonian systems we will then explore how chaos arises. More specifically, we will consider two examples: solar system orbits and a zero-sum two-player game.

1. Introduction

Hamiltonian Mechanics is a derivation of classical mechanics offering a deeper insight into the connection between classic and Lagrangian mechanics. There are many applications, to name a few: simple harmonic oscillators, planet orbits, and the weather. Perturbations in Hamiltonian systems give rise to chaos, which can be explained through KAM theory. Although the study of Hamiltonian Mechanics has been around since the 1800's, chaos is only now beginning to be understood.

2. History

Hamiltonian Mechanics was introduced in 1833 by the Irish mathematician William Rowan Hamilton. As a man inspired with knowledge at a very young age Hamilton was able to accomplish many tasks in only the 60 years he lived. He attended Trinity College in Dublin, which is where he would spend his entire academic career. While there, he studied optics, classical mechanics, dynamic methods, quaternions, conjugate algebraic couple functions, and many other topics. Hamilton remained active in the mathematical society until his death in 1865 from a severe case of gout, due to excessive drinking [18].

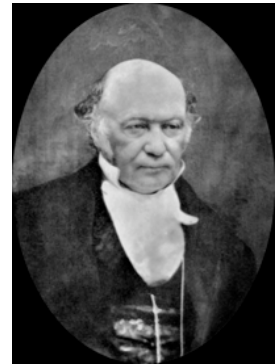


Figure 1: William Hamilton [19]

3. Overview of Hamiltonian Systems

Hamiltonian systems satisfy a number of properties, but before addressing those, an overview of what a Hamiltonian system is needs to be addressed. A Hamiltonian system with n degrees of freedom on an open subset E of \mathfrak{R}^{2n} must satisfy the following:

Let $H \in C^2(E)$ where $H = H(x, y)$ with $x, y \in \mathfrak{R}^n$. Then the system

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

satisfies

$$\frac{\partial H}{\partial x} = \left(\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n} \right)^T \quad \text{and} \quad \frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \dots, \frac{\partial H}{\partial y_n} \right)^T$$

where x represents a generalized coordinate vector and y represents a momentum vector [11]. But why classify these systems? Why not just consider Lagrangian systems? The reason to step away from Lagrangian systems is that Hamiltonian systems link classical and quantum mechanics. More specifically, Hamiltonian systems are conservative, which can easily be extended to many physical systems. Furthermore, the biggest difference between Hamiltonian systems and Lagrangian systems is the fact that Hamiltonian equations have a larger set of symmetries. By applying these equations to physical systems, it is easier to exploit and manipulate the symmetries. Thus, Hamiltonian systems provide more tools to implement when trying to understand a physical system.

One of the most basic examples of a Hamiltonian system is the ideal pendulum. In this case, there is a mass m attached to a rigid, massless rod of length L in a “vacuum.” The momentum of this system is in terms of the angular momentum, namely, $p = mL^2\dot{\theta}$, and the potential energy is given by $-mgL\cos(\theta)$ where g is the gravitational force inside the vacuum. These two equations lead to the Hamiltonian system:

$$H(\theta, p) = \frac{p^2}{2mL^2} - mgL\cos(\theta)$$

where $\dot{\theta} = \frac{p}{mL^2}$ and $\dot{p} = -mgL\sin(\theta)$ [7].

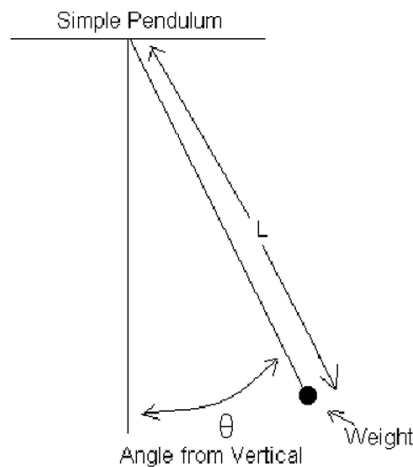


Figure 2: Ideal Pendulum [10]

In fact, the pendulum is a more specific type of a Hamiltonian system. The class of Hamiltonian systems in which the pendulum fits into is called the Newtonian system. Newtonian systems have one degree of freedom and can be written in the form

$$\ddot{x} = f(x)$$

where $f \in C^1(a, b)$. In turn, this differential equation can be re-written as a system in \mathfrak{R}^2 where

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= f(x)\end{aligned}$$

Then the total energy is given by $H(x, y) = \frac{y^2}{2} + U(x)$ where $\frac{y^2}{2}$ is the kinetic energy and

$U(x) = -\int_{x_0}^x f(s)ds$ is the potential energy [11]. And clearly, when rewriting the Newtonian

system in this manner, it can be seen that it is, in fact, a Hamiltonian system. An interesting fact about Newtonian systems is that all the critical points lie on the x -axis, which is further supported by the pendulum example [11].

Hamiltonian systems are also related to gradient systems. A gradient system is defined as follows: Let E be an open subset of \mathfrak{R}^n and let $V \in C^2(E)$. A system of the form

$\dot{x} = -grad(V(x))$ where $grad(V) = \left(\frac{dV}{dx_1}, \dots, \frac{dV}{dx_n}\right)^T$ is called a gradient system on E [11]. But

how is this related to Hamiltonian systems? The system of the form $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ (a planar system) is a Hamiltonian system if and only if the system, $\dot{x} = Q(x, y), \dot{y} = -P(x, y)$, orthogonal to the planar system is a gradient system [11]. This connection is another tool to be used when investigating Hamiltonian systems.

3.1 Properties

Hamiltonian systems consist of numerous properties. The most well-known property is that Hamiltonian systems conserve at least one particular quantity, namely energy. More specifically, the total energy H of the Hamiltonian system remains constant along its trajectories. This can be proven by taking the derivative of H with respect to time t . In doing so, we get that it equals zero, which proves that H is constant along any solution curve. Therefore, the trajectories of the system are on the surfaces of H , which is a constant. In particular, the ideal pendulum system conserves energy. It is important to note that there are systems that conserve energy but are non-Hamiltonian. Thus, conserving energy is not the only key property of Hamiltonian systems. Furthermore, Hamiltonian systems are frictionless and the equations are time-reversible. Also, the motion of the system occurs in a hypersurface and the flow is incompressible. Incompressible flow is characterized by the unchanged density of a system. Hamiltonian systems also possess other properties, but those will not be addressed in this paper [1].

Now properties of Hamiltonian systems as they relate to an entry level ordinary differential equation course will be addressed. In particular, equilibrium points will be investigated. The first thing to note is that the critical points of H correspond to the equilibrium points of the system. Furthermore, the equilibrium points are non-degenerate if the determinant of the second derivative of H is nonzero when evaluated at the equilibrium points. Stability can

also be determined using the second derivative of H . When evaluating the second derivative at an equilibrium point, if all the eigenvalues have a positive real part, then that point is stable. If we consider Hamiltonian systems of degree one, then non-degenerate equilibrium points can be classified more readily. If an equilibrium point is non-degenerate, then we have that that point is a saddle of the Hamiltonian system if and only if it is a saddle of H . Furthermore, it is a center if and only if it is a local maximum or a local minimum of H [8]. In the pendulum case, we can see from the following phase space trajectories that we get centers and saddles.

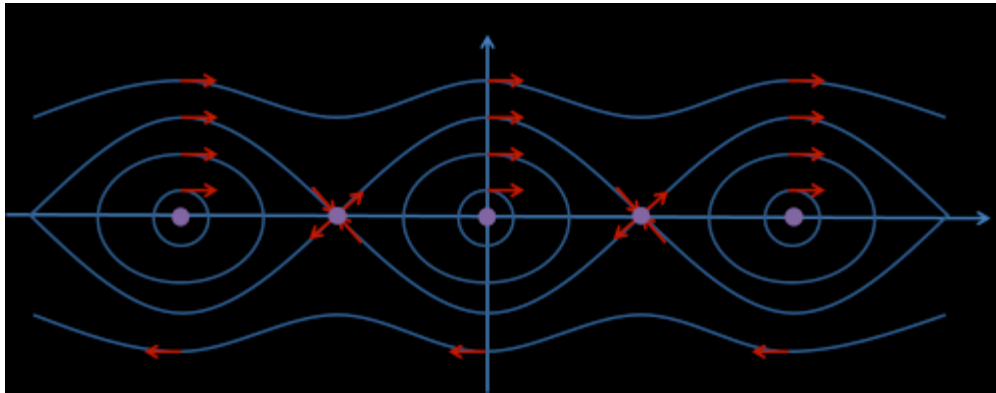


Figure 3: Ideal Pendulum Phase Portrait [9]

3.2 Examples

In order to see how the above properties apply to Hamiltonian systems, a more complicated example will be explored. One such example is that of charged particles in a magnetic field. More specifically, the Kepler problem will be investigated. In the Kepler system there are two particles in R^3 . This system has one particle fixed at the origin, while the other moves based on the gravitational field created by the fixed particle. What is being investigated in the Kepler problem is the movement of the non-fixed particle. The Hamiltonian system is as follows:

$$H(q, p) = \frac{1}{2} \langle p, p \rangle - \frac{u}{|q|}$$

where \langle, \rangle is the Euclidean inner product, $|q|$ is the length of the vector q , and u is the attractive force ($u > 0$). Based on this system, we obtain the following two equations:

$$\dot{q} \doteq p, \dot{p} = -u \frac{q}{|q|^3} [16].$$

This system conserves angular momentum and occurs in many cases. It arises in celestial mechanics; more specifically, it deals with satellites moving about planets or a planet about the sun. The Kepler problem also arises in electrostatics, especially with hydrogen atoms, positronium, and muonium. While the Kepler system is a bit more complicated system than that of the pendulum, it is the other foundation to classical mechanics (the other being the pendulum). In fact, in these two cases, they both have closed orbits for every possible set of initial conditions [4]. The following image is a prototype of the Kepler problem.

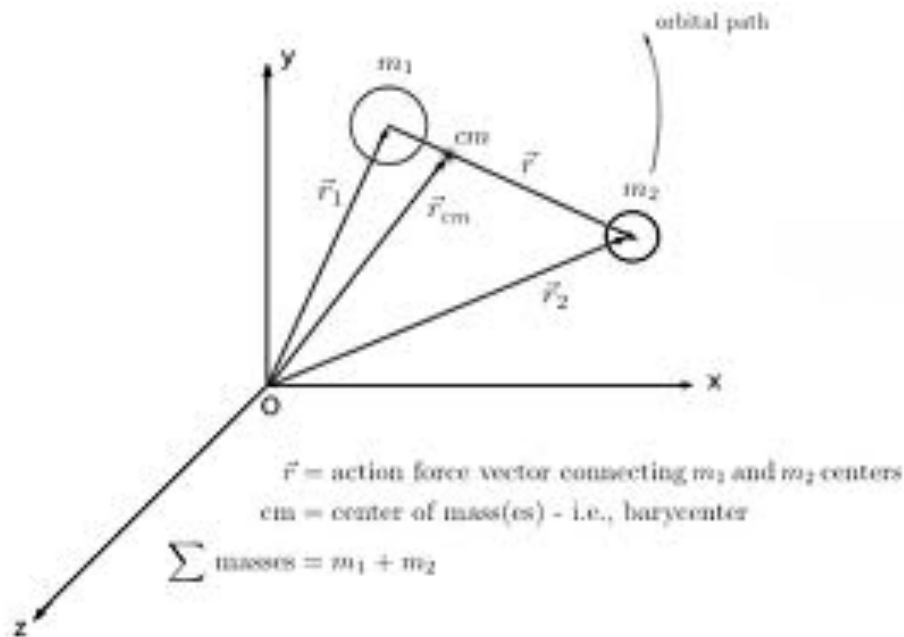


Figure 4: An example of a Kepler problem [12]

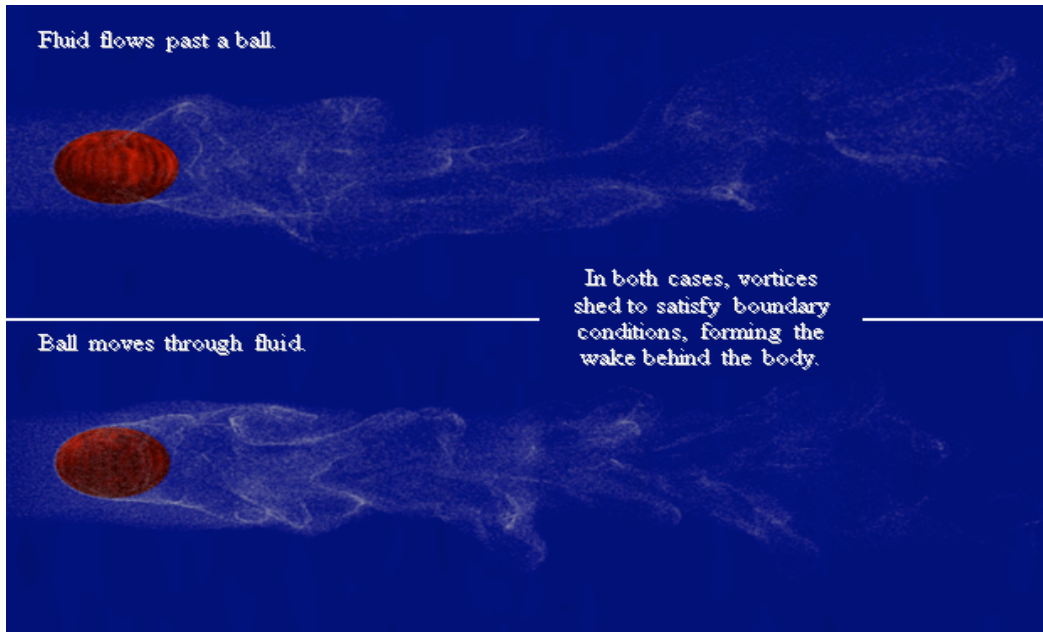
Another example that is timeless and has numerous applications is the field line behavior. Consider a vector field $v(\vec{x})$ where $x \in \mathbf{R}^3$. Then the field line behavior in space can be written as follows:

$$\frac{\partial x}{v_x} = \frac{\partial y}{v_y} = \frac{\partial z}{v_z}$$

But this formula can be re-written (and simplified) in parametric form to obtain the following:

$$\dot{x} = \frac{\partial \vec{x}}{\partial t} = v(\vec{x})$$

where t represents the length of the field line. This parametric form is then used for various applications. One application is that for advected particles in fluids, also known as tracers. The following figure is a neat, specific example of where tracers are actually used in the real world. This example is used to help design video games from Intel by simulating a ball flowing through a fluid and a fluid flowing past a ball. Another application for field line behavior is in magnetic field lines with respect to fusion devices [20].



3.3 Integrability versus Non-Integrability

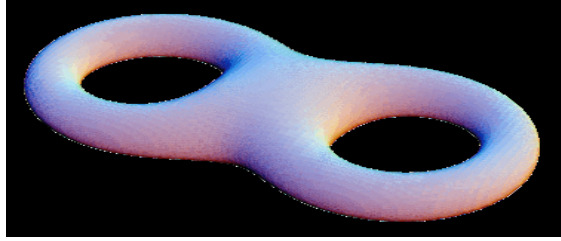
Integrability of a Hamiltonian system plays an enormous role in the perturbation of that system. In fact, as it will be discussed in the following section, chaos can occur when a system becomes perturbed. Thus, integrability of a system is a necessary factor when studying Hamiltonian systems. Furthermore, the definition of integrability is not unique, so this paper will present a variety of definitions. The reader may choose to use which ever definition suites her best. Lastly, before presenting these definitions, it is important to realize that these definitions have a specific name. Specifically, we say a Hamilton system is Liouville integrable if any of the following definitions is satisfied. One definition of integrability is as follows: a system is integrable if its solutions can be expressed through quadratures [20]. But the word quadrature has at least three distinct meanings in mathematics, so this definition is a bit ambiguous. Therefore, the following is another definition of integrability: a system is integrable if the number M of independent commuting integrals satisfy the condition $M \leq N$ (N is the degrees of freedom), but the family of trajectories of the Hamiltonian system cannot be displayed on an invariant N -torus. Or rather, a system is integrable if it can be written in the following form:

$$\dot{I}_j = -\frac{\partial H}{\partial v_j}, \quad \dot{v}_j = \frac{\partial H}{\partial I_j} \quad \forall j$$

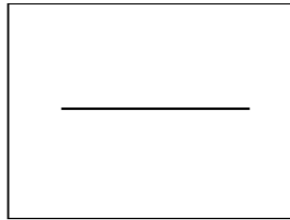
$$H = H(I_1, \dots, I_n) = H(I)$$

where I is a function of actions and v are phases canonically conjugated to I [20].

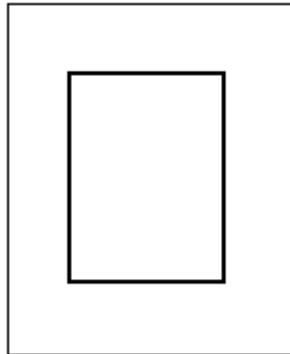
But, these definitions of integrability could be negated to give us the definition of a Hamiltonian system being non-integrable. We will now consider several non-integrable systems with two degrees of freedom.



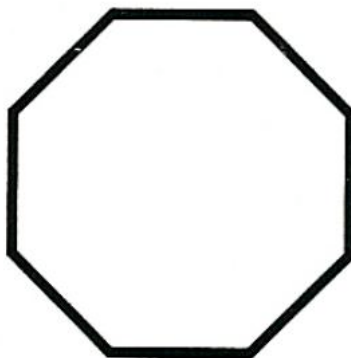
Double Torus: Non-Integrable



2-Dimension Billiard: Non-Integrable



2-Dimension Billiard: Non-Integrable



2-Dimension Billiard: Non-Integrable

These four systems are non-integrable due to the trajectories that wind about each surface. These trajectories are sensitive to small perturbations. Thus, the systems cannot be integrated.

4. Hamiltonian Chaos

4.1 Definition and Properties

In a non-chaotic Hamiltonian system, the motion is oscillatory. Thus, geometrically we get that the orbits of the system move on tori. But, for chaos to be introduced into the system, the tori need to be destroyed. And by destroying the invariant tori, the system in turn creates a cantori. Chirikov first discovered that for local chaos to occur in a Hamiltonian system, stable and

unstable manifolds had to intersect. And this chaos occurs when $S^2 > 1$, where $S = \frac{\Delta\omega_r}{\Omega_d}$, $\Delta\omega_r$

is the frequency and Ω_d is the distance frequency between two unperturbed resonances.

Furthermore, Chirikov found that global chaos occurs when $2.5S^2 > 1$ [15]. Poincaré labeled this occurrence “homoclinic trellis.” This idea can be seen through the work of John Greene in 1968. His work dealt with further perturbing a KAM torus. He found that if one continually perturbs a KAM torus, it reaches a point where the phase space has fractal structure. Upon reaching this point, if the torus is perturbed anymore, the torus will be destroyed, creating a cantorus [7].

But the question should be, how do these tori become perturbed? It should be noted that the tori are not perturbed in the integrable Hamiltonian system H that was defined earlier. Rather, the system becomes perturbed when a nonintegrable Hamiltonian perturbation is added. In doing so, the following equation is obtained:

$$H = H_0(y) + \varepsilon H_1(x, y).$$

In adding the nonintegrable function, tori begin to deform, and those that survive are “sufficiently irrational.” In fact, according to KAM theory, tori survive for C^r perturbations if

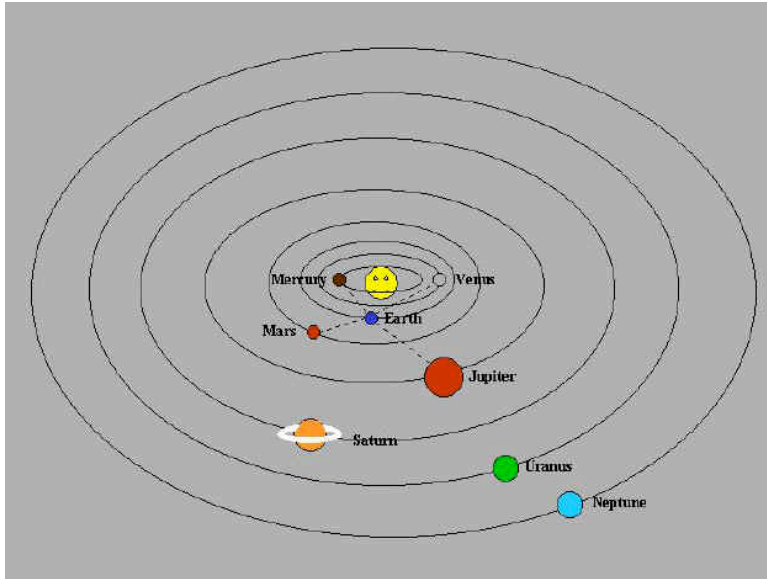
$$\left| \frac{dH_0}{dy_0} * k \right| > \lambda |k|^{-\tau} \quad \text{for } N-1 < \tau < \frac{1}{2}r-1 \text{ and } \lambda \text{ is of order } \sqrt{\varepsilon} \text{ for small } \varepsilon, \text{ where } \lambda \text{ is based}$$

on frequency. But for chaos to occur, even these “sufficiently irrational” tori become perturbed [1].

4.2 Chaos Examples

One common and interesting example of Hamiltonian chaos is the problem of stability of the solar system. Is the solar system stable over long periods of time? This can be modeled as a Hamiltonian system by considering the force between each pair of massive bodies, or planets. The problem is simple to set up but very difficult to solve, so difficult, in fact, that it hasn’t been answered and continues to stump mathematicians. So, mathematicians considered a more simple

system using only the sun and two planets. This has become famously known as the “three body problem”. In the mid-20th Century three Russian mathematicians found the solution, known today as the KAM theorem. The theorem is based on invariant tori and answers the question, what happens to the invariant tori as the nonlinearity of the system increases? How does this answer our solar system question? Well, the planets motion represent that of a torus, so over a long period of time if we know what happens to a torus then we know what happens to the solar



system. The KAM theorem concludes that tori go unstable in order of their degree of irrationality. So, for the “three body problem” consider the Earth, Jupiter, and the Sun. The full phase space would be 18 dimensional and the motion is represented as a torus. The winding numbers is the same as the ratio of orbit periods which happens to be about 11.862972. This seems fairly irrational, so we could expect Earth’s orbit to be fairly stable. This is of course a simple model and becomes more difficult when considering larger and more complicated systems [6]. But this theory tells us how Hamiltonian

Figure 5: Solar system and chaos [16] systems become chaotic, which is what we’re interested in.

Other interesting examples arise in economics and simple two-person games. Consider the zero-sum game of rock, paper, scissors. The zero-sum game can be represented as a Hamiltonian system since the learning dynamics have a conserved quantity. We would expect the game to converge to the Nash Equilibrium, since this would imply that at every step of the game, a player tries to better his/her position in the game. However, this is not always true and when a player does not take the Nash Equilibrium mixed strategy, which is to choose the move randomly with all choices having equal probability, the game results in chaos. This can be modeled by:

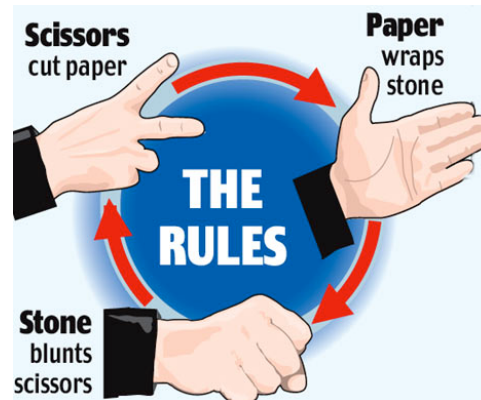


Figure 6: Rules of Rock, Paper, Scissors [13]

$$\dot{x}_i = x_i[(Ay)_i - xAy](i = 1, \dots, m)$$

$$\dot{y}_j = y_j[(Bx)_j - yBx](j = 1, \dots, n)$$

Player one chooses from m different strategies with frequency $x = (x_1, \dots, x_m)$ and player two chooses from n different strategies with frequency $y = (y_1, \dots, y_n)$, where A is the payoff matrix for player one and B is the payoff matrix for player two.

$$A = \begin{bmatrix} \varepsilon_x & -1 & 1 \\ 1 & \varepsilon_x & -1 \\ -1 & 1 & \varepsilon_x \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon_y & -1 & 1 \\ 1 & \varepsilon_y & -1 \\ -1 & 1 & \varepsilon_y \end{bmatrix}$$

Note that $-1 \leq \varepsilon_x \leq 1$ and $-1 \leq \varepsilon_y \leq 1$ are the payoffs for a tie. Since $A = -B^t$, the dynamics are Hamiltonian. Choosing $(x_1, x_2, x_3, y_1, y_2, y_3) = (0.5, 0.01k, 0.5 - 0.01k, 0.5, 0.25, 0.25)$ with $k = 1, 2, \dots, 25$ and varying ε from 0 to 0.5 results in chaos [14]. Nash strategy appears often in economics and has similar outcomes to the rock, paper, scissors game [14].

5. Conclusion

There is still much to be known about chaos, as seen in the example of the solar system. The neat thing about these systems is that they apply to many areas. We would be interested in seeing if Hamiltonian chaos arises in an n -person ($n > 2$) zero-sum game. And we also look forward to the discoveries that will be made in the study of chaos with regards to the solar system. This paper is just a brief overview of Hamiltonian chaos and the interested reader should seek further references.

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