Handbook of Mathematics, Physics and Astronomy Data

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1.1 Physical Constants

Symbol	Quantity	Value
c	Speed of light in free space	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
h	Planck constant	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
$\hbar$	$h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
G	Universal gravitation constant	$6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
e	Electron charge	$1.602 \times 10^{-19}\,\mathrm{C}$
m_e	Electron rest mass	$9.109 \times 10^{-31}\,\mathrm{kg}$
m_p	Proton rest mass	$1.673 \times 10^{-27}\,\mathrm{kg}$
m_n	Neutron rest mass	$1.675 \times 10^{-27}\,\mathrm{kg}$
u	Atomic mass unit	$(\frac{1}{12}$ mass of $^{12}\mathrm{C})$
		$= 1.661 \times 10^{-27}\,\mathrm{kg}$
N_A	Avogadro's constant	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
		$= 6.022 \times 10^{26}\,(\text{kg-mole})^{-1}$
k_B	Boltzmann constant	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
R	Gas constant $= Nk$	8.314×10^{3} J $\mathrm{K^{-1}}\,(\text{kg-mole})^{-1}$
		8.314 J $\mathrm{K^{-1}\,mol^{-1}}$
μ_B	Bohr magneton	$9.274 \times 10^{-24}\,\mathrm{J\,T^{-1}}$ (or $\mathrm{A\,m^2}$)
μ_N	Nuclear magneton	$5.051 \times 10^{-27}\,\mathrm{J\,T^{-1}}$
R_∞	Rydberg constant	$10973732\,\mathrm{m^{-1}}$
a_0	Bohr radius	$5.292 \times 10^{-11}\,\mathrm{m}$
σ	Stefan-Boltzmann constant	$5.670 \times 10^{-8}\,\mathrm{J\,K^{-4}\,m^{-2}\,s^{-1}}$
α	Fine structure constant	$1/137.04$
σ_T	Thomson cross-section	$6.652 \times 10^{-29}\,\mathrm{m^2}$
μ_0	Permeability of free space	$4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$
ϵ_0	Permittivity of free space	$1/(\mu_0 c^2)$
		$= 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
eV	Electron volt	1.602×10^{-19} J
g	Standard acceleration of gravity	9.807 m $\mathrm{s^{-2}}$
atm	Standard atmosphere	101325 N $\mathrm{m^{-2}} = 101325$ Pa

1.2 Astrophysical Quantities

Symbol	Quantity	Value
$\mathrm{M}_\odot$	Mass of Sun	1.989×10^{30} kg
$\mathrm{R}_\odot$	Radius of Sun	6.955×10^{8} m
$\mathrm{L}_\odot$	Bolometric luminosity of Sun	3.846×10^{26} W
$M_{\mathrm{bol}}^{\odot}$	Absolute bolometric magnitude of Sun	+4.75
$M_{\mathrm{vis}}^{\odot}$	Absolute visual magnitude of Sun	+4.83
$\mathrm{T}_\odot$	Effective temperature of Sun	5778 K
	Spectral type of Sun	G2 V
M_J	Mass of Jupiter	1.899×10^{27} kg
R_J	Equatorial radius of Jupiter	71492 km
$\mathrm{M}_\oplus$	Mass of Earth	5.974×10^{24} kg
$\mathrm{R}_\oplus$	Equatorial radius of Earth	6378 km
$\mathrm{M}_{☾}$	Mass of Moon	7.348×10^{22} kg
$\mathrm{R}_{☾}$	Equatorial radius of Moon	1738 km
	Sidereal year	3.156×10^{7} s
AU	Astronomical Unit	1.496×10^{11} m
ly	Light year	9.461×10^{15} m
pc	Parsec	3.086×10^{16} m
Jy	Jansky	10^{-26} W m^{-2} Hz^{-1}
H_0	Hubble constant	72 ± 5 km s^{-1} Mpc^{-1}

1.3 Periodic Table

Periodic Table of the Elements

1999 IUPAC Values (limited to 0.001 atomic mass units)
Complete values with errors available at
www.chem.qmw.ac.uk/iupac/AtWt/

Hydrogen 1.0079 H 1																	Helium 4.003 He 2
Lithium 6.941 Li 3	Beryllium 9.012 Be 4											Boron 10.811 B 5	Carbon 12.011 C 6	Nitrogen 14.007 N 7	Oxygen 15.999 O 8	Fluorine 18.998 F 9	Neon 20.180 Ne 10
Sodium 22.990 Na 11	Magnesium 24.305 Mg 12											Aluminum 26.982 Al 13	Silicon 28.086 Si 14	Phosphorus 30.974 P 15	Sulfur 32.065 S 16	Chlorine 35.453 Cl 17	Argon 39.948 Ar 18
Potassium 39.098 K 19	Calcium 40.078 Ca 20	Scandium 44.956 Sc 21	Titanium 47.867 Ti 22	Vanadium 50.942 V 23	Chromium 51.996 Cr 24	Manganese 54.938 Mn 25	Iron 55.845 Fe 26	Cobalt 58.933 Co 27	Nickel 58.693 Ni 28	Copper 63.546 Cu 29	Zinc 65.39 Zn 30	Gallium 69.723 Ga 31	Germanium 72.64 Ge 32	Arsenic 74.922 As 33	Selenium 78.96 Se 34	Bromine 79.904 Br 35	Krypton 83.80 Kr 36
Rubidium 85.468 Rb 37	Strontium 87.62 Sr 38	Yttrium 88.906 Y 39	Zirconium 91.224 Zr 40	Niobium 92.906 Nb 41	Molybdenum 95.94 Mo 42	Technetium {98} Tc 43	Ruthenium 101.07 Ru 44	Rhodium 102.906 Rh 45	Palladium 106.42 Pd 46	Silver 107.868 Ag 47	Cadmium 112.411 Cd 48	Indium 114.818 In 49	Tin 118.710 Sn 50	Antimony 121.760 Sb 51	Tellurium 127.60 Te 52	Iodine 126.904 I 53	Xenon 131.293 Xe 54
Cesium 132.905 Cs 55	Barium 137.327 Ba 56	Lutetium 174.967 Lu 71	Hafnium 178.49 Hf 72	Tantalum 180.948 Ta 73	Tungsten 183.84 W 74	Rhenium 186.207 Re 75	Osmium 190.23 Os 76	Iridium 192.217 Ir 77	Platinum 195.078 Pt 78	Gold 196.967 Au 79	Mercury 200.59 Hg 80	Thallium 204.383 Tl 81	Lead 207.2 Pb 82	Bismuth 208.980 Bi 83	Polonium {209} Po 84	Astatine {210} At 85	Radon {222} Rn 86
Francium {223} Fr 87	Radium {226} Ra 88	Lawrencium {262} Lr 103	Rutherfordium {261} Rf 104	Dubnium {262} Db 105	Seaborgium {266} Sg 106	Bohrium {264} Bh 107	Hassium {277} Hs 108	Meitnerium {268} Mt 109	{281} 110 110	{272} 111 111	{285} 112 112	113	{289} 114 114	115	116	117	118

Lanthanum 138.906 La 57	Cerium 140.116 Ce 58	Praseodymium 140.908 Pr 59	Neodymium 144.24 Nd 60	Promethium {145} Pm 61	Samarium 150.36 Sm 62	Europium 151.964 Eu 63	Gadolinium 157.25 Gd 64	Terbium 158.925 Tb 65	Dysprosium 162.50 Dy 66	Holmium 164.930 Ho 67	Erbium 167.259 Er 68	Thulium 168.934 Tm 69	Ytterbium 173.04 Yb 70
Actinium {227} Ac 89	Thorium 232.038 Th 90	Protactinium 231.036 Pa 91	Uranium 238.029 U 92	Neptunium {237} Np 93	Plutonium {244} Pu 94	Americium {243} Am 95	Curium {247} Cm 96	Berkelium {247} Bk 97	Californium {251} Cf 98	Einsteinium {252} Es 99	Fermium {257} Fm 100	Mendelevium {258} Md 101	Nobelium {259} No 102

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Solids
Liquids
Gases

1.4 Electron Configurations of the Elements

Z	Element	Electron configuration													
		1s	2s	2p	3s	3p	3d	4s	4p	4d	4f	5s	5p	5d	5f
1	H	1													
2	He	2													
3	Li	2	1												
4	Be	2	2												
5	B	2	2	1											
6	C	2	2	2											
7	N	2	2	3											
8	O	2	2	4											
9	F	2	2	5											
10	Ne	2	2	6											
11	Na	2	2	6	1										
12	Mg	2	2	6	2										
13	Al	2	2	6	2	1									
14	Si	2	2	6	2	2									
15	P	2	2	6	2	3									
16	S	2	2	6	2	4									
17	Cl	2	2	6	2	5									
18	Ar	2	2	6	2	6									
19	K	2	2	6	2	6	-	1							
20	Ca	2	2	6	2	6	-	2							
21	Sc	2	2	6	2	6	1	2							
22	Ti	2	2	6	2	6	2	2							
23	V	2	2	6	2	6	3	2							
24	Cr	2	2	6	2	6	5	1							
25	Mn	2	2	6	2	6	5	2							
26	Fe	2	2	6	2	6	6	2							
27	Co	2	2	6	2	6	7	2							
28	Ni	2	2	6	2	6	8	2							
29	Cu	2	2	6	2	6	10	1							
30	Zn	2	2	6	2	6	10	2							
31	Ga	2	2	6	2	6	10	2	1						
32	Ge	2	2	6	2	6	10	2	2						
33	As	2	2	6	2	6	10	2	3						
34	Se	2	2	6	2	6	10	2	4						
35	Br	2	2	6	2	6	10	2	5						
36	Kr	2	2	6	2	6	10	2	6						
37	Rb	2	2	6	2	6	10	2	6	-	-	1			
38	Sr	2	2	6	2	6	10	2	6	-	-	2			
39	Y	2	2	6	2	6	10	2	6	1	-	2			
40	Zr	2	2	6	2	6	10	2	6	2	-	2			
41	Nb	2	2	6	2	6	10	2	6	4	-	1			
42	Mo	2	2	6	2	6	10	2	6	5	-	1			
43	Tc	2	2	6	2	6	10	2	6	6	-	1			
44	Ru	2	2	6	2	6	10	2	6	7	-	1			
45	Rh	2	2	6	2	6	10	2	6	8	-	1			
46	Pd	2	2	6	2	6	10	2	6	10	-	-			
47	Ag	2	2	6	2	6	10	2	6	10	-	1			
48	Cd	2	2	6	2	6	10	2	6	10	-	2			
49	In	2	2	6	2	6	10	2	6	10	-	2	1		
50	Sn	2	2	6	2	6	10	2	6	10	-	2	2		
51	Sb	2	2	6	2	6	10	2	6	10	-	2	3		
52	Te	2	2	6	2	6	10	2	6	10	-	2	4		
53	I	2	2	6	2	6	10	2	6	10	-	2	5		
54	Xe	2	2	6	2	6	10	2	6	10	-	2	6		

1.5 Greek Alphabet and SI Prefixes

The Greek alphabet

A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ϵ, ε	epsilon	P	ρ, ϱ	rho
Z	ζ	zeta	Σ	σ, ς	sigma
H	η	eta	T	τ	tau
Θ	θ, ϑ	theta	Y	υ	upsilon
I	ι	iota	Φ	ϕ, φ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

SI Prefixes

Name	Prefix	Factor
yotta	Y	10^{24}
zetta	Z	10^{21}
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deca	da	10^{1}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}
zepto	z	10^{-21}
yocto	y	10^{-24}

Mathematics

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2.1 Mathematical Constants and Notation

Constants

$$\begin{aligned}
\pi &= 3.141592654\ldots \qquad \text{(N.B.} \ \ \pi^2 \simeq 10) \\
e &= 2.718281828\ldots \\
\ln 10 &= 2.302585093\ldots \\
\log_{10} e &= 0.434294481\ldots \\
\ln x &= 2.302585093 \log_{10} x \\
1 \text{ radian} &= 180/\pi \simeq 57.2958 \text{ degrees} \\
1 \text{ degree} &= \pi/180 \simeq 0.0174533 \text{ radians}
\end{aligned}$$

Notation

$$\begin{aligned}
\text{Factorial } n = n! &= n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 \\
& \qquad\qquad\qquad (\text{N.B. } 0! = 1) \\
\textsc{Stirling's approximation} \qquad n! &\simeq \left(\frac{n}{e}\right)^n (2\pi n)^{1/2} \quad (n \gg 1) \\
\ln n! &\simeq n \ln n - n \qquad (\text{Error } \lesssim 4\% \text{ for } n \geq 15)
\end{aligned}$$

$$\text{Double Factorial } n!! = \begin{cases} n \times (n-2) \times \cdots \times 5 \times 3 \times 1 & \text{for } n > 0 \text{ odd} \\ n \times (n-2) \times \cdots \times 6 \times 4 \times 2 & \text{for } n > 0 \text{ even} \\ 1 & \text{for } n = -1, 0 \end{cases}$$

$$\begin{aligned}
\exp(x) &= e^x \\
\ln x &= \log_e x \\
\arcsin x &= \sin^{-1} x \\
\arccos x &= \cos^{-1} x \\
\arctan x &= \tan^{-1} x \\
\sum_{i=1}^{n} A_i &= A_1 + A_2 + A_3 + \cdots + A_n = \text{ sum of } n \text{ terms} \\
\prod_{i=1}^{n} A_i &= A_1 \times A_2 \times A_3 \times \cdots \times A_n = \text{ product of } n \text{ terms}
\end{aligned}$$

$$\text{Sign function: sgn } x = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

2.2 Algebra

Polynomial expansions

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(ax+b)^2 = a^2x^2 + 2abx + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(ax+b)^3 = a^3x^3 + 3a^2bx^2 + 3ab^2x + b^3$$

Quadratic equations

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Logarithms and Exponentials

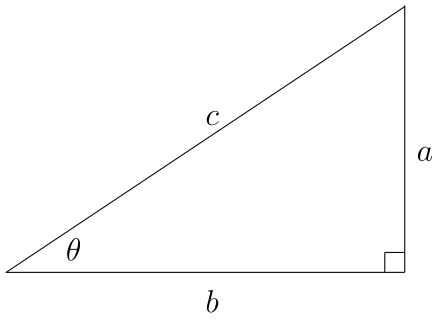
$$\text{If } y = a^x \quad \text{then} \quad y = e^{x \ln a} \quad \text{and} \quad \log_a y = x$$

$$\begin{aligned} a^0 &= 1 \\ a^{-x} &= 1/a^x \\ a^x \times a^y &= a^{x+y} \\ a^x/a^y &= a^{x-y} \\ (a^x)^y &= (a^y)^x = a^{xy} \\ \ln 1 &= 0 \\ \ln(1/x) &= -\ln x \\ \ln(xy) &= \ln x + \ln y \\ \ln(x/y) &= \ln x - \ln y \\ \ln x^y &= y \ln x \end{aligned}$$

$$\begin{aligned} \text{Change of base: } \log_a y &= \frac{\log_b y}{\log_b a} \\ \text{and in particular } \ln y &= \frac{\log_{10} y}{\log_{10} e} \simeq 2.303 \log_{10} y \end{aligned}$$

2.3 Trigonometrical Identities

Trigonometric functions



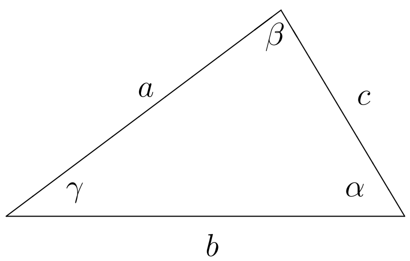
$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c} \qquad \tan\theta = \frac{a}{b}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Basic relations

$$\begin{aligned}(\sin\theta)^2 + (\cos\theta)^2 \equiv \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \\ \frac{\sin\theta}{\cos\theta} &= \tan\theta\end{aligned}$$

Sine and Cosine Rules



Sine Rule $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

Cosine Rule $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Expansions for compound angles

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\sin(A-B) &= \sin A\cos B - \cos A\sin B \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\cos(A-B) &= \cos A\cos B + \sin A\sin B \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A\tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A\tan B}
\end{aligned}$$

$$\sin\left(\theta+\frac{\pi}{2}\right) = +\cos\theta \qquad \sin\left(\frac{\pi}{2}-\theta\right) = +\cos\theta$$

$$\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right) = +\sin\theta$$

$$\sin(\pi+\theta) = -\sin\theta \qquad \sin(\pi-\theta) = +\sin\theta$$

$$\cos(\pi+\theta) = -\cos\theta \qquad \cos(\pi-\theta) = -\cos\theta$$

$$\begin{aligned}
\cos A\cos B &= \frac{1}{2}[\cos(A+B)+\cos(A-B)] \\
\sin A\sin B &= \frac{1}{2}[\cos(A-B)-\cos(A+B)] \\
\sin A\cos B &= \frac{1}{2}[\sin(A+B)+\sin(A-B)] \\
\cos A\sin B &= \frac{1}{2}[\sin(A+B)-\sin(A-B)] \\
\sin 2A &= 2\sin A\cos A \\
\cos 2A = \cos^2 A - \sin^2 A &= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A \\
\tan 2A &= \frac{2\tan A}{1-\tan^2 A}
\end{aligned}$$

Factor formulae

$$\begin{aligned}
\sin A + \sin B &= +2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\sin A - \sin B &= +2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \\
\cos A + \cos B &= +2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\cos A - \cos B &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)
\end{aligned}$$

2.4 Hyperbolic Functions

Definitions and basic relations

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2}\\ \cosh x &= \frac{e^x + e^{-x}}{2}\\ \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^{2x}-1}{e^{2x}+1}\end{aligned}$$

$$\begin{aligned}\text{sech }x &= 1/\cosh x\\ \text{cosech }x &= 1/\sinh x\\ \coth x &= 1/\tanh x\end{aligned}\qquad\qquad \begin{aligned}\cosh^2 x - \sinh^2 x &= 1\\ 1-\tanh^2 x &= \text{sech}^2 x\\ \coth^2 x - 1 &= \text{cosech}^2 x\end{aligned}$$

$$\begin{aligned}\sinh^{-1} x &= \log_e[x+\sqrt{x^2+1}]\\ \cosh^{-1} x &= \pm\log_e[x+\sqrt{x^2-1}]\\ \tanh^{-1} x &= \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)\qquad (x^2<1)\end{aligned}$$

Expansions for compound arguments

$$\begin{aligned}\sinh(A\pm B) &= \sinh A\cosh B \pm \cosh A \sinh B\\ \cosh(A\pm B) &= \cosh A\cosh B \pm \sinh A\sinh B\\ \tanh(A\pm B) &= \frac{\tanh A \pm \tanh B}{(1+\tanh A\tanh B)}\\ \sinh 2A &= 2\sinh A\cosh A\\ \cosh 2A &= \cosh^2 A + \sinh^2 A = 2\cosh^2 A - 1 = 1 + 2\sinh^2 A\\ \tanh 2A &= \frac{2\tanh A}{(1+\tanh^2 A)}\end{aligned}$$

Factor formulae

$$\begin{aligned}\sinh A + \sinh B &= 2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)\\ \sinh A - \sinh B &= 2\cosh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)\\ \cosh A + \cosh B &= 2\cosh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)\\ \cosh A - \cosh B &= 2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)\end{aligned}$$

2.5 Differentiation

Definition

$$f'(x) \equiv \frac{d}{dx}\, f(x) = \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x} \right]$$

$$f''(x) \equiv \frac{d^2}{dx^2}\, f(x) = \frac{d}{dx} f'(x)$$

$$f^n(x) \equiv \frac{d^n}{dx^n}\, f(x) = \text{ the } n^{th} \text{ order differential,}$$

obtained by taking n successive differentiations of $f(x)$.

The overdot notation is often used to indicate a derivative taken with respect to time:

$$\dot{y} \equiv \frac{dy}{dt},\ \ddot{y} \equiv \frac{d^2y}{dt^2}, \quad \text{etc.}$$

Rules of differentiation

If $u = u(x)$ and $v = v(x)$ then:

SUM RULE $$\frac{d}{dx}\,(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

FACTOR RULE $$\frac{d}{dx}\,(ku) = k\frac{du}{dx} \quad \text{where } k \text{ is any constant}$$

PRODUCT RULE $$\frac{d}{dx}\,(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

QUOTIENT RULE $$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \Big/ v^2$$

CHAIN RULE $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Leibnitz' formula

Leibnitz' formula for the n^{th} derivative of a product of two functions $u(x)$ and $v(x)$:

$$[uv]_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3 + \cdots + u v_n,$$

where $u_n = d^n u/dx^n$ etc.

2.6 Standard Derivatives

$$\begin{aligned}
\frac{d}{dx}\left(x^{n}\right) &= nx^{n-1} \\
\frac{d}{dx}\left(\exp[ax]\right) &= a\exp[ax] \\
\frac{d}{dx}\left(a^{x}\right) &= a^{x}\ln a \\
\frac{d}{dx}\left(\ln x\right) &= x^{-1} \\
\frac{d}{dx}\left(\ln(ax+b)\right) &= \frac{a}{(ax+b)} \\
\frac{d}{dx}\left(\log_a x\right) &= x^{-1}\log_a e \\
\frac{d}{dx}\left(\sin(ax+b)\right) &= a\cos(ax+b) \\
\frac{d}{dx}\left(\cos(ax+b)\right) &= -a\sin(ax+b) \\
\frac{d}{dx}\left(\tan(ax+b)\right) &= a\sec^{2}(ax+b) \\
\frac{d}{dx}\left(\sinh(ax+b)\right) &= a\cosh(ax+b) \\
\frac{d}{dx}\left(\cosh(ax+b)\right) &= a\sinh(ax+b) \\
\frac{d}{dx}\left(\tanh(ax+b)\right) &= a\,\mathrm{sech}^{2}(ax+b) \\
\frac{d}{dx}\left(\arcsin(ax+b)\right) &= a\left[1-(ax+b)^{2}\right]^{-1/2} \\
\frac{d}{dx}\left(\arccos(ax+b)\right) &= -a\left[1-(ax+b)^{2}\right]^{-1/2} \\
\frac{d}{dx}\left(\arctan(ax+b)\right) &= a[1+(ax+b)^{2}]^{-1} \\
\frac{d}{dx}\left(\exp\left[ax^{n}\right]\right) &= anx^{(n-1)}\exp\left[ax^{n}\right] \\
\frac{d}{dx}\left(\sin^{2}x\right) &= 2\sin x\cos x \\
\frac{d}{dx}\left(\cos^{2}x\right) &= -2\sin x\cos x
\end{aligned}$$

2.7 Integration

Definitions

The area (A) under a curve is given by

$$A = \lim_{dx_i \to 0} \sum f(x_i) dx_i = \int f(x)\, dx$$

The Indefinite Integral is

$$\int f(x)\, dx = F(x) + C$$

where $F(x)$ is a function such that $F'(x) = f(x)$ and C is the constant of integration.
The Definite Integral is

$$\int_a^b f(x)\, dx = F(b) - F(a) = \Big[F(x) \Big]_a^b$$

where a is the lower limit of integration and b the upper limit of integration.

Rules of integration

SUM RULE $\quad \int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$

FACTOR RULE $\quad \int k f(x)\, dx = k \int f(x)\, dx$ where k is any constant

SUBSTITUTION $\quad \int f(x)\, dx = \int f(x) \frac{dx}{du}\, du$ where $u = g(x)$ is any function of x

N.B. for definite integrals you must also substitute the values of u into the limits of the integral.

Integration by parts

An integral of the form $\int u(x) q(x)\, dx$ can sometimes be solved if $q(x)$ can be integrated and $u(x)$ differentiated. So if we let $q(x) = \frac{dv}{dx}$, so $v = \int q(x)\, dx$, then the *Integration by Parts* formula is

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

Note that if you pick u and $\frac{dv}{dx}$ the wrong way round you will end up with an integral that is even more complex than the initial one. The aim is to pick u and $\frac{dv}{dx}$ such that $\frac{du}{dx}$ is simplified.

2.8 Standard Indefinite Integrals

In the following table C is the constant of integration.

$$
\begin{aligned}
\int x^n\,dx &= \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \\
\int x^{-1}\,dx &= \ln|x| + C \\
\int \ln|x|\,dx &= x\ln x - x + C \\
\int \sin x\,dx &= -\cos x + C \\
\int \cos x\,dx &= \sin x + C \\
\int \tan x\,dx &= -\ln|\cos x| + C \\
\int \cot x\,dx &= \ln|\sin x| + C \\
\int \sec^2 x\,dx &= \tan x + C \\
\int \csc^2 x\,dx &= -\cot x + C \\
\int \cos^2 x\,dx &= \frac{1}{2}x + \frac{1}{2}\sin x\cos x + C \\
\int \sin^2 x\,dx &= \frac{1}{2}x - \frac{1}{2}\sin x\cos x + C \\
\int \sin^n x\,dx &= -\frac{\sin^{n-1} x\cos x}{n} + \frac{(n-1)}{n}\int \sin^{n-2} x\,dx + C \\
\int \cos^n x\,dx &= \frac{\cos^{n-1} x\sin x}{n} + \frac{(n-1)}{n}\int \cos^{n-2} x\,dx + C \\
\int \sin x\cos x\,dx &= \frac{1}{2}\sin^2 x + C \\
\int \cos mx\cos nx\,dx &= \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C \quad (m^2 \neq n^2) \\
\int \sin mx\sin nx\,dx &= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C \quad (m^2 \neq n^2) \\
\int \sin mx\cos nx\,dx &= -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C \quad (m^2 \neq n^2) \\
\int x^2\cos x\,dx &= (x^2-2)\sin x + 2x\cos x + C \\
\int x^2\sin x\,dx &= (2-x^2)\cos x + 2x\sin x + C \\
\int x\cos nx\,dx &= \frac{x\sin nx}{n} + \frac{\cos nx}{n^2} + C \\
\int x\sin nx\,dx &= -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2} + C \\
\int e^{ax}\,dx &= \frac{e^{ax}}{a} + C
\end{aligned}
$$

$$
\begin{aligned}
\int x e^{ax}\, dx &= e^{ax}(x - 1/a)/a + C \\
\int x e^{-inx}\, dx &= \frac{1}{n}\left(\frac{1}{n} + ix\right) e^{-inx} + C \\
\int e^{ax} \sin kx\, dx &= \frac{e^{ax}(a \sin kx - k \cos kx)}{(a^2 + k^2)} + C \\
\int e^{ax} \cos kx\, dx &= \frac{e^{ax}(a \cos kx + k \sin kx)}{(a^2 + k^2)} + C \\
\int \sinh x\, dx &= \cosh x + C \\
\int \cosh x\, dx &= \sinh x + C \\
\int \tanh x\, dx &= \ln \cosh x + C \\
\int \mathrm{sech}^2 x\, dx &= \tanh x + C \\
\int \mathrm{csch}^2 x\, dx &= \coth x + C \\
\int \frac{1}{a^2 + x^2}\, dx &= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \\
\int \frac{1}{a^2 - x^2}\, dx &= \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) \\
&= \frac{1}{2a} \ln\left(\frac{a + x}{a - x}\right) + C \\
\int \frac{1}{(a^2 - x^2)^{1/2}}\, dx &= \arcsin\left(\frac{x}{a}\right) + C \\
&= -\arccos\left(\frac{x}{a}\right) + C \\
\int \frac{1}{(x^2 - a^2)^{1/2}}\, dx &= \cosh^{-1}\left(\frac{x}{a}\right) + C \\
\int \frac{x^2}{(a^2 + x^2)}\, dx &= x - a \arctan\left(\frac{x}{a}\right) + C \\
\int \frac{1}{(a^2 + x^2)^{1/2}}\, dx &= \ln[x + (x^2 + a^2)^{1/2}] + C \\
&= \sinh^{-1}\left(\frac{x}{a}\right) + C \\
\int \frac{1}{(a^2 + x^2)^{3/2}}\, dx &= \sin\left[\arctan\left(\frac{x}{a}\right)\right] / a^2 + C \\
&= \frac{1}{a^2} \frac{x}{(a^2 + x^2)^{1/2}} + C \\
\int \frac{x^{1/2}}{(a - x)^{1/2}}\, dx &= a \arcsin\left(\sqrt{}(x/a)\right) - a\sqrt{x/a - (x/a)^2} + C \\
\int \frac{x}{(a^2 + x^2)^{1/2}}\, dx &= (a^2 + x^2)^{1/2} + C \\
\int \frac{1}{(a + bx^2)^2}\, dx &= \frac{x}{2a(a + bx^2)} + \frac{1}{2a\sqrt{}(ab)} \arctan[x\sqrt{}(b/a)] + C
\end{aligned}
$$

2.9 Definite Integrals

$$\int_0^\infty x^{1/2}e^{-x}\,dx = \frac{\sqrt{\pi}}{2} \qquad\qquad \int_0^\infty \frac{x^{1/2}}{(e^x-1)}\,dx = \frac{2.61\sqrt{\pi}}{2}$$

$$\int_0^\infty x^n e^{-x}\,dx = \int_0^1 \left(\ln\frac{1}{x}\right)^n = \Gamma(n+1) \qquad \text{the Gamma function}$$

Note: for n an integer greater than 0, $\Gamma(n+1) = n!$, $\Gamma(1) = 0! = 1$

$$\frac{2}{\sqrt{\pi}}\int_0^u e^{-x^2}dx = \operatorname{erf}(u) \qquad \text{the Error function}$$

Note that $\operatorname{erf}(\infty) = 1$, so that $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^\infty e^{-ax^2}\,dx = \frac{1}{\sqrt{a}}$.

$$\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x^{2n}e^{-ax^2}dx = \frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}} \equiv S \qquad (n = 1, 2, 3\ldots)$$

$$\int_{-\infty}^\infty x^{2n}e^{-ax^2}\,dx = 2S$$

$$\int_0^\infty x^{2n+1}e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}} \qquad (a > 0; n = 0, 1, 2\ldots)$$

$$\int_{-\infty}^{+\infty} x^{2n+1}e^{-ax^2}dx = 0$$

$$\int_0^\infty x^2\ln(1-e^{-x})dx = \frac{-\pi^4}{45}$$

$$\int_0^\infty e^{-ax}\cos(kx)dx = \frac{a}{a^2+k^2}$$

$$\int_0^\infty \frac{1}{1+x^{2n}}\,dx = \frac{\pi/2n}{\sin(\pi/2n)}$$

2.10 Curvilinear Coordinate Systems

Definitions

Spherical Coordinates

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

Cylindrical Coordinates

$$\begin{aligned} x &= r \cos\phi \\ y &= r \sin\phi \\ z &= z \end{aligned}$$

Elements of area and volume

Elements of Area

Cartesian (x, y)	$dS = dx\, dy$
Plane polar (r, θ)	$dS = r\, dr\, d\theta$

Elements of Volume

Cartesian (x, y, z)	$dV = dx\, dy\, dz$
Spherical polar (r, θ, ϕ)	$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$
Cylindrical polar (r, ϕ, z)	$dV = r\, dr\, d\phi\, dz$

MISCELLANEOUS

Area of elementary circular annulus, width dr, centred on the origin: $dS = 2\pi r dr$

Volume of elementary cylindrical annulus of height dz and thickness dr: $dV = 2\pi r dr dz$

Volume of elementary spherical shell of thickness dr, centred on the origin: $dV = 4\pi r^2 dr$

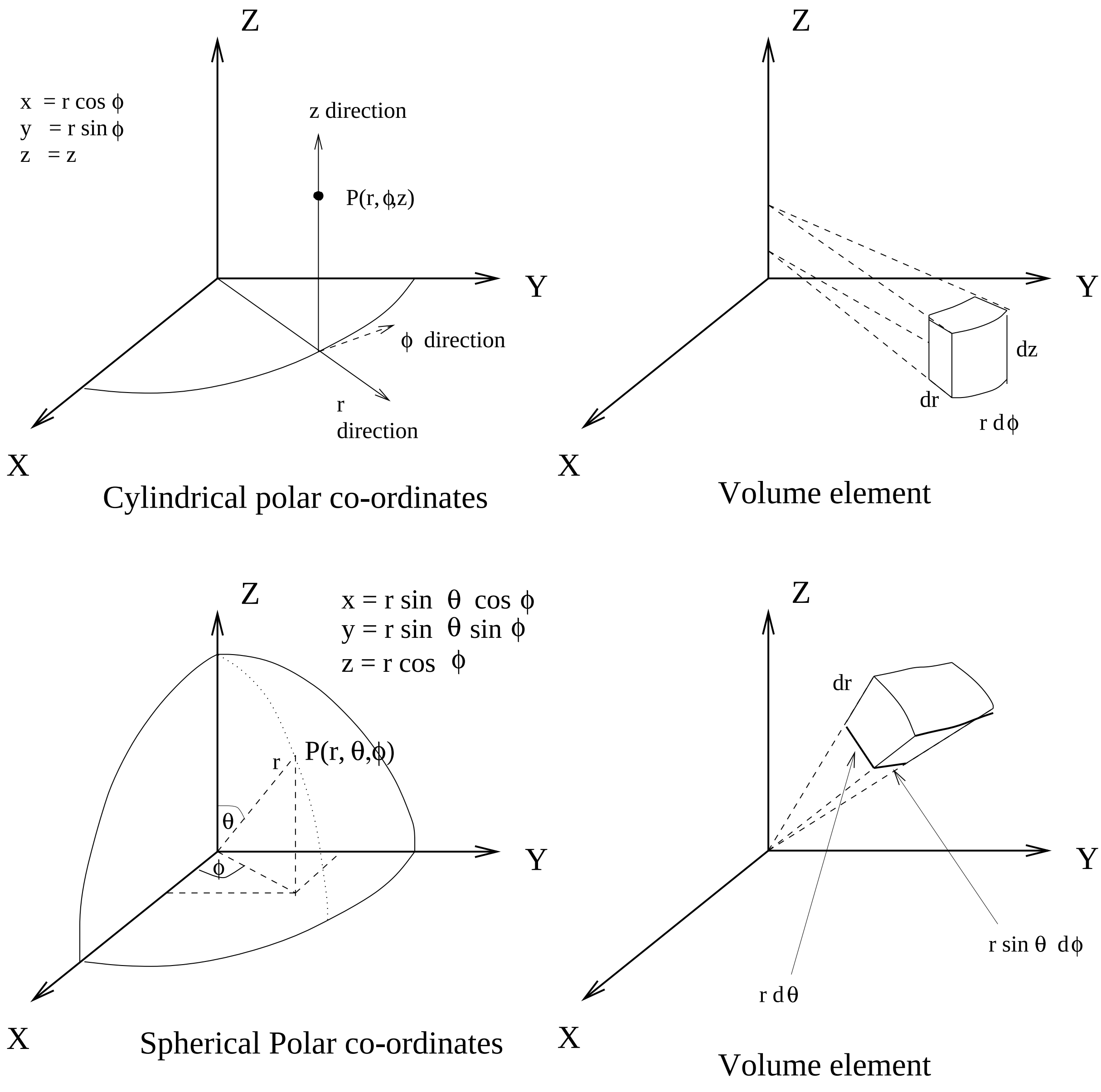


Figure 2.1: Coordinate Systems and Elements of volume.

Solid angle

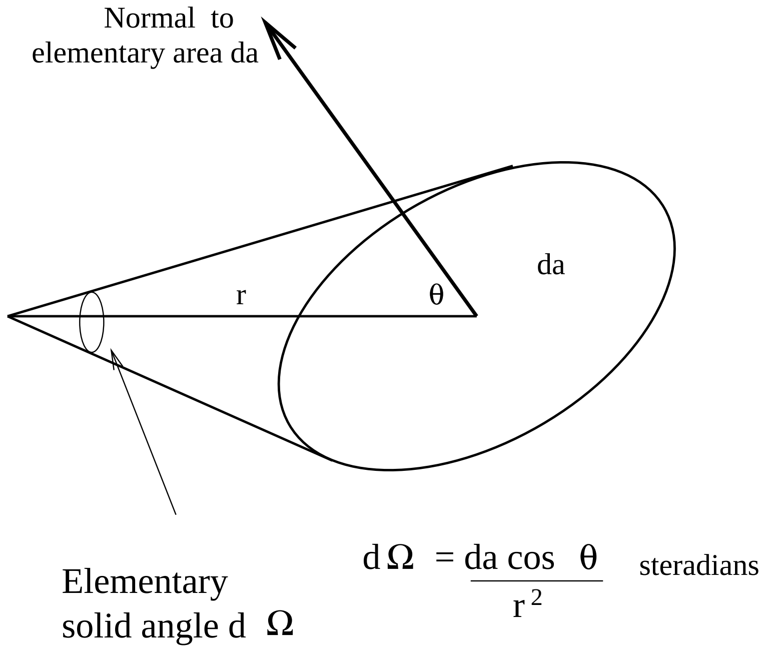


Figure 2.2: Solid angle.

1. The solid angle subtended by any closed surface at any point inside the surface is 4π;
2. The solid angle subtended by any closed surface at a point outside the surface is zero.

2.11 Vectors and Vector Algebra

Vectors are quantities with *both* magnitude and direction; they are combined by the triangle rule (see Fig. 2.3).

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} = \mathbf{C}$$

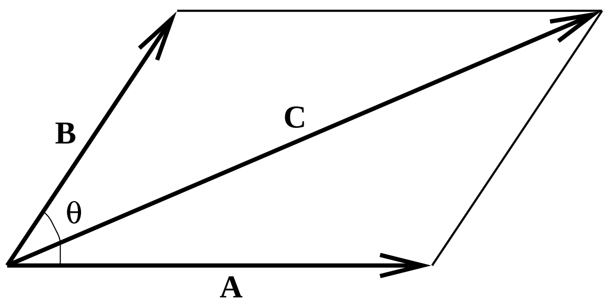


Figure 2.3: Vector addition.

Vectors may be denoted by bold type $\mathbf{A}$, by putting a little arrow over the symbol $\vec{A}$, or by underlining the symbol $\underline{A}$. Unit vectors are usually denoted by a circumflex accent (e.g. $\hat{\mathbf{i}}$).

Magnitude etc.

$$|\mathbf{A}| = \surd(\mathbf{A} \cdot \mathbf{A}) = \surd(A_x^2 + A_y^2 + A_z^2)$$

The angle θ between two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{(A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2)}}$$

Unit vectors

Unit vector in the direction of $\mathbf{A} = \mathbf{A}/|\mathbf{A}|$
Cartesian co-ordinates: $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors in the directions of the x, y, z cartesian axes respectively.
If A_x, A_y, A_z are the cartesian components of $\mathbf{A}$ then

$$\mathbf{A} = \hat{\mathbf{i}}A_x + \hat{\mathbf{j}}A_y + \hat{\mathbf{k}}A_z$$

Addition and subtraction

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{(Commutative law)}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \quad \text{(Associative law)}$$

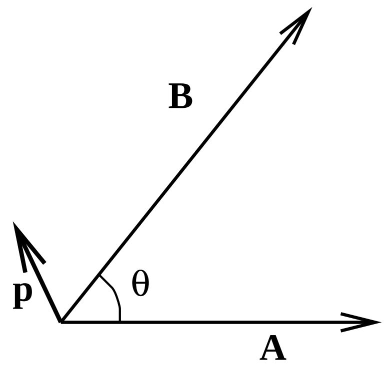


Figure 2.4: Vector (or Cross) Product; the vector $\mathbf{p}$ is directed out of the page.

Products

SCALAR PRODUCT
$\mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}||\mathbf{B}| \cos\theta = \mathbf{B} \cdot \mathbf{A}$ (a *scalar*)

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$
$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$
$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$
$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

VECTOR (OR CROSS) PRODUCT
See Fig. 2.4
$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} = (|\mathbf{A}||\mathbf{B}| \sin\theta)\hat{\mathbf{p}}$ (a *vector*)
where $\hat{\mathbf{p}}$ is a *unit* vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. Note that the vector product is non-commutative.

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$$

Also, in cartesian co-ordinates,

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_y B_z - A_z B_y)\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \end{aligned}$$

Scalar triple product

$$
\begin{aligned}
(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} &= (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A} = (\mathbf{C}\times\mathbf{A})\cdot\mathbf{B} \quad \text{(a } \textit{scalar}\text{)} \\
& \quad \text{(Note the cyclic order: } \mathbf{A}\!\rightarrow\!\mathbf{B}\!\rightarrow\!\mathbf{C}\text{)} \\
&= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\
&= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x)
\end{aligned}
$$

Vector triple product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \qquad \text{(a } \textit{vector}\text{)}$$

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$ (Note the cyclic order: $\mathbf{A}\rightarrow\mathbf{B}\rightarrow\mathbf{C}$)

2.12 Complex Numbers

$z = a + ib$ is a complex number where a, b are *real* and $i = \sqrt{-1}$ (N.B. sometimes j is used instead of i).
a is the *real* part of z and b is the *imaginary* part. Sometimes the real part of a complex quantity z is denoted by $\Re(z)$, the imaginary part by $\Im(z)$.

If $a_1 + ib_1 = a_2 + ib_2$ then $a_1 = a_2$ and $b_1 = b_2$.

Modulus and argument

The modulus of $z \equiv |z| = \sqrt{a^2 + b^2}$.
The argument of $z = \theta = \arctan(b/a)$

Complex conjugate

To form the complex conjugate of any complex number simply replace i by $-i$ *wherever* it occurs in the number. Thus if $z = a + ib$ then the complex conjugate is $z^* = a - ib$.
If $z = Ae^{-ix}$ then $z^* = A^* e^{+ix}$.
Note: $|z| = \sqrt{zz^*} = \sqrt{(a + ib)(a - ib)} = \sqrt{a^2 + b^2}$

Rationalization

If $z = A/B$, where A and B are both complex numbers, then the quotient can be 'rationalized' as follows:

$$z = \frac{A}{B} = \frac{AB^*}{BB^*} = \frac{AB^*}{|B|^2}$$

and the *denominator* is now real.

Polar form

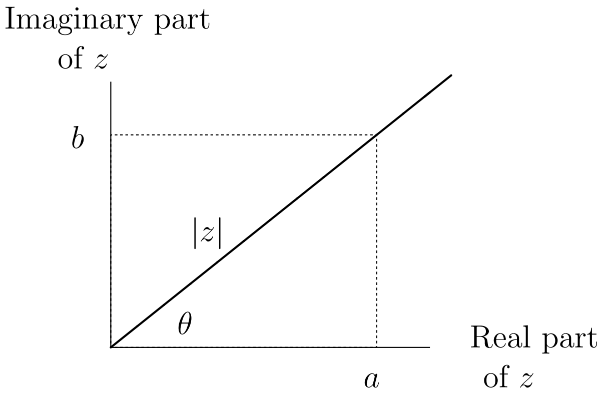
See Fig. 2.5. If $z = a + ib$ then $|z| = \sqrt{a^2 + b^2}$ and $\theta = \arctan(b/a)$. Note when evaluating $\arctan(b/a)$, θ must be put in the correct quadrant (see Fig 2.6).

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{(Euler's identity)}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\begin{aligned} z &= a + ib \\ &= |z|\cos\theta + i|z|\sin\theta \\ &= |z|\exp[i\theta] \end{aligned}$$

$$\begin{aligned} \text{If} \quad z_1 &= |z_1|\exp[i\theta_1] \text{ and } z_2 = |z_2|\exp[i\theta_2] \\ \text{then} \quad z_1 z_2 &= |z_1||z_2|\exp i[\theta_1 + \theta_2] \\ \text{and} \quad \frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|}\exp i[\theta_1 - \theta_2] \end{aligned}$$



$$a = |z|\cos\theta$$
$$b = |z|\sin\theta$$

Figure 2.5: Argand diagram.

$$\begin{aligned}
\text{If} \quad z^n &= w \quad \text{where } w = |w|\exp[i\theta] \\
\text{then} \quad z &= |w|^{1/n}\exp[i(\theta + 2k\pi)/n] \quad \text{where } k = 0, 1, 2 \ldots (n-1) \\
|z^n| &= |z|^n \\
|z|^m|z|^n &= |z|^{m+n} \\
\left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|}
\end{aligned}$$

DeMoivre's theorem

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{where n is an integer}$$

Trigonometric and hyperbolic functions

$$\begin{aligned}
\sinh(i\theta) &= i\sin\theta \\
\cosh(i\theta) &= \cos\theta \\
\tanh(i\theta) &= i\tan\theta
\end{aligned}$$

$$\begin{aligned}
\sin(i\theta) &= i\sinh\theta \\
\cos(i\theta) &= \cosh\theta \\
\tan(i\theta) &= i\tanh\theta
\end{aligned}$$

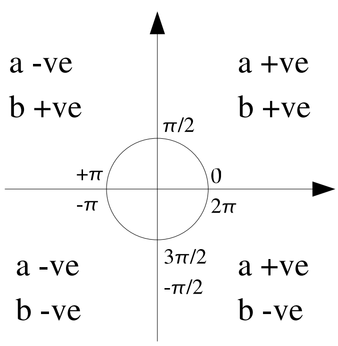


Figure 2.6: Selecting the correct quadrant for $\theta = \arctan(b/a)$

2.13 Series

Arithmetic progression (A.P.)

$$S = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a + [n - 1]d)$$

Sum over n terms is

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Geometric progression (G.P.)

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Sum over n terms is

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

If $|r| < 1$ the sum to infinity is

$$S_\infty = \frac{a}{(1 - r)}$$

Binomial theorem

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$$

If n is a *positive integer* the series contains $(n + 1)$ terms. If n is a *negative integer* or a positive or negative *fraction* the series is infinite. The series converges if $|b/a| < 1$.
Special cases:

$$\begin{aligned}
(1 \pm x)^n &= 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \quad \text{Valid for all } n. \\
(1 \pm x)^{-1} &= 1 \mp x + x^2 \mp x^3 + x^4 \mp \cdots \\
(1 \pm x)^{-2} &= 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp \cdots \\
(1 \pm x)^{\frac{1}{2}} &= 1 \pm \frac{x}{2} - \frac{x^2}{8} \pm \frac{x^3}{16} - \frac{5x^4}{128} \pm \cdots \\
(1 \pm x)^{-\frac{1}{2}} &= 1 \mp \frac{x}{2} + \frac{3x^2}{8} \mp \frac{5x^3}{16} + \frac{35x^4}{128} \mp \cdots
\end{aligned}$$

Maclaurin's theorem

$$f(x) = f(0) + x\left.\frac{df}{dx}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=0} + \frac{x^3}{3!}\left.\frac{d^3f}{dx^3}\right|_{x=0} + \cdots + \frac{x^n}{n!}\left.\frac{d^nf}{dx^n}\right|_{x=0} + \cdots$$

where, for example $d^2f/dx^2|_{x=0}$ means the result of forming the second derivative of $f(x)$ with respect to x and *then* setting $x = 0$.

Taylor's theorem

$$f(x) = f(a) + (x-a)\left.\frac{df}{dx}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{d^3f}{dx^3}\right|_{x=a} + \cdots$$
$$\cdots + \frac{(x-a)^n}{n!}\left.\frac{d^nf}{dx^n}\right|_{x=a} + \cdots$$

where, for example $d^2f/dx^2|_{x=a}$ again means the result of forming the second derivative of $f(x)$ with respect to x and *then* setting $x = a$.

Series expansions of trigonometric functions

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

For θ in *radians* and small (i.e. $\theta \ll 1$):

$\sin\theta \simeq \theta$ Error $\lesssim 4$ % for $\theta \lesssim 30° \simeq 0.52$ radians

$\tan\theta \simeq \theta$ Error $\lesssim 4$ % for $\theta \lesssim 30° \simeq 0.52$ radians

$\cos\theta \simeq 1$ Error $\lesssim 4$ % for $\theta \lesssim 16° \simeq 0.28$ radians

Series expansions of exponential functions

$$e^{\pm x} = 1 \pm x + \frac{x^2}{2!} \pm \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{Convergent for all values of } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{Convergent for } -1 < x \leq 1$$

For x small (i.e. $x \ll 1$):

$$e^{\pm x} \equiv \exp[\pm x] \simeq 1 \pm x \cdots$$
$$\ln(1 \pm x) \simeq \pm x \cdots$$

Series expansions of hyperbolic functions

$$\sinh x \equiv \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
$$\cosh x \equiv \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

L'Hôpital's rule

If two functions $f(x)$ and $g(x)$ are both zero or infinite at $x = a$ the ratio $f(a)/g(a)$ is undefined. However the limit of $f(x)/g(x)$ as x approaches a may exist. This may be found from

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

where $f'(a)$ means the result of differentiating $f(x)$ with respect to x and *then* putting $x = a$.

Convergence Tests

D'Alembert's ratio test

In a series, $\sum_{n=1}^{\infty} a_n$, let the ratio $R = \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n} \right)$.

- If $R < 1$ the series is *convergent*
- If $R > 1$ the series is *divergent*
- If $R = 1$ the test fails.

The Integral Test

A sum to infinity of a_n converges if $\int_1^{\infty} a_n \, dn$ is finite. This can only be applied to series where a_n is positive and decreasing as n gets larger.

2.14 Ordinary Differential Equations

General points

1. In general, finding a function which is 'a solution of' (i.e. satisfies) any particular differential equation is a trial and error process. It involves inductive not deductive reasoning, comparable with integration as opposed to differentiation.

2. If the highest differential coefficient in the equation is the n^{th} then the general solution must contain n arbitrary constants.

3. The known physical conditions–the *boundary conditions*–may enable one particular solution or a set of solutions to be selected from the infinite family of possible mathematical solutions; that is boundary conditions may allow specific values to be assigned to the arbitrary constants in the general solution.

4. Virtually all the ordinary differential equations met in basic physics are *linear*, that is the differential coefficients occur to the *first power* only.

Definitions

ORDER OF A DIFFERENTIAL EQUATION
The *order* of a differential equation is the order of the highest differential coefficient it contains.

DEGREE OF A DIFFERENTIAL EQUATION
The *degree* of a differential equation is the power to which the *highest order differential coefficient* is raised.

DEPENDENT AND INDEPENDENT VARIABLES
Ordinary differential equations involve only two variables, one of which is referred to as the *dependent* variable and the other as the *independent* variable. It is usually clear from the nature of the physical problem which is the independent and which is the dependent variable.

First Order Differential Equations

Direct Integration

The equation $\dfrac{dy}{dx} = f(x)$ has the solution $y = \int f(x)dx$. Thus it can be solved (in principle) by direct integration.

Separable Variables

First order differential equations of the form $\dfrac{dy}{dx} = f(x)g(y)$, where $f(x)$ is a function of x only and $g(y)$ is a function of y only. Dividing both sides by $g(y)$ and integrating gives $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx + C$, which can be used to obtain the solution of $y(x)$.

The linear equation

A general first order linear equation of the form $\dfrac{dy}{dx} + P(x)y = Q(x)$
This can be solved by multiplying through by an 'integrating factor' e^I, where $I = \int P(x)dx$, so that the original equation can be rewritten as

$$\frac{d}{dx}\left(ye^I\right) = e^I Q(x)$$

Since Q and I are only functions of x we can integrate both sides to obtain

$$ye^I = \int Q(x)e^I dx$$

Second Order Differential Equations

Direct Integration

Equations of the form $\dfrac{d^2y}{dx^2} = f(x)$, can be solved by integrating twice:

$$y = \int \left[\int f(x)dx + C\right] dx = \int \left[\int f(x)dx\right] dx + Cx + D$$

Note that there are two arbitrary constants, C and D.

Homogeneous Second Order Differential Equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where a, b and c are constants. Letting $y = Ae^{\alpha x}$, gives the auxiliary equation

$$a\alpha^2 + b\alpha + c = 0$$

This is solved for α using the quadratic equation, which gives two values for α, α_1 and α_2. The general solution is the combination of the two, $y = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$.

The auxiliary equation has real roots When $b^2 > 4ac$, both α_1 and α_2 are real. The general solution is $y = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$.

The auxiliary equation has complex roots When $b^2 < 4ac$, both α_1 and α_2 are complex. Using Euler's Equation, substituting $C = A+B$ and $D = i(A-B)$, the general solution can be written as

$$y = e^{\alpha x}\left(C\cos(\beta x) + D\sin(\beta x)\right)$$

where $\alpha = -b/(2a)$ and $\beta = \sqrt{(4ac - b^2)}/(2a)$.

The auxiliary equation has equal roots When $b^2 = 4ac$, there is only one α. The general solution is given by $y = (A + Bx)e^{\alpha x}$

Non-homogeneous Second Order Differential Equations

Non-homogeneous second order differential equations are of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

To solve, first solve the homogeneous equation (i.e. for right-hand side = 0),

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

using the method given above to get the solution

$$y = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$$

which is known as the *complementary function (CF)*. Then we find a *particular solution (PS)* for the whole equation. The *general solution* is CF + PS.

The particular solution is taken to be the same form as the function $f(x)$.

$$\begin{array}{rll} f(x) = k & \text{assume} & y = C \\ f(x) = kx & \text{assume} & y = Cx + D \\ f(x) = kx^2 & \text{assume} & y = Cx^2 + Dx + E \\ f(x) = k\sin x \text{ or } k\cos x & \text{assume} & y = C\cos x + D\sin x \\ f(x) = e^{kx} & \text{assume} & y = Ce^{kx} \end{array}$$

2.15 Partial Differentiation

Definition

If $f = f(x, y)$ with x and y independent, then

$$\begin{aligned}
\left(\frac{\partial f}{\partial x}\right)_y &\equiv \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} \\
&= \text{derivative with respect to } x \text{ with } y \text{ kept constant}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial f}{\partial y}\right)_x &\equiv \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \\
&= \text{derivative with respect to } y \text{ with } x \text{ kept constant}
\end{aligned}$$

The rules of partial differentiation are the same as differentiation, always bearing in mind which term is varying and which are constant.

Convenient notation

$$f_x = \frac{\partial f}{\partial x}, f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

Note that for functions with continuous derivatives $f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = f_{yx}$

Total Derivatives

Total change in f due to infinitesimal changes in both x and y:

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

$\dfrac{df}{dx} = \left(\dfrac{\partial f}{\partial x}\right)_y + \left(\dfrac{\partial f}{\partial y}\right)_x \dfrac{dy}{dx}$ is the *total derivative* of f with respect to x.

$\dfrac{df}{dy} = \left(\dfrac{\partial f}{\partial y}\right)_x + \left(\dfrac{\partial f}{\partial x}\right)_y \dfrac{dx}{dy}$ is the *total derivative* of f with respect to y.

For a function where each variable depends upon a third parameter, such as $f(x, y)$ where x and y depend on time (t):

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dt}$$

Maxima and Minima with two or more variables

If f is a function of two or more variables we can still find the maximum and minimum points of the function. Consider a 3-d surface given by $f = f(x, y)$. We can identify the following types of *stationary points* where gradients are zero:

peak – a *local maximum*

pit – a *local minimum*

pass or *saddle point* – minimum in one direction, maximum in the other.

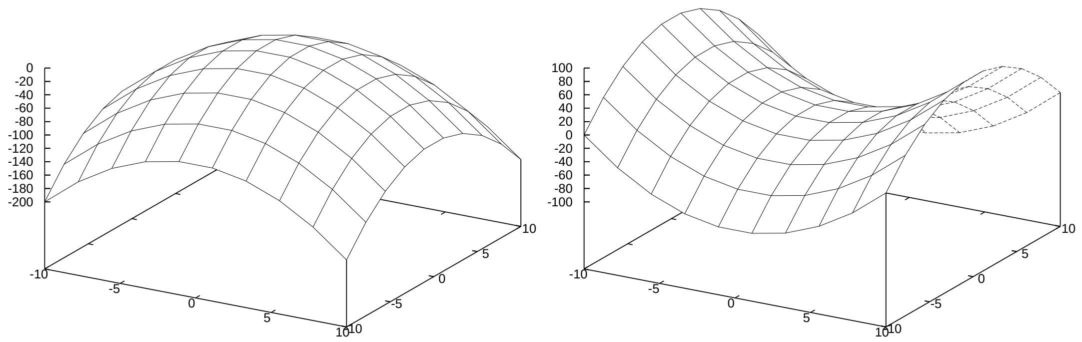


Figure 2.7: Surface plots showing a peak (left) and a saddle Point (right)

At each peak, pit or pass, the function f is stationary, i.e.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

Let $f(x_0, y_0)$ be a stationary point and define the *second derivative test discriminant* as

$$D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = f_{xx} f_{yy} - f_{xy}^2$$

which is evaluated at (x_0, y_0) and,

if $D > 0$ and $f_{xx} > 0$ we have a pit (minimum)

if $D > 0$ and $f_{xx} < 0$ we have a peak (maximum)

if $D < 0$ we have a pass (saddle point)

if $D = 0$ we do not know, have to test further comparing $f(x_0, y_0)$, $f(x_0 \pm dx, y_0)$, $f(x_0, y_0 \pm dy)$, i.e. compare with values close to $f(x_0, y_0)$.

2.16 Partial Differential Equations

The following partial differential equations are basic to physics:

One dimension	Three dimensions
$--$	$\nabla^2\phi = 0$ Laplace's equation
$--$	$\nabla^2\phi = \text{constant}$ Poisson's equation
$\dfrac{\partial^2\phi}{\partial x^2} = D\dfrac{\partial\phi}{\partial t}$	$\nabla^2\phi = D\dfrac{\partial\phi}{\partial t}$ Diffusion equation
$\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$	$\nabla^2\phi = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial^2 t}$ Wave equation

In general a partial differential equation can be satisfied by a wide variety of different functions, i.e. if $\phi = f(x,t)$ or $\phi = f(x,y,z)$, f may take many different forms which are *not* equivalent ways of representing the same set of surfaces. For example, *any* continuous, differentiable function of $(x \pm ct)$ will fit the one-dimensional wave equation.
'Solving' these partial differential equations in a particular physical context therefore involves choosing not just constants but also the functions which fit the boundary conditions. Equations involving three or four independent variables, e.g. (x,y,t) or (x,y,z,t) can be solved only when the 'boundaries' are surfaces of some simple co-ordinate system, such as rectangular, polar, cylindrical polar, spherical polar. The partial differential equations can then be separated into a number of *ordinary* differential equations in the separate co-ordinates, and solutions can be expressed as expansions of various classical mathematical functions. This is analogous to the general representation of the solution $f(x \pm ct)$ of the one-dimensional wave equation by a Fourier series of sine and cosine functions.

2.17 Determinants and Matrices

Determinants

The general set of simultaneous linear equations may be written as:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= y_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= y_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= y_m
\end{aligned}
$$

The solutions of these equations are:

$$x_k = \frac{1}{D}\sum_{j=1}^{n} y_j D_{jk} \qquad \text{(Cramer's rule)}$$

where

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}$$

is the *determinant* of the coefficients of the x_i and where

$D_{jk} = (-1)^{j+k} \quad \times \quad$ [determinant obtained by suppressing the j^{th} row and the k^{th} column of D]

D_{jk} is called the *co-factor* of a_{jk}. The determinant D can be expanded, and ultimately evaluated, as follows:

$$D = a_{11}D_{11} + a_{12}D_{12} + \cdots + a_{1n}D_{1n} \qquad \text{(‘expansion by the first row’)}$$

or

$$D = a_{11}D_{11} + a_{21}D_{21} + \cdots + a_{m1}D_{m1} \qquad \text{(‘expansion by the first column’)}$$

The expansion procedure is repeated for D_{1n} etc. until the remaining determinants have dimensions 2×2. If

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \text{then } D = (ad - bc)$$

Note: this method is very tedious for $m, n > 3$ and it may be better to use a ‘condensation’ procedure (see text books).

EXAMPLE

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Note that the value of a determinant is unaltered if the rows and columns are interchanged. See Sections on Vectors and Vector Calculus, where vector product and the curl of a vector are expressed as determinants.

Matrices

The general set of simultaneous linear equations above can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

(an $[m \times n]$ matrix) (column vectors)

where the arrays of ordered coefficients are called *matrices*. One of the coefficients, or terms, is called an 'element' and a matrix is often denoted by the general element $[a_{ij}]$, where i indicates the row and j the column.

Addition of matrices

If two matrices are of the *same* order $m \times n$ then

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Scalar multiplication

If λ is a *scalar* number then

$$\lambda[a_{ij}] = [\lambda a_{ij}]$$

Matrix multiplication

Multiplication of two matrices $[a_{ij}], [b_{ij}]$ is possible *only* if the number of *columns* in $[a_{ij}]$ is the same as the number of *rows* in $[b_{ij}]$. The product $[c_{ij}]$ is given by

$$[c_{ij}] = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Note

1. Matrix multiplication is not defined unless the two matrices have the appropriate number of rows and columns.
2. Matrix multiplication is generally non-commutative: $AB \neq BA$

The unit matrix

The unit (or identity) matrix, denoted by I, is a square $(n \times n)$ matrix with its diagonal elements equal to unity and all other elements zero. For example the 3×3 unit matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we have a square matrix A of order n and the unit matrix of the same order then

$$IA = AI = A$$

and in general, provided the matrix product is defined (see above), multiplying *any* matrix A by a unit matrix leaves A unchanged.

The transpose of a matrix

If the rows and columns of a matrix are interchanged, a new matrix, called the *transposed matrix*, is obtained. The transpose of a matrix A is denoted by A^{T}. For example if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

then

$$A^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

The adjoint matrix

The adjoint of a matrix (denoted by $\operatorname{adj} A$) is defined as the *transpose of the matrix of the cofactors*, where the cofactors are as defined above (see Section on Determinants, p. 36).

The inverse of a matrix

The inverse A^{-1} of a matrix A has the property that

$$A^{-1}A = AA^{-1} = I,$$

the unit matrix. It is evaluated as follows:

$$A^{-1} = \frac{\operatorname{adj} A}{|A|},$$

where $|A|$ is the determinant of A.

Hermitian and unitary matrices

If a matrix A contains complex elements then the complex conjugate of A is found by taking the complex conjugate of the individual elements. A matrix A is said to be *Hermitian* if

$$\widetilde{A^*} = A$$

A *unitary* matrix is defined by the condition

$$A\widetilde{A^*} = I$$

2.18 Vector Calculus

Differentiation of vectors (non-rotating axes)

$$\frac{d\mathbf{A}}{dt} = \hat{\mathbf{i}}\frac{dA_x}{dt} + \hat{\mathbf{j}}\frac{dA_y}{dt} + \hat{\mathbf{k}}\frac{dA_z}{dt}$$

$$\frac{d(\mathbf{A}\cdot\mathbf{B})}{dt} = \left(\mathbf{A}\cdot\frac{d\mathbf{B}}{dt}\right) + \left(\frac{d\mathbf{A}}{dt}\cdot\mathbf{B}\right)$$

$$\frac{d(\mathbf{A}\times\mathbf{B})}{dt} = \left(\mathbf{A}\times\frac{d\mathbf{B}}{dt}\right) + \left(\frac{d\mathbf{A}}{dt}\times\mathbf{B}\right)$$

Gradient of a scalar function

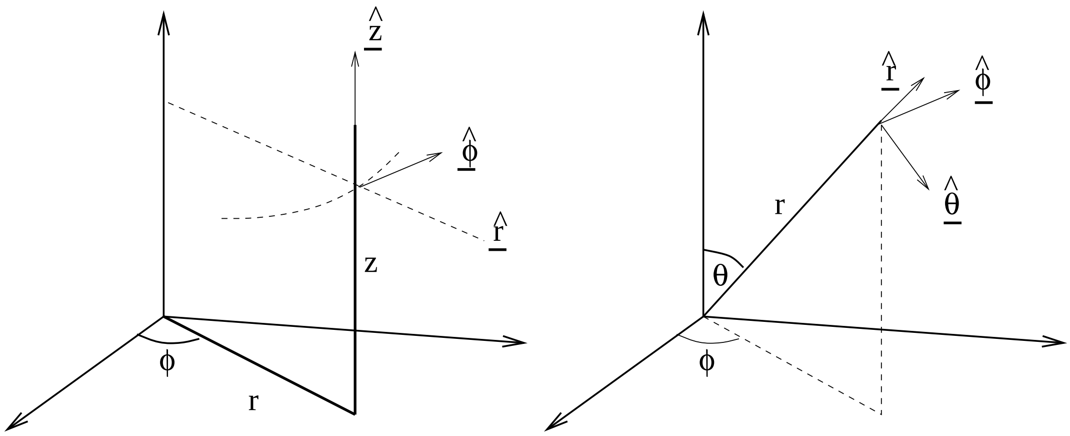


Figure 2.8: Cylindrical (left) and Spherical (right) polars.

CARTESIAN CO-ORDINATES

$$\nabla \equiv \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \qquad \text{(a \textit{vector} operator).}$$

$$\nabla U = \text{ grad } U\text{, where } U \text{ is a scalar. } \nabla U \text{ is a vector.}$$

$$\nabla(U+V) = \nabla U + \nabla V \qquad U, V \text{ scalars.}$$

$$\nabla(UV) = V(\nabla U) + (\nabla U)V$$

CYLINDRICAL CO-ORDINATES

$$\nabla \equiv \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

SPHERICAL POLAR CO-ORDINATES

$$\nabla \equiv \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Divergence of a vector function

CARTESIAN CO-ORDINATES

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\\ &\equiv \text{div}\mathbf{A} \qquad \text{a } \textit{scalar}\end{aligned}$$

CYLINDRICAL POLAR CO-ORDINATES

$$\nabla\cdot\mathbf{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$$

SPHERICAL POLAR CO-ORDINATES

$$\nabla\cdot\mathbf{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Curl of a vector function

CARTESIAN CO-ORDINATES

$$\begin{aligned}\nabla\times\mathbf{A}\equiv\text{curl }\mathbf{A} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}\\ &\equiv \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{\mathbf{i}}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{\mathbf{j}}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{\mathbf{k}}\end{aligned}$$

CYLINDRICAL POLAR CO-ORDINATES

$$\begin{aligned}\nabla\times\mathbf{A} &= \frac{1}{r}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{vmatrix}\\ &\equiv \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\hat{\mathbf{r}}+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\phi}}+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right)\hat{\mathbf{z}}\end{aligned}$$

SPHERICAL POLAR CO-ORDINATES

$$\begin{aligned}\nabla\times\mathbf{A} &= \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & rA_\phi\sin\theta\end{vmatrix}\\ &\equiv \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\hat{\mathbf{r}}+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right)\hat{\boldsymbol{\theta}}\\ &\quad+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}\end{aligned}$$

Relations

$$
\begin{aligned}
\nabla \times (\nabla U) &\equiv 0 \\
\nabla \cdot (\nabla \times \mathbf{A}) &\equiv 0 \\
\nabla \cdot (\nabla U) &\equiv \nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \\
\nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\end{aligned}
$$

The Laplacian operator ∇^2

CARTESIAN CO-ORDINATES

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

CYLINDRICAL POLAR CO-ORDINATES

$$
\begin{aligned}
\nabla^2 &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\right)\right] \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

SPHERICAL POLAR CO-ORDINATES

$$
\begin{aligned}
\nabla^2 &= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right] \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}
\end{aligned}
$$

Integral theorems

DIVERGENCE/GAUSS' THEOREM

$$\oiint_S \mathbf{A} \cdot d\mathbf{s} = \iiint_V (\nabla \cdot \mathbf{A})\, dV$$

STOKES' THEOREM

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$$

GREEN'S THEOREM

$$\oiint_S (\theta \nabla \phi - \phi \nabla \theta) \cdot d\mathbf{s} = \iiint_V (\theta \nabla^2 \phi - \phi \nabla^2 \theta) dV$$

2.19 Fourier Series

If a function $f(t)$ is periodic in t with period T, (i.e. $f(t+nT) = f(t)$ for any integer n and all t), then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right]$$

where

$$a_0 = \frac{2}{T} \int_0^T f(t)\, dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2\pi nt}{T}\right)\, dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi nt}{T}\right)\, dt$$

Notes:

1. t can be any continuous variable, *not necessarily time.*

2. The function $f(t)$ is a continuous function from $t = -\infty$ to $t = +\infty$. For some functions $t = 0$ may be so chosen as to produce a simpler series in which either *all* $a_n = 0$ or *all* $b_n = 0$, e.g. a 'square' wave. See Fig. 2.9.

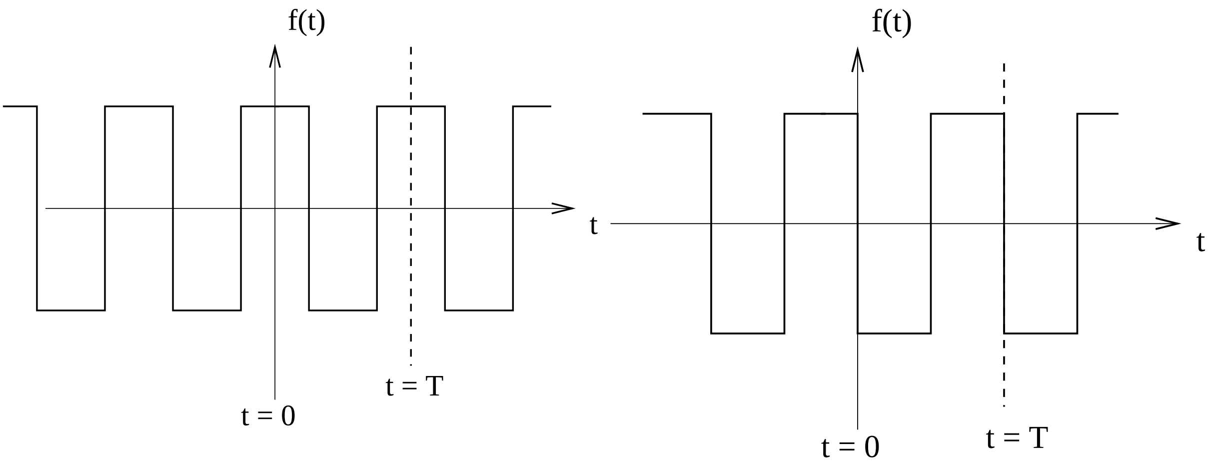


Figure 2.9: Even function (left), $f(+t) = f(-t)$ so $b_n = 0$. Odd function (right), $f(+t) = -f(-t)$ so $a_n = 0$

Complex form of the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \exp\left(i\frac{2\pi nt}{T}\right)$$

where

$$\begin{aligned} C_n &= \frac{1}{T}\int_0^T f(t)\exp\left(-i\frac{2\pi nt}{T}\right)\,dt \\ &= \frac{1}{2}(a_n - ib_n) \text{ for } n > 0 \\ &= \frac{1}{2}(a_n + ib_n) \text{ for } n < 0 \end{aligned}$$

$$\Re[C_n] = \frac{1}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)\,dt$$

$$\Im[C_n] = -\frac{1}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)\,dt$$

Average value of the product of two periodic functions

$$\overline{f_1(t)f_2(t)} = \sum_{n=-\infty}^{\infty} (C_1)_n (C_2)_n$$

where the 'bar' means 'averaged over a complete period'.

$$\begin{aligned} \overline{\{f(t)\}^2} &= \sum_{n=-\infty}^{\infty} C_n C_{-n} = \sum_n C_n C_n^* = \sum_n |C_n|^2 \\ &= \frac{a_0^2}{4} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n^2 + b_n^2) \end{aligned}$$

Fourier transforms

For non-periodic functions:

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(t)\exp[-i\omega t]\,dt \qquad \text{Fourier transform}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\omega)\exp[i\omega t]\,d\omega \qquad \text{inverse Fourier transform}$$

The functions $f(t)$ and $\mathcal{F}(\omega)$ are called a Fourier transform pair. Some examples are given in Figure 2.10.

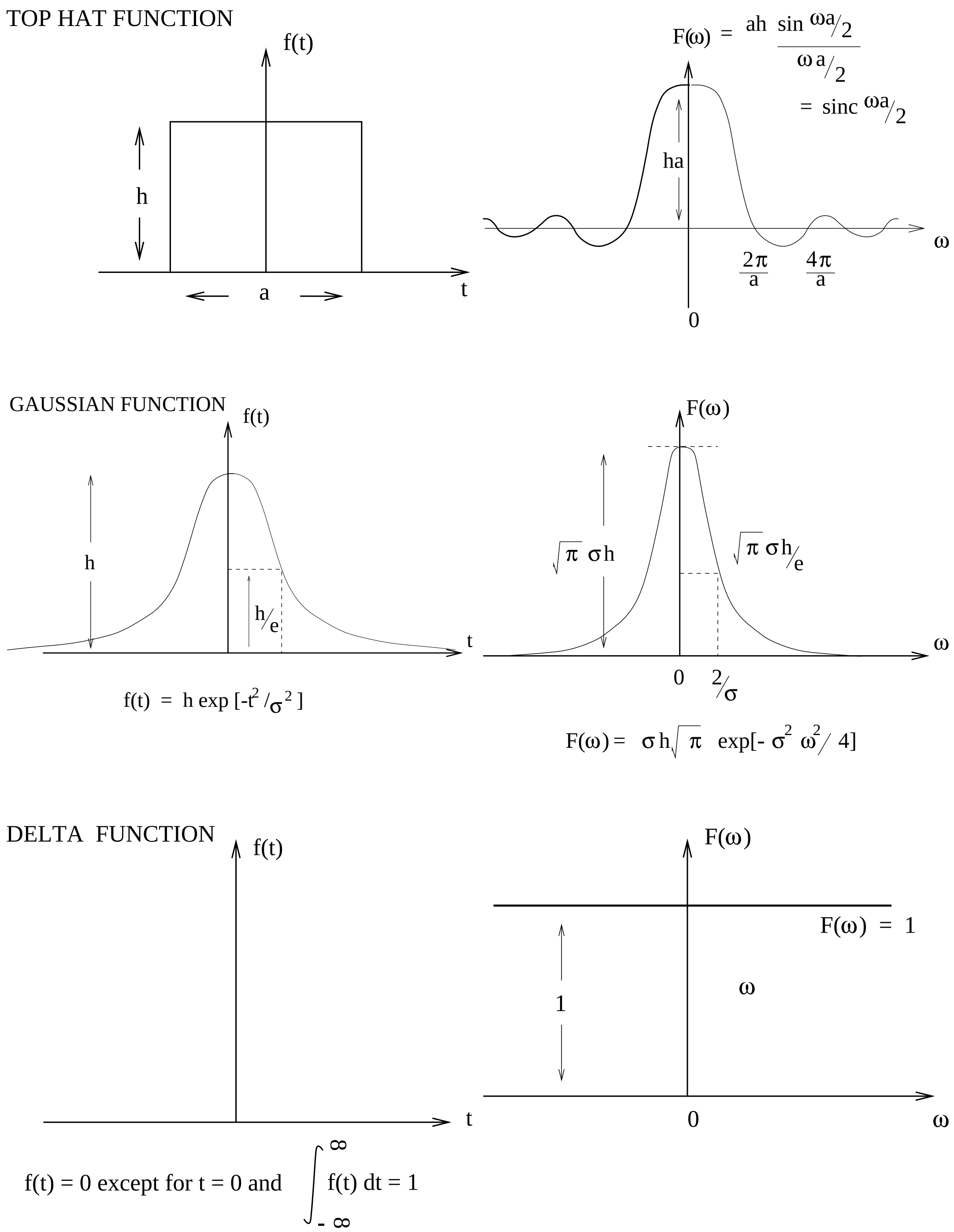


Figure 2.10: Examples of Fourier transforms

2.20 Statistics

Mean and RMS

If $x_1, x_2, \cdots x_n$ are n values of some quantity, then

The *arithmetic mean* is

$$\overline{x} = \frac{1}{n}\sum_i^n x_i = \frac{1}{n}\left(x_1 + x_2 + \cdots + x_n\right)$$

The *geometric mean* is

$$x_{\mathrm{GM}} = \left(\prod_i^n x_i\right)^{1/n} = \sqrt[n]{x_1 \times x_2 \times \cdots \times x_n}$$

The *root-mean-square* (RMS) is

$$RMS = \sqrt{\frac{1}{n}\sum_i x_i^2} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$$

Permutations

Permutations of n things taken r at a time

$$= n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} \equiv {}^{n}P_r$$

Combinations

Combinations of n things taken r at a time

$$= \frac{n!}{r!(n-r)!} \equiv {}^{n}C_r$$

Note that in a permutation the *order* in which selection is made is significant. Thus $A\underbrace{BC}DEFG$ is a different *permutation*, but the same *combination*, as $A\underbrace{CB}DEFG$.

Binomial distribution

Random variables can have *two* values A or B (e.g. heads or tails in the case of a toss of a coin). Let the probability of A occurring $= p$, the probability of B occurring $= (1-p) = q$. The probability of A occurring m times in n trials is

$$p_m(\mathrm{A}) = \frac{n!}{m!(n-m)!}p^m q^{n-m} \qquad \text{N.B. } 0! = 1$$

This is the m^{th} term in the binomial expansion of $(p+q)^n$.

Poisson distribution

Events occurring with average frequency ν but randomly distributed, in time for example (e.g. radioactive decay of nuclei, shot noise, goals, floods and horse kicks!). Probability of m events occurring in time interval T is

$$p_m(T) = \frac{(\nu T)^m}{m!} \exp(-\nu T)$$

Normal distribution (Gaussian)

For a continuous variable which is randomly distributed about a mean value μ with standard deviation σ, (e.g. random experimental errors of measurement), the probability that a measurement lies between x and $x + dx$ is

$$p(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx.$$

The quantity σ^2 is also known as the *variance*.

Given a sample of N measurements, the mean value of the sample, $\bar{x}$, is an unbiased estimator of μ, and the *sample standard deviation*, $s_{N-1} = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{N-1}}$ is an unbiased estimator of σ.

The *standard error on the mean* (SEM) is:

$$\sigma_{\bar{x}} = \frac{s_{N-1}}{\sqrt{N}}.$$

For large N, the difference $\bar{x} - \mu$ is itself a normal distribution with mean 0 and standard deviation $\sigma_{\bar{x}}$.

Selected Physics Formulae

3.1 Equations of Electromagnetism 48
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3.3 Thermodynamics and Statistical Physics 50

3.1 Equations of Electromagnetism

Definitions

$$
\begin{aligned}
\mathbf{B} &= \mu_{\mathrm{r}}\mu_0\mathbf{H} = \text{ magnetic field} \\
\mathbf{D} &= \epsilon_{\mathrm{r}}\epsilon_0\mathbf{E} = \text{ electric displacement} \\
\mathbf{E} &= \text{ electric field} \\
\mathbf{J} &= \text{ conduction current density} \\
\rho &= \text{ charge density}
\end{aligned}
$$

where ϵ_{r} and μ_{r} are the relative permittivity and permeability respectively. The definitions for **B** and **D** are for linear, isotropic, homogeneous media.

Biot-Savart law

$$
d\mathbf{B} = \frac{\mu_0}{4\pi}\; I\; \frac{d\mathbf{l}\times\mathbf{r}}{r^3}
$$

Maxwell's equations

These are four differential equations linking the space- and time-derivatives of the electromagnetic field quantities:

$$
\begin{aligned}
\nabla\cdot\mathbf{B} &= 0 \qquad & \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0 \\
\nabla\cdot\mathbf{D} &= \rho \qquad & \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} &= \mathbf{J}
\end{aligned}
$$

They can also be expressed in integral form:

$$
\begin{aligned}
\int_S \mathbf{B}\cdot d\mathbf{S} &= 0 \\
\int_S \mathbf{D}\cdot d\mathbf{S} &= \int_\tau \rho\, d\tau \\
\oint \mathbf{E}\cdot d\mathbf{l} &= -\int \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S} \\
\oint \mathbf{H}\cdot d\mathbf{l} &= \int_S \left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}
\end{aligned}
$$

$$
\begin{aligned}
\text{Energy density in an electric field} &= \frac{\epsilon_{\mathrm{r}}\epsilon_0 E^2}{2} \\
\text{Energy density in a magnetic field} &= \frac{\mu_{\mathrm{r}}\mu_0 H^2}{2} \\
\text{Velocity of plane waves in a linear, homogeneous and isotropic medium} \quad u &= (\mu_{\mathrm{r}}\epsilon_{\mathrm{r}}\mu_0\epsilon_0)^{-1/2}
\end{aligned}
$$

3.2 Equations of Relativistic Kinematics and Mechanics

Definitions

E = energy;
m_0 = rest mass
p = linear momentum
v = relative velocity of reference frames in x, x' direction
$\gamma = 1/\sqrt{(1 - v^2/c^2)}$

Lorentz transformations

Two inertial frames, S and S', are such that S' moves relative to S along the positive x direction, with velocity v as measured in S; the origins coincide at time $t = t' = 0$. The Lorentz transformations are:

$$x = \gamma(x' + vt') \qquad x' = \gamma(x - vt)$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$E^2 = (pc)^2 + (m_0c^2)^2$$

3.3 Thermodynamics and Statistical Physics

Maxwell speed distribution

For a gas, molecular weight m, in thermodynamic equilibrium at temperature T, the *fraction* $f(v)\,dv$ of molecules with speed in the range $v \to v + dv$ is

$$f(v)\,dv = 4\pi v^2 \left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} \exp\left[-\frac{mv^2}{2k_{\mathrm{B}}T}\right] dv$$

Thermodynamic variables

Helmholtz free energy:	$F = U - TS$
Gibbs function:	$G = U - TS + PV$
Enthalpy:	$H = U + PV$

Maxwell's thermodynamic relations

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V \qquad \left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P$$

$$\left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial S}{\partial P}\right)_T \qquad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

Statistical physics

Partition function: $$Z = \sum_i e^{-\beta E_i} = \sum_i e^{-E_i/kT}$$

Helmholtz free energy: $$F = -kT \ln Z$$

Entropy: $$S = k \ln \Omega = -k \sum_i p_i \ln p_i$$

Blackbody radiation

The energy emitted per unit area, per unit time, into unit solid angle, in the frequency range $\nu \to \nu + d\nu$ is

$$B(T,\nu) = \frac{2h\nu^3}{c^2} \frac{1}{(\exp[h\nu/kT] - 1)}$$

Quantum statistics

Distribution function:

$$f(E_{\mathrm{s}}) = \left[\exp\left(\frac{(E_{\mathrm{s}} - \mu)}{kT}\right) \pm 1\right]^{-1}$$

Fermi-Dirac: $+$ sign; $\mu = E_{\mathrm{F}}$
Bose-Einstein: $-$ sign
N.B. for photons $\mu = 0$.
For high energies $E \gg kT$ both distributions reduce to the classical Maxwell-Boltzmann distribution.

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