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Harmonic functions on manifolds

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0. Introduction

Around 1974 Yau ([Y1]) generalized the classical Liouville theorem of complex analysis to open manifolds with nonnegative Ricci curvature. Specifically, he proved that a positive harmonic function on such a manifold must be constant. Yau's Liouville theorem was considerably generalized by Cheng-Yau (see [CgY]) by means of a gradient estimate which implies the Harnack inequality. As a consequence of this gradient estimate (see [Cg]) on such a manifold, nonconstant harmonic functions must grow at least linearly. Some time later Yau made the following conjecture (see [Y3], [Y4], [Y5], and the survey article by Peter Li [L1]):

Conjecture 0.1 (Yau). *For an open manifold with nonnegative Ricci curvature the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional.*

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We recall the definition of polynomial growth.

Definition 0.2. For an open (complete noncompact) manifold, M^n, given a point $p \in M$ let r be the distance from p. Define $\mathcal{H}_d(M)$ to be the linear space of harmonic functions with order of growth at most d. This means that $u \in \mathcal{H}_d$ if u is harmonic and there exists some $C < \infty$ so that $|u| \leq C(1 + r^d)$.

The main result of this paper is the following.

THEOREM 0.3. *Conjecture* 0.1 *is true.*

We show Theorem 0.3 by giving an explicit bound on the dimension of $\mathcal{H}_d(M)$ depending only on n and d.

The case $n = 2$ of Conjecture 0.1 was done earlier by Peter Li and L. F. Tam [LT2] (in fact, for surfaces with finite total curvature). For another proof in the case $n = 2$ using nodal sets, see Harold Donnelly and Charles Fefferman [DF]. In [LT1], Peter Li and L. F. Tam settled the case $d = 1$. See the end of this introduction for additional references to results related to this conjecture.

In fact, Theorem 0.3 will be a consequence of a much more general result. In order to state this we need to recall the definition of some basic analytic inequalities on Riemannian manifolds.

Let (M^n, g) be a complete Riemannian manifold.

Doubling property. We say that M^n has the doubling property if there exists $C_D < \infty$ such that for all $p \in M^n$ and $r > 0$

$$\mathrm{Vol}\,(B_{2r}(p)) \leq C_D \,\mathrm{Vol}\,(B_r(p))\,. \tag{0.4}$$

Neumann-Poincaré inequality. We say that M^n satisfies a uniform Neumann-Poincaré inequality if there exists $C_N < \infty$ such that for all $p \in M^n$, $r > 0$ and $f \in W^{2,1}_{\mathrm{loc}}(M)$

$$\int_{B_r(p)} (f - \mathcal{A})^2 \leq C_N\, r^2 \int_{B_r(p)} |\nabla f|^2\,, \tag{0.5}$$

where $\mathcal{A} = \frac{1}{\mathrm{Vol}(B_r(p))} \int_{B_r(p)} f$.

Note that if M^n is a manifold with nonnegative Ricci curvature, then M^n has the doubling property with doubling constant $C_D = 2^n$ by the classical relative volume comparison theorem. Observe also that in [Bu] Peter Buser showed that these manifolds satisfy a uniform Neumann-Poincaré inequality with $C_N = C_N(n) < \infty$.

In [Y2] (see also [ScY]), Yau proved the following reverse Poincaré inequality. This is the only place where harmonicity is used.

Reverse Poincaré inequality. If $\Omega > 1$, u is harmonic on M^n, $p \in M$, and $r > 0$, then there exists $C_R = C_R(\Omega) < \infty$ such that

$$r^2 \int_{B_r(p)} |\nabla u|^2 \le C_R \int_{B_{\Omega r}(p)} u^2 . \tag{0.6}$$

We can now state our more general result.

THEOREM 0.7. *If M^n is an open manifold which has the doubling property and satisfies a uniform Neumann-Poincaré inequality, then for all $d > 0$, there exists $C = C(d, C_D, C_N) < \infty$ such that* $\dim \mathcal{H}_d(M) \le C$.

Remark 0.8. A particular consequence of our proof is that we need very little regularity of the metric (only enough to use Stokes' theorem in the proof of the reverse Poincaré inequality). For example, it suffices that the metric is locally Lipschitz. Of course in this case, harmonic is in the weak sense.

An immediate consequence of Theorem 0.7, in addition to Theorem 0.3, is the following.

COROLLARY 0.9. *If M^n is an open manifold which is quasi isometric to an open manifold with nonnegative Ricci curvature then* $\dim \mathcal{H}_d(M) < \infty$ *for all $d > 0$.*

Recall that two metric spaces are said to be quasi isometric if they are bilipschitz.

In [Ly], Lyons gave examples showing that, in general, Liouville properties are not stable under quasi isometric changes. However, Saloff-Coste [Sa1], and Grigor'yan [G] independently showed that if an open manifold has nonnegative Ricci curvature then any other quasi isometric manifold does not admit any nonconstant bounded harmonic function. In fact, Saloff-Coste [Sa2], and Grigor'yan [G], gave a lower bound in terms of n, C_D and C_N for the rate of growth of a nonconstant harmonic function on a manifold which has a uniform Neumann-Poincaré inequality and has the doubling property. This theorem of Saloff-Coste and Grigor'yan generalizes Yau's Liouville theorem mentioned earlier.

Prior to the results of [G], [Sa1], [Sa2], Lyons and Sullivan [LySu], and Guivarc'h [Gu], showed that a normal cover with nilpotent deck group of a closed manifold does not admit a nonconstant bounded (or more generally positive) harmonic function. As a generalization of this we have the following immediate consequence of Theorem 0.7. (See for instance [Gr] for the definition of polynomial growth of a finitely generated group.)

COROLLARY 0.10. *Suppose that M^n is a closed manifold and $\bar{M}$ is a normal cover of M with deck group of polynomial growth. For all $d > 0$,* $\dim \mathcal{H}_d(\bar{M}) < \infty$.

For elliptic operators in divergence form on $\mathbf{R}^n$ we get the following theorem.

THEOREM 0.11. *If L is a quasi uniformly elliptic operator in divergence form on $\mathbf{R}^n$ then the linear space of L-harmonic functions on $\mathbf{R}^n$, with polynomial growth of a fixed rate is finite dimensional.*

We refer to Section 6 for the exact definitions involved in the statement of Theorem 0.11. Here we will only note that quasi uniformly elliptic is more general than uniformly elliptic.

For area minimizing hypersurfaces, we have the following application of Theorem 0.7 (see §7 for the relevant definitions). In [CM6], we will give an extension of this result, and some geometric applications, to a more general class of minimal submanifolds.

THEOREM 0.12. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a complete area minimizing hypersurface without boundary, L a uniformly elliptic divergence form operator on Σ, and $d > 0$. If Σ has Euclidean volume growth, then $\dim \mathcal{H}_d(\Sigma, L) < \infty$.*

Finally, in the last section, Section 8, we discuss some of the applications of Theorem 0.7 to linear subelliptic second order operators.

In our earlier paper [CM2], we proved Conjecture 0.1 under the additional assumption that M^n has Euclidean volume growth.

Recall that M^n is said to have Euclidean volume growth if there exists $p \in M$ and a positive constant V such that $\text{Vol}(B_r(p)) \geq \text{V}r^n$ for all $r > 0$. Note that by the Bishop volume comparison theorem (see [BCr]) we have that $\text{Vol}(B_r(p)) \leq \text{V}_0^n(1)\, r^n$ for $r > 0$. Here, as in the rest of this paper, $\text{V}_\Lambda^n(r)$ denotes the volume of the geodesic ball of radius r in the n-dimensional space form of constant sectional curvature Λ.

The proof given in [CM2] relied in an essential way on the Euclidean cone structure at infinity of these manifolds which was proven in [ChC1]. If one only assumes that M^n has nonnegative Ricci curvature then tangent cones at infinity may not be Euclidean cones; see the examples given in [ChC2]. In addition to this essential difficulty of extending the approach of [CM2] to the general case, there is another key point. Namely, if M^n does not have Euclidean volume growth (the so-called collapsed case), then rescaled harmonic functions do not necessarily converge to harmonic functions on the tangent cones at infinity. In fact, the measured Hausdorff convergence poses an additional obstacle; see [ChC2] for the concept of measured convergence.

It should however be pointed out that in [CM2] we described, in addition, the asymptotics of these harmonic functions. This important structure question in the general case will not be dealt with in the present paper. See also the conjecture about quantitative strong unique continuation raised in [CM3].

We note that F. H. Lin [Ln], proved, independently of [CM2], Conjecture 0.1 under the additional assumption that M^n has Euclidean volume growth, quadratic curvature decay, and the tangent cone at infinity of M^n is unique. Note that tangent cones at infinity are not unique in general even under these additional assumptions; see for instance [ChC2].

Important contributions on this conjecture of Yau and related problems, in addition to the ones mentioned above, have been made by Cheeger-Colding-Minicozzi, Christiansen-Zworski, Donnelly-Fefferman, Kasue, Kazdan, Li, Li-Tam, Wang, and Wu (see [ChCM], [CnZ], [DF], [K1], [K2], [Kz], [L1], [L2], [LT1], [LT2], [W], and [Wu]).

Most of the results of this paper were announced in [CM4]. See also [CM5], [CM6] for related results.

Finally, in [CM5] we will show Weyl type asymptotic bounds (sharp in the rate of growth) for the dimension of $\mathcal{H}_d$ on manifolds which have a uniform Neumann-Poincaré inequality and have the doubling property.

1. Definitions and notation

From now on let M^n be an open n-dimensional manifold. Fix a point $p \in M$, and let $B_r = B_r(p)$ denote the ball of radius r centered at p. For a locally square integrable function u on M^n, we define the quantity

$$I_u(r) = \int_{B_r} u^2. \tag{1.1}$$

Note that this differs from the definition in [CM2].

Observe that by definition $I_u(r)$ is monotone nondecreasing for all functions u on M^n.

Further, we will use that for each r, $I_u(r)$ defines a quadratic form on the linear space of square integrable functions on B_r. The associated bilinear form J_r is given by

$$J_r(u, v) = \int_{B_r} u\, v, \tag{1.2}$$

for functions u and v. Note, in fact, that (1.2) defines an inner product on the space of square integrable functions on B_r and a positive semi-definite bilinear form on $L^2_{\text{loc}}(M)$.

If M^n has polynomial volume growth of degree at most $n_0 > 0$, that is,

$$\text{Vol}(B_r(p)) \leq \text{V}\, (r^{n_0} + 1) \tag{1.3}$$

for some $\text{V} > 0$, then for $d > 0$ we let $\mathcal{P}_d(M^n)$ denote the linear space of functions, u, on M such that $u \in \mathcal{P}_d(M^n)$ if there exists some $K > 0$ so that

$$I_u(r) \leq K\, (r^{2d+n_0} + 1). \tag{1.4}$$

Further, we let $\mathcal{HP}_d(M)$ denote the space of functions u on M such that $u \in \mathcal{P}_d(M^n)$ and u is harmonic.

Note in particular that if M^n has nonnegative Ricci curvature, then by the Bishop volume comparison theorem,

$$\mathrm{Vol}(B_r(p)) \leq \mathrm{V}_0^n(1)\, r^n. \tag{1.5}$$

More generally, if M^n has the doubling property with doubling constant C_D then for any integer $s > 0$

$$\mathrm{Vol}(B_{2^s}(p)) \leq C_D^s \mathrm{Vol}(B_1(p)). \tag{1.6}$$

Hence M^n has polynomial volume growth of degree at most

$$n_0 = \frac{\log C_D}{\log 2} \quad \text{and} \quad \mathrm{V} = C_D\, \mathrm{Vol}(B_1(p)). \tag{1.7}$$

If $u \in \mathcal{H}_d$ then there exists a constant C such that $|u| \leq C(r^d + 1)$. Using this and (1.7) we obtain

$$I_u(r) \leq C^2(r^d+1)^2 \mathrm{V}\,(r^{n_0}+1) \leq 4C^2 \mathrm{V}\,(r^{2d+n_0}+1). \tag{1.8}$$

It follows in particular that $\mathcal{H}_d \subset \mathcal{HP}_d \subset \mathcal{P}_d$.

It is clear that, in the case where harmonic functions on M^n satisfy a meanvalue inequality (which is the case when M^n has the doubling property and satisfies a uniform Neumann-Poincaré inequality [G], [Sa2]), $\mathcal{H}_d \subset \mathcal{HP}_d \subset \mathcal{H}_{d+n_0}$. However neither the meanvalue inequality nor the Harnack inequality is ever used in our proof; see Section 6.

2. Bounding the number of orthonormal functions with bounded energy

In this section we will give bounds on the dimension of the space of L^2-orthonormal functions with a given energy bound under very general conditions.

Since the arguments involved are so flexible, we will state the main result of this section, Proposition 2.5, for complete metric spaces equipped with a locally finite positive Borel measure and a notion of *gradient squared* of a function. In the applications, we will have a manifold structure and it will be clear what is meant by the energy of a function.

Definition 2.1. Let (Y, d, μ) be a complete metric space with a locally finite positive Borel measure μ and for $p \in Y$ let $B_r = B_r(p)$ be a metric ball. Let $W^{2,1}(B_r)$ be the $(2,1)$-Sobolev space on B_r. We define the set $W_{k^2}(B_r)$ to be $\{u \in W^{2,1}(B_r) \mid \int_{B_r} u^2 + r^2 \int_{B_r} |\nabla u|^2 \leq k^2\}$.

Definition 2.2 (η-almost orthonormal functions). Let (X,μ) be a measure space with a probability measure, μ, and suppose that $f_1,\dots,f_m$ are L^2 functions on X. We say that the f_i are η-almost orthonormal if

$$\int_X f_i^2 = 1, \tag{2.3}$$

and for $i \neq j$

$$\left| \int_X f_i f_j \right| < \eta. \tag{2.4}$$

In the next proposition, we think of r as the scaling factor and k as the parameter.

PROPOSITION 2.5. *Let (Y,d,μ) be a complete metric space with a locally finite positive Borel measure μ. For $p \in Y$ let $X = B_r(p)$ be a metric ball with $\mu(X) = 1$. Suppose that Y satisfies a uniform Neumann-Poincaré inequality with constant C_N and has the doubling property with constant C_D. Given $k > 0$, there exist at most $\mathcal{N} - 1$ $\frac{1}{2}$-almost orthonormal (on $B_r(p)$) functions in $W_{k^2}(B_{2r}(p))$ where $\mathcal{N} = \mathcal{N}(k^2,C_D,C_N)$.*

Proof. Let $f_1,\dots,f_n$ be such functions. We let $B_{r_0}(x_1),\dots,B_{r_0}(x_\nu)$ be a maximal set of disjoint balls with centers in X and with radius

$$r_0 = \frac{r}{20\, C_D^{\frac{3}{2}}\, C_N^{\frac{1}{2}}\, k} = \frac{r}{c\,k}. \tag{2.6}$$

First, we note that since $\mu(X) = 1$,

$$\frac{1}{C_D} \leq \mu(B_r(x_j)) \leq \mu(B_{2r}(p)) \leq C_D; \tag{2.7}$$

therefore

$$\frac{1}{C_D} \leq \mu(B_r(x_j)) \leq C_D^{\frac{\log ck}{\log 2}+1} \mu(B_{r_0}(x_j)). \tag{2.8}$$

Since the $B_{r_0}(x_j)$ are disjoint, (2.8) implies

$$\nu \leq C_D^{\frac{\log ck}{\log 2}+3}. \tag{2.9}$$

It follows from maximality that double the balls covers X. We now partition X into ν (disjoint) subsets $S_1,\dots,S_\nu$, where $B_{r_0}(x_j)\cap X \subset S_j \subset B_{2r_0}(x_j)$.

Let $\eta(y)$ be the number of j such that $y \in B_{2r_0}(x_j)$ and let $\bar{C} = \max_Y \eta$. If $y \in \cap_{m=1}^{\eta(y)} B_{2r_0}(x_{j_m})$, it follows that $B_{3r_0}(y)$ contains all of the balls $B_{r_0}(x_{j_1})$, $B_{r_0}(x_{j_2})$, $\dots$, $B_{r_0}(x_{j_{\eta(y)}})$. Since these are disjoint,

$$\sum_{m=1}^{\eta(y)} \mu(B_{r_0}(x_{j_m})) \leq \mu(B_{3r_0}(y)). \tag{2.10}$$

Also, for each $m = 1, \dots, \eta(y)$, the doubling property together with the triangle inequality yields

$$\mu(B_{3r_0}(y)) \leq \mu(B_{5r_0}(x_{j_m})) \leq \mu(B_{8r_0}(x_{j_m})) \leq C_D^3\, \mu(B_{r_0}(x_{j_m})). \tag{2.11}$$

Combining (2.10) and (2.11), we see that $\eta(y) \leq C_D^3$; hence,

$$\bar{C} \leq C_D^3. \tag{2.12}$$

Let (P, μ') denote the (finite) set of points $\{x_j\}$ with probability measure μ', where $\mu'(x_j) = \mu(S_j)$. We can therefore identify functions on P with functions on X which are constant on each S_j.

Set

$$\mathcal{A}_{i,j} = \frac{1}{\mu(B_{2r_0}(x_j))} \int_{B_{2r_0}(x_j)} f_i. \tag{2.13}$$

It follows from the Cauchy-Schwarz inequality, together with $f_i \in W_{k^2}(B_{2r})$, and (2.8) that

$$|\mathcal{A}_{i,j}|^2 \leq \frac{1}{\mu(B_{2r_0}(x_j))} \int_{B_{2r_0}(x_j)} f_i^2 \leq \frac{1}{\mu(B_{2r_0}(x_j))} \int_{B_{2r}(p)} f_i^2 \leq C_D^{\frac{\log ck}{\log 2}+1} k^2. \tag{2.14}$$

Let Λ denote the set $\{\frac{s}{10} \mid s \in \mathbf{Z}, |s| \leq k\, C_D^{\frac{\log ck}{2\log 2}+\frac{1}{2}}\}$. We will now construct an injective map $\mathcal{M}$ from the orthonormal set of functions, $\{f_i\}$, to the set of maps from P (the points $\{x_j\}$) to Λ: let $\mathcal{M}(f_i)(x_j) \in \Lambda$ be any closest point of Λ to $\mathcal{A}_{i,j}$ (there are at most two possibilities). Note that by (2.14)

$$|\mathcal{A}_{i,j} - \mathcal{M}(f_i)(x_j)|^2 \leq \frac{1}{400}. \tag{2.15}$$

By the Neumann-Poincaré inequality and (2.15),

$$\begin{aligned} \int_{S_j} |f_i - \mathcal{M}(f_i)(x_j)|^2 &\leq 2 \int_{B_{2r_0}(x_j)} |f_i - \mathcal{A}_{i,j}|^2 + 2 \int_{S_j} |\mathcal{A}_{i,j} - \mathcal{M}(f_i)(x_j)|^2 \\ &\leq 8\, r_0^2\, C_N \int_{B_{2r_0}(x_j)} |\nabla f_i|^2 + 2\, \mu(S_j)\, |\mathcal{A}_{i,j} - \mathcal{M}(f_i)(x_j)|^2 \\ &\leq 8\, r_0^2\, C_N \int_{B_{2r_0}(x_j)} |\nabla f_i|^2 + \mu(S_j)\, \frac{1}{200}; \end{aligned} \tag{2.16}$$

hence, since $f_i \in W_{k^2}(B_{2r})$,

$$\begin{aligned} \int_X |f_i - \mathcal{M}(f_i)|^2 &\leq \bar{C}\, 8\, r_0^2\, C_N \int_{B_{2r}(p)} |\nabla f_i|^2 + \frac{1}{200} \\ &\leq 2\, \bar{C}\, r_0^2\, C_N\, k^2\, r^{-2} + \frac{1}{200} \leq \frac{1}{100}. \end{aligned} \tag{2.17}$$

By the triangle inequality, together with (2.17), for $i \neq j$ we get,

$$\left(\int_X |f_i - f_j|^2\right)^{\frac{1}{2}} - \left(\int_X |\mathcal{M}(f_i) - \mathcal{M}(f_j)|^2\right)^{\frac{1}{2}} \tag{2.18}$$
$$\leq \left(\int_X |f_i - \mathcal{M}(f_i)|^2\right)^{\frac{1}{2}} + \left(\int_X |f_j - \mathcal{M}(f_j)|^2\right)^{\frac{1}{2}} \leq \frac{1}{5}.$$

Furthermore, since the f_i are $\frac{1}{2}$-almost orthonormal, we see that

$$\int_X |f_i|^2 = 1, \tag{2.19}$$

and for $i \neq j$,

$$\left|\int_X f_i f_j\right| < \frac{1}{2}. \tag{2.20}$$

Consequently, for $i \neq j$,

$$1 < \left(\int_X |f_i - f_j|^2\right)^{\frac{1}{2}}. \tag{2.21}$$

Combining (2.18) and (2.21), for $i \neq j$ we obtain,

$$0 < \frac{4}{5} < \left(\int_X |\mathcal{M}(f_i) - \mathcal{M}(f_j)|^2\right)^{\frac{1}{2}}. \tag{2.22}$$

Hence, $\mathcal{M}$ is injective. The proposition follows by counting the cardinality of the set of maps between two finite point sets (in fact, $\mathcal{N} \leq (20\, k\, C_D^{\frac{\log ck}{2\log 2}+\frac{1}{2}} + 1)^{\nu}$, where $c = 20\, C_D^{\frac{3}{2}}\, C_N^{\frac{1}{2}}$). □

Remark 2.23. Note that in Proposition 2.5 we need only assume that the doubling property and the weak Neumann-Poincaré inequality hold for $s \geq \frac{r}{20k}\, C_D^{-\frac{3}{2}}\, C_N^{-\frac{1}{2}}$. That is, we need only assume that for $y \in Y$, $s \geq \frac{r}{20k}\, C_D^{-\frac{3}{2}}\, C_N^{-\frac{1}{2}}$, and $f \in W^{2,1}(B_{2r})$,

$$\mathrm{Vol}(B_{2s}(y)) \leq C_D\, \mathrm{Vol}(B_s(y)), \tag{2.24}$$

and the weak Neumann-Poincaré inequality

$$\int_{B_s(y)} (f - \mathcal{A})^2 \leq C_N\, s^2 \int_{B_{\frac{3}{2}s}(y)} |\nabla f|^2. \tag{2.25}$$

3. Growth properties of functions of one variable

In this section, we will slightly generalize some elementary results from [CM2] for functions of a single variable with polynomial growth. These results show the existence of infinitely many annuli with bounded growth.

The basic idea is that for any set of $2k$ functions with polynomial growth of degree at most d, we can find a subset of k functions and infinitely many annuli for which the degree of growth from the inner radius to the outer radius of each of the functions in the subset is at most $2d$.

We will think of this elementary fact as a weak version of a uniform Harnack inequality for a set of functions with polynomial growth.

In the next section, we will produce functions of one variable with the properties of the functions of this section.

LEMMA 3.1 (cf. Lemma 7.1 of [CM2]). *Suppose that $f_1,\dots,f_l$ are nonnegative nondecreasing functions on $(0,\infty)$ such that none of the f_i vanishes identically, and for some $d,K>0$ and all i,*

$$f_i(r) \leq K(r^d+1). \tag{3.2}$$

For all $\Omega > 1$, $k \leq l$, and any $C > \Omega^{\frac{ld}{l-k+1}}$, there exist k of these functions $f_{\alpha_1},\dots,f_{\alpha_k}$ and infinitely many integers, $m \geq 1$, such that for $i=1,\dots,k$,

$$f_{\alpha_i}(\Omega^{m+1}) \leq C f_{\alpha_i}(\Omega^m). \tag{3.3}$$

Proof. Since the functions are nondecreasing and none of them vanish identically, we may suppose that for some $R>0$ and any $r>R$, $f_i(r)>0$ for all i.

We will show that there are infinitely many m such that there is some rank k subset of $\{f_i\}$ (where the subset could vary with m) satisfying (3.3). This will suffice to prove the lemma; since there are only finitely many rank k subsets of the l functions, one of these rank k subsets must have been repeated infinitely often.

Set for $r>R$,

$$g(r) = \prod_{i=1}^{l} f_i(r); \tag{3.4}$$

note that

$$g(r) \leq K^l (r^d+1)^l, \tag{3.5}$$

and g is a positive nondecreasing function. Assume that there are only finitely many such $m \geq \frac{\log R}{\log \Omega}$. Let m_0-1 be the largest such m; for all $j \geq 1$ we have that

$$g(\Omega^{m_0+j}) > C^{l-k+1} g(\Omega^{m_0+j-1}). \tag{3.6}$$

Iterating this, we obtain

$$g(\Omega^{m_0+j}) > C^{j(l-k+1)} g(\Omega^{m_0}). \tag{3.7}$$

From the upper bound on g, equation (3.5), for all $j > m_0$ we see that

$$\tilde{c}\left(\Omega^j\right)^{dl} \geq C^{j(l-k+1)} g(\Omega^{m_0}), \tag{3.8}$$

where $\tilde{c} = \tilde{c}(l, m_0, \Omega, K)$. Since $C > \Omega^{\frac{ld}{l-k+1}}$ and $g(\Omega^{m_0}) > 0$ this is impossible, yielding the contradiction. □

COROLLARY 3.9 (Weak version of a uniform Harnack inequality for a set of functions with polynomial growth (cf. Cor. 7.9 of [CM2])). *Suppose that* $f_1, \ldots, f_{2k}$ *are nonnegative nondecreasing functions on* $(0,\infty)$ *such that none of the* f_i *vanishes identically and for some* $d, K > 0$ *and all* i,

$$f_i(r) \leq K(r^d + 1). \tag{3.10}$$

For all $\Omega > 1$, *there exist* k *functions* $f_{\alpha_1}, \ldots, f_{\alpha_k}$ *and infinitely many integers,* $m \geq 1$, *such that for* $i = 1, \ldots, k$,

$$f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{2d} f_{\alpha_i}(\Omega^m). \tag{3.11}$$

Proof. This is an immediate consequence of Lemma 3.1 with $l = 2k$. □

Remark 3.12. That the upper bound in Corollary 3.9 for the degree of growth of $f_{\alpha_1}, \ldots, f_{\alpha_k}$ from Ω^m to Ω^{m+1} can be made independent of k, Ω, and K is crucial for the applications.

In the proof of Theorem 0.7, we will use Corollary 3.9 to get an annulus on which we have some growth control (see Cor. 4.14). Henceforth, we will work on an annulus where we have this control on the growth.

4. Constructing functions with good properties from given ones

From now on M^n will denote an open manifold with at most polynomial volume growth; i.e.,

$$\mathrm{Vol}(B_r(p)) \leq \mathrm{V}(r^{n_0} + 1), \tag{4.1}$$

for some $\mathrm{V} > 0$ and some $n_0 > 0$.

In this section, given a linearly independent set of functions in $\mathcal{P}_d(M)$, we will construct functions of one variable which reflect the growth and independence properties of this set. In particular, we shall establish that these functions of one variable satisfy the conditions of Section 3.

We begin with two definitions. The first constructs the functions whose growth properties will be studied.

Definition 4.2 ($w_{i,r}$ and f_i). Suppose that $u_1, \dots, u_k$ are linearly independent functions on M. For each $r > 0$ we will now define an orthogonal spanning set $w_{i,r}$ with respect to the inner product

$$J_r(u,v) = \int_{B_r} u\,v, \tag{4.3}$$

and functions f_i. Set $w_{1,r} = w_1 = u_1$ and $f_1(r) = I_{u_1}(r)$. Define $w_{i,r}$ by requiring it to be orthogonal to $u_j|B_r$ for $j < i$ with respect to the inner product (4.3); hence we have

$$u_i = \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j + w_{i,r}. \tag{4.4}$$

Note that $\lambda_{ij}(r)$ is not uniquely defined if $u_i|B_r$ are linearly dependent. However, since u_i are linearly independent on M, for r sufficiently large we see that $\lambda_{i,j}(r)$ will be uniquely defined. In any case, for all $r > 0$ $w_{i,r}$ is well defined and so is the following quantity (which is in fact positive for r sufficiently large)

$$f_i(r) = \int_{B_r} w_{i,r}^2. \tag{4.5}$$

Definition 4.6 (Barrier). We will say that a function f is a (left) barrier for a function g at r if $f(r) = g(r)$ and for $s < r$, $f(s) \leq g(s)$.

We will use the barrier property to conclude that the growth of g from s to r is not larger than the growth of f from s to r.

In the next proposition (cf. Prop. 8.6 of [CM2]), we will establish some key properties of the functions f_i from Definition 4.2.

PROPOSITION 4.7 (Properties of f_i). *If $u_1, \dots, u_k \in \mathcal{P}_d(M)$ are linearly independent, then the f_i from Definition* 4.2 *have the following four properties. There exists a constant $K > 0$ (depending on the set $\{u_i\}$) such that*

$$f_i(r) \leq K(r^{2d+n_0} + 1), \tag{4.8}$$

$$f_i \text{ is a nondecreasing function,} \tag{4.9}$$

$$f_i \text{ is nonnegative and positive for } r \text{ sufficiently large,} \tag{4.10}$$

and

$$f_i \text{ is a barrier for } I_{w_{i,r}} \text{ at } r. \tag{4.11}$$

Proof. First note that

$$f_i(r) \leq I_{u_i}(r), \tag{4.12}$$

which implies (4.8). Furthermore, for $s < r$

$$
\begin{aligned}
(4.13) \qquad f_i(s) &= \int_{B_s} \left| u_i - \sum_{j=1}^{i-1} \lambda_{ji}(s)\, u_j \right|^2 = I_{w_{i,s}}(s) \\
&\leq \int_{B_s} \left| u_i - \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j \right|^2 = I_{w_{i,r}}(s) \\
&\leq \int_{B_r} \left| u_i - \sum_{j=1}^{i-1} \lambda_{ji}(r)\, u_j \right|^2 = I_{w_{i,r}}(r) \\
&= f_i(r),
\end{aligned}
$$

where the first inequality of (4.13) follows from the orthogonality of $w_{i,r}$ to u_j for $j < i$, and the second inequality of (4.13) follows from the monotonicity of I. From (4.13), and since u_i are linearly independent, we get (4.9) and (4.10).

By (4.13), we also see that f_i is a barrier for $I_{w_{i,r}}$ at r; this shows (4.11). □

The following corollary of Corollary 3.9 and the properties of the f_i will be used to get control of the growth in the proof of Theorem 0.7.

COROLLARY 4.14. *Suppose that $u_1, \ldots, u_{2k} \in \mathcal{P}_d(M)$ are linearly independent. Given $\Omega > 1$ and $m_0 > 0$, there exist $m \geq m_0$ and a subset $f_{\alpha_1}, \ldots, f_{\alpha_k}$ such that for $i = 1, \ldots, k$*

$$(4.15) \qquad 0 < f_{\alpha_i}(\Omega^{m+1}) \leq \Omega^{4d+2n_0} f_{\alpha_i}(\Omega^m).$$

Proof. This follows immediately by combining Corollary 3.9 and Proposition 4.7. □

Note in particular that the functions $w_{\alpha_i,\Omega^{m+1}}|B_{\Omega^{m+1}}$ given by Corollary 4.14 are linearly independent.

Further we have the following:

PROPOSITION 4.16. *Suppose that $u_1, \ldots, u_{2k} \in \mathcal{P}_d(M)$ are linearly independent. Given $\Omega > 1$ and $m_0 > 0$, there exist $m \geq m_0$, $\ell \geq \frac{k}{2}\,\Omega^{-4d-2n_0}$, and functions $v_1, \ldots, v_\ell$ in the linear span of u_i such that*

$$(4.17) \qquad 2\,\Omega^{4d+2n_0} = 2\,\Omega^{4d+2n_0}\, I_{v_i}(\Omega^m) \geq I_{v_i}(\Omega^{m+1}),$$

and

$$(4.18) \qquad J_{\Omega^m}(v_i, v_j) = \delta_{i,j}.$$

Proof. Let m and $w_{\alpha_i,\Omega^{m+1}}$ be given by Corollary 4.14.

Consider the k-dimensional linear space spanned by the functions $w_{\alpha_i,\Omega^{m+1}}$ with inner product $J_{\Omega^{m+1}}$. On this space there is also the positive semi-definite bilinear form J_{Ω^m}. Let $v_1,\dots,v_k$ be an orthonormal basis for $J_{\Omega^{m+1}}$ which diagonalizes J_{Ω^m}. We will now evaluate the trace of J_{Ω^m} with respect to these two bases. First with respect to the orthogonal basis $w_{\alpha_i,\Omega^{m+1}}$, by (4.11) (the barrier property) and (4.15), we get

$$\sum_{i=1}^{k}\frac{I_{w_{\alpha_i},\Omega^{m+1}}(\Omega^m)}{I_{w_{\alpha_i},\Omega^{m+1}}(\Omega^{m+1})}\geq k\,\Omega^{-4d-2n_0}. \tag{4.19}$$

Since the trace is independent of the choice of basis we get when we evaluate this on the orthonormal basis v_i,

$$\sum_{i=1}^{k} I_{v_i}(\Omega^m)\geq k\,\Omega^{-4d-2n_0}. \tag{4.20}$$

Combining this with

$$1\geq I_{v_i}(\Omega^m)\geq 0, \tag{4.21}$$

which follows from the monotonicity of I, we get that there exist at least $\ell\geq\frac{k}{2}\,\Omega^{-4d-2n_0}$ of the v_i such that for each of these

$$1=I_{v_i}(\Omega^{m+1})\geq I_{v_i}(\Omega^m)\geq\frac{1}{2}\,\Omega^{-4d-2n_0}. \tag{4.22}$$

With slight abuse of notation we renormalize these ℓ functions to have

$$I_{v_i}(\Omega^m)=1, \tag{4.23}$$

and denote them by $v_1,\dots,v_\ell$. This shows the proposition. □

5. Harmonic functions with polynomial growth

As before, let M be an open n-dimensional Riemannian manifold which has a uniform Neumann-Poincaré inequality with constant C_N and has the doubling property with doubling constant C_D. Let $p\in M$ be fixed.

Proof (of Th. 0.7). Fix $\Omega>2$. For $m>0$ let $X_m=B_{\Omega^m}(p)$. Let $\mu=\mu_m$ be the measure on M given by

$$\mu(A)=\frac{\mathrm{Vol}(A)}{\mathrm{Vol}(X_m)}. \tag{5.1}$$

Applying Proposition 2.5 to X_m with the measure μ, we get a constant $\mathcal{N}=\mathcal{N}((8\,C_R+2)\Omega^{4d+2n_0},C_D,C_N)$ (independent of m) such that any set of $L^2(X_m)$-orthonormal functions in $W_{(8\,C_R+2)\Omega^{4d+2n_0}}(B_{2\Omega^m}(p))$ has at most $\mathcal{N}-1$ elements. Here $C_R=C_R(\frac{\Omega}{2})>0$.

We will show that if $\dim \mathcal{HP}_d \geq \mathcal{N}_0 = 4\Omega^{4d+2n_0}\mathcal{N}$, then we can for all $m_0 > 0$ find an $m > m_0$ and $\mathcal{N}$ functions in $\mathcal{HP}_d \cap W_{(8\,C_R+2)\Omega^{4d+2n_0}}(B_{2\Omega^m}(p))$ which are $L^2(X_m)$-orthonormal. This contradiction yields the result.

Suppose therefore that $u_1, \ldots, u_{\mathcal{N}_0} \in \mathcal{HP}_d(M)$ are linearly independent.

Given this set of harmonic functions, we will now proceed to construct for all $m_0 > 0$ an $m > m_0$ and a set, $\{v_i\}$, of orthonormal harmonic functions on the ball B_{Ω^m}. Note also that for different m the v_i may be different. By Proposition 4.16 together with the reverse Poincaré inequality we have that there exist some $m \geq m_0$ and $v_1, \ldots, v_\ell$ harmonic functions with

$$\ell \geq \frac{1}{4}\mathcal{N}_0\,\Omega^{-4d-2n_0} = \mathcal{N}, \tag{5.2}$$

$$\int_{B_{\Omega^m}} v_i\, v_j = \delta_{i,j}, \tag{5.3}$$

$$\int_{B_{\Omega^{m+1}}} v_i^2 \leq 2\,\Omega^{4d+2n_0}, \tag{5.4}$$

and

$$\int_{B_{2\Omega^m}} |\nabla v_i|^2 \leq C_R\,\Omega^{-2m} \int_{B_{\Omega^{m+1}}} v_i^2 \leq 2\,C_R\,\Omega^{4d+2n_0-2m}. \tag{5.5}$$

Note, finally, that since I is monotone and $\Omega > 2$, (5.4) implies that

$$\int_{B_{2\Omega^m}} v_i^2 \leq 2\,\Omega^{4d+2n_0}. \tag{5.6}$$

The theorem therefore follows from Proposition 2.5. □

Remark 5.7. Note that while our construction produced infinitely many good balls (i.e. infinitely many m), we needed just a single ball for the proof.

Remark 5.8. Observe also that the only place where we used that the functions u_i are harmonic was in the application of the reverse Poincaré inequality, that is (5.5). In fact, our result applies for polynomial growth L-harmonic functions whenever these satisfy a reverse Poincaré inequality. For example, the result holds for $L = L_0 + X + V$ where L_0 is a second order uniformly elliptic divergence form operator, X is a vector field with $\lim_{r\to\infty} r\,|X| \to 0$, and V is a nonpositive function; compare Section 6.

Remark 5.9. Finally we note that we need only assume that the uniform Neumann-Poincaré inequality holds for harmonic functions.

6. Quasi uniformly elliptic operators

In the proof of Theorem 0.7, very little was used about the harmonic functions themselves. The necessary ingredients were the geometry of the manifold (i.e., the doubling property and the Neumann-Poincaré inequality), the linearity of the space of solutions, and Yau's reverse Poincaré inequality. It is then clear from the proof of the reverse Poincaré inequality that the arguments carry over for any uniformly elliptic linear second order divergence form operator L with symmetric coefficients. Hence Theorem 0.7 holds for such L.

In this section, we will show that, in fact, our arguments work for a more general class of operators.

We say that a second order elliptic operator, L, on $\mathbf{R}^n$ is quasi uniformly elliptic and in divergence form if for some measurable functions $a_{i,j} = a_{j,i}$

$$Lu = \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_j}\right), \tag{6.1}$$

with

$$c_1\, I \leq (a_{i,j}(x)), \tag{6.2}$$

and

$$\sum_{i,j} a_{i,j}(x)\, x_i\, x_j \leq c_2\, |x|^2, \tag{6.3}$$

for positive constants c_1 and c_2.

Note that these operators are more general than uniformly elliptic operators in divergence form as the following example shows.

Example 6.4 (Quasi uniformly elliptic but not uniformly elliptic). Fix $c_2 \geq c_1 > 0$ and let λ and μ be any two measurable functions on $\mathbf{R}^2$ with $c_1 \leq \lambda$ and $c_1 \leq \mu \leq c_2$. Set

$$Lu = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_j}\right), \tag{6.5}$$

where

$$a_{1,2}(x) = a_{2,1}(x) = \frac{(\mu-\lambda)x_1x_2}{x_1^2+x_2^2}, \tag{6.6}$$

$$a_{1,1}(x) = \frac{\mu x_1^2 + \lambda x_2^2}{x_1^2+x_2^2} \quad \text{and} \quad a_{2,2}(x) = \frac{\mu x_2^2 + \lambda x_1^2}{x_1^2+x_2^2}. \tag{6.7}$$

Then L is quasi uniformly elliptic with constants c_1 and c_2. However it is only uniformly elliptic if λ is bounded.

We say that u is L-harmonic if $Lu = 0$ in the weak sense. Further we define for $d > 0$ in an obvious way the space $\mathcal{H}_d(\mathbf{R}^n, L)$. With this notation we have the following theorem (Th.0.11 from the Introduction).

THEOREM 6.8. *If L is a quasi uniformly elliptic operator in divergence form on $\mathbf{R}^n$, then for all $d > 0$, $\dim \mathcal{H}_d(\mathbf{R}^n, L) \leq C(d, \frac{c_2}{c_1}, n) < \infty$.*

Proof. From the proof of Theorem 0.7 it follows that it suffices to show that any L harmonic function on $\mathbf{R}^n$ satisfies a reverse Poincaré inequality and that the energy of L bounds the energy of Δ. The second fact follows immediately from (6.2).

To see that L satisfies a reverse Poincaré inequality, let ϕ be the radial cut-off function with $\phi|B_r \equiv 1$, support of ϕ contained in B_{2r}, and with $|\nabla\phi| \leq \frac{1}{r}$. Here $B_r = B_r(0)$. Observe that by the Cauchy-Schwarz inequality we have for any function v

$$\left|\sum_{i,j} a_{i,j} \frac{\partial\phi}{\partial x_i}\frac{\partial v}{\partial x_j}\right| \leq \left(\sum_{i,j} a_{i,j}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\right)^{\frac{1}{2}} \left(\sum_{i,j} a_{i,j}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\right)^{\frac{1}{2}}. \tag{6.9}$$

From (6.9), integration by parts, and the Cauchy-Schwarz inequality, for any L-harmonic function u we get,

$$\begin{aligned} 0 &= \int \phi^2\, u\, Lu = -2\sum_{i,j}\int \phi\, u\, a_{i,j}\,\frac{\partial\phi}{\partial x_i}\,\frac{\partial u}{\partial x_j} - \sum_{i,j}\int a_{i,j}\,\phi^2\,\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \\ &\leq 2\left(\int_{B_{2r}\setminus B_r} u^2 \sum_{i,j} a_{i,j}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\right)^{\frac{1}{2}} \left(\sum_{i,j}\int \phi^2\, a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)^{\frac{1}{2}} \\ &\quad -\sum_{i,j}\int \phi^2\, a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \end{aligned} \tag{6.10}$$

By (6.3), since $r\,|x|\,\frac{\partial\phi}{\partial x_i} = -x_i$ on $B_{2r}\setminus B_r$, (6.10) yields
(6.11)

$$\sum_{i,j}\int \phi^2\, a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \leq \frac{2\sqrt{c_2}}{r}\left(\int_{B_{2r}\setminus B_r} u^2\right)^{\frac{1}{2}} \left(\sum_{i,j}\int \phi^2\, a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)^{\frac{1}{2}}.$$

From this we see that

$$\sum_{i,j}\int_{B_r} a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \leq \sum_{i,j}\int \phi^2\, a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \leq \frac{4\,c_2}{r^2}\int_{B_{2r}\setminus B_r} u^2. \tag{6.12}$$

This shows the reverse Poincaré inequality for the operator L, and completes the proof of the theorem. □

Remark 6.13. It is interesting to note that there are well-known examples of Plis ([Pl]) that show that for any $0 < \alpha < 1$ there are uniformly elliptic operators in divergence form on $\mathbf{R}^n$ with C^α coefficients that do not satisfy the unique continuation property. Therefore, one cannot get a uniform frequency bound in this situation (cf. [CM2] and [CM3]). As a result, it is not possible to prove Theorem 0.11 by bounding the order of vanishing at a single point.

Next we will discuss quasi uniformly elliptic operators on general Riemannian manifolds.

Suppose that (M^n, g) is a Riemannian manifold. We then say that a second order elliptic operator, L, on M^n is quasi uniformly elliptic and in divergence form if for some measurable section A of the bundle of symmetric automorphisms of the tangent bundle, TM,

$$Lu = \operatorname{div}(A\,\nabla u), \tag{6.14}$$

with $X \in TM$,

$$c_1\, g(X,X) \leq g(A\,X,X), \tag{6.15}$$

and some $p \in M$,

$$g(A\,\nabla r, \nabla r) \leq c_2, \tag{6.16}$$

for positive constants c_1 and c_2. Here r is the distance function to p.

With this definition we have the following theorem.

THEOREM 6.17. *Suppose that (M^n,g) satisfies a uniform Neumann-Poincaré inequality with Neumann constant C_N and M has the doubling property with doubling constant C_D. If L is a quasi uniformly elliptic operator in divergence form on M then for all $d > 0$, $\dim \mathcal{H}_d(M,L) \leq C(d,\frac{c_2}{c_1},C_D,C_N) < \infty$.*

Proof. The proof is a slight generalization of that of Theorem 6.8 and is therefore left for the reader. □

7. Area minimizing hypersurfaces

In this section we will give an application of Theorem 0.7 to function theory on area minimizing hypersurfaces in Euclidean space. Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a complete minimal hypersurface without boundary with the intrinsic Riemannian metric. For $x, y \in \mathbf{R}^{n+1}$, $|x-y|$ denotes the Euclidean distance from x to y. We will consider uniformly elliptic divergence form operators L on Σ. A particular example is the Laplacian for the intrinsic metric.

For $d > 0$, we define the spaces $\mathcal{H}_d(\Sigma, L)$ with respect to the Euclidean norm (instead of the induced Riemannian distance). Notice that with this definition, the coordinate functions x_i are in $\mathcal{H}_1$. In particular, on the catenoid in $\mathbf{R}^3$ (which is rotationally symmetric about the x_3-axis), the function x_3

grows slower than any power of the geodesic distance; however, it is not in $\mathcal{H}_d$ for any $d < 1$.

The proof of Theorem 0.12 will be a slight modification of the proof of Theorem 0.7. Note that we make no assumptions about the uniqueness of the tangent cone at infinity of Σ.

Recall the following classical facts about minimal hypersurfaces in Euclidean space (For instance, see [Si]).

Given any $x \in \mathbf{R}^{n+1}$ and $r > 0$, the density is defined by

$$\Theta_\Sigma(x, r) = \frac{\mathrm{Vol}(\Sigma \cap B_r(x))}{\mathrm{V}_0^n(1)\, r^n}. \tag{7.1}$$

LEMMA 7.2 (Monotonicity of volume). *If Σ is any complete minimal submanifold in $\mathbf{R}^{n+1}$ and $x \in \mathbf{R}^{n+1}$, then $\Theta_\Sigma(x,r)$ is monotone nondecreasing in r.*

We say that a minimal hypersurface Σ has Euclidean volume growth if given any $x \in \mathbf{R}^{n+1}$ there exists $\mathrm{V} < \infty$ such that

$$\Theta_\Sigma(x, r) \leq \mathrm{V} \tag{7.3}$$

for all $r > 0$. For any minimal hypersurface Σ with Euclidean volume growth, any $x \in \Sigma$, and any $r > 0$, it follows easily from Lemma 7.2 that

$$1 \leq \Theta_\Sigma(x, r) \leq \mathrm{V}. \tag{7.4}$$

Therefore, Σ has the volume doubling property with $C_D = 2^n \mathrm{V}$.

Note that for $x \in \mathbf{R}^{n+1}$, the function

$$y \to |x - y| \tag{7.5}$$

is Lipschitz with Lipschitz constant 1. Therefore, Yau's reverse Poincaré inequality [Y2], carries over to this setting. Let L be a second order divergence form operator on Σ which is uniformly elliptic with respect to the intrinsic metric. Given an L-harmonic function u, $\Omega > 1$, and $r > 0$, then

$$r^2 \int_{B_r(x)\cap\Sigma} |\nabla u|^2 \leq C_R \int_{B_{\Omega r}(x)\cap\Sigma} u^2, \tag{7.6}$$

where $C_R = C_R(\Omega, L) < \infty$.

Finally, if Σ is an area minimizing hypersurface, Bombieri-Giusti proved the following uniform Neumann-Poincaré inequality [BoG]. Note that this is the only place where we need Σ to be area minimizing rather than just minimal.

LEMMA 7.7 (Neumann-Poincaré inequality for area minimizing hypersurfaces [BoG]). *Let Σ be a complete area minimizing hypersurface. There exists*

$C_N < \infty$ such that for all $x \in \Sigma$, $r > 0$ and $f \in W^{2,1}_{\text{loc}}(\Sigma)$

$$\int_{B_r(x)\cap\Sigma} (f - \mathcal{A})^2 \leq C_N\, r^2 \int_{B_r(x)\cap\Sigma} |\nabla f|^2, \tag{7.8}$$

where $\mathcal{A} = \frac{1}{\text{Vol}(B_r(x)\cap\Sigma)} \int_{B_r(x)\cap\Sigma} f$.

We now have all the ingredients needed to apply Theorem 0.7; hence Theorem 0.12 follows.

In [CM6], we will give generalizations of Theorem 0.12 to more general classes of minimal submanifolds. We will also give some applications of our function theoretic results to the geometry of minimal submanifolds.

8. Subelliptic second order operators

In this section, we will give some applications of Theorem 0.7 to the study of uniformly subelliptic operators. Let M be a smooth manifold and μ a smooth positive measure.

Let L be a linear second order subelliptic operator on M which is symmetric and satisfies $L1 = 0$; i.e. L has no zero order term. Let ρ be the associated Carnot distance on M and let ∇ denote the associated gradient. Assume that (M, ρ) is a complete metric space and take balls to be Carnot balls. See, for instance, Fefferman-Phong [FP], and Saloff-Coste [Sa2], for background.

Let $\mathcal{H}_d(M, L)$ be the space of L-harmonic functions on M with polynomial growth of rate at most d (here the distance is the Carnot distance).

With M and L as above, Saloff-Coste showed that the doubling property and the uniform Neumann-Poincaré inequality on balls (for the associated gradient) imply the Harnack inequality. This of course implies that positive L-harmonic functions on such an M must be constant.

As a generalization of this, a straightforward application of Theorem 0.7 yields the following.

THEOREM 8.1. *Let M and L be as above. If μ satisfies the doubling property and we have an uniform Neumann-Poincaré inequality on balls (for the L-gradient), then for any $d > 0$, $\dim \mathcal{H}_d(M,L) < \infty$.*

Recall that a family of vector fields on a manifold is said to have the Hörmander property if, under Lie bracketing, it generates the full tangent space at each point.

In [SaSt], Saloff-Coste and Stroock proved Harnack inequalities and corresponding Liouville theorems, for uniformly subelliptic operators on polynomial growth Lie groups. Applying Theorem 8.1 to that situation, we get the following:

COROLLARY 8.2. *Let G be a Lie group with polynomial volume growth, let $X_1, \ldots, X_k$ be a family of left-invariant vector fields having the Hörmander property, and let $L = \sum_{i,j} X_i a_{i,j} X_j$, where $(a_{i,j})$ is a smooth symmetric matrix valued function on G such that $c_1 I \leq (a_{i,j}) \leq c_2 I$. Then for any $d > 0$, $\dim \mathcal{H}_d(G,L) < \infty$.*

Proof. The doubling property is immediate for any polynomial growth Lie group. The uniform Neumann-Poincaré inequality follows from a generalization of Jerison's work in [J]; see [SaSt] for details. □

As an example, we note that Corollary 8.2 applies to uniformly subelliptic operators on the Heisenberg group. Theorem 8.1 also applies to the following more general situation (see [Sa2] for references).

Suppose that M^n is a closed manifold and $\tilde{M}$ is a normal cover of M with deck group of polynomial growth, and let L be a subelliptic operator with the Hörmander property. If L is uniformly subelliptic with respect to the Laplace operator, μ is uniformly equivalent to the Riemannian measure, and the norm of the gradient associated to L is dominated by a constant times the Riemannian norm of the Riemannian gradient, then for all $d > 0$, $\dim \mathcal{H}_d(\tilde{M}) < \infty$.

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