Head Loss in Pipe Systems
Laminar Flow and Introduction to Turbulent Flow
ME 322 Lecture Slides, Winter 2007

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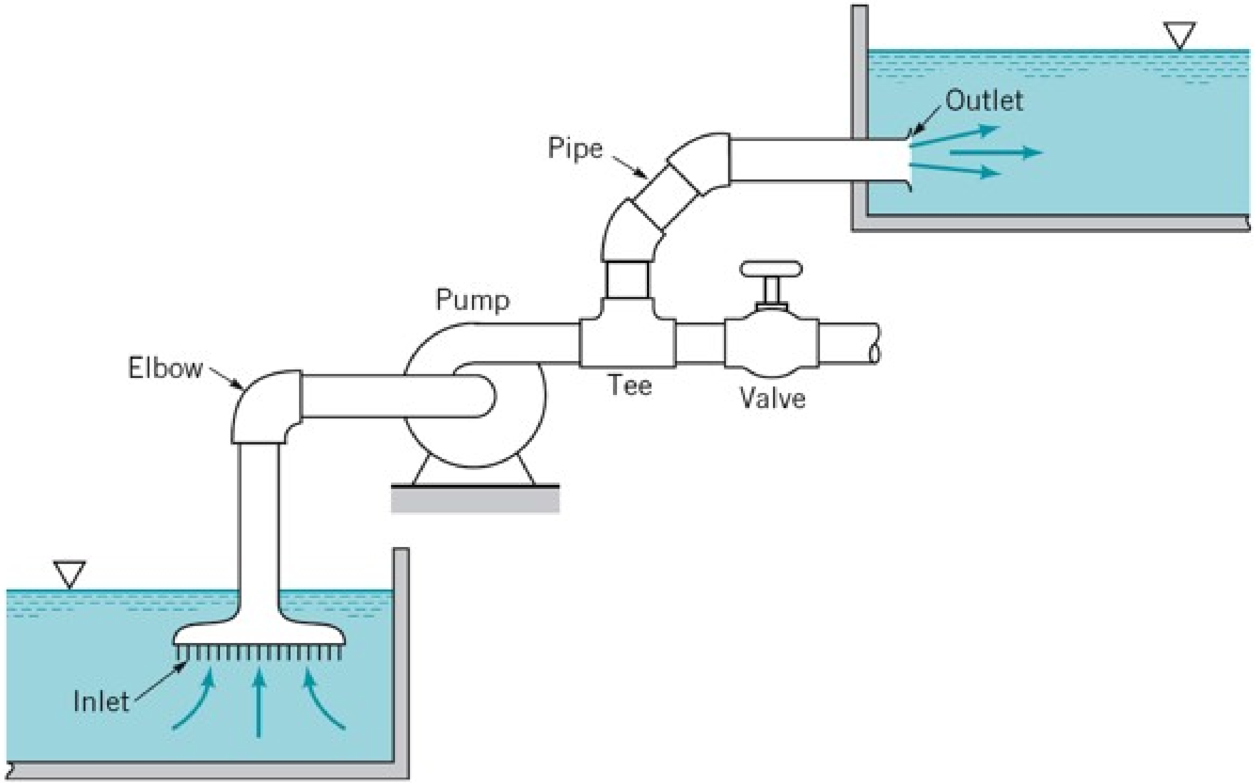
Typical Pipe System



Source: Munson, Young and Okiishi, Figure 8.1, p. 402

Classification of Pipe flows

- Open channel versus Pipe Flow
- Fully Developed flow
- Laminar versus Turbulent flow.

Laminar:	$\mathrm{Re} < 2000$
Turbulent:	$\mathrm{Re} > 2000$

$$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

V is a characteristic velocity
L is a characteristic length

Conditions for an (Easy) Analytical Solution

Analytical solution is possible for following reasonable assumptions

- Steady
- Incompressible
- Pipe cross-section doesn't change with axial position
- Flow is fully-developed

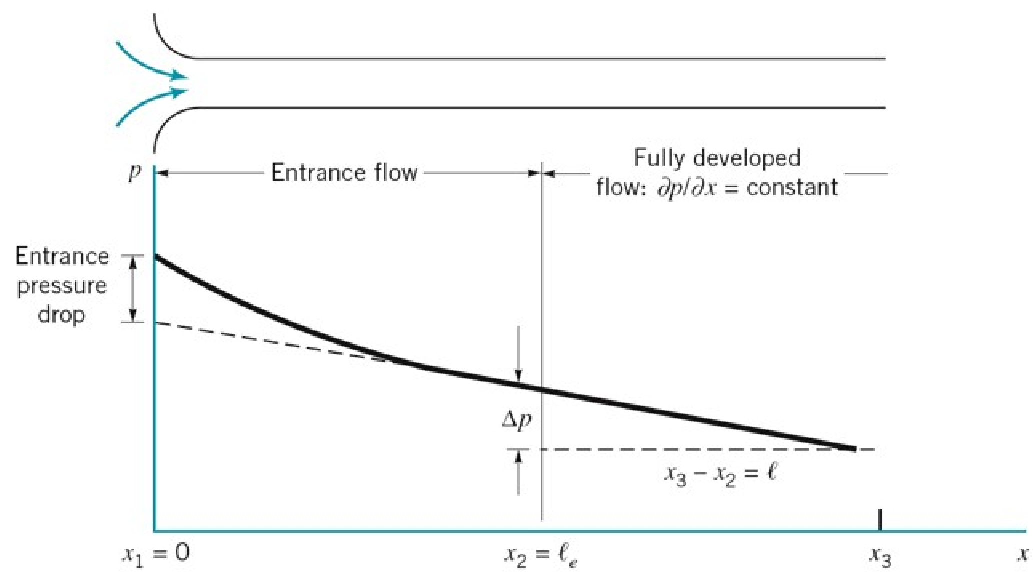
Developing Flow in a Pipe

- Flow becomes fully developed after an entrance length
- Velocity *profile* is independent of axial position
- Pressure gradient is constant

$$\frac{dp}{dx} = \text{constant}$$

$$\Longrightarrow \quad p(x) \text{ is linear in } x$$

- Applies to laminar and turbulent flow



Source: Munson, Young and Okiishi, Figure 8.6, p. 407

Entrance Length

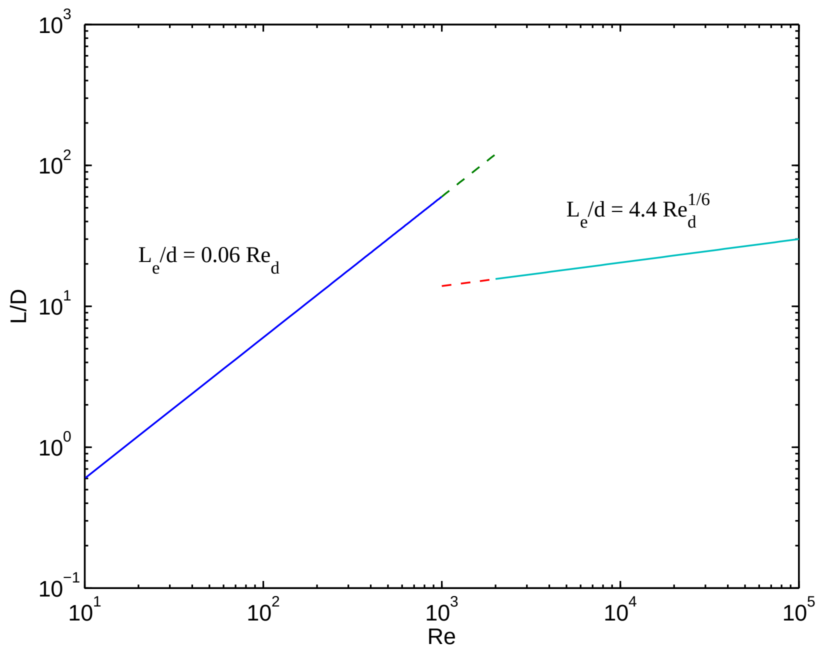
Laminar flow: $(\mathrm{Re}_d < 2000)$

$$\frac{L_e}{d} \approx 0.06\mathrm{Re}_d$$

Turbulent flow: $(\mathrm{Re}_d > 2000)$

$$\frac{L_e}{d} \approx 4.4\mathrm{Re}_d^{1/6}$$

See Munson, Young and Okiishi §8.1.2, pp. 405–406



In most design calculations, the flow in straight sections is assumed to be fully developed. The entrance length correlations are used to check to see whether this is a good assumption.

Tools

- Mass conservation for incompressible flow

$$\sum Q_i = 0 \qquad Q_i = V_i A_i$$

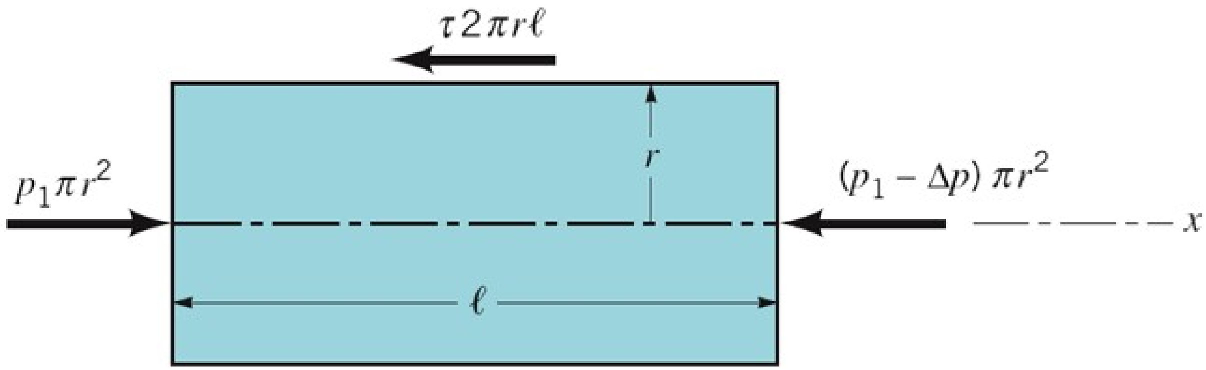
- Energy conservation (MYO, Equation (5.84), p. 280)

$$\left[\frac{p}{\gamma} + \frac{V^2}{2g} + z\right]_{\text{out}} = \left[\frac{p}{\gamma} + \frac{V^2}{2g} + z\right]_{\text{in}} + h_s - h_L$$

  NOTE: All "h" terms on right hand side are positive.

- Semi-empirical information: Darcy-Weisbach equation and Moody chart

Force Balance on a Control Volume in a Pipe (1)



Source: Munson, Young and Okiishi, Figure 8.8, p. 409

Force balance on a plug-shaped element of fluid gives

$$(p_1)\pi r^2 - (p_1 - \Delta p)\pi r^2 - (\tau)2\pi r\ell = 0 \tag{1}$$

The pressure is assumed to *decrease* in the flow direction, hence the pressure on the right hand side is $p_1 - \Delta p$.

Force Balance on a Control Volume in a Pipe (2)

Rearrange Equation (1)

$$\frac{\Delta p}{\ell} = \frac{2\tau}{r} \tag{2}$$

Equation (1) shows that the pressure drop exists because of shear stress on the circumpherential surface of the fluid element. Ultimately this shear stress is transmitted to the wall of the pipe.

Solve Equation (2) for τ

$$\tau = \frac{\Delta p}{2\ell} r \tag{3}$$

Thus, if $\Delta p/\ell$ is constant, i.e. if the flow is fully-developed then the shear stress varies linearly with r. This result applies for laminar or turbulent flow.

Force Balance on a Control Volume in a Pipe (3)

Evaluate Equation (3) at $r = R$, i.e. at the wall

$$\tau_w = \frac{\Delta p}{2\ell} R \tag{4}$$

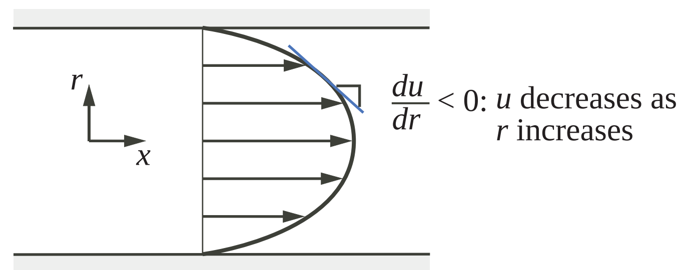
Summary so far:

- Force balance applies to laminar or turbulent flow
- For fully-developed flow, dp/dx is constant. As a consequence the shear stress profile is linear: $\tau = 0$ at the centerline and $\tau = \tau_w$ at $r = R$.
- We need a relationship between τ and u to obtain the velocity profile.

Analytical Solution for Laminar Flow (1)

Introduce modified form of Newton's Law. This applies to laminar flow only

$$\tau = -\mu \frac{du}{dr} \tag{5}$$



The minus sign gives positive τ (as shown in original control volume) when $du/dr < 0$.

Since the pressure gradient is constant for fully-developed flow

$$\frac{\Delta p}{\ell} = \frac{dp}{dx} \tag{6}$$

Analytical Solution for Laminar Flow (2)

Substitute formulas for τ and dp/dr into Equation (3)

$$-\mu\frac{du}{dr} = \frac{1}{2}\frac{dp}{dx}r$$

Since dp/dx is constant (for fully developed flow), the preceding ODE can be rearranged and integrated

$$\frac{du}{dr} = -\frac{1}{2\mu}\frac{dp}{dx}r = Kr$$

where $K = -(1/2\mu)(dp/dx)$.

Integrate twice

$$\frac{du}{dr} = Kr \quad \Longrightarrow \quad u = \frac{1}{2}Kr^2 + C_1$$

Apply B.C. that $u = 0$ and $r = D/2$ to get $C_1 = \dfrac{D^2}{16\mu}\dfrac{dp}{dx}$

Analytical Solution for Laminar Flow (3)

The analytical solution for velocity profile in laminar flow is

$$\boxed{u(r) = -\frac{D^2}{16\mu}\frac{dp}{dx}\left[1 - \left(\frac{r}{R}\right)^2\right]}$$

Note that $dp/dx < 0$ for flow in the positive x direction.

Also remember that dp/dx is constant. In MYO, $dp/dx = \Delta p/\ell$.

Summary so far:

- Apply a force balance to a differential control volume to get an ODE.
- Integrate the ODE analytically to get the velocity profile.

Next: Use the velocity profile to derive formulas useful for practical engineering design.

Analytical Solution for Laminar Flow (4)

The solution to the velocity profile enables us to compute some very important practical quantities

The maximum velocity in the pipe is at the centerline

$$V_c = u(0) = -\frac{D^2}{16\mu}\frac{dp}{dx}$$

$$\Longrightarrow \boxed{u(r) = V_c\left[1 - \left(\frac{r}{R}\right)^2\right]}$$

Analytical Solution for Laminar Flow (5)

The average velocity in the pipe

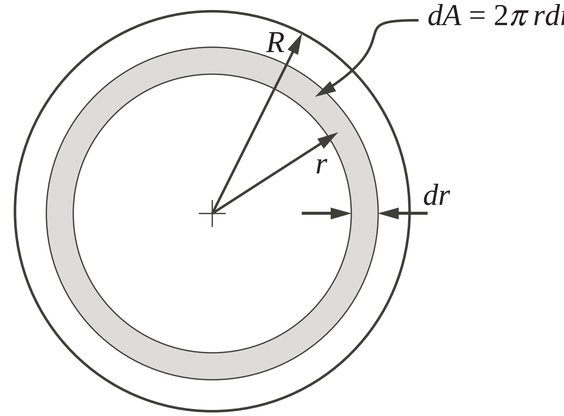
$$Q = VA \qquad V = \text{average velocity} \qquad A = \text{cross sectional area}$$

The total flow rate, and hence the average velocity, can be computed exactly because the formula for the velocity profile is known.

$$Q = \int_0^R u(r)\, dA \qquad \Longrightarrow \qquad V = \frac{Q}{A} = \frac{1}{A}\int_0^R u(r)\, dA$$

Analytical Solution for Laminar Flow (6)

Carry out the integration. It's easy!



$$Q = \int_0^R u(r)\, 2\pi r\, dr$$

$$= 2\pi V_c \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r\, dr$$

$$= 2\pi V_c \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = \frac{\pi R^2 V_c}{2}$$

Therefore

$$V = \frac{Q}{A} = \frac{V_c}{2}$$

Analytical Solution for Laminar Flow (7)

Total flow rate is

$$Q = \frac{\pi R^2 V_c}{2} = \frac{\pi R^2}{2}\frac{D^2}{16\mu}\left(-\frac{dp}{dx}\right) = \frac{\pi D^4}{128\mu}\left(-\frac{dp}{dx}\right)$$

Now, for convenience define Δp as the *pressure drop* that occurs over a length of pipe L.

In other words, let $\quad -\frac{dp}{dx} \equiv \frac{\Delta p}{L}$

Then

$$\boxed{Q = \frac{\pi D^4}{128\mu}\frac{\Delta p}{L}} \tag{7}$$

So, *for laminar flow*, once we know Q and L, we can easily compute Δp and vice versa.

Implications of the Energy Equation

Apply the steady-flow energy equation

$$\left[\frac{p}{\gamma} + \frac{V^2}{2g} + z\right]_{\text{out}} = \left[\frac{p}{\gamma} + \frac{V^2}{2g} + z\right]_{\text{in}} + h_s - h_L$$

For a horizontal pipe $(z_{\text{out}} - z_{\text{in}})$ with no pump $(h_s = 0)$, and constant cross section $(V_{\text{out}} = \text{in})$, the energy equation reduces to

$$h_L = \frac{p_{\text{in}} - p_{\text{out}}}{\gamma} = \frac{\Delta p}{\gamma}$$

Solve Equation (7) for Δp

$$\Delta p = \frac{128\mu Q L}{\pi D^4} \qquad \Longrightarrow \qquad h_L = \frac{128\mu Q L}{\pi \gamma D^4} \tag{8}$$

These formulas only apply to laminar flow. We need a more general approach.

Dimensional Analysis

The pressure drop for laminar flow in a pipe is

$$\Delta p = \frac{128\mu Q L}{\pi D^4} \tag{8}$$

or, in the form of a dimensional analysis

$$\Delta p = \phi(V, L, D, \mu)$$

where $\phi(\ )$ is the function in Equation (8). Note that ρ does not appear. For turbulent flow, fluid density *does* influence pressure drop.

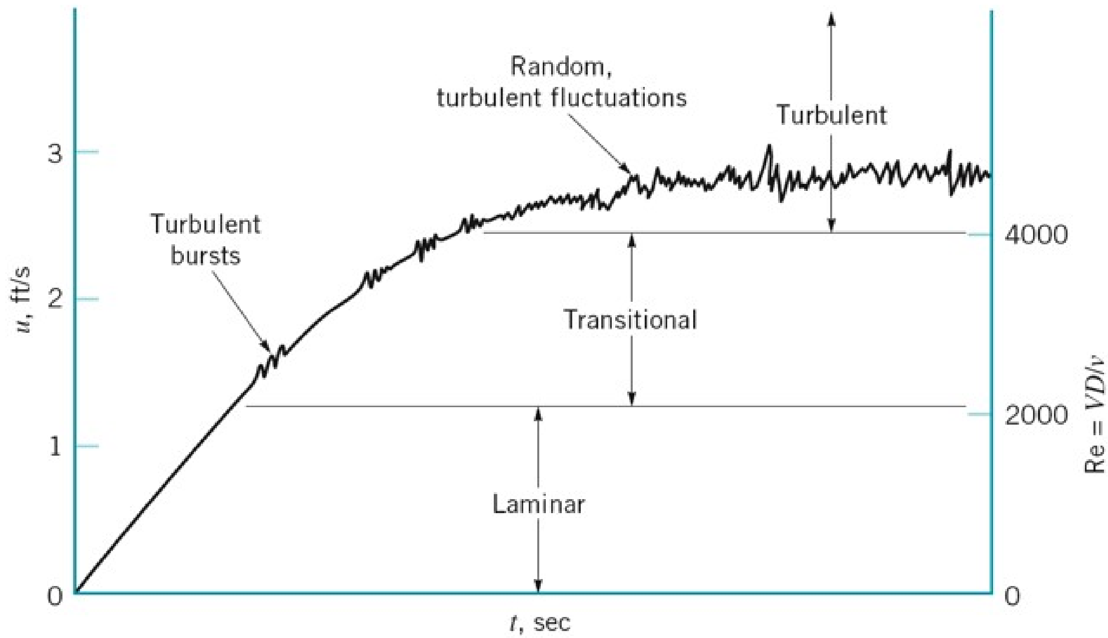
For turbulent flow the dimensional form of the equation for pressure drop is

$$\Delta p = \phi(V, L, D, \mu, \rho, \varepsilon)$$

where ε is the length scale determining the wall roughness.

Turbulent Flow in Pipes (1)

Consider the instantaneous velocity at a point in a pipe when the flow rate is increased from zero up to a constant value such that the flow is eventually turbulent.



Source: Munson, Young and Okiishi, Figure 8.11, p. 418

Turbulent Flow in Pipes (2)

Reality:

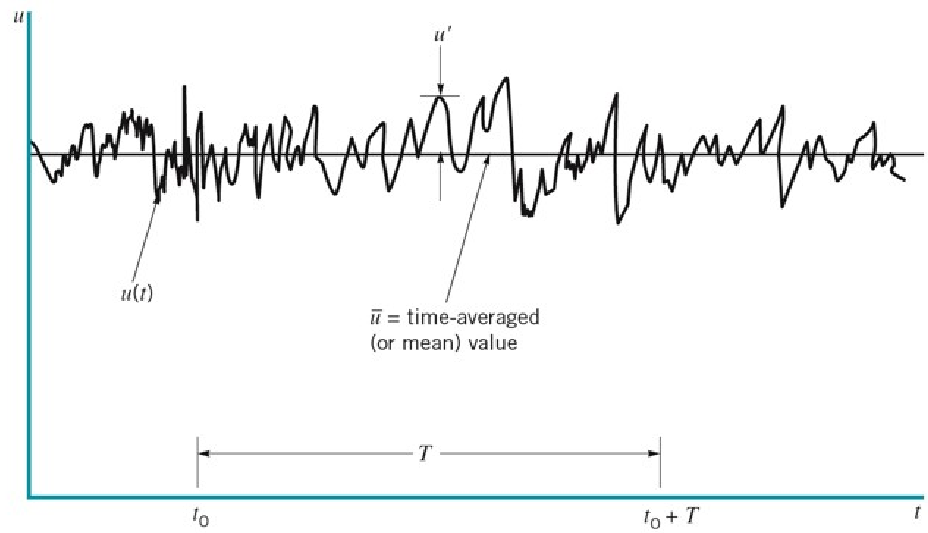
Turbulent flows are unsteady: fluctuations at a point are caused by convection of eddies of many sizes. As eddies move through the flow the velocity field at a fixed point changes.

Model:

When measured with a "slow" sensor (e.g. Pitot tube) the velocity at a point is apparently steady. Treat flow variables (velocitie components, pressure, temperature) as time averages (or ensemble averages). These averages are steady.

Engineering Model:

Flow is "Steady-in-the-Mean"



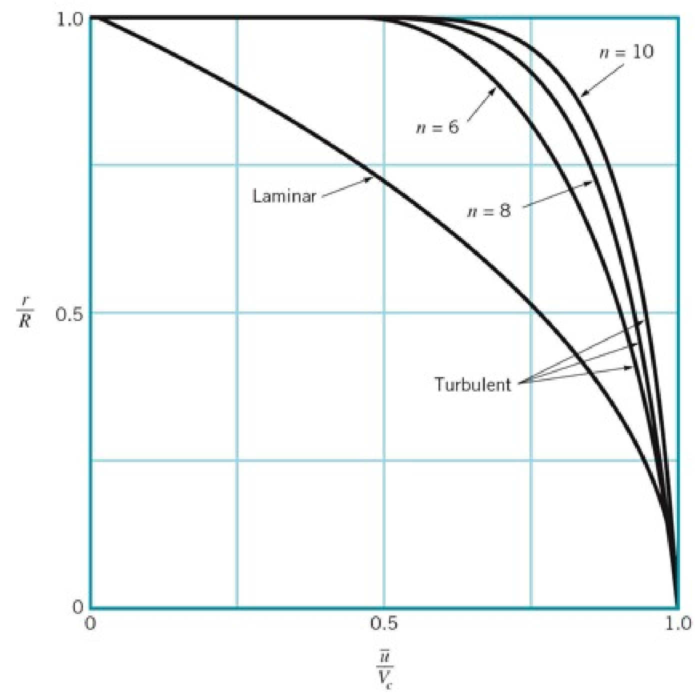
Source: Munson, Young and Okiishi, Figure 8.1, p. 402

Turbulent Velocity Profiles in a Pipe

A power-law function fits the shape of the turbulent velocity profile

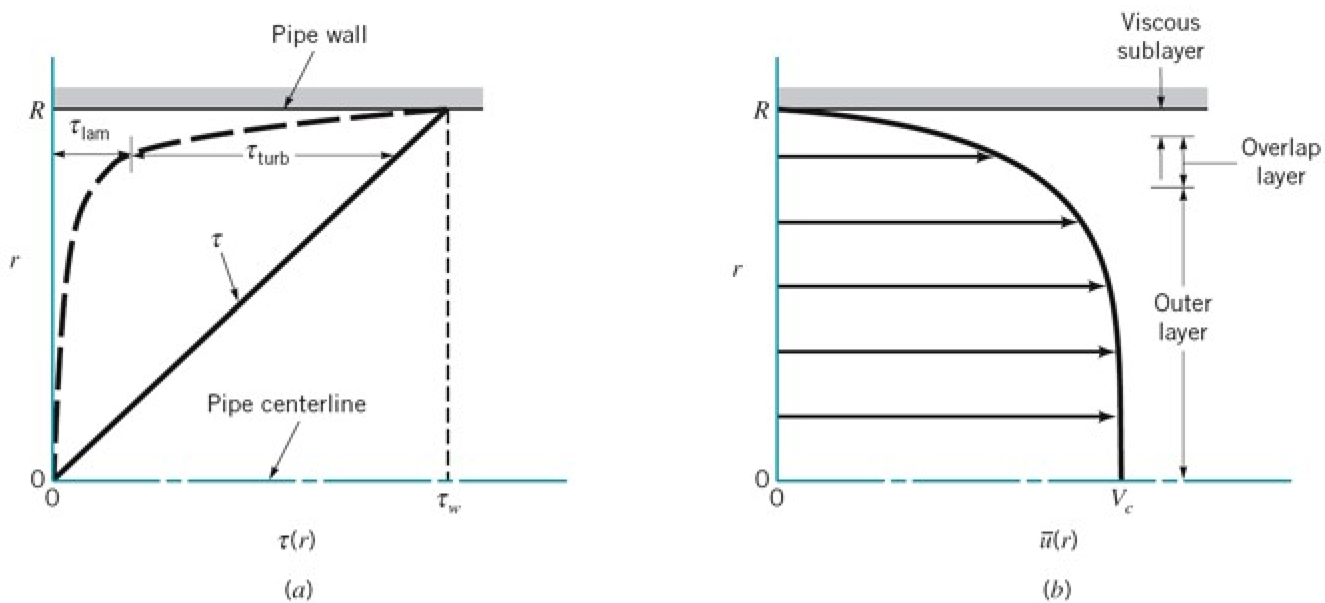
$$\frac{\bar{u}}{V_c} = \left(1 - \frac{r}{R}\right)^{1/n}$$

where $\bar{u} = \bar{u}(r)$ is the mean axial velocity, V_c is the centerline velocity, and $n = f(\mathrm{Re})$. See Figure 8.17, p. 4.26



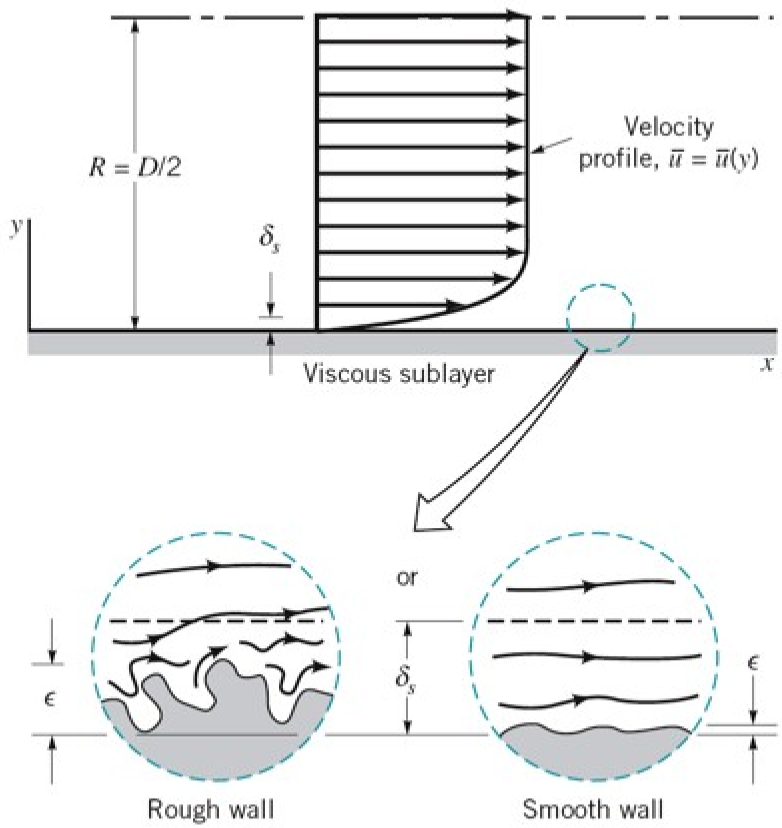
Source: Munson, Young and Okiishi, Figure 8.18, p. 427

Structure of Turbulence in a Pipe Flow



Source: Munson, Young and Okiishi, Figure 8.15, p. 424

Roughness and the Viscous Sublayer



Source: Munson, Young and Okiishi, Figure 8.19, p. 431

Head Loss Correlations (1)

For turbulent flow the dimensional form of the equation for pressure drop is

$$\Delta p = \phi(V, L, D, \mu, \rho, \varepsilon)$$

where ε is the length scale determining the wall roughness.

Form dimensionless groups to get

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \tilde{\phi}\left(\frac{\rho V D}{\mu}, \frac{L}{D}, \frac{\varepsilon}{D}\right)$$

Head Loss Correlations (2)

From practical experience we know that the pressure drop increases linearly with pipe length, as long as the entrance effects are negligible.

Factor out L/D

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{L}{D}\tilde{\phi}_2\left(\frac{\rho V D}{\mu}, \frac{\varepsilon}{D}\right) \tag{9}$$

The $\tilde{\phi}_2$ function is universal: it applies to all pipes. It's called the friction factor, and given the symbol f

$$f = \tilde{\phi}_2\left(\frac{\rho V D}{\mu}, \frac{\varepsilon}{D}\right) \tag{10}$$

Combine Equation (9) and Equation (10) to get a working formula for the *Darcy friction factor*

$$f = \frac{\Delta p}{\frac{1}{2}\rho V^2}\frac{D}{L} \tag{11}$$

Friction Factor for Laminar Flow

The pressure drop for fully-developed laminar flow in a pipe is

$$\Delta p = \frac{128\mu Q L}{\pi D^4} \tag{8}$$

Divide both sides by $(1/2)\rho V^2$ and rearrange

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{1}{\frac{1}{2}\rho V^2}\frac{128\mu Q L}{\pi D^4} = \frac{1}{\frac{1}{2}\rho V^2}\frac{128\mu V(\pi/4)D^2 L}{\pi D^4} = \frac{64\mu}{\rho V D}\frac{L}{D} = \frac{64}{\mathrm{Re}_D}\frac{L}{D}$$

Therefore, for laminar flow in a pipe

$$\boxed{f_{\mathsf{lam}} = \frac{64}{\mathrm{Re}_D}}$$

Colebrook Equation

Nikuradse did experiments with artificially roughened pipes

Colebrook and Moody put Nikuradse's data into a form useful for engineering calculations.

The Colebrook equation

$$\boxed{\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)} \tag{12}$$

Moody Diagram