Hierarchical Modeling for Large Univariate Areal Data

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Areal data

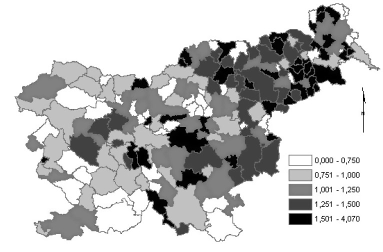
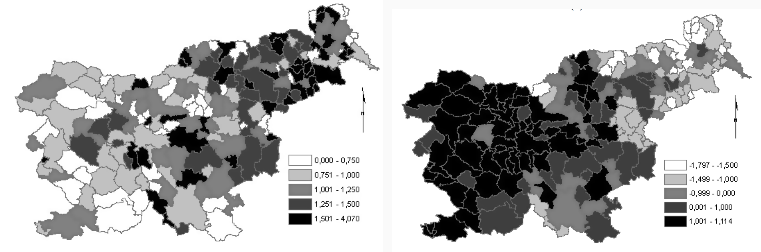


Figure: Standardized stomach cancer incidence in 194 municipalities in Slovenia

- Each datapoint is associated with a region like state, county, municipality etc.
- Usually a result of aggregating point level data

Spatial disease mapping



Standardized cancer incidence

Socio-economic score

Figure: Slovenia stomach cancer data

- Goal: Identify factors (covariates) associated with the disease
- Goal: Identify spatial pattern, if any, and smooth spatially
- Inference is often restricted only to the given set of regions

GLM for Spatial disease mapping

- At unit (region) i, we observe response y_i and covariate x_i
- $g(E(y_i)) = x_i'\beta + w_i$ where $g(\cdot)$ denotes a suitable link function

Hierarchical areal model:

$$\prod_{i=1}^{k} p_1(y_i | x_i'\beta + w_i) \times N^{-1}(w \,|\, 0, \tau_w Q(\rho)) \times p_2(\beta, \tau_w, \rho)$$

- Notation: $N^{-1}(m, Q)$ denotes normal distribution with mean m and precision (inverse covariance) Q
- p_1 denotes the functional form of the density corresponding to the link $g(\cdot)$

How to model $Q(\rho)$

- Choice of $Q(\rho)$ should enable spatial smoothing
- One possibility: Represent each region by a single point and use Gaussian Process covariance i.e. $Q(\rho)_{ij}^{-1} = C(m(i), m(j))$
- Many possible choices to map the region i into a Euclidean coordinate $m(i)$
- Is it appropriate to represent a large area with a single point?
- Also GP approach is computationally very expensive
- Alternate approach: Represent spatial information in terms of a graph depicting the relative orientation of the regions

CAR models

- Conditional autoregressive (CAR) model (Besag, 1974; Clayton and Bernardinelli, 1992)
- Areal data modeled as a graph or network: V is the set of vertices (regions)
- $i \sim j$ if regions i and j share a common border
- Adjacency matrix $A = (a_{ij})$ such that $a_{ij} = I(i \sim j)$
- n_i is the number of neighbors of i
- CAR model:

$$w_i \,|\, w_{-i} \sim N^{-1}\left(\frac{\rho}{n_i} \sum_{j \,|\, i \sim j} w_j, \tau_w n_i\right)$$

CAR models

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$$w_i \mid w_{-i} \sim N^{-1}(\frac{1}{n_i} \sum_{j \mid i \sim j} w_j, \tau_w n_i)$$

- $w = (w_1, w_2, \ldots, w_k)' \sim N^{-1}(0, \tau_w(D - \rho A))$ where $D = diag(n_1, n_2, \ldots, n_k)$
- $\rho = 1 \Rightarrow$ Improper distribution as $(D - A)1 = 0$ (ICAR)
  - Can be still used as a prior for random effects
  - Cannot be used directly as a data generating model

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  - Cannot be used directly as a data generating model
- $\rho < 1 \Rightarrow$ Proper distribution with added parameter flexibility

SAR models

- Simultaneous Autoregressive (SAR) model (Whittle, 1954)
- Instead of taking the conditional route, SAR model proceeds by simultaneously modeling the random effects

$$w_i = \rho \sum_{i \neq j} b_{ij} w_j + \epsilon_i \text{ for } i = 1, 2, \dots, k$$

- $\epsilon_i \overset{ind}{\sim} N^{-1}(0, \tau_i)$ are errors independent of w
- A common choice is to define $b_{ij} = I(i \sim j)/n_i$
- Joint distribution: $w \sim N^{-1}(0, (I - \rho B)' F (I - \rho B))$, $B = (b_{ij})$ and $F = diag(\tau_1, \tau_2, \dots, \tau_k)$
- $\rho = 1 \Rightarrow$ Improper distribution

Interpretation of ρ in proper CAR and SAR models

- Calibration of ρ as a correlation, e.g., (as reported in Banerjee et al. 2014)

$$\rho = 0.80 \text{ yields } 0.1 \leq \text{Moran's } I \leq 0.15,$$
$$\rho = 0.90 \text{ yields } 0.2 \leq \text{Moran's } I \leq 0.25,$$
$$\rho = 0.99 \text{ yields Moran's } I \leq 0.5$$

- So, used with random effects, scope of spatial pattern may be limited

Interpretation of ρ in proper CAR and SAR models

- ρ cannot be interpreted as correlation between neighboring w_i's (Wall, 2004; Assuncao and Krainski, 2009)

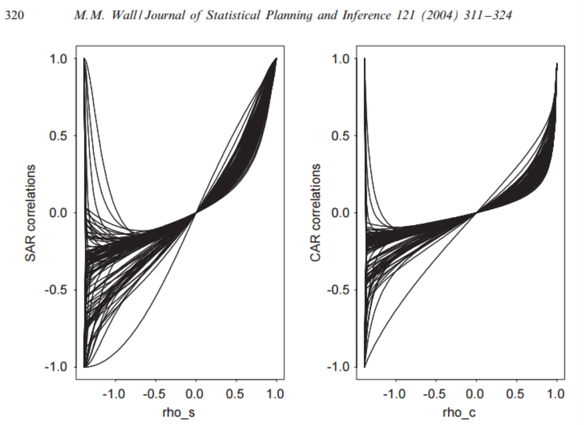

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Figure: Neighbor pair correlations as a function of ρ for proper CAR and SAR models over the graph of US states

SAR model and Cholesky factors

- General SAR model:

$$w_i = \sum_{i \neq j} b_{ij} w_j + \epsilon_i \text{ for } i = 1, 2, \ldots, k$$

- $w \sim N^{-1}(0, (I - B)' F (I - B))$ where $F = diag(\tau_1, \tau_2, \ldots, \tau_k)$
- Only proper when $I - B$ is invertible which is not guaranteed for arbitrary B
- SAR is essentially modeling the precision matrix through the Cholesky factor $I - B$

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- SAR is essentially modeling the precision matrix through the Cholesky factor $I - B$
- Cholesky factors are not unique
- We can always choose a lower triangular Cholesky factor

New model

$$\begin{aligned}
w_1 &= \epsilon_1 \\
w_2 &= b_{21} w_1 + \epsilon_2 \\
w_3 &= b_{31} w_1 + b_{32} w_2 + \epsilon_3 \\
&\vdots \\
w_k &= b_{k1} w_1 + b_{k1} w_2 + \ldots + b_{k,k-1} w_{k-1} + \epsilon_k
\end{aligned}$$

- $B = (b_{ij})$ is now a strictly lower triangular matrix.

New model

- Advantages of lower triangular B:
  - $w \sim N^{-1}(0, (I - B)' F (I - B))$ is a proper distribution for any choice of lower triangular B
  - $\det(L' F L) = \prod_{i=1}^{n} \tau_i$ where $F = diag(\tau_1, \dots, \tau_k)$ and $L = I - B$
  - $w' L' F L w = \tau_1 w_1^2 \; + \sum_{i=2}^{k} \tau_i (w_i - \sum_{\{j<i\}} w_j b_{ij})^2$
  - Likelihood $N^{-1}(w \,|\, 0, (I - B)' F (I - B))$ can be computed using $O(k + s)$ flops where s denotes the sparsity (number of non-zero entries) of B.
  - Even if k is large, evaluation of likelihood is fast if each region only shares border with a few others

Choice of B and F

- How to specify B and F?
- Sparsity of B is desirable
- If data had replicates for each region, there is large literature on fully data driven estimation of sparse Cholesky factors (Wu and Pourahmadi, 2003; Huang et al., 2006; Rothman et al., 2008; Levina et al., 2008; Wagaman and Levina, 2009; Lam and Fan, 2009)
- Unfortunately many areal datasets lack replication

Choice of B and F

- How to specify B and F?
- Sparsity of B is desirable
- Like in NNGP set $b_{ij} = 0$ for j outside neighbor sets $N(i)$
  - Pros: For graphs neighbor sets are naturally chosen: $N(i) = \{j \,|\, j \sim i, j < i\}$
  - Cons: There is no covariance function on arbitrary graphs from which we can obtain non-zero b_{ij}'s and F

Autoregressive models on trees

- $D = (d_{ij})$ is the shortest distance matrix on the graph
- If the graph was a tree (no loops), then $\rho^D = (\rho^{d_{ij}}$ is then a valid *autoregressive* correlation matrix (AR(1) model on a tree, Basseville et al., 2006).
- Areal graphs are loopy and are not usually trees

Local embedded spanning trees

- Embedded spanning trees (EST) of a graph G is a subgraph of G which is a tree and spans all the vertices of G
- Note that to specify $w_i = \sum_{j \in N(i)} b_{ij} w_j + \epsilon_i$ we only need a joint distribution on $\{i\} \cup N(i)$
- Let G_i denote the subgraph of G which includes vertices $\{i\} \cup N(i)$ and the edges among them
- The subgraph T_i of G_i which only contains the edges $\{i \sim j \,|\, j \in N(i)\}$ is an embedded spanning tree of G_i
- Use the local embedded spanning trees T_i to specify the b_{ij}'s and τ_i

Directed acyclic graph autoregressive (DAGAR) model

- AR_i denotes the $AR(1)$ distribution on T_i
- Solve for b_{ij} and τ_i such that $E_{AR_i}(w_i \mid w_{N(i)}) = \sum_{j \in N(i)} b_{ij} w_j$ and $\tau_i = 1/Var_{AR_i}(w_i \mid w_{N(i)})$
- No edge is left out !

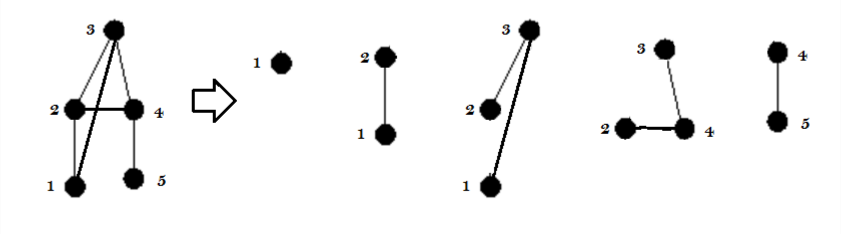


Figure: Decomposing a graph into a sequence of embedded spanning trees

Properties of DAGAR models

- $b_{ij} = b_i = \rho/(1 + (|N(i)| - 1)\rho^2)$
- $\tau_i = (1 + (|N(i)| - 1)\rho^2)/(1 - \rho^2)$
- $\det(Q_{DAGAR}) = \prod_{i=1}^{k} \tau_i$
- Positive definite for any $0 \leq \rho \leq 1$
- Interpretability of ρ:
    - If the graph is a tree, then DAGAR model is same as the AR(1) model on the tree i.e. correlation between d^{th} order neighbors is ρ^d for $d = 1, 2, \ldots$
    - If the graph is a closed two-dimensional grid, then each neighbor pair correlation is ρ
- $p_{DAGAR}(w)$ can be stored and evaluated using $O(e + k)$ flops where e is the total number of neighbor pairs

Dependence on ordering

- DAGAR model depends on the ordering of the regions when decomposing into local trees
- We can define a DAGAR model for every ordering
- Spatial regions do not have natural ordering
- How to choose the ordering?
- Coordinate based orderings were used in Datta et al., 2016; Stein, 2004; Vecchia, 1988
- Model averaging over orderings ? Too many possibilities (k!)

Order-free model

- Let Q be the average over DAGAR precision matrices corresponding to all $k!$ possible orderings

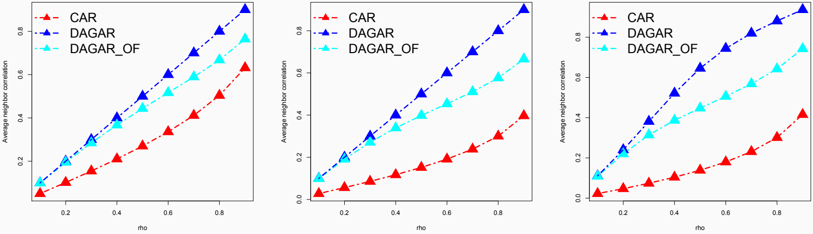
Order-free model

- Let Q be the average over DAGAR precision matrices corresponding to all $k!$ possible orderings
- Q is is free of ordering and available in closed form
- $Q(i,j)$ is non-zero if and only if either $i \sim j$ or $i \approx j$

Order-free model

- Sparsity of Q is e_2 where e_2 is the number of edges in the second order graph (moral graph) created from G
- As $e_2 > e$, Q is less sparse than the CAR model or the ordered DAGAR model precision matrix and has higher flop count
- Total computational total cost for evaluating Q is $O(e_2 n_{\max})$
- $e_2 < kn_{\max}(n_{\max} + 1)/2$ where $n_{\max} = \max(n_i)$
- If $n_{\max}$ is small, i.e., as long as each region only shares border with a few others (which is often the case), Q is still quite sparse even for large k

Interpretation of ρ



path graph

grid graph

USA state map

Figure: Average neighbor pair correlations as a funcion of ρ for proper CAR and DAGAR models

Simulated data analysis

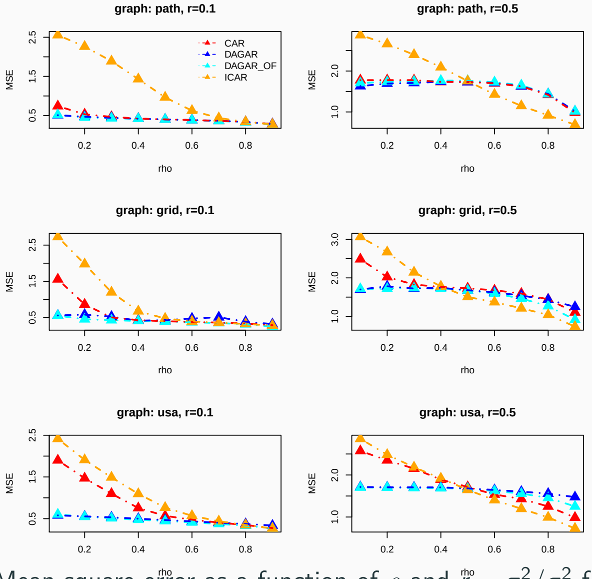
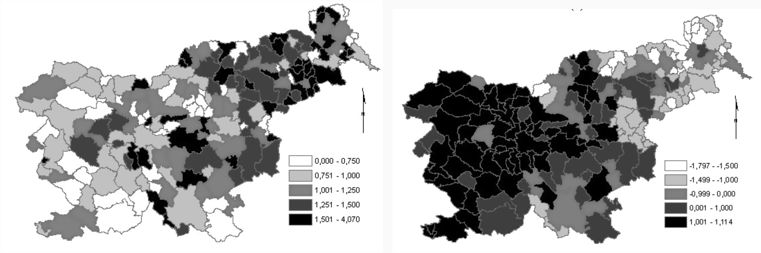


Figure: Mean square error as a function of ρ and $r = \tau^2/\sigma^2$ for DAGAR and CAR models

Slovenia stomach cancer data



Standardized cancer incidence

Socio-economic score

Figure: Slovenia stomach cancer data

- Observed (O_i) and expected (E_i) number of cancer counts for each of the 194 municipalities of the country
- $O_i \sim Poisson(E_i \exp(\alpha + \beta SE_i + w_i))$ where $w \sim N^{-1}(0, \tau_w Q(\rho))$

Slovenia stomach cancer data

Table: Parameter estimates with confidence intervals and model comparison metrics

	α	β	ρ	DIC	LPPD_{LOOCV}[1]
CAR	0.09 (0.02, 0.16)	-0.12 (-0.19, -0.04)	0.33 (0.02, 0.86)	1097	1170
DAGAR	0.11 (0.03, 0.18)	-0.12 (-0.19, -0.06)	0.08 (0.004, 0.24)	1091	1127
DAGAR_{OF}	0.11 (0.05, 0.17)	-0.12 (-0.18, -0.06)	0.06 (0.003, 0.2)	1090	1133

- Zadnik and Reich (2006) observed spatial confounding with ICAR model ($\hat{\beta}_{ICAR} = -0.02(-0.10, 0.06)$)
- Here for all three models the CIs for β lie outside zero
- Estimates of ρ are much smaller than 1
- Estimates of β here are closer to those obtained in the non-spatial (NS) analysis ($\hat{\beta}_{NS} = -1.4(-0.17, -0.10)$)

[1] Log-predictive posterior density using Leave one out cross validation

Summary

- DAGAR models for areal data constructed from sparse Cholesky factors
- Scalability for large areal data
- Ordered vs order-free DAGAR
    - For all analysis, ordered model performed very similar to the order-free model
    - Ordered model is faster with theoretical results about interpretability of ρ
- DAGAR models are positive definite and can be directly used to model or simulate any multivariate data on graphs (like imaging or social network data)
- Better performance than CAR modes for many scenarios
- DAGAR available at https://arxiv.org/pdf/1704.07848.pdf