International Journal of Mathematics and Statistics Studies
Vol.7, No.3, pp.1-11, June 2019
Published by European Centre for Research Training and Development UK (www.eajournals.org)

HIGH ORDER COMPACT FINITE DIFFERENCE TECHNIQUES FOR STOCHASTIC ADVECTION DIFFUSION EQUATIONS

Oyakhire Friday Ighaghai
Mathematics/ Statstics Department, Akanu Ibiam Federal Potytechnic, Unwana.
Ebonyi State, Nigeria.

ABSTRACT: *High order compact finite difference scheme for stochastic advection – diffusion equations (SCDEs) of Ito type is designed. Firstly, Modified Mathematical formulation of the stochastic advection – diffusion was developed, followed by the derivation of stochastic differential advection – diffusion using compact finite difference schemes. Explicit- implicit Euler's scheme was adopted to established the stability criteria in the resulting linear stochastic system of differential equations. The stability criterion was investigated using Fourier mode. Numerical examples were conducted to test the validity, efficient, accuracy and robustness of the derived schemes.*

KEYWORDS: Stochastic differential advection– diffusion equation, explicit- implicit Euler's method, compact finite difference schemes, stability, Numerical examples.

INTRODUCTION

In recent years, there has been interested regarding the study of stochastic advection - diffusion equations. Stochastic advection- diffusion equation is one of the most important parts of partial differential equations, observed in a wide range of engineering, mathematical sciences, and practical industrial application. Due to the importance of stochastic advection – diffusion the present paper, solves and analyzes these problems using a new finite difference method called compact finite difference schemes as well as numerical experience. The developed scheme is based on Ito type of stochastic differential equation. Stochastic advection- diffusion equations is used to describes dynamic of stochastic process defined on space and time Continuum, heat transfer in a draining film, water Transfer in soil dispersion of tracers in porous media, contaminant dispersion in shallow lakes, the spread of solute in a liquid flowing through tube, long range transport of pollutants in the atmosphere and dispersion of dissolved salts in groundwater. Analytic solution can be obtained for very few stochastic advection – diffusion equations and some authors have studied then theoretically and these includes the initial works of Ben berg and Gut finger (1992), presented that while obtaining the analytical solutions of dispersion problem in the ideal conditions, the basic approach was to reduce the advection- diffusion equation into a diffusion equation by eliminating the advection terms. Ahmed, S.G (2012), A numerical algorithms for solving advection- diffusion equation with constant and variable coefficients. Dai,W and Nassir

Print ISSN: 2053-2229 (Print), Online ISSN: 2053-2210 (Online)

(2002) finite difference methods (FDM), Lele, S.K (1992) $\rightarrow$ (CFDS), Bertram D and F. Michel (2008) demonstrated in his paper titled stochastic differential equation using compact finite difference schemes (SDEUCFDS). Oyakhire et al. (2016), presented paper on high order compact finite difference schemes for Poisson equation using Pade approximation (HOCFDSPUPA), where stochastic differential compact finite difference schemes was treated using Taylor series approach. Wade, W .R (2001), presented a paper titled Linear programming stochastic partial differential equations (LPSPDEs), Kalita, J and Dala (2002),demonstrated paper on explicit and implicit finite difference method. The outline of the paper is as follows: in section 2, we stated the mathematical formulation of the stochastic advection- diffusion equations by integral form, derivation of stochastic advection – diffusion equation using compact finite difference schemes for discretizing spatial and time in section 3 , in section 4 treats the establishment of stability criteria by applying the explicit- implicit Euler's method and Fourier mode, numerical examples in section 5 and finally conclusion.

MODIFIED MATHEMATICAL FORMULATION

In this section, we consider the mathematical formulation of one dimensional stochastic advection- diffusion equations

$$u_t(x,t) = \alpha u_{xx}(x,t) + \beta u_x(x,t) + \sigma u(x,t)\dot{W}(t)$$

$$u(x,t) = u_0(x),$$

$$u(0,t) = f_1(t), \qquad u(L,t) = f_2(t) \tag{1}$$

where $t \in [t_0, T], \quad x = [0, L]$. In equation (1) $\alpha > 0, \ \beta, \ \sigma$ are constants and $w'(t)$ is a random noise which related to the Brownian notation $w'(t)$. Equation (1) can be rewritten as:

$$u(x,t) = u(x,0) + \int_0^1 (\alpha u_{xx}(x,s) + \beta u_x(x,s))ds + \int_0^1 \sigma u(x,s)dw(s) \tag{2}$$

The stochastic integral is then the Ito integral with respect to $R'-$ valued Wiener process $(w(f), F_t)_{t\in[0,T]}$ defined on a complete probability space (Ω, F, ρ) , adapted to the standard filtration.

DERIVATION OF (SADEs) COMPACT FINITE DIFERENCE SCHEMES

Here, we introduce the standard compact approximations for the spatial derivatives of equation (1). Considering the following differential equations:

$$\alpha u_{xx} + \beta u_x + \sigma u \dot{W}(t) = f(t), \quad x \in [0, X] \tag{3}$$

If we denote the central difference scheme of order two for standard and first order derivatives of u as

$$\delta_x^2 u_i = \frac{u_{i=1} - 2u_i + u_{i-1}}{\Delta x^2}, \qquad \delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} \tag{4}$$

Respectively, then we obtain the following approximation for equation (3) at point x_i:

$$\alpha \delta_x^2 u_i + \beta \delta_x u_i + \sigma u_i \dot{w}(t) - \tau_i = f_i \tag{5}$$

where τ_i is the local truncation error in one dimension in which

$$\tau_i = \frac{\Delta x^2}{12}(\alpha \frac{\partial^4 u}{\partial x^4} + 2\beta \frac{\partial^3 u}{\partial x^3})_i \tag{6}$$

In order to obtain a higher –order scheme, the fourth and the third derivatives of u can be approximated as in Lele (1992).

$$\frac{\partial^2 u}{\partial x^2}\Big|_i = \frac{1}{\alpha}(f - \beta u_x - \sigma u \dot{w}(t))\Big|_i$$

$$= \frac{1}{\alpha}(f_i - \beta \delta_x u_i - \sigma u_i \dot{w}(t))$$

$$\frac{\partial^3 u}{\partial x^3}\Big|_i = \frac{1}{\alpha}(f_x - \beta u_{xx} - \sigma u_x \dot{w}(t))\Big|_i$$

$$= \frac{1}{\alpha}(\delta_x f_i - \beta \delta_x^2 u_i - \sigma \delta_x u \;\; \dot{w}(t))$$

$$\frac{\partial^4 u}{\partial x^4}\Big|_i = \frac{1}{\alpha}(f_{xx} - \beta u_{xxx} - \sigma u_{xx} \dot{w}(t))\Big|_i$$

$$= \frac{1}{\alpha}\left(\delta_x^2 f_i - \frac{\beta}{\alpha}(\delta_x f_i - \beta \delta_x^2 u_i - \sigma \delta_x^2 u_i \dot{w}) - \sigma \delta_x^2 u_i \dot{w}\right) \tag{7}$$

Substituting the above derivatives into equation (5):

$$\alpha \delta_x^2 u_i + \beta \delta_x u_i + \sigma u_i w(t)$$

$$-\frac{\Delta x^2}{12}\left[\begin{array}{l}\left\{\delta_x^2 f_i - \frac{\beta}{\alpha}\left(\delta_x f_i - \beta\delta_x^2 u_i - \sigma\delta_x u_i \dot{w}\right) - \sigma\delta_x^2 u_i \dot{w}\right\} \\ + 2\frac{\alpha}{\beta}\left\{\delta_x f_i - \beta\delta_x^2 u_i - \sigma\delta_x u_i \dot{w}\right\}\end{array}\right] = f_i$$

(8)

Equation (8) can be rewritten as

$$\alpha\delta_x^2 u_i + \beta\delta_x u_i + \sigma u_i \dot{w}(t)$$
$$-\frac{\Delta x^2}{12}\left[\frac{\beta}{\alpha}\delta_x^2 u_i + \frac{\beta}{\alpha}\sigma\delta_x u_i \dot{w} - \sigma\delta_x^2 u_i \dot{w} - 2\frac{\beta}{\alpha}\delta_x^2 \dot{u}_i - 2\frac{\beta}{\alpha}\sigma\delta_x u_i \dot{w}\right]$$

$$= f_i + \frac{\Delta x^2}{12}\left(\delta_x^2 f_i - \frac{\beta}{\alpha}\delta_x f_i + 2\frac{\beta}{\alpha}\delta_x f_i\right) \qquad (9)$$

For an integer positive M if $\Delta x = \frac{X}{M}$ and time step size respectively, and define

$$x_j = j\Delta x, \qquad j = 0,1,\ldots, M$$

$$t_j = j\Delta t, \qquad j = 0,1,2,\ldots$$

In order derivative of high-order difference algorithm, equation (1) must be discrete in space at point x_i according to equation (9) to obtain a system of stochastic differential equation as follows:

$$\left[\frac{\alpha}{\Delta x^2} + \frac{\beta^2}{12\alpha} - \frac{\beta}{2\Delta x} - \frac{\beta\delta\Delta x}{24\alpha}\dot{w} + \frac{\sigma}{12}\dot{w}\right]u_{i-1}$$
$$+\left[\frac{5}{6}\sigma\dot{w} - \frac{2\alpha}{\Delta x^2} - \frac{\beta^2}{6\alpha}\right]u_i + \left[\frac{\alpha}{\Delta x^2} + \frac{\beta^2}{12\alpha} - \frac{\beta}{2\Delta x} - \frac{\beta\delta\Delta x}{24\alpha}\dot{w} + \frac{\sigma}{12}\dot{w}\right]u_{i+1} \qquad (10)$$

Further simplification of equation (8) gives,

$$= \left[\frac{1}{12} - \frac{\beta\Delta x}{24\alpha}\right]u_{i-1} + \frac{5}{6}u_i + \left[\frac{1}{12} + \frac{\beta\Delta x}{24\alpha}\right]u_{i+1} \qquad (11)$$

Print ISSN: 2053-2229 (Print), Online ISSN: 2053-2210 (Online)

Let the boundary conditions be homogenous, then our system can be written as:

$$AU' = \left(B + \sigma \dot{W} A\right)U \tag{12}$$

In which A and B are Tridiagonal matrices as follows:

$$A = Tri\left[\frac{1}{12} - \frac{\beta\Delta x}{24\alpha}, \; \frac{5}{6}, \; \frac{1}{12} + \frac{\beta\Delta x}{24\alpha}\right] \tag{13}$$

$$B = Tri\left[\sum_{1}, \ldots, \sum_{2}, \ldots, \sum_{3}\right]$$

(14)

where

$$\sum_{1} = \frac{\alpha}{\Delta x^2} + \frac{\beta^2}{12\alpha} - \frac{\beta}{2\Delta x}$$

$$\sum_{2} = \frac{-2\alpha}{\Delta x^2} - \frac{\beta^2}{6\alpha}$$

$$\sum_{3} = \frac{\alpha}{\Delta x^2} + \frac{\beta^2}{12\alpha} + \frac{\beta}{2\Delta x}$$

(15)

and

$$U = \left[U_i(t), \ldots, U_{M-1}(t)\right]^T$$

$$U' = \left[U_i'(t), \ldots, U_{M-1}'(t)\right]^T \tag{16}$$

STABILITY ANALYSIS

Considering the advection-diffusion test problem $u_t = au_{xx} + \beta u_x$ with periodicity condition at $x = 0$ where both of these terms appear , then we integrate the advective term using different time stepping schemes and such approach is called splitting and is used to apply the appropriate method for each term in the advection – diffusion equation. Considering the splitting method. First we consider a comparison of cases where the advective term is integrated in time with the explicit Euler but the diffusive term is integrated in time with the Euler's method (FTCS) or the implicit Euler's method. The standard semi- discrete system is

Print ISSN: 2053-2229 (Print), Online ISSN: 2053-2210 (Online)

$$w_j(t)=\frac{1}{\Delta t}(w_{j+1}(t)-w_j(t)) \;,\;\; w'_j(t)=\frac{\beta}{2h}(w_{j-1}(t)+w_{j+1}(t)) \text{ and}$$

$$w''_j(t)=\frac{1}{h^2}(w_{j-1}(t)-2w_j(t)+w_{j+1}(t), \quad j=1,2,..,m \tag{17}$$

With $w_0=w_m$ and $w_{m+1}=w_1$. First, consider utilizing the Euler's method for both for both the advective and diffusive term (FTCS) . With the discretization of the original equation we obtain

$$\frac{w_j^{n+1}-w_j^n}{\Delta t}=\frac{\beta}{2h}(w_{j-1}^n-w_{j+1}^n)+\frac{c}{h^2}(w_{j-1}^n-2w_j^n+w_{j+1}^n)=g(x,t) \tag{18}$$

Application of the explicit Euler-method now gives the fully discrete scheme

$$w_j^{n+1}=w_j^n+\frac{\beta\Delta t}{2h}(w_{j-1}^n-w_{j+1}^n)+\frac{c\Delta t}{h^2}(w_{j-1}^n-2w_j^n+w_{j+1}^n)=\Delta t g(x,t) \tag{19}$$

the stability criteria can be established by inserting the discrete Fourier modes.Thus, we put $w_j^0=(\phi_k)=e^{2\pi ikx_j}$ and we make the "Ansatz" $w_j^{n+1}=rw_j^n$, that is $w_j^n=A^ne^{2\pi ikx_j}$ $or\, w_j^n=A^n\exp(ik\Delta_j)$ for $n\geq 0$.

Inserting into equation (19) yields,

$$\frac{A^{n+1}e^{ik\Delta(j+1)}-A^ne^{ik\Delta j}}{\Delta t}=-\beta\frac{-A^ne^{ik\Delta(j-1)}+A^ne^{ik\Delta(j+1)}}{2h}+c\frac{A^ne^{ik\Delta(j-1)}-2A^ne^{ik\Delta j}+A^ne^{ik\Delta(j+1)}}{h^2} \tag{20}$$

Solving the equation for the amplification factor A, we discovered that

$$A=1-iC\sin k\Delta-2\frac{C}{R}(1-\cos k\Delta) \text{ or } A=1-iC\sin kh-2\frac{C}{R}(1-\cos kh) \tag{21}$$

Where C is the courant number we saw with discussion of the advection equation, R is the non dimensional number that compares the viscous and interval effects over grid.

$$R=\frac{C\Delta}{A}$$

which is called the cell Reynolds numbers. Then the magnitude of A can be expressed as

$$|A|^2=\left[1-2\frac{C}{R}(1-\cos k\Delta\right]^2+C^2\sin^2 k\Delta \tag{22}$$

When $k\Delta = 0$ and π, the stability of scheme become critical $(|A| \approx 1)$. Expanding equation (22) using Taylor series about these critical points of $k\Delta$, we observe that

$$|A|^2 \to 1 + C^2 (k\Delta)^2 - 2\frac{C}{R}(k\Delta)^2 + \sigma((k\Delta x))^4 \text{ for } k\Delta \to 0 \tag{23}$$

$$|A|^2 \to 1 - 8\frac{C}{R} + 16\left(\frac{C}{R}\right)^2 + 0((K\Delta + \pi^2)) \text{ for } k\Delta \to \pi \tag{24}$$

which we need then to satisfy $|A|^2 \leq 1$ for stability. Therefore, we find that $C \leq \frac{2}{R}$ and $C \leq \frac{R}{2}$ must be satisfied for the stable numerical integration based on the explicit Euler's method for both advective and diffusive stochastic terms. Considering the implicit Euler's scheme or method to demonstrate the changes in stability characteristics from equation (18) as

$$\frac{A^n e^{ik\Delta(j-1)} - A^n e^{ik\Delta j}}{\Delta t} = -\beta \frac{-A^n e^{ik\Delta(j-1)} + A^n e^{ik\Delta(j+1)}}{2\Delta x} + c\frac{A^n e^{ik\Delta(j-i)} - 2A^n e^{ik\Delta j} + A^n e^{ik\Delta(j+1)}}{\Delta x^2} \tag{25}$$

Utilizing the Fourier interpretation in the above equation we can find

$$A = \frac{1 - iC\sin k\Delta}{1 + 2\frac{C}{R}(1 - \cos k\Delta)} \tag{26}$$

For a stable calculation, we require that

$$|A| = \frac{\sqrt{1 + C^2 \sin k\Delta}}{1 + 4\frac{C}{R}\sin^2\frac{k\Delta}{2}} \tag{25}$$

Therefore, the stability condition indicates :

(1) $|\lambda| \leq 1$ where the system is symmetric. (2) $\lambda \leq 1$ must be. Since our schemes are tridiagonal matrix we obtain $1 - 4r\sin^2\frac{8\pi}{2j}$. $s = 1, 2, \ldots j-1$ and $j\Delta x = 1$.

NUMERICAL ANALYSIS

Some numerical examples were conducted to test the validity, accuracy, efficient and robustness of the schemes derived.

Print ISSN: 2053-2229 (Print), Online ISSN: 2053-2210 (Online)

Problem A:

$$\frac{\partial u(x,t)}{\partial t} = -c\frac{\partial^2 u)x,t)}{\partial x^2} + \beta\frac{\partial u(x,t)}{\partial x}, \qquad 0 \le x \le 1$$

The exact solution

$$u^{ex}(x,t) = a(\exp(bt + C_x x), \quad C_x = \frac{v_1 \pm \sqrt{v_x^2 + 4k_x}}{2k_x}$$

The initial and boundary conditions are obtained from the analytic solution as

$$u(x,0) = u^{ex}(x,0), \qquad u(0,t) = u^{ex}(0,t) \text{ and } u(1,t) = u^{ex}(1,t)$$

Considering different times $t = 0.1, 0.5, 0.9$ Secs .As it clear from these figures that a good agreement between the two results are obtained.

Problem B:

Considering one dimensional stochastic advection – diffusion equation

$$C = 0.01, \quad \beta = 1.0$$

$$u(x,0) = f(x) = \exp\left(\frac{(-x+0.5)^2}{0.00125}\right),$$

$$u(x=0,t) = g_0(x) = \frac{0.025}{\sqrt{0.000625 + 0.02t}}\exp\left(\frac{-(0.5-t)^2}{0.00125+0.04t}\right)$$

$$u(x=1,t) = g_1(x) = \frac{0.025}{\sqrt{0.000625 + 0.02t}}\exp\left(\frac{-(1.5-t)^2}{0.00125+0.04t}\right)$$

Exact solution is given by

$$u(x,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}}\exp\left(\frac{-(x+0.5-t)^2}{0.00125+0.04t}\right)$$

The problem is solved at different cases for space size step and time step.

Problem C:

Print ISSN: 2053-2229 (Print), Online ISSN: 2053-2210 (Online)

Considering one dimensional stochastic advection – diffusion equation with $C = 1.0,\ \beta = 2.0$ with initial and boundary condition respectively is given by

$$ux,t) = 0.0, \quad u(x = 0, t) = a\exp(bt)), \quad (0 < t < T)$$

$$u(x = 1, t) = a\exp(bt + C_x x)$$

$$u(x., t = 0) = a\exp(C_X)$$

The analytic solution of the problem is given by

$$u(x,t) = a\exp(bt + C_x), \quad C_x = \frac{v_x \pm \sqrt{v_x^2 + 4\beta b}}{2\beta}$$

Table 1: Assume values for the parameters

Parameters	symbols	Assumed value
Dispersion coefficients	v	1
Velocity of the water flow	β	1
Gradient interval	Δx	2.5
Decay coefficient	γ	0.2
Concentration at the inlet	C_{in}	100
Length of the channel	L	10

Table 2: Computation of the exact solution for the concentration of the reaction at x=0.0m, 2.5m, 5.0m, 7.5m and 10m.

C(ml/dm^3)	$x = 0.0$	$x = 2.5m$	$x = 5m$	$x = 7.5m$	$x = 10m$
Exact solution	85.4102	55.7245	36.3626	23.8408	17.7334
$\Delta x = 0.0125$	85.4100	55.7259	36.3644	23.8417	17.7247
$\Delta x = 0.25$	85.4095	55.7299	36.3697	23.8443	17.6987
$\Delta x = 2.5$	85.3426	56.2335	37.0471	24.4888	17.0750

Table 2, shows the results for difference space size step and difference time size step. It is clear that a good agreement between the analytic solution and scheme derived because results obtained converges to the same point thereby reducing the errors obtained,

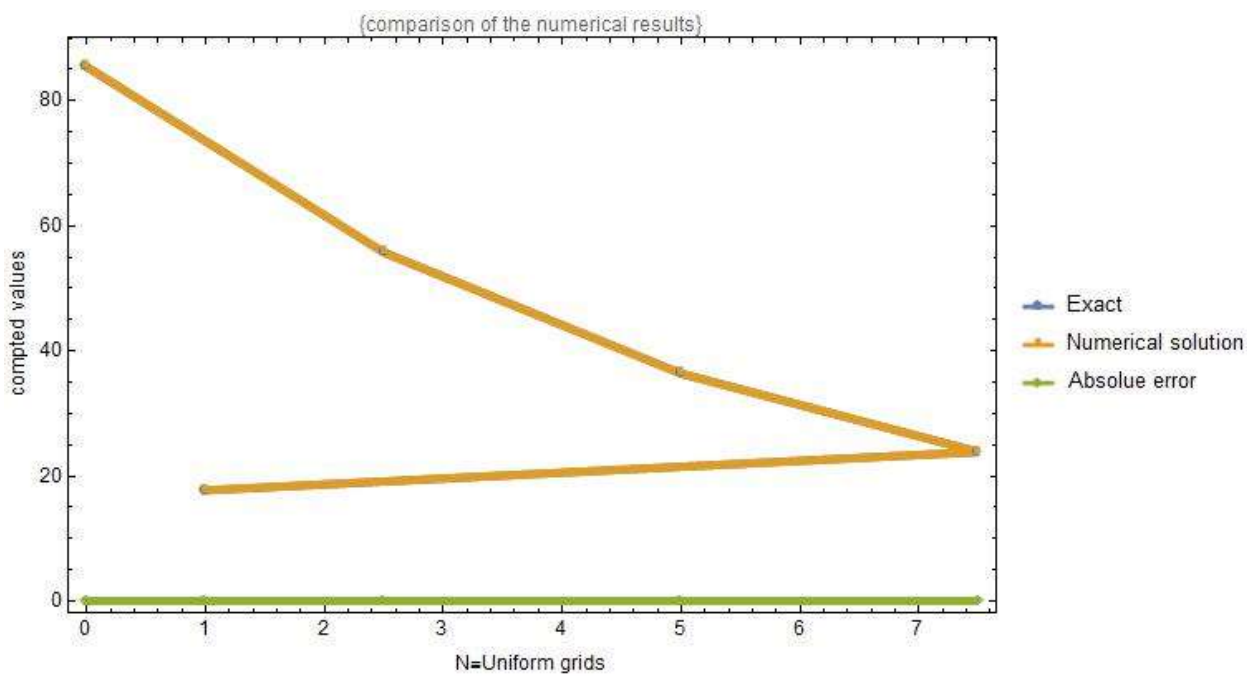
Print ISSN: 2053-2229 (Print), Online ISSN: 2053-2210 (Online)

Table 3: Comparison of the numerical results

x	Simple explicit-implicit method	Exact solution	Absolute Error
0.0	85.40949391791340	85.410217907988ii	0,00072399007471
2.5	55.72990523497476	55.72454779633919	0.00535743863357
5	36.36967495688857	36.36263229460928	0.00704266227029
7.5	23.84930920620534	23.84077640806307	0.00353279814227
10	1.69873908593070	17.73340643352621	0.03466734729551

It is clear in table 3 that there is a good agreement between the analytic solution and scheme derived with minimum error obtained, and the error become clear when using larger size step for time and space.

Figure 1:Comparison of the numerical results



It is clear in figure 1 that there is a good agreement between the exact solution and numerical solution with minimum error obtained, and the error become clear when using larger size step for time and space. Finally errors obtain are less than zero.

CONCLUSION

In this paper, we solved stochastic advection- diffusion differential equation using compact finite difference techniques, explicit – implicit Euler's method and investigated the stability condition theoretically and numerically. Mathematical software was used in the implementation .Numerical experiment conducted show that the proposed schemes are unconditionally stable for .

$$\frac{1}{2} \leq \theta$$

REFFERENCES

Ahmed, S. G, (2012),A numerical algorithm for solving advection – diffusion equation with constant and variable coefficients, *The open numerical method Journal,* Vol. 4, pp.1-7

Barles, G and Soner, H; (1998), Option pricing with transaction costs and nonlinear Black - Scholes equation, Finance stochastic, Vol. 2, pp. 369 - 397.

Ben berg, J .and Gut finger , C (1992), Heat transfer to a draining film, *International Journal of Heat Transfer*, Vol.16, pp.505- 512.

Bertram, D and F, Michel (2008), High –order compact finite difference scheme for option pricing in stochastic volatility models, *Institute of mathematics and information France.*

Bishehniassar, M .and Soheili, A. R (2013) Approximation of stochastic advection – diffusion equations (SADEs) using compact finite difference technique, *Iranian Journal of science and technology (IJST*) 37a3 (special issues on mathematics, pp. 327- 333.

Dai , W, and Nassir, R (2002) A compact finite difference for solving a three dimensional heat transport equation in a thin film, *Numerical Method partial differential equation* . Vol.16, pp. 441 – 458.

Kalita , J ; Dala , D and Dass, A ; (2002), A Class of high –order compact finite scheme for the unsteady two dimensional convection – diffusion equation with variable coefficients, *International Journal for methods in fluids*, Vol.38, pp 781 -800.

Lele, S.K (1992) Compact finite difference schemes with spectral- Like resolution, *Journal of computational physics,* 103, pp.16-42.

Oyakhire,F.I, OKORO, F.M and Asibor, R.E (2016), High order compact finite difference schemes for solving Poisson equation using Pade approximation, *Journal of the Nigerian Association of Mathematical Physics*, V0l. 38, pp.117-126.

Print ISSN: 2053-2229 (Print), Online ISSN: 2053-2210 (Online)