Higher Inductive Types as Homotopy-Initial Algebras

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Venované mojej babke, Márii Sojákovej, za jej podporu a lásku. Babka, mám Ťa rada.

Abstract

Homotopy type theory is a new field of mathematics based on the recently-discovered correspondence between Martin-Löf's constructive type theory and abstract homotopy theory. We have a powerful interplay between these disciplines - we can use geometric intuition to formulate new concepts in type theory and, conversely, use type-theoretic machinery to verify and often simplify existing mathematical proofs. Higher inductive types form a crucial part of this new system since they allow us to represent mathematical objects from a variety of domains, such as spheres, tori, pushouts, and quotients, in the type theory.

In this thesis we formulated and investigated a class of higher inductive types we call W-quotients which generalize Martin-Löf's well-founded trees to a higher-dimensional setting. We have shown that a propositional variant of W-quotients, whose computational behavior is determined up to a higher path, is characterized by the universal property of being a homotopy-initial algebra. As a corollary we derived that W-quotients in the strict form are homotopy-initial. A number of other interesting higher inductive types (such as spheres, suspensions, and type quotients) arise as special cases of W-quotients, and the characterization of these as homotopy-initial algebras thus follows as a corollary to our main result.

We have investigated further higher inductive types - arbitrary truncations, set quotients, and groupoid quotients - and established analogous characterizations in terms of the universal property of homotopy-initiality. Furthermore, we showed that set and groupoid quotients can be recovered from W-quotients and truncations.

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1

Introduction

The mathematical discipline of type theory was developed by B. Russell as an alternative to set theory which would not suffer from Russell's paradox. A type theory is a deductive formal system consisting of inference rules which specify how to assign a *type* to a *term*. In relation to set theory, a type can thought of as a set and a term of a particular type as an element of this set.

Since its invention, type theory has been widely used as a basis for programming languages, proof assistants, and formalization of mathematics (SML [24], Haskell [19], Coq [8, 32], Agda [26], NuPRL [12], Twelf [27]). Among the most studied type theories is Martin-Löf's intuitionistic type theory ([20, 22]), also known as constructive or dependent type theory. It comes in two general flavors - *intensional* and *extensional* - which differ in their treatment of equality between terms. In the intensional system, we have two notions of equality: there is *definitional equality*, which is syntax-based and can only be reasoned about on the meta-level, and *propositional equality*, which may be asserted and reasoned about from within the system. In the extensional system, these two notions of equality coincide, which means that any two terms which are provably equal in the theory are considered syntactically interchangeable. Proofs of equality only serve to establish this identification and do not hold any computational content of their own. These features make extensional Martin-Löf's type theory easier to use and more suited for implementing everyday set-level mathematics.

On the other hand, the newly-developed field of homotopy type theory uncovers deep connections between (an extension of) intensional Martin-Löf's type theory and the fields of abstract homotopy theory, higher categories, and algebraic topology ([2, 6, 7, 9, 10, 14, 16, 34, 35, 37, 38]). Insights from homotopy theory are used to add new concepts to the type theory, such as the representation of various geometric objects as higher inductive types. Conversely, type theory is used to formalize and verify existing mathematical proofs using proof assistants such as Coq and Agda. Moreover, type-theoretic insights often help us discover novel proofs of known results which are simpler than their homotopy-theoretic versions: the calculation of $\pi_n(\mathbf{S}^n)$ ([13, 15]); the Freudenthal Suspension Theorem [33]; the Blakers-Massey Theorem [33], etc.

As a formal system, homotopy type theory (HoTT) [33] extends intensional Martin-Löf type theory with two features motivated by abstract homotopy theory: Voevodsky's *univalence axiom* ([10, 37]) and *higher inductive types* ([17, 30]). The slogan in HoTT is that *types are topological spaces, terms are points, and proofs of identity are paths between points*. The structure of an identity type in HoTT is thus far more complex than just consisting of reflexivity paths [16, 34], despite the definition of $\mathsf{Id}_A(x, y)$ as an inductive type with a single constructor $1_x : \mathsf{Id}_A(x, x)$. It is a beautiful, and perhaps surprising, fact that not only does this richer theory admit an interpretation into homotopy theory ([2], [10]) but that many fundamental concepts and results from mathematics arise naturally as constructions and theorems of HoTT.

For example, the circle $\mathbf{S}^1$ (see section 2.4) is defined as a higher inductive type with a point base and a path loop going from base to itself. It comes with a recursion principle which says that to construct a function $f : \mathbf{S}^1 \to X$, it suffices to supply a point $x : X$ and a loop based at x. The value $f(\mathsf{base})$ then computes to x. Such definitional computation rules are convenient to work with but also pose some conceptual difficulties. For instance, an alternative encoding of the circle as a higher inductive type $\mathbf{S}^1_a$ specifies two points south, north and two paths from north to south, called east and west. The recursion principle then says that in order to construct a function $f : \mathbf{S}^1_a \to X$, it suffices to supply two points $x, y : X$ and two paths between them. The values $f(\mathsf{north})$ and $f(\mathsf{south})$ then compute to x and y respectively.

We have a natural way of relating these two representations via an equivalence, *i.e.*, a function which has an inverse up to propositional equality: in one direction, map base to north and loop to east; in the other direction, map both north and south to base and map east to loop and west to the identity path at base. Unfortunately, the types $\mathbf{S}^1$, $\mathbf{S}^1_a$ related this way do *not* satisfy the same definitional laws, which poses a compatibility issue. Even more importantly, we do not have a way of *internalizing* these notions of a circle and working with them inside the type theory, since we can only talk about definitional equalities on the meta-level.

In this work we thus study higher inductive types abstractly, as arbitrary types endowed with certain constructors and *propositional* computation behavior: in the case of $\mathbf{S}^1$, for example, we say that a type C with constructors $b : C$ and $l : \mathsf{Id}_C(c, c)$ satisfies the recursion principle for a circle if for any other type X, point $x : X$ and loop based at x, there exists a function $f : C \to X$ for which there is a *path between* $f(b)$ *and* x (and which satisfies a higher coherence condition). We note that we require *no* change to the underlying type theory: the particular higher inductive type $\mathbf{S}^1$ just becomes a specific instance of the abstract definition of a circle, one whose computation rules happen to hold definitionally.

A major advantage of types with propositional computation rules is that we can internalize the definitions and reason about them within the type theory - and in particular, use proof assistants to verify the results. In this respect, our work is complementary to [18], which gives an external, category-theoretic semantics for a certain class of higher inductive types. Another advantage of propositional computation behavior is portability: relaxing the computation laws satisfied by the types $\mathbf{S}^1$ and $\mathbf{S}^1_a$ to their propositional counterparts results in two notions of a circle that are equivalent, in a precise sense. This in particular means that any type C which is a circle according to one definition is also a circle according to the alternate definition. We can thus state and prove results about either of these specifications, knowing that the proofs carry over to any particular implementation - be it $\mathbf{S}^1$, $\mathbf{S}^1_a$, or a third one.

It further turns out that types with propositional rules tend to keep many of their desirable properties; for instance, it can be shown that the main result of [15], that the fundamental group of the circle is the group of integers, carries over to the case when *both* the circle and the integer types have propositional computational behavior. In addition, we can now show that higher inductive types are characterized by the universal property of being a *homotopy-initial algebra*. This notion was first introduced in [3], where an analogous result was established for the "ordinary" inductive type of well-founded trees (Martin-Löf's W-types [21]). In the higher setting, an *algebra* is a type X together with a number of finitary operations $f, g, h \ldots$, which are allowed to act not only on X but also on any higher identity type over X, for example $\mathsf{Id}_X(\ldots,\ldots)$ or $\mathsf{Id}_{\mathsf{Id}_X(\ldots,\ldots)}(\ldots,\ldots)$. An *algebra morphism* has to preserve all operations up to propositional equality. Finally, just as for W-types, we say that an algebra $\mathcal{X}$ is *homotopy-initial* if the type of algebra morphisms from $\mathcal{X}$ to any other algebra $\mathcal{Y}$ is contractible, *i.e.*, there exists a morphism from $\mathcal{X}$ to $\mathcal{Y}$ which is unique up to propositional equality.

Our main theorem is stated for a class of higher inductive types which we call W-quotients; they generalize ordinary W-types by allowing an arbitrary number of *path* constructors (which construct canonical terms of the identity type over the higher inductive type X) in addition to the ordinary *point* constructors (which construct canonical terms of the higher inductive type X itself). We show that the induction principle for W-quotients is equivalent (as a type) to homotopy-initiality. This extends the main result of [3] for "ordinary" inductive types to the important, and much more difficult, higher case. A number of other interesting higher inductive types, such as spheres, suspensions, and type quotients (see 2.4) arise as special cases of W-quotients, and the characterization of these as homotopy-initial algebras thus follows as a corollary to our main result.

The chief significance of homotopy-initiality lies in its simplicity; the induction principle, which involves dependent types, can be rather complicated to state and use even for higher inductive types with relatively simple data, such as the torus. On the other hand, proving that an algebra is homotopy-initial is generally a more pleasant endeavor, since we only have to care about satisfying the universal property with respect to non-dependent types. Moreover, our treatment is internal to the type theory and hence fully constructive and formalizable; this in particular means that once we prove that an algebra is homotopy-initial, we can run our algorithm to automatically recover the term witnessing the induction principle.

We also investigate further higher inductive types - truncations 4.1, set quotients 4.2, and groupoid quotients 4.3 - which do not arise as W-quotients in an obvious way, and establish analogous characterizations in terms of homotopy-initiality. Furthermore, we show that set and groupoid quotients can be recovered from W-quotients and truncations. Since recent work by E. Rijke and F. van Doorn shows that truncations can be recovered from type quotients (which are themselves subsumed by W-quotients) our result implies that set and groupoid quotients are special cases of W-quotients. We conjecture that this holds for all higher inductive types described in the first 9 chapters of [33].

If this is case, we would have one possible convincing answer to the still-open question of "what is a higher inductive type". As mentioned above, our approach is complementary to the one pursued by M. Shulman and P. Lumsdaine [18], who aim to develop a unifying schema for higher inductive types; the possible relationship and/or interplay between the two approaches is unknown at this time. On the other hand, there is a strong relationship between our approach and the work of van Doorn, Rijke, and others, on reducing general higher inductive types to a combination of a few small "building blocks". Finally, we designed all the proofs in this thesis with the intent of formalization in the Coq proof assistant, which is a work in progress.

The remainder of the thesis is organized as follows. We first provide some background in chapter 2, which summarizes the basic concepts in homotopy type theory and describes higher inductive types. We also review the well-known result of P. Dybjer, which characterizes inductive types as initial algebras (in the setting of extensional type theory). In chapter 3, we introduce the notions of algebras, morphisms, and homotopy-initiality for higher inductive types. We give a formal definition of W-quotients and prove our main result - the characterization of W-quotients as homotopy-initial algebras. We conclude the chapter by showing how to use the main result to derive analogous characterizations for higher inductive types definable as W-quotients, for example the circle. Finally, in chapter 4 we establish the characterization of truncations as homotopy-initial algebras and present the reduction of set and groupoid quotients to W-quotients (plus truncations). We derive the characterization of set and groupoid quotients as homotopy-initial algebras as a corollary to this reduction.

2

Background

We now review some background needed for the main chapters 3 and 4. In the first section, we briefly describe the extensional and intensional varieties of dependent type theory and indicate where homotopy type theory fits in the picture. The second section is devoted to the basic concepts of homotopy type theory. In the third section, we discuss Martin-Löf's W-types, a well-studied class of inductive types which in the extensional setting subsumes many other inductive types of interest such as natural numbers and lists. We give a syntactic presentation of Dybjer's result characterizing inductive types as initial algebras [5]; this sets the stage for the definitions and results of chapter 3, where we move away from considering ordinary inductive types in the extensional setting into the land of higher inductive types in the homotopy type-theoretic setting. Finally, in the last section we describe higher inductive types by giving a number of specific examples, which we will later revisit to indicate how they arise as special cases of our main construction, the W-quotient.

2.1 Intensional, Extensional, and Homotopy Type Theory

The core of Martin-Löf's intensional type theory [20], denoted $\mathcal{H}_{\text{int}}$, is a dependent type theory with the following:

- Two main forms of judgment:

$$\Gamma \vdash a : A \qquad\qquad \Gamma \vdash a \equiv b : A$$

  where the former stands for term membership, the latter for the *definitional equality* of two terms, and Γ denotes a context of assumptions.

- A cumulative hierarchy of universes $\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \ldots$ in the style of Russell [28].
- Dependent pair types $\Sigma_{x:A}B(x)$ and dependent function types $\Pi_{x:A}B(x)$ (with the non-dependent versions $A \times B$ and $A \to B$). We assume definitional η-conversion for functions and pairs, both for the sake of simplicity and to stay close to the Coq [32] proof assistant, which we intend to use for formalization.
- Identity types $\mathsf{Id}_A(x, y)$, also denoted $x =_A y$, obeying the following rules:
  - Id-formation rule.

$$\frac{\Gamma \vdash A : \mathcal{U}_i \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathsf{Id}_A(a, b) : \mathcal{U}_i}$$

  - Id-introduction rule.

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{1}_a : \mathsf{Id}_A(a, a)}$$

  - Id-elimination rule.

$$\frac{\begin{array}{c}\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash q : \mathsf{Id}_A(a, b) \\ \Gamma, x : A, y : A, p : \mathsf{Id}_A(x, y) \vdash E(x, y, p) : \mathcal{U}_j \qquad \Gamma, x : A \vdash d(x) : E(a, a, \mathsf{1}_a)\end{array}}{\Gamma \vdash \mathsf{J}(x.y.p.E(x, y, p), x.d(x), a, b, q) : E(a, b, q)}$$

  - Id-computation rule.

$$\frac{\begin{array}{c}\Gamma \vdash a : A \\ \Gamma, x : A, y : A, p : \mathsf{Id}_A(x, y) \vdash E(x, y, p) : \mathcal{U}_j \qquad \Gamma, x : A \vdash d(x) : E(a, a, \mathsf{1}_a)\end{array}}{\Gamma \vdash \mathsf{J}(x.y.p.E(x, y, p), x.d(x), a, a, \mathsf{1}_a) \equiv d(a) : E(a, a, \mathsf{1}_a)}$$

  If the type $\mathsf{Id}_A(x, y)$ is inhabited, we say that x and y are *(propositionally) equal*. If we do not care about the specific equality witness, we often simply say that $x =_A y$ or if the type A is clear, $x = y$.

Martin-Löf's extensional type theory [22], denoted $\mathcal{H}_{\text{ext}}$, is obtained by extending $\mathcal{H}_{\text{int}}$ with the following rule:

- Identity reflection.

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash p : a =_A b}{\Gamma \vdash a \equiv b : A}$$

Furthermore, the identity reflection rule implies the following:

- Uniqueness-of-identity-proofs principle (UIP) aka Streicher's K [31]:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash p : a =_A b \qquad \Gamma \vdash q : a =_A b}{\Gamma \vdash p =_{\mathsf{Id}_A(a,b)} q}$$

Another consequence of identity reflection is the principle of *function extensionality*, which states that two pointwise-equal functions are equal as maps. This justifies the usage of the terminology *extensional type theory* for $\mathcal{H}_{\text{ext}}$.

In practice, we often extend the type theories $\mathcal{H}_{\text{int}}$ and $\mathcal{H}_{\text{ext}}$ with specific inductive types, such as finite types, coproducts, natural numbers, well-founded trees and so on. These type theories admit a natural computational interpretation: for instance, it is possible to show that in the empty context, every term of type $\mathbb{N}$ is definitionally equal to a numeral, a property known as the *canonicity for natural numbers*. Intuitively, this is not surprising - since the only constructor for identity types that we have available is reflexivity, we should always be able to reduce any term involving an identity eliminator by using the computation rule for identity types (and similarly for other type constructors). Translating this idea into a proof however requires a significant amount of machinery [23].

In extensional type theory, the identity reflection rule makes propositional and definitional equalities coincide. This makes the system significantly easier to work with and particularly suitable for reasoning about *sets* as found in everyday mathematics. Here a proof of equality does not carry any computational content - two objects can be equal in at most one way. The fact that equality is no longer tied to the syntax leads to undecidable type checking, which, however, does not have to prevent automation - the proof assistant NuPRL [12], which is also based on an undecidable type theory, manages this by working with derivations of the typing judgment rather than the terms themselves.

Inductive types in an extensional setting behave particularly nicely: for instance, they can be characterized as the *initial* objects among certain *algebras*: the type of natural numbers $\mathbb{N}$ together with zero and the successor function is precisely the initial object among algebras formed by a type C with a term $c_0 : C$ and a function $c_s : C \to C$.

This correspondence breaks down in intensional type theory as inductive types in this setting tend to be rather poorly behaved. For instance, it is not true that there is a definitionally unique function out of the empty type $\mathbf{0}$ into an arbitrary codomain C. The best we can do is to show that any two such functions must be *pointwise propositionally equal*, which, in the absence of function extensionality, does not even give us equality as functions. Right away we see that the empty type cannot be characterized as being initial among all types and this pattern of course carries over to other inductive types as well. As another consequence of this, we get that natural numbers can no longer be encoded as Martin-Löf's well-founded trees in any obvious way [5].

Thus, most of the work on the theory of inductive types has traditionally been in the extensional setting (*e.g.* [5],[25],[1]). However, the technical difficulties associated with inductive types in intensional type theory do not necessarily pose a problem when inductive types are used as a device for *programming*. Furthermore, type checking in the absence of identity reflection remains decidable, which makes variants of intensional type theory a suitable choice for application-oriented proof assistants such as Coq [32] and Agda [26].

Without identity reflection, two proofs of equality between a and b may or may not be equal themselves. Similarly, two proofs of equality between two proofs of equality between a and b may or may not be equal, and so on. Thus, in intensional type theory, proofs of equality carry *content* and are hence better understood as *paths* between *points* (terms) in *spaces* (types). This intuition is further justified by the fact that much like paths in topological spaces, proofs of identity can be reversed, concatenated, transported along mappings, and so on. Furthermore, it makes sense to talk about the *space of all paths between* a *and* b, *the space of all paths between* p *and* q, where p, q are paths between a and b, and so on.

Despite the novel way of viewing proofs of equality as paths, as first observed independently by Awodey/Warren, and Voevodsky, the purely intensional type theory discussed so far is rather limiting since we have no other path constructors besides reflexivity. In particular, we have no explicit way of constructing a path between two pointwise-equal functions or between two isomorphic types in the same universe. Homotopy type theory provides us with both - and then some. The core, denoted by $\mathcal{H}$, is an extension of the theory $\mathcal{H}_{\text{int}}$ with the *univalence axiom* [37], which can roughly be paraphrased as saying that *isomorphic types are equal*. We often extend this system further with specific *higher inductive types*, which are a generalization of ordinary inductive types. We will discuss these concepts in more detail in Sections 2.2 and 2.4.

The univalence axiom as well as (proper) higher inductive types give us new ways of constructing paths. This in particular leads to numerous examples of paths whose endpoints coincide but which are provably not equal. Thus, the new system is inconsistent with UIP and the traditional set semantics is no longer sound. Instead, we can use the *simplicial set model* [10], the cubical set model [4], or other related models, which are all highly nontrivial.

There are other far-reaching consequences of having nontrivial path constructors around: for instance, what is the justification for the principle of identity elimination if the identity type has other constructors besides reflexivity? In particular, what happens if we apply the eliminator to a path which is not reflexivity? Can a certain form of canonicity still be recovered? These are some of the more pressing open questions at the moment and are the subjects of intense ongoing research, which we do not pursue in this thesis.

2.2 Homotopy Type Theory

In extensional type theory, a type can be interpreted as a set consisting of definitionally distinct (equivalence classes of) terms with no non-trivial identity paths among them. In homotopy type theory, this discrete interpretation is no longer valid. Instead, a type is better understood as a structure loosely termed "∞-*groupoid*" [9, 16, 34], where at the lowest dimension we have all the different terms ("points"), at the next dimension up we have all the different paths between these points, a yet higher dimension has all the different paths between paths between points, and so on.

A key notion in homotopy type theory [33] is that of an *equivalence of types*: two types are called equivalent if there exists a function between them which has both a left and a right inverse up to propositional equality (an *equivalence*). The *univalence axiom* then postulates that the path space between any two types $A, B : \mathcal{U}_i$ in the universe can be characterized as the type of equivalences between A and B. In particular, equivalent types are equal and are thus indistinguishable from within the type theory. On the semantic side, examples of models consisting of "∞-groupoids" which in addition satisfy univalence are provided by the simplicial [10] and cubical [4] set models.

2.2.1 Groupoid Laws

Proofs of identity behave much like paths in topological spaces: they can be reversed, concatenated, mapped along functions, etc. Below we summarize a few of these properties:

- For any path $p : x =_A y$ there is a path $p^{-1} : y =_A x$, and we have $\left(1_x\right)^{-1} \equiv 1_x$.
- For any paths $p : x =_A y$ and $q : y =_A z$ there is a path $p \cdot q : x =_A z$, and $1_x \cdot 1_x \equiv 1_x$.
- Associativity of composition: for any paths $p : x =_A y$, $q : y =_A z$, and $r : z =_A u$ we have $(p \cdot q) \cdot r = p \cdot (q \cdot r)$.
- We have $1_x \cdot p = p$ and $p \cdot 1_y = p$ for any $p : x =_A y$.
- For any $p : x =_A y$, $q : y =_A z$ we have $p \cdot p^{-1} = 1_x$, $p^{-1} \cdot p = 1_y$, and $(p^{-1})^{-1} = p$, $(p \cdot q)^{-1} = q^{-1} \cdot p^{-1}$.
- For any type family $P : A \to \mathcal{U}_i$ and path $p : x =_A y$ there are functions $p_*^P : P(x) \to P(y)$ and $p^*_P : P(y) \to P(x)$, called the *covariant transport* and *contravariant transport*, respectively. We furthermore have $(1_x)_*^P \equiv (1_x)^*_P \equiv \mathsf{id}_{P(x)}$.
- We have $(p^{-1})_*^P = p^*_P$, $(p^{-1})^*_P = p_*^P$ and $(p \cdot q)_*^P = q_*^P \circ p_*^P$, $(p \cdot q)^*_P = p^*_P \circ q^*_P$ for any family $P : A \to \mathcal{U}_i$ and paths $p : x =_A y$, $q : y =_A z$.
- For any function $f : A \to B$ and path $p : x =_A y$, there is a path $\mathsf{ap}_f(p) : f(x) =_B f(y)$ and we have $\mathsf{ap}_f(1_x) \equiv 1_{f(x)}$.
- We have $\mathsf{ap}_f(p^{-1}) = \mathsf{ap}_f(p)^{-1}$ and $\mathsf{ap}_f(p \cdot q) = \mathsf{ap}_f(p) \cdot \mathsf{ap}_f(q)$ for any $f : A \to B$ and $p : x =_A y$, $q : y =_A z$.
- We have $\mathsf{ap}_{g \circ f}(p) = \mathsf{ap}_g(\mathsf{ap}_f(p))$ for any $f : A \to B$, $g : B \to C$ and $p : x =_A y$.
- For any $f : \Pi_{x:A} B(x)$ and path $p : x =_A y$, there are paths $\mathsf{dap}_f(p) : p_*^B(f(x)) =_{B(y)} f(y)$

and $\mathsf{dap}^f(p) : p_B^*(f(y)) =_{B(x)} f(x)$. We have $\mathsf{dap}_f(1_x) \equiv \mathsf{dap}^f(1_x) \equiv 1_{f(x)}$.

- All constructs respect propositional equality.

2.2.2 Truncation Levels

In general, the structure of paths on a type A can be rather complex - we can have many distinct *0-cells* $x, y, \ldots : A$; there can be many distinct *1-cells* $p, q, \ldots : x =_A y$; there can be many distinct *2-cells* $\gamma, \delta, \ldots : p =_{x=_A y} q$; ad infinitum. The hierarchy of truncation levels describes those types which are, informally speaking, trivial beyond a certain dimension: a type A of truncation level n, also called an n-type, can be characterized by the property that all m-cells for $m > n$ with the same source and target are equal. From this intuitive description we can see that the hierarchy of truncation levels is cumulative, in the sense that if a type if of truncation level n, then it is also of truncation level $n + 1$.

It is customary to also speak of truncation levels -2 and -1, called *contractible types* and *mere propositions* respectively:

Definition 1. *A type $A : \mathcal{U}_i$ is called* contractible *if there exists a point $a : A$ such that any other point $x : A$ is equal to a:*

$$\mathsf{isContr}(A) :\equiv \Sigma_{a:A}\Pi_{x:A}(a =_A x)$$

A type $A : \mathcal{U}_i$ is called a mere proposition *if all its inhabitants are equal:*

$$\mathsf{isProp}(A) :\equiv \Pi_{x,y:A}(x =_A y)$$

A contractible type can be seen as having exactly one inhabitant, up to equality; a mere proposition can be seen as having at most one inhabitant, up to equality. Clearly:

Lemma 2. *If $A : \mathcal{U}_i$ is contractible then $A : \mathcal{U}_i$ is a mere proposition.*

The existence of a path between any two points implies more than just path-connectedness:

Lemma 3. *If $A : \mathcal{U}_i$ is a mere proposition, then $x =_A y$ is contractible for any $x, y : A$.*

Thus, contractible types are in a sense the “nicest” possible: any two points are equal up to a *1-cell*, which itself is unique up to a *2-cell*, which itself is unique up to a *3-cell*, and so on. Mere propositions are the “nicest” ones after contractible types.

We can generalize the idea of the previous lemma to give a recursive definition of an n-type, for $n \geq -2$ (since we recurse on the natural numbers, we subtract 2):

Definition 4. *We define a predicate* $\mathsf{is}\text{-}(n-2)\text{-}\mathsf{type} : \mathcal{U}_i \to \mathcal{U}_i$ *by recursion on $n : \mathbb{N}$ as follows:*

$$\begin{aligned}&\mathsf{is}\text{-}(-2)\text{-}\mathsf{type}(A) :\equiv \mathsf{isContr}(A)\\&\mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}(A) :\equiv \Pi_{x,y:A}\mathsf{is}\text{-}(n-2)\text{-}\mathsf{type}(x =_A y)\end{aligned}$$

Using lemma 3, we can easily show the following:

Corollary 5. *For $n : \mathbb{N}$, if $A : \mathcal{U}_i$ is an $(n-2)$-type then it is also an $(n-1)$-type.*

Corollary 6. *For $n : \mathbb{N}$, if $A : \mathcal{U}_i$ is an $(n-2)$-type then for any $x, y : A$, the type $x = y$ is also an $(n-2)$-type.*

2.2.3 Homotopies

A homotopy between two functions is in a sense a "natural transformation":

Definition 7. *For $f, g : \Pi_{x:A} B(x)$, we define the type*

$$f \sim g :\equiv \Pi_{a:A} \mathsf{Id}_{B(a)}(f(a), g(a))$$

and call it the type of homotopies *between f and g.*

Definition 8. *For $f : A \to B$ and $g : A' \to B$, we define the type*

$$f \sim_{\mathcal{H}} g :\equiv \Pi_{a:A} \Pi_{a':A'} \mathsf{Id}_B(f(a), g(a'))$$

and call it the type of heterogeneous homotopies *between f and g.*

We now observe the following:

- For any $f, g : X \to Y$, $p : x =_X y$, $\alpha : f \sim g$, there is a path

  $$\mathsf{nat}(\alpha, p) : \alpha(x) \cdot \mathsf{ap}_g(p) = \mathsf{ap}_f(p) \cdot \alpha(y)$$

  defined in the obvious way by induction on p and referred to as the naturality of the homotopy α. Pictorially, we have

  $$\begin{array}{ccc} f(x) & \overset{\alpha(x)}{\rule{4em}{0.4pt}} & g(x) \\ \mathsf{ap}_f(p) \Big| & \mathsf{nat}(\alpha, p) & \Big| \mathsf{ap}_g(p) \\ f(y) & \underset{\alpha(y)}{\rule{4em}{0.4pt}} & g(y) \end{array}$$

- For any $f, g : \Pi_{x:X} Y(x)$, $p : x =_X y$, $\alpha : f \sim g$, there is a path

  $$\mathsf{nat}_{\mathcal{F}}(\alpha, p) : \mathsf{ap}_{p_*^Y}(\alpha(x)) \cdot \mathsf{dap}_g(p) = \mathsf{dap}_f(p) \cdot \alpha(y)$$

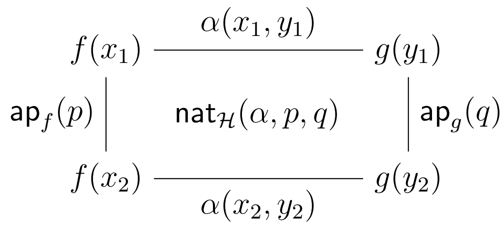
  defined in the obvious way by induction on p and referred to as the naturality of the "fibered" homotopy α. Pictorially, we have

  $$\begin{array}{ccc} p_*^Y(f(x)) & \overset{\mathsf{ap}_{p_*^Y}(\alpha(x))}{\rule{4em}{0.4pt}} & p_*^Y(g(x)) \\ \mathsf{dap}_f(p) \Big| & \mathsf{nat}_{\mathcal{F}}(\alpha, p) & \Big| \mathsf{dap}_g(p) \\ f(y) & \underset{\alpha(y)}{\rule{6em}{0.4pt}} & g(y) \end{array}$$

- For any $f : X \to Z$, $g : Y \to Z$, $p : x_1 =_X x_2$, $q : y_1 =_Y y_2$, $\alpha : f \sim_{\mathcal{H}} g$, there is a path

  $$\mathsf{nat}_{\mathcal{H}}(\alpha, p, q) : \alpha(x_1, y_1) \cdot \mathsf{ap}_g(q) = \mathsf{ap}_f(p) \cdot \alpha(x_2, y_2)$$

  defined in the obvious way by induction on p and q and referred to as the naturality of the heterogeneous homotopy α. Pictorially, we have



2.2.4 Equivalences

Definition 9. *A map $f : A \to B$ is called an* equivalence *if it has both a left and a right inverse:*

$$\mathsf{isEq}(f) :\equiv \left(\Sigma_{g:B\to A}(g \circ f \sim \mathsf{id}_A)\right) \times \left(\Sigma_{h:B\to A}(f \circ h \sim \mathsf{id}_B)\right)$$

We define

$$(A \simeq B) :\equiv \Sigma_{f:A\to B}\mathsf{isEq}(f)$$

and call A and B equivalent *if the above type is inhabited.*

Unsurprisingly, we can prove that A and B are equivalent by constructing functions going back and forth, which compose to identity on both sides[1]; this is also a necessary condition.

Lemma 10. *Two types A and B are equivalent if and only if there exist functions $f : A \to B$ and $g : B \to A$ such that $g \circ f \sim \mathsf{id}_A$ and $f \circ g \sim \mathsf{id}_B$.*

We will refer to such functions f and g as forming a *quasi-equivalence* and say that f and g are *quasi-inverses* of each other. From this we can easily show:

Lemma 11. *Equivalence of types is an equivalence relation.*

We call A and B *logically equivalent* if there are exist functions $f : A \to B$, $g : B \to A$. It is immediate that if both types are mere propositions then logical equivalence implies $A \simeq B$. For example:

Corollary 12. *For any A,* $\mathsf{isContr}(A) \simeq A \times \mathsf{isProp}(A)$.

Corollary 13. *For any A,* $\mathsf{is}\text{-}(-1)\text{-}\mathsf{type}(A) \simeq \mathsf{isProp}(A)$.

2.2.5 Structure of Path Types

Let us first consider the product type $A \times B$. We would like for two pairs $c, d : A \times B$ to be equal precisely when their first and second projections are equal. By path induction we can easily construct a function

$$^{=}\mathsf{E}^{\times}_{c,d} : (c = d) \to (\pi_1(c) = \pi_1(d)) \times (\pi_2(c) = \pi_2(d))$$

We can show:

Lemma 14. *The map $^{=}\mathsf{E}^{\times}_{c,d}$ is an equivalence for any $c, d : A \times B$.*

[1]Although the type of such functions itself is not equivalent to $A \simeq B$, see chapter 4 of [33].

We will denote the quasi-inverse of ${}^{=}\mathsf{E}^{\times}_{c,d}$ by ${}^{\times}\mathsf{E}^{=}_{c,d}$. We have a similar correspondence for dependent pairs; however, the second projections of $c, d : \Sigma_{x:A}B(x)$ now lie in different fibers of B and we employ (covariant) transport. By path induction we can define a map

$$^{=}\mathsf{E}^{\Sigma}_{c,d} : (c = d) \to \Sigma_{(p:\pi_1(c)=\pi_1(d))}(p^B_*(\pi_2(c)) = \pi_2(d))$$

Lemma 15. *The map* ${}^{=}\mathsf{E}^{\Sigma}_{c,d}$ *is an equivalence for any* $c, d : \Sigma_{x:A}B(x)$.

We will denote the quasi-inverse of ${}^{=}\mathsf{E}^{\Sigma}_{c,d}$ by ${}^{\Sigma}\mathsf{E}^{=}_{c,d}$. We also have an analogous correspondence using a contravariant transport.

We would like for two types $A, B : \mathcal{U}_i$ to be equal precisely when they are equivalent. As before, we can easily obtain a function

$$^{=}\mathsf{E}^{\simeq}_{A,B} : (A = B) \to (A \simeq B)$$

The univalence axiom now states that this map is an equivalence:

Axiom 1 (Univalence)**.** *The map* ${}^{=}\mathsf{E}^{\simeq}_{A,B}$ *is an equivalence for any* $A, B : \mathcal{U}_i$.

We will denote the quasi-inverse of ${}^{=}\mathsf{E}^{\simeq}_{A,B}$ by ${}^{\simeq}\mathsf{E}^{=}_{A,B}$. It follows from univalence that *equivalent types are equal* and hence they satisfy the same properties:

Lemma 16. $(\mathcal{H})$ *For any type family* $P : \mathcal{U}_i \to \mathcal{U}_j$*, and types* $A, B : \mathcal{U}_i$ *with* $A \simeq B$*, we have that* $P(A) \simeq P(B)$*. Thus in particular,* $P(A)$ *is inhabited precisely when* $P(B)$ *is.*

Finally, two functions $f, g : \Pi_{x:A}B(x)$ should be equal precisely when there exists a homotopy between them. Constructing a map

$$^{=}\mathsf{E}^{\Pi}_{f,g} : (f = g) \to (f \sim g)$$

is easy. Showing that this map is an equivalence (or even constructing a map in the opposite direction) is much harder, and is in fact among the chief consequences of univalence:

Lemma 17. $(\mathcal{H}, \mathcal{H}_{\text{ext}})$ *The map* ${}^{=}\mathsf{E}^{\Pi}_{f,g}$ *is an equivalence for any* $f, g : \Pi_{x:A}B(x)$.

Proof. See chapter 4.9 of [33]. □

We will denote the quasi-inverse of ${}^{=}\mathsf{E}^{\Pi}_{f,g}$ by ${}^{\Pi}\mathsf{E}^{=}_{f,g}$. Function extensionality will turn out to be crucial for most of the development in later chapters. As an example, we note that using lemma 3 and function extensionality, we can show the following:

Corollary 18. $(\mathcal{H}, \mathcal{H}_{\text{ext}})$ *For* $n : \mathbb{N}$*,* $A : \mathcal{U}_i$*, the types* $\mathsf{isContr}(A)$*,* $\mathsf{isProp}(A)$*,* $\mathsf{is}\text{-}(n-2)\text{-}\mathsf{type}(A)$ *are all mere propositions.*

2.3 Inductive Types

In this section we give an overview of Martin-Löf's well-founded trees [21], also known as W-types, as a prominent example of an inductive type. The classical result by Dybjer [5] characterizes W-types as initial algebras of polynomial endofunctors; here we present the type-theoretic version of this correspondence, where we internalize the notions of algebras, morphisms, and initiality using the propositions-as-types principle.

Formally, given a type $A : \mathcal{U}_i$ and a type family $B : A \to \mathcal{U}_i$, the W-type $\mathsf{W}(A, B) : \mathcal{U}_i$ is the inductive type generated by the single constructor

$$\mathsf{sup} : \Pi_{a:A}(B(a) \to \mathsf{W}(A, B)) \to \mathsf{W}(A, B)$$

W-types can be seen informally as the free algebras for signatures with operations of possibly infinite arity, but no equations. Indeed, the parameters A and B can be thought of as specifying a signature that has the elements of A as operations and where for any $a : A$, the type $B(a)$ represents the arity of the operation a. For instance, the obvious way to encode the type $\mathbb{N}$ of natural numbers is to take $A :\equiv \mathbf{2}$ and $B(\top) :\equiv \mathbf{0}$, $B(\bot) :\equiv \mathbf{1}$, where $\top$ represents the natural number zero and $\bot$ represents the successor operator. This encoding is adequate within extensional type theory but, as remarked earlier, fails to exhibit the right computational behavior even up to propositional equality when working in the purely intensional setting. Within homotopy type theory, the computation laws can be recovered up to propositional equality.

The recursion principle for $\mathsf{W}(A, B)$ says that given terms

- $E : \mathcal{U}_j$,
- $e : \Pi_{a:A}(B(a) \to E) \to E$,

there is a recursor $\mathsf{rec}_{\mathsf{W}}(E, e) : \mathsf{W}(A, B) \to E$. We usually omit the parameters E and e if they are clear from the context. The recursor satisfies the computation law

- $\mathsf{rec}_{\mathsf{W}}(\mathsf{sup}(a, t)) \equiv e\big(a, \mathsf{rec}_{\mathsf{W}} \circ t\big)$ for any $a : A, t : B(a) \to \mathsf{W}(A, B)$

Similarly, we have an induction principle: given terms

- $E : \mathsf{W}(A, B) \to \mathcal{U}_j$,
- $e : \Pi_{a:A}\Pi_{t:B(a)\to\mathsf{W}(A,B)}\big(\Pi_{b:B(a)}E(t\ b)\big) \to E(\mathsf{sup}(a, t))$,

there is an inductor $\mathsf{ind}_{\mathsf{W}}(E, e) : \Pi_{w:\mathsf{W}(A,B)}E(w)$. As before, we usually omit E and e if they can be inferred. The inductor satisfies the computation law

- $\mathsf{ind}_{\mathsf{W}}(\mathsf{sup}(a, t)) \equiv e\big(a, t, \mathsf{ind}_{\mathsf{W}} \circ t\big)$ for any $a : A, t : B(a) \to \mathsf{W}(A, B)$

To present Dybjer's result, for the remainder of the section we work in extensional type theory. We first note that the induction principle implies the following uniqueness principle: given terms

- $E : \mathcal{U}_j$,
- $e : \Pi_{a:A}(B(a) \to E) \to E$,
- $f : \mathsf{W}(A, B) \to E$,
- $\beta : \Pi_{a:A}\Pi_{t:B(a)\to\mathsf{W}(A,B)}\big(f(\mathsf{sup}(a, t)) = e(a, f \circ t)\big)$,

- $g : \mathsf{W}(A,B) \to E$,
- $\gamma : \Pi_{a:A}\Pi_{t:B(a)\to\mathsf{W}(A,B)}\big(g(\mathsf{sup}(a,t)) = e(a, g\circ t)\big)$,

we have $f = g$. In other words, any two functions out of $\mathsf{W}(A,B)$ which satisfy the same recurrence are equal. We have an analogous dependent version of the above uniqueness principle: given terms

- $E : \mathsf{W}(A,B) \to \mathcal{U}_j$,
- $e : \Pi_{a:A}\Pi_{t:B(a)\to\mathsf{W}(A,B)}\big(\Pi_{b:B(a)}E(t\ b)\big) \to E(\mathsf{sup}(a,t))$,
- $f : \Pi_{w:\mathsf{W}(A,B)}E(w)$,
- $\beta : \Pi_{a:A}\Pi_{t:B(a)\to\mathsf{W}(A,B)}\big(f(\mathsf{sup}(a,t)) = e(a,t,f\circ t)\big)$,
- $g : \Pi_{w:\mathsf{W}(A,B)}E(w)$,
- $\gamma : \Pi_{a:A}\Pi_{t:B(a)\to\mathsf{W}(A,B)}\big(g(\mathsf{sup}(a,t)) = e(a,t,g\circ t)\big)$,

we have $f = g$.

To show this, we use induction with the type family $w \mapsto f(w) = g(w)$. We need a term of type $\Pi_{a:A}\Pi_{t:B(a)\to\mathsf{W}(A,B)}\big(\Pi_{b:B(a)}(f(t\ b) = g(t\ b))\big) \to \big(f(\mathsf{sup}(a,t)) = g(\mathsf{sup}(a,t))\big)$. Fix the parameters a, t and the induction hypothesis $u : \Pi_{b:B(a)}(f(t\ b) = g(t\ b))$. Using u with identity reflection gives us $f\circ t \equiv g\circ t$. Together with the premises of the uniqueness rule (and identity reflection again) this gives us $f(\mathsf{sup}(a,t)) \equiv g(\mathsf{sup}(a,t))$ as desired. The induction principle thus establishes a pointwise equality between f and g and we appeal to function extensionality to finish the proof. The simple uniqueness principle follows from the dependent one.

Based on the recursion and computation rules, we can internalize the well-known notions of W-type *algebras* and *morphisms* as follows:

Definition 19. *For $A : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, define the type of* W-algebras *on a universe $\mathcal{U}_j$ as*

$$\mathsf{WAlg}_{\mathcal{U}_j}(A,B) :\equiv \Sigma_{C:\mathcal{U}_j}\Pi_{a:A}(B(a) \to C) \to C$$

Definition 20. *For algebras $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$ and $\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A,B)$, define the type of* W-morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathsf{WMor}\ (C,c)\ (D,d) :\equiv \Sigma_{f:C\to D}\Pi_{a:A}\Pi_{t:B(a)\to C}\big(f(c(a,t)) = d(a, f\circ t)\big)$$

We note that in order to form the type of morphisms, we had to use propositional rather than definitional equality. Of course, in the setting of an extensional type theory this distinction is immaterial; however, it will become important in section 3. The recursion principle now becomes a property internal to the type theory and can be expressed compactly as saying that there is a morphism into any other algebra $\mathcal{Y}$:

Definition 21. *An algebra $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$* satisfies the recursion principle *on a universe $\mathcal{U}_k$ if for any algebra $\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A,B)$ there exists a morphism from $\mathcal{X}$ to $\mathcal{Y}$:*

$$\mathsf{hasWRec}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \big(\Pi\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A,B)\big)\ \mathsf{WMor}\ \mathcal{X}\ \mathcal{Y}$$

To express the induction principle in a similar fashion, we first need to introduce dependent or *fibered* versions of algebras and algebra morphisms:

Definition 22. *For an algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$*, define the type of* fibered W-algebras *over* $\mathcal{X}$ *on a universe* $\mathcal{U}_k$ *by*

$$\mathsf{WFibAlg}_{\mathcal{U}_k}\ (C,c) :\equiv \Sigma_{E:C\to\mathcal{U}_k}\Pi_{a:A}\Pi_{t:B(a)\to C}\big(\Pi_{b:B(a)}E(t\ b)\big) \to E(c(a,t))$$

Definition 23. *For algebras* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$ *and* $\mathcal{Y} : \mathsf{WFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$*, define the type of* fibered W-morphisms *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *by*

$$\mathsf{WFibMor}\ (C,c)\ (E,e) :\equiv \Sigma_{f:(\Pi x:C)E(x)}\Pi_{a:A}\Pi_{t:B(a)\to C}\big(f(c(a,t)) = e(a,t,f\circ t)\big)$$

Definition 24. *An algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$ satisfies the induction principle *on a universe* $\mathcal{U}_k$ *if for any fibered algebra* $\mathcal{Y} : \mathsf{WFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *there exists a fibered morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasWInd}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \big(\Pi\mathcal{Y} : \mathsf{WFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\big)\ \mathsf{WFibMor}\ \mathcal{X}\ \mathcal{Y}$$

Furthermore, the uniqueness principles motivate the following definitions:

Definition 25. *An algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$ satisfies the recursion uniqueness principle *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A,B)$ *any two morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *are equal:*

$$\mathsf{hasWRecUniq}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \big(\Pi\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A,B)\big)\ \mathsf{isProp}(\mathsf{WMor}\ \mathcal{X}\ \mathcal{Y})$$

Definition 26. *An algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$ satisfies the induction uniqueness principle *on a universe* $\mathcal{U}_k$ *if for any fibered algebra* $\mathcal{Y} : \mathsf{WFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *any two fibered morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *are equal:*

$$\mathsf{hasWIndUniq}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \big(\Pi\mathcal{Y} : \mathsf{WFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\big)\ \mathsf{isProp}(\mathsf{WFibMor}\ \mathcal{X}\ \mathcal{Y})$$

The uniqueness principles as in Defs. 25, 26 require that any two morphisms (f,β) and (g,γ) be equal as *pairs*; however, in the presence of UIP this is the same as saying that their first components agree, i.e., that $f = g$ (and hence $f \equiv g$).

Definition 27. *An algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$ *is* initial *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A,B)$ *there exists a unique morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{isWInit}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \big(\Pi\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A,B)\big)\ \mathsf{isContr}(\mathsf{WMor}\ \mathcal{X}\ \mathcal{Y})$$

The contractibility requirement precisely captures the notion of initiality: in the presence of the identity reflection rule, a contractible type is one which contains a definitionally unique element.

Note: We have used the concepts of contractibility and mere propositions, which were introduced in section 2.2 in the context of homotopy type theory; however, these definitions are perfectly applicable in the setting of extensional/intensional type theory as well. Clearly:

Lemma 28. *For an algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$ *we have*

$$\mathsf{isWInit}_{\mathcal{U}_k}(\mathcal{X}) \quad\leftrightarrow\quad \mathsf{hasWRec}_{\mathcal{U}_k}(\mathcal{X}) \times \mathsf{hasWRecUniq}_{\mathcal{U}_k}(\mathcal{X})$$

We have the following relationship between the fibered and non-fibered versions of W-algebras and morphisms:

Lemma 29. *For an algebra $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ we have a function*

$$\mathsf{WAlgToFibAlg}_{\mathcal{U}_k}(\mathcal{X}) \quad : \quad \mathsf{WAlg}_{\mathcal{U}_k}(A, B) \quad \to \quad \mathsf{WFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$$

Proof. Let algebras $(C, c) : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ and $(D, d) : \mathsf{WAlg}_{\mathcal{U}_k}(A, B)$ be given. We turn (D, d) into the desired fibered algebra $(E, e) : \mathsf{WFibAlg}_{\mathcal{U}_k}\ (C, c)$ by putting $E(x) :\equiv D$ and $e(a, t, u) :\equiv d(a, u)$. □

Remark 30. *We note that for any algebras $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ and $\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A, B)$ we have*

$$\mathsf{WMor}\ \mathcal{X}\ \mathcal{Y} \quad \equiv \quad \mathsf{WFibMor}\ \mathcal{X}\ \Big(\mathsf{WAlgToFibAlg}_{\mathcal{U}_k}(\mathcal{X})\ \mathcal{Y}\Big)$$

The previous two observations immediately imply the following:

Lemma 31. *If an algebra $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ satisfies the induction principle on the universe $\mathcal{U}_k$, then it satisfies the recursion principle on $\mathcal{U}_k$. In other words, we have*

$$\mathsf{hasWInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasWRec}_{\mathcal{U}_k}(\mathcal{X})$$

Lemma 32. *If an algebra $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ satisfies the induction uniqueness principle on the universe $\mathcal{U}_k$, then it satisfies the recursion uniqueness principle on $\mathcal{U}_k$. In other words, we have*

$$\mathsf{hasWIndUniq}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasWRecUniq}_{\mathcal{U}_k}(\mathcal{X})$$

The next lemma in particular shows that it is not necessary to have a "fibered" version of the initiality property, which quantifies over all fibered algebras $\mathcal{Y} : \mathsf{WFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$.

Lemma 33. $(\mathcal{H}_{\text{ext}})$ *If an algebra $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ satisfies the induction principle on the universe $\mathcal{U}_k$, then it satisfies the induction uniqueness principle on $\mathcal{U}_k$. In other words, we have*

$$\mathsf{hasWInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasWIndUniq}_{\mathcal{U}_k}(\mathcal{X})$$

Proof. The proof is analogous to the one showing that the induction principle implies the dependent uniqueness principle, except as in the proof of the previous lemma, we use the language of algebras and morphisms.

Fix an algebra $(C, c) : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ and assume that $\mathsf{hasWInd}_{\mathcal{U}_k}(C, c)$ holds. To prove that $\mathsf{hasWIndUniq}_{\mathcal{U}_k}(C, c)$ holds, take any fibered algebra $(E, e) : \mathsf{WFibAlg}_{\mathcal{U}_k}\ (C, c)$ and morphisms $(f, \beta), (g, \gamma) : \mathsf{WFibMor}\ (C, c)\ (E, e)$. Because of UIP, showing $(f, \beta) = (g, \gamma)$ is equivalent to showing $f = g$.

To do this, we use the induction principle with the fibered algebra $(E', e') : \mathsf{WFibAlg}_{\mathcal{U}_k}\ (C, c)$ where $E'(x) :\equiv (f(x) = g(x))$ and $e'(a, t, u) :\equiv 1_{f(c(a,t))}$. This is indeed well-typed since $\beta(a, t)$ gives us $f(c(a, t)) \equiv e(a, t, f \circ t)$ and $\gamma(a, t)$ gives us $g(c(a, t)) \equiv e(a, t, g \circ t)$ and u gives us $f \circ t \equiv g \circ t$, hence we have $f(c(a, t)) \equiv g(c(a, t))$.

The first component of the resulting morphism from (C, c) to (E', e') together with function extensionality then gives us $f = g$ as desired. □

Corollary 34. $(\mathcal{H}_{\text{ext}})$ *If an algebra $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ satisfies the induction principle on the universe $\mathcal{U}_k$, then it is initial on $\mathcal{U}_k$. In other words, we have*

$$\mathsf{hasWInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{isWInit}_{\mathcal{U}_k}(\mathcal{X})$$

Lemma 35. $(\mathcal{H}_{\text{ext}})$ *If an algebra $\mathcal{X}$: $\mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ satisfies the recursion and recursion uniqueness principles on the universe $\mathcal{U}_k$ and $k \geq j$, then it satisfies the induction principle on $\mathcal{U}_k$. In other words, we have*

$$\mathsf{hasWRec}_{\mathcal{U}_k}(\mathcal{X}) \times \mathsf{hasWRecUniq}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasWInd}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$.

Proof. Let an algebra (C, c) : $\mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ be given and assume that $\mathsf{hasWRec}_{\mathcal{U}_k}(C, c)$ and $\mathsf{hasWRecUniq}_{\mathcal{U}_k}(C, c)$ hold. To prove that $\mathsf{hasWInd}_{\mathcal{U}_k}(C, c)$ holds, fix a fibered algebra (E, e) : $\mathsf{WFibAlg}_{\mathcal{U}_k}(C, c)$. We use the recursion principle with the algebra (D, d) : $\mathsf{WAlg}_{\mathcal{U}_k}(A, B)$ where $D :\equiv \Sigma_{x:C} E(x)$ and $d(a, u) :\equiv \big(c(a, \pi_1 \circ u), e(a, \pi_1 \circ u, \pi_2 \circ u)\big)$. We note that the type D belongs to $\mathcal{U}_k$ as $j \leq k$. The recursion principle then gives us a morphism (f, β) : $\mathsf{WMor}\ (C, c)\ (D, d)$, where $f : C \to \Sigma_{x:C} E(x)$ and $\beta : \Pi_{a:A} \Pi_{t:B(a) \to C} \big(f(c(a, t)) = d(a, f \circ t)\big)$. Hence, for any a, t we have

$$f(c(a, t)) \equiv \big(c(a, \pi_1 \circ f \circ t), e(a, \pi_1 \circ f \circ t, \pi_2 \circ f \circ t)\big) \tag{$\star$}$$

We now want to show that the function $\pi_1 \circ f : C \to C$ is in fact the identity on C. We can do this by endowing both of the functions $\pi_1 \circ$ and id_C with a morphism structure on the algebra (C, c); by the recursion uniqueness principle it will follow that these morphisms are equal, and in particular they are equal as maps. For the identity function, this reduces to showing that for any a, t, we have $c(a, t) = c(a, t)$, which is obvious. For the function $\pi_1 \circ f$, we need to show that for any a, t, we have $\pi_1(f(c(a, t))) = c(a, \pi_1 \circ f \circ t)$. But this follows by applying the first projection to ($\star$).

The recursion uniqueness principle tells us that the two morphisms just constructed are equal and in particular, we have $\pi_1 \circ f \equiv \mathsf{id}_C$. This means that the map $\pi_2 \circ f$ has the desired type $\Pi_{x:C} E(x)$ and it remains to show that it can be endowed with a fibered morphism structure on (E, e), *i.e.*, that for any a, t we have $\pi_2(f(c(a, t))) = e(a, t, \pi_2 \circ f \circ t)$. This follows by applying the second projection to ($\star$) and using the aforementioned result that $\pi_1 \circ f \equiv \mathsf{id}_C$. □

Corollary 36. $(\mathcal{H}_{\text{ext}})$ *For $A : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, the following conditions on an algebra $\mathcal{X}$: $\mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ are logically equivalent:*

- *$\mathcal{X}$ satisfies the induction principle on the universe $\mathcal{U}_k$*
- *$\mathcal{X}$ is initial on the universe $\mathcal{U}_k$*

for $k \geq j$. In other words, we have

$$\mathsf{hasWInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \leftrightarrow \quad \mathsf{isWInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$.

Corollary 37. $(\mathcal{H}_{\text{ext}} + \mathsf{W})$ *For $A : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, the algebra*

$$\Big(\mathsf{W}(A, B), \mathsf{sup}\Big) : \mathsf{WAlg}_{\mathcal{U}_i}(A, B)$$

is initial on any universe $\mathcal{U}_j$.

2.4 Higher Inductive Types

An inductive type X can be understood as being *freely generated* by a collection of constructors: in the familiar case of natural numbers, we have the two constructors for zero and successor. The property of being freely generated can be roughly stated as an induction principle: in order to show that a property $P : \mathbb{N} \to \mathcal{U}_i$ holds for all $n : \mathbb{N}$, it suffices to show that it holds for zero and is preserved by the successor operation. As a special case, we get the recursion principle: in order to define a map $f : \mathbb{N} \to C$, is suffices to determine its value at zero and its behavior with respect to successor.

Moreover, the induction principle (with function extensionality) implies that any two functions out of $\mathbb{N}$ which satisfy the same recurrence are equal. This suggests another, perhaps more familiar notion of being freely-generated, in the sense that there is an essentially unique homomorphism from X to any other structure having the same form - in the case of natural numbers, the structures are triples (C, z, s) where C is a type and $z : C$, $s : C \to C$ are terms. In [3], we showed that these two notions of freeness coincide for ordinary inductive types in the homotopy-type-theoretic setting.

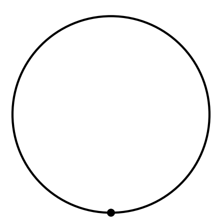
Higher inductive types generalize ordinary inductive types by allowing constructors involving *path spaces of* X rather than just X itself, as the next example shows.

2.4.1 The Circle

The circle $\mathbf{S^1}$ ([15, 17, 30], chapter 6.2 of [33]) is represented as an inductive type $\mathbf{S} : \mathcal{U}_0$ with two constructors:

$$\begin{aligned} &\mathsf{base} : \mathbf{S} \\ &\mathsf{loop} : \mathsf{base} =_{\mathbf{S}} \mathsf{base} \end{aligned}$$

pictured as



This in particular means that we have further paths, such as $\mathsf{loop}^{-1} \cdot \mathsf{loop} \cdot \mathsf{loop} \cdot 1_{\mathsf{base}}$ (which is equal to loop). We can reason about the circle using the principle of *circle recursion*, which tells us that given terms

- $C : \mathcal{U}_i$,
- $c : C$,
- $s : c = c$,

there is a recursor $\mathsf{rec}_{\mathbf{S}}(C, c, s) : \mathbf{S} \to C$. The recursor satisfies the computation laws

- $\mathsf{rec}_{\mathbf{S}}(\mathsf{base}) \equiv c$,

- $\mathsf{ap}_{\mathsf{rec}_{\mathbf{S}}}(\mathsf{loop}) = s$.

The second rule type-checks by virtue of the first one. We note that in order to record the effect of the recursor on the path loop, we use the "action-on-paths" construct ap. Since this is a derived notion rather than a primitive one, we state the rule as a propositional rather than definitional equality.

We also have the more general principle of *circle induction*, which subsumes recursion. Instead of a type $C : \mathcal{U}_i$ we now have a type family $E : \mathbf{S} \to \mathcal{U}_i$. Where previously we required a point $c : C$, we now need a point $e : E(\mathsf{base})$. Finally, an obvious generalization of needing a loop $s : c =_C c$ would be to ask for a loop $d : e =_{E(\mathsf{base})} e$. However, this would be incorrect: once we have our desired inductor of type $\Pi_{x:\mathbf{S}} E(x)$, its effect on loop is not a loop at e in the fiber $E(\mathsf{base})$ but a path from $\mathsf{loop}_*^E(e)$ to e in $E(\mathsf{base})$ (or its contravariant version). The induction principle thus takes the following form: given terms

- $E : \mathbf{S} \to \mathcal{U}_i$,
- $e : E(\mathsf{base})$,
- $d : \mathsf{loop}_*^E(e) = e$,

there is an inductor $\mathsf{ind}_{\mathbf{S}}(E, e, d) : \Pi_{x:\mathbf{S}} E(x)$. The inductor satisfies the computation laws

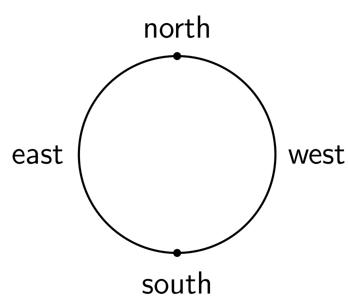
- $\mathsf{ind}_{\mathbf{S}}(\mathsf{base}) \equiv e$,
- $\mathsf{dap}_{\mathsf{ind}_{\mathbf{S}}}(\mathsf{loop}) = d$.

2.4.2 The Circle, Round Two

We could have alternatively represented $\mathbf{S^1}$ as an inductive type $\mathbb{S} : \mathcal{U}_0$ with four constructors (chapter 6.4 of [33]):

$$\begin{aligned} \mathsf{north} &: \mathbb{S} \\ \mathsf{south} &: \mathbb{S} \\ \mathsf{east} &: \mathsf{north} =_{\mathbb{S}} \mathsf{south} \\ \mathsf{west} &: \mathsf{north} =_{\mathbb{S}} \mathsf{south} \end{aligned}$$

pictured as



The recursion principle now says that given terms

- $C : \mathcal{U}_i$,
- $c : C$,

- $d : C$,
- $p : c = d$,
- $q : c = d$,

there is a recursor $\mathsf{rec}_{\mathbb{S}}(C, c, d, p, q) : \mathbb{S} \to C$. The recursor satisfies the computation laws

- $\mathsf{rec}_{\mathbb{S}}(\mathsf{north}) \equiv c$,
- $\mathsf{rec}_{\mathbb{S}}(\mathsf{south}) \equiv d$,
- $\mathsf{ap}_{\mathsf{rec}_{\mathbb{S}}}(\mathsf{east}) = p$,
- $\mathsf{ap}_{\mathsf{rec}_{\mathbb{S}}}(\mathsf{west}) = q$.

The corresponding induction principle says that given terms

- $E : \mathbb{S} \to \mathcal{U}_i$,
- $u : E(\mathsf{north})$,
- $v : E(\mathsf{north})$,
- $\mu : \mathsf{east}_*^E(u) = v$,
- $\nu : \mathsf{west}_*^E(u) = v$,

there is an inductor $\mathsf{ind}_{\mathbb{S}}(E, u, v, \mu, \nu) : \Pi_{x:\mathbb{S}} E(x)$. The inductor satisfies the computation laws

- $\mathsf{ind}_{\mathbb{S}}(\mathsf{north}) \equiv u$,
- $\mathsf{ind}_{\mathbb{S}}(\mathsf{south}) \equiv v$,
- $\mathsf{dap}_{\mathsf{ind}_{\mathbb{S}}}(\mathsf{east}) = \mu$,
- $\mathsf{dap}_{\mathsf{ind}_{\mathbb{S}}}(\mathsf{west}) = \nu$.

In section 3.1, we will return to the two different definitions of a circle and examine how they relate to each other.

2.4.3 The Interval

The interval type $\mathbb{I} : \mathcal{U}_0$ ([17, 29, 30], chapter 6.3 of [33]) is a simple higher inductive type generated by two points and a path connecting them:

$$0_{\mathbb{I}} : \mathbb{I}$$
$$1_{\mathbb{I}} : \mathbb{I}$$
$$\mathsf{seg} : 0_{\mathbb{I}} =_{\mathbb{I}} 1_{\mathbb{I}}$$

pictured as

$$0_{\mathbb{I}} \overset{\mathsf{seg}}{\text{———}} 1_{\mathbb{I}}$$

The recursion principle now says that given terms

- $C : \mathcal{U}_i$,
- $c : C$,
- $d : C$,

- $p : c = d$,

there is a recursor $\mathsf{rec}_{\mathbb{I}}(C, c, d, p) : \mathbb{I} \to C$. The recursor satisfies the computation laws

- $\mathsf{rec}_{\mathbb{I}}(0_{\mathbb{I}}) \equiv c$,
- $\mathsf{rec}_{\mathbb{I}}(1_{\mathbb{I}}) \equiv d$,
- $\mathsf{ap}_{\mathsf{rec}_{\mathbb{I}}}(\mathsf{seg}) = p$.

The corresponding induction principle says that given terms

- $E : \mathbb{I} \to \mathcal{U}_i$,
- $u : E(0_{\mathbb{I}})$,
- $v : E(1_{\mathbb{I}})$,
- $\mu : \mathsf{seg}_*^E(u) = v$,

there is an inductor $\mathsf{ind}_{\mathbb{I}}(E, u, v, \mu) : \Pi_{x:\mathbb{I}} E(x)$. The inductor satisfies the computation laws

- $\mathsf{ind}_{\mathbb{I}}(0_{\mathbb{I}}) \equiv u$,
- $\mathsf{ind}_{\mathbb{I}}(1_{\mathbb{I}}) \equiv v$,
- $\mathsf{dap}_{\mathsf{ind}_{\mathbb{I}}}(\mathsf{seg}) = \mu$.

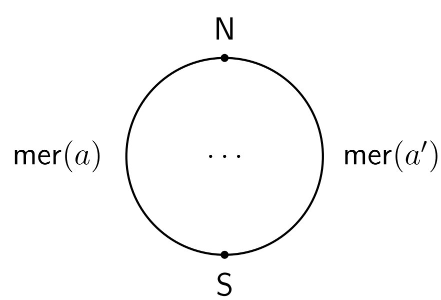
It is easy to show that the interval is contractible (see chapter 6.3 of [33]); however, it still has some interesting properties. For instance, the existence of an interval by itself implies function extensionality ([29], chapter 6.3 of [33]).

2.4.4 Suspensions

If we look at the specification of the higher inductive types $\mathbb{S}$ and $\mathbb{I}$, we see that they are both generated by two points (north, south for $\mathbb{S}$ and $0_{\mathbb{I}}$, $1_{\mathbb{I}}$ for $\mathbb{I}$) and a given number of paths between them (the two paths east, west for $\mathbb{S}$ and the single path seg for $\mathbb{I}$). Suspensions ([17], chapter 6.5 of [33]) generalize this observation by allowing an arbitrary number of path generators between the distinguished points. Formally, given a type $A : \mathcal{U}_i$, the suspension $\Sigma A : \mathcal{U}_i$ is the higher inductive type generated by the two constructors

$$\begin{aligned} \mathsf{N} &: \Sigma A \\ \mathsf{S} &: \Sigma A \\ \mathsf{mer} &: \Pi_{a:A}\big(\mathsf{N} =_{\Sigma A} \mathsf{S}\big) \end{aligned}$$

pictured as



Hence, for each $a : A$ we get one "meridian" $\mathsf{mer}(a)$ running from the "north" N to the "south" S. The recursion principle for the suspension ΣA says that given terms

- $C : \mathcal{U}_j$,
- $c : C$,
- $d : C$,
- $p : \Pi_{a:A}(c = d)$,

there is a recursor $\mathsf{rec}_{\Sigma A}(C, c, d, p) : \Sigma A \to C$. The recursor satisfies the computation laws

- $\mathsf{rec}_{\Sigma A}(\mathsf{N}) \equiv c$,
- $\mathsf{rec}_{\Sigma A}(\mathsf{S}) \equiv d$,
- $\mathsf{ap}_{\mathsf{rec}_{\Sigma A}}(\mathsf{mer}(a)) = p(a)$ for any $a : A$.

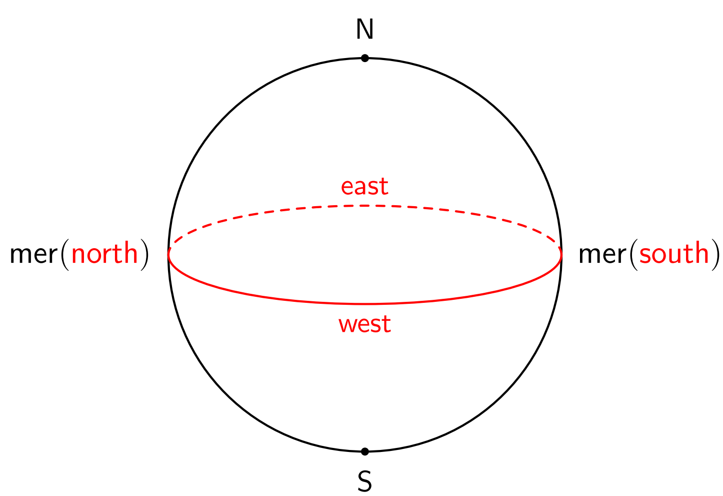
The corresponding induction principle says that given terms

- $E : \Sigma A \to \mathcal{U}_j$,
- $u : E(\mathsf{N})$,
- $v : E(\mathsf{S})$,
- $\mu : \Pi_{a:A}\big(\mathsf{seg}_*^E(u) = v\big)$,

there is an inductor $\mathsf{ind}_{\Sigma A}(E, u, v, \mu) : \Pi_{x:\Sigma A} E(x)$. The inductor satisfies the computation laws

- $\mathsf{ind}_{\Sigma A}(\mathsf{N}) \equiv u$,
- $\mathsf{ind}_{\Sigma A}(\mathsf{S}) \equiv v$,
- $\mathsf{dap}_{\mathsf{ind}_{\Sigma A}}(\mathsf{mer}(a)) = \mu$ for any $a : A$.

It is easy to see that we can describe the types $\mathbb{I}$ and $\mathbb{S}$ as the suspensions $\Sigma\mathbf{1}$ and $\Sigma\mathbf{2}$ respectively, and that the type $\mathbf{2}$ itself arises as the suspension $\Sigma\mathbf{0}$. Moreover, it turns out the suspension $\Sigma\mathbb{S}$ (or $\Sigma(\Sigma\mathbf{2})$) looks very much like the ordinary 3-dimensional sphere $\mathbf{S}^2$:



As we can see in the above picture, for any point $x : \mathbb{S}$, and in particular for north and south, we get a path $\mathsf{mer}(x)$ running from N to S, giving us the full sphere $\mathbf{S}^2$. Using this idea ([17] chapter 6.5 of [33]), we can define the n-sphere recursively:

Definition 38. *We define a type $\mathbf{S}^{n-1}$ by recursion on $n : \mathbb{N}$ as follows:*

$$\mathbf{S}^{-1} :\equiv \mathbf{0}$$
$$\mathbf{S}^{n} :\equiv \Sigma \mathbf{S}^{n-1}$$

One can show that spheres defined as above indeed have many of the familiar mathematical properties that one would expect (see [13, 15], notes in chapter 8 of [33]).

2.4.5 Type Quotients

As we saw in the previous section, suspensions allow us to specify an arbitrary number of path constructors between the two points N and S. We can take this idea further by also allowing an arbitrary number of point constructors. Type quotients [36] accomplish just that: formally, for a type $A : \mathcal{U}_i$ and a type-family $R : A \to A \to \mathcal{U}_i$ we define $A/R : \mathcal{U}_i$ to be the higher inductive type generated by the constructors

$$\begin{aligned} \mathsf{point} &: A \to A/R \\ \mathsf{cell} &: \Pi_{a,b:A} R(a,b) \to (\mathsf{point}(a) = \mathsf{point}(b)) \end{aligned}$$

The type A can be understood as the type of labels for point constructors and for each $a, b : A$, the type $R(a, b)$ can be understood as the type of labels for path constructors between the points labeled by a and b. The recursion principle for type quotients says that given terms

- $E : \mathcal{U}_j$,
- $f : A \to E$,
- $p : \Pi_{a,b:A} R(a,b) \to (f(a) = f(b))$,

there is a recursor $\mathsf{rec}_{./.}(E, f, p) : A/R \to E$. The recursor satisfies the computation laws

- $\mathsf{rec}_{./.}(\mathsf{point}(a)) \equiv f(a)$ for any $a : A$,
- $\mathsf{ap}_{\mathsf{rec}_{./.}}(\mathsf{cell}(z)) = p(z)$ for any $a : A, b : A, z : R(a,b)$.

Similarly, we have an induction principle: given terms

- $E : A/R \to \mathcal{U}_j$,
- $f : \Pi_{a:A} E(\mathsf{point}(a))$,
- $p : \Pi_{a,b:A} \Pi_{z:R(a,b)} \big(\mathsf{cell}(z)_*^E\ f(x) = f(y)\big)$,

there is an inductor $\mathsf{ind}_{./.}(E, f, p) : \Pi_{x:A/R} E(x)$. The inductor satisfies the computation laws

- $\mathsf{ind}_{./.}(\mathsf{point}(a)) \equiv f(a)$ for any $a : A$,
- $\mathsf{dap}_{\mathsf{ind}_{./.}}(\mathsf{cell}(z)) = p(z)$ for any $a : A, b : A, z : R(a,b)$.

Recent work by F. van Doorn [36], Egbert Rijke, and others shows that despite their relative simplicity, type quotients are a quite general class of higher inductive types, in the sense that many other higher inductive types (such as any of the previous ones presented in this section and propositional truncations), can be reduced to special cases of type quotients, although this reduction can be highly non-trivial).

3

W-quotients and Homotopy-initiality

In this chapter we describe the main contributions, most of which revolve around the notion of *homotopy-initiality*, originally introduced in [3]. As the name suggests, this universal property generalizes the category-theoretic concept of "initiality" to the homotopy-type-theoretic setting. Its major significance is that it is equivalent to the induction principle but often simpler to state and prove: the induction principle for a higher (or even an ordinary) inductive type involves dependent types and as such can be rather hard to understand (let alone establish in a model). Even for higher inductive types generated by relatively simple data, such as the torus, the full induction principle can be rather convoluted and tedious to work with. Working with homotopy-initiality, on the other hand, tends to be simpler since we only have to care about satisfying the universal property with respect to non-dependent types. Moreover, all proofs we give are internal to the type theory and hence fully constructive and formalizable; this in particular means that once we prove that an algebra is homotopy-initial, we can run our algorithm to automatically recover the term witnessing the induction principle.

The first section of this chapter serves to introduce the notion of algebras, morphisms, and the associated recursion, induction, and homotopy-initiality principles for the higher inductive types introduced in chapter 2.4. We start by showing how this generalized setting allows us to overcome certain drawbacks introduced by relying on specific definitional behavior of some higher inductive types (corollaries 61, 62).

The second section describes W-quotients, which are the central construction in the thesis. The third section is entirely devoted to proving the main result (theorem). The proof itself relies on a series of steps, which we give as separate lemmas but which are useful on their own: for example, lemmas 91 and 101 characterize the path space between two algebra morphisms as the type of *cells*, and can be thought of as principles of "univalence" for the type of morphisms.

The last section focuses on showing that the higher inductive types introduced in chapter 2.4 are really just special cases of W-quotients. We show this in a precise sense and in the process derive the homotopy-initiality characterization for these higher inductive types (*e.g.*, corollaries 113, 114, 115).

3.1 Algebras for Higher Inductive Types

In section 2.3 we introduced the type-theoretic counterparts of the notions of algebras, morphisms, and initiality for W-types [3], and used them to establish a syntactic version of the well-known result by Dybjer [5], which characterizes W-types in extensional type theory as precisely the initial algebras. As given, our definitions make perfect sense even in the absence of identity reflection and UIP - except, of course, they no longer internalize the concepts of (strict) morphisms and initiality, but rather their "up-to-homotopy" counterparts. In the intensional setting, we still want to refer to maps which preserve the algebra structure as morphisms, albeit in a more general sense; however, we replace initiality by *homotopy-initiality*, which internalizes the notion of existence plus uniqueness as contractibility:

Definition 39. *An algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ *is* homotopy-initial *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A, B)$*, the type of* W*-morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is contractible:*

$$\mathsf{isWHInit}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \big(\Pi \mathcal{Y} : \mathsf{WAlg}_{\mathcal{U}_k}(A, B)\big)\ \mathsf{isContr}(\mathsf{WMor}\ \mathcal{X}\ \mathcal{Y})$$

As noted in chapter 2, the contractibility requirement implies that there exists a morphism from $\mathcal{X}$ to $\mathcal{Y}$ which is unique up to a higher path, which is itself unique up to a yet higher path, and so on. As we showed in [3], this is precisely the universal property characterizing W-types in homotopy type theory; in fact, the notions of what it means to be a W-type and a homotopy-intial algebra are equivalent mere propositions. Our goal in this chapter is to establish an analogous equivalence for a reasonably large class of higher inductive types. We start by revisiting our running example of the circle.

3.1.1 Circle Algebras

In section 2.4, we gave two definitions of the unit circle as a higher inductive type. It is not hard to show that the two types are equivalent (chapter 6.5 of [33]):

Lemma 40. $(\mathcal{H}_{\mathrm{int}} + \mathbf{S} + \mathbb{S})$ *We have* $\mathbf{S} \simeq \mathbb{S}$.

Proof sketch. From left to right, map base to north and loop to $\mathsf{east} \cdot \mathsf{west}^{-1}$. From right to left, map both north and south to base, east to loop, and west to 1_{base}. Using the respective induction principles, show that these two mappings compose to identity on both sides. ☐

In particular, the types $\mathbf{S}$ and $\mathbb{S}$ satisfy the same properties (see Lem. 16). We would thus expect the induction principle for $\mathbf{S}$ to carry over to $\mathbb{S}$, and vice versa. Indeed, with a little effort we can show the former:

Lemma 41. $(\mathcal{H}_{\mathrm{int}} + \mathbb{S})$ *The type* $\mathbb{S}$ *satisfies the induction and computation laws for* $\mathbf{S}$*, with* north *acting as the constructor* base *and* $\mathsf{east} \cdot \mathsf{west}^{-1}$ *acting as the constructor* loop.

In the other direction, though, we hit a snag - the only obvious choice we have is to define both points north and south to be base, one of the paths west and east to be loop, and the other one the identity path at base. This, however, does not give us the desired induction principle: unless the two given points $u : E(\mathsf{base})$ and $v : E(\mathsf{base})$ happen to be definitionally equal, we will not be able to map base to both of them, as required by the computation rules.

This poses more than just a conceptual problem - in mathematics, we often have several possible definitions of a given notion, all of which are interchangeable from the point of view of a "user". Having two definitions of a circle which are not (known to be) interchangeable, however, can be problematic: any theorem we establish about or by appealing to $\mathbb{S}$ might no longer hold - or even type-check! - when using $\mathbf{S}$ instead. As an example, take the second computation law for $\mathbb{S}$, $\mathsf{dap}_{\mathsf{ind}_{\mathbb{S}}(E,u,v,\mu,\nu)}(\mathsf{west}) = \nu$. If we attempt to "implement" $\mathbb{S}$ using the circle $\mathbf{S}$ instead - by taking $\mathsf{north}, \mathsf{south} :\equiv \mathsf{base}$, $\mathsf{east} :\equiv \mathsf{loop}$, $\mathsf{west} :\equiv 1_{\mathsf{base}}$ - the computation law is no longer well-typed since the left-hand side reduces to a reflexivity path whereas the right hand side is a path from u to v. The issue associated with having different representations is not specific to higher inductive types: a similar problem arises when we encode natural numbers as a W-types. In this representation, we can show that the computation rule for the successor holds up to propositional equality but have no (known) way to make it hold definitionally.

This is one of the motivations for considering inductive types with *propositional* computation behavior: we now want to investigate types which "act like the circle" up to propositional equality. In the case of $\mathbf{S}$, such a type $C : \mathcal{U}_i$ should come with a point $b : C$ and loop $l : c =_C c$. In the case of $\mathbb{S}$, such a type should come with two points $n, s : C$ and two paths $e, w : n =_C s$. We can express this more concisely as follows:

Definition 42. *Define the type of* $\mathbf{S}$-algebras *on a universe* $\mathcal{U}_i$ *as*

$$\mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i} :\equiv \Sigma_{C:\mathcal{U}_i}\Sigma_{b:C}(b = b)$$

Definition 43. *Define the type of* $\mathbb{S}$-algebras *on a universe* $\mathcal{U}_i$ *as*

$$\mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i} :\equiv \Sigma_{C:\mathcal{U}_i}\Sigma_{n:C}\Sigma_{s:C}(n = s) \times (n = s)$$

We are now interested in maps between algebras which in a suitable sense preserve the distinguished points and paths, i.e., *algebra morphisms*. A morphism between two $\mathbf{S}$-algebras (C, c, p) and (D, d, q) should be a function $f : C \to D$ for which we have a path $\beta : f(c) = d$. Furthermore, f should also appropriately relate p and q. To figure out what this means, we observe that if we map p along f, we obtain a path $\mathsf{ap}_f(p) : f(c) = f(c)$. Each of the (identical) endpoints is equal to d, via the path β. Thus, we now have another path $\beta^{-1} \cdot \mathsf{ap}_f(p) \cdot \beta : d = d$. It is reasonable to require that this path be equal to q, i.e., that the following diagram commutes:

$$\begin{array}{ccc} f(c) & \xrightarrow{\mathsf{ap}_f(p)} & f(c) \\ {\scriptstyle\beta}\big| & & \big|{\scriptstyle\beta} \\ d & \underset{q}{\longrightarrow} & d \end{array}$$

Likewise, a morphism between two $\mathbb{S}$-algebras (C, a, b, p, q) and (D, c, d, r, s) should be a function $f : C \to D$ for which we have paths $\beta : f(a) = c$, $\gamma : f(b) = d$ and for which the following diagrams commute:

$$\begin{array}{ccc} f(a) & \xrightarrow{\mathsf{ap}_f(p)} & f(b) \\ {\scriptstyle\beta}\big| & & \big|{\scriptstyle\gamma} \\ c & \underset{r}{\longrightarrow} & d \end{array} \qquad\qquad \begin{array}{ccc} f(a) & \xrightarrow{\mathsf{ap}_f(q)} & f(b) \\ {\scriptstyle\beta}\big| & & \big|{\scriptstyle\gamma} \\ c & \underset{s}{\longrightarrow} & d \end{array}$$

In other words, an $\mathbf{S}$- or $\mathbb{S}$-morphism behaves just like a function constructed by the appropriate circle recursion, albeit with propositional computation laws for points and paths. We can express this as follows:

Definition 44. *For algebras $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ and $\mathcal{Y} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}$, define the type of* $\mathbf{S}$-morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathbf{S}\text{-}\mathsf{Mor}\ (C, c, p)\ (D, d, q) :\equiv \Sigma_{f:C\to D}\Sigma_{\beta:f(c)=d}\big(\mathsf{ap}_f(p) = \beta \cdot q \cdot \beta^{-1}\big)$$

Definition 45. *For algebras $\mathcal{X} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ and $\mathcal{Y} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}$, define the type of* $\mathbb{S}$-morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\begin{aligned}\mathbb{S}\text{-}\mathsf{Mor}\ (C, a, b, p, q)\ (D, c, d, r, s) &:\equiv \Sigma_{f:C\to D}\Sigma_{\beta:f(a)=c}\Sigma_{\gamma:f(b)=d}\\ &\big(\mathsf{ap}_f(p) = \beta \cdot r \cdot \gamma^{-1}\big) \times \big(\mathsf{ap}_f(q) = \beta \cdot s \cdot \gamma^{-1}\big)\end{aligned}$$

We note that as in Sect. 2.3, we needed propositional computation laws to be able to form the type of morphisms. The recursion principle is again a property internal to the type theory and can be expressed analogously as saying that there is a morphism into any other algebra $\mathcal{Y}$:

Definition 46. *An algebra $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$* satisfies the $\mathbf{S}$-recursion principle *on a universe $\mathcal{U}_j$ if for any algebra $\mathcal{Y} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}$ there exists a morphism from $\mathcal{X}$ to $\mathcal{Y}$:*

$$\mathsf{has}\text{-}\mathbf{S}\text{-}\mathsf{Rec}_{\mathcal{U}_j}(\mathcal{X}) :\equiv \big(\Pi\mathcal{Y} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}\big)\ \mathbf{S}\text{-}\mathsf{Mor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 47. *An algebra $\mathcal{X} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$* satisfies the $\mathbb{S}$-recursion principle *on a universe $\mathcal{U}_j$ if for any algebra $\mathcal{Y} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}$ there exists a morphism from $\mathcal{X}$ to $\mathcal{Y}$:*

$$\mathsf{has}\text{-}\mathbb{S}\text{-}\mathsf{Rec}_{\mathcal{U}_j}(\mathcal{X}) :\equiv \big(\Pi\mathcal{Y} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}\big)\ \mathbb{S}\text{-}\mathsf{Mor}\ \mathcal{X}\ \mathcal{Y}$$

As in Sect. 2.3, to express the induction principle in a similar fashion we introduce the fibered versions of algebras and algebra morphisms:

Definition 48. *For an algebra $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$, define the type of* fibered $\mathbf{S}$-algebras *over $\mathcal{X}$ on a universe $\mathcal{U}_j$ by*

$$\mathbf{S}\text{-}\mathsf{FibAlg}_{\mathcal{U}_j}\ (C, c, p) :\equiv \Sigma_{E:C\to\mathcal{U}_j}\Sigma_{e:E(c)}\big(p^E_*(e) = e\big)$$

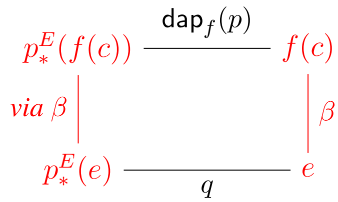
Definition 49. *For an algebra $\mathcal{X} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$, define the type of* fibered $\mathbb{S}$-algebras *over $\mathcal{X}$ on a universe $\mathcal{U}_j$ by*

$$\mathbb{S}\text{-}\mathsf{FibAlg}_{\mathcal{U}_j}\ (C, c, d, p, q) :\equiv \Sigma_{E:C\to\mathcal{U}_j}\Sigma_{u:E(c)}\Sigma_{v:E(d)}\big(p^E_*(u) = v\big) \times \big(q^E_*(u) = v\big)$$

Definition 50. *For algebras $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ and $\mathcal{Y} : \mathbf{S}\text{-}\mathsf{FibAlg}_{\mathcal{U}_j}\ \mathcal{X}$, we define the type of* fibered $\mathbf{S}$-morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathbf{S}\text{-}\mathsf{FibMor}\ (C, c, p)\ (E, e, q) :\equiv \Sigma_{f:(\Pi x:C)E(x)}\Sigma_{\beta:f(c)=e}\big(\mathsf{dap}_f(p) = \mathsf{ap}_{p^E_*}(\beta) \cdot q \cdot \beta^{-1}\big)$$

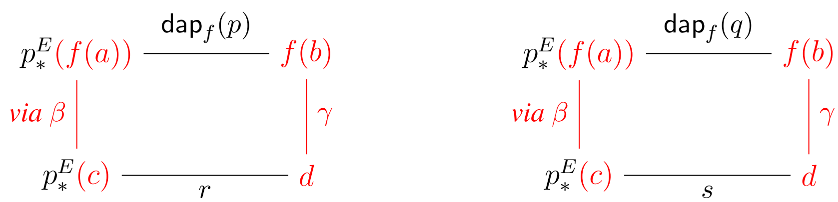
Pictorially, the last component of an $\mathbf{S}$-morphism witnesses the commuting diagram



Definition 51. *For algebras* $\mathcal{X} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ *and* $\mathcal{Y} : \mathbb{S}\text{-}\mathsf{FibAlg}_{\mathcal{U}_j}\ \mathcal{X}$*, we define the type of* fibered $\mathbb{S}$-morphisms *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *by*

$$\mathbb{S}\text{-}\mathsf{FibMor}\ (C, a, b, p, q)\ (D, c, d, r, s) :\equiv \Sigma_{f:(\Pi x:C)E(x)} \Sigma_{\beta:f(a)=c} \Sigma_{\gamma:f(b)=d}$$
$$\left(\mathsf{dap}_f(p) = \mathsf{ap}_{p_*^E}(\beta) \cdot r \cdot \gamma^{-1}\right) \times \left(\mathsf{dap}_f(q) = \mathsf{ap}_{q_*^E}(\beta) \cdot s \cdot \gamma^{-1}\right)$$

Pictorially, the last two components of an $\mathbb{S}$-morphism witness the commuting diagrams



The induction principle can now be expressed as saying that there is a fibered morphism into any fibered algebra $\mathcal{Y}$:

Definition 52. *An algebra* $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ satisfies the $\mathbf{S}$-induction principle *on a universe* $\mathcal{U}_j$ *if for any fibered algebra* $\mathcal{Y} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}\ \mathcal{X}$ *there exists a fibered morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{has}\text{-}\mathbf{S}\text{-}\mathsf{Ind}_{\mathcal{U}_j}(\mathcal{X}) :\equiv \left(\Pi\mathcal{Y} : \mathbf{S}\text{-}\mathsf{FibAlg}_{\mathcal{U}_j}\right)\ \mathbf{S}\text{-}\mathsf{FibMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 53. *An algebra* $\mathcal{X} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ satisfies the $\mathbb{S}$-induction principle *on universe* $\mathcal{U}_j$ *if for any fibered algebra* $\mathcal{Y} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}\ \mathcal{X}$ *there exists a fibered morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{has}\text{-}\mathbb{S}\text{-}\mathsf{Ind}_{\mathcal{U}_j}(\mathcal{X}) :\equiv \left(\Pi\mathcal{Y} : \mathbb{S}\text{-}\mathsf{FibAlg}_{\mathcal{U}_j}\right)\ \mathbb{S}\text{-}\mathsf{FibMor}\ \mathcal{X}\ \mathcal{Y}$$

The homotopy-initiality principle for circles states that there is a propositionally unique morphism into any other algebra $\mathcal{Y}$:

Definition 54. *An algebra* $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ *is* homotopy-initial *on a universe* $\mathcal{U}_j$ *if for any other algebra* $\mathcal{Y} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}$ *the type of* $\mathbf{S}$*-morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is contractible:*

$$\mathsf{is}\text{-}\mathbf{S}\text{-}\mathsf{HInit}_{\mathcal{U}_j}(\mathcal{X}) :\equiv \left(\Pi\mathcal{Y} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}\right)\ \mathsf{isContr}(\mathbf{S}\text{-}\mathsf{Mor}\ \mathcal{X}\ \mathcal{Y})$$

Definition 55. *An algebra* $\mathcal{X} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ *is* homotopy-initial *on a universe* $\mathcal{U}_j$ *if for any other algebra* $\mathcal{Y} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}$ *the type of* $\mathbb{S}$*-morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is contractible:*

$$\mathsf{is}\text{-}\mathbb{S}\text{-}\mathsf{HInit}_{\mathcal{U}_j}(\mathcal{X}) :\equiv \left(\Pi\mathcal{Y} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}\right)\ \mathsf{isContr}(\mathbb{S}\text{-}\mathsf{Mor}\ \mathcal{X}\ \mathcal{Y})$$

3.1.2 Relating The Two Circles

We first note that the notions of **S**-algebras and $\mathbb{S}$-algebras are in fact the same:

Lemma 56. *We have a function*

$$\mathsf{S\text{-}To\text{-}\mathbb{S}\text{-}Alg} : \mathsf{S\text{-}Alg}_{\mathcal{U}_i} \to \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$$

which is an equivalence.

Proof. Follows immediately from the fact that for any $C : \mathcal{U}_i$, $c : C$, we have

$$\begin{aligned} & c = c \\ \simeq\ & \Big(\Sigma \mathbf{r} : \Sigma_{d:C}(d = c)\Big)\big(c = \pi_1(\mathbf{r})\big) \\ \simeq\ & \Sigma_{d:C}(c = d) \times (c = d) \end{aligned}$$

where the first equivalence follows from the fact that the type $\Sigma_{d:C}(c = d)$ is contractible with the center of contraction (*i.e.*, its propositionally unique term) $(c, 1_c)$. □

Next, we note that the notions of fibered **S**-algebras and fibered $\mathbb{S}$-algebras are the same, in the following sense:

Lemma 57. *For any algebra* $\mathcal{X} : \mathsf{S\text{-}Alg}_{\mathcal{U}_i}$ *we have a function*

$$\mathsf{S\text{-}To\text{-}\mathbb{S}\text{-}FibAlg}(\mathcal{X}) : \mathsf{S\text{-}FibAlg}_{\mathcal{U}_j}\ \mathcal{X} \to \mathbb{S}\text{-}\mathsf{FibAlg}_{\mathcal{U}_j}\ \Big(\mathsf{S\text{-}To\text{-}\mathbb{S}\text{-}Alg}\ \mathcal{X}\Big)$$

which is an equivalence.

Proof. Fix an algebra $(C, c, p) : \mathsf{S\text{-}Alg}_{\mathcal{U}_i}$. Then $\mathsf{S\text{-}To\text{-}\mathbb{S}\text{-}Alg}\ (C, c, p)$ is the algebra $(C, c, c, 1_c, p)$. The desired equivalence now follows exactly as in the non-fibered case. □

The notions of (fibered) **S**-morphisms and $\mathbb{S}$-morphisms also coincide:

Lemma 58. *For any algebras* $\mathcal{X} : \mathsf{S\text{-}Alg}_{\mathcal{U}_i}$ *and* $\mathcal{Y} : \mathsf{S\text{-}FibAlg}_{\mathcal{U}_j}\ \mathcal{X}$ *we have*

$$\mathsf{S\text{-}FibMor}\ \mathcal{X}\ \mathcal{Y}\ \simeq\ \mathbb{S}\text{-}\mathsf{FibMor}\ \Big(\mathsf{S\text{-}To\text{-}\mathbb{S}\text{-}Alg}\ \mathcal{X}\Big)\ \Big(\mathsf{S\text{-}To\text{-}\mathbb{S}\text{-}FibAlg}(\mathcal{X})\ \mathcal{Y}\Big)$$

Proof. Let algebras $(C, c, p) : \mathsf{S\text{-}Alg}_{\mathcal{U}_i}$ and $(D, d, q) : \mathsf{S\text{-}FibAlg}_{\mathcal{U}_j}\ (D, d, p)$ be given. Then as before, $\mathsf{S\text{-}To\text{-}\mathbb{S}\text{-}Alg}\ (C, c, p)$ is the algebra $(C, c, c, 1_c, p)$ and $\mathsf{S\text{-}To\text{-}\mathbb{S}\text{-}FibAlg}(C, c, p)\ (D, d, q)$ is the algebra $(D, d, d, 1_d, q)$. The desired equivalence now follows immediately from the fact that for any $f : \Pi_{x:C} D(x)$, $\beta : f(c) = d$, we have

$$\begin{aligned} & \mathsf{dap}_f(p) = \mathsf{ap}_{p_*^E}(\beta) \cdot q \cdot \beta^{-1} \\ \simeq\ & \Big(\Sigma \mathbf{r} : \Sigma_{\gamma : f(c) = d}(\gamma = \beta)\Big)\Big(\mathsf{dap}_f(p) = \mathsf{ap}_{p_*^E}(\beta) \cdot q \cdot \pi_1(\mathbf{r})^{-1}\Big) \\ \simeq\ & \big(\Sigma \gamma : f(c) = d\big)(\gamma = \beta) \times \Big(\mathsf{dap}_f(p) = \mathsf{ap}_{p_*^E}(\beta) \cdot q \cdot \gamma^{-1}\Big) \\ \simeq\ & \big(\Sigma \gamma : f(c) = d\big)\Big(\mathsf{dap}_f(1_c) = \mathsf{ap}_{(1_c)_*^E}(\beta) \cdot 1_d \cdot \gamma^{-1}\Big) \times \Big(\mathsf{dap}_f(p) = \mathsf{ap}_{p_*^E}(\beta) \cdot q \cdot \gamma^{-1}\Big) \end{aligned}$$

where the first equivalence follows from the fact that the type $\Sigma_{\gamma : f(c) = d}(\gamma = \beta)$ is contractible with the center of contraction $(\beta, 1_\beta)$. □

Lemma 59. *For algebras $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ and $\mathcal{Y} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}$ we have*

$$\mathbf{S}\text{-}\mathsf{Mor}\ \mathcal{X}\ \mathcal{Y} \;\simeq\; \mathbb{S}\text{-}\mathsf{Mor}\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathbb{S}\text{-}\mathsf{Alg}\ \mathcal{X}\right)\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathbb{S}\text{-}\mathsf{Alg}\ \mathcal{Y}\right)$$

Proof. Analogously to the fibered case. □

We can now show that $\mathbf{S}$-recursion is the same as $\mathbb{S}$-recursion, and likewise for induction and homotopy-initiality:

Lemma 60. *For an algebra $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ we have*

$$\begin{aligned}
\mathsf{has}\text{-}\mathbf{S}\text{-}\mathsf{Rec}_{\mathcal{U}_j}(\mathcal{X}) &\simeq \mathsf{has}\text{-}\mathbb{S}\text{-}\mathsf{Rec}_{\mathcal{U}_j}\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathbb{S}\text{-}\mathsf{Alg}(\mathcal{X})\right)\\
\mathsf{has}\text{-}\mathbf{S}\text{-}\mathsf{Ind}_{\mathcal{U}_j}(\mathcal{X}) &\simeq \mathsf{has}\text{-}\mathbb{S}\text{-}\mathsf{Ind}_{\mathcal{U}_j}\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathbb{S}\text{-}\mathsf{Alg}(\mathcal{X})\right)\\
\mathsf{is}\text{-}\mathbf{S}\text{-}\mathsf{HInit}_{\mathcal{U}_j}(\mathcal{X}) &\simeq \mathsf{is}\text{-}\mathbb{S}\text{-}\mathsf{HInit}_{\mathcal{U}_j}\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathbb{S}\text{-}\mathsf{Alg}(\mathcal{X})\right)
\end{aligned}$$

Corollary 61. $(\mathcal{H}_{\mathrm{int}} + \mathbf{S})$ *The $\mathbb{S}$-algebra $(\mathbf{S}, \mathsf{base}, \mathsf{base}, \mathsf{loop}, 1_{\mathsf{base}})$ satisfies the $\mathbb{S}$-induction principle on any universe $\mathcal{U}_j$.*

Corollary 62. $(\mathcal{H}_{\mathrm{int}} + \mathbb{S})$ *The $\mathbf{S}$-algebra $(\mathbb{S}, \mathsf{north}, \mathsf{east} \cdot \mathsf{west}^{-1})$ satisfies the $\mathbf{S}$-induction principle on any universe $\mathcal{U}_j$.*

We note that the equivalence between $\mathbf{S}$ and $\mathbb{S}$ established in 40 together with the univalence axiom give us an equality of types $\mathbf{S} = \mathbb{S}$. Hence we have another way to turn the type $\mathbb{S}$ it into an $\mathbf{S}$-algebra: we simply "carry over" each of the constructors base and loop along the equality $\mathbf{S} = \mathbb{S}$, to get constructors base' and loop' operating on $\mathbb{S}$ instead of $\mathbf{S}$. However, to understand what base' and loop' in fact are, we need to "unwrap" the application of the univalence axiom and understand how it acts on this specific equivalence. In our case, it is not hard to show that base' is equal to north and loop' is appropriately related to $\mathsf{east} \cdot \mathsf{west}^{-1}$; this gives us an insight into the computational content of univalence in this particular scenario.

3.1.3 Type Quotient Algebras

Just like we did in the preceding section for circles, we can define the notions of algebra, morphism, and homotopy-initiality for any of the higher inductive types described in section 2.4. However, it is not hard to see that the type quotients A/R subsume all the other ones as special cases: for example, we can encode the circle $\mathbf{S}^1$ by putting $A :\equiv \mathbf{1}$ and $R(-,-) :\equiv \mathbf{1}$, and the suspension ΣB by putting $A :\equiv \mathbf{2}$ and $R(\bot,\top), R(\bot,\bot), R(\top,\top) :\equiv \mathbf{0}$, $R(\top,\bot) :\equiv B$. For this reason we will only focus on type quotients in this section.

Definition 63. *For $A : \mathcal{U}_i$ and $R : A \to A \to \mathcal{U}_i$, define the type of* type quotient algebras *on a universe $\mathcal{U}_j$ as*

$$\mathsf{TQAlg}_{\mathcal{U}_j}(A, R) :\equiv \Sigma_{C:\mathcal{U}_j}\Sigma_{c:A\to C}\Pi_{a,b:A}R(a,b) \to (c(a) = c(b))$$

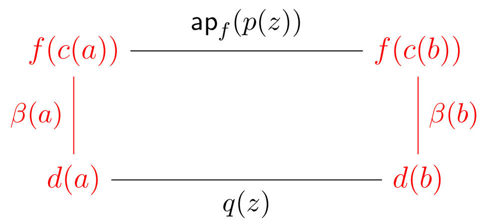
Definition 64. *Given an algebra $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$, we define the type of* fibered type quotient algebras *on a universe $\mathcal{U}_k$ by*

$$\mathsf{TQFibAlg}_{\mathcal{U}_k}\ (C, c, p) :\equiv \Sigma_{E:C\to\mathcal{U}_k}\Sigma_{e:(\Pi a:A)E(c(a))}\Pi_{a,b:A}\Pi_{z:R(a,b)}(p(z)_*^E\ e(a) = e(b))$$

Definition 65. *Given algebras* $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ *and* $\mathcal{Y} : \mathsf{TQAlg}_{\mathcal{U}_k}(A, R)$*, define the type of* type quotient morphisms *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *by*

$$\mathsf{TQMor}\ (C, c, p)\ (D, d, q) :\equiv \Sigma_{f:C\to D}\Sigma_{\beta:(\Pi a:A)(f(c(a))=d(a))}\Pi_{a,b:A}\Pi_{z:R(a,b)} \Big(\mathsf{ap}_f(p(z)) = \beta(a) \cdot q(z) \cdot \beta(b)^{-1}\Big)$$

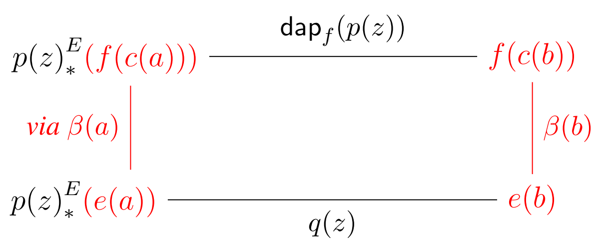
Pictorially, the last component of a type quotient morphism witnesses the following commuting diagram for any a, b, z:



Definition 66. *Given algebras* $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ *and* $\mathcal{Y} : \mathsf{TQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$*, we define the type of* fibered type quotient morphisms *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *by*

$$\mathsf{TQFibMor}\ (C, c, p)\ (E, e, q) :\equiv \Sigma_{f:(\Pi x:C)E(x)}\Sigma_{\beta:(\Pi a:A)(f(c(a))=e(a))}\Pi_{a,b:A}\Pi_{z:R(a,b)} \Big(\mathsf{dap}_f(p(z)) = \mathsf{ap}_{p(z)^E_*}(\beta(a)) \cdot q(z) \cdot \beta(b)^{-1}\Big)$$

Pictorially, the last component of a fibered type quotient morphism witnesses the following commuting diagram for any a, b, z:



Definition 67. *An algebra* $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ satisfies the recursion principle *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{TQAlg}_{\mathcal{U}_k}(A, R)$ *there exists a morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasTQRec}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi\mathcal{Y} : \mathsf{TQAlg}_{\mathcal{U}_k}(A, R)\Big)\ \mathsf{TQMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 68. *An algebra* $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ satisfies the induction principle *on a universe* $\mathcal{U}_k$ *if for any fibered algebra* $\mathcal{Y} : \mathsf{TQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *there exists a fibered morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasTQInd}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi\mathcal{Y} : \mathsf{TQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\Big)\ \mathsf{TQFibMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 69. *An algebra* $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ *is* homotopy-initial *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{TQAlg}_{\mathcal{U}_k}(A, R)$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is contractible:*

$$\mathsf{isTQHInit}_{\mathcal{U}_k}(\mathcal{X}) \ :\equiv \Big(\Pi\mathcal{Y} : \mathsf{TQAlg}_{\mathcal{U}_k}(A, R)\Big)\ \mathsf{TQMor}\ \mathcal{X}\ \mathcal{Y}$$

Now that we have introduced all the relevant notions, we can ask whether the analogue of corollary 36 holds for higher inductive types. For this, we would like to study a class of higher inductive types which is as general as we can make it, so that the special cases for the circle, etc. would arise as simple corollaries. As we pointed out at the beginning of this section, type quotients subsume all of the higher inductive types presented so far. However, they do not provide us with a way to do nontrivial recursion, so constructions such as the W-types from section 2.3 do not obviously arise as special cases of type quotients. This raises a natural question of whether we can come up with a useful class of higher inductive types that combines higher dimensional structure with proper recursion. The answer is yes, as we will see shortly.

3.2 W-quotients

Here we consider a class of higher inductive types which we call W-*quotients*; informally, they combine Martin-Löf's W-types [21] with a certain form of type quotients ([36]). Ordinary W-types allow proper induction on the level of points but have no higher-dimensional constructors. Type quotients, on the other hand, only provide vacuous induction on the point level, in the form of the $[-]$ constructor; however, they allow us to specify an arbitrary number of path constructors between the points thus obtained. A suitable combination of these two classes of types keeps the of induction and higher-dimensionality orthogonal, which gives us a well-behaved elimination principle.

Formally, given types $A, C : \mathcal{U}_i$, a type family $B : A \to \mathcal{U}_i$, and functions $\mathtt{l}, \mathtt{r} : C \to A$, the W-quotient $\mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r}) : \mathcal{U}_i$ is the higher inductive type generated by the constructors

$$\begin{aligned}
\mathsf{point}_{\mathsf{W}} &: \Pi_{a:A}\big(B(a) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r})\big) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r}) \\
\mathsf{cell}_{\mathsf{W}} &: \Pi_{c:C}\Pi_{t:B(\mathtt{l}\ c)\to\mathsf{WQ}(A,B,C,\mathtt{l},\mathtt{r})}\Pi_{s:B(\mathtt{r}\ c)\to\mathsf{WQ}(A,B,C,\mathtt{l},\mathtt{r})}\ \big(\mathsf{point}_{\mathsf{W}}(\mathtt{l}\ c, t) = \mathsf{point}_{\mathsf{W}}(\mathtt{r}\ c, s)\big)
\end{aligned}$$

As in the case of ordinary W-types, the type A can be thought of as the type of operations and for any $a : A$, the type $B(a)$ represents the arity of the operation a, ***i.e.***, it is the index type for the arguments of a. Similarly, the type C represents the type of labels for paths between points. For any $c : C$, the terms $\mathtt{l}(c)$ and $\mathtt{r}(c)$ determine the respective labels of the left and right endpoints of the paths labeled by c. As can be read off from the type of the constructor $\mathsf{cell}_{\mathsf{W}}$, each label $c : C$ determines a family of paths in $\mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r})$, one for each pair of terms $t : B(\mathtt{l}\ c) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r})$ and $s : B(\mathtt{r}\ c) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r})$.

An ordinary W-type $\mathsf{W}(A, B)$ arises as a W-quotient in the obvious way by taking $A :\equiv A$, $B :\equiv B$, $C :\equiv \mathbf{0}$, and letting both $\mathtt{l}$ and $\mathtt{r}$ be the canonical function from $\mathbf{0}$ into A. The type quotient A/R arises if we take $A :\equiv A$, $B(-) :\equiv \mathbf{0}$, $C :\equiv \Sigma_{a,b:A}R(a, b)$, $\mathtt{l}(a, b, z) :\equiv a$, $\mathtt{r}(a, b, z) :\equiv b$. We can encode the circle $\mathbf{S}$ by taking $A, C :\equiv \mathbf{1}$, $B(-) :\equiv \mathbf{0}$, $\mathtt{l}(-) :\equiv \star$, $\mathtt{r}(-) :\equiv \star$. The circle $\mathbb{S}$ arises when we take $A, C :\equiv \mathbf{2}$, $B(-) :\equiv \mathbf{0}$, $\mathtt{l}(-) :\equiv \top$, $\mathtt{r}(-) :\equiv \bot$. Of course, we can also represent the special cases of type quotients and W-types: the interval, suspensions - in particular all the higher spheres $\mathbf{S}^n$ - natural numbers, lists, and so on. We remark, however, that in most of these cases, the higher inductive types encoded as W-quotients will satisfy the computation rules *up to propositional equality* rather than definitionally; this goes back to the issue of different representations mentioned in section 3.1.

As another example we consider positive integers modulo two. Let $\mathbf{4}$ be the inductive type with constructors $\mathsf{tt}, \mathsf{tf}, \mathsf{ft}, \mathsf{ff} : \mathbf{4}$. We put $A :\equiv \mathbf{4}$; $B(\mathsf{tt}) :\equiv \mathbf{0}$, $B(\mathsf{ff}) :\equiv \mathbf{0}$, $B(\mathsf{tf}) :\equiv \mathbf{1}$, $B(\mathsf{ft}) :\equiv \mathbf{1}$; $C :\equiv \mathbf{2}$; $\mathtt{l}(\top) :\equiv \mathsf{tt}$, $\mathtt{l}(\bot) :\equiv \mathsf{ff}$; $\mathtt{r}(\top) :\equiv \mathsf{tf}$, $\mathtt{r}(\bot) :\equiv \mathsf{ft}$. The nullary point labels tt and ff encode the positive integers one and two, respectively. The unary point label tf represents the function $n \mapsto 2n + 1$ and the unary point label ft represents the function $n \mapsto 2(n + 1)$. The path label $\top$ represents equations of the form $(\mathsf{tt}, -) = (\mathsf{tf}, -)$, to equate all odd positive integers and the path label $\bot$ represents equations of the form $(\mathsf{ff}, -) = (\mathsf{ft}, -)$, to equate all even positive integers. We can see this more clearly in figure 3.1, where the positive integer represented is highlighted in red, and $\star$ stands for the canonical function out of the empty type.

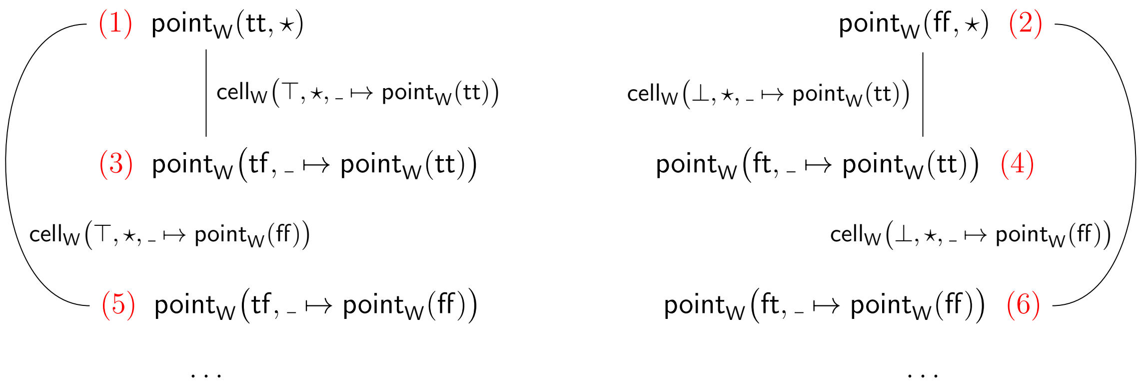


Figure 3.1: Positive integers modulo 2 as a W-quotient

W-quotients come with the expected recursion principle: given terms

- $E : \mathcal{U}_j$,
- $e : \Pi_{a:A}(B(a) \to E) \to E$,
- $q : \Pi_{c:C}\Pi_{u:B(\mathtt{l}\ c)\to E}\Pi_{v:B(\mathtt{r}\ c)\to E}\big(e(\mathtt{l}\ c, u) = e(\mathtt{r}\ c, v)\big)$,

there is a recursor $\mathsf{rec}_{\mathsf{WQ}}(E, e, q) : \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r}) \to E$. The recursor satisfies the computation laws

- $\mathsf{rec}_{\mathsf{WQ}}(\mathsf{point}_{\mathsf{W}}(a, t)) \equiv e\big(a, \mathsf{rec}_{\mathsf{WQ}} \circ t\big)$ for any $a : A, t : B(a) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r})$,
- $\mathsf{ap}_{\mathsf{rec}_{\mathsf{WQ}}}(\mathsf{cell}_{\mathsf{W}}(c, t, s)) = q\Big(c, \mathsf{rec}_{\mathsf{WQ}} \circ t, \mathsf{rec}_{\mathsf{WQ}} \circ s\Big)$
  for any $c : C, t : B(\mathtt{l}\ c) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r}), s : B(\mathtt{r}\ c) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r})$.

Similarly, we have an induction principle: given terms

- $E : \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r}) \to \mathcal{U}_j$,
- $e : \Pi_{a:A}\Pi_{t:B(a)\to\mathsf{WQ}(A,B,C,\mathtt{l},\mathtt{r})}\big(\Pi_{b:B(a)}E(t\ b)\big) \to E(\mathsf{point}_{\mathsf{W}}(a, t))$,
- $q : \Pi_{c:C}\Pi_{t:B(\mathtt{l}\ c)\to\mathsf{WQ}(A,B,C,\mathtt{l},\mathtt{r})}\Pi_{s:B(\mathtt{r}\ c)\to\mathsf{WQ}(A,B,C,\mathtt{l},\mathtt{r})}\Pi_{u:(\Pi b:B(\mathtt{l}\ c))E(t\ b)}\Pi_{v:(\Pi b:B(\mathtt{r}\ c))E(s\ b)}$
  $\big(\mathsf{cell}_{\mathsf{W}}(c, t, s)_*^E\ e(\mathtt{l}\ c, t, u) = e(\mathtt{r}\ c, s, v)\big)$,

there is an inductor $\mathsf{ind}_{\mathsf{WQ}}(E, e, q) : \Pi_{w:\mathsf{WQ}(A,B,C,\mathtt{l},\mathtt{r})}E(w)$. The inductor satisfies the computation laws

- $\mathsf{ind}_{\mathsf{WQ}}(\mathsf{point}_{\mathsf{W}}(a, t)) \equiv e\big(a, t, \mathsf{ind}_{\mathsf{WQ}} \circ t\big)$ for any $a : A, t : B(a) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r})$,
- $\mathsf{dap}_{\mathsf{ind}_{\mathsf{WQ}}}(\mathsf{cell}_{\mathsf{W}}(c, t, s)) = q\Big(c, t, s, \mathsf{ind}_{\mathsf{WQ}} \circ t, \mathsf{ind}_{\mathsf{WQ}} \circ s\Big)$
  for any $c : C, t : B(\mathtt{l}\ c) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r}), s : B(\mathtt{r}\ c) \to \mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r})$.

Following the now-familiar pattern, we define W-quotient algebras and morphisms, together with their fibered counterparts. For convenience, we will utilize the corresponding notions for ordinary W-types $\mathsf{W}(A, B)$ introduced in section 2.3. For notational convenience, we capture the additional structure of a W-quotient algebra or morphism in a type family defined over the corresponding W-type algebra or morphism.

Definition 70. *For* $A, C : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, $\mathtt{l}, \mathtt{r} : C \to A$, *we define a type family over the type* $\mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ *by*

$$\mathsf{WAlgFam}(A, B, C, \mathtt{l}, \mathtt{r})\ (D, d) :\equiv \Pi_{c:C} \Pi_{u:B(\mathtt{l}\ c) \to D} \Pi_{v:B(\mathtt{r}\ c) \to D} \big(d(\mathtt{l}\ c, u) = d(\mathtt{r}\ c, v)\big)$$

Definition 71. *For* $A, C : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, $\mathtt{l}, \mathtt{r} : C \to A$, *we define the type of* W-quotient algebras *on a universe* $\mathcal{U}_j$ *by*

$$\mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r}) :\equiv \Big(\Sigma \mathcal{X}_0 : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)\Big)\ \mathsf{WAlgFam}(A, B, C, \mathtt{l}, \mathtt{r})\ \mathcal{X}_0$$

Definition 72. *For an algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$, *we define a type family over the type* $\mathsf{WFibAlg}_{\mathcal{U}_k}\ \pi_1(\mathcal{X})$ *by*

$$\begin{aligned}\mathsf{WFibAlgFam}\ (D, d, p)\ (E, e) :\equiv\ & \Pi_{c:C} \Pi_{t:B(\mathtt{l}\ c) \to D} \Pi_{s:B(\mathtt{r}\ c) \to D} \\ & \Pi_{u:(\Pi b:B(\mathtt{l}\ c))E(t\ b)} \Pi_{v:(\Pi b:B(\mathtt{r}\ c))E(s\ b)} \Big(p(c, t, s)_*^E\ e(\mathtt{l}\ c, t, u) = e(\mathtt{r}\ c, s, v)\Big)\end{aligned}$$

Definition 73. *Given an algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$, *we define the type of* fibered W-quotient algebras *on a universe* $\mathcal{U}_k$ *over* $\mathcal{X}$ *by*

$$\mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X} :\equiv \Big(\Sigma \mathcal{Y}_0 : \mathsf{WFibAlg}_{\mathcal{U}_k}\ \pi_1(\mathcal{X})\Big)\ \mathsf{WFibAlgFam}\ \mathcal{X}\ \mathcal{Y}_0$$

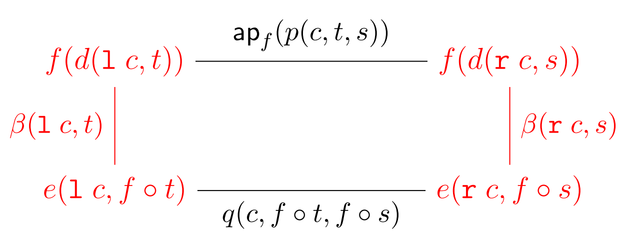
Definition 74. *Given algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$, *define a type family over the type* $\mathsf{WMor}\ \pi_1(\mathcal{X})\ \pi_1(\mathcal{Y})$ *by*

$$\begin{aligned}\mathsf{WMorFam}\ (D, d, p)\ (E, e, q)\ (f, \beta) :\equiv\ & \Pi_{c:C} \Pi_{t:B(\mathtt{l}\ c) \to D} \Pi_{s:B(\mathtt{r}\ c) \to D} \\ & \Big(\mathsf{ap}_f(p(c, t, s)) = \beta(\mathtt{l}\ c, t) \cdot q(c, f \circ t, f \circ s) \cdot \beta(\mathtt{r}\ c, s)^{-1}\Big)\end{aligned}$$

Definition 75. *Given algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$, *define the type of* W-quotient morphisms *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *by*

$$\mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y} :\equiv \Big(\Sigma \mu_0 : \mathsf{WMor}\ \pi_1(\mathcal{X})\ \pi_1(\mathcal{Y})\Big)\ \mathsf{WMorFam}\ \mathcal{X}\ \mathcal{Y}\ \mu_0$$

Pictorially, the last component of a W-quotient morphism witnesses the following commuting diagram for any c, t, s:



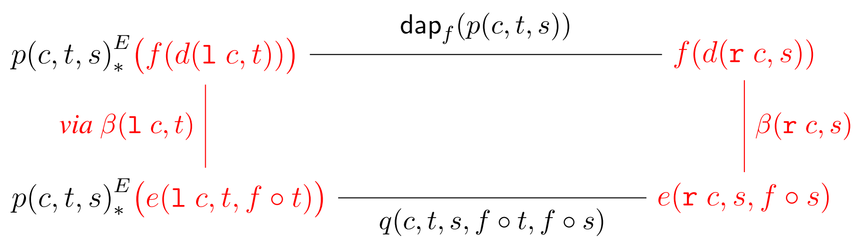
Definition 76. *For algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$*, we define a type family over the type* $\mathsf{WFibMor}\ \pi_1(\mathcal{X})\ \pi_1(\mathcal{Y})$ *by*

$$\mathsf{WFibMorFam}\ (D, d, p)\ (E, e, q)\ (f, \beta) :\equiv \Pi_{c:C}\Pi_{t:B(\mathtt{l}\ c)\to D}\Pi_{s:B(\mathtt{r}\ c)\to D}$$
$$\left(\mathsf{dap}_f(p(c,t,s)) = \mathsf{ap}_{p(c,t,s)^E_*}(\beta(\mathtt{l}\ c, t)) \cdot q(c,t,s,f\circ t, f\circ s) \cdot \beta(\mathtt{r}\ c, s)^{-1}\right)$$

Definition 77. *For algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$*, we define the type of* fibered W-quotient morphisms *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *by*

$$\mathsf{WQFibMor}\ \mathcal{X}\ \mathcal{Y} :\equiv \left(\Sigma\mu_0 : \mathsf{WFibMor}\ \pi_1(\mathcal{X})\ \pi_1(\mathcal{Y})\right)\ \mathsf{WFibMorFam}\ \mathcal{X}\ \mathcal{Y}\ \mu_0$$

Pictorially, the last component of a fibered W-quotient morphism witnesses the following commuting diagram for any c, t, s:



The recursion and induction principles for W-quotients can now be defined as usual:

Definition 78. *An algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ satisfies the recursion principle *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ *there exists a morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasWQRec}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \left(\Pi\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})\right)\ \mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 79. *An algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ satisfies the induction principle *on a universe* $\mathcal{U}_k$ *if for any fibered algebra* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *there is a fibered morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \left(\Pi\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\right)\ \mathsf{WQFibMor}\ \mathcal{X}\ \mathcal{Y}$$

We will also need the following uniqueness properties which state that any two (fibered) morphisms into any (fibered) algebra $\mathcal{Y}$ are equal:

Definition 80. *An algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ satisfies the recursion uniqueness principle *on a universe* $\mathcal{U}_k$ *if for any other algebra* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is a mere proposition:*

$$\mathsf{hasWQRecUniq}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \left(\Pi\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})\right)\ \mathsf{isProp}(\mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y})$$

Definition 81. *An algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ satisfies the induction uniqueness principle *on a universe* $\mathcal{U}_k$ *if for any fibered algebra* $\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is a mere proposition:*

$$\mathsf{hasWQIndUniq}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \left(\Pi\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\right)\ \mathsf{isProp}(\mathsf{WQFibMor}\ \mathcal{X}\ \mathcal{Y})$$

The homotopy-initiality condition again says that there is a propositionally unique morphism into any other algebra $\mathcal{Y}$:

Definition 82. *An algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *is* homotopy-initial *on a universe* $\mathcal{U}_k$ *if for any other algebra* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is contractible:*

$$\mathsf{isWQHInit}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})\Big)\ \mathsf{isContr}(\mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y})$$

3.3 Homotopy-initiality for W-quotients

Our main result establishes the equivalence between the universal property of being homotopy-initial and the satisfaction of the induction principle:

Theorem 83. ($\mathcal{H}$) *For $A, C : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, $\mathtt{l}, \mathtt{r} : C \to A$, the following conditions on an algebra $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ are equivalent:*

- *$\mathcal{X}$ satisfies the induction principle on the universe $\mathcal{U}_k$*
- *$\mathcal{X}$ is homotopy-initial on the universe $\mathcal{U}_k$*

for $k \geq j$. In other words, we have

$$\mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \simeq \quad \mathsf{isWQHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. Moreover, the two types above are mere propositions.

We note that since universe levels are cumulative, the technical restriction that $k \geq j$ does not pose a problem. It is easy to see that homotopy-initiality is equivalent to the principles of recursion plus recursion uniqueness (a fact recorded as lemma 84 later in this section). The uniqueness condition is necessary since as is well known, in general the recursion principle does not fully determine an inductive type: the recursion principle for the circle, for example, is also satisfied by the disjoint union of *two* circles.

Proof outline A crucial step of the proof is the characterization of the path space $\mu = \nu$ between two (fibered) W-quotient morphisms $\mu, \nu : \mathcal{X} \to \mathcal{Y}$ in a more explicit form. For simplicity we only consider the non-fibered case here. We recall that a morphism between two algebras (D, d, p), (E, e, q) is a triple (f, β, θ), where $f : C \to D$ is a function between the carrier types, β specifies the behavior of f on the 0-cells, i.e., the value of $f(d(a, t))$, and θ specifies the behavior of f on the 1-cells, i.e., the value of $\mathsf{ap}_f(p(c, t, s))$.

Using the characterization of paths between tuples together with function extensionality, the path space $(f, \beta, \theta) = (g, \gamma, \phi)$ between two morphisms should be equivalent to a type of triples (α, η, ψ), where $\alpha : f \sim g$ is a homotopy relating the two underlying mappings, and η, ψ relate β to γ resp. θ to ϕ in an appropriate way. We will call such a triple (α, η, ψ) a W*-quotient cell*. The recursion uniqueness condition on an algebra $\mathcal{X}$ can then be equivalently expressed as saying that for any algebra $\mathcal{Y}$ and morphisms μ, ν from $\mathcal{X}$ to $\mathcal{Y}$, there exists a W-quotient cell between μ and ν.

We point out that this uniqueness condition can itself be understood as a certain form of induction, albeit a very specific one. The existence of a W-quotient cell between any two morphisms (f, β, θ), (g, γ, ϕ) in particular guarantees the existence of a dependent function $\alpha : \Pi_{x:X}(f(x) = g(x))$ - the "inductor". The behavior of α on the 0-cells, i.e., the value of $\alpha(d(a, t))$, is specified by the term η, which thus serves as a witness for the first "computation rule". Finally, the behavior of α on the 1-cells, i.e., the value of $\mathsf{dap}_\alpha(p(c, t, s))$, is specified by the term ψ, which hence serves as a witness for the second "computation rule."

We can thus see why the full induction principle for W-quotients gives us homotopy-initiality: the latter essentially amounts to the recursion principle plus a specific form of induction, both of which are implied by the general induction principle. The hardest part of the proof is showing the converse, i.e., that the general induction principle can be recovered from homotopy-initiality.

We now proceed to the formal proof of the main theorem. We have the following analogues to the lemmas presented in section 2.3 for ordinary W-types in extensional type theory:

Lemma 84. *For an algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *we have*

$$\mathsf{isWQHInit}_{\mathcal{U}_k}(\mathcal{X}) \;\simeq\; \mathsf{hasWQRec}_{\mathcal{U}_k}(\mathcal{X}) \times \mathsf{hasWQRecUniq}_{\mathcal{U}_k}(\mathcal{X})$$

Lemma 85. *For an algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *we have a function*

$$\mathsf{WQAlgToFibAlg}_{\mathcal{U}_k}(\mathcal{X}) \;:\; \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r}) \;\to\; \mathsf{WQFibAlg}_{\mathcal{U}_k}\,\mathcal{X}$$

Proof. Fix algebras $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ and $(E, e, q) : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$. We turn (E, e, q) into the desired fibered algebra $(E', e', q') : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ (D, d, p)$ by putting $E'(x) :\equiv E$ and $e'(a, t, u) :\equiv e(a, u)$ and letting $q'(c, t, s, u, v)$ be the path

$$p(c,t,s)_*^{-\mapsto E}\big(e(\mathtt{l}\ c,u)\big) \;=\!=\!=\; e(\mathtt{l}\ c,u) \;\overset{q(c,u,v)}{=\!=\!=\!=}\; e(\mathtt{r}\ c,v)$$

where the first equality follows from the straightforward fact that the transport between any two fibers of a constant type family is the identity function. ∎

Lemma 86. $(\mathcal{H})$ *For algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ *we have*

$$\mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y} \;\simeq\; \mathsf{WQFibMor}\ \mathcal{X}\ \Big(\mathsf{WQAlgToFibAlg}_{\mathcal{U}_k}(\mathcal{X})\ \mathcal{Y}\Big)$$

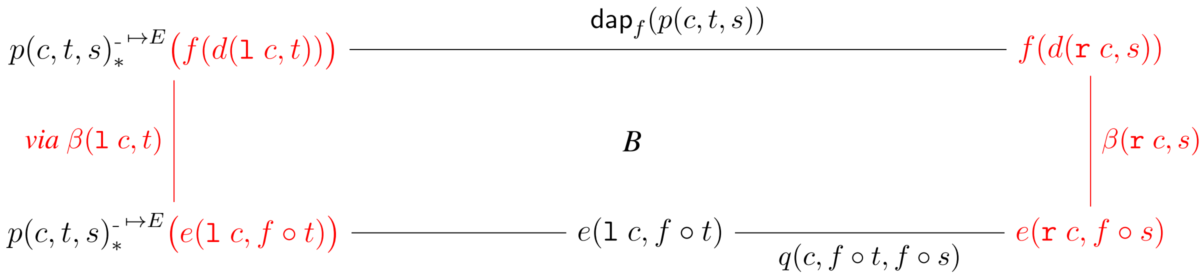
Proof. Let algebras $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ and $(E, e, q) : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ be given and let $(E', e', q') :\equiv \mathsf{WQAlgToFibAlg}_{\mathcal{U}_k}(D, d, p)\ (E, e, q)$. We aim to show

$$\mathsf{WQMor}\ (D, d, p)\ (E, e, q) \;\simeq\; \mathsf{WQFibMor}\ (D, d, p)\ (E', e', q')$$

As observed in remark 30 we have $\mathsf{WMor}\ (D, d)\ (E, e) \equiv \mathsf{WFibMor}\ (D, d)\ (E', e')$. It thus suffices to show that for any $(f, \beta) : \mathsf{WMor}\ (D, d)\ (E, e)$ and (by function extensionality) any c, t, s, the commutativity of the diagram

$$\begin{array}{ccc}
f(d(\mathtt{l}\ c,t)) & \overset{\mathsf{ap}_f(p(c,t,s))}{=\!=\!=\!=\!=\!=} & f(d(\mathtt{r}\ c,s)) \\
\beta(\mathtt{l}\ c,t)\ \Big\| & A & \Big\|\ \beta(\mathtt{r}\ c,s) \\
e(\mathtt{l}\ c, f\circ t) & \underset{q(c, f\circ t, f\circ s)}{=\!=\!=\!=\!=\!=} & e(\mathtt{r}\ c, f\circ s)
\end{array}$$

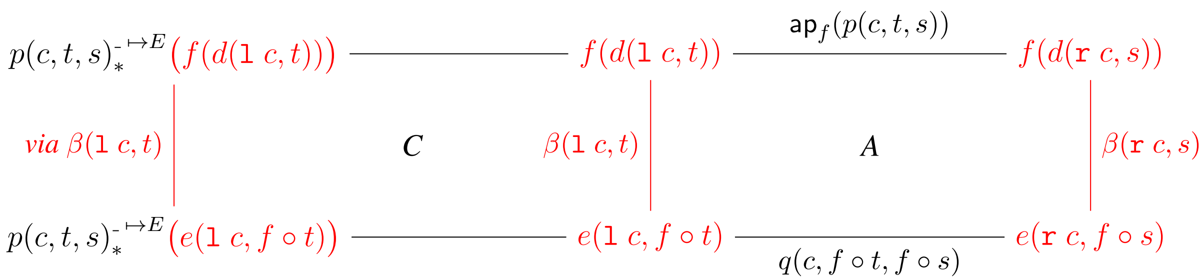
is equivalent to the commutativity of the diagram



We note that by a straightforward path induction we can express $\mathsf{dap}_f(p(c,t,s))$ equivalently as the path

$$p(c,t,s)_*^{-\mapsto E}\big(f(d(\mathtt{l}\ c,t))\big) \;\text{———}\; f(d(\mathtt{l}\ c,t)) \;\overset{\mathsf{ap}_f(p(c,t,s))}{\text{————}}\; f(d(\mathtt{r}\ c,s))$$

Thus the commutativity of *B* is equivalent to the commutativity of the outer rectangle in the diagram below:



But rectangle *C* clearly commutes by an easy generalization and subsequent path induction on $\beta(\mathtt{l}\ c,t)$. Hence the commutativity of the outer rectangle is equivalent to the commutativity of *A* and we are done. □

As in section 2.3, the previous two lemmas immediately imply the following:

Lemma 87. $(\mathcal{H})$ *For any algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A,B,C,\mathtt{l},\mathtt{r})$ *we have*

$$\mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X}) \;\;\to\;\; \mathsf{hasWQRec}_{\mathcal{U}_k}(\mathcal{X})$$

Lemma 88. $(\mathcal{H})$ *For any algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A,B,C,\mathtt{l},\mathtt{r})$ *we have*

$$\mathsf{hasWQIndUniq}_{\mathcal{U}_k}(\mathcal{X}) \;\;\to\;\; \mathsf{hasWQRecUniq}_{\mathcal{U}_k}(\mathcal{X})$$

As discussed earlier, in order to establish the analogues of lemmas 33 and 35 we use the auxiliary notion of a (fibered) W-quotient cell. For simplicity we first introduce the corresponding notions for ordinary W-types and then proceed to the general case of W-quotients.

Intuitively, a (fibered) W-cell between two (fibered) W-morphisms (f,β) and (g,γ) is a homotopy $\alpha : f \sim g$, together with a proof that α behaves as expected on canonical elements, *i.e.*, that the value of $\alpha(d(a,t))$ is the "obvious" one obtained recursively by combining β, γ, and the values of $\alpha(t\ b)$ for $b : B(a)$.

Definition 89. $(\mathcal{H})$ *For algebras* $\mathcal{X}_0 : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$, $\mathcal{Y}_0 : \mathsf{WFibAlg}_{\mathcal{U}_k}(A,B)\ \mathcal{X}_0$, *and fibered morphisms* $\mu_0, \nu_0 : \mathsf{WFibMor}\ \mathcal{X}_0\ \mathcal{Y}_0$, *define the type of* fibered W-cells *between* μ_0 *and* ν_0 *by*

$$\begin{aligned}&\mathsf{WFibCell}\ (D,d)\ (E,e)\ (f,\beta)\ (g,\gamma) :\equiv \\ &\quad \Sigma_{\alpha:f\sim g}\Pi_{a:A}\Pi_{t:B(a)\to D}\Big(\alpha(d(a,t)) = \beta(a,t) \cdot \mathsf{ap}_{e(a,t)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha\circ t)\big) \cdot \gamma(a,t)^{-1}\Big)\end{aligned}$$

Pictorially, the second component of a fibered W-cell witnesses the commutativity of the following diagram for any a, t:

$$\begin{array}{ccc} f(d(a,t)) & \xrightarrow{\quad \alpha(d(a,t)) \quad} & g(d(a,t)) \\ \beta(a,t) \Big| & & \Big| \gamma(a,t) \\ e(a,t,f\circ t) & \xrightarrow[\mathsf{ap}_{e(a,t)}({}^{\Pi}\mathsf{E}^{=}(\alpha\circ t))]{} & e(a,t,g\circ t) \end{array}$$

As observed in remark 30, a non-fibered W-morphism is just a special case of a fibered one. We define a non-fibered W-cell analogously:

Definition 90. $(\mathcal{H})$ *For algebras* $\mathcal{X}_0 : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$, $\mathcal{Y}_0 : \mathsf{WAlg}_{\mathcal{U}_k}(A,B)$, *and morphisms* $\mu_0, \nu_0 : \mathsf{WMor}\ \mathcal{X}_0\ \mathcal{Y}_0$, *define the type of* W-cells *between* μ_0 *and* ν_0 *by*

$$\mathsf{WCell}\ \mathcal{X}_0\ \mathcal{Y}_0\ \mu_0\ \nu_0 :\equiv \mathsf{WFibCell}\ \mathcal{X}\ \Big(\mathsf{WAlgToFibAlg}_{\mathcal{U}_k}\ \mathcal{Y}\Big)\ \mu_0\ \nu_0$$

The right-hand side is well-typed precisely due to the fact recalled in remark 30. The second component of a W-cell witnesses the commutativity of the following diagram for any a, t:

$$\begin{array}{ccc} f(d(a,t)) & \xrightarrow{\quad \alpha(d(a,t)) \quad} & g(d(a,t)) \\ \beta(a,t) \Big| & & \Big| \gamma(a,t) \\ e(a,f\circ t) & \xrightarrow[\mathsf{ap}_{e(a)}({}^{\Pi}\mathsf{E}^{=}(\alpha\circ t))]{} & e(a,g\circ t) \end{array}$$

We note that a (fibered) W-cell is nothing more than a fibered W-morphism of a special form; it is thus easy to see why the induction principle for W-types implies the uniqueness principle, and hence homotopy-initiality. From now on we will omit all but the last two arguments to WCell and WFibCell.

Our next order of business is to show that this definition of a (fibered) W-cell is indeed the right one, *i.e.*, that the type of (fibered) W-cells between two (fibered) morphisms μ_0, ν_0 is equivalent to the type of paths between μ_0 and ν_0.

Lemma 91. $(\mathcal{H})$ *For algebras* $\mathcal{X}_0 : \mathsf{WAlg}_{\mathcal{U}_j}(A,B)$, $\mathcal{Y}_0 : \mathsf{WFibAlg}_{\mathcal{U}_k}\ \mathcal{X}_0$, *and fibered morphisms* $\mu_0, \nu_0 : \mathsf{WFibMor}\ \mathcal{X}_0\ \mathcal{Y}_0$, *we have an equivalence*

$$\mathsf{WFibMorPathToCell}_{\mu_0,\nu_0} \quad : \quad (\mu_0 = \nu_0) \ \simeq\ \mathsf{WFibCell}\ \mu_0\ \nu_0$$

Proof. Let algebras $(D, d) : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ and $(E, e) : \mathsf{WFibAlg}_{\mathcal{U}_k}\ (D, d)$ and fibered morphisms $(f, \beta), (g, \gamma) : \mathsf{WFibMor}\ (D, d)\ (E, e)$ be given. We establish the following chain of equivalences:

$$
\begin{aligned}
& (f, \beta) = (g, \gamma) \\
\simeq\ & \Big(\Sigma\alpha : f = g\Big)\ (\alpha)_*^{h \mapsto (\Pi a:A)(\Pi t:B(a)\to D)\left(h(d(a,t)) = e(a,t,h\circ t)\right)}(\beta) = \gamma \\
\simeq\ & \Big(\Sigma\alpha : f = g\Big)\ \Pi_{a:A}\Pi_{t:B(a)\to D}\Big(\big({}^{=}\mathsf{E}^{\Pi}(\alpha)\big)(d(a,t)) = \beta_{a,t} \cdot \mathsf{ap}_{e(a,t)}\big({}^{\Pi}\mathsf{E}^{=}\big({}^{=}\mathsf{E}^{\Pi}(\alpha) \circ t\big)\big) \cdot \gamma_{a,t}^{-1}\Big) \\
\simeq\ & \Big(\Sigma\alpha : f \sim g\Big)\ \Pi_{a:A}\Pi_{t:B(a)\to D}\Big(\alpha(d(a,t)) = \beta_{a,t} \cdot \mathsf{ap}_{e(a,t)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha \circ t)\big) \cdot \gamma_{a,t}^{-1}\Big) \\
\equiv\ & \mathsf{WFibCell}\ (f, \beta)\ (g, \gamma)
\end{aligned}
$$

The first equivalence follows by the characterization of paths in dependent product spaces. The third equivalence follows from the fact that the map ${}^{=}\mathsf{E}^{\Pi} : (f = g) \to (f \sim g)$ is itself an equivalence. Finally, to prove the second equivalence it suffices to show that for any $\alpha : f = g$, the respective fibers over α are equivalent. To do so, we generalize γ and perform a one-sided path induction on α, keeping the left endpoint f fixed. This leaves us to prove that for any $\gamma : \Pi_{a:A}\Pi_{t:B(a)\to D}\big(f(d(a,t)) = e(a, t, f \circ t)\big)$ we have

$$\beta = \gamma \ \ \simeq\ \ \Pi_{a:A}\Pi_{t:B(a)\to D}\Big(1_{d(a,t)} = \beta_{a,t} \cdot \mathsf{ap}_{e(a,t)}\big({}^{\Pi}\mathsf{E}^{=}\big({}^{=}\mathsf{E}^{\Pi}(1_{f\circ t})\big)\big) \cdot \gamma_{a,t}^{-1}\Big)$$

By function extensionality it suffices to prove that for any $a : A, t : B(a) \to D$, we have

$$\beta_{a,t} = \gamma_{a,t} \ \ \simeq\ \ \Big(1_{d(a,t)} = \beta_{a,t} \cdot \mathsf{ap}_{e(a,t)}\big({}^{\Pi}\mathsf{E}^{=}\big({}^{=}\mathsf{E}^{\Pi}(1_{f\circ t})\big)\big) \cdot \gamma_{a,t}^{-1}\Big)$$

This follows from the following chain of equivalences:

$$
\begin{aligned}
& \beta_{a,t} = \gamma_{a,t} \\
\simeq\ & \gamma_{a,t} = \beta_{a,t} \\
\simeq\ & 1_{d(a,t)} \cdot \gamma_{a,t} = \beta_{a,t} \cdot 1_{e(a,t,f\circ t)} \\
\simeq\ & 1_{d(a,t)} = \beta_{a,t} \cdot 1_{e(a,t,f\circ t)} \cdot \gamma_{a,t}^{-1} \\
\equiv\ & 1_{d(a,t)} = \beta_{a,t} \cdot \mathsf{ap}_{e(a,t)}(1_{f\circ t}) \cdot \gamma_{a,t}^{-1} \\
\simeq\ & 1_{d(a,t)} = \beta_{a,t} \cdot \mathsf{ap}_{e(a,t)}\big({}^{\Pi}\mathsf{E}^{=}\big({}^{=}\mathsf{E}^{\Pi}(1_{f\circ t})\big)\big) \cdot \gamma_{a,t}^{-1}
\end{aligned}
$$

□

Corollary 92. $(\mathcal{H})$ *For algebras* $\mathcal{X}_0 : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$, $\mathcal{Y}_0 : \mathsf{WAlg}_{\mathcal{U}_k}(A, B)$, *and morphisms* $\mu_0, \nu_0 : \mathsf{WMor}\ \mathcal{X}_0\ \mathcal{Y}_0$, *we have an equivalence*

$$\mathsf{WMorPathToCell}_{\mu_0,\nu_0} \ :\ (\mu_0 = \nu_0) \ \simeq\ \mathsf{WCell}\ \mu_0\ \nu_0$$

We are now ready to define W-quotient cells. Following the same methodology as before, we postulate that a W-quotient cell between (f, β, θ) and (g, γ, ϕ) should consist of a W-cell (α, η)

together with a proof that the value of $\mathsf{dap}_\alpha(p(c,t,s))$ is the "obvious" one. However, the type of the term $\mathsf{dap}_\alpha(p(c,t,s))$ involves a transport along the fibers of the type family $x \mapsto f(x) = g(x)$, making it unwieldy to work with. Instead, we axiomatize the value of $\mathsf{nat}(\alpha, p(c,t,s))$, which nevertheless specifies the value of $\mathsf{dap}_\alpha(p(c,t,s))$ uniquely as the latter term is expressible using the former.

Determining (and stating!) what the so-called obvious value of $\mathsf{nat}(\alpha, p(c,t,s))$ is requires a little work; this is expected since we are now working with paths on a higher level. To state the crucial definitions more compactly, we introduce the following notations:

- For any $u : a =_X b$, $v : b =_X d$, $w : a =_X c$, $z : c =_X d$, as in the diagram

$$\begin{array}{ccc} a & \overset{u}{\text{———}} & b \\ w \Big| & & \Big| v \\ c & \underset{z}{\text{———}} & d \end{array}$$

  we have maps

$$\mathbf{I}^1_\Box : (u = w \cdot z \cdot v^{-1}) \to (u \cdot v = w \cdot z)$$
$$\mathbf{I}^2_\Box : (u = w \cdot z \cdot v^{-1}) \to (w^{-1} \cdot u = z \cdot v^{-1})$$

  defined by induction on w, z, v; for the base case, given $\theta : u = 1_a$ we let $\mathbf{I}^1_\Box(\theta)$ and $\mathbf{I}^2_\Box(\theta)$ be the respective paths

$$u \cdot 1_a \;\text{———}\; u \;\overset{\theta}{\text{———}}\; 1_a \qquad \text{and} \qquad 1_a \cdot u \;\text{———}\; u \;\overset{\theta}{\text{———}}\; 1_a$$

  These maps are equivalences and we denote their quasi-inverses by $\mathbf{I}^{-1}_\Box$ and $\mathbf{I}^{-2}_\Box$.

Definition 93. *Given*

- *functions* $e_1 : X_1 \to Y$, $e_2 : X_2 \to Y$,
- *a heterogeneous homotopy* $q : e_1 \sim_{\mathcal{H}} e_2$,
- *paths* $r_1 : a_1 =_{X_1} b_1$, $r_2 : a_2 =_{X_2} b_2$ *and* $\delta_1 : c_1 =_Y c_2$, $\delta_2 : d_1 =_Y d_2$,
- *paths* $\beta_1 : c_1 =_Y e_1(a_1)$, $\beta_2 : c_2 =_Y e_2(a_2)$ *and* $\gamma_1 : d_1 =_Y e_1(b_1)$, $\gamma_2 : d_2 =_Y e_2(b_2)$,
- *higher paths* $\Theta : \delta_1 = \beta_1 \cdot q(a_1, a_2) \cdot \beta_2^{-1}$ *and* $\Phi : \delta_2 = \gamma_1 \cdot q(b_1, b_2) \cdot \gamma_2^{-1}$,

we let $\mathcal{P}(e_1, e_2, q, \Theta, \Phi, r_1, r_2)$ *be the higher path in figure 3.2.*

Definition 94. *Given*

- *a function* $F : Y_1 \to Y_2$,
- *functions* $e_1 : X_1 \to Y_1$, $e_2 : X_2 \to Y_2$,
- *a heterogeneous homotopy* $q : (F \circ e_1) \sim_{\mathcal{H}} e_2$,
- *paths* $r_1 : a_1 =_{X_1} b_1$, $r_2 : a_2 =_{X_2} b_2$ *and* $\delta_1 : F(c_1) =_{Y_2} c_2$, $\delta_2 : F(d_1) =_{Y_2} d_2$,
- *paths* $\beta_1 : c_1 =_{Y_1} e_1(a_1)$, $\beta_2 : c_2 =_{Y_2} e_2(a_2)$ *and* $\gamma_1 : d_1 =_{Y_1} e_1(b_1)$, $\gamma_2 : d_2 =_{Y_2} e_2(b_2)$,
- *higher paths* $\Theta : \delta_1 = \mathsf{ap}_F(\beta_1) \cdot q(a_1, a_2) \cdot \beta_2^{-1}$ *and* $\Phi : \delta_2 = \mathsf{ap}_F(\gamma_1) \cdot q(b_1, b_2) \cdot \gamma_2^{-1}$,

we let $\mathcal{Q}(e_1, e_2, q, \Theta, \Phi, r_1, r_2)$ *be the higher path in figure 3.3.*

$$\left(\beta_1 \cdot \mathsf{ap}_{e_1}(r_1) \cdot \gamma_1^{-1}\right) \cdot \delta_2$$

$$\Big| \; \mathcal{R}\Big(\beta_1, \mathsf{ap}_{e_1}(r_1), \mathbf{I}^2_{\square}(\Phi)\Big)$$

$$\beta_1 \cdot \left(\mathsf{ap}_{e_1}(r_1) \cdot q(b_1, b_2)\right) \cdot \gamma_2^{-1}$$

$$\Big| \; \textit{via } \mathsf{nat}_{\mathcal{H}}(q, r_1, r_2)^{-1}$$

$$\beta_1 \cdot \left(q(a_1, a_2) \cdot \mathsf{ap}_{e_2}(r_2)\right) \cdot \gamma_2^{-1}$$

$$\Big| \; \mathcal{S}\Big(\mathsf{ap}_{e_2}(r_2), \gamma_2^{-1}, \mathbf{I}^1_{\square}(\Theta)\Big)^{-1}$$

$$\delta_1 \cdot \left(\beta_2 \cdot \mathsf{ap}_{e_2}(r_2) \cdot \gamma_2^{-1}\right)$$

Figure 3.2: The path $\mathcal{P}(e_1, e_2, q, \Theta, \Phi, r_1, r_2)$

$$\mathsf{ap}_F\left(\beta_1 \cdot \mathsf{ap}_{e_1}(r_1) \cdot \gamma_1^{-1}\right) \cdot \delta_2$$

$$\Big|$$

$$\Big(\mathsf{ap}_F(\beta_1) \cdot \mathsf{ap}_{F \circ e_1}(r_1) \cdot (\mathsf{ap}_F(\gamma_1))^{-1}\Big) \cdot \delta_2$$

$$\Big| \; \mathcal{P}(F \circ e_1, e_2, q, \Theta, \Phi, r_1, r_2)$$

$$\delta_1 \cdot \left(\beta_2 \cdot \mathsf{ap}_{e_2}(r_2) \cdot \gamma_2^{-1}\right)$$

Figure 3.3: The path $\mathcal{Q}(e_1, e_2, q, \Theta, \Phi, r_1, r_2)$

$$\left(\alpha_1 \cdot \alpha_2 \cdot \alpha_3\right) \cdot \alpha_4$$

$$\Big|$$

$$\left(\alpha_1 \cdot \alpha_2\right) \cdot \left(\alpha_3 \cdot \alpha_4\right)$$

$$\Big| \; \textit{via } \Psi$$

$$\left(\alpha_1 \cdot \alpha_2\right) \cdot \left(\alpha_5 \cdot \alpha_6\right)$$

$$\Big|$$

$$\alpha_1 \cdot \left(\alpha_2 \cdot \alpha_5\right) \cdot \alpha_6$$

Figure 3.4: The path $\mathcal{R}(\alpha_1, \alpha_2, \Psi)$

$$\alpha_1 \cdot \left(\alpha_2 \cdot \alpha_3 \cdot \alpha_4\right)$$

$$\Big|$$

$$\left(\alpha_1 \cdot \alpha_2\right) \cdot \left(\alpha_3 \cdot \alpha_4\right)$$

$$\Big| \; \textit{via } \Psi$$

$$\left(\alpha_4 \cdot \alpha_5\right) \cdot \left(\alpha_3 \cdot \alpha_4\right)$$

$$\Big|$$

$$\alpha_4 \cdot \left(\alpha_5 \cdot \alpha_3\right) \cdot \alpha_4$$

Figure 3.5: The path $\mathcal{S}(\alpha_3, \alpha_4, \Psi)$

Definition 95. $(\mathcal{H})$ *Given*

- *algebras* $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $(E, e, q) : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$*,*
- *morphisms* $(f, \beta, \theta), (g, \gamma, \phi) : \mathsf{WQMor}\ (D, d, p)\ (E, e, q)$*,*
- *a* W*-cell* $(\alpha, \eta) : \mathsf{WCell}\ (f, \beta)\ (g, \gamma)$*,*

we let $\mathcal{M}\Big((D, d, p), (E, e, q), (f, \beta, \theta), (g, \gamma, \phi), (\alpha, \eta)\Big)$ *be the higher path in figure 3.6.*

Definition 96. $(\mathcal{H})$ *Given*

- *algebras* $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $(E, e, q) : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ (D, d, p)$*,*
- *fibered morphisms* $(f, \beta, \theta), (g, \gamma, \phi) : \mathsf{WQFibMor}\ (D, d, p)\ (E, e, q)$*,*
- *a fibered* W*-cell* $(\alpha, \eta) : \mathsf{WFibCell}\ (f, \beta)\ (g, \gamma)$*,*

we let $\mathcal{N}\Big((D, d, p), (E, e, q), (f, \beta, \theta), (g, \gamma, \phi), (\alpha, \eta)\Big)$ *be the higher path in figure 3.7.*

The higher paths $\mathcal{M}$ and $\mathcal{N}$ in figures 3.6 and 3.7 are the aforementioned "obvious" values of $\mathsf{nat}(\alpha, p(c, t, s))$ and $\mathsf{nat}_{\mathcal{F}}(\alpha, p(c, t, s))$, obtained by combining $\beta, \theta, \gamma, \phi$ and η in a suitable fashion.

Definition 97. $(\mathcal{H})$ *For algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$*,* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ *and morphisms* $\mu, \nu : \mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y}$*, define a type family over the type* $\mathsf{WCell}\ \pi_1(\mu)\ \pi_1(\nu)$ *by*

$$\begin{aligned}\mathsf{WCellFam}\ (D, d, p)\ (E, e, q)\ (f, \beta, \theta)\ (g, \gamma, \phi)\ (\alpha, \eta) &:\equiv \Pi_{c:C}\Pi_{t:B(\mathtt{l}\ c)\to D}\Pi_{s:B(\mathtt{r}\ c)\to D} \\ \mathsf{nat}(\alpha, p(c, t, s)) = \mathcal{M}\Big((D, d, p), (E, e, q), (f, \beta, \theta), (g, \gamma, \phi), (\alpha, \eta)\Big)\end{aligned}$$

Definition 98. $(\mathcal{H})$ *For algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$*,* $\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$*, and fibered morphisms* $\mu, \nu : \mathsf{WQFibMor}\ \mathcal{X}\ \mathcal{Y}$*, define a type family over* $\mathsf{WFibCell}\ \pi_1(\mu)\ \pi_1(\nu)$ *by*

$$\begin{aligned}\mathsf{WFibCellFam}\ (D, d, p)\ (E, e, q)\ (f, \beta, \theta)\ (g, \gamma, \phi)\ (\alpha, \eta) &:\equiv \Pi_{c:C}\Pi_{t:B(\mathtt{l}\ c)\to D}\Pi_{s:B(\mathtt{r}\ c)\to D} \\ \mathsf{nat}_{\mathcal{F}}(\alpha, p(c, t, s)) = \mathcal{N}\Big((D, d, p), (E, e, q), (f, \beta, \theta), (g, \gamma, \phi), (\alpha, \eta)\Big)\end{aligned}$$

We will usually leave out all but the last three arguments to $\mathsf{WCellFam}$ and $\mathsf{WFibCellFam}$.

Definition 99. $(\mathcal{H})$ *Given algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$*,* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ *and morphisms* $\mu, \nu : \mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y}$*, define the type of* W-quotient cells *between* μ *and* ν *by*

$$\mathsf{WQCell}\ \mu\ \nu :\equiv \Big(\Sigma\mathcal{C}_0 : \mathsf{WCell}\ \pi_1(\mu)\ \pi_1(\nu)\Big)\ \mathsf{WCellFam}\ \mu\ \nu\ \mathcal{C}_0$$

Definition 100. $(\mathcal{H})$ *Given algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$*,* $\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *and fibered morphisms* $\mu, \nu : \mathsf{WQFibMor}\ \mathcal{X}\ \mathcal{Y}$*, define the type of* fibered W-quotient cells *between* μ *and* ν *by*

$$\mathsf{WQFibCell}\ \mu\ \nu :\equiv \Big(\Sigma\mathcal{C}_0 : \mathsf{WFibCell}\ \pi_1(\mu)\ \pi_1(\nu)\Big)\ \mathsf{WFibCellFam}\ \mu\ \nu\ \mathcal{C}_0$$

$$\alpha(d(\mathtt{l}\ c,t)) \cdot \mathsf{ap}_g(p(c,t,s))$$

$$\Big| \ \textit{via}\ \eta(\mathtt{l}\ c,t)$$

$$\Big(\beta(\mathtt{l}\ c,t) \cdot \mathsf{ap}_{e(\mathtt{l}\ c)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha\circ t)\big) \cdot \gamma(\mathtt{l}\ c,t)^{-1}\Big) \cdot \mathsf{ap}_g(p(c,t,s))$$

$$\Big| \ \mathcal{P}\Big(e(\mathtt{l}\ c), e(\mathtt{r}\ c), q(c), \theta(c,t,s), \phi(c,t,s), {}^{\Pi}\mathsf{E}^{=}(\alpha\circ t), {}^{\Pi}\mathsf{E}^{=}(\alpha\circ s)\Big)$$

$$\mathsf{ap}_f(p(c,t,s)) \cdot \Big(\beta(\mathtt{r}\ c,s) \cdot \mathsf{ap}_{e(\mathtt{r}\ c)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha\circ s)\big) \cdot \gamma(\mathtt{r}\ c,s)^{-1}\Big)$$

$$\Big| \ \textit{via}\ \eta(\mathtt{r}\ c,t)^{-1}$$

$$\mathsf{ap}_f(p(c,t,s)) \cdot \alpha(d(\mathtt{r}\ c,s))$$

Figure 3.6: The path $\mathcal{M}\Big((D,d,p),(E,e,q),(f,\beta,\theta),(g,\gamma,\phi),(\alpha,\eta)\Big)$

$$\mathsf{ap}_{p(c,t,s)^E_*}\big(\alpha(d(\mathtt{l}\ c,t))\big) \cdot \mathsf{dap}_g(p(c,t,s))$$

$$\Big| \ \textit{via}\ \eta(\mathtt{l}\ c,t)$$

$$\mathsf{ap}_{p(c,t,s)^E_*}\Big(\beta(\mathtt{l}\ c,t) \cdot \mathsf{ap}_{e(\mathtt{l}\ c,t)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha\circ t)\big) \cdot \gamma(\mathtt{l}\ c,t)^{-1}\Big) \cdot \mathsf{dap}_g(p(c,t,s))$$

$$\Big| \ \mathcal{Q}\Big(e(\mathtt{l}\ c,t), e(\mathtt{r}\ c,s), q(c,t,s), \theta(c,t,s), \phi(c,t,s), {}^{\Pi}\mathsf{E}^{=}(\alpha\circ t), {}^{\Pi}\mathsf{E}^{=}(\alpha\circ s)\Big)$$

$$\mathsf{dap}_f(p(c,t,s)) \cdot \Big(\beta(\mathtt{r}\ c,s) \cdot \mathsf{ap}_{e(\mathtt{r}\ c,s)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha\circ s)\big) \cdot \gamma(\mathtt{r}\ c,s)^{-1}\Big)$$

$$\Big| \ \textit{via}\ \eta(\mathtt{r}\ c,t)^{-1}$$

$$\mathsf{dap}_f(p(c,t,s)) \cdot \alpha(d(\mathtt{r}\ c,s))$$

Figure 3.7: The path $\mathcal{N}\Big((D,d,p),(E,e,q),(f,\beta,\theta),(g,\gamma,\phi),(\alpha,\eta)\Big)$

We have the following analogue of lemma 91:

Lemma 101. $(\mathcal{H})$ *For algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$*,* $\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *and fibered morphisms* $\mu, \nu : \mathsf{WQFibMor}\ \mathcal{X}\ \mathcal{Y}$*, we have*

$$\mu = \nu \quad \simeq \quad \mathsf{WQFibCell}\ \mu\ \nu$$

Proof. Fix algebras $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ and $(E, e, q) : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ (D, d, p)$, and fibered homomorphisms $(\mu_0, \theta), (\nu_0, \phi) : \mathsf{WQFibMor}\ (D, d, p)\ (E, e, q)$. We establish the following chain of equivalences:

$$\begin{aligned}
& (\mu_0, \theta) = (\nu_0, \phi) \\
\simeq\ & \left(\Sigma \mathcal{C}_0 : \mu_0 = \nu_0\right)\ (\mathcal{C}_0)_*^{\mathsf{WFibMorFam}\ (D,d,p)\ (E,e,q)}(\theta) = \phi \\
\simeq\ & \left(\Sigma \mathcal{C}_0 : \mu_0 = \nu_0\right)\ \mathsf{WFibCellFam}\ (\mu_0, \theta)\ (\nu_0, \phi)\ \big(\mathsf{WFibMorPathToCell}(\mathcal{C}_0)\big) \\
\simeq\ & \left(\Sigma \mathcal{C}_0 : \mathsf{WFibCell}\ \mu_0\ \nu_0\right)\ \mathsf{WFibCellFam}\ (\mu_0, \theta)\ (\nu_0, \phi)\ \mathcal{C}_0 \\
\equiv\ & \mathsf{WQFibCell}\ (\mu_0, \theta)\ (\nu_0, \phi)
\end{aligned}$$

The first equivalence follows by the characterization of paths in dependent product spaces. The third equivalence follows as $\mathsf{WFibMorPathToCell} : (\mu_0 = \nu_0) \to (\mathsf{WFibCell}\ \mu_0\ \nu_0)$ is itself an equivalence. Finally, to prove the second equivalence it suffices to show that for any $\mathcal{C}_0 : \mu_0 = \nu_0$, the respective fibers over $\mathcal{C}_0$ are equivalent. To do so, we generalize ϕ and perform a one-sided path induction on $\mathcal{C}_0$, keeping the left endpoint μ_0 fixed. This leaves us to prove that for any $\phi : \mathsf{WFibMorFam}\ (D, d, p)\ (E, e, q)\ \mu_0$ we have

$$\theta = \phi \quad \simeq \quad \mathsf{WFibCellFam}\ (\mu_0, \theta)\ (\mu_0, \phi)\ \big(\mathsf{WFibMorPathToCell}(1_{\mu_0})\big)$$

Writing μ_0 as a pair, we can reformulate our current goal as showing that for any $(f, \beta) : \mathsf{WFibMor}\ (D, d)\ (E, e)$ and any $\theta, \phi : \mathsf{WFibMorFam}\ (D, d, p)\ (E, e, q)\ (f, \beta)$ we have

$$\theta = \phi \quad \simeq \quad \mathsf{WFibCellFam}\ (f, \beta, \theta)\ (f, \beta, \phi)\ \big(\mathsf{WFibMorPathToCell}(1_{(f,\beta)})\big)$$

Examining the definition of the map $\mathsf{WFibMorPathToCell}$ given in the proof of lemma 91, we see that $\mathsf{WFibMorPathToCell}(1_{(f,\beta)})$ is equal to a pair $(x \mapsto 1_{f(x)}, \eta)$, where $\eta(a, t)$ is the path

$$\begin{array}{c}
1_{f(d(a,t))} \\
\Big| \ \mathbf{I}_{\square}^{-1}(\mathcal{I}_{1,1}(1_{\beta(a,t)})) \\
\beta(a,t) \cdot \mathsf{ap}_{e(a,t)}\left(1_{f \circ t}\right) \cdot \beta(a,t)^{-1} \\
\Big| \\
\beta(a,t) \cdot \mathsf{ap}_{e(a,t)}\left({}^{\Pi}\mathsf{E}^{=}\left({}^{=}\mathsf{E}^{\Pi}(1_{f \circ t})\right)\right) \cdot \beta(a,t)^{-1}
\end{array}$$

and for any $u, v : a =_X b$, the map $\mathcal{I}_{1,1} : (u = v) \to (1_a \cdot u = v \cdot 1_b)$ is an equivalence defined by mapping $\theta : u = v$ to the path below:

$$1_a \cdot u \;\frac{\quad\quad}{}\; u \;\frac{\theta}{\quad\quad}\; v \;\frac{\quad\quad}{}\; v \cdot 1_a$$

Plugging in the above pair, we note that by function extensionality it suffices to show that for any c, t, s, the type $\theta(c, t, s) = \phi(c, t, s)$ is equivalent to

$$\mathsf{nat}_{\mathcal{F}}(x \mapsto 1_{f(x)}, p(c, t, s)) = \mathcal{N}\Big((D, d, p), (E, e, q), (f, \beta, \theta), (f, \beta, \phi), (x \mapsto 1_{f(x)}, \eta)\Big)$$

Fix c, t, s. We now slightly generalize our goal, which also helps to keep the notation in check. The desired equivalence will follow if we can show that given terms

- $x, y : D$,
- $e_1 : \left(\Pi_{b:B(\mathtt{l}\ c)} E(t\ b)\right) \to E(x)$ and $e_2 : \left(\Pi_{b:B(\mathtt{r}\ c)} E(s\ b)\right) \to E(y)$,
- $\beta_1 : f(x) = e_1(f \circ t)$ and $\beta_2 : f(y) = e_2(f \circ s)$,
- $\mathsf{p} : x = y$ and $\mathsf{q} : \mathsf{p}_*^E \circ e_1 \sim_{\mathcal{H}} e_2$,
- $\Theta, \Phi : \mathsf{dap}_f(\mathsf{p}) = \mathsf{ap}_{\mathsf{p}_*^E}(\beta_1) \cdot \mathsf{q}(f \circ t, f \circ s) \cdot \beta_2^{-2}$,
- $r_1 : f \circ t = f \circ t$ and $r_2 : f \circ s = f \circ s$,
- $r_1^* : 1_{f \circ t} = r_1$ and $r_2^* : 1_{f \circ s} = r_2$,

the type $\Theta = \Phi$ is equivalent to the commutativity of the diagram below

$$\begin{array}{ccc}
\mathsf{ap}_{\mathsf{p}_*^E}(1_{f(x)}) \cdot \mathsf{dap}_f(\mathsf{p}) & \xrightarrow{\ \mathsf{nat}_{\mathcal{F}}(x \mapsto 1_{f(x)}, \mathsf{p})\ } & \mathsf{dap}_f(\mathsf{p}) \cdot 1_{f(x)} \\
\Big| \ \textit{via } \eta_1 & & \Big| \ \textit{via } \eta_2 \\
\mathsf{ap}_{\mathsf{p}_*^E}\left(\beta_1 \cdot \mathsf{ap}_{e_1}(r_1) \cdot \beta_1^{-1}\right) \cdot \mathsf{dap}_f(\mathsf{p}) & \xrightarrow[\ \mathcal{Q}(e_1, e_2, \mathsf{q}, \Theta, \Phi, r_1, r_2)\ ]{} & \mathsf{dap}_f(\mathsf{p}) \cdot \left(\beta_2 \cdot \mathsf{ap}_{e_2}(r_2) \cdot \beta_2^{-1}\right)
\end{array}$$

where η_1, η_2 are the following two paths:

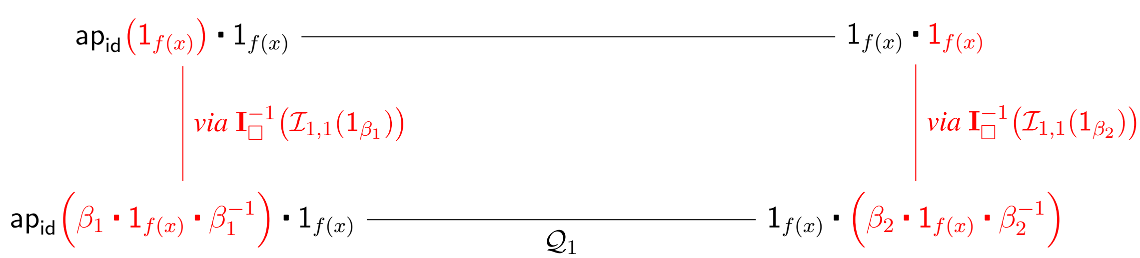
$$\begin{array}{ccc}
1_{f(x)} & \qquad & 1_{f(x)} \\
\Big| \ \mathbf{I}_\square^{-1}(\mathcal{I}_{1,1}(1_{\beta_1})) & & \Big| \ \mathbf{I}_\square^{-1}(\mathcal{I}_{1,1}(1_{\beta_2})) \\
\beta_1 \cdot \mathsf{ap}_{e_1}(1_{f \circ t}) \cdot \beta_1^{-1} & & \beta_2 \cdot \mathsf{ap}_{e_2}(1_{f \circ s}) \cdot \beta_2^{-1} \\
\Big| \ \textit{via } r_1^* & & \Big| \ \textit{via } r_2^* \\
\beta_1 \cdot \mathsf{ap}_{e_1}(r_1) \cdot \beta_1^{-1} & & \beta_2 \cdot \mathsf{ap}_{e_2}(r_2) \cdot \beta_2^{-1}
\end{array}$$

To prove that this generalized statement implies our original goal, we just let $x :\equiv d(\mathtt{l}\ c, t)$, $y :\equiv d(\mathtt{r}\ c, s)$, $e_1 :\equiv e(\mathtt{l}\ c, t)$, $e_2 :\equiv e(\mathtt{r}\ c, s)$, $\beta_1 :\equiv \beta(\mathtt{l}\ c, t)$, $\beta_2 :\equiv \beta(\mathtt{r}\ c, t)$, $\mathtt{p} :\equiv p(c, t, s)$, $\mathtt{q} :\equiv q(c, t, s)$, $\Theta :\equiv \theta(c, t, s)$, $\Phi :\equiv \phi(c, t, s)$, $r_1 :\equiv {}^{\Pi}\mathsf{E}^{=}\big({}^{=}\mathsf{E}^{\Pi}(1_{f \circ t})\big)$, $r_2 :\equiv {}^{\Pi}\mathsf{E}^{=}\big({}^{=}\mathsf{E}^{\Pi}(1_{f \circ s})\big)$, and let r_1^*, r_2^* be the obvious paths. The paths η_1, η_2 then become $\eta(\mathtt{l}\ c, t)$ and $\eta(\mathtt{r}\ c, s)$, which finishes the proof of the implication.

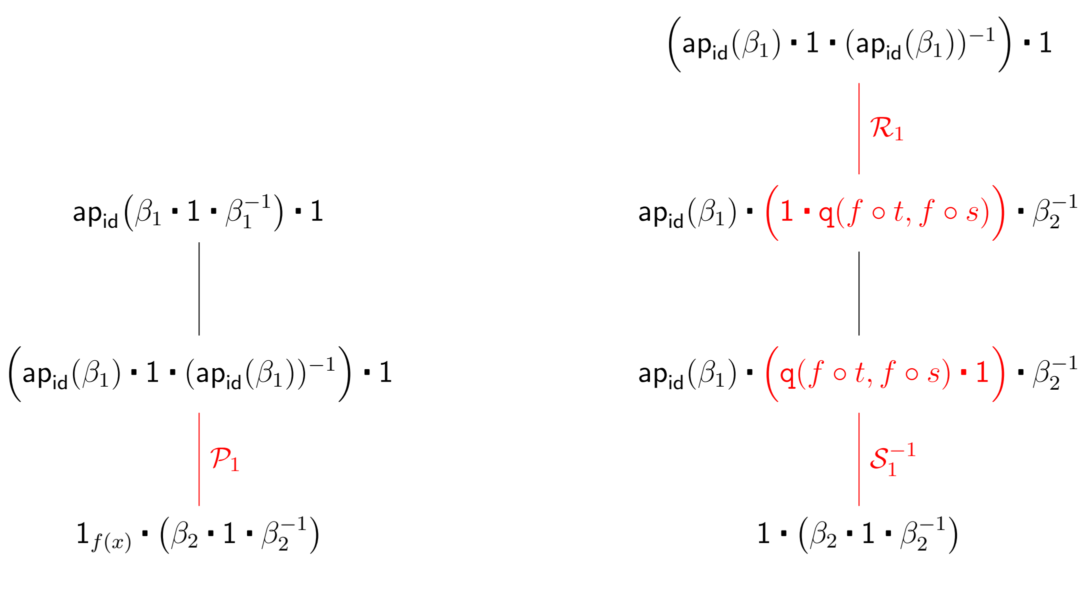
Working towards our generalized goal, we first note that we can now perform the usual path induction on $\mathtt{p}$ and one-sided path induction on r_1^*, r_2^* (with the right endpoint fixed). It thus suffices to show that given terms

- $x : D$,
- $e_1 : \big(\Pi_{b:B(\mathtt{l}\ c)} E(t\ b)\big) \to E(x)$ and $e_2 : \big(\Pi_{b:B(\mathtt{r}\ c)} E(s\ b)\big) \to E(x)$,
- $\beta_1 : f(x) = e_1(f \circ t)$ and $\beta_2 : f(x) = e_2(f \circ s)$,
- $\mathtt{q} : e_1 \sim_{\mathcal{H}} e_2$,
- $\Theta, \Phi : 1_{f(x)} = \mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot \mathtt{q}(f \circ t, f \circ s) \cdot \beta_2^{-2}$,

the type $\Theta = \Phi$ is equivalent to the commutativity of the following diagram:



where the path $\mathcal{Q}_1$ and its components look as follows:



a$_1$) The path $\mathcal{Q}_1$ **b**$_1$) The path $\mathcal{P}_1$

$$
\begin{array}{c}
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1 \cdot (\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1}\right) \cdot 1 \\
\Big| \\
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1\right) \cdot \left((\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1} \cdot 1\right) \\
\Big| \; \textit{via } \mathbf{I}^2_{\square}(\Phi) \\
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1\right) \cdot \left(\mathsf{q}(f \circ t, f \circ s) \cdot \beta_2^{-1}\right) \\
\Big| \\
\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot \left(1 \cdot \mathsf{q}(f \circ t, f \circ s)\right) \cdot \beta_2^{-1}
\end{array}
$$

c$_1$**)** The path $\mathcal{R}_1$

$$
\begin{array}{c}
1 \cdot \left(\beta_2 \cdot 1 \cdot \beta_2^{-1}\right) \\
\Big| \\
\left(1 \cdot \beta_2\right) \cdot \left(1 \cdot \beta_2^{-1}\right) \\
\Big| \; \textit{via } \mathbf{I}^1_{\square}(\Theta) \\
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot \mathsf{q}(f \circ t, f \circ s)\right) \cdot \left(1 \cdot \beta_2^{-1}\right) \\
\Big| \\
\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot \left(\mathsf{q}(f \circ t, f \circ s) \cdot 1\right) \cdot \beta_2^{-1}
\end{array}
$$

d$_1$**)** The path $\mathcal{S}_1$

We note that the path $\mathsf{nat}_{\mathcal{F}}(x \mapsto 1_{f(x)}, \mathsf{p})$ and the two paths involving r_1^* and r_2^* have reduced to reflexivities. Furthermore, in the path $\mathcal{P}_1$ we no longer make use of the naturality of the heterogeneous homotopy q, since the term $\mathsf{nat}_{\mathcal{H}}(\mathsf{q}, 1_{f \circ t}, 1_{f \circ s})$ reduces to the obvious path from $\mathsf{q}(f \circ t, f \circ s) \cdot 1_{e_2(f \circ s)}$ to $1_{e_1(f \circ t)} \cdot \mathsf{q}(f \circ t, f \circ s)$. The only way we do make use of the homotopy q is by applying it to the two arguments $f \circ t$, $f \circ s$. A similar observation applies to the functions e_1, e_2: the only way we make use of them is by referring to the values $e_1(f \circ t)$ and $e_2(f \circ s)$. This suggests the following generalization of our current goal: given terms

- $x : D$,
- $\mathsf{e}_1, \mathsf{e}_2 : E(x)$ and $\mathbf{q} : \mathsf{e}_1 = \mathsf{e}_2$,
- $\beta_1 : f(x) = \mathsf{e}_1$ and $\beta_2 : f(x) = \mathsf{e}_2$,
- $\Theta, \Phi : 1_{f(x)} = \mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot \mathbf{q} \cdot \beta_2^{-2}$,

the type $\Theta = \Phi$ is equivalent to the commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathsf{ap}_{\mathsf{id}}\left(1_{f(x)}\right) \cdot 1_{f(x)} & \text{———————} & 1_{f(x)} \cdot 1_{f(x)} \\
\Big| \; \textit{via } \mathbf{I}^{-1}_{\square}(\mathcal{I}_{1,1}(1_{\beta_1})) & & \Big| \; \textit{via } \mathbf{I}^{-1}_{\square}(\mathcal{I}_{1,1}(1_{\beta_2})) \\
\mathsf{ap}_{\mathsf{id}}\left(\beta_1 \cdot 1_{f(x)} \cdot \beta_1^{-1}\right) \cdot 1_{f(x)} & \underset{\mathcal{Q}_2}{\text{———————}} & 1_{f(x)} \cdot \left(\beta_2 \cdot 1_{f(x)} \cdot \beta_2^{-1}\right)
\end{array}
$$

where the path $\mathcal{Q}_2$ and its components now look as follows:

$$\mathsf{ap}_{\mathsf{id}}\left(\beta_1 \cdot 1 \cdot \beta_1^{-1}\right) \cdot 1$$

$$\Big|$$

$$\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1 \cdot (\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1}\right) \cdot 1$$

$$\Big| \; \mathcal{P}_2$$

$$1 \cdot \left(\beta_2 \cdot 1 \cdot \beta_2^{-1}\right)$$

a$_2$) The path $\mathcal{Q}_2$

$$\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1 \cdot (\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1}\right) \cdot 1$$

$$\Big| \; \mathcal{R}_2$$

$$\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot (1 \cdot \mathbf{q}) \cdot \beta_2^{-1}$$

$$\Big|$$

$$\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot (\mathbf{q} \cdot 1) \cdot \beta_2^{-1}$$

$$\Big| \; \mathcal{S}_2^{-1}$$

$$1 \cdot \left(\beta_2 \cdot 1 \cdot \beta_2^{-1}\right)$$

b$_2$) The path $\mathcal{P}_2$

$$\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1 \cdot (\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1}\right) \cdot 1$$

$$\Big|$$

$$\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1\right) \cdot \left((\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1} \cdot 1\right)$$

$$\Big| \; \textit{via } \mathbf{I}^2_{\square}(\Phi)$$

$$\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1\right) \cdot \left(\mathbf{q} \cdot \beta_2^{-1}\right)$$

$$\Big|$$

$$\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot (1 \cdot \mathbf{q}) \cdot \beta_2^{-1}$$

c$_2$) The path $\mathcal{R}_2$

$$1 \cdot \left(\beta_2 \cdot 1 \cdot \beta_2^{-1}\right)$$

$$\Big|$$

$$(1 \cdot \beta_2) \cdot (1 \cdot \beta_2^{-1})$$

$$\Big| \; \textit{via } \mathbf{I}^1_{\square}(\Theta)$$

$$\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot \mathbf{q}\right) \cdot (1 \cdot \beta_2^{-1})$$

$$\Big|$$

$$\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot (\mathbf{q} \cdot 1) \cdot \beta_2^{-1}$$

d$_2$) The path $\mathcal{S}_2$

We can now perform path induction on $\mathbf{q}$ and our goal becomes to show that given terms

- $x : D$ and $\mathsf{e}_1 : E(x)$,
- $\beta_1, \beta_2 : f(x) = \mathsf{e}_1$,
- $\Theta, \Phi : 1_{f(x)} = \mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1_{\mathsf{e}_1} \cdot \beta_2^{-2}$,

the type $\Theta = \Phi$ is equivalent to the commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathsf{ap}_{\mathsf{id}}\left(1_{f(x)}\right) \cdot 1_{f(x)} & \text{———} & 1_{f(x)} \cdot 1_{f(x)} \\
\Big| \ \textit{via } \mathbf{I}_{\square}^{-1}(\mathcal{I}_{1,1}(1_{\beta_1})) & & \Big| \ \textit{via } \mathbf{I}_{\square}^{-1}(\mathcal{I}_{1,1}(1_{\beta_2})) \\
\mathsf{ap}_{\mathsf{id}}\left(\beta_1 \cdot 1_{f(x)} \cdot \beta_1^{-1}\right) \cdot 1_{f(x)} & \underset{\mathcal{Q}_3}{\text{———}} & 1_{f(x)} \cdot \left(\beta_2 \cdot 1_{f(x)} \cdot \beta_2^{-1}\right)
\end{array}
$$

where the path $\mathcal{Q}_3$ and its components now look as follows:

$$
\begin{array}{c}
\mathsf{ap}_{\mathsf{id}}\left(\beta_1 \cdot 1 \cdot \beta_1^{-1}\right) \cdot 1 \\
\Big| \\
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1 \cdot (\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1}\right) \cdot 1 \\
\Big| \ \mathcal{P}_3 \\
1 \cdot \left(\beta_2 \cdot 1 \cdot \beta_2^{-1}\right)
\end{array}
$$

a$_3$) The path $\mathcal{Q}_3$

$$
\begin{array}{c}
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1 \cdot (\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1}\right) \cdot 1 \\
\Big| \ \mathcal{R}_3 \\
\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot (1 \cdot 1) \cdot \beta_2^{-1} \\
\Big| \ \mathcal{S}_3^{-1} \\
1 \cdot \left(\beta_2 \cdot 1 \cdot \beta_2^{-1}\right)
\end{array}
$$

b$_3$) The path $\mathcal{P}_3$

$$
\begin{array}{c}
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1 \cdot (\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1}\right) \cdot 1 \\
\Big| \\
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1\right) \cdot \left((\mathsf{ap}_{\mathsf{id}}(\beta_1))^{-1} \cdot 1\right) \\
\Big| \ \textit{via } \mathbf{I}_{\square}^{2}(\Phi) \\
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1\right) \cdot \left(1 \cdot \beta_2^{-1}\right) \\
\Big| \\
\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot (1 \cdot 1) \cdot \beta_2^{-1}
\end{array}
$$

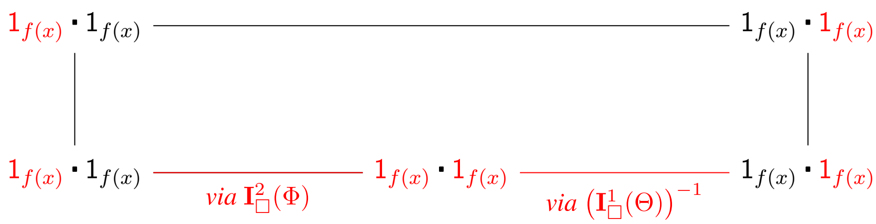
c$_3$) The path $\mathcal{R}_3$

$$
\begin{array}{c}
1_{f(x)} \cdot \left(\beta_2 \cdot 1 \cdot \beta_2^{-1}\right) \\
\Big| \\
\left(1_{f(x)} \cdot \beta_2\right) \cdot \left(1 \cdot \beta_2^{-1}\right) \\
\Big| \ \textit{via } \mathbf{I}_{\square}^{1}(\Theta) \\
\left(\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot 1\right) \cdot \left(1_{e_1} \cdot \beta_2^{-1}\right) \\
\Big| \\
\mathsf{ap}_{\mathsf{id}}(\beta_1) \cdot \left(1 \cdot 1\right) \cdot \beta_2^{-1}
\end{array}
$$

d$_3$) The path $\mathcal{S}_3$

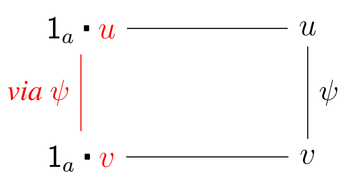
We now note that either one of the two assumptions Θ, Φ implies $\beta_1 = \beta_2$. Thus, it is enough to prove our goal under the additional assumption $\psi : \beta_1 = \beta_2$. But now we can perform ordinary path induction on ψ, which replaces β_2 with β_1, and a subsequent one-sided path induction on $\beta_1 : f(x) = \mathtt{e}_1$ (with the left endpoint fixed).

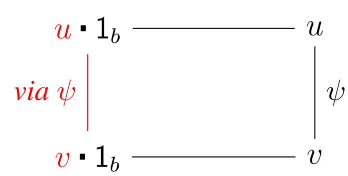
It now suffices to show the following: given terms $x : D$ and $\Theta, \Phi : 1_{f(x)} = 1_{f(x)}$, the type $\Theta = \Phi$ is equivalent to the commutativity of the following diagram:



In particular, we note that both of the vertical paths have reduced to reflexivities. The commutativity of the above diagram is of course equivalent to the commutativity of the diagram below:

Now we note that for any $u, v : a =_X b$ and $\psi : u = v$, the following two diagrams commute:





In the case when $\psi :\equiv \mathbf{I}^1_\square \Theta)$ and $\psi :\equiv \mathbf{I}^2_\square(\Phi)$, all of the horizontal paths in the above two diagrams become reflexivities; hence we can reformulate our goal as showing that given terms $x : D$ and $\Theta, \Phi : 1_{f(x)} = 1_{f(x)}$, we have $\big(\Theta = \Phi\big) \simeq \big(\mathbf{I}^1_\square(\Theta) = \mathbf{I}^2_\square(\Phi)\big)$. However, when examining the definitions of $\mathbf{I}^1_\square$ and $\mathbf{I}^2_\square$, we clearly see that $\mathbf{I}^1_\square(\Theta) = \Theta$ and $\mathbf{I}^2_\square(\Phi) = \Phi$, which means we are done. □

Lemma 102. *($\mathcal{H}$) For algebras $\mathcal{X}$: $\mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$, $\mathcal{Y}$: $\mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ and morphisms μ, ν : $\mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y}$, we have*

$$\mu = \nu \quad \simeq \quad \mathsf{WQCell}\ \mu\ \nu$$

Proof. By an entirely analogous argument as in the proof of 101. □

We are now ready to establish the analogues of lemmas 33 and 35.

Lemma 103. $(\mathcal{H})$ *For any algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *we have*

$$\mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasWQIndUniq}_{\mathcal{U}_k}(\mathcal{X})$$

Proof. Fix an algebra $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ and assume that $\mathsf{hasWQInd}_{\mathcal{U}_k}(D, d, p)$ holds. In order to show that $\mathsf{hasWQIndUniq}_{\mathcal{U}_k}(D, d, p)$ holds, take any fibered algebra $(E, e, q) : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ (D, d, p)$ and fibered morphisms $(f, \beta, \theta), (g, \gamma, \phi) : \mathsf{WQFibMor}\ (D, d, p)\ (E, e, q)$. By lemma 101, to show $(f, \beta, \theta) = (g, \gamma, \phi)$ it suffices to exhibit a fibered W-quotient cell between (f, β, θ) and (g, γ, ϕ).

To do so, we use the induction principle with an appropriate fibered algebra $(E', e', q') : \mathsf{WQFibAlg}\ (D, d, p)$. Defining the first component is easy: we put $E' :\equiv x \mapsto f(x) = g(x)$, which clearly still belongs to $\mathcal{U}_k$ fiberwise. For the second component, we put

$$e'(a, t, u) :\equiv \beta(a, t) \cdot \mathsf{ap}_{e(a,t)}({}^{\Pi}\mathsf{E}^{=}(u)) \cdot \gamma(a, t)^{-1}$$

Finally, we let $q'(c, t, s, u, v)$ be the path

$$p(c, t, s)_*^{x \mapsto f(x) = g(x)}\big(e'(\mathtt{l}\ c, t, u)\big)$$

$$\Big|\ \mathcal{T}_{\mathcal{F}}\Big(f, g, p(c, t, s), e'(\mathtt{l}\ c, t, u)\Big)$$

$$\mathsf{dap}_f(p(c, t, s))^{-1} \cdot \Big(\mathsf{ap}_{p(c,t,s)_*^E}(e'(\mathtt{l}\ c, t, u)) \cdot \mathsf{dap}_g(p(c, t, s))\Big)$$

$$\Big|\ \textit{via}\ \mathcal{Q}\Big(e(\mathtt{l}\ c, t), e(\mathtt{r}\ c, s), q(c, t, s), \theta(c, t, s), \phi(c, t, s), {}^{\Pi}\mathsf{E}^{=}(u), {}^{\Pi}\mathsf{E}^{=}(v)\Big)$$

$$\mathsf{dap}_f(p(c, t, s))^{-1} \cdot \Big(\mathsf{dap}_f(p(c, t, s)) \cdot e'(\mathtt{r}\ c, s, v)\Big)$$

$$\Big|$$

$$e'(\mathtt{r}\ c, s, v)$$

where for any $h, i : \Pi_{x:X} Y(x)$ and $w : a =_X b$, $z : h(a) = i(a)$, the path

$$\mathcal{T}_{\mathcal{F}}\big(h, i, w, z\big) : w_*^{x \mapsto h(x) = i(x)}(z) = \mathsf{dap}_h(w)^{-1} \cdot \Big(\mathsf{ap}_{w_*^E}(z) \cdot \mathsf{dap}_i(w)\Big)$$

is defined by path induction on w in an obvious way. The induction principle then gives us a fibered morphism $(\alpha, \eta, \psi) : \mathsf{WFibMor}\ (D, d, p)\ (E', e', q')$, where $\alpha : f \sim g$ and

$$\eta(a, t) : \alpha(d(a, t)) = e'(a, t, \alpha \circ t)$$
$$\psi(c, t, s) : \mathsf{dap}_\alpha(p(c, t, s)) = \mathsf{ap}_{p(c,t,s)_*^{E'}}\big(\eta(\mathtt{l}\ c, t)\big) \cdot q'(c, t, s, \alpha \circ t, \alpha \circ s) \cdot \eta(\mathtt{r}\ c, s)^{-1}$$

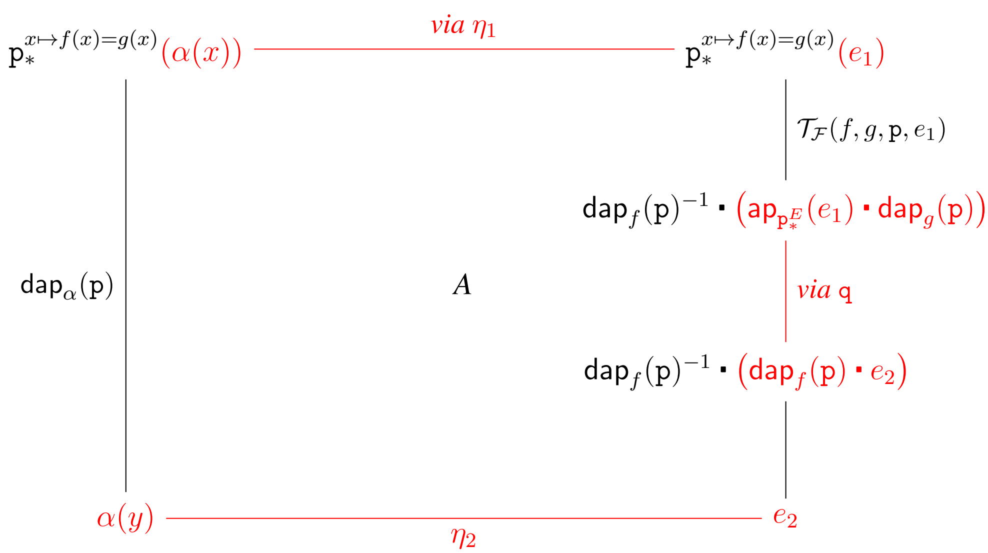
The pair (α, η) : $\mathsf{WFibCell}\ (f, \beta)\ (g, \gamma)$ forms the first component of our desired W-quotient cell between (f, β, θ) and (g, γ, ϕ). It remains to show that for any c, t, s, we have

$$\mathsf{nat}_{\mathcal{F}}(\alpha, p(c,t,s)) = \mathcal{N}\Big((D,d,p), (E,e,q), (f,\beta,\theta), (g,\gamma,\phi), (\alpha,\eta)\Big)$$

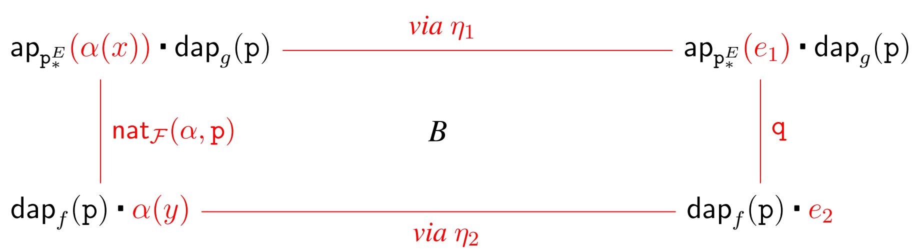
Fix c, t, s. The desired equality will follow if we can show that given terms

- $x, y : D$,
- $e_1 : E(x)$ and $e_2 : E(y)$,
- $\eta_1 : \alpha(x) = e_1$ and $\eta_2 : \alpha(y) = e_2$,
- $\mathtt{p} : x = y$ and $\mathtt{q} : \mathsf{ap}_{\mathtt{p}_*^E}(e_1) \cdot \mathsf{dap}_g(\mathtt{p}) = \mathsf{dap}_f(\mathtt{p}) \cdot e_2$,

the commutativity of the diagram



implies the commutativity of the following diagram:



To prove that this generalized statement implies our original goal, we just let $x :\equiv d(\mathtt{l}\ c, t)$, $y :\equiv d(\mathtt{r}\ c, s)$, $e_1 :\equiv e'(\mathtt{l}\ c, t, \alpha \circ t)$, $e_2 :\equiv e'(\mathtt{r}\ c, s, \alpha \circ s)$, $\eta_1 :\equiv \eta(\mathtt{l}\ c, t)$, $\eta_2 :\equiv \eta(\mathtt{r}\ c, s)$, $\mathtt{p} :\equiv p(c,t,s)$, $\mathtt{q} :\equiv \mathcal{Q}\Big(e(\mathtt{l}\ c, t), e(\mathtt{r}\ c, s), q(c,t,s), \theta(c,t,s), \phi(c,t,s), {}^{\Pi}\mathsf{E}^{=}(\alpha \circ t), {}^{\Pi}\mathsf{E}^{=}(\alpha \circ s)\Big)$. Finally, $\psi(c,t,s)$ implies the commutativity of *A*, which finishes the proof of the implication.

Working towards our new goal, we note that by a straightforward path induction we can express $\mathsf{dap}_{\alpha}(\mathtt{p})$ equivalently as the path

$$\begin{array}{c}
\mathtt{p}_*^{x\mapsto f(x)=g(x)}(\alpha(x)) \\
\Big| \; \mathcal{T}_{\mathcal{F}}(f,g,\mathtt{p},\alpha(x)) \\
\mathsf{dap}_f(\mathtt{p})^{-1} \cdot \big(\mathsf{ap}_{\mathtt{p}_*^E}(\alpha(x)) \cdot \mathsf{dap}_g(\mathtt{p})\big) \\
\Big| \; \textit{via } \mathsf{nat}_{\mathcal{F}}(\alpha,\mathtt{p}) \\
\mathsf{dap}_f(\mathtt{p})^{-1} \cdot \big(\mathsf{dap}_f(\mathtt{p}) \cdot \alpha(y)\big) \\
\Big| \\
\alpha(y)
\end{array}$$

Thus the commutativity of A is equivalent to the commutativity of the outer rectangle in the diagram below:

$$\begin{array}{ccc}
\mathtt{p}_*^{x\mapsto f(x)=g(x)}(\alpha(x)) & \xrightarrow{\;\textit{via } \eta_1\;} & \mathtt{p}_*^{x\mapsto f(x)=g(x)}(e_1) \\
\Big| \; \mathcal{T}_{\mathcal{F}}(f,g,\mathtt{p},\alpha(x)) & C & \Big| \; \mathcal{T}_{\mathcal{F}}(f,g,\mathtt{p},e_1) \\
\mathsf{dap}_f(\mathtt{p})^{-1} \cdot \big(\mathsf{ap}_{\mathtt{p}_*^E}(\alpha(x)) \cdot \mathsf{dap}_g(\mathtt{p})\big) & \xrightarrow[\textit{via } \eta_1]{} & \mathsf{dap}_f(\mathtt{p})^{-1} \cdot \big(\mathsf{ap}_{\mathtt{p}_*^E}(e_1) \cdot \mathsf{dap}_g(\mathtt{p})\big) \\
\Big| \; \textit{via } \mathsf{nat}_{\mathcal{F}}(\alpha,\mathtt{p}) & D & \Big| \; \textit{via } \mathsf{q} \\
\mathsf{dap}_f(\mathtt{p})^{-1} \cdot \big(\mathsf{dap}_f(\mathtt{p}) \cdot \alpha(y)\big) & \xrightarrow[\textit{via } \eta_2]{} & \mathsf{dap}_f(\mathtt{p})^{-1} \cdot \big(\mathsf{dap}_f(\mathtt{p}) \cdot e_2\big) \\
\Big| & E & \Big| \\
\alpha(y) & \xrightarrow[\eta_2]{} & e_2
\end{array}$$

Rectangles C and E clearly commute by an easy generalization and subsequent path induction on η_1 and η_2. Hence the commutativity of the outer rectangle is equivalent to the commutativity of D. But the commutativity of D is equivalent to the commutativity of B and we are done. □

Corollary 104. $(\mathcal{H})$ *For any algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A,B,C,\mathtt{l},\mathtt{r})$ *we have*

$$\mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X}) \;\;\to\;\; \mathsf{isWQHInit}_{\mathcal{U}_k}(\mathcal{X})$$

Lemma 105. $(\mathcal{H})$ *For any algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *we have*

$$\mathsf{hasWQRec}_{\mathcal{U}_k}(\mathcal{X}) \times \mathsf{hasWQRecUniq}_{\mathcal{U}_k}(\mathcal{X}) \;\; \to \;\; \mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$.

Proof. Fix an algebra $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ and assume that $\mathsf{hasWQRec}_{\mathcal{U}_k}(D, d, p)$ and $\mathsf{hasWQRecUniq}_{\mathcal{U}_k}(D, d, p)$ hold. To show that $\mathsf{hasWQInd}_{\mathcal{U}_k}(D, d, p)$ holds, fix any fibered algebra $(E, e, q) : \mathsf{WQFibAlg}_{\mathcal{U}_k}\ (D, d, p)$. In order to apply the recursion principle, we need to turn this into a non-fibered algebra (E', e', q'). The first component is easy: the only reasonable choice we have is to put $E' :\equiv \Sigma_{x:D} E(x)$; we note that since $D : \mathcal{U}_j$, $E : D \to \mathcal{U}_k$, and $j \leq k$, E' belongs to $\mathcal{U}_k$ as needed. For the second component, we put

$$e'(a, u) :\equiv \Big(d\big(a, \pi_1 \circ u\big), e\big(a, \pi_1 \circ u, \pi_2 \circ u\big)\Big)$$

Finally, we let $q'(c, u, v)$ be the path

$$\begin{array}{l} \Big(d\big(\mathtt{l}\ c, \pi_1 \circ u\big), e\big(\mathtt{l}\ c, \pi_1 \circ u, \pi_2 \circ u\big)\Big) \\ \qquad \Big|\ {}^{\Sigma}\mathsf{E}^{=}\Big(p\big(c, \pi_1 \circ u, \pi_1 \circ v\big), q\big(c, \pi_1 \circ u, \pi_1 \circ v, \pi_2 \circ u, \pi_2 \circ v\big)\Big) \\ \Big(d\big(\mathtt{r}\ c, \pi_1 \circ v\big), e\big(\mathtt{r}\ c, \pi_1 \circ v, \pi_2 \circ v\big)\Big) \end{array}$$

The recursion principle then gives us a morphism $(f, \beta, \varphi) : \mathsf{WQMor}\ (D, d, p)\ (E', e', q')$, where $f : D \to \Sigma_{x:D} E(x)$ and

$$\begin{aligned} \beta(a, t) &: f(d(a, t)) = e'(a, f \circ t) \\ \varphi(c, t, s) &: \mathsf{ap}_f(p(c, t, s)) = \beta(\mathtt{l}\ c, t) \cdot q'(c, f \circ t, f \circ s) \cdot \beta(\mathtt{r}\ c, s)^{-1} \end{aligned}$$

We now want to show that the function $\pi_1 \circ f : D \to D$ is in fact the identity on D (up to a homotopy, of course). We can do this by endowing both of the functions $\pi_1 \circ f$ and id_D with a morphism structure on the algebra (D, d, p); by the recursion uniqueness principle it will follow that these morphisms are equal, and in particular they are equal as maps.

We start with the easier case: we turn the identity map id_D into a morphism $\big(\mathsf{id}_D, \delta, \phi\big)$: $\mathsf{WQMor}\ (D, d, p)\ (D, d, p)$ by defining $\delta(a, t) :\equiv 1_{d(a,t)}$ and $\phi(c, t, s) :\equiv \mathbf{I}_{\square}^{-2}\big(\Phi(p(c, t, s))\big)$, where for any $r : x =_X y$, $\Phi(r) : 1_x \cdot \mathsf{ap}_{\mathsf{id}}(r) = r \cdot 1_y$ is the obvious path.

We turn the composition $\pi_1 \circ f$ into a morphism $\big(\pi_1 \circ f, \gamma, \theta\big) : \mathsf{WQMor}\ (D, d, p)\ (D, d, p)$ as follows. We note that $\beta(a, t) : f(d(a, t)) = \big(d(a, \pi_1 \circ f \circ t), \dots\big)$. We need a path between the respective first components of the endpoints; to simplify the notation, we let $\pi_1^{=}$ and $\pi_2^{=}$ denote the compositions $\pi_1 \circ {}^{=}\mathsf{E}^{\Sigma}$ and $\pi_2 \circ {}^{=}\mathsf{E}^{\Sigma}$, and define $\gamma(a, t) :\equiv \pi_1^{=}(\beta(a, t))$.

Finally, we define $\theta(c,t,s) :\equiv \mathbf{I}_{\square}^{-1}(\Theta(c,t,s))$, where $\Theta(c,t,s)$ is the path

$$\begin{array}{c}
\mathsf{ap}_{\pi_1 \circ f}(p(c,t,s)) \centerdot \pi_1^{=}(\beta(\mathtt{r}\ c,s)) \\
\Big| \ \mathcal{U}(c,t,s) \\
\pi_1^{=}\Big(\mathsf{ap}_f(p(c,t,s)) \centerdot \beta(\mathtt{r}\ c,s)\Big) \\
\Big| \ \textit{via } \mathbf{I}_{\square}^{1}(\varphi(c,t,s)) \\
\pi_1^{=}\Big(\beta(\mathtt{l}\ c,t) \centerdot q'(c, f \circ t, f \circ s)\Big) \\
\Big| \ \mathcal{V}(c,t,s)^{-1} \\
\pi_1^{=}(\beta(\mathtt{l}\ c,t)) \centerdot p(c, \pi_1 \circ f \circ t, \pi_1 \circ f \circ s)
\end{array}$$

and $\mathcal{U}(c,t,s)$, $\mathcal{V}(c,t,s)$ are the two paths below:

$$\begin{array}{c}
\mathsf{ap}_{\pi_1 \circ f}(p(c,t,s)) \centerdot \pi_1^{=}(\beta(\mathtt{r}\ c,s)) \\
\Big| \\
\pi_1^{=}(\mathsf{ap}_f(p(c,t,s))) \centerdot \pi_1^{=}(\beta(\mathtt{r}\ c,s)) \\
\Big| \\
\pi_1^{=}\Big(\mathsf{ap}_f(p(c,t,s)) \centerdot \beta(\mathtt{r}\ c,s)\Big)
\end{array}
\qquad
\begin{array}{c}
\pi_1^{=}(\beta(\mathtt{l}\ c,t)) \centerdot p(c, \pi_1 \circ f \circ t, \pi_1 \circ f \circ s) \\
\Big| \\
\pi_1^{=}(\beta(\mathtt{l}\ c,t)) \centerdot \pi_1^{=}\big(q'(c, f \circ t, f \circ s)\big) \\
\Big| \\
\pi_1^{=}\Big(\beta(\mathtt{l}\ c,t) \centerdot q'(c, f \circ t, f \circ s)\Big)
\end{array}$$

By the recursion uniqueness rule, the morphisms $(\pi_1 \circ f, \gamma, \theta)$ and $(\mathsf{id}_D, \delta, \phi)$ are equal. By lemma 102 there exists a W-quotient cell (α, η, ψ) between them, where $\alpha : \pi_1 \circ f \sim \mathsf{id}_D$ and

$$\begin{aligned}
\eta(a,t) &: \alpha(d(a,t)) = \pi_1^{=}(\beta(a,t)) \centerdot \mathsf{ap}_{d(a)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha \circ t)\big) \centerdot 1_{d(a,t)} \\
\psi(c,t,s) &: \mathsf{nat}(\alpha, p(c,t,s)) = \mathcal{M}\Big((D,d,p),(D,d,p),(\pi_1 \circ f, \gamma, \theta),(\mathsf{id}_D, \delta, \phi),(\alpha,\eta)\Big)
\end{aligned}$$

Our desired fibered homomorphism $(f_D, \beta_D, \theta_D) : \mathsf{WQFibMor}\ (D,d,p)\ (E,e,q)$ will be constructed as follows. Given a homotopy $H : h_1 \sim h_2$ between two maps $h_1, h_2 : X \to Y$ and a function $g : \Pi_{x:X} Z(h_1(x))$, let $H \circ_{\mathcal{H}} g$ denote the function $x \mapsto H(x)_*^Z\ g(x)$, which has type $\Pi_{x:X} Z(h_2(x))$. We can now define $f_D :\equiv \alpha \circ_{\mathcal{H}} (\pi_2 \circ f)$, which has the required type $\Pi_{x:D} E(x)$, since we have $\alpha : \pi_1 \circ f \sim \mathsf{id}_D$ and $\pi_2 \circ f : \Pi_{x:D} E\big(\pi_1(f(x))\big)$.

For the second component, we need a path $\beta_D(a,t) : f_D(d(a,t)) = e(a,t,f_D \circ t)$. To this end we introduce the following auxiliary definitions:

- For any $a : A$, $t : B(a) \to D$, let

$$\kappa(a,t) : \left({}^{\Pi}\mathsf{E}^{=}\left({}^{=}\mathsf{E}^{\Pi}(\alpha \circ t)\right)\right) \circ_{\mathcal{H}} (\pi_2 \circ f \circ t) = (\alpha \circ t) \circ_{\mathcal{H}} (\pi_2 \circ f \circ t)$$

  be the obvious equality.

- For any $a : A$, $t_1, t_2 : B(a) \to D$, $r : t_1 = t_2$, $u_1 : \Pi_{b:B(a)} E(t_1\ b)$, let

$$\iota(a,r,u_1) : \left(\mathsf{ap}_{d(a)}(r)\right)_*^E\ e(a,t_1,u_1) = e\Big(a, t_2, \left({}^{=}\mathsf{E}^{\Pi}(r)\right) \circ_{\mathcal{H}} u_1\Big)$$

  be the obvious equality defined by path induction on r.

- For any $a : A$, $t_1, t_2 : B(a) \to D$, $r : t_1 = t_2$, $u_1 : \Pi_{b:B(a)} E(t_1\ b)$, let

$$\varepsilon(a,r,u_1) : \Big(d(a,t_1), e(a,t_1,u_1)\Big) = \Big(d(a,t_2), e\Big(a, t_2, \left({}^{=}\mathsf{E}^{\Pi}(r)\right) \circ_{\mathcal{H}} u_1\Big)\Big)$$

  be the path defined by $\varepsilon(a,r,u_1) :\equiv {}^{\Sigma}\mathsf{E}^{=}\big(\mathsf{ap}_{d(a)}(r), \iota(a,r,u_1)\big)$.

- For any $a : A$, $t : B(a) \to D$, $u_1, u_2 : \Pi_{b:B(a)} E(t\ b)$, $r : u_1 = u_2$, let

$$\upsilon(a,t,r) : \Big(d(a,t), e(a,t,u_1)\Big) = \Big(d(a,t), e(a,t,u_2)\Big)$$

  be the path defined by $\upsilon(a,t,r) :\equiv {}^{\Sigma}\mathsf{E}^{=}\big(1_{d(a,t)}, \mathsf{ap}_{e(a,t)}(r)\big)$.

At this point we can define $\beta_D(a,t)$ as the path

$$\alpha(d(a,t))_*^E\ \pi_2\big(f(d(a,t))\big)$$

$$\Big|\ \textit{via } \mathcal{B}(a,t)$$

$$\Big(\pi_1^{=}\Big(\beta(a,t) \bullet \varepsilon\big(a, {}^{\Pi}\mathsf{E}^{=}(\alpha \circ t), \pi_2 \circ f \circ t\big) \bullet \upsilon\big(a,t,\kappa(a,t)\big)\Big)\Big)_*^E\ \pi_2\big(f(d(a,t))\big)$$

$$\Big|\ \pi_2^{=}\Big(\beta(a,t) \bullet \varepsilon\big(a, {}^{\Pi}\mathsf{E}^{=}(\alpha \circ t), \pi_2 \circ f \circ t\big) \bullet \upsilon\big(a,t,\kappa(a,t)\big)\Big)$$

$$e\Big(a, t, (\alpha \circ t) \circ_{\mathcal{H}} (\pi_2 \circ f \circ t)\Big)$$

where $\mathcal{B}(a,t)$ is the path

$$
\begin{array}{c}
\alpha(d(a,t)) \\
\Big| \; \eta(a,t) \\
\pi_1^{=}(\beta(a,t)) \cdot \mathsf{ap}_{d(a)}\left({}^{\Pi}\mathsf{E}^{=}(\alpha \circ t)\right) \cdot 1_{d(a,t)} \\
\Big| \\
\pi_1^{=}(\beta(a,t)) \cdot \pi_1^{=}\Big(\varepsilon\big(a, {}^{\Pi}\mathsf{E}^{=}(\alpha \circ t), \pi_2 \circ f \circ t\big)\Big) \cdot \pi_1^{=}\Big(\upsilon\big(a, t, \kappa(a,t)\big)\Big) \\
\Big| \\
\pi_1^{=}\Big(\beta(a,t) \cdot \varepsilon\big(a, {}^{\Pi}\mathsf{E}^{=}(\alpha \circ t), \pi_2 \circ f \circ t\big) \cdot \upsilon\big(a, t, \kappa(a,t)\big)\Big)
\end{array}
$$

For the last component, we need a path

$$\theta_D(c,t,s) : \mathsf{dap}_{f_D}(p(c,t,s)) = \mathsf{ap}_{p(c,t,s)_*^E}\big(\beta_D(\mathtt{l}\ c, t)\big) \cdot q(c,t,s,f_D \circ t, f_D \circ s) \cdot \beta_D(\mathtt{r}\ c, s)^{-1}$$

To do so, we generalize our goal as follows: let $i \in \{1,2\}$. Then given terms

- $x_i : A$,
- $s_i : B(x_i) \to D$ and $u_i : \Pi_{b:B(x_i)} E(s_i\ b)$,
- $t_i : B(x_i) \to D$ and $v_i : \Pi_{b:B(x_i)} E(t_i\ b)$,
- $r_i : s_i = t_i$ and $\omega_i : \big({}^{=}\mathsf{E}^{\Pi}(r_i)\big) \circ_{\mathcal{H}} u_i = v_i$,
- $\beta_i : f(d(x_i,t_i)) = \big(d(x_i,s_i), e(x_i,s_i,u_i)\big)$,
- $\eta_i : \alpha(d(x_i,t_i)) = \pi_1^{=}(\beta_i) \cdot \mathsf{ap}_{d(x_i)}(r_i) \cdot 1_{d(x_i,t_i)}$,
- $\mathtt{p} : d(x_1) \sim_{\mathcal{H}} d(x_2)$,
- $\mathtt{q} : \Pi_{t:B(x_1)\to D}\Pi_{s:B(x_2)\to D}\Big(\big(\mathtt{p}(t,s)_*^E \circ e(x_1,t)\big) \sim_{\mathcal{H}} e(x_2,s)\Big)$,
- $\varphi_* : \mathsf{ap}_f(\mathtt{p}(t_1,t_2)) \cdot \beta_2 = \beta_1 \cdot {}^{\Sigma}\mathsf{E}^{=}\Big(\mathtt{p}(s_1,s_2), \mathtt{q}(s_1,s_2,u_1,u_2)\Big)$,

the commutativity of the diagram

$$
\begin{array}{ccc}
\alpha(d(x_1,t_1)) \cdot \mathsf{ap}_{\mathsf{id}}(\mathtt{p}(t_1,t_2)) & \xrightarrow{\quad via\ \eta_1 \quad} & \Big(\pi_1^{=}(\beta_1) \cdot \mathsf{ap}_{d(x_1)}(r_1) \cdot 1\Big) \cdot \mathsf{ap}_{\mathsf{id}}(\mathtt{p}(t_1,t_2)) \\
\Big| \ \mathsf{nat}(\alpha, \mathtt{p}(t_1,t_2)) & & \Big| \ \mathcal{P}_1 \\
\mathsf{ap}_{\pi_1 \circ f}(\mathtt{p}(t_1,t_2)) \cdot \alpha(d(x_2,t_2)) & \xrightarrow[\quad via\ \eta_2 \quad]{} & \mathsf{ap}_{\pi_1 \circ f}(\mathtt{p}(t_1,t_2)) \cdot \Big(\pi_1^{=}(\beta_2) \cdot \mathsf{ap}_{d(x_2)}(r_2) \cdot 1\Big)
\end{array}
$$

implies the commutativity of the following diagram:

$$
\begin{array}{ccc}
\mathsf{p}(t_1,t_2)_*^E\big(f_D(d(x_1,t_1))\big) & \overset{\mathsf{dap}_{f_D}(\mathsf{p}(t_1,t_2))}{\rule{8em}{0.4pt}} & f_D(d(x_2,t_2)) \\
\Big| \; \textit{via } \mathcal{D}_1^1 & & \Big| \; \mathcal{D}_1^2 \\
\mathsf{p}(t_1,t_2)_*^E\big(e(x_1,t_1,v_1)\big) & \underset{\mathsf{q}(t_1,t_2,u_1,u_2)}{\rule{8em}{0.4pt}} & e(x_2,t_2,v_2)
\end{array}
$$

The paths $\mathcal{D}_1^i$, $\mathcal{P}_1$ and their components are defined below.

$$
\begin{array}{c}
\alpha(d(x_i,t_i))_*^E \; \pi_2\big(f(d(x_i,t_i))\big) \\
\Big| \; \textit{via } \mathcal{B}_1^i \\
\Big(\pi_1^{=}\big(\beta_i \cdot \varepsilon(x_i,r_i,u_i) \cdot \upsilon(x_i,t_i,\omega_i)\big)\Big)_*^E \; \pi_2\big(f(d(x_i,t_i))\big) \\
\Big| \; \pi_2^{=}\big(\beta_i \cdot \varepsilon(x_i,r_i,u_i) \cdot \upsilon(x_i,t_i,\omega_i)\big) \\
e(x_i,t_i,v_i)
\end{array}
$$

a$_1$) The path $\mathcal{D}_1^i$

$$
\begin{array}{c}
\alpha(d(x_i,t_i)) \\
\Big| \; \eta_i \\
\pi_1^{=}(\beta_i) \cdot \mathsf{ap}_{d(x_i)}(r_i) \cdot \mathsf{1} \\
\Big| \\
\pi_1^{=}(\beta_i) \cdot \pi_1^{=}(\varepsilon(x_i,r_i,u_i)) \cdot \pi_1^{=}(\upsilon(x_i,t_i,\omega_i)) \\
\Big| \\
\pi_1^{=}\Big(\beta_i \cdot \varepsilon(x_i,r_i,u_i) \cdot \upsilon(x_i,t_i,\omega_i)\Big)
\end{array}
$$

b$_1$) The path $\mathcal{B}_1^i$

$$\left(\pi_1^{=}(\beta_1) \cdot \mathsf{ap}_{d(x_1)}(r_1) \cdot 1\right) \cdot \mathsf{ap}_{\mathsf{id}}(\mathsf{p}(t_1,t_2))$$

$$\Big|\ \mathcal{R}_1$$

$$\pi_1^{=}(\beta_1) \cdot \left(\mathsf{ap}_{d(x_1)}(r_1) \cdot \mathsf{p}(t_1,t_2)\right) \cdot 1$$

$$\Big|\ \textit{via}\ \mathsf{nat}_{\mathcal{H}}(\mathsf{p}, r_1, r_2)^{-1}$$

$$\pi_1^{=}(\beta_1) \cdot \left(\mathsf{p}(s_1,s_2) \cdot \mathsf{ap}_{d(x_2)}(r_2)\right) \cdot 1$$

$$\Big|\ \mathcal{S}_1^{-1}$$

$$\mathsf{ap}_{\pi_1 \circ f}(\mathsf{p}(t_1,t_2)) \cdot \left(\pi_1^{=}(\beta_2) \cdot \mathsf{ap}_{d(x_2)}(r_2) \cdot 1\right)$$

c$_1$**)** The path $\mathcal{P}_1$

$$\left(\pi_1^{=}(\beta_1) \cdot \mathsf{ap}_{d(x_1)}(r_1) \cdot 1\right) \cdot \mathsf{ap}_{\mathsf{id}}(\mathsf{p}(t_1,t_2))$$

$$\Big|$$

$$\pi_1^{=}(\beta_1) \cdot \mathsf{ap}_{d(x_1)}(r_1) \cdot \left(1 \cdot \mathsf{ap}_{\mathsf{id}}(\mathsf{p}(t_1,t_2))\right)$$

$$\Big|\ \textit{via}\ \mathbf{I}_{\square}^{2}\left(\mathbf{I}_{\square}^{-2}\left(\Phi(\mathsf{p}(t_1,t_2))\right)\right)$$

$$\pi_1^{=}(\beta_1) \cdot \mathsf{ap}_{d(x_1)}(r_1) \cdot \left(\mathsf{p}(t_1,t_2) \cdot 1\right)$$

$$\Big|$$

$$\pi_1^{=}(\beta_1) \cdot \left(\mathsf{ap}_{d(x_1)}(r_1) \cdot \mathsf{p}(t_1,t_2)\right) \cdot 1$$

d$_1$**)** The path $\mathcal{R}_1$

$$\mathsf{ap}_{\pi_1 \circ f}(\mathsf{p}(t_1,t_2)) \cdot \left(\pi_1^{=}(\beta_2) \cdot \mathsf{ap}_{d(x_2)}(r_2) \cdot 1\right)$$

$$\Big|$$

$$\left(\mathsf{ap}_{\pi_1 \circ f}(\mathsf{p}(t_1,t_2)) \cdot \pi_1^{=}(\beta_2)\right) \cdot \mathsf{ap}_{d(x_2)}(r_2) \cdot 1$$

$$\Big|\ \textit{via}\ \mathbf{I}_{\square}^{1}(\mathbf{I}_{\square}^{-1}(\Theta_1))$$

$$\left(\pi_1^{=}(\beta_1) \cdot \mathsf{p}(s_1,s_2)\right) \cdot \mathsf{ap}_{d(x_2)}(r_2) \cdot 1$$

$$\Big|$$

$$\pi_1^{=}(\beta_1) \cdot \left(\mathsf{p}(s_1,s_2) \cdot \mathsf{ap}_{d(x_2)}(r_2)\right) \cdot 1$$

e$_1$**)** The path $\mathcal{S}_1$

$$\mathsf{ap}_{\pi_1 \circ f}(\mathsf{p}(t_1,t_2)) \cdot \pi_1^{=}(\beta_2)$$

$$\Big|\ \mathcal{U}_1$$

$$\pi_1^{=}\left(\mathsf{ap}_f(\mathsf{p}(t_1,t_2)) \cdot \beta_2\right)$$

$$\Big|\ \textit{via}\ \varphi_*$$

$$\pi_1^{=}\left(\beta_1 \cdot {}^{\Sigma}\mathsf{E}^{=}\left(\mathsf{p}(s_1,s_2), \mathsf{q}(s_1,s_2,u_1,u_2)\right)\right)$$

$$\Big|\ \mathcal{V}_1^{-1}$$

$$\pi_1^{=}(\beta_1) \cdot \mathsf{p}(s_1,s_2)$$

f$_1$**)** The path Θ_1

We note that in the paths $\mathcal{R}_1$ and $\mathcal{S}_1$ we choose to use the equalities $\mathbf{I}_{\square}^{2}\left(\mathbf{I}_{\square}^{-2}\left(\Phi(\mathsf{p}(t_1,t_2))\right)\right)$ and $\mathbf{I}_{\square}^{1}(\mathbf{I}_{\square}^{-1}(\Theta_1))$ instead of the more economical $\Phi(\mathsf{p}(t_1,t_2))$ and Θ_1. This is due to the fact that in our chosen form, $\mathcal{R}_1$ and $\mathcal{S}_1$ are direct generalizations of the respective paths in our original goal, which would not be the case had we opted for the more concise version.

$$\mathsf{ap}_{\pi_1\circ f}(\mathsf{p}(t_1,t_2)) \bullet \pi_1^{=}(\beta_2)$$

$$\pi_1^{=}\big(\mathsf{ap}_f(\mathsf{p}(t_1,t_2))\big) \bullet \pi_1^{=}(\beta_2)$$

$$\pi_1^{=}\Big(\mathsf{ap}_f(\mathsf{p}(t_1,t_2)) \bullet \beta_1\Big)$$

g$_1$**)** The path $\mathcal{U}_1$

$$\pi_1^{=}(\beta_1) \bullet \mathsf{p}(s_1,s_2)$$

$$\pi_1^{=}(\beta_1) \bullet {}^{\Sigma}\mathsf{E}^{=}\Big(\mathsf{p}(s_1,s_2),\mathsf{q}(s_1,s_2,u_1,u_2)\Big)$$

$$\pi_1^{=}\Big(\beta_1 \bullet {}^{\Sigma}\mathsf{E}^{=}\Big(\mathsf{p}(s_1,s_2),\mathsf{q}(s_1,s_2,u_1,u_2)\Big)\Big)$$

h$_1$**)** The path $\mathcal{V}_1$

To prove that this generalized statement implies our original goal, fix c, t, s and let $x := \mathtt{l}(c)$, $y := \mathtt{r}(c)$, $s_1 := \pi_1 \circ f \circ t$, $s_2 := \pi_1 \circ f \circ s$, $u_1 := \pi_2 \circ f \circ t$, $u_2 := \pi_2 \circ f \circ s$, $t_1 := t$, $t_2 := s$, $v_1 := f_D \circ t$, $v_2 := f_D \circ s$, $r_1 := {}^{\Pi}\mathsf{E}^{=}(\alpha \circ t)$, $r_2 := {}^{\Pi}\mathsf{E}^{=}(\alpha \circ s)$, $\omega_1 := \kappa(\mathtt{l}\ c, t)$, $\omega_2 := \kappa(\mathtt{r}\ c, s)$, $\beta_1 := \beta(\mathtt{l}\ c, t)$, $\beta_2 := \beta(\mathtt{r}\ c, s)$, $\eta_1 := \eta(\mathtt{l}\ c, t)$, $\beta_2 := \eta(\mathtt{r}\ c, s)$, $\mathtt{p} := p(c)$, $\mathtt{q} := q(c)$, $\varphi_* := \mathbf{I}^1_{\square}(\varphi(c,t,s))$. The paths $\mathcal{B}^1_1$, $\mathcal{B}^2_1$, $\mathcal{D}^1_1$, $\mathcal{D}^2_1$, $\mathcal{U}_1$, $\mathcal{V}_1$, Θ_1 then become $\mathcal{B}(\mathtt{l}\ c, t)$, $\mathcal{B}(\mathtt{r}\ c, s)$, $\beta_D(\mathtt{l}\ c, t)$, $\beta_D(\mathtt{r}\ c, s)$, $\mathcal{U}(c,t,s)$, $\mathcal{V}(c,t,s)$, $\Theta(c,t,s)$.

The path $\mathcal{R}_1$ becomes $\mathcal{R}\Big(\gamma(\mathtt{l}\ c, t), \mathsf{ap}_{d(\mathtt{l}\ c)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha\circ t)\big), \mathbf{I}^2_{\square}\big(\phi(c,t,s)\big)\Big)$, where we refer to the definitions of γ and ϕ made when endowing the map $\pi_1 \circ f$ with a morphism structure. Similarly, $\mathcal{S}_1$ becomes $\mathcal{S}\Big(\mathsf{ap}_{d(\mathtt{r}\ c)}\big({}^{\Pi}\mathsf{E}^{=}(\alpha\circ s)\big), \delta(\mathtt{r}\ c, s)^{-1}, \mathbf{I}^1_{\square}\big(\theta(c,t,s)\big)\Big)$; this makes sense since δ was defined as a point-wise reflexivity. Finally, $\mathcal{P}_1$ becomes $\mathcal{P}\Big(d(\mathtt{l}\ c), d(\mathtt{r}\ c), p(c), \theta(c,t,s), \phi(c,t,s)\Big)$ and the higher path $\psi(c,t,s)$ then proves the commutativity of the diagram required in the hypotheses, which finishes the proof of the implication.

Working towards our generalized goal, we note that we can now perform one-sided path induction (with the right endpoint fixed) on r_i and consequently on ω_i. This leads to the following goal: given terms

- $x_i : A$,
- $t_i : B(x_i) \to D$ and $v_i : \Pi_{b:B(x_i)} E(t_i\ b)$,
- $\beta_i : f(d(x_i,t_i)) = \big(d(x_i,t_i), e(x_i,t_i,v_i)\big)$,
- $\eta_i : \alpha(d(x_i,t_i)) = \pi_1^{=}(\beta_i) \bullet 1_{d(x_i,t_i)} \bullet 1_{d(x_i,t_i)}$,
- $\mathtt{p} : d(x_1) \sim_{\mathcal{H}} d(x_2)$,
- $\mathtt{q} : \Pi_{t:B(x_1)\to D}\Pi_{s:B(x_2)\to D}\Big(\big(\mathtt{p}(t,s)^E_* \circ e(x_1,t)\big) \sim_{\mathcal{H}} e(x_2,s)\Big)$,
- $\varphi_* : \mathsf{ap}_f(\mathtt{p}(t_1,t_2)) \bullet \beta_2 = \beta_1 \bullet {}^{\Sigma}\mathsf{E}^{=}\Big(\mathtt{p}(t_1,t_2), \mathtt{q}(t_1,t_2,v_1,v_2)\Big)$,

the commutativity of the diagram

$$\begin{array}{ccc}
\alpha(d(x_1,t_1)) \bullet \mathsf{ap}_{\mathsf{id}}(\mathsf{p}(t_1,t_2)) & \overset{\textit{via } \eta_1}{\rule{3em}{0.4pt}} & \big(\pi_1^{=}(\beta_1) \bullet \mathsf{1} \bullet \mathsf{1}\big) \bullet \mathsf{ap}_{\mathsf{id}}(\mathsf{p}(t_1,t_2)) \\
\mathsf{nat}(\alpha,\mathsf{p}(t_1,t_2)) \Big| & & \Big| \mathcal{P}_2 \\
\mathsf{ap}_{\pi_1 \circ f}(\mathsf{p}(t_1,t_2)) \bullet \alpha(d(x_2,t_2)) & \underset{\textit{via } \eta_2}{\rule{3em}{0.4pt}} & \mathsf{ap}_{\pi_1 \circ f}(\mathsf{p}(t_1,t_2)) \bullet \big(\pi_1^{=}(\beta_2) \bullet \mathsf{1} \bullet \mathsf{1}\big)
\end{array}$$

implies the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathsf{p}(t_1,t_2)_*^E\big(f_D(d(x_1,t_1))\big) & \overset{\mathsf{dap}_{f_D}(\mathsf{p}(t_1,t_2))}{\rule{3em}{0.4pt}} & f_D(d(x_2,t_2)) \\
\textit{via } \mathcal{D}_2^1 \Big| & & \Big| \mathcal{D}_2^2 \\
\mathsf{p}(t_1,t_2)_*^E\big(e(x_1,t_1,v_1)\big) & \underset{\mathsf{q}(t_1,t_2,v_1,v_2)}{\rule{3em}{0.4pt}} & e(x_2,t_2,v_2)
\end{array}$$

The paths $\mathcal{D}_2^i$, $\mathcal{P}_2$ and their components are defined below.

$$\begin{array}{c}
\alpha(d(x_i,t_i))_*^E\ \pi_2\big(f(d(x_i,t_i))\big) \\
\Big| \ \textit{via } \mathcal{B}_2^i \\
\big(\pi_1^{=}(\beta_i \bullet \mathsf{1} \bullet \mathsf{1})\big)_*^E\ \pi_2\big(f(d(x_i,t_i))\big) \\
\Big| \ \pi_2^{=}(\beta_i \bullet \mathsf{1} \bullet \mathsf{1}) \\
e(x_i,t_i,v_i)
\end{array}$$

a$_2$) The path $\mathcal{D}_2^i$

$$\begin{array}{c}
\alpha(d(x_i,t_i)) \\
\Big| \ \eta_i \\
\pi_1^{=}(\beta_i) \bullet \mathsf{1} \bullet \mathsf{1} \\
\Big| \\
\pi_1^{=}(\beta_i \bullet \mathsf{1} \bullet \mathsf{1})
\end{array}$$

b$_2$) The path $\mathcal{B}_2^i$

$$\left(\pi_1^{=}(\beta_1) \cdot 1 \cdot 1\right) \cdot \mathsf{ap}_{\mathsf{id}}(\mathrm{p}(t_1, t_2))$$
$$\Big|\ \mathcal{R}_2$$
$$\pi_1^{=}(\beta_1) \cdot \left(1 \cdot \mathrm{p}(t_1, t_2)\right) \cdot 1$$
$$\Big|$$
$$\pi_1^{=}(\beta_1) \cdot \left(\mathrm{p}(t_1, t_2) \cdot 1\right) \cdot 1$$
$$\Big|\ \mathcal{S}_2^{-1}$$
$$\mathsf{ap}_{\pi_1 \circ f}(\mathrm{p}(t_1, t_2)) \cdot \left(\pi_1^{=}(\beta_2) \cdot 1 \cdot 1\right)$$

c$_2$**)** The path $\mathcal{P}_2$

$$\left(\pi_1^{=}(\beta_1) \cdot 1 \cdot 1\right) \cdot \mathsf{ap}_{\mathsf{id}}(\mathrm{p}(t_1, t_2))$$
$$\Big|$$
$$\pi_1^{=}(\beta_1) \cdot 1 \cdot \left(1 \cdot \mathsf{ap}_{\mathsf{id}}(\mathrm{p}(t_1, t_2))\right)$$
$$\Big|\ \textit{via } \mathbf{I}_{\square}^{2}(\mathbf{I}_{\square}^{-2}(\Phi(\mathrm{p}(t_1, t_2))))$$
$$\pi_1^{=}(\beta_1) \cdot 1 \cdot \left(\mathrm{p}(t_1, t_2) \cdot 1\right)$$
$$\Big|$$
$$\pi_1^{=}(\beta_1) \cdot \left(1 \cdot \mathrm{p}(t_1, t_2)\right) \cdot 1$$

d$_2$**)** The path $\mathcal{R}_2$

$$\mathsf{ap}_{\pi_1 \circ f}(\mathrm{p}(t_1, t_2)) \cdot \left(\pi_1^{=}(\beta_2) \cdot 1 \cdot 1\right)$$
$$\Big|$$
$$\left(\mathsf{ap}_{\pi_1 \circ f}(\mathrm{p}(t_1, t_2)) \cdot \pi_1^{=}(\beta_2)\right) \cdot 1 \cdot 1$$
$$\Big|\ \textit{via } \mathbf{I}_{\square}^{1}(\mathbf{I}_{\square}^{-1}(\Theta_2))$$
$$\left(\pi_1^{=}(\beta_1) \cdot \mathrm{p}(t_1, t_2)\right) \cdot 1 \cdot 1$$
$$\Big|$$
$$\pi_1^{=}(\beta_1) \cdot \left(\mathrm{p}(t_1, t_2) \cdot 1\right) \cdot 1$$

e$_2$**)** The path $\mathcal{S}_2$

$$\mathsf{ap}_{\pi_1 \circ f}(\mathrm{p}(t_1, t_2)) \cdot \pi_1^{=}(\beta_2)$$
$$\Big|\ \mathcal{U}_2$$
$$\pi_1^{=}\left(\mathsf{ap}_f(\mathrm{p}(t_1, t_2)) \cdot \beta_2\right)$$
$$\Big|\ \textit{via } \varphi_*$$
$$\pi_1^{=}\left(\beta_1 \cdot {}^{\Sigma}\mathsf{E}^{=}\left(\mathrm{p}(t_1, t_2), \mathsf{q}(t_1, t_2, v_1, v_2)\right)\right)$$
$$\Big|\ \mathcal{V}_2^{-1}$$
$$\pi_1^{=}(\beta_1) \cdot \mathrm{p}(t_1, t_2)$$

f$_2$**)** The path Θ_2

$$
\begin{array}{c}
\mathsf{ap}_{\pi_1 \circ f}(\mathsf{p}(t_1, t_2)) \cdot \pi_1^{=}(\beta_2) \\
\Big| \\
\pi_1^{=}\big(\mathsf{ap}_f(\mathsf{p}(t_1, t_2))\big) \cdot \pi_1^{=}(\beta_2) \\
\Big| \\
\pi_1^{=}\Big(\mathsf{ap}_f(\mathsf{p}(t_1, t_2)) \cdot \beta_1\Big)
\end{array}
\qquad\qquad
\begin{array}{c}
\pi_1^{=}(\beta_1) \cdot \mathsf{p}(t_1, t_2) \\
\Big| \\
\pi_1^{=}(\beta_1) \cdot {}^{\Sigma}\mathsf{E}^{=}\Big(\mathsf{p}(t_1, t_2), \mathsf{q}(t_1, t_2, v_1, v_2)\Big) \\
\Big| \\
\pi_1^{=}\Big(\beta_1 \cdot {}^{\Sigma}\mathsf{E}^{=}\Big(\mathsf{p}(t_1, t_2), \mathsf{q}(t_1, t_2, v_1, v_2)\Big)\Big)
\end{array}
$$

g$_2$) The path $\mathcal{U}_2$ **h**$_2$) The path $\mathcal{V}_2$

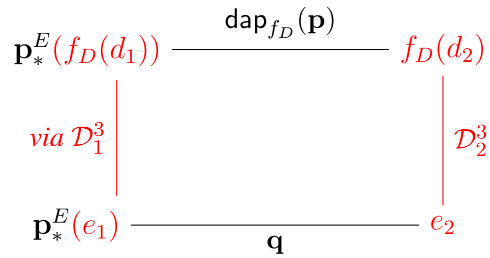
We note that in the paths $\mathcal{B}_2^i$ and $\mathcal{D}_2^i$ we no longer refer to the terms ε and υ, since $\varepsilon(x_i, 1_{t_1}, v_i)$ and $\upsilon(x_i, t_i, 1_{v_i})$ have reduced to the identity on the pair $\big(d(x_i, t_i), e(x_i, t_i, v_i)\big)$. Furthermore, in the path $\mathcal{P}_2$ we no longer make use of the naturality of the heterogeneous homotopy p, as the term $\mathsf{nat}_{\mathcal{H}}(\mathsf{p}, 1_{t_1}, 1_{t_2})$ reduces to the obvious path from $\mathsf{p}(t_1, t_2) \cdot 1_{d(x_2,t_2)}$ to $1_{d(x_1,t_1)} \cdot \mathsf{p}(t_1, t_2)$. The only way we do make use of the homotopy p is by applying it to the two arguments t_1, t_2. Similarly, we only use the homotopy q when applying it to the arguments t_1, t_2, v_1, v_2. An analogous observation applies to the functions d and e: the only way we utilize them is by referring to the values $d(x_1, t_1)$, $d(x_2, t_2)$, $e(x_1, t_1, v_1)$, $e(x_2, t_2, v_2)$. This suggests the following generalization of our current goal: given terms

- $d_i : D$ and $e_i : E(d_i)$,
- $\beta_i : f(d_i) = (d_i, e_i)$,
- $\eta_i : \alpha(d_i) = \pi_1^{=}(\beta_i) \cdot 1_{d_i} \cdot 1_{d_i}$,
- $\mathbf{p} : d_1 = d_2$ and $\mathbf{q} : \mathbf{p}_*^E\, e_1 = e_2$,
- $\varphi_* : \mathsf{ap}_f(\mathbf{p}) \cdot \beta_2 = \beta_1 \cdot {}^{\Sigma}\mathsf{E}^{=}(\mathbf{p}, \mathbf{q})$,

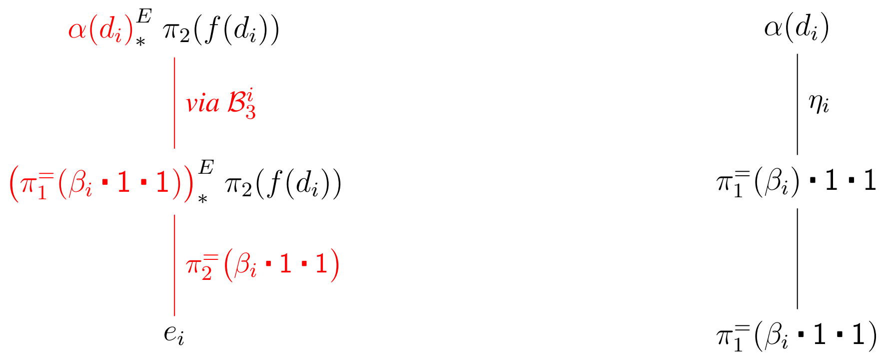
the commutativity of the diagram

$$
\begin{array}{ccc}
\alpha(d_1) \cdot \mathsf{ap}_{\mathsf{id}}(\mathbf{p}) & \xrightarrow{\quad \textit{via } \eta_1 \quad} & \big(\pi_1^{=}(\beta_1) \cdot 1 \cdot 1\big) \cdot \mathsf{ap}_{\mathsf{id}}(\mathbf{p}) \\
\Big| \, \mathsf{nat}(\alpha, \mathbf{p}) & & \Big| \, \mathcal{P}_3 \\
\mathsf{ap}_{\pi_1 \circ f}(\mathbf{p}) \cdot \alpha(d_2) & \xrightarrow[\quad \textit{via } \eta_2 \quad]{} & \mathsf{ap}_{\pi_1 \circ f}(\mathbf{p}) \cdot \big(\pi_1^{=}(\beta_2) \cdot 1 \cdot 1\big)
\end{array}
$$

implies the commutativity of the following diagram:

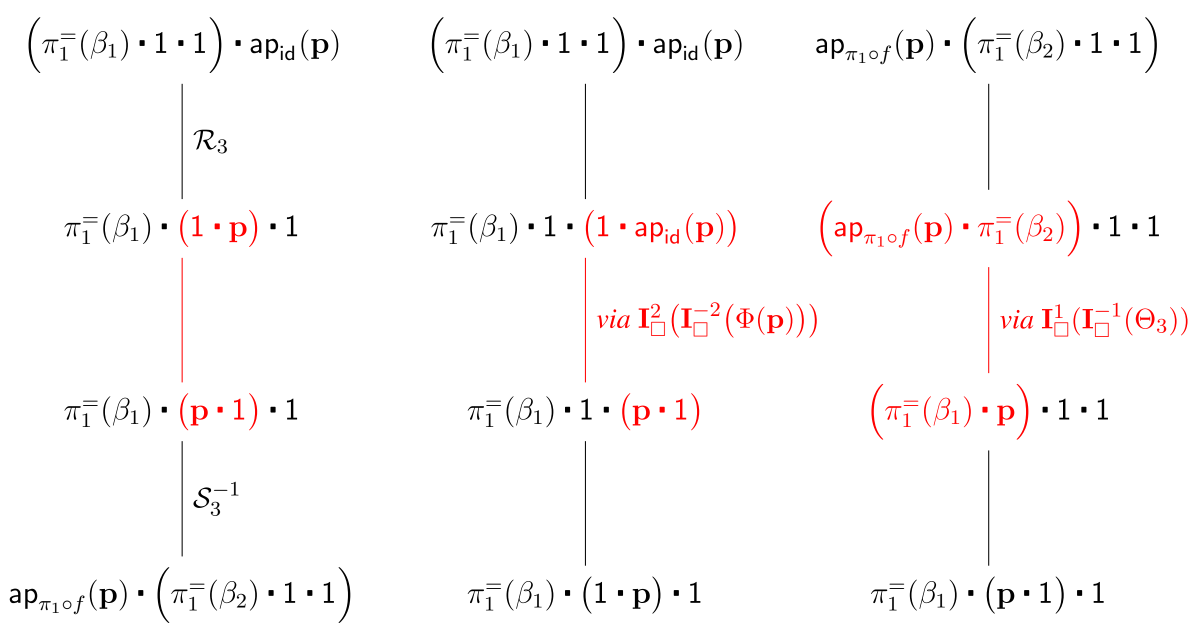


The paths $\mathcal{D}^i_3$, $\mathcal{P}_3$ and their components are as follows:



a$_3$**)** The path $\mathcal{D}^i_3$

b$_3$**)** The path $\mathcal{B}^i_3$



c$_3$**)** The path $\mathcal{P}_3$

d$_3$**)** The path $\mathcal{R}_3$

e$_3$**)** The path $\mathcal{S}_3$

$$
\begin{array}{c}
\mathsf{ap}_{\pi_1 \circ f}(\mathbf{p}) \cdot \pi_1^{=}(\beta_2) \\
\Big| \; \mathcal{U}_3 \\
\pi_1^{=}\Big(\mathsf{ap}_f(\mathbf{p}) \cdot \beta_2\Big) \\
\Big| \; \textit{via } \varphi_* \\
\pi_1^{=}\Big(\beta_1 \cdot {}^{\Sigma}\mathsf{E}^{=}(\mathbf{p}, \mathbf{q})\Big) \\
\Big| \; \mathcal{V}_3^{-1} \\
\pi_1^{=}(\beta_1) \cdot \mathbf{p}
\end{array}
\qquad
\begin{array}{c}
\mathsf{ap}_{\pi_1 \circ f}(\mathbf{p}) \cdot \pi_1^{=}(\beta_2) \\
\Big| \\
\pi_1^{=}\big(\mathsf{ap}_f(\mathbf{p})\big) \cdot \pi_1^{=}(\beta_2) \\
\Big| \\
\pi_1^{=}\Big(\mathsf{ap}_f(\mathbf{p}) \cdot \beta_1\Big)
\end{array}
\qquad
\begin{array}{c}
\pi_1^{=}(\beta_1) \cdot \mathbf{p} \\
\Big| \\
\pi_1^{=}(\beta_1) \cdot {}^{\Sigma}\mathsf{E}^{=}(\mathbf{p}, \mathbf{q}) \\
\Big| \\
\pi_1^{=}\Big(\beta_1 \cdot {}^{\Sigma}\mathsf{E}^{=}(\mathbf{p}, \mathbf{q})\Big)
\end{array}
$$

f$_3$**)** The path Θ_3 **g**$_3$**)** The path $\mathcal{U}_3$ **h**$_3$**)** The path $\mathcal{V}_3$

We can now perform the usual path induction on $\mathbf{p}$ and consequently on $\mathbf{q}$. It thus suffices to show that given terms

- $d_1 : D$ and $e_1 : E(d_1)$,
- $\beta_i : f(d_1) = (d_1, e_1)$,
- $\eta_i : \alpha(d_1) = \pi_1^{=}(\beta_i) \cdot 1_{d_1} \cdot 1_{d_1}$,
- $\varphi_* : 1_{f(d_1)} \cdot \beta_2 = \beta_1 \cdot 1_{(d_1, e_1)}$,

the commutativity of the diagram

$$
\begin{array}{ccc}
\alpha(d_1) \cdot 1 & \xrightarrow{\textit{via } \eta_1} & \Big(\pi_1^{=}(\beta_1) \cdot 1 \cdot 1\Big) \cdot 1 \\
\Big| & & \Big| \; \mathcal{P}_4 \\
1 \cdot \alpha(d_1) & \xrightarrow[\textit{via } \eta_2]{} & 1 \cdot \Big(\pi_1^{=}(\beta_2) \cdot 1 \cdot 1\Big)
\end{array}
$$

implies the commutativity of the following diagram:

$$
\begin{array}{ccc}
f_D(d_1) & \text{———} & f_D(d_2) \\
\mathcal{D}_4^1 \; \Big| & & \Big| \; \mathcal{D}_4^2 \\
e_1 & \text{———} & e_1
\end{array}
$$

We note that the two horizontal paths in the above diagram have reduced to reflexivities; furthermore, we no longer refer to the naturality of α since the term $\mathsf{nat}(\alpha, 1_{d_1})$ reduces to the obvious path from $\alpha(d_1) \cdot 1_{d_1}$ to $1_{d_1} \cdot \alpha(d_1)$. The paths $\mathcal{D}_4^i$, $\mathcal{P}_4$ and their components are as follows:

$$\begin{array}{c} \alpha(d_1)_*^E\ \pi_2(f(d_1)) \\ \Big| \ \textit{via } \mathcal{B}_4^i \\ \left(\pi_1^=(\beta_i \cdot 1 \cdot 1)\right)_*^E\ \pi_2(f(d_1)) \\ \Big| \ \pi_2^=(\beta_i \cdot 1 \cdot 1) \\ e_1 \end{array}$$

a$_4$) The path $\mathcal{D}_4^i$

$$\begin{array}{c} \alpha(d_1) \\ \Big| \ \eta_i \\ \pi_1^=(\beta_i) \cdot 1 \cdot 1 \\ \Big| \\ \pi_1^=(\beta_i \cdot 1 \cdot 1) \end{array}$$

b$_4$) The path $\mathcal{B}_4^i$

$$\begin{array}{c} \left(\pi_1^=(\beta_1) \cdot 1 \cdot 1\right) \cdot 1 \\ \Big| \ \mathcal{R}_4 \\ \pi_1^=(\beta_1) \cdot (1 \cdot 1) \cdot 1 \\ \Big| \ \mathcal{S}_4^{-1} \\ 1 \cdot \left(\pi_1^=(\beta_2) \cdot 1 \cdot 1\right) \end{array}$$

c$_4$) The path $\mathcal{P}_4$

$$\begin{array}{c} \left(\pi_1^=(\beta_1) \cdot 1 \cdot 1\right) \cdot 1 \\ \Big| \\ \pi_1^=(\beta_1) \cdot 1 \cdot (1 \cdot 1) \\ \Big| \\ \pi_1^=(\beta_1) \cdot (1 \cdot 1) \cdot 1 \end{array}$$

d$_4$) The path $\mathcal{R}_4$

$$\begin{array}{c} 1 \cdot \left(\pi_1^=(\beta_2) \cdot 1 \cdot 1\right) \\ \Big| \\ \left(1 \cdot \pi_1^=(\beta_2)\right) \cdot 1 \cdot 1 \\ \Big| \ \textit{via } \mathbf{I}_\square^1(\mathbf{I}_\square^{-1}(\Theta_4)) \\ \left(\pi_1^=(\beta_1) \cdot 1\right) \cdot 1 \cdot 1 \\ \Big| \\ \pi_1^=(\beta_1) \cdot (1 \cdot 1) \cdot 1 \end{array}$$

e$_4$) The path $\mathcal{S}_4$

$$\begin{array}{c} 1 \cdot \pi_1^=(\beta_2) \\ \Big| \\ \pi_1^=(1 \cdot \beta_2) \\ \Big| \ \textit{via } \varphi_* \\ \pi_1^=(\beta_1 \cdot 1) \\ \Big| \\ \pi_1^=(\beta_1) \cdot 1 \end{array}$$

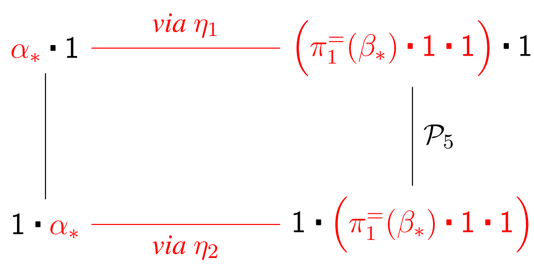
f$_4$) The path Θ_4

We note that in the path $\mathcal{R}_4$ we no longer refer to the term Φ, since the term $\mathbf{I}_\square^2\left(\mathbf{I}_\square^{-2}\left(\Phi(1_{d_1})\right)\right)$ reduces to reflexivity. Furthermore, the only way we make use of the homotopy α is by referring to the value $\alpha(d_1)$.

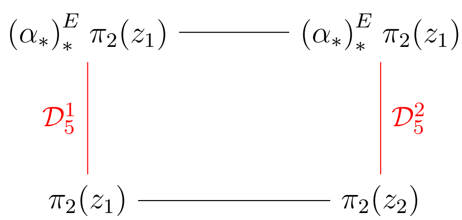
To prove our newest goal, we observe that assuming a higher path $\varphi_* : 1_{f(d_1)} \cdot \beta_2 = \beta_1 \cdot 1_{(d_1,e_1)}$ is equivalent to assuming a path $\varphi_*^* : \beta_2 = \beta_1$ instead and replacing the occurrence of φ_* in Θ_4 by $\mathcal{I}_{1,1}(\varphi_*^*)$. The assumption φ_*^* has the advantage that we can immediately perform path induction on it (keeping the right endpoint fixed), which results in replacing β_2 with β_1. Moreover, at this point we can generalize not only the paths $\alpha(d_1)$ and β_1, but also the points $f(d_1)$ and (d_1, e_1), replacing every occurrence of d_1 and e_1 with the appropriate first or second projection. This leads to the following goal: given terms

- $z_1, z_2 : \Sigma_{x:D} E(x)$,
- $\alpha_* : \pi_1(z_1) = \pi_1(z_2)$ and $\beta_* : z_1 = z_2$,
- $\eta_i : \alpha_* = \pi_1^{=}(\beta_*) \cdot 1_{\pi_1(z_2)} \cdot 1_{\pi_1(z_2)}$,

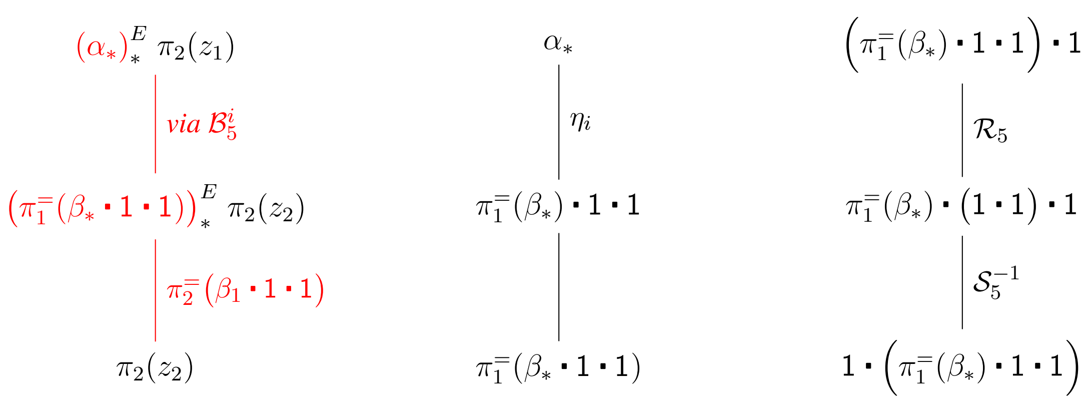
the commutativity of the diagram



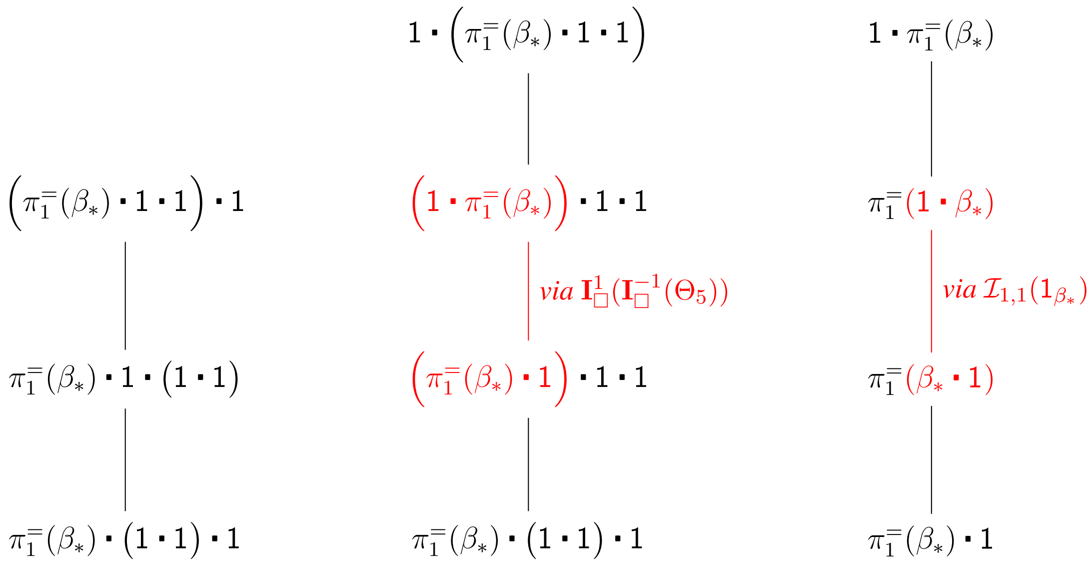
implies the commutativity of the following diagram:



The paths $\mathcal{D}_5^i$, $\mathcal{P}_5$ and their components are as follows:

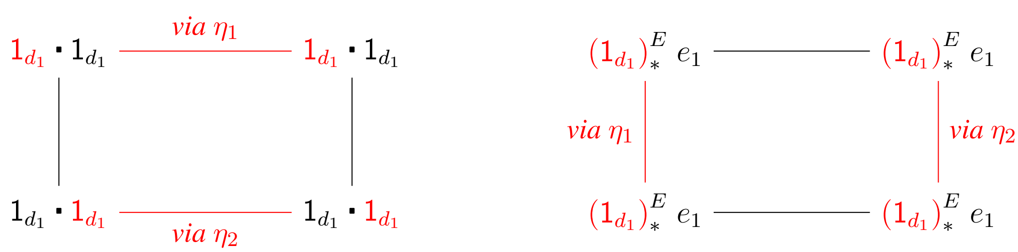


a$_5$) The path $\mathcal{D}_5^i$ **b**$_5$) The path $\mathcal{B}_5^i$ **c**$_5$) The path $\mathcal{P}_5$



$\mathbf{d}_5$) The path $\mathcal{R}_5$ $\quad$ $\mathbf{e}_5$) The path $\mathcal{S}_5$ $\quad$ $\mathbf{f}_5$) The path Θ_5

Of course, either one of the two assumptions η_1, η_2 implies $\alpha_* = \pi_1^{=}(\beta_*)$. Thus, it is enough to prove our goal under the additional assumption $\xi : \alpha_* = \pi_1^{=}(\beta_*)$. But now we can perform one-sided path induction on ξ, which replaces α_* with $\pi_1^{=}(\beta_*)$, and a subsequent path induction on β_*. It now suffices to show the following: given terms $d_1 : D, e_1 : E(d_1)$ and $\eta_1, \eta_2 : 1_{d_1} = 1_{d_1}$, the commutativity of the diagram on the left implies the commutativity of the diagram on the right:



In particular, we note that the path $\mathcal{P}_5$ and all of its components have reduced to reflexivity and the paths $\mathcal{D}_5^1$ and $\mathcal{D}_5^2$ have become equal to η_1 and η_2.

As observed in the proof of lemma 101, for any $u, v : a =_X b$ and $\xi : u = v$, the following two diagrams commute:

In the case when $\xi :\equiv \eta_1$ and $\psi :\equiv \eta_2$, all of the horizontal paths in the above two diagrams become reflexivities; hence it suffices to show that given terms $d_1 : D, e_1 : E(d_1)$ and $\eta_1, \eta_2 : 1_{d_1} = 1_{d_1}$, the equality $\eta_1 = \eta_2$ implies the commutativity of the diagram below:

$$\begin{array}{ccc} (1_{d_1})^E_* \, e_1 & \text{———} & (1_{d_1})^E_* \, e_1 \\ \textit{via } \eta_1 \Big| & & \Big| \textit{ via } \eta_2 \\ (1_{d_1})^E_* \, e_1 & \text{———} & (1_{d_1})^E_* \, e_1 \end{array}$$

But this is clear and we are done. □

Corollary 106. $(\mathcal{H})$ *For any algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *we have*

$$\mathsf{isWQHInit}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$.

At this point, the main theorem 83 is an easy corollary, which we restate here:

Corollary 107. $(\mathcal{H})$ *For* $A, C : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, $\mathtt{l}, \mathtt{r} : C \to A$, *the following conditions on an algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_k$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_k$

for $k \geq j$. *In other words, we have*

$$\mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \simeq \quad \mathsf{isWQHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. *Moreover, the two types above are mere propositions.*

Proof. Using corollaries 104 and 106, we have a logical equivalence between $\mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X})$ and $\mathsf{isWQHInit}_{\mathcal{U}_k}(\mathcal{X})$. It remains to show that both of these types are mere propositions. The latter is a mere proposition by lemma 18. To show that $\mathsf{hasWQInd}_{\mathcal{U}_k}(\mathcal{X})$ is a mere proposition, it is sufficient to do so under the assumption that it is inhabited. Since $\mathcal{X}$ satisfies the induction principle, by lemma 103 it satisfies the induction uniqueness principle. This means that for any fibered algebra $\mathcal{Y}$, the type $\mathsf{WQFibMor}\ \mathcal{X}\ \mathcal{Y}$ is a mere proposition. Since taking a Π of a family of mere propositions results again in a mere proposition, this finishes the proof. □

Furthermore:

Corollary 108. $(\mathcal{H} + \mathsf{WQ})$ *For* $A, C : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, $\mathtt{l}, \mathtt{r} : C \to A$, *the algebra*

$$\Big(\mathsf{WQ}(A, B, C, \mathtt{l}, \mathtt{r}), \mathsf{point}, \mathsf{cell}\Big) : \mathsf{WQAlg}_{\mathcal{U}_i}(A, B, C, \mathtt{l}, \mathtt{r})$$

is homotopy-initial on any universe $\mathcal{U}_j$.

3.4 Definability

We now show how to construct the circles $\mathbf{S}$ and $\mathbb{S}$, type quotients A/R, and W-types $\mathsf{W}(A, B)$ as specific W-quotients. As a consequence, we derive the analogue of our main result for these higher inductive types as a corollary to theorem 3. The interval $\mathbb{I}$ and suspensions ΣA follow the same methodology.

3.4.1 Homotopy-initiality for W-types

In this section, fix $A :\equiv A$, $B :\equiv B$, $C :\equiv 0$ and let $\mathtt{l}, \mathtt{r} :\equiv \mathsf{rec}_{\mathbf{0}}(A)$, *i.e.*, the canonical function from $\mathbf{0}$ to A.

Lemma 109. $(\mathcal{H})$ *We have a function*

$$\mathsf{WQToWAlg} \;\; : \;\; \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r}) \;\; \rightarrow \;\; \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$$

which is an equivalence.

Proof. This follows immediately from the fact that for any $\mathcal{X}_0 : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$, we have

$$\mathsf{WAlgFam}(A, B, C, \mathtt{l}, \mathtt{r})\; \mathcal{X}_0 \;\; \simeq \;\; \mathbf{1}$$

□

Lemma 110. $(\mathcal{H})$ *For an algebra* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *we have a function*

$$\mathsf{WQToWFibAlg}(\mathcal{X}) \;\; : \;\; \mathsf{WQFibAlg}_{\mathcal{U}_k}\; \mathcal{X} \;\; \rightarrow \;\; \mathsf{WFibAlg}_{\mathcal{U}_k} \Big(\mathsf{WQToWAlg}\; \mathcal{X}\Big)$$

which is an equivalence.

Proof. Fix an algebra $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$. Then $\mathsf{WQToWAlg}\; (D, d, p)$ is just the algebra (D, d). The desired equivalence follows easily since for any $\mathcal{X}_0 : \mathsf{WFibAlg}_{\mathcal{U}_k}\; (D, d)$, we have

$$\mathsf{WFibAlgFam}\; (D, d, p)\; \mathcal{X}_0 \;\; \simeq \;\; \mathbf{1}$$

□

Lemma 111. $(\mathcal{H})$ *For algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $\mathcal{Y} : \mathsf{WQFibAlg}_{\mathcal{U}_k}\; \mathcal{X}$ *we have*

$$\mathsf{WQFibMor}\; \mathcal{X}\; \mathcal{Y} \;\; \simeq \;\; \mathsf{WFibMor} \Big(\mathsf{WQToWAlg}\; \mathcal{X}\Big) \Big(\mathsf{WQToWFibAlg}(\mathcal{X})\; \mathcal{Y}\Big)$$

Proof. Fix algebras $(D, d, p) : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ and $(E, e, q) : \mathsf{WQFibAlg}_{\mathcal{U}_k}\; (D, d, p)$. Then $\mathsf{WQToWAlg}\; (D, d, p)$ is the algebra (D, d) and $\mathsf{WQToWFibAlg}(D, d, p)\; (E, e, q)$ is the algebra (E, e). The equivalence follows easily since for any $\mu_0 : \mathsf{WFibMor}\; (D, d)\; (E, e)$, we have

$$\mathsf{WFibMorFam}\; (D, d, p)\; (E, e, q)\; \mu_0 \;\; \simeq \;\; \mathbf{1}$$

□

Corollary 112. ($\mathcal{H}$) *For algebras* $\mathcal{X} : \mathsf{WQAlg}_{\mathcal{U}_j}(A, B, C, \mathtt{l}, \mathtt{r})$ *and* $\mathcal{Y} : \mathsf{WQAlg}_{\mathcal{U}_k}(A, B, C, \mathtt{l}, \mathtt{r})$ *we have*

$$\mathsf{WQMor}\ \mathcal{X}\ \mathcal{Y} \ \simeq\ \mathsf{WMor}\left(\mathsf{WQToWAlg}\ \mathcal{X}\right)\left(\mathsf{WQToWAlg}\ \mathcal{Y}\right)$$

Proof. Using lemma 86. □

Corollary 113. ($\mathcal{H}$) *For an algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ *we have*

$$\begin{aligned}
\mathsf{hasWRec}_{\mathcal{U}_k}(\mathcal{X}) &\ \simeq\ \mathsf{hasWQRec}_{\mathcal{U}_k}\left(\mathsf{WQToWAlg}^{-1}(\mathcal{X})\right)\\
\mathsf{hasWInd}_{\mathcal{U}_k}(\mathcal{X}) &\ \simeq\ \mathsf{hasWQInd}_{\mathcal{U}_k}\left(\mathsf{WQToWAlg}^{-1}(\mathcal{X})\right)\\
\mathsf{isWHInit}_{\mathcal{U}_k}(\mathcal{X}) &\ \simeq\ \mathsf{isWQHInit}_{\mathcal{U}_k}\left(\mathsf{WQToWAlg}^{-1}(\mathcal{X})\right)
\end{aligned}$$

Corollary 114. ($\mathcal{H}$) *For* $A : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, *the following conditions on an algebra* $\mathcal{X} : \mathsf{WAlg}_{\mathcal{U}_j}(A, B)$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_k$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_k$

for $k \geq j$. *In other words, we have*

$$\mathsf{hasWInd}_{\mathcal{U}_k}(\mathcal{X}) \ \simeq\ \mathsf{isWHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. *Moreover, the two types above are mere propositions.*

Corollary 115. ($\mathcal{H} + \mathsf{W}$) *For* $A : \mathcal{U}_i$, $B : A \to \mathcal{U}_i$, *the algebra*

$$\left(\mathsf{W}(A, B), \mathsf{sup}\right) : \mathsf{WAlg}_{\mathcal{U}_i}(A, B)$$

is homotopy-initial on any universe $\mathcal{U}_j$.

3.4.2 Homotopy-initiality for The Circle

We first treat the $\mathbf{S}$-case, for which we define $A, C :\equiv \mathbf{1}$, $B(-) :\equiv \mathbf{0}$, $\mathtt{l}(-) :\equiv \star$, $\mathtt{r}(-) :\equiv \star$.

Lemma 116. ($\mathcal{H}$) *We have a function*

$$\mathbf{S}\text{-}\mathsf{To}\text{-}\mathsf{WQ}\text{-}\mathsf{Alg} \ :\ \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i} \ \to\ \mathsf{WQAlg}_{\mathcal{U}_i}(A, B, C, \mathtt{l}, \mathtt{r})$$

which is an equivalence.

Proof. This follows easily from the observations that for any $D : \mathcal{U}_i$, we have the equivalence

$$d \mapsto \lambda_{a:\mathbf{1}}\lambda_{t:\mathbf{0}\to D} d \ :\ D \ \simeq\ \Pi_{a:\mathbf{1}}(\mathbf{0} \to D) \to D$$

and for any $D : \mathcal{U}_i$, $d : D$, we have the equivalence

$$p \mapsto \lambda_{c:\mathbf{1}}\lambda_{u:\mathbf{0}\to D}\lambda_{v:\mathbf{0}\to D} p \ :\ d = d \ \simeq\ \mathsf{WAlgFam}(A, B, C, \mathtt{l}, \mathtt{r})\left(D, \lambda_{a:\mathbf{1}}\lambda_{t:\mathbf{0}\to D} d\right)$$

□

Lemma 117. ($\mathcal{H}$) *For an algebra* $\mathcal{X} : \textsf{S-Alg}_{\mathcal{U}_i}$ *we have a function*

$$\textsf{S-To-WQ-FibAlg}(\mathcal{X}) \quad : \quad \textsf{S-FibAlg}_{\mathcal{U}_j}\ \mathcal{X} \quad \to \quad \textsf{WQFibAlg}_{\mathcal{U}_j}\ \Big(\textsf{S-To-WQ-Alg}\ \mathcal{X}\Big)$$

which is an equivalence.

Proof. Fix an algebra $(D, d, p) : \textsf{S-FibAlg}_{\mathcal{U}_i}$. Let $(D, d', p') :\equiv \textsf{S-To-WQ-Alg}\ (D, d, p)$. Then

$$d'(a, t) :\equiv d$$
$$p'(c, t, s) :\equiv p$$

The desired equivalence follows easily from the observations that for any $E : D \to \mathcal{U}_j$, we have the equivalence

$$e \mapsto \lambda_{a:\mathbf{1}}\lambda_{t:\mathbf{0}\to D}\lambda_{u:(\Pi b:\mathbf{0})E(t\ b)}{\color{red}e} \quad : \quad E(d) \quad \simeq \quad \Pi_{a:\mathbf{1}}\Pi_{t:\mathbf{0}\to D}\big(\Pi_{b:\mathbf{0}}E(t\ b)\big) \to E(d)$$

and for any $E : D \to \mathcal{U}_j$, $e : E(d)$, we have the equivalence

$$q \mapsto \lambda_{c:\mathbf{1}}\lambda_{t:\mathbf{0}\to D}\lambda_{s:\mathbf{0}\to D}\lambda_{u:(\Pi b:\mathbf{0})E(t\ b)}\lambda_{v:(\Pi b:\mathbf{0})E(s\ b)}{\color{red}q} \quad :$$
$$p_*^E(e) = e \quad \simeq \quad \textsf{WFibAlgFam}\ (D, d', p')\ \Big({\color{red}E}, \lambda_{a:\mathbf{1}}\lambda_{t:\mathbf{0}\to D}\lambda_{u:(\Pi b:\mathbf{0})E(t\ b)}{\color{red}e}\Big)$$

□

Lemma 118. ($\mathcal{H}$) *For algebras* $\mathcal{X} : \textsf{S-Alg}_{\mathcal{U}_i}$ *and* $\mathcal{Y} : \textsf{S-FibAlg}_{\mathcal{U}_j}\ \mathcal{X}$ *we have*

$$\textsf{S-FibMor}\ \mathcal{X}\ \mathcal{Y} \quad \simeq \quad \textsf{WQFibMor}\ \Big(\textsf{S-To-WQ-Alg}\ \mathcal{X}\Big)\ \Big(\textsf{S-To-WQ-FibAlg}(\mathcal{X})\ \mathcal{Y}\Big)$$

Proof. Fix algebras $(D, d, p) : \textsf{S-Alg}_{\mathcal{U}_i}$ and $(E, e, q) : \textsf{S-FibAlg}_{\mathcal{U}_j}\ (D, d, p)$. As before, we let $(D, d', p') :\equiv \textsf{S-To-WQ-Alg}\ (D, d, p)$. Then

$$d'(a, t) :\equiv d$$
$$p'(c, t, s) :\equiv p$$

Similarly, let $(E, e', q') :\equiv \textsf{S-To-WQ-FibAlg}(D, d, p)\ (E, e, q)$. Then

$$e'(a, t) :\equiv e$$
$$q'(c, t, s) :\equiv q$$

The desired equivalence follows easily from the observations that for any $f : \Pi_{x:D}E(x)$, we have the equivalence

$$\beta \mapsto \lambda_{a:\mathbf{1}}\lambda_{t:\mathbf{0}\to D}{\color{red}\beta} \quad : \quad f(d) = e \quad \simeq \quad \Pi_{a:\mathbf{1}}\Pi_{t:\mathbf{0}\to D}(f(d) = e)$$

and for any $f : \Pi_{x:D}E(x)$, $\beta : f(d) = e$, we have the equivalence

$$\theta \mapsto \lambda_{c:\mathbf{1}}\lambda_{t:\mathbf{0}\to D}\lambda_{s:\mathbf{0}\to D}{\color{red}\theta} \quad :$$
$$\textsf{dap}_f(p) = \textsf{ap}_{p_*^E}(\beta) \cdot q \cdot \beta^{-1} \quad \simeq \quad \textsf{WFibMorFam}\ (D, d', p')\ (E, e', q')\ \Big({\color{red}f}, \lambda_{a:\mathbf{1}}\lambda_{t:\mathbf{0}\to D}{\color{red}\beta}\Big)$$

□

Lemma 119. $(\mathcal{H})$ *For algebras* $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ *and* $\mathcal{Y} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_j}$ *we have*

$$\mathbf{S}\text{-}\mathsf{Mor}\ \mathcal{X}\ \mathcal{Y} \;\simeq\; \mathsf{WQMor}\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathsf{WQ}\text{-}\mathsf{Alg}\ \mathcal{X}\right)\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathsf{WQ}\text{-}\mathsf{Alg}\ \mathcal{Y}\right)$$

Proof. Exactly as in the fibered case. □

Corollary 120. $(\mathcal{H})$ *For an algebra* $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ *we have*

$$\begin{aligned}
\mathsf{has}\text{-}\mathbf{S}\text{-}\mathsf{Rec}_{\mathcal{U}_j}(\mathcal{X}) &\;\simeq\; \mathsf{hasWQRec}_{\mathcal{U}_j}\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathsf{WQ}\text{-}\mathsf{Alg}(\mathcal{X})\right)\\
\mathsf{has}\text{-}\mathbf{S}\text{-}\mathsf{Ind}_{\mathcal{U}_j}(\mathcal{X}) &\;\simeq\; \mathsf{hasWQInd}_{\mathcal{U}_j}\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathsf{WQ}\text{-}\mathsf{Alg}(\mathcal{X})\right)\\
\mathsf{is}\text{-}\mathbf{S}\text{-}\mathsf{HInit}_{\mathcal{U}_j}(\mathcal{X}) &\;\simeq\; \mathsf{isWQHInit}_{\mathcal{U}_j}\left(\mathbf{S}\text{-}\mathsf{To}\text{-}\mathsf{WQ}\text{-}\mathsf{Alg}(\mathcal{X})\right)
\end{aligned}$$

Corollary 121. $(\mathcal{H})$ *The following conditions on an algebra* $\mathcal{X} : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_j$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_j$

for $j \geq i$*. In other words, we have*

$$\mathsf{has}\text{-}\mathbf{S}\text{-}\mathsf{Ind}_{\mathcal{U}_j}(\mathcal{X}) \;\simeq\; \mathsf{is}\text{-}\mathbf{S}\text{-}\mathsf{HInit}_{\mathcal{U}_j}(\mathcal{X})$$

provided for $j \geq i$*. Moreover, the two types above are mere propositions.*

Corollary 122. $(\mathcal{H} + \mathbf{S})$ *The algebra*

$$\left(\mathbf{S}, \mathsf{base}, \mathsf{loop}\right) : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_0}$$

is homotopy-initial on any universe $\mathcal{U}_i$*.*

Corollary 123. $(\mathcal{H})$ *The following conditions on an algebra* $\mathcal{X} : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_i}$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_j$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_j$

for $j \geq i$*. In other words, we have*

$$\mathsf{has}\text{-}\mathbb{S}\text{-}\mathsf{Ind}_{\mathcal{U}_j}(\mathcal{X}) \;\simeq\; \mathsf{is}\text{-}\mathbb{S}\text{-}\mathsf{HInit}_{\mathcal{U}_j}(\mathcal{X})$$

provided for $j \geq i$*. Moreover, the two types above are mere propositions.*

Proof. Using corollary 60. □

Corollary 124. $(\mathcal{H} + \mathbb{S})$ *The algebra*

$$\left(\mathbb{S}, \mathsf{north}, \mathsf{east}, \mathsf{west}\right) : \mathbb{S}\text{-}\mathsf{Alg}_{\mathcal{U}_0}$$

is homotopy-initial on any universe $\mathcal{U}_i$*.*

3.4.3 Homotopy-initiality for Type Quotients

For a type quotient A/R, define $A :\equiv A$, $B(-) :\equiv \mathbf{0}$, $C :\equiv \Sigma_{a,b:A}R(a,b)$, $\mathtt{l} :\equiv \pi_1$, $\mathtt{r} :\equiv \pi_1 \circ \pi_2$.

Lemma 125. $(\mathcal{H})$ *We have a function*

$$\mathsf{TQToWAlg} \quad : \quad \mathsf{TQAlg}_{\mathcal{U}_j}(A,R) \quad \to \quad \mathsf{WQAlg}_{\mathcal{U}_j}(A,B,C,\mathtt{l},\mathtt{r})$$

which is an equivalence.

Proof. This follows easily from the observations that for any $D : \mathcal{U}_j$, we have the equivalence

$$d \mapsto \lambda_{a:A}\lambda_{t:\mathbf{0}\to D}d(a) \quad : \quad A \to D \quad \simeq \quad \Pi_{a:A}(\mathbf{0} \to D) \to D$$

and for any $D : \mathcal{U}_j$, $d : A \to D$, we have the equivalence

$$p \mapsto \lambda_{c:(\Sigma a,b:A)R(a,b)}\lambda_{u:\mathbf{0}\to D}\lambda_{v:\mathbf{0}\to D}p\Big(\pi_1(c), \pi_1(\pi_2(c)), \pi_2(\pi_2(c))\Big) \quad :$$
$$\Pi_{a,b:A}R(a,b) \to (d(a) = d(b)) \quad \simeq \quad \mathsf{WAlgFam}(A,B,C,\mathtt{l},\mathtt{r})\ \Big(D, \lambda_{a:A}\lambda_{t:\mathbf{0}\to D}d(a)\Big)$$

□

Lemma 126. $(\mathcal{H})$ *For an algebra* $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A,R)$ *we have a function*

$$\mathsf{TQToWFibAlg}(\mathcal{X}) \quad : \quad \mathsf{TQFibAlg}_{\mathcal{U}_k}\ \mathcal{X} \quad \to \quad \mathsf{WQFibAlg}_{\mathcal{U}_k}\ \Big(\mathsf{TQToWAlg}\ \mathcal{X}\Big)$$

which is an equivalence.

Proof. Fix an algebra $(D,d,p) : \mathsf{TQAlg}_{\mathcal{U}_j}(A,R)$. Let $(D,d',p') :\equiv \mathsf{TQToWAlg}\ (D,d,p)$. Then

$$d'(a,t) :\equiv d(a)$$
$$p'(c,t,s) :\equiv p\Big(\pi_1(c), \pi_1(\pi_2(c)), \pi_2(\pi_2(c))\Big)$$

The desired equivalence follows easily from the observations that for any $E : D \to \mathcal{U}_k$, we have the equivalence

$$e \mapsto \lambda_{a:A}\lambda_{t:\mathbf{0}\to D}\lambda_{u:(\Pi b:\mathbf{0})E(t\ b)}e(a) \quad : \quad \Pi_{a:A}E(d(a)) \quad \simeq \quad \Pi_{a:A}\Pi_{t:\mathbf{0}\to D}\Big(\Pi_{b:\mathbf{0}}E(t\ b)\Big) \to E(d(a))$$

and for any $E : D \to \mathcal{U}_j$, $e : \Pi_{a:A}E(d(a))$, we have the equivalence

$$q \mapsto \lambda_{c:(\Sigma a,b:A)R(a,b)}\lambda_{t:\mathbf{0}\to D}\lambda_{s:\mathbf{0}\to D}\lambda_{u:(\Pi b:\mathbf{0})E(t\ b)}\lambda_{v:(\Pi b:\mathbf{0})E(s\ b)}q\Big(\pi_1(c), \pi_1(\pi_2(c)), \pi_2(\pi_2(c))\Big) \quad :$$
$$\Pi_{a,b:A}\Pi_{z:R(a,b)}\Big(p(a,b,z)_*^E\ e(a) = e(b)\Big) \quad \simeq$$
$$\mathsf{WFibAlgFam}\ (D,d',p')\ \Big(E, \lambda_{a:A}\lambda_{t:\mathbf{0}\to D}\lambda_{u:(\Pi b:\mathbf{0})E(t\ b)}e(a)\Big)$$

□

Lemma 127. *($\mathcal{H}$) For algebras $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ and $\mathcal{Y} : \mathsf{TQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ we have*

$$\mathsf{TQFibMor}\ \mathcal{X}\ \mathcal{Y} \simeq \mathsf{WQFibMor}\ \Big(\mathsf{TQToWAlg}\ \mathcal{X}\Big)\ \Big(\mathsf{TQToWFibAlg}(\mathcal{X})\ \mathcal{Y}\Big)$$

Proof. Fix algebras $(D, d, p) : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ and $(E, e, q) : \mathsf{TQFibAlg}_{\mathcal{U}_k}\ (D, d, p)$. As before, let $(D, d', p') :\equiv \mathsf{TQToWAlg}\ (D, d, p)$. Then

$$\begin{aligned} d'(a, t) &:\equiv d(a) \\ p'(c, t, s) &:\equiv p\Big(\pi_1(c), \pi_1(\pi_2(c)), \pi_2(\pi_2(c))\Big) \end{aligned}$$

Similarly, let $(E, e', q') :\equiv \mathsf{TQToWFibAlg}(D, d, p)\ (E, e, q)$. Then

$$\begin{aligned} e'(a, t) &:\equiv e(a) \\ q'(c, t, s, u, v) &:\equiv q\Big(\pi_1(c), \pi_1(\pi_2(c)), \pi_2(\pi_2(c))\Big) \end{aligned}$$

The desired equivalence follows easily from the observations that for any $f : \Pi_{x:D} E(x)$, we have the equivalence

$$\beta \mapsto \lambda_{a:A} \lambda_{t:\mathbf{0}\to D} \beta(a) \ \ : \ \ \Pi_{a:A}(f(d(a)) = e(a)) \ \simeq \ \Pi_{a:A}\Pi_{t:\mathbf{0}\to D}(f(d(a)) = e(a))$$

and for any $f : \Pi_{x:D} E(x)$, $\beta : \Pi_{a:A}(f(d(a)) = e(a))$, we have the equivalence

$$\begin{aligned} &\theta \mapsto \lambda_{c:(\Sigma a,b:A)R(a,b)} \lambda_{t:\mathbf{0}\to D} \lambda_{s:\mathbf{0}\to D} \theta\Big(\pi_1(c), \pi_1(\pi_2(c)), \pi_2(\pi_2(c))\Big) \ \ : \\ &\quad \Pi_{a,b:A}\Pi_{z:R(a,b)}\Big(\mathsf{dap}_f(p(a, b, z)) = \mathsf{ap}_{p(a,b,z)^E_*}(\beta(a)) \cdot q(a, b, z) \cdot \beta(b)^{-1}\Big) \ \simeq \\ &\quad \mathsf{WFibMorFam}\ (D, d', p')\ (E, e', q')\ \Big(f, \lambda_{a:A}\lambda_{t:\mathbf{0}\to D}\beta(a)\Big) \end{aligned}$$

□

Lemma 128. *($\mathcal{H}$) For algebras $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ and $\mathcal{Y} : \mathsf{TQAlg}_{\mathcal{U}_k}(A, R)$ we have*

$$\mathsf{TQMor}\ \mathcal{X}\ \mathcal{Y} \simeq \mathsf{WQMor}\ \Big(\mathsf{TQToWAlg}\ \mathcal{X}\Big)\ \Big(\mathsf{TQToWAlg}\ \mathcal{Y}\Big)$$

Proof. Exactly as in the fibered case. □

Corollary 129. *($\mathcal{H}$) For an algebra $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ we have*

$$\begin{aligned} \mathsf{hasTQRec}_{\mathcal{U}_k}(\mathcal{X}) &\simeq \mathsf{hasWQRec}_{\mathcal{U}_k}\Big(\mathsf{TQToWAlg}(\mathcal{X})\Big) \\ \mathsf{hasTQInd}_{\mathcal{U}_k}(\mathcal{X}) &\simeq \mathsf{hasWQInd}_{\mathcal{U}_k}\Big(\mathsf{TQToWAlg}(\mathcal{X})\Big) \\ \mathsf{isTQHInit}_{\mathcal{U}_k}(\mathcal{X}) &\simeq \mathsf{isWQHInit}_{\mathcal{U}_k}\Big(\mathsf{TQToWAlg}(\mathcal{X})\Big) \end{aligned}$$

Corollary 130. $(\mathcal{H})$ *For* $A : \mathcal{U}_i$, $R : A \to A \to \mathcal{U}_i$, *the following conditions on an algebra* $\mathcal{X} : \mathsf{TQAlg}_{\mathcal{U}_j}(A, R)$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_k$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_k$

for $k \geq j$. *In other words, we have*

$$\mathsf{hasTQInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \simeq \quad \mathsf{isTQHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. *Moreover, the two types above are mere propositions.*

Corollary 131. $(\mathcal{H} + /)$ *For* $A : \mathcal{U}_i$, $R : A \to A \to \mathcal{U}_i$, *the algebra*

$$\Big(A/R, [-], \mathsf{c}\Big) : \mathsf{TQAlg}_{\mathcal{U}_i}(A, R)$$

is homotopy-initial on any universe $\mathcal{U}_j$.

4

Homotopy-initiality for Further Higher Inductive Types

As we saw in the previous chapter, all the higher inductive types described in section 2.4 are special cases of W-quotients and as such admit a simple characterization as homotopy-initial algebras of a certain form. A natural question to ask at this point would be whether we can characterize *all* higher inductive types in a similar fashion. As of now, this is not a mathematically precise statement because homotopy type theory does not yet have a universally agreed-upon definition of what a higher inductive type should be (although recent unpublished work by Lumsdaine and Shulman offers significant progress in this direction). Instead, one generally works with specific examples that everybody agrees are well-behaved higher inductive types, such as the ones we saw in section 2.4.

In this chapter we will study two further classes of higher inductive types - *truncations* (chapters 6.9 and 7.3 of [33]) and *set/groupoid quotients* (chapters 6.10 and 9.9 of [33]) and show that they too can be characterized as appropriate homotopy-initial algebras. The proof we give for truncations is not a corollary of our main theorem 83 since truncations do not arise as W-quotients *in an obvious way*, the way a type quotient or a W-type does. Recent work by Rijke and van Doorn [36] shows that truncations *can* in fact be recovered from type quotients (and hence from W-quotients) but the reduction is highly nontrivial. On the other hand, we present the reduction of groupoid (and, rather trivially, set) quotients to W-quotients plus truncations (which by Rijke's result implies a reduction to W-quotients themselves) in this section as a new result and derive the analogue of theorem 83 for set and groupoid quotients as a consequence of this reduction. We also remark that while the definition of a group quotient as a higher inductive type appears in [33] (chapter 9.9, page 333), the associated recursion and induction rules are not described in the book and hence we give our own version here.

4.1 Homotopy-initiality for Truncations

4.1.1 Truncations

The *n-truncation* $||A||_n$ of a type A for $n \geq -2$ should intuitively be the "best approximation" of A by an n-type. For $n :\equiv -2$, the only choice is of course the unit type $\mathbf{1}$. For $n :\equiv -1$, we could define $||A||_{-1}$ to be the higher inductive type generated by the point constructor $|\cdot| : A \to ||A||_{-1}$, and a path constructor ensuring that the resulting higher inductive type will indeed be a mere proposition, for example $\mathtt{t} : \big(\Pi x, y : ||A||_{-1}\big)(x = y)$. For $n :\equiv 0$, we would modify the path constructor to $\mathtt{t} : \big(\Pi x, y : ||A||_0\big)\big(\Pi p, q : x = y\big)(p = q)$, for $n :\equiv 1$ to $\mathtt{t} : \big(\Pi x, y : ||A||_1\big)\big(\Pi p, q : x = y\big)\big(\Pi \gamma, \delta : p = q\big)(\gamma = \delta)$ and so on. Clearly, this does not scale very well to higher n; in fact, even for $n :\equiv 1$ the associated induction principle is too ugly to write down. Instead, we will describe n-truncations compactly by using n-spheres, exactly as done in [33], although our motivation and subsequent justification will be slightly different.

Since we will be working with n-spheres, let us remind ourselves of the following universal properties of suspensions, whose proofs are simple exercises:

Lemma 132. $(\mathcal{H} + \Sigma)$ *For any $A, D : \mathcal{U}_i$ we have*

$$\Sigma A \to D \quad \simeq \quad \Sigma_{n,s:D} A \to (n = s)$$

Proof. The equivalence is given by the following quasi-inverses:

$$f \mapsto \Big(f(\mathsf{N}), f(\mathsf{S}), \lambda_{a:A}\mathsf{ap}_f(\mathsf{mer}(a))\Big)$$
$$(n, s, m) \mapsto \mathsf{rec}_{\Sigma A}(D, n, s, m)$$

□

Lemma 133. $(\mathcal{H} + \Sigma)$ *For any $A : \mathcal{U}_i$, $E : A \to \mathcal{U}_i$, we have*

$$\Pi_{x:\Sigma A} E(x) \quad \simeq \quad \Sigma_{n:E(\mathsf{N})}\Sigma_{s:E(\mathsf{S})}\Pi_{a:A}\big(\mathsf{mer}(a)_*^E(n) = s\big)$$

Proof. The equivalence is given by the following quasi-inverses:

$$f \mapsto \Big(f(\mathsf{N}), f(\mathsf{S}), \lambda_{a:A}\mathsf{dap}_f(\mathsf{mer}(a))\Big)$$
$$(n, s, m) \mapsto \mathsf{ind}_{\Sigma A}(E, n, s, m)$$

□

Our next goal is to establish a different characterization of n-types, for $n \geq -1$, as precisely those types $A : \mathcal{U}_i$ for which any function $f : \mathbf{S}^{n+1} \to A$ is constant up to propositional equality (see [11] on various notions of "constant" maps):

Definition 134. *A function $f : A \to B$ is said to be* homotopy-constant *if it is propositionally equal to a constant function:*

$$\mathsf{HConst}(f) :\equiv \Sigma_{b:B}\Pi_{a:A}(f(a) = b)$$

Lemma 135. $(\mathcal{H}+\Sigma)$ *For any* $n:\mathbb{N}$ *and* $f:\mathbf{S}^n\to A$*, we have*

$$\mathsf{HConst}(f) \quad\simeq\quad \mathsf{HConst}\big((\lambda x:\mathbf{S}^{n-1})\mathsf{ap}_f(\mathsf{mer}(x))\big)$$

Proof. We have the following chain of equivalences:

$$\begin{aligned}
& \mathsf{HConst}(f) \\
\equiv\;& \big(\Sigma a:A\big)\big(\Pi x:\mathbf{S}^n\big)\big(f(x)=a\big) \\
(2)\quad \simeq\;& \big(\Sigma a:A\big)\big(\Sigma\alpha:f(\mathsf{N})=a\big)\big(\Sigma\beta:f(\mathsf{S})=a\big)\big(\Pi x:\mathbf{S}^{n-1}\big)\Big(\mathsf{mer}(x)_*^{z\mapsto f(z)=a}(\alpha)=\beta\Big) \\
(3)\quad \simeq\;& \big(\Sigma a:A\big)\big(\Sigma\alpha:f(\mathsf{N})=a\big)\big(\Sigma\beta:f(\mathsf{S})=a\big)\big(\Pi x:\mathbf{S}^{n-1}\big)\Big(\mathsf{ap}_f(\mathsf{mer}(x))=\alpha\cdot\beta^{-1}\Big) \\
\simeq\;& \big(\Sigma\mathbf{r}:\Sigma_{a:A}(a=f(\mathsf{N}))\big)\big(\Sigma\beta:f(\mathsf{S})=\pi_1(\mathbf{r})\big)\big(\Pi x:\mathbf{S}^{n-1}\big)\Big(\mathsf{ap}_f(\mathsf{mer}(x))=\pi_2(\mathbf{r})\cdot\beta^{-1}\Big) \\
(5)\quad \simeq\;& \big(\Sigma\beta:f(\mathsf{S})=f(\mathsf{N})\big)\big(\Pi x:\mathbf{S}^{n-1}\big)\Big(\mathsf{ap}_f(\mathsf{mer}(x))=1_{f(\mathsf{N})}\cdot\beta^{-1}\Big) \\
\simeq\;& \big(\Sigma\beta:f(\mathsf{N})=f(\mathsf{S})\big)\big(\Pi x:\mathbf{S}^{n-1}\big)\Big(\mathsf{ap}_f(\mathsf{mer}(x))=1_{f(\mathsf{N})}\cdot\beta\Big) \\
\simeq\;& \big(\Sigma\beta:f(\mathsf{N})=f(\mathsf{S})\big)\big(\Pi x:\mathbf{S}^{n-1}\big)\Big(\mathsf{ap}_f(\mathsf{mer}(x))=\beta\Big) \\
\equiv\;& \mathsf{HConst}\big((\lambda x:\mathbf{S}^{n-1})\mathsf{ap}_f(\mathsf{mer}(x))\big)
\end{aligned}$$

The second equivalence follows from lemma 133, the third by a suitable generalization and path induction on $\mathsf{mer}(x)$, and the fifth from the fact that the type $\Sigma_{a:A}(a=f(\mathsf{N}))$ is contractible with the center of contraction $\big(f(\mathsf{N}),1_{f(\mathsf{N})}\big)$. □

Lemma 136. $(\mathcal{H}+\Sigma)$ *For any* $n:\mathbb{N}$ *and* $A:\mathcal{U}_i$*, we have*

$$\big(\Pi f:\mathbf{S}^n\to A\big)\ \mathsf{HConst}(f) \quad\simeq\quad \big(\Pi a,b:A\big)\big(\Pi f:\mathbf{S}^{n-1}\to(a=b)\big)\ \mathsf{HConst}(f)$$

Proof. We have the following chain of equivalences:

$$\begin{aligned}
& \big(\Pi f:\mathbf{S}^n\to A\big)\ \mathsf{HConst}(f) \\
\simeq\;& \Big(\Pi\mathbf{r}:\Sigma_{a,b:A}\mathbf{S}^{n-1}\to(a=b)\Big)\ \mathsf{HConst}\Big(\mathsf{rec}_{\mathbf{S}^n}\big(A,\pi_1(\mathbf{r}),\pi_1(\pi_2(\mathbf{r})),\pi_2(\pi_2(\mathbf{r}))\big)\Big) \\
\simeq\;& \big(\Pi a,b:A\big)\big(\Pi f:\mathbf{S}^{n-1}\to(a=b)\big)\ \mathsf{HConst}\big(\mathsf{rec}_{\mathbf{S}^n}(A,a,b,f)\big) \\
\simeq\;& \big(\Pi a,b:A\big)\big(\Pi f:\mathbf{S}^{n-1}\to(a=b)\big)\ \mathsf{HConst}(f)
\end{aligned}$$

Here the first equivalence follows from lemma 132 and the third from lemma 135. □

Corollary 137. $(\mathcal{H}+\Sigma)$ *For any* $n:\mathbb{N}$ *and* $A:\mathcal{U}_i$*, we have*

$$\big(\Pi f:\mathbf{S}^n\to A\big)\ \mathsf{HConst}(f) \quad\simeq\quad \mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}(A)$$

Proof. By induction on n. For the base case, we have $\mathsf{is}\text{-}(-1)\text{-}\mathsf{type}(A)\simeq\mathsf{isProp}(A)$ and

$$\big(\Pi f:\mathbf{S}^0\to A\big)\ \mathsf{HConst}(f) \quad\simeq\quad \big(\Pi a,b:A\big)\big(\Pi f:\mathbf{0}\to(a=b)\big)\ \mathsf{HConst}(f)$$

by lemma 136. For any $a, b : A$, we have

$$\begin{aligned}
& \Pi_{f:\mathbf{0}\to(a=b)}\ \mathsf{HConst}(f) \\
\equiv\ & \Pi_{f:\mathbf{0}\to(a=b)}\ \Sigma_{p:a=b}\Pi_{x:\mathbf{0}}(f(x)=p) \\
\simeq\ & \Pi_{f:\mathbf{0}\to(a=b)}\ (a=b) \\
\simeq\ & (a=b)
\end{aligned}$$

This finishes the base step; the inductive step follows easily from lemma 136. □

We are now ready to formally define truncations. For $n : \mathbb{N}$ and a type $A : \mathcal{U}_i$, we define the truncation $||A||_{n-2} : \mathcal{U}_i$ by case analysis on n. For $n :\equiv \mathbf{0}$, we put $||A||_{-2} :\equiv \mathbf{1}$. For the successor case, just like in chapter 7.3 of [33] we define $||A||_{n-1} : \mathcal{U}_i$ to be the higher inductive type generated by the constructors

$$\begin{aligned}
|\cdot|_{n-1} &: A \to ||A||_{n-1} \\
\mathsf{hub} &: \big(\mathbf{S}^n \to ||A||_{n-1}\big) \to ||A||_{n-1} \\
\mathsf{spoke} &: \big(\Pi r : \mathbf{S}^n \to ||A||_{n-1}\big)\big(\Pi x : \mathbf{S}^n\big)(r(x) = \mathsf{hub}(r))
\end{aligned}$$

The recursion principle for truncations $||A||_{n-1}$ says that given terms

- $E : \mathcal{U}_j$,
- $e : A \to E$,
- $h : (\mathbf{S}^n \to E) \to E$,
- $s : \Pi_{u:\mathbf{S}^n\to E}\Pi_{x:\mathbf{S}^n}(u(x) = h(u))$,

there is a recursor $\mathsf{rec}_{||\cdot||}(E, e, h, s) : ||A||_{n-1} \to E$. The recursor satisfies the following computation laws:

- $\mathsf{rec}_{||\cdot||}(|a|_{n-1}) \equiv f(a)$ for any $a : A$,
- $\mathsf{rec}_{||\cdot||}(\mathsf{hub}(r)) = h\big(\mathsf{rec}_{||\cdot||} \circ r\big)$ for any $r : \mathbf{S}^n \to ||A||_{n-1}$,
- $\mathsf{ap}_{\mathsf{rec}_{||\cdot||}}(\mathsf{spoke}(r, x))$ is equal to the path below for any $r : \mathbf{S}^n \to ||A||_{n-1}, x : \mathbf{S}^n$, where equality (1) uses the computation rule for the hub constructor:

$$\mathsf{rec}_{||\cdot||}(r(x)) \xrightarrow{\ s(\mathsf{rec}_{||\cdot||}\circ r,\, x)\ } h\big(\mathsf{rec}_{||\cdot||}\circ r\big) \xrightarrow{\ (1)\ } \mathsf{rec}_{||\cdot||}(\mathsf{hub}(r))$$

As we see, the computation law for the hub constructor is propositional rather than definitional. There is little incentive for stating it definitionally; the hub and spoke constructors only serve to ensure that $||A||_{n-1}$ is indeed an $(n-1)$-type and are not intended for purposes of computation. Furthermore, the propositional form of the above law will turn out to be quite useful, as we will see shortly.

The induction principle says that given terms

- $E : ||A||_{n-1} \to \mathcal{U}_j$,
- $e : \Pi_{a:A}E(|a|_{n-1})$,

- $h : \big(\Pi r : \mathbf{S}^n \to ||A||_{n-1}\big)\big(\Pi_{x:\mathbf{S}^n} E(r(x))\big) \to E(\mathsf{hub}(r))$,
- $s : \big(\Pi r : \mathbf{S}^n \to ||A||_{n-1}\big)\big(\Pi u : \Pi_{x:\mathbf{S}^n} E(r(x))\big)\big(\Pi x : \mathbf{S}^n\big)\big(\mathsf{spoke}(r,x)_*^E\ u(x) = h(r,u)\big)$,

there is an inductor $\mathsf{ind}_{||\cdot||}(E,e,h,s) : \Pi_{x:||A||_{n-1}} E(x)$. The inductor satisfies the computation laws

- $\mathsf{ind}_{||\cdot||}(|a|_{n-1}) \equiv e(a)$ for any $a : A$,
- $\mathsf{ind}_{||\cdot||}(\mathsf{hub}(r)) \equiv h\big(r, \mathsf{ind}_{||\cdot||} \circ r\big)$ for any $r : \mathbf{S}^n \to ||A||_{n-1}$,
- $\mathsf{dap}_{\mathsf{ind}_{||\cdot||}}(\mathsf{spoke}(r,x))$ is equal to the path below for any $r : \mathbf{S}^n \to ||A||_{n-1}, x : \mathbf{S}^n$, where equality (1) uses the computation rule for the hub constructor:

$$\mathsf{spoke}(r,x)_*^E\ \big(\mathsf{ind}_{||\cdot||}(r(x))\big) \xrightarrow{\ s(r,\mathsf{ind}_{||\cdot||}\circ r, x)\ } h\big(r, \mathsf{ind}_{||\cdot||}\circ r\big) \xrightarrow{\ (1)\ } \mathsf{ind}_{||\cdot||}(\mathsf{hub}(r))$$

Before we proceed further, we establish a couple of technical lemmas:

Lemma 138. $(\mathcal{H} + \Sigma)$ *For any* $n : \mathbb{N}$ *and* $f : \mathbf{S}^{n-1} \to A$*, we have*

$$\mathsf{is}\text{-}(n-2)\text{-}\mathsf{type}(A) \quad \to \quad \mathsf{isContr}\big(\mathsf{HConst}(f)\big)$$

Proof. By induction on n. For the base step, assume A is contractible. Then

$$\mathsf{HConst}(f) \quad \equiv \quad \Sigma_{a:A}\Pi_{x:\mathbf{0}}(f(x) = a) \quad \simeq \quad A$$

is also contractible. The inductive step follows easily from lemma 135. □

Lemma 139. $(\mathcal{H} + \Sigma)$ *For any* $n : \mathbb{N}$ *and* $D : \mathcal{U}_j$ *we have*

$$\big(\Sigma h : (\mathbf{S}^n \to D) \to D\big)\big(\Pi u : \mathbf{S}^n \to D\big)\big(\Pi x : \mathbf{S}^n\big)(u(x) = h(u)) \quad \simeq \quad \mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}(D)$$

Proof. We have the following chain of equivalences:

$$\begin{aligned}
&\big(\Sigma h : (\mathbf{S}^n \to D) \to D\big)\big(\Pi u : \mathbf{S}^n \to D\big)\big(\Pi x : \mathbf{S}^n\big)(u(x) = h(u)) \\
\simeq\ &\big(\Pi u : \mathbf{S}^n \to D\big)\big(\Sigma d : D\big)\big(\Pi x : \mathbf{S}^n\big)(u(x) = d) \\
\equiv\ &\big(\Pi u : \mathbf{S}^n \to D\big)\ \mathsf{HConst}(u) \\
\simeq\ &\mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}(D)
\end{aligned}$$

□

Lemma 140. $(\mathcal{H} + \Sigma)$ *For any* $n : \mathbb{N}$ *and terms*

- $D : \mathcal{U}_j$,
- $h : (\mathbf{S}^n \to D) \to D$,
- $s : \big(\Pi u : \mathbf{S}^n \to D\big)\big(\Pi x : \mathbf{S}^n\big)(u(x) = h(u))$,
- $E : D \to \mathcal{U}_k$,

the type

$$\Big(\Sigma i : \big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi_{x:\mathbf{S}^n} E(r(x))\big) \to E(h(r))\Big)$$
$$\big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi u : \Pi_{x:\mathbf{S}^n} E(r(x))\big)\big(\Pi x : \mathbf{S}^n\big)\big(s(r,x)_*^E\ u(x) = i(r,u)\big)$$

is equivalent to

$$\big(\Pi y : D\big)\ \mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}(E(y))$$

Proof. The former type is clearly equivalent to

$$\big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi u : \Pi_{x:\mathbf{S}^n} E(r(x))\big)\big(\Sigma i : E(h(r))\big)\big(\Pi x : \mathbf{S}^n\big)\big(s(r,x)_*^E\ u(x) = i\big)$$

Furthermore, for any $r : \mathbf{S}^n \to D$, the types $\Pi_{x:\mathbf{S}^n} E(r(x))$ and $\mathbf{S}^n \to E(h(r))$ are equivalent, via the quasi-equivalences

$$u \mapsto \lambda_{x:\mathbf{S}^n}\big(s(r,x)_*^E\ u(x)\big)$$
$$u \mapsto \lambda_{x:\mathbf{S}^n}\big(s(r,x)_E^*\ u(x)\big)$$

Hence we have

$$\begin{aligned}
& \big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi u : \Pi_{x:\mathbf{S}^n} E(r(x))\big)\big(\Sigma i : E(h(r))\big)\big(\Pi x : \mathbf{S}^n\big)\big(s(r,x)_*^E\ u(x) = i\big) \\
\simeq\ & \big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi u : \mathbf{S}^n \to E(h(r))\big)\big(\Sigma i : E(h(r))\big)\big(\Pi x : \mathbf{S}^n\big)\big(u(x) = i\big) \\
\equiv\ & \big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi u : \mathbf{S}^n \to E(h(r))\big)\ \mathsf{HConst}(u) \\
\simeq\ & \big(\Pi r : \mathbf{S}^n \to D\big)\ \mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}\big(E(h(r))\big)
\end{aligned}$$

where the last equivalence follows from corollary 137. Finally, it is not hard to see that

$$\big(\Pi r : \mathbf{S}^n \to D\big)\ \mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}\big(E(h(r))\big) \quad\simeq\quad \big(\Pi y : D\big)\ \mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}(E(y))$$

To show this, we first note that both types are mere propositions. Furthermore, the latter clearly implies the former. To show the converse, take any $y : D$ and define $r(x) :\equiv y$. Then we get $\mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}\big(E(h(r))\big)$. Now we have $\mathsf{spoke}(r, \mathsf{N}) : y = h(r)$, hence $E(y) = E(h(r))$ and in particular $\mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}(E(y))$ as desired. □

Lemma 141. $(\mathcal{H} + \Sigma)$ *For any* $n : \mathbb{N}$ *and terms*

- $D : \mathcal{U}_j$,
- $h : (\mathbf{S}^n \to D) \to D$,
- $s : \big(\Pi u : \mathbf{S}^n \to D\big)\big(\Pi x : \mathbf{S}^n\big)(u(x) = h(u))$,
- $E : D \to \mathcal{U}_k$,
- $i : \big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi_{x:\mathbf{S}^n} E(r(x))\big) \to E(h(r))$,
- $t : \big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi u : \Pi_{x:\mathbf{S}^n} E(r(x))\big)\big(\Pi x : \mathbf{S}^n\big)\big(s(r,x)_*^E\ u(x) = i(r,u)\big)$,
- $f : \Pi_{x:D} E(x)$,

the type

$$\Big(\Sigma\gamma : \big(\Pi r : \mathbf{S}^n \to D\big)\big(f(h(r)) = i(r, f \circ r)\big)\Big)$$
$$\big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi x : \mathbf{S}^n\big)\Big(\mathsf{dap}_f(s(r,x)) = t(r, f \circ r, x) \cdot \gamma(r)^{-1}\Big)$$

is contractible.

Proof. It is not hard to see that the type in question is equivalent to

$$\big(\Pi r : \mathbf{S}^n \to D\big)\big(\Sigma\gamma : f(h(r)) = i(r, f \circ r)\big)\big(\Pi x : \mathbf{S}^n\big)\Big(\mathsf{dap}_f(s(r,x)) = t(r, f \circ r, x) \cdot \gamma^{-1}\Big)$$

which in turn is equivalent to

$$\big(\Pi r : \mathbf{S}^n \to D\big)\big(\Sigma\gamma : f(h(r)) = i(r, f \circ r)\big)\big(\Pi x : \mathbf{S}^n\big)\Big(\mathsf{dap}_f(s(r,x))^{-1} \cdot t(r, f \circ r, x) = \gamma\Big)$$

This is definitionally the same as

$$\big(\Pi r : \mathbf{S}^n \to D\big)\ \mathsf{HConst}\Big((\lambda x : \mathbf{S}^n)\mathsf{dap}_f(s(r,x))^{-1} \cdot t(r, f \circ r, x)\Big)$$

Using lemma 140, we see that the pair (i, t) implies that each fiber of E, and in particular $E(h(r))$, is an $(n-1)$-type. Hence $f(h(r)) = i(r, f \circ r)$ is also an $(n-1)$-type. From lemma 138 we can thus conclude that our type is contractible as desired. ☐

Lemma 142. $(\mathcal{H} + \Sigma)$ *For any* $n : \mathbb{N}$ *and terms*

- $D : \mathcal{U}_j$,
- $h : (\mathbf{S}^n \to D) \to D$,
- $s : \big(\Pi u : \mathbf{S}^n \to D\big)\big(\Pi x : \mathbf{S}^n\big)(u(x) = h(u))$,
- $E : \mathcal{U}_k$,
- $i : (\mathbf{S}^n \to E) \to E$,
- $t : \big(\Pi u : \mathbf{S}^n \to E\big)\big(\Pi x : \mathbf{S}^n\big)(u(x) = i(u))$,
- $f : D \to E$,

the type

$$\Big(\Sigma\gamma : \big(\Pi r : \mathbf{S}^n \to D\big)\big(f(h(r)) = i(f \circ r)\big)\Big)$$
$$\big(\Pi r : \mathbf{S}^n \to D\big)\big(\Pi x : \mathbf{S}^n\big)\Big(\mathsf{ap}_f(s(r,x)) = t(f \circ r, x) \cdot \gamma(r)^{-1}\Big)$$

is contractible.

Proof. Exactly as in the fibered case, we show that the type in question is equivalent to

$$\big(\Pi r : \mathbf{S}^n \to D\big)\ \mathsf{HConst}\Big((\lambda x : \mathbf{S}^n)\mathsf{ap}_f(s(r,x))^{-1} \cdot t(f \circ r, x)\Big)$$

By lemma 139, we see that the pair (i, t) implies E is an $(n-1)$-type. Hence $f(h(r)) = i(f \circ r)$ is also an $(n-1)$-type. From lemma 138 we can thus conclude that our type is contractible as desired. ☐

Lemmas 139, 140, 141, 142 show that the $(n-1)$-truncation of $A : \mathcal{U}_i$ for $n : \mathbb{N}$ can be equivalently presented as a type $||A||_{n-1} : \mathcal{U}_i$ endowed with constructors

$$|\cdot|_{n-1} : A \to ||A||_{n-1}$$
$$\mathsf{tr} : \mathsf{is}\text{-}(n-1)\text{-}\mathsf{type}(||A||_{n-1})$$

such that that given terms

- $E : \mathcal{U}_j$,
- $t : \mathsf{is}\text{-}(n-2)\text{-}\mathsf{type}(E)$,
- $e : A \to E$,

there is a recursor $\mathsf{rec}_{||\cdot||}(E, t, e) : ||A||_{n-1} \to E$ satisfying the computation law

- $\mathsf{rec}_{||\cdot||}(|a|_{n-1}) \equiv e(a)$ for any $a : A$,

and for any terms

- $E : ||A||_{n-1} \to \mathcal{U}_j$,
- $t : \Pi_{y:D}\mathsf{is}\text{-}(n-2)\text{-}\mathsf{type}(E(y))$,
- $e : \Pi_{a:A}E(|a|_{n-1})$,

there is an inductor $\mathsf{ind}_{||\cdot||}(E, t, e) : \Pi_{x:||A||_{n-1}}E(x)$ satisfying the computation law

- $\mathsf{ind}_{||\cdot||}(|a|_{n-1}) \equiv e(a)$ for any $a : A$.

We are now ready to define truncation algebras and morphisms. We note that the definitions presented also subsume the case of -2:

Definition 143. *For $n : \mathbb{N}$, $A : \mathcal{U}_i$, let the type of* truncation algebras *on a universe $\mathcal{U}_j$ be*

$$\mathsf{TrAlg}_{\mathcal{U}_j}(n, A) :\equiv \Sigma_{D:\mathcal{U}_j}\mathsf{is}\text{-}(n-2)\text{-}\mathsf{type}(D) \times (A \to D)$$

Definition 144. *For an algebra $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$, define the type of* fibered truncation algebras *over $\mathcal{X}$ on a universe $\mathcal{U}_k$ by*

$$\mathsf{TrFibAlg}_{\mathcal{U}_k}\ (D, -, d) :\equiv \Sigma_{E:D\to\mathcal{U}_k}\Big(\Pi_{y:D}\mathsf{is}\text{-}(n-2)\text{-}\mathsf{type}(E(y))\Big) \times \big(\Pi_{a:A}E(d(a))\big)$$

Definition 145. *For algebras $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ and $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)$, we define the type of* truncation morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathsf{TrMor}\ (D, -, d)\ (E, -, e) :\equiv \Sigma_{f:D\to E}\Pi_{a:A}(f(d(a)) = e(a))$$

Definition 146. *For algebras $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ and $\mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$, define the type of* fibered truncation morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathsf{TrFibMor}\ (D, -, d)\ (E, -, e) :\equiv \Sigma_{f:(\Pi x:D)E(x)}\Pi_{a:A}(f(d(a)) = e(a))$$

As before, we can define the recursion and induction principles, the associated uniqueness principles, and homotopy-initiality.

Definition 147. *An algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ satisfies the recursion principle *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}$ *there exists a morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasTrRec}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)\Big)\ \mathsf{TrMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 148. *An algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ satisfies the induction principle *on a universe* $\mathcal{U}_k$ *if for any fibered algebra* $\mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *there exists a fibered morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasTrInd}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\Big)\ \mathsf{TrFibMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 149. *An algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ satisfies the recursion uniqueness principle *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is a mere proposition:*

$$\mathsf{hasTrRecUniq}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)\Big)\ \mathsf{isProp}(\mathsf{TrMor}\ \mathcal{X}\ \mathcal{Y})$$

Definition 150. *An algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ satisfies the induction uniqueness principle *on a universe* $\mathcal{U}_k$ *if for any fibered algebra* $\mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is a mere proposition:*

$$\mathsf{hasTrIndUniq}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\Big)\ \mathsf{isProp}(\mathsf{TrFibMor}\ \mathcal{X}\ \mathcal{Y})$$

Definition 151. *An algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ *is* homotopy-initial *on a universe* $\mathcal{U}_k$ *if for any other algebra* $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(A, n)$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is contractible:*

$$\mathsf{isTrHInit}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)\Big)\ \mathsf{isContr}(\mathsf{TrMor}\ \mathcal{X}\ \mathcal{Y})$$

4.1.2 Homotopy-initiality for Truncations

We aim to show the following analogue to our main theorem for W-quotients:

Theorem 152. *($\mathcal{H}$) For $n : \mathbb{N}$, $A : \mathcal{U}_i$, the following conditions on an algebra $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ are equivalent:*

- *$\mathcal{X}$ satisfies the induction principle on the universe $\mathcal{U}_k$*
- *$\mathcal{X}$ is homotopy-initial on the universe $\mathcal{U}_k$*

for $k \geq j$. In other words, we have

$$\mathsf{hasTrInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \simeq \quad \mathsf{isTrHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. Moreover, the two types above are mere propositions.

In principle, one could recover the characterization of truncations as homotopy-initial algebras from the recently-discovered conjectured reduction of truncations to type quotients, due to Rijke and van Doorn [36], and our main theorem for W-quotients. However, at this point this is not needed due to the work done in the previous section, where we "polished up" the definitions of truncation algebras and morphisms to the degree that most of the proofs are now straightforward.

Lemma 153. *For an algebra $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ we have*

$$\mathsf{isTrHInit}_{\mathcal{U}_k}(\mathcal{X}) \quad \simeq \quad \mathsf{hasTrRec}_{\mathcal{U}_k}(\mathcal{X}) \times \mathsf{hasTrRecUniq}_{\mathcal{U}_k}(\mathcal{X})$$

Lemma 154. *For an algebra $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ we have a function*

$$\mathsf{TrAlgToFibAlg}_{\mathcal{U}_k}(\mathcal{X}) \quad : \quad \mathsf{TrAlg}_{\mathcal{U}_k}(n, A) \quad \to \quad \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$$

Proof. Fix algebras $(D, p, d) : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ and $(E, q, e) : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)$. We turn (E, q, e) into the desired fibered algebra $(E', q', e') : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ (D, p, d)$ in the expected way by defining $E'(x) :\equiv E$, $q'(y) :\equiv q$, $e'(a) :\equiv e(a)$. □

Remark 155. *We note that for any algebras $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ and $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)$ we have*

$$\mathsf{TrMor}\ \mathcal{X}\ \mathcal{Y} \quad \equiv \quad \mathsf{TrFibMor}\ \mathcal{X}\ \Big(\mathsf{TrAlgToFibAlg}_{\mathcal{U}_k}(\mathcal{X})\ \mathcal{Y}\Big)$$

Lemma 156. *For any algebra $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ we have*

$$\mathsf{hasTrInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasTrRec}_{\mathcal{U}_k}(\mathcal{X})$$

Lemma 157. *For any algebra $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ we have*

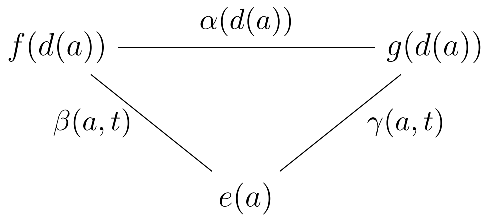
$$\mathsf{hasTrIndUniq}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasTrRecUniq}_{\mathcal{U}_k}(\mathcal{X})$$

The notion of a truncation cell is now particularly simple, since the morphisms have no higher dimensional computation rules:

Definition 158. *For algebras* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$*,* $\mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}(n, A)\ \mathcal{X}$*, and fibered morphisms* $\mu, \nu : \mathsf{TrFibMor}\ \mathcal{X}\ \mathcal{Y}$*, define the type of* fibered truncation cells *from* μ *to* ν *by*

$$\mathsf{TrFibCell}\ (D, p, d)\ (E, q, e)\ (f, \beta)\ (g, \gamma) :\equiv \Sigma_{\alpha : f \sim g} \Pi_{a:A} \Big(\alpha(d(a)) = \beta(a) \cdot \gamma(a)^{-1} \Big)$$

Pictorially, the second component of a fibered truncation cell witnesses the commutativity of the following diagram for any a:



Definition 159. *For algebras* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$*,* $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)$*, and morphisms* $\mu, \nu : \mathsf{TrMor}\ \mathcal{X}\ \mathcal{Y}$*, define the type of* $(n-1)$-truncation cells *between* μ *and* ν *by*

$$\mathsf{TrCell}\ \mathcal{X}\ \mathcal{Y}\ \mu\ \nu :\equiv \mathsf{TrFibCell}\ \mathcal{X}\ \Big(\mathsf{TrAlgToFibAlg}_{\mathcal{U}_k}\ \mathcal{Y}\Big)\ \mu\ \nu$$

Lemma 160. $(\mathcal{H})$ *For algebras* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$*,* $\mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$*, and fibered morphisms* $\mu, \nu : \mathsf{TrFibMor}\ \mathcal{X}\ \mathcal{Y}$*, we have an equivalence*

$$(\mu = \nu) \ \simeq \ \mathsf{TrFibCell}\ \mu\ \nu$$

Proof. Let algebras $(D, p, d) : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ and $(E, q, e) : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ (D, p, d)$ and fibered morphisms $(f, \beta), (g, \gamma) : \mathsf{TrFibMor}\ (D, p, d)\ (E, q, e)$ be given. We establish the following chain of equivalences:

$$\begin{aligned}
& (f, \beta) = (g, \gamma) \\
\simeq\ & \Big(\Sigma \alpha : f = g\Big)\ (\alpha)_*^{h \mapsto (\Pi a:A)(h(d(a)) = e(a))}(\beta) = \gamma \\
\simeq\ & \Big(\Sigma \alpha : f = g\Big)\ \Pi_{a:A} \Big(\big({}^{=}\mathsf{E}^{\Pi}(\alpha)\big)(d(a)) = \beta(a) \cdot \gamma(a)^{-1} \Big) \\
\simeq\ & \Big(\Sigma \alpha : f \sim g\Big)\ \Pi_{a:A} \Big(\alpha(d(a)) = \beta(a) \cdot \gamma(a)^{-1} \Big) \\
\equiv\ & \mathsf{TrFibCell}\ (f, \beta)\ (g, \gamma)
\end{aligned}$$

The first equivalence follows by the characterization of paths in dependent product spaces. The second equivalence follows by induction on α and function extensionality. Finally, the third equivalence follows from the fact that the map ${}^{=}\mathsf{E}^{\Pi} : (f = g) \to (f \sim g)$ is itself an equivalence. □

Corollary 161. $(\mathcal{H})$ *Given algebras* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$*,* $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(n, A)$*, and morphisms* $\mu, \nu : \mathsf{TrMor}\ \mathcal{X}\ \mathcal{Y}$*, we have an equivalence*

$$(\mu = \nu) \ \simeq \ \mathsf{TrCell}\ \mu\ \nu$$

Lemma 162. ($\mathcal{H}$) *For any algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ *we have*

$$\mathsf{hasTrInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasTrIndUniq}_{\mathcal{U}_k}(\mathcal{X})$$

Proof. Fix an algebra $(D, p, d) : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ and assume that $\mathsf{hasTrInd}_{\mathcal{U}_k}(D, p, d)$ holds. To show that $\mathsf{hasTrIndUniq}_{\mathcal{U}_k}(D, p, d)$ holds, take a fibered algebra $(E, q, e) : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ (D, p, d)$ and fibered morphisms $(f, \beta), (g, \gamma) : \mathsf{TrFibMor}\ (D, p, d)\ (E, q, e)$. By lemma 160, to show $(f, \beta) = (g, \gamma)$ it suffices to exhibit a fibered truncation cell between (f, β) and (g, γ).

To do so, we use the induction principle with an appropriate fibered algebra $(E', q', e') : \mathsf{TrFibAlg}\ (D, p, d)$. To this end, we put $E' :\equiv x \mapsto f(x) = g(x)$, which clearly still belongs to $\mathcal{U}_k$ fiberwise. For the third component, we put $e'(a) :\equiv \beta(a) \cdot \gamma(a)^{-1}$. Finally, to construct q' we need to show that each fiber of E', namely $f(x) =_{E(x)} g(x)$ for $x : D$, is an $(n-2)$-type. It thus suffices to show $E(x)$ is an $(n-2)$-type; but this is exactly assumption $q(x)$.

The induction principle gives us a fibered morphism $(\alpha, \eta) : \mathsf{TrFibMor}\ (D, p, d)\ (E', q', e')$, which is exactly our desired truncation cell between (f, β) and (g, γ). □

Corollary 163. ($\mathcal{H}$) *For any algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ *we have*

$$\mathsf{hasTrInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{isTrHInit}_{\mathcal{U}_k}(\mathcal{X})$$

Lemma 164. *For any algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ *we have*

$$\mathsf{hasTrRec}_{\mathcal{U}_k}(\mathcal{X}) \times \mathsf{hasTrRecUniq}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasTrInd}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$.

Proof. Let an algebra $(D, p, d) : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ be given and assume that $\mathsf{hasTrRec}_{\mathcal{U}_k}(D, p, d)$ and $\mathsf{hasTrRecUniq}_{\mathcal{U}_k}(D, p, d)$ hold. To show that $\mathsf{hasTrInd}_{\mathcal{U}_k}(D, p, d)$ holds, fix any fibered algebra $(E, q, e) : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ (D, p, d)$. In order to apply the recursion principle, we need to turn this into a non-fibered algebra (E', q', e'). We put $E' :\equiv \Sigma_{x:D} E(x)$; we note that since $D : \mathcal{U}_j$, $E : D \to \mathcal{U}_k$, and $j \leq k$, E' belongs to $\mathcal{U}_k$ as needed. For the third component, we put $e'(a) :\equiv (a, e(a))$. Finally, to construct q' we need to show that $\Sigma_{x:D} E(x)$ is an $(n-2)$-type. But we have that D is an $(n-2)$-type by the assumption p and for any x, $E(x)$ is an $(n-2)$-type by the assumption $q(x)$. Since taking a Σ of a family of $(n-2)$-types over an $(n-2)$-type results again in an $(n-2)$-type, we are done.

The recursion principle then gives us a morphism $(f, \beta) : \mathsf{TrMor}\ (D, p, d)\ (E', q', e')$, where $f : D \to \Sigma_{x:D} E(x)$ and $\beta(a) : f(d(a)) = (a, e(a))$.

We now want to show that the function $\pi_1 \circ f : D \to D$ is in fact the identity on D (up to a homotopy, of course). We can do this by endowing both of the functions $\pi_1 \circ f$ and id_D with a morphism structure on the algebra (D, p, d); by the recursion uniqueness principle it will follow that these morphisms are equal, and in particular they are equal as maps.

We turn the identity map id_D into the morphism $\big(\mathsf{id}_D, a \mapsto 1_{d(a)}\big)$ and the composition $\pi_1 \circ f$ into the morphism $\big(\pi_1 \circ f, a \mapsto \pi_1^{=}(\beta(a))\big)$. By the recursion uniqueness rule, these morphisms are equal and by corollary 161, there exists a truncation cell (α, η) between them, where $\alpha : \pi_1 \circ f \sim \mathsf{id}_D$ and $\eta(a) : \alpha(d(a)) = \pi_1^{=}(\beta(a)) \cdot 1_{d(a)}$.

Our desired fibered homomorphism $(f_D, \beta_D) : \mathsf{TrFibMor}\ (D, p, d)\ (E, q, e)$ is now constructed by putting $f_D :\equiv \alpha \circ_{\mathcal{H}} (\pi_2 \circ f)$ and defining $\beta_D(a)$ as the path

$$
\begin{array}{c}
\alpha(d(a))_*^E\ \pi_2\big(f(d(a))\big) \\
\Big|\ \textit{via}\ \mathcal{B}(a) \\
\pi_1^=(\beta(a))_*^E\ \pi_2\big(f(d(a,t))\big) \\
\Big|\ \pi_2^=(\beta(a)) \\
e(a)
\end{array}
$$

where $\mathcal{B}(a,t)$ is the path

$$\alpha(d(a)) \overset{\eta(a)}{\text{———}} \pi_1^=(\beta(a,t)) \cdot 1_{d(a)} \text{———} \pi_1^=(\beta(a,t))$$

□

Corollary 165. $(\mathcal{H})$ *For any algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ *we have*

$$\mathsf{isTrHInit}_{\mathcal{U}_k}(\mathcal{X}) \quad \to \quad \mathsf{hasTrInd}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$.

Corollary 166. $(\mathcal{H})$ *For* $n : \mathbb{N}$, $A : \mathcal{U}_i$, *the following conditions on an algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_k$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_k$

for $k \geq j$. *In other words, we have*

$$\mathsf{hasTrInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \simeq \quad \mathsf{isTrHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. *Moreover, the two types above are mere propositions.*

Proof. Exactly as in the proof of 107. □

Corollary 167. $(\mathcal{H} + ||\cdot||)$ *For* $n : \mathbb{N}$, $A : \mathcal{U}_i$, *the algebra*

$$\Big(||A||_{n-2}, p, |\cdot|\Big) : \mathsf{TrAlg}_{\mathcal{U}_i}(n, A)$$

is homotopy-initial for any p *on any universe* $\mathcal{U}_j$.

Finally, we note that homotopy-initial truncation algebras satisfy the following universal properties, whose proofs are simple exercises:

Lemma 168. $(\mathcal{H})$ *For any algebra* $\mathcal{X} :\equiv (D, -, |\cdot|_D) : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ *and type* $E : \mathcal{U}_k$, *if* $\mathcal{X}$ *is homotopy-initial on* $\mathcal{U}_k$ *and* E *is an* $(n-2)$*-type, then we have*

$$D \to E \quad \simeq \quad A \to E$$

provided $k \geq j$.

Proof. The quasi-inverse from left to right is given by precomposition with $|\cdot|_D$; the quasi-inverse in the opposite direction is given by the recursion principle. $\square$

Lemma 169. $(\mathcal{H})$ *For an algebra* $\mathcal{X} :\equiv (D, -, |\cdot|_D) : \mathsf{TrAlg}_{\mathcal{U}_j}(n, A)$ *and type family* $E : D \to \mathcal{U}_k$*, if* $\mathcal{X}$ *is homotopy-initial on* $\mathcal{U}_k$ *and each fiber of* E *is an* $(n-2)$*-type, then we have*

$$\Pi_{x:D} E(x) \;\; \simeq \;\; \Pi_{a:A} E(|a|_D)$$

provided $k \geq j$.

Proof. The quasi-inverse from left to right is given by precomposition with $|\cdot|_D$; the quasi-inverse in the opposite direction is given by the induction principle. $\square$

4.2 Homotopy-initiality for Set Quotients

4.2.1 Set Quotients

In mathematics, we very often need to take the quotient of a set by an equivalence relation, *e.g.*, when constructing the rational numbers from the integers. The notion of an equivalence relation in the type-theoretic setting is straightforward; we start off with the following definition.

Definition 170. *For a general type-valued relation $R : A \to A \to \mathcal{U}_i$ on $A : \mathcal{U}_i$ we define*

$$\begin{aligned}\mathsf{id}(R) &:\equiv \Pi_{a:A} R(a,a)\\ \mathsf{inv}(R) &:\equiv \Pi_{a,b:A} R(a,b) \to R(b,a)\\ \mathsf{comp}(R) &:\equiv \Pi_{a,b,c:A} (R(a,b) \times R(b,c)) \to R(a,c)\end{aligned}$$

In the special case when $R(a,b)$ is a mere proposition for all $a, b : A$, the above types can be understood as statements of reflexivity, symmetry, and transitivity of R. Thus, we have the following:

Definition 171. *For $A : \mathcal{U}_i$, we define the type of* equivalence relations *on A as*

$$\mathsf{EqRel}(A) :\equiv \Sigma_{R:A\to A\to\mathcal{U}_i} \big(\Pi_{a,b:A} \mathsf{isProp}(R(a,b))\big) \times \mathsf{id}(R) \times \mathsf{inv}(R) \times \mathsf{comp}(R)$$

We note that any equivalence relation on A can be thought of as specifying the path structure of a 0-type, also known as a *set*, where the 0-cells are induced by the points of A and the 1-cells with source $a : A$ and target $b : A$ are induced by the terms of $R(a,b)$. The notion of a *set quotient* is meant to capture this intuition. We follow the definition in [33], chapter 6.10, except we do not require the type $A : \mathcal{U}_i$ itself to be a set (as mentioned in remark 6.10.1, this is unnecessary for the general theory of set quotients).

Formally, given $A : \mathcal{U}_i$ and an equivalence relation $\mathbf{R} :\equiv (R, -, -, -, -) : \mathsf{EqRel}(A)$, the set quotient $A/_0\mathbf{R}$ of A by $\mathbf{R}$ is the higher inductive type generated by the following constructors:

$$\begin{aligned}\mathsf{point}_0 &: A \to A/_0\mathbf{R}\\ \mathsf{cell}_0 &: \Pi_{a,b:A} R(a,b) \to (\mathsf{point}_0(a) = \mathsf{point}_0(b))\\ \mathsf{hub} &: \big(\mathbf{S}^1 \to A/_0\mathbf{R}\big) \to A/_0\mathbf{R}\\ \mathsf{spoke} &: \big(\Pi r : \mathbf{S}^1 \to A/_0\mathbf{R}\big)\big(\Pi x : \mathbf{S}^1\big)(r(x) = \mathsf{hub}(r))\end{aligned}$$

The constructor point_0 can be understood as taking $a : A$ to its equivalence class under $\mathbf{R}$. The constructor cell_0 is meant to identify the two equivalence classes $\mathsf{point}_0(a)$ and $\mathsf{point}_0(b)$ whenever we have $R(a,b)$. Finally, as it was in the case of truncations, the purpose of the constructors hub and spoke is to ensure that the resulting set quotient is a 0-type.

How do we know that this definition of set a quotient is indeed correct? In mathematics, the most important property of a quotient by an equivalence relation is the fact that two elements belong to the same equivalence class if and only if they are related; in our setting, this means $(\mathsf{point}_0(a) = \mathsf{point}_0(b)) \simeq R(a,b)$ for all $a, b : A$, a property also known as *effectiveness*. This is lemma 10.1.8 of [33]; we will show this in a more general setting at the end of this section.

The recursion principle for quotients says that given terms

- $E : \mathcal{U}_j$,
- $e : A \to E$,
- $p : \Pi_{a,b:A} R(a,b) \to (e(x) = e(y))$,
- $h : (\mathbf{S}^1 \to E) \to E$,
- $s : \Pi_{u:\mathbf{S}^1 \to E} \Pi_{x:\mathbf{S}^1} (u(x) = h(u))$,

there is a recursor $\mathsf{rec}_{\cdot/_0\cdot}(E, e, p, h, s) : A/_0\mathbf{R} \to E$. The recursor satisfies the computation laws

- $\mathsf{rec}_{\cdot/_0\cdot}(\mathsf{point}_0(a)) \equiv e(a)$ for any $a : A$,
- $\mathsf{ap}_{\mathsf{rec}_{\cdot/_0\cdot}}(\mathsf{cell}_0(z)) = p(z)$ for any $a : A, b : A, z : R(a,b)$,
- $\mathsf{rec}_{\cdot/_0\cdot}(\mathsf{hub}(r)) = h\big(\mathsf{rec}_{\cdot/_0\cdot} \circ r\big)$ for any $r : \mathbf{S}^1 \to A/_0\mathbf{R}$,
- $\mathsf{ap}_{\mathsf{rec}_{\cdot/_0\cdot}}(\mathsf{spoke}(r,x))$ is equal to the path below for any $r : \mathbf{S}^1 \to A/_0\mathbf{R}$ and $x : \mathbf{S}^1$, where equality (1) uses the computation rule for the hub constructor:

$$\mathsf{rec}_{\cdot/_0\cdot}(r(x)) \xrightarrow{\;s\left(\mathsf{rec}_{\cdot/_0\cdot} \circ r,\, x\right)\;} h\big(\mathsf{rec}_{\cdot/_0\cdot}(s) \circ r\big) \xrightarrow{\;(1)\;} \mathsf{rec}_{\cdot/_0\cdot}(\mathsf{hub}(r))$$

Similarly, we have an induction principle: given terms

- $E : A/_0\mathbf{R} \to \mathcal{U}_j$,
- $e : \Pi_{a:A} E(\mathsf{point}_0(a))$,
- $p : \Pi_{a,b:A} \Pi_{z:R(a,b)} \big(\mathsf{cell}_0(z)_*^E\ e(a) = e(b)\big)$,
- $h : \big(\Pi r : \mathbf{S}^1 \to A/_0\mathbf{R}\big)\big(\Pi_{x:\mathbf{S}^1} E(r(x))\big) \to E(\mathsf{hub}(r))$,
- $s : \big(\Pi r : \mathbf{S}^1 \to A/_0\mathbf{R}\big)\big(\Pi u : \Pi_{x:\mathbf{S}^1} E(r(x))\big)\big(\Pi x : \mathbf{S}^1\big)\big(\mathsf{spoke}(r,x)_*^E\ u(x) = h(r,u)\big)$,

there is an inductor $\mathsf{ind}_{\cdot/_0\cdot}(E, e, p, h, s) : \Pi_{x:A/_0\mathbf{R}} E(x)$. The inductor satisfies the following computation laws:

- $\mathsf{ind}_{\cdot/_0\cdot}(\mathsf{point}_0(a)) \equiv e(a)$ for any $a : A$,
- $\mathsf{dap}_{\mathsf{ind}_{\cdot/_0\cdot}}(\mathsf{cell}_0(z)) = p(z)$ for any $a : A, b : A, z : R(a,b)$,
- $\mathsf{ind}_{\cdot/_0\cdot}(\mathsf{hub}(r)) = h\big(r, \mathsf{ind}_{\cdot/_0\cdot} \circ r\big)$ for any $r : \mathbf{S}^1 \to A/_0\mathbf{R}$,
- $\mathsf{dap}_{\mathsf{ind}_{\cdot/_0\cdot}}(\mathsf{spoke}(r,x))$ is equal to the path below for any $r : \mathbf{S}^1 \to A/_0\mathbf{R}$ and $x : \mathbf{S}^1$, where equality (1) uses the computation rule for the hub constructor:

$$\mathsf{spoke}(r,x)_*^E\ \big(\mathsf{ind}_{\cdot/_0\cdot}(r(x))\big) \xrightarrow{\;s\left(r,\, \mathsf{ind}_{\cdot/_0\cdot} \circ r,\, x\right)\;} h\big(r, \mathsf{ind}_{\cdot/_0\cdot} \circ r\big) \xrightarrow{\;(1)\;} \mathsf{ind}_{\cdot/_0\cdot}(\mathsf{hub}(r))$$

At this point we note that all but the first computation rules are unnecessary, both in the recursive and the inductive case: as in the case of truncations, the hypotheses h and s are equivalent to asserting that E is (fiberwise) a 0-type (lemmas 139 and 140); the computation laws for cell_0 are then redundant by lemma 3 and the laws for hub and spoke were redundant to begin with, by lemmas 142 and 141.

Hence, the set quotient of $A : \mathcal{U}_i$ by an equivalence relation $\mathbf{R} :\equiv (R, \ldots) : \mathsf{EqRel}(A)$ can be equivalently presented as a type $A/_0\mathbf{R} : \mathcal{U}_i$ endowed with constructors

$$\begin{aligned} \mathsf{point}_0 &: A \to A/_0\mathbf{R} \\ \mathsf{cell}_0 &: \Pi_{a,b:A} R(a, b) \to (\mathsf{point}_0(a) = \mathsf{point}_0(b)) \\ \mathsf{tr} &: \mathsf{isSet}(A/_0\mathbf{R}) \end{aligned}$$

where we use the abbreviation $\mathsf{isSet}(X) :\equiv \textsf{is-0-type}(X)$, such that given terms

- $E : \mathcal{U}_j$,
- $t : \mathsf{isSet}(E)$,
- $e : A \to E$,
- $p : \Pi_{a,b:A} R(a, b) \to (e(x) = e(y))$,

there is a recursor $\mathsf{rec}_{\cdot/_0\cdot}(E, t, e, p) : A/_0\mathbf{R} \to E$ satisfying the computation law

- $\mathsf{rec}_{\cdot/_0\cdot}(\mathsf{point}_0(a)) \equiv e(a)$ for any $a : A$,

and for any terms

- $E : A/_0\mathbf{R} \to \mathcal{U}_j$,
- $t : \Pi_{y:D}\mathsf{isSet}(E(y))$,
- $e : \Pi_{a:A} E(\mathsf{point}_0(a))$,
- $p : \Pi_{a,b:A}\Pi_{z:R(a,b)}\big(\mathsf{cell}_0(z)_*^E\ e(a) = e(b)\big)$,

there is an inductor $\mathsf{ind}_{\cdot/_0\cdot}(E, t, e, p) : \Pi_{x:A/_0\mathbf{R}} E(x)$ satisfying the computation law

- $\mathsf{ind}_{\cdot/_0\cdot}(\mathsf{point}_0(a)) \equiv e(a)$ for any $a : A$.

Definition 172. *For $A : \mathcal{U}_i$, $\mathbf{R} :\equiv (R, \ldots) : \mathsf{EqRel}(A)$, we define the type of* set quotient algebras *on a universe $\mathcal{U}_j$ be*

$$\mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R}) :\equiv \Sigma_{D:\mathcal{U}_j}\mathsf{isSet}(D) \times \Big(\Sigma_{e:A\to D}\Pi_{a,b:A} R(a, b) \to (e(x) = e(y))\Big)$$

Definition 173. *For an algebra $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$ with $\mathbf{R} :\equiv (R, \ldots)$, define the type of* fibered set quotient algebras *over $\mathcal{X}$ on a universe $\mathcal{U}_k$ by*

$$\begin{aligned} \mathsf{SQFibAlg}_{\mathcal{U}_k}\ (D, -, d, p) :\equiv \Sigma_{E:D\to\mathcal{U}_k}\Big(\Pi_{y:D}\mathsf{isSet}(E(y))\Big) \times \\ \Big(\Sigma_{e:(\Pi a:A)E(d(a))}\Pi_{a,b:A}\Pi_{z:R(a,b)}\big(p(a, b, z)_*^E\ e(a) = e(b)\big)\Big) \end{aligned}$$

Definition 174. *For algebras $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$ and $\mathcal{Y} : \mathsf{SQAlg}_{\mathcal{U}_k}(A, \mathbf{R})$, we define the type of* set quotient morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathsf{SQMor}\ (D, -, d, -)\ (E, -, e, -) :\equiv \Sigma_{f:D\to E}\Pi_{a:A}\big(f(d(a)) = e(a)\big)$$

Definition 175. *For algebras $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$ and $\mathcal{Y} : \mathsf{SQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$, we define the type of* fibered set quotient morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathsf{SQFibMor}\ (D, -, d, -)\ (E, -, e, -) :\equiv \Sigma_{f:(\Pi x:D)E(x)}\Pi_{a:A}\big(f(d(a)) = e(a)\big)$$

Definition 176. *An algebra* $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$ satisfies the recursion principle *on a universe* $\mathcal{U}_k$ *if for any algebra* $\mathcal{Y} : \mathsf{SQAlg}_{\mathcal{U}_k}(A, \mathbf{R})$ *there exists a morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasSQRec}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R}\Big)\ \mathsf{SQMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 177. *An algebra* $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$ satisfies the induction principle *on a universe* $\mathcal{U}_k$ *if for any fibered algebra* $\mathcal{Y} : \mathsf{SQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *there exists a fibered morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$*:*

$$\mathsf{hasSQInd}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{SQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\Big)\ \mathsf{SQFibMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 178. *An algebra* $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$ *is* homotopy-initial *on a universe* $\mathcal{U}_k$ *if for any other algebra* $\mathcal{Y} : \mathsf{SQAlg}_{\mathcal{U}_k}(A, \mathbf{R})$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is contractible:*

$$\mathsf{isSQHInit}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R}\Big)\ \mathsf{isContr}(\mathsf{SQMor}\ \mathcal{X}\ \mathcal{Y})$$

4.2.2 Homotopy-initiality for Set Quotients

We aim to show the following analogue to our main theorem for W-quotients:

Theorem 179. $(\mathcal{H} + /)$ *For* $A : \mathcal{U}_i$, $\mathbf{R} : \mathsf{EqRel}(A)$, *the following conditions on an algebra* $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_k$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_k$

for $k \geq j$. *In other words, we have*

$$\mathsf{hasSQInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \simeq \quad \mathsf{isSQHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. *Moreover, the two types above are mere propositions.*

Fix $k \geq j$, type $A : \mathcal{U}_i$, and an equivalence relation $\mathbf{R} :\equiv (R, \ldots)$. We aim to establish the above theorem by encoding each $\mathcal{X}$ as a 0-truncation of a suitable type quotient algebra and invoking theorem 152. Let us fix an algebra $(T, \mathsf{p}_T, \mathsf{c}_T) : \mathsf{TQAlg}_{\mathcal{U}_i}(A, R)$ which satisfies the induction principle on $\mathcal{U}_k$. We now show that set quotient algebras are really the same thing as 0-truncation algebras over the type quotient T. We note that we need to use the parameter $n :\equiv 2$ to obtain a 0-truncation.

Lemma 180. $(\mathcal{H} + /)$ *We have a function*

$$\mathsf{TrToSQAlg} \quad : \quad \mathsf{TrAlg}_{\mathcal{U}_j}(2, T) \quad \to \quad \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$$

which is an equivalence.

Proof. We proceed in four steps:

Step 1 First we define TrToSQAlg; for this, take an algebra $(X, t_X, |\cdot|_X) : \mathsf{TrAlg}_{\mathcal{U}_j}(2, T)$. To construct a set quotient algebra, we can use the same underlying type X, with t_X showing it is a 0-type as desired. For the third component, we need a function $\mathsf{p}_X : A \to X$. The only possibility we have is to define $\mathsf{p}_X(a) :\equiv |\mathsf{p}_T(a)|_X$. To obtain the fourth component, we need a function c_X mapping each $a, b : A$, $z : R(a, b)$ to a path from $|\mathsf{p}_T(a)|_X$ to $|\mathsf{p}_T(b)|_X$. Again, the only choice we have is to define $\mathsf{c}_X(z) :\equiv \mathsf{ap}_{|-|_X}(\mathsf{c}_T(z))$.

Step 2 Define the intended quasi-inverse SQToTrAlg; for this, take an algebra $(X, t_X, \mathsf{p}_X, \mathsf{c}_X) : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$. To construct a truncation algebra, we can use the same underlying type X, with t_X showing it is a 0-type as desired. To obtain the last component, we need a function $T\text{-to-}X : T \to X$. We proceed by recursion (we can do this since $j \leq k$), mapping $\mathsf{p}_T(a) \mapsto \mathsf{p}_X(a)$ for any $a : A$ and $\mathsf{c}_T(z) \mapsto \mathsf{c}_X(z)$ for any $a, b : A$, $z : R(a, b)$. The first computation rule then gives us a family of paths $\beta_{T\text{-to-}X}(a) : T\text{-to-}X(\mathsf{p}_T(a)) = \mathsf{p}_X(a)$ for $a : A$.

Step 3 We now want to show that for any set quotient algebra $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$, we have $\mathsf{TrToSQAlg}(\mathsf{SQToTrAlg}(\mathcal{X})) = \mathcal{X}$. Let such an algebra $(X, t_X, \mathsf{p}_X, \mathsf{c}_X)$ be given. The first and second components of $\mathsf{TrToSQAlg}\big(\mathsf{SQToTrAlg}(X, t_X, \mathsf{p}_X, \mathsf{c}_X)\big)$ are X and t_X themselves. The third component is the map $a \mapsto T\text{-to-}X(\mathsf{p}_T(a))$. Since the type of the fourth component is a mere proposition, all we need is a path γ equating the third component with p_X. But we can just put $\gamma :\equiv {}^{\Pi}\mathsf{E}^{=}(\beta_{T\text{-to-}X})$.

Step 4 Finally, we want to show that for any truncation algebra $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(2, T)$, we have $\mathsf{SQToTrAlg}(\mathsf{TrToSQAlg}(\mathcal{X})) = \mathcal{X}$. Let such an algebra $(X, t_X, |\cdot|_X)$ be given. The first and second components of $\mathsf{SQToTrAlg}\big(\mathsf{TrToSQAlg}(X, t_X, |\cdot|_X)\big)$ are X and t_X themselves. The third component is the map T-to-X. We thus need to show that for any $t : T$, we have $T\text{-to-}X(t) = |t|_X$. We proceed by induction, mapping $\mathsf{p}_T(a) \mapsto \beta_{T\text{-to-}X}(a)$ for any $a : A$. Since the type $T\text{-to-}X(t) = |t|_X$ is a mere proposition for any t, we do not have to provide a mapping for $\mathsf{c}_T(z)$, which means we are done. □

Lemma 181. $(\mathcal{H} + /)$ *For an algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(2, T)$ *we have a function*

$$\mathsf{TrToSQFibAlg} \;:\; \mathsf{TrFibAlg}_{\mathcal{U}_k}\, \mathcal{X} \;\to\; \mathsf{SQFibAlg}_{\mathcal{U}_k}\Big(\mathsf{TrToSQAlg}\; \mathcal{X}\Big)$$

which is an equivalence.

Proof. Fix an algebra $\mathcal{X} :\equiv (X, t_X, |\cdot|_X) : \mathsf{TrAlg}_{\mathcal{U}_j}(2, T)$. We recall that $\mathsf{TrToSQAlg}\ \mathcal{X}$ is the algebra $(X, t_X, \mathsf{p}_X, \mathsf{c}_X)$, where

$$\begin{aligned} \mathsf{p}_X(a) &:\equiv |\mathsf{p}_T(a)|_X && \text{for } a : A \\ \mathsf{c}_X(z) &:\equiv \mathsf{ap}_{|-|_X}(\mathsf{c}_T(z)) && \text{for } a, b : A,\ z : R(a, b) \end{aligned}$$

We now proceed in four steps:

Step 1 First we define $\mathsf{TrToSQFibAlg}$; for this, take an algebra $(E, t_E, |\cdot|_E) : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$. To construct a fibered set quotient algebra, we can use the same underlying type family E, with t_E showing it is fiberwise a 0-type as desired. For the third component, we need a function $\mathsf{p}_E : \Pi_{a:A} E(|\mathsf{p}_T(a)|_X)$. The only possibility we have is to define $\mathsf{p}_E(a) :\equiv |\mathsf{p}_T(a)|_E$. For the fourth component, we need a function c_E mapping each $a, b : A$ and $z : R(a, b)$ to a path from $\Big(\mathsf{ap}_{|-|_X}(\mathsf{c}_T(z))\Big)^E_* |\mathsf{p}_T(a)|_E$ to $|\mathsf{p}_T(b)|_E$. Again, the only reasonable choice we have is to define $\mathsf{c}_E(z)$ to be the following path:

$$\begin{aligned} &\Big(\mathsf{ap}_{|-|_X}(\mathsf{c}_T(z))\Big)^E_*\ |\mathsf{p}_T(a)|_E \\ &= \mathsf{c}_T(z)^{E \circ |-|_X}_*\ |\mathsf{p}_T(a)|_E \\ &= |\mathsf{p}_T(b)|_E \quad \text{by } \mathsf{dap}_{|-|_E}(\mathsf{c}_T(z)) \end{aligned}$$

Step 2 Define the intended quasi-inverse $\mathsf{SQToTrFibAlg}$; for this, fix algebra $(E, t_E, \mathsf{p}_E, \mathsf{c}_E) : \mathsf{SQFibAlg}_{\mathcal{U}_k}\ \big(\mathsf{TrToSQAlg}\ \mathcal{X}\big)$. To construct a fibered truncation algebra, we can use the same underlying type family E, with t_E showing it is fiberwise a 0-type as desired. To obtain the last component, we need a function $T\text{-to-}E : \Pi_{t:T} E(|t|_X)$. We proceed by induction, mapping $\mathsf{p}_T(a) \mapsto \mathsf{p}_E(a)$ for any $a : A$ and $\mathsf{c}_T(z)$ to the path below for any $a, b : A$, $z : R(a, b)$:

$$\begin{array}{c} \mathsf{c}_T(z)_*^{E \circ |-|_X}\ \mathsf{p}_E(a) \\ \Big| \\ \Big(\mathsf{ap}_{|-|_X}(\mathsf{c}_T(z))\Big)_*^{E}\ \mathsf{p}_E(a) \\ \Big|\ \mathsf{c}_E(z) \\ |b|_E \end{array}$$

The first computation rule then gives us a family of paths $\beta_{T\text{-to-}E}(a) : T\text{-to-}E(\mathsf{p}_T(a)) = \mathsf{p}_E(a)$ for $a : A$.

Steps 3 + 4 These are entirely analogous to the non-fibered case. □

Lemma 182. $(\mathcal{H} + /)$ *For algebras* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(2, T)$ *and* $\mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *we have*

$$\mathsf{TrFibMor}\ \mathcal{X}\ \mathcal{Y} \ \simeq\ \mathsf{SQFibMor}\ \Big(\mathsf{TrToSQAlg}\ \mathcal{X}\Big)\ \Big(\mathsf{TrToSQFibAlg}(\mathcal{X})\ \mathcal{Y}\Big)$$

Proof. Fix algebras $\mathcal{X} :\equiv (X, t_X, |\cdot|_X) : \mathsf{TrAlg}_{\mathcal{U}_j}(2, T)$ and $\mathcal{Y} :\equiv (E, t_E, |\cdot|_E) : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$. To construct a map from left to right, take a morphism $(f, \beta) : \mathsf{TrFibMor}\ \mathcal{X}\ \mathcal{Y}$. To turn f into an appropriate set quotient morphism, we show that for each $a : A$ we have $f(|\mathsf{p}_T(a)|_X) = |\mathsf{p}_T(a)|_E$. But this is exactly the equality witnessed by $\beta(\mathsf{p}_T(a))$. For a map in the opposite direction, take a morphism $(f, \beta) : \mathsf{SQFibMor}\ \big(\mathsf{TrToSQAlg}\ \mathcal{X}\big)\ \big(\mathsf{TrToSQFibAlg}(\mathcal{X})\ \mathcal{Y}\big)$. To turn f into a truncation morphism, we need to show that for any $t : T$, we have $f(|t|_X) = |t|_E$. For this, we proceed by induction. Since the type $f(|t|_X) = |t|_E$ is a mere proposition for any $t : T$, we only need to show that it is inhabited for $t :\equiv \mathsf{p}_T(a)$, *i.e.*, that for each $a : A$ we have $f(|\mathsf{p}_T(a)|_X) = |\mathsf{p}_T(a)|_E$. But this is exactly the equality observed by $\beta(a)$.

Showing that these two functions compose to identity on both sides is trivial since they both preserve the underlying map f and for each morphism involved, the type of its second component is a mere proposition. □

Corollary 183. $(\mathcal{H} + /)$ *For algebras* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(2, T)$ *and* $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_j}(2, T)$ *we have*

$$\mathsf{TrMor}\ \mathcal{X}\ \mathcal{Y} \ \simeq\ \mathsf{SQFibMor}\ \Big(\mathsf{TrToSQAlg}\ \mathcal{X}\Big)\ \Big(\mathsf{TrToSQAlg}\ \mathcal{Y}\Big)$$

Proof. Exactly as in the fibered case. □

Corollary 184. *$(\mathcal{H} + /)$ For an algebra $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{G})$ we have*

$$\begin{aligned} \mathsf{hasSQRec}_{\mathcal{U}_k}(\mathcal{X}) &\simeq \mathsf{hasTrRec}_{\mathcal{U}_k}\Big(\mathsf{TrToSQAlg}^{-1}(\mathcal{X})\Big) \\ \mathsf{hasSQInd}_{\mathcal{U}_k}(\mathcal{X}) &\simeq \mathsf{hasTrInd}_{\mathcal{U}_k}\Big(\mathsf{TrToSQAlg}^{-1}(\mathcal{X})\Big) \\ \mathsf{isSQHInit}_{\mathcal{U}_k}(\mathcal{X}) &\simeq \mathsf{isTrHInit}_{\mathcal{U}_k}\Big(\mathsf{TrToSQAlg}^{-1}(\mathcal{X})\Big) \end{aligned}$$

Corollary 185. *$(\mathcal{H} + /)$ For $A : \mathcal{U}_i$, $\mathbf{R} : \mathsf{EqRel}(A)$, the following conditions on an algebra $\mathcal{X} : \mathsf{SQAlg}_{\mathcal{U}_j}(A, \mathbf{R})$ are equivalent:*

- *$\mathcal{X}$ satisfies the induction principle on the universe $\mathcal{U}_k$*
- *$\mathcal{X}$ is homotopy-initial on the universe $\mathcal{U}_k$*

for $k \geq j$. In other words, we have

$$\mathsf{hasSQInd}_{\mathcal{U}_k}(\mathcal{X}) \simeq \mathsf{isSQHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. Moreover, the two types above are mere propositions.

Corollary 186. *$(\mathcal{H} + / + \cdot/_0\cdot)$ For $A : \mathcal{U}_i$, $\mathbf{R} : \mathsf{EqRel}(A)$, the algebra*

$$\Big(A/_0\mathbf{R}, p, \mathsf{point}_0, \mathsf{cell}_0\Big) : \mathsf{SQAlg}_{\mathcal{U}_i}(A, \mathbf{R})$$

is homotopy-initial for any p on any universe $\mathcal{U}_j$.

Homotopy-initial set quotient algebras enjoy the property of effectiveness:

Lemma 187. *$(\mathcal{H})$ For $A : \mathcal{U}_i$, $\mathbf{R} : \mathsf{EqRel}(A)$, and algebra $\mathcal{X} :\equiv (D, t_D, \mathsf{p}, \mathsf{c}) : \mathsf{SQAlg}_{\mathcal{U}_i}(A, \mathbf{R})$, where $\mathbf{R} :\equiv (R, t_R, \mathbf{r}, \mathbf{i}, \mathbf{c})$, if $\mathcal{X}$ is homotopy-initial on the universe $\mathcal{U}_{i+1}$, then for any $a, b : A$ we have*

$$(\mathsf{p}(a) = \mathsf{p}(b)) \simeq R(a, b)$$

Proof. We employ the "encode-decode" method introduced by D. Licata. That is, we first define a function $\mathsf{C} : D \to D \to \Sigma_{X:\mathcal{U}_i}\mathsf{isProp}(X)$, where $\mathsf{C}(x, y)$ is meant to explicitly describe the path type $x = y$, and then show that we indeed have $(x = y) \simeq \pi_1(\mathsf{C}(x, y))$. Of course, we will construct C in such a way that $\pi_1\big(\mathsf{C}(\mathsf{p}(a), \mathsf{p}(b))\big)$ is equivalent to $R(a, b)$, which will finish the proof. Since for any $x, y : D$ the types $(x = y)$ and $\pi_1(\mathsf{C}(x, y))$ are mere propositions, it suffices to exhibit a logical equivalence between them, *i.e.*, construct maps

$$\begin{aligned} &\mathsf{e}(x, y) : (x = y) \to \pi_1(\mathsf{C}(x, y)) \\ &\mathsf{d}(x, y) : \pi_1(\mathsf{C}(x, y)) \to (x = y) \end{aligned}$$

Step 1 To define C, we proceed by recursion on the first argument. In other words, we construct a set quotient algebra on $\mathcal{U}_{i+1}$ whose carrier type is $D \to \Sigma_{X:\mathcal{U}_i}\mathsf{isProp}(X)$. The function C will then be the underlying map of the propositionally unique morphism into this algebra, the existence of which is guaranteed by the fact that $\mathcal{X}$ satisfies the recursion principle on $\mathcal{U}_{i+1}$ (being homotopy-initial on $\mathcal{U}_{i+1}$). To obtain the aforementioned set quotient algebra, we first need to verify that the type $D \to \Sigma_{X:\mathcal{U}_i}\mathsf{isProp}(X)$ indeed belongs to $\mathcal{U}_{i+1}$, which is easy to check. Next we need to show that the type $D \to \Sigma_{X:\mathcal{U}_i}\mathsf{isProp}(X)$ is in fact a set. For this it suffices to show that $\Sigma_{X:\mathcal{U}_i}\mathsf{isProp}(X)$ is a set, which is a simple exercise (or see theorem 7.1.11 in [33]).

Now we need to define a function $F : A \to D \to \Sigma_{X:\mathcal{U}_i}\mathsf{isProp}(X)$. We do this by recursion on the second argument. So we fix $a : A$ and map $\mathsf{p}(b) \mapsto (R(a,b), t_R(a,b))$. To finish the definition, we need to show that for any $b_1, b_2 : A$, and $z : R(b_1, b_2)$, there is a path from $(R(a,b_1), t_R(a,b_1))$ to $(R(a,b_2), t_R(a,b_2))$ in the type $\Sigma_{X:\mathcal{U}_j}\mathsf{isProp}(X)$. Since $\mathsf{isProp}(X)$ is a mere proposition for any X, this is equivalent to showing $R(a,b_1) = R(a,b_2)$ (the map ap_{π_1} serves as the equivalence). By univalence, this is equivalent to showing $R(a,b_1) \simeq R(a,b_2)$. But this clearly holds since we have the quasi-inverses $u \mapsto \mathbf{c}(u,z)$ and $v \mapsto \mathbf{c}(v, \mathbf{i}(z))$ (these maps are necessarily quasi-inverse to each other since $R(a,b_1)$ and $R(a,b_2)$ are mere propositions). This finishes the definition of $F(a)$. We note that by the first computation rule, we get a family of paths $\beta_F(a,b) : F(a, \mathsf{p}(b)) = (R(a,b), t_R(a,b))$.

To finish the definition of C, we must show that for any $a_1, a_2 : A$ and $v : R(a_1,a_2)$, we have $F(a_1) = F(a_2)$. By function extensionality, it suffices to show that for any $x : D$, we have $F(a_1, x) = F(a_2, x)$. We now proceed by induction on x. It is clear that for any x, the type $F(a_1,x) = F(a_2,x)$ belongs to $\mathcal{U}_{i+1}$. We also need to show that it is a set, but it is even a mere proposition, so it only remains to establish the case $F(a_1, \mathsf{p}(b)) = F(a_2, \mathsf{p}(b))$ for $b : A$. By $\beta_F(a_1, b)$ and $\beta_F(a_2, b)$, it suffices to show that $(R(a_1,b), t_R(a_1,b))$ is equal to $(R(a_2,b), t_R(a_2,b))$. For this it suffices to exhibit an equivalence $R(a_1,b) \simeq R(a_2,b)$. This clearly holds since we have the quasi-inverses $u \mapsto \mathbf{c}(\mathbf{i}(v), u)$ and $w \mapsto \mathbf{c}(v,w)$. This finishes the definition of C and we note that by the first computation rule, we get a family of paths $\beta_{\mathsf{C}}(a) : \mathsf{C}(\mathsf{p}(a)) = F(a)$ for any $a : A$.

Step 2 To define $\mathsf{d}(x,y)$, we proceed by induction on x. We can do this since the type $\Pi_{y:D}\pi_1(\mathsf{C}(x,y)) \to (x = y)$ belongs to $\mathcal{U}_{i+1}$ and is a mere proposition for any $x : D$. The latter implies that it suffices to show that $\Pi_{y:D}\pi_1(\mathsf{C}(\mathsf{p}(a),y)) \to (\mathsf{p}(a) = y)$ is inhabited for any $a : A$. To show this, fix $a : A$ and proceed by induction on y. It suffices to show that the type $\pi_1\big(\mathsf{C}(\mathsf{p}(a), \mathsf{p}(b))\big) \to (\mathsf{p}(a) = \mathsf{p}(b))$ is inhabited for any $b : A$. To this end, let $\varepsilon(a,b)$ be the path below:

$$\mathsf{C}(\mathsf{p}(a), \mathsf{p}(b))$$

$$\Big| \; {}^{=}\mathsf{E}^{\Pi}\big(\beta_{\mathsf{C}}(a), \mathsf{p}(b)\big)$$

$$F(a, \mathsf{p}(b))$$

$$\Big| \; \beta_F(a,b)$$

$$(R(a,b), t_R(a,b))$$

Then $\mathsf{c} \circ \pi_1\big({}^{=}\mathsf{E}^{\simeq}(\mathsf{ap}_{\pi_1}(\varepsilon(a,b)))\big)$ is our desired function from $\pi_1\big(\mathsf{C}(\mathsf{p}(a), \mathsf{p}(b))\big)$ to $\mathsf{p}(a) = \mathsf{p}(b)$, and the definition of d is complete.

Step 3 To define $\mathsf{e}(x, y, u)$, we perform path induction on u. To construct a function $G(x) : \pi_1(\mathsf{C}(x, x))$, we proceed by induction on x. Since the type $\pi_1(\mathsf{C}(x, x))$ is a mere proposition for any $x : D$, it suffices to show that it is inhabited when $x :\equiv \mathsf{p}(a)$ for $a : A$. We map $\mathsf{p}(a) \mapsto \pi_1\big({}^{=}\mathsf{E}^{\simeq}(\mathsf{ap}_{\pi_1}(\varepsilon(a, a)))\big)\ \mathbf{r}(a)$, where ε was defined in the previous step. This finishes the definition of e. □

4.3 Homotopy-initiality for Groupoid Quotients

4.3.1 Groupoid Quotients

We can take the underlying idea of set quotients one level higher to obtain *groupoid quotients*. In homotopy theory, the canonical example of a groupoid quotient is the classifying space BG of a group G. Type-theoretically, a groupoid on $A : \mathcal{U}_i$ is a structure that "looks like" the path structure of a 1-type; namely, it is a set-valued relation with identity, inverse, and composition operators which satisfy the usual algebraic laws.

Definition 188. *For $A : \mathcal{U}_i$, we define the type of* groupoids *on A as*

$$\begin{aligned}\mathsf{Grp}(A) :\equiv \Sigma_{R:A\to A\to\mathcal{U}_i}\big(\Pi_{a,b:A}\mathsf{isSet}(R(a,b))\big) \times \Sigma_{\mathbf{r}:\mathsf{id}(R)}\Sigma_{\mathbf{i}:\mathsf{inv}(R)}\Sigma_{\mathbf{c}:\mathsf{comp}(R)}\\ \Big(\Pi_{a,b:A}\Pi_{z:R(a,b)}\big(\mathbf{c}(\mathbf{r}(a),z)=z\big)\Big) \times \Big(\Pi_{a,b:A}\Pi_{z:R(a,b)}\big(\mathbf{c}(z,\mathbf{r}(b))=z\big)\Big) \times\\ \Big(\Pi_{a,b:A}\Pi_{z:R(a,b)}\big(\mathbf{c}(z,\mathbf{i}(z))=\mathbf{r}(a)\big)\Big) \times \Big(\Pi_{a,b:A}\Pi_{z:R(a,b)}\big(\mathbf{c}(\mathbf{i}(z),z)=\mathbf{r}(b)\big)\Big) \times\\ \Big(\Pi_{a,b,c,d:A}\Pi_{v:R(a,b)}\Pi_{w:R(b,c)}\Pi_{z:R(c,d)}\big(\mathbf{c}(v,\mathbf{c}(w,z))=\mathbf{c}(\mathbf{c}(v,w),z)\big)\Big)\end{aligned}$$

Given $A : \mathcal{U}_i$ and a groupoid $\mathbf{G} :\equiv (R, -, \mathbf{r}, \mathbf{i}, \mathbf{c}, \ldots) : \mathsf{Grp}(A)$, the groupoid quotient $A/_1\mathbf{G}$ of A by $\mathbf{G}$ is the higher inductive type generated by the following constructors:

$$\begin{aligned}\mathsf{point}_1 &: A \to A/_1\mathbf{G}\\ \mathsf{cell}_1 &: \Pi_{a,b:A}R(a,b) \to (\mathsf{point}_1(a) = \mathsf{point}_1(b))\\ \mathsf{pres}_{\mathsf{c}} &: \Pi_{a,b,c:A}\Pi_{v:R(a,b)}\Pi_{w:R(b,c)}\big(\mathsf{cell}_1(\mathbf{c}(v,w)) = \mathsf{cell}_1(v) \cdot \mathsf{cell}_1(w)\big)\\ \mathsf{hub} &: \big(\mathbf{S}^2 \to A/_1\mathbf{G}\big) \to A/_1\mathbf{G}\\ \mathsf{spoke} &: \big(\Pi r : \mathbf{S}^2 \to A/_1\mathbf{G}\big)\big(\Pi x : \mathbf{S}^2\big)(r(x) = \mathsf{hub}(r))\end{aligned}$$

The only constructor which is not analogous to the set quotient case is $\mathsf{pres}_{\mathsf{c}}$. It asserts that the composition operator given by the groupoid structure mirrors the one given by the path structure of $A/_1\mathbf{G}$. In other words, the mapping cell_1 carries the composition of two "arrows" $v : R(a,b)$ and $w : R(b,c)$ to the actual composition of the component paths $\mathsf{cell}_1(v)$ and $\mathsf{cell}_1(w)$.

Of course, we could have also included constructors asserting that identities and inverses are preserved in a similar way, *e.g.*, the constructors $\mathsf{pres}_{\mathsf{r}} : \Pi_{a:A}\big(\mathsf{cell}_1(\mathbf{r}(a)) = 1_{[a]_1}\big)$ and $\mathsf{pres}_{\mathsf{i}} : \Pi_{a,b:A}\Pi_{z:R(a,b)}\big(\mathsf{cell}_1(\mathbf{i}(z)) = \mathsf{cell}_1(z)^{-1}\big)$. However, this is unnecessary: the preservation of composition automatically implies the preservation of identities and inverses; that is, we are able to construct terms having the same types as $\mathsf{pres}_{\mathsf{r}}$ and $\mathsf{pres}_{\mathsf{i}}$ just by using the constructors given above. Moreover, such terms are necessarily unique (up to propositional equality) since $A/_1\mathbf{G}$ is a 1-type (as ensured by the hub and spoke constructors). We also note that a constructor such as $\mathsf{pres}_{\mathsf{c}}$ was unnecessary in the set quotient case since in a 0-type any two paths with the same endpoints are automatically equal.

The recursion principle for groupoid quotients says that given terms

- $E : \mathcal{U}_j$,
- $e : A \to E$,
- $p : \Pi_{a,b:A} R(a,b) \to (e(a) = e(b))$,
- $m : \Pi_{a,b,c:A} \Pi_{v:R(a,b)} \Pi_{w:R(b,c)} \big(p(\mathbf{c}(v,w)) = p(v) \cdot p(w)\big)$,
- $h : (\mathbf{S}^2 \to E) \to E$,
- $s : \Pi_{u:\mathbf{S}^2 \to E} \Pi_{x:\mathbf{S}^2} (u(x) = h(u))$,

there is a recursor $\mathsf{rec}_{\cdot/_1\cdot}(E,e,p,m,h,s) : A/_1\mathbf{G} \to E$. The recursor satisfies the computation laws

- $\mathsf{rec}_{\cdot/_1\cdot}(\mathsf{point}_1(a)) \equiv e(a)$ for any $a : A$,
- $\mathsf{ap}_{\mathsf{rec}_{\cdot/_1\cdot}}(\mathsf{cell}_1(z)) = p(z)$ for any $a : A, b : A, z : R(a,b)$,
- $\mathsf{ap}_{\mathsf{ap}_{\mathsf{rec}_{\cdot/_1\cdot}}}(\mathsf{pres_c}(v,w))$ is equal to the path below for any $a,b,c : A$, $v : R(a,b)$, $w : R(b,c)$, where equalities (1) and (2) use the computation rule for the cell_1 constructor:

$$\begin{aligned}
&\mathsf{ap}_{\mathsf{rec}_{\cdot/_1\cdot}}\big(\mathsf{cell}_1(\mathbf{c}(v,w))\big)\\
&\quad \overset{(1)}{=} p(\mathbf{c}(v,w))\\
&\quad \overset{\color{red}{m(v,w)}}{=} p(v) \cdot p(w)\\
&\quad \overset{(2)}{=} \mathsf{ap}_{\mathsf{rec}_{\cdot/_1\cdot}}(\mathsf{cell}_1(v)) \cdot \mathsf{ap}_{\mathsf{rec}_{\cdot/_1\cdot}}(\mathsf{cell}_1(w))\\
&\quad = \mathsf{ap}_{\mathsf{rec}_{\cdot/_1\cdot}}\big(\mathsf{cell}_1(v) \cdot \mathsf{cell}_1(w)\big)
\end{aligned}$$

- $\mathsf{rec}_{\cdot/_1\cdot}(\mathsf{hub}(r)) = h\big(\mathsf{rec}_{\cdot/_1\cdot} \circ r\big)$ for any $r : \mathbf{S}^2 \to A/_1\mathbf{G}$,
- $\mathsf{ap}_{\mathsf{rec}_{\cdot/_1\cdot}}(\mathsf{spoke}(r,x))$ is equal to the path below for any $r : \mathbf{S}^2 \to A/_1\mathbf{G}$ and $x : \mathbf{S}^2$, where equality (1) uses the computation rule for the hub constructor:

$$\mathsf{rec}_{\cdot/_1\cdot}(r(x)) \overset{\color{red}{s(\mathsf{rec}_{\cdot/_1\cdot} \circ r,\, x)}}{=\!=\!=\!=\!=} h\big(\mathsf{rec}_{\cdot/_1\cdot} \circ r\big) \overset{(1)}{=\!=\!=} \mathsf{rec}_{\cdot/_1\cdot}(\mathsf{hub}(r))$$

To express the induction rule concisely, we introduce the following notation. For any type $E : X \to \mathcal{U}_k$, paths $v : a =_X b$, $w : b =_X c$, terms $e_a : E(a)$, $e_b : E(b)$, $e_c : E(c)$, and paths $\mu : v_*^E\ e_a = e_b$, $\nu : w_*^E\ e_b = e_c$, we denote by $\mathcal{C}(\mu,\nu)$ the path

$$(v \cdot w)_*^E\ e_a =\!=\!=\!= w_*^E\ (v_*^E\ e_a) \overset{\color{red}{\textit{via}\ \mu}}{=\!=\!=\!=} w_*^E\ e_b \overset{\color{red}{\nu}}{=\!=\!=} e_c$$

The induction principle for groupoid quotients then says that given terms

- $E : A/_1\mathbf{G} \to \mathcal{U}_k$,
- $e : \Pi_{a:A} E(\mathsf{point}_1(a))$,
- $p : \Pi_{a,b:A}\Pi_{z:R(a,b)}\big(\mathsf{cell}_1(z)_*^E\ e(a) = e(b)\big)$,
- $m : \Pi_{a,b,c:A}\Pi_{v:R(a,b)}\Pi_{w:R(b,c)}$
  $\Big(p(\mathbf{c}(v,w)) = {}^{=}\mathsf{E}^{\Pi}\big(\mathsf{ap}_{(-)_*^E}(\mathsf{pres}_{\mathsf{c}}(v,w)), e(a)\big) \cdot \mathcal{C}(p(v), p(w))\Big)$,
- $h : \big(\Pi r : \mathbf{S}^2 \to A/_1\mathbf{G}\big)\big(\Pi_{x:\mathbf{S}^2} E(r(x))\big) \to E(\mathsf{hub}(r))$,
- $s : \big(\Pi r : \mathbf{S}^2 \to A/_1\mathbf{G}\big)\big(\Pi u : \Pi_{x:\mathbf{S}^2} E(r(x))\big)\big(\Pi x : \mathbf{S}^2\big)\big(\mathsf{spoke}(r,x)_*^E\ u(x) = h(r,u)\big)$,

there is an inductor $\mathsf{ind}_{\cdot/_1\cdot}(E, e, p, m, h, s) : \Pi_{x:A/_1\mathbf{G}} E(x)$ satisfying the computation laws

- $\mathsf{ind}_{\cdot/_1\cdot}(\mathsf{point}_1(a)) \equiv e(a)$ for any $a : A$,
- $\mathsf{dap}_{\mathsf{ind}_{\cdot/_1\cdot}}(\mathsf{cell}_1(z)) = p(z)$ for any $a : A, b : A, z : R(a,b)$,
- $\mathsf{dap}_{\mathsf{dap}_{\mathsf{ind}_{\cdot/_1\cdot}}}(\mathsf{pres}_{\mathsf{c}}(v,w))$ is equal to the path below for any $a, b, c : A$ and $v : R(a,b)$, $w : R(b,c)$. Here equalities (1) and (3) use the computation rule for the cell_1 constructor, (2) is given by the characterization of transport between the fibers of the type family $q \mapsto q_*^E\ e(a) = e(c)$ (easily shown by path induction), and (4) is an analogue of functoriality for the dap operator (also easily shown by path induction).

$$\begin{aligned}
&\mathsf{pres}_{\mathsf{c}}(v,w)_*^{q \mapsto q_*^E\ e(a) = e(c)}\ \Big(\mathsf{dap}_{\mathsf{ind}_{\cdot/_1\cdot}}(\mathsf{cell}_1(\mathbf{c}(u,w)))\Big)\\
&\quad = \text{(1)}\\
&\mathsf{pres}_{\mathsf{c}}(v,w)_*^{q \mapsto q_*^E\ e(a) = e(c)}\ p(\mathbf{c}(v,w))\\
&\quad = \text{(2)}\\
&\Big({}^{=}\mathsf{E}^{\Pi}\big(\mathsf{ap}_{(-)_*^E}(\mathsf{pres}_{\mathsf{c}}(v,w)), e(a)\big)\Big)^{-1} \cdot p(\mathbf{c}(v,w))\\
&\quad = \textit{via } m(v,w)\\
&\Big({}^{=}\mathsf{E}^{\Pi}\big(\mathsf{ap}_{(-)_*^E}(\mathsf{pres}_{\mathsf{c}}(v,w)), e(a)\big)\Big)^{-1} \cdot \Big({}^{=}\mathsf{E}^{\Pi}\big(\mathsf{ap}_{(-)_*^E}(\mathsf{pres}_{\mathsf{c}}(v,w)), e(a)\big) \cdot \mathcal{C}(p(v), p(w))\Big)\\
&\quad =\\
&\mathcal{C}\big(p(v), p(w)\big)\\
&\quad = \text{(3)}\\
&\mathcal{C}\Big(\mathsf{dap}_{\mathsf{ind}_{\cdot/_1\cdot}}(\mathsf{cell}_1(v)), \mathsf{dap}_{\mathsf{ind}_{\cdot/_1\cdot}}(\mathsf{cell}_1(w))\Big)\\
&\quad = \text{(4)}\\
&\mathsf{dap}_{\mathsf{ind}_{\cdot/_1\cdot}}\big(\mathsf{cell}_1(v) \cdot \mathsf{cell}_1(w)\big)
\end{aligned}$$

- $\mathsf{ind}_{\cdot/_1\cdot}(\mathsf{hub}(r)) = h\big(r, \mathsf{ind}_{\cdot/_1\cdot} \circ r\big)$ for any $r : \mathbf{S}^2 \to A/_1\mathbf{G}$,
- $\mathsf{dap}_{\mathsf{ind}_{\cdot/_1\cdot}}(\mathsf{spoke}(r,x))$ is equal to the path below for any $r : \mathbf{S}^2 \to A/_1\mathbf{G}$ and $x : \mathbf{S}^2$, where equality (1) uses the computation rule for the hub constructor:

$$\mathsf{spoke}(r,x)^E_*\ \big(\mathsf{ind}_{\cdot/_1\cdot}(r(x))\big) \xrightarrow{\ s(r,\, \mathsf{ind}_{\cdot/_1\cdot} \circ r,\, x)\ } h\big(r, \mathsf{ind}_{\cdot/_1\cdot} \circ r\big) \xrightarrow{\ (1)\ } \mathsf{ind}_{\cdot/_1\cdot}(\mathsf{hub}(r))$$

To give some intuition for the induction principle, let us look at how we obtained the type for the hypothesis m. For any $a, b, c : A$ and $v : R(a,b)$, $w : R(b,c)$, the term $p(\mathbf{c}(v,w))$ gives us a path from $\mathsf{cell}_1(\mathbf{c}(v,w))^E_*\ e(a)$ to $e(c)$. However, the constructor pres_c allows us to construct another path with the same endpoints: namely, we first apply congruence to pres_c to take us from $\mathsf{cell}_1(\mathbf{c}(v,w))^E_*\ e(a)$ to $\big(\mathsf{cell}_1(v) \cdot \mathsf{cell}_1(w)\big)^E_*\ e(a)$ and then appeal to $\mathcal{C}(p(v), p(w))$ to take us all the way to $e(c)$. We want these two paths to coincide, which is what m asserts.

The computation rule associated to pres_c ensures that the inductor behaves as expected when applied to pres_c; "applied" here means in the higher-dimensional sense of course. Fortunately for us, the presence of the hypotheses h and s means that this rule is redundant since they imply that E is fiberwise a 1-type (lemma 140) and in a 1-type, any two paths between paths are equal provided the endpoints agree. However, the above computation rule for pres_c serves to illustrate the difficulty involved in developing a unifying theory of higher inductive types: had we not included constructors ensuring that the resulting type is appropriately truncated, we would have been forced to carry this rule around.

In light of the above discussion, the groupoid quotient of $A : \mathcal{U}_i$ by a groupoid $\mathbf{G} :\equiv (R, -, \mathbf{r}, \mathbf{i}, \mathbf{c}, \ldots) : \mathsf{Grp}(A)$ can be equivalently presented as a type $A/_1\mathbf{G} : \mathcal{U}_i$ endowed with constructors

$$\begin{aligned}
\mathsf{point}_1 &: A \to A/_1\mathbf{G} \\
\mathsf{cell}_1 &: \Pi_{a,b:A} R(a,b) \to (\mathsf{point}_1(a) = \mathsf{point}_1(b)) \\
\mathsf{pres}_\mathsf{c} &: \Pi_{a,b,c:A}\Pi_{v:R(a,b)}\Pi_{w:R(b,c)}\big(\mathsf{cell}_1(\mathbf{c}(v,w)) = \mathsf{cell}_1(v) \cdot \mathsf{cell}_1(w)\big) \\
\mathsf{tr} &: \mathsf{isGrp}(A/_1\mathbf{G})
\end{aligned}$$

where we use the abbreviation $\mathsf{isGrp}(X) :\equiv \mathsf{is\text{-}1\text{-}type}(X)$, such that that given terms

- $E : \mathcal{U}_j$,
- $t : \mathsf{isGrp}(E)$,
- $e : A \to E$,
- $p : \Pi_{a,b:A} R(a,b) \to (e(a) = e(b))$,
- $m : \Pi_{a,b,c:A}\Pi_{v:R(a,b)}\Pi_{w:R(b,c)}\big(p(\mathbf{c}(v,w)) = p(v) \cdot p(w)\big)$,

there is a recursor $\mathsf{rec}_{\cdot/_1\cdot}(E, t, e, p, m) : A/_1\mathbf{G} \to E$ satisfying the computation laws

- $\mathsf{rec}_{\cdot/_1\cdot}(\mathsf{point}_1(a)) \equiv e(a)$ for any $a : A$,
- $\mathsf{ap}_{\mathsf{rec}_{\cdot/_1\cdot}}(\mathsf{cell}_1(z)) = p(z)$ for any $a : A, b : A, z : R(a,b)$,

and for any terms

- $E : A/_1\mathbf{G} \to \mathcal{U}_k$,
- $t : \Pi_{y:D}\mathsf{isGrp}(E(y))$,
- $e : \Pi_{a:A}E(\mathsf{point}_1(a))$,
- $p : \Pi_{a,b:A}\Pi_{z:R(a,b)}\big(\mathsf{cell}_1(z)_*^E\ e(a) = e(b)\big)$,
- $m : \Pi_{a,b,c:A}\Pi_{v:R(a,b)}\Pi_{w:R(b,c)}$
  $\Big(p(\mathbf{c}(v,w)) = {}^{=}\mathsf{E}^{\Pi}\big(\mathsf{ap}_{(-)_*^E}(\mathsf{pres}_{\mathsf{c}}(v,w)), e(a)\big) \cdot \mathcal{C}(p(v), p(w))\Big)$,

there is an inductor $\mathsf{ind}_{\cdot/_1\cdot}(E,t,e,p,m) : \Pi_{x:A/_1\mathbf{G}}E(x)$ satisfying the computation laws

- $\mathsf{ind}_{\cdot/_1\cdot}(\mathsf{point}_1(a)) \equiv e(a)$ for any $a : A$,
- $\mathsf{dap}_{\mathsf{ind}_{\cdot/_1\cdot}}(\mathsf{cell}_1(z)) = p(z)$ for any $a : A, b : A, z : R(a,b)$.

Definition 189. *For $A : \mathcal{U}_i$, $\mathbf{G} :\equiv (R, -, \mathbf{r}, \mathbf{i}, \mathbf{c}, \ldots) : \mathsf{Grp}(A)$, we define the type of* groupoid quotient algebras *on a universe $\mathcal{U}_j$ to be*

$$\mathsf{GQAlg}_{\mathcal{U}_j}(A,\mathbf{G}) :\equiv \Sigma_{D:\mathcal{U}_j}\mathsf{isGrp}(D) \times \Big(\Sigma_{e:A\to D}\Sigma_{p:(\Pi a,b:A)R(a,b)\to(e(x)=e(y))} \Pi_{a,b,c:A}\Pi_{v:R(a,b)}\Pi_{w:R(b,c)}\big(p(\mathbf{c}(v,w)) = p(v) \cdot p(w)\big)\Big)$$

Definition 190. *For an algebra $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A,\mathbf{G})$ with $\mathbf{G} :\equiv (R, -, \mathbf{r}, \mathbf{i}, \mathbf{c}, \ldots)$, define the type of* fibered groupoid quotient algebras *on a universe $\mathcal{U}_k$ by*

$$\mathsf{GQFibAlg}_{\mathcal{U}_k}\ (D,-,d,p,m) :\equiv \Sigma_{E:D\to\mathcal{U}_k}\Big(\Pi_{y:D}\mathsf{isGrp}(E(y))\Big) \times \Big(\Sigma_{e:(\Pi a:A)E(d(a))} \Sigma_{q:(\Pi a,b:A)(\Pi z:R(a,b))(p(z)_*^E\ e(a)=e(b))}\Pi_{a,b,c:A}\Pi_{v:R(a,b)}\Pi_{w:R(b,c)} \big(q(\mathbf{c}(v,w)) = {}^{=}\mathsf{E}^{\Pi}\big(\mathsf{ap}_{(-)_*^E}(\mathsf{pr}_{\mathsf{c}}(v,w)), e(a)\big) \cdot \mathcal{C}(q(v), q(w))\big)\Big)$$

Definition 191. *For algebras $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A,\mathbf{G})$ and $\mathcal{Y} : \mathsf{GQAlg}_{\mathcal{U}_k}(A,\mathbf{G})$, with $\mathbf{G} :\equiv (R, \ldots)$, we define the type of* groupoid quotient morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathsf{GQMor}\ (D,-,d,p,-)\ (E,-,e,q,-) :\equiv \mathsf{TQMor}\ (D,d,q)\ (E,e,q)$$

Definition 192. *For algebras $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A,\mathbf{G})$ with $\mathcal{Y} : \mathsf{GQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$, with $\mathbf{G} :\equiv (R, \ldots)$, we define the type of* fibered groupoid quotient morphisms *from $\mathcal{X}$ to $\mathcal{Y}$ by*

$$\mathsf{GQFibMor}\ (D,-,d,p,-)\ (E,-,e,q,-) :\equiv \mathsf{TQFibMor}\ (D,d,q)\ (E,e,q)$$

Definition 193. *An algebra $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A,\mathbf{G})$* satisfies the recursion principle *on a universe $\mathcal{U}_k$ if for any algebra $\mathcal{Y} : \mathsf{GQAlg}_{\mathcal{U}_k}(A,\mathbf{G})$ there exists a morphism from $\mathcal{X}$ to $\mathcal{Y}$:*

$$\mathsf{hasGQRec}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi\mathcal{Y} : \mathsf{GQAlg}_{\mathcal{U}_j}(A,\mathbf{G})\Big)\ \mathsf{GQMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 194. *An algebra $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A,\mathbf{G})$* satisfies the induction principle *on a universe $\mathcal{U}_k$ if for any fibered algebra $\mathcal{Y} : \mathsf{GQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ there exists a fibered morphism from $\mathcal{X}$ to $\mathcal{Y}$:*

$$\mathsf{hasGQInd}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi\mathcal{Y} : \mathsf{GQFibAlg}_{\mathcal{U}_k}\ \mathcal{X}\Big)\ \mathsf{GQFibMor}\ \mathcal{X}\ \mathcal{Y}$$

Definition 195. *An algebra* $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A, \mathbf{G})$ *is* homotopy-initial *on a universe* $\mathcal{U}_k$ *if for any other algebra* $\mathcal{Y} : \mathsf{SQAlg}_{\mathcal{U}_k}(A, \mathbf{G})$ *the type of morphisms from* $\mathcal{X}$ *to* $\mathcal{Y}$ *is contractible:*

$$\mathsf{isGQHInit}_{\mathcal{U}_k}(\mathcal{X}) :\equiv \Big(\Pi \mathcal{Y} : \mathsf{GQAlg}_{\mathcal{U}_j}(A, \mathbf{G})\Big)\ \mathsf{isContr}(\mathsf{GQMor}\ \mathcal{X}\ \mathcal{Y})$$

4.3.2 Homotopy-initiality for Groupoid Quotients

We aim to show the following analogue to our main theorem for W-quotients:

Theorem 196. $(\mathcal{H} + \mathsf{W})$ *For* $A : \mathcal{U}_i$, $\mathbf{G} : \mathsf{Grp}(A)$, *the following conditions on an algebra* $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A, \mathbf{G})$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_k$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_k$

for $k \geq j$. *In other words, we have*

$$\mathsf{hasGQInd}_{\mathcal{U}_k}(\mathcal{X}) \quad \simeq \quad \mathsf{isGQHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. *Moreover, the two types above are mere propositions.*

Fix $k \geq j$, type $A : \mathcal{U}_i$, and groupoid $\mathbf{G} :\equiv (R, -, \mathbf{r}, \mathbf{i}, \mathbf{c}, \ldots)$. We aim to establish the above theorem by encoding each $\mathcal{X}$ as a 1-truncation of a suitable W-quotient algebra and invoking theorem 152. We first need to fix the parameters for the W-quotient, which already requires some work. Let us fix an algebra $(S, \mathsf{b}, \mathsf{lp}) : \mathbf{S}\text{-}\mathsf{Alg}_{\mathcal{U}_0}$ which satisfies the induction principle on $\mathcal{U}_{\max(i,k)}$ and an algebra $(T, \mathsf{p}_T, \mathsf{c}_T) : \mathsf{TQAlg}_{\mathcal{U}_i}(A, R)$ which satisfies the induction principle on $\mathcal{U}_k$. Let $D :\equiv \Sigma_{a,b,c:A} R(a,b) \times R(b,c)$.

Define a map $\mathbf{f} : D \to S \to T$ by recursion on the second argument, with $\mathsf{b} \mapsto \mathsf{p}_T(c)$ and $\mathsf{lp} \mapsto \big(\mathsf{c}_T(v) \cdot \mathsf{c}_T(w)\big)^{-1} \cdot \mathsf{c}_T(\mathbf{c}(v,w))$ for any $d :\equiv (a,b,c,v,w) : D$. Hence we have a family of paths $\beta_{\mathbf{f}}(d) : \mathbf{f}(d, \mathsf{b}) = \mathsf{p}_T(c)$ and a family of higher paths $\theta_{\mathbf{f}}(d)$ witnessing the following commuting diagram:

$$\begin{array}{ccc}
\mathbf{f}(d,\mathsf{b}) & \xrightarrow{\quad \mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}) \quad} & \mathbf{f}(d,\mathsf{b}) \\
{\scriptstyle \beta_{\mathbf{f}}(d)} \Big| & & \Big| {\scriptstyle \beta_{\mathbf{f}}(d)} \\
\mathsf{p}_T(c) & \xrightarrow[\big(\mathsf{c}_T(v) \cdot \mathsf{c}_T(w)\big)^{-1} \cdot \mathsf{c}_T(\mathbf{c}(v,w))]{} & \mathsf{p}_T(c)
\end{array}$$

The intuition behind this definition is to help us construct the "$\mathsf{pres}_{\mathsf{c}}$" component of a groupoid quotient algebra, as we will see shortly.

The parameters for our W-quotient will be as follows. For the "A" parameter, which encodes the type of labels for points, we will use $T + D$. The left component serves to embed the type quotient T into our W-quotient and the right component introduces a "hub" point for each $d : D$, which will be used to ensure that $\mathbf{f}$ composed with the embedding of T into the W-quotient is homotopy-constant. This will imply that $\big(\mathsf{c}_T(v) \cdot \mathsf{c}_T(w)\big)^{-1} \cdot \mathsf{c}_T(\mathbf{c}(v,w))$ when mapped into the W-quotient becomes the identity path, which is what we want for the construction of $\mathsf{pres}_{\mathsf{c}}$.

Since all of labels described above are nullary, the "B" parameter, which encodes the arity of each point label, will always return $\mathbf{0}$. For the "C" parameter, which encodes the type of labels for paths, we will use $D \times S$. A term (d, x) will stand for the "spoke" going from the embedding of $\mathbf{f}(d, x)$ to the hub point. The maps $\mathtt{l}, \mathtt{r} : D \times S \to T + D$ encode this intuition: we define the map $\mathtt{l}$ by $(d, x) \mapsto \mathsf{inl}(\mathbf{f}(d, x))$ and the map $\mathtt{r}$ by $(d, -) \mapsto \mathsf{inr}(d)$.

Fix an algebra $(W, \mathsf{p}_W, \mathsf{c}_W) : \mathsf{WQAlg}_{\mathcal{U}_i}\big(T + D, \_ \mapsto \mathbf{0}, D \times S, \mathtt{l}, \mathtt{r}\big)$ which satisfies the induction principle on $\mathcal{U}_k$. Finally, define a function $\mathbf{h} : T \to W$, which will be the aforementioned embedding of T into W, by $y \mapsto \mathsf{p}_W(\mathsf{inl}(y), f_\mathbf{0})$, where $f_\mathbf{0}$ is the canonical function out of $\mathbf{0}$ to W.

We are now ready to prove that the groupoid quotient algebras are really the same thing as 1-truncation algebras over the W-quotient W. We note that we need to use the parameter $n :\equiv 3$ to obtain a 1-truncation.

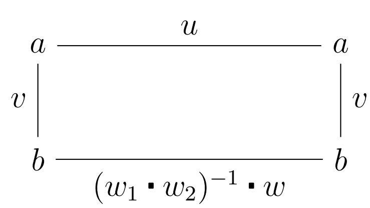
Lemma 197. $(\mathcal{H} + \mathsf{W})$ *We have a function*

$$\mathsf{TrToGQAlg} \quad : \quad \mathsf{TrAlg}_{\mathcal{U}_j}(3, W) \quad \to \quad \mathsf{GQAlg}_{\mathcal{U}_j}(A, \mathbf{G})$$

which is an equivalence.

Proof. We first introduce some notation that will be used throughout:

- For any $t : \mathbf{0} \to W$, we have a path $\alpha_\star(t) : f_\mathbf{0} = t$ witnessing the fact that $f_\mathbf{0}$ is the unique function out of the empty type.
- For any $u : a =_Y b$ and $c : Z$, we have a path
$$\mathsf{apc}(c, u) : \mathsf{ap}_{\_ \mapsto c}(u) = 1_c$$
defined by induction on u.
- For any $f, g : Y \to Z$, and $u, v : a =_Y a$, and $\gamma : f = g$, $\delta : u = v$, we have a function
$$\mathsf{idc}(\gamma, \delta) : (\mathsf{ap}_g(v) = 1_{g(a)}) \to (\mathsf{ap}_f(u) = 1_{f(a)})$$
defined by induction on γ and δ.
- For any $u, v : a =_Y b$, we have equivalences
$$\mathcal{I}_l : (1_a \cdot u = v) \to (u = v)$$
$$\mathcal{I}_r : (u = v \cdot 1_b) \to (u = v)$$
- For a function $f : Y \to Z$ and paths $u : a =_Y a$, $v : a =_Y b$, $w : d =_Y b$, $w_1 : d =_Y c$, $w_2 : c =_Y b$, as in the diagram



we have a map
$$\mathcal{H}_\mathsf{c} : \Big(v \cdot \big((w_1 \cdot w_2)^{-1} \cdot w\big) = u \cdot v\Big) \to$$
$$(\mathsf{ap}_f(u) = 1_{f(a)}) \to \Big(\mathsf{ap}_f(w) = \mathsf{ap}_f(w_1) \cdot \mathsf{ap}_f(w_2)\Big)$$
defined by induction on v, w_1, w_2, and subsequently mapping any $\theta : 1_a \cdot (1_a \cdot w) = u \cdot 1_a$ to the term $\mathsf{idc}\big(1_f, \mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta)))\big)$.

- For $d : D$, let $\mathbf{f}_\star(d) :\equiv \mathsf{p}_W(\mathsf{inr}(d), f_\mathbf{0})$. Then the composition $\mathbf{h} \circ \mathbf{f}(d)$ is equal to the constant map on $\mathbf{f}_\star(d)$, as evidenced by the path

$$\eta_\mathbf{f}(d) :\equiv {}^{\Pi}\mathsf{E}^{=}\big(x \mapsto \mathsf{c}_W((d,x), f_\mathbf{0}, f_\mathbf{0})\big)$$

This implies that the term $\mathsf{ap}_{\mathbf{h}\,\circ\,\mathbf{f}(d)}(\mathsf{lp})$ is equal to reflexivity, as witnessed by the path

$$\upsilon_\mathbf{f}(d) :\equiv \mathsf{idc}\Big(\eta_\mathbf{f}(d), 1_{\mathsf{lp}}, \mathsf{apc}\big(\mathbf{f}_\star(d), \mathsf{lp}\big)\Big)$$

Of course, the term $\mathsf{ap}_\mathbf{h}(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}))$ is then also equal to reflexivity, as witnessed by the obvious path $\vartheta_\mathbf{f}(d) :\equiv$

$$\mathsf{ap}_\mathbf{h}(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})) \;\frac{\quad\quad}{}\; \mathsf{ap}_{\mathbf{h}\,\circ\,\mathbf{f}(d)}(\mathsf{lp}) \;\frac{\upsilon_\mathbf{f}(d)}{\quad\quad}\; 1$$

Finally, it follows that the term $\mathsf{ap}_{|-|_X \circ\, \mathbf{h}}\big(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})\big)$ is equal to reflexivity, as evidenced by the path $\varepsilon_\mathbf{f}(d) :\equiv$

$$\mathsf{ap}_{|-|_X \circ\, \mathbf{h}}\big(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})\big) \;\frac{\quad\quad}{}\; \mathsf{ap}_{|-|_X}\big(\mathsf{ap}_\mathbf{h}(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}))\big) \;\frac{\textit{via } \vartheta_\mathbf{f}(d)}{\quad\quad}\; \mathsf{ap}_{|-|_X}(1)$$

Furthermore, we will use the following observation:

- **Observation:** Given a type family $Y : W \to \mathcal{U}_k$ and function $g : \Pi_{y:T} Y(\mathbf{h}(t))$, if each fiber of Y is a set then there is a function $f : \Pi_{w:W} Y(w)$ such that $f(\mathbf{h}(y)) = g(y)$ for any $y : T$.

  To prove this claim, we construct the desired function f by induction. For this we first need to define a map

$$e : \Pi_{x:T+D}\Pi_{t:\mathbf{0}\to W}\big(\Pi_{b:\mathbf{0}} Y(t\ b)\big) \to Y(\mathsf{p}_W(x,t))$$

  We define $e(x,t,-) :\equiv \mathbf{e}(x,t,\alpha_\star(t))$, where for any $x : T + D$, $t : \mathbf{0} \to W$, $\alpha : t = f_\mathbf{0}$, the term

$$\mathbf{e}(x,t,\alpha) : E(\mathsf{p}_W(x,t))$$

  is defined by one-sided path induction on α and the subsequent mapping

$$\begin{aligned} \mathsf{inl}(y) &\mapsto g(y) \\ \mathsf{inr}(d) &\mapsto \mathsf{c}_W((d,\mathsf{b}), f_\mathbf{0}, f_\mathbf{0})^Y_*\ g(\mathbf{f}(d,\mathsf{b})) \end{aligned}$$

  To complete the inductive definition we need to construct a map

$$\begin{gathered} q : \Pi_{c:D\times S}\Pi_{t:\mathbf{0}\to W}\Pi_{s:\mathbf{0}\to W}\Pi_{u:(\Pi b:\mathbf{0})Y(t\ b)}\Pi_{v:(\Pi b:\mathbf{0})E(s\ b)} \\ \Big(\mathsf{c}_W(c,t,s)^Y_*\ e(\mathtt{l}(c),t,u) = e(\mathtt{r}(c),s,v)\Big) \end{gathered}$$

We define $q(c, t, s, -, -) :\equiv \mathbf{q}(c, t, \alpha_\star(t), s, \alpha_\star(s))$, where for $c : D \times S$, $t : \mathbf{0} \to W$, $\alpha_t : t = f_{\mathbf{0}}$, $s : \mathbf{0} \to W$, $\alpha_s : s = f_{\mathbf{0}}$, the term

$$\mathbf{q}(c, t, \alpha_t, s, \alpha_s) : \mathsf{c}_W(c, t, s)_*^Y\ \mathbf{e}(\mathtt{l}(c), t, \alpha_t) = \mathbf{e}(\mathtt{r}(c), s, \alpha_s)$$

is defined by one-sided path induction on α_t, α_s, and the subsequent mapping $(d, x) \mapsto \mathbf{q}'(d, x)$. Here for any $d : D$, $x : S$, the path family

$$\mathbf{q}'(d, x) : \mathsf{c}_W((d, x), f_{\mathbf{0}}, f_{\mathbf{0}})_*^Y\ g(\mathbf{f}(d, x)) = \mathsf{c}_W((d, \mathsf{b}), f_{\mathbf{0}}, f_{\mathbf{0}})_*^Y\ g(\mathbf{f}(d, \mathsf{b}))$$

is defined by induction on x (we can do this since $k \leq \max(i, k)$), mapping $\mathsf{b} \mapsto 1$. We do not have to supply a mapping for the path constructor lp since by assumption, each fiber of Y is a set and this makes the type of $\mathbf{q}'(d, x)$ a mere proposition.

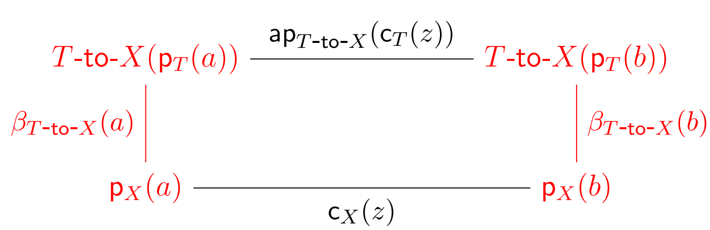
We now proceed in four steps:

Step 1 First we define $\mathsf{TrToGQAlg}$; for this, take an algebra $(X, t_X, |\cdot|_X) : \mathsf{TrAlg}_{\mathcal{U}_j}(3, W)$. To construct a group quotient algebra, we can use the same underlying type X, with t_X showing it is a 1-type as desired. For the third component, we need a function $\mathsf{p}_X : A \to X$. The only real possibility we have is to define $\mathsf{p}_X(a) :\equiv |\mathbf{h}(\mathsf{p}_T(a))|_X$. To obtain the fourth component, we need a function c_X mapping each $a, b : A$, $z : R(a, b)$ to a path from $|\mathbf{h}(\mathsf{p}_T(a))|_X$ to $|\mathbf{h}(\mathsf{p}_T(b))|_X$. Again, the only obvious choice we have (up to homotopy) is to define $\mathsf{c}_X(z) :\equiv \mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))$. To obtain the final component, we need to exhibit a path

$$\mathsf{m}_X(v, w) : \mathsf{c}_X\big(\mathbf{c}(v, w)\big) = \mathsf{c}_X(v) \cdot \mathsf{c}_X(w)$$

for any $a, b, c : A$, and $v : R(a, b)$, $w : R(b, c)$. For this, we define $\mathsf{m}_X(v, w) :\equiv \mathcal{H}_{\mathsf{c}}(\theta_{\mathbf{f}}(d), \varepsilon_{\mathbf{f}}(d))$, where $d :\equiv (a, b, c, v, w)$.

Step 2 We define the intended quasi-inverse $\mathsf{GQToTrAlg}$; fix an algebra $(X, t_X, \mathsf{p}_X, \mathsf{c}_X, \mathsf{m}_X) : \mathsf{GQAlg}_{\mathcal{U}_j}(A, \mathbf{G})$. To construct a truncation algebra, we can use the same underlying type X, with t_X showing it is a 1-type as desired. To obtain the last component, we need a function $W\text{-to-}X : W \to X$. For this we will first need a function $T\text{-to-}X : T \to X$. We proceed by recursion (we can do this since $j \leq k$), mapping $\mathsf{p}_T(a) \mapsto \mathsf{p}_X(a)$ for any $a : A$ and $\mathsf{c}_T(z) \mapsto \mathsf{c}_X(z)$ for any $a, b : A$, $z : R(a, b)$. The first computation rule then gives us a family of paths $\beta_{T\text{-to-}X}(a) : T\text{-to-}X(\mathsf{p}_T(a)) = \mathsf{p}_X(a)$ for $a : A$, and the second computation rule implies the commutativity of the following diagram for any $a, b : A$, $z : R(a, b)$:



To obtain W-to-X we again proceed by recursion. For this we first need to define a function $e : T + D \to (\mathbf{0} \to X) \to X$; we do so by mapping $\mathsf{inl}(y), \_ \mapsto T\text{-to-}X(y)$ and $\mathsf{inr}(d), \_ \mapsto T\text{-to-}X(\mathbf{f}(d, \mathsf{b}))$. It remains to construct a $q : \Pi_{c:D\times S}\Pi_{u:\mathbf{0}\to X}\Pi_{v:\mathbf{0}\to X}(e(\mathtt{l}(c), u) = e(\mathtt{r}(c), v))$. For this, it suffices to prove the following:

- **Goal:** For any $d : D$, $x : S$, there exists a path family

$$c_{\mathbf{f}}(d, x) : T\text{-to-}X(\mathbf{f}(d, x)) = T\text{-to-}X(\mathbf{f}(d, \mathsf{b}))$$

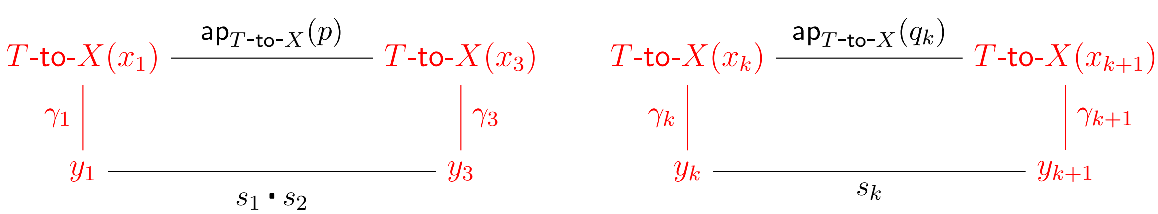
We can then define $q((d, x), -, -) :\equiv c_{\mathbf{f}}(d, x)$. To prove the existence of the function $c_{\mathbf{f}}$, we will make use of the following easy claim:

- **Claim:** For a type $Z : \mathcal{U}_k$ and map $f : S \to Z$, if $\mathsf{ap}_f(\mathsf{lp}) = 1$ then there is a type family $c(x) : f(x) = f(\mathsf{b})$.

By the above claim it suffices to show that $\mathsf{ap}_{T\text{-to-}X \circ \mathbf{f}(d)}(\mathsf{lp}) = 1$ for any $d : D$. Fix $d :\equiv (a, b, c, v, w) : D$. It now suffices to establish the following generalization: given terms

- $x_k : T$ and $y_k : X$ for $k \in \{1, 2, 3\}$,
- $\gamma_k : T\text{-to-}X(x_k) = y_k$ for $k \in \{1, 2, 3\}$,
- $p : x_1 = x_3$ and $q_k : x_k = x_{k+1}$ for $k \in \{1, 2\}$,
- $\beta : \mathbf{f}(d, \mathsf{b}) = x_3$,
- $s_k : y_k = y_{k+1}$ for $k \in \{1, 2\}$,
- $\theta : \beta \cdot \big((q_1 \cdot q_2)^{-1} \cdot p\big) = \mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}) \cdot \beta$,

we have $\mathsf{ap}_{T\text{-to-}X \circ \mathbf{f}(d)}(\mathsf{lp}) = 1$ provided the diagrams below commute for $k \in \{1, 2\}$:



We instantiate this generalization by $\gamma_1 :\equiv \beta_{T\text{-to-}X}(a)$, $\gamma_2 :\equiv \beta_{T\text{-to-}X}(b)$, $\gamma_3 :\equiv \beta_{T\text{-to-}X}(c)$, $p :\equiv \mathsf{c}_T(\mathbf{c}(v, w))$, $q_1 :\equiv \mathsf{c}_T(v)$, $q_2 :\equiv \mathsf{c}_T(w)$, $\beta :\equiv \beta_{\mathbf{f}}(d)$, $s_1 :\equiv \mathsf{c}_X(v)$, $s_2 :\equiv \mathsf{c}_X(w)$, $\theta :\equiv \theta_{\mathbf{f}}(d)$. The commutativity of the diagrams in the hypotheses is implied by the second computation rule for T-to-X and $\mathsf{m}_X(v, w)$.

Finally, to prove this generalization we perform one-sided path induction on β, γ_1, γ_2, γ_3, q_1, q_2. The commutativity of the diagrams in the hypothesis becomes equivalent to the conditions $\mathsf{ap}_{T\text{-to-}X}(p) = s_1 \cdot s_2$ for the first diagram and $s_1 = 1$, $s_2 = 1$ for the remaining two. Performing one-sided path induction on these last two conditions then replaces s_1 and s_2 with reflexivities, and the first condition thus reduces to $\mathsf{ap}_{T\text{-to-}X}(p) = 1$.

Moreover, we now have an assumption $\theta : 1 \cdot p = \mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}) \cdot 1$, which is equivalent to assuming $p = \mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})$. Performing a one-sided path induction on this latter assumption replaces

p with $\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})$, giving us $\mathsf{ap}_{T\text{-to-}X}(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})) = 1$. This implies $\mathsf{ap}_{T\text{-to-}X \circ \mathbf{f}(d)}(\mathsf{lp}) = 1$ as desired.

Although we are done with this step, it will be useful to note that since W-to-X was defined by recursion, the first computation rule gives us a family of paths

$$\beta_{W\text{-to-}X}(y) : W\text{-to-}X(\mathsf{h}(y)) = T\text{-to-}X(y)$$

for any $y : T$.

Step 3 We now want to show that for any groupoid quotient algebra $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A, \mathbf{G})$, we have $\mathsf{TrToGQAlg}(\mathsf{GQToTrAlg}(\mathcal{X})) = \mathcal{X}$. Let such an algebra $(X, t_X, \mathsf{p}_X, \mathsf{c}_X, \mathsf{m}_X)$ be given. The first and second components of $\mathsf{TrToGQAlg}\big(\mathsf{GQToTrAlg}(X, t_X, \mathsf{p}_X, \mathsf{c}_X, \mathsf{m}_X)\big)$ are X and t_X themselves. The third and fourth components are the maps $a \mapsto W\text{-to-}X(\mathbf{h}(\mathsf{p}_T(a)))$ and $a, b, z \mapsto \mathsf{ap}_{W\text{-to-}X \circ \mathbf{h}}(\mathsf{c}_T(z))$ respectively.

The type of the final component is a mere proposition, so all we need to show is that there is a path γ equating the third component with p_X, such that the fourth component transported along γ among the fibers of the type family $f \mapsto \Pi_{a,b:A}\Pi_{z:R(a,b)}(f(a) =_X f(b))$ is equal to c_X. An easy generalization and path induction, with an appeal to function extensionality, shows that the latter condition is equivalent to the assertion that the diagram below commutes for all a, b, z:

$$\begin{array}{ccc}
W\text{-to-}X\big(\mathbf{h}(\mathsf{p}_T(a))\big) & \xrightarrow{\;\mathsf{ap}_{W\text{-to-}X \circ \mathbf{h}}(\mathsf{c}_T(z))\;} & W\text{-to-}X\big(\mathbf{h}(\mathsf{p}_T(b))\big) \\
{}^{=}\mathsf{E}^{\Pi}(\gamma, a) \Big| & & \Big| {}^{=}\mathsf{E}^{\Pi}(\gamma, b) \\
\mathsf{p}_X(a) & \xrightarrow[\;\mathsf{c}_X(z)\;]{} & \mathsf{p}_X(b)
\end{array}$$

To construct γ, we put

$$\gamma :\equiv {}^{\Pi}\mathsf{E}^{=}\Big(a \mapsto \beta_{W\text{-to-}X}(\mathsf{p}_T(a)) \cdot \beta_{T\text{-to-}X}(a)\Big)$$

It now suffices to show that the outer rectangle in the diagram below commutes:

$$\begin{array}{ccc}
W\text{-to-}X\big(\mathbf{h}(\mathsf{p}_T(a))\big) & \xrightarrow{\;\mathsf{ap}_{W\text{-to-}X \circ \mathbf{h}}(\mathsf{c}_T(z))\;} & W\text{-to-}X\big(\mathbf{h}(\mathsf{p}_T(b))\big) \\
\beta_{W\text{-to-}X}(\mathsf{p}_T(a)) \Big| & A & \Big| \beta_{W\text{-to-}X}(\mathsf{p}_T(b)) \\
T\text{-to-}X(\mathsf{p}_T(a)) & \xrightarrow[\;\mathsf{ap}_{T\text{-to-}X}(\mathsf{c}_T(z))\;]{} & T\text{-to-}X(\mathsf{p}_T(b)) \\
\beta_{T\text{-to-}X}(a) \Big| & B & \Big| \beta_{T\text{-to-}X}(b) \\
\mathsf{p}_X(a) & \xrightarrow[\;\mathsf{c}_X(z)\;]{} & \mathsf{p}_X(b)
\end{array}$$

But rectangle A commutes by an easy path induction and rectangle B commutes by the second computation rule for T-to-X, so we are done.

Step 4 Finally, we want to show that for any truncation algebra $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(3, W)$, we have $\mathsf{GQToTrAlg}(\mathsf{TrToGQAlg}(\mathcal{X})) = \mathcal{X}$. Let such an algebra $(X, t_X, |\cdot|_X)$ be given. The first and second components of $\mathsf{GQToTrAlg}\big(\mathsf{TrToGQAlg}(X, t_X, |\cdot|_X)\big)$ are X and t_X themselves. The third component is the map $W\text{-to-}X$. We need to show that for any $w : W$ we have $W\text{-to-}X(w) = |w|_X$. Since the type $W\text{-to-}X(w) = |w|_X$ is a set for any $w : W$, by an earlier observation at the beginning of the proof of lemma 197 it suffices to show that for any $y : T$ we have $W\text{-to-}X(\mathbf{h}(y)) = |\mathbf{h}(y)|_X$. However, we recall that we have the path $\beta_{W\text{-to-}X}(y) : W\text{-to-}X(\mathbf{h}(y)) = T\text{-to-}X(y)$. Hence it suffices to show that $T\text{-to-}X(y) = |\mathbf{h}(y)|_X$ for any $y : T$.

We proceed by induction, mapping $\mathsf{p}_T(a) \mapsto \beta_{T\text{-to-}X}(a)$ for any $a : A$. To map $\mathsf{c}_T(z)$ for $a, b : A$, $z : R(a, b)$, we need to show that $\beta_{T\text{-to-}X}(a)$ transported along $\mathsf{c}_T(z)$ among the fibers of the type family $x \mapsto T\text{-to-}X(x) = |\mathbf{h}(x)|_X$ is equal to $\beta_{T\text{-to-}X}(b)$. An easy generalization and path induction shows that this is equivalent to the assertion that the diagram below commutes:

$$\begin{array}{ccc} T\text{-to-}X(\mathsf{p}_T(a)) & \xrightarrow{\ \mathsf{ap}_{T\text{-to-}X}(\mathsf{c}_T(z))\ } & T\text{-to-}X(\mathsf{p}_T(b)) \\ {\scriptstyle \beta_{T\text{-to-}X}(a)} \Big| & & \Big| {\scriptstyle \beta_{T\text{-to-}X}(b)} \\ |\mathbf{h}(\mathsf{p}_T(a))|_X & \xrightarrow[\ \mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))\ ]{} & |\mathbf{h}(\mathsf{p}_T(b))|_X \end{array}$$

But this is implied by the second computation rule for $T\text{-to-}X$, as observed earlier. □

Lemma 198. $(\mathcal{H} + \mathsf{W})$ *For an algebra* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(3, W)$ *we have a function*

$$\mathsf{TrToGQFibAlg} \ : \ \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X} \ \to \ \mathsf{GQFibAlg}_{\mathcal{U}_k}\Big(\mathsf{TrToGQAlg}\ \mathcal{X}\Big)$$

which is an equivalence.

Proof. Fix an algebra $\mathcal{X} :\equiv (X, t_X, |\cdot|_X) : \mathsf{TrAlg}_{\mathcal{U}_j}(3, W)$. We recall that $\mathsf{TrToGQAlg}\ \mathcal{X}$ is the algebra $(X, t_X, \mathsf{p}_X, \mathsf{c}_X, \mathsf{m}_X)$, where

$\mathsf{p}_X(a) :\equiv |\mathbf{h}(\mathsf{p}_T(a))|_X$	for $a : A$
$\mathsf{c}_X(z) :\equiv \mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))$	for $a, b : A$, $z : R(a, b)$
$\mathsf{m}_X(v, w) :\equiv \mathcal{H}_\mathsf{c}(\theta_\mathbf{f}(d), \varepsilon_\mathbf{f}(d))$	for $a, b, c : A$, $v : R(a, b)$, $w : R(b, c)$ with $d :\equiv (a, b, c, v, w)$

We will also make use of the following notation:

- For a map $f : U \to Z$, type family $Y : V \to \mathcal{U}_m$, terms $y_1 : Y(f(x_1))$ and $y_2 : Y(f(x_2))$, and paths $p : x_1 =_U x_2$ and $q : p_*^{Y \circ f}\ y_1 = y_2$ we have an equivalence

  $$\mathcal{D}_f(q) : \left(p_*^{Y \circ f}\ y_1 = y_2\right) \simeq \left(\big(\mathsf{ap}_f(p)\big)_*^Y\ y_1 = y_2\right)$$

  defined simply by mapping q to the path

$$\big(\mathsf{ap}_f(p)\big)^Y_* \, y_1 \;\;\text{———}\;\; p^{Y\circ f}_* \, y_1 \;\;\overset{q}{\text{———}}\;\; y_2$$

We note that we will only be interested in the specific case when $f :\equiv |\cdot|_X \circ \mathbf{h}$ and will thus omit the subscript to $\mathcal{D}_f(q)$. We now proceed in four steps:

Step 1 First we define $\mathsf{TrToGQFibAlg}$; for this, take an algebra $(E, t_E, |\cdot|_E) : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$. To construct a fibered group quotient algebra, we can use the same underlying type E, with t_E showing it is fiberwise a 1-type as desired. For the third component, we need a function $\mathsf{p}_E : \Pi_{a:A} E(|\mathbf{h}(\mathsf{p}_T(a))|_X)$. The only possibility we have is to define $\mathsf{p}_E(a) :\equiv |\mathbf{h}(\mathsf{p}_T(a))|_E$. To obtain the fourth component, we need a function c_E mapping each $a, b : A$, $z : R(a,b)$ to a path from $\big(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))\big)^E_* \ |\mathbf{h}(\mathsf{p}_T(a))|_E$ to $|\mathbf{h}(\mathsf{p}_T(b))|_E$. Using the notation from above, we define $\mathsf{c}_E(z) :\equiv \mathcal{D}\big(\mathsf{dap}_{|-|_E \circ \mathbf{h}}(\mathsf{c}_T(z))\big)$, which makes sense since the term $\mathsf{dap}_{|-|_E \circ \mathbf{h}}(\mathsf{c}_T(z))$ provides us with a path from $\mathsf{c}_T(z)^{E\circ|-|_X\circ\mathbf{h}}_* \ |\mathbf{h}(\mathsf{p}_T(a))|_E$ to $|\mathbf{h}(\mathsf{p}_T(b))|_E$. To obtain the final component of our desired fibered groupoid quotient algebra, we need to show that for any $a, b, c : A$ and $v : R(a,b)$, $w : R(b,c)$, the following diagram commutes:

$$\begin{array}{ccc}
\big(\mathsf{c}_X(\mathbf{c}(v,w))\big)^E_* \ \mathsf{p}_E(a) & \overset{\textit{via } \mathsf{m}_X(v,w)}{\text{—————}} & \big(\mathsf{c}_X(v) \cdot \mathsf{c}_X(w)\big)^E_* \ \mathsf{p}_E(a) \\
\Big| \ \mathsf{c}_E(\mathbf{c}(v,w)) & & \diagup \ \mathcal{C}\big(\mathsf{c}_E(v), \mathsf{c}_E(w)\big) \\
[c]_E & &
\end{array}$$

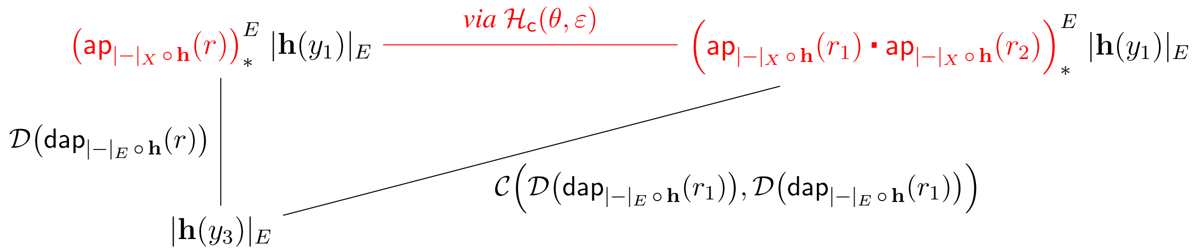
To show this, fix a, b, c, v, w with $d :\equiv (a, b, c, v, w)$, and consider the following generalization: given terms

- $p : x =_T x$, $q : x =_T y_3$, $r : y_1 =_T y_3$, $r_1 : y_1 =_T y_2$, $r_2 : y_2 =_T y_3$, as in the diagram

$$\begin{array}{ccc}
x & \overset{p}{\text{—————}} & x \\
q \Big| & & \Big| q \\
y_3 & \underset{(r_1 \cdot r_2)^{-1} \cdot r}{\text{—————}} & y_3
\end{array}$$

- $\theta : q \cdot \big((r_1 \cdot r_2)^{-1} \cdot r\big) = p \cdot q$,
- $\gamma : \mathsf{ap}_{\mathbf{h}}(p) = 1$,

the following diagram commutes

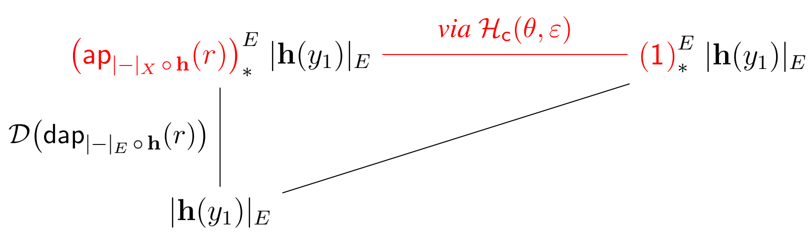


where ε denotes the path

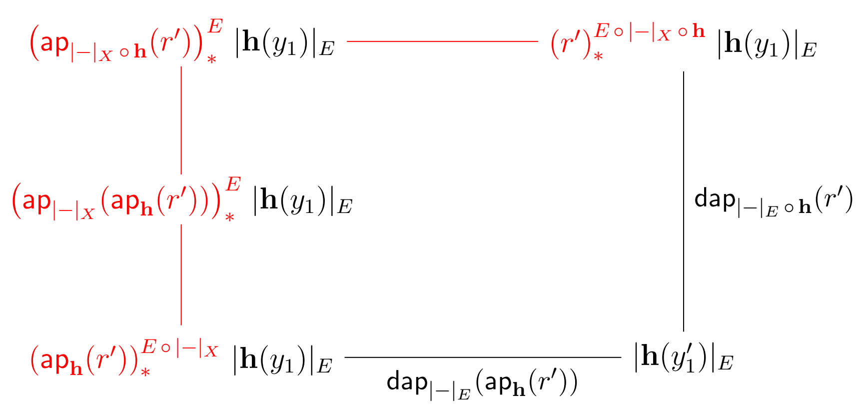
$$\mathsf{ap}_{|-|_X \circ \mathbf{h}}(p) \longrightarrow \mathsf{ap}_{|-|_X}(\mathsf{ap}_{\mathbf{h}}(p)) \xrightarrow{\textit{via } \gamma} \mathsf{ap}_{|-|_X}(1)$$

To show that the above generalization implies our original goal, we instantiate $p :\equiv \mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})$, $q :\equiv \beta_{\mathbf{f}}(d)$, $r :\equiv \mathsf{c}_T(\mathbf{c}(v, w))$, $r_1 :\equiv \mathsf{c}_T(v)$, $r_2 :\equiv \mathsf{c}_T(w)$, $\theta :\equiv \theta_{\mathbf{f}}(d)$, $\gamma :\equiv \vartheta_{\mathbf{f}}(d)$.

To prove our new goal, we perform path induction on r_1 and r_2. This reduces the above diagram to



To show that this simplified diagram commutes, we first note that for a general $r' : y_1 = y_1'$ we can express the path $\mathcal{D}\left(\mathsf{dap}_{|-|_E \circ \mathbf{h}}(r')\right)$ is an equivalent way, as justified by the diagram below which commutes by an obvious path induction:



Specializing this situation to $r' :\equiv r$, it suffices to show that the outer rectangle in the diagram below commutes:

$$
\begin{array}{ccc}
\left(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(r)\right)_*^E \; |\mathbf{h}(y_1)|_E & \overset{\textit{via } \mathcal{H}_{\mathsf{c}}(\theta, \varepsilon)}{=\!=\!=\!=\!=\!=\!=} & (1)_*^E \; |\mathbf{h}(y_1)|_E \\
\Big| & A & \Big| \\
\left(\mathsf{ap}_{|-|_X}(\mathsf{ap}_{\mathbf{h}}(r))\right)_*^E \; |\mathbf{h}(y_1)|_E & \underset{\textit{via } \mathcal{H}_{\mathsf{c}}(\theta, \gamma)}{=\!=\!=\!=\!=\!=\!=} & \left(\mathsf{ap}_{|-|_X}(1)\right)_*^E \; |\mathbf{h}(y_1)|_E \\
\Big| & B & \Big| \\
(\mathsf{ap}_{\mathbf{h}}(r))_*^{E \circ |-|_X} \; |\mathbf{h}(y_1)|_E & \underset{\mathsf{dap}_{|-|_E}(\mathsf{ap}_{\mathbf{h}}(r))}{=\!=\!=\!=\!=\!=\!=} & |\mathbf{h}(y_1)|_E
\end{array}
$$

It suffices to show that the rectangles *A* and *B* commute. The commutativity of *B* follows by generalizing $\mathcal{H}_{\mathsf{c}}(\theta, \gamma)$ together with its left endpoint $\mathsf{ap}_{\mathbf{h}}(r)$ and performing a one-sided path induction. To show the commutativity of *A*, we first perform a path induction on q. This reduces the terms $\mathcal{H}_{\mathsf{c}}(\theta, \varepsilon)$ and $\mathcal{H}_{\mathsf{c}}(\theta, \gamma)$ to $\mathsf{idc}\big(1_{|-|_X \circ \mathbf{h}}, \mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta))), \varepsilon\big)$ and $\mathsf{idc}\big(1_{\mathbf{h}}, \mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta))), \gamma\big)$ respectively. Since θ now has type $1 \cdot (1 \cdot r) = p \cdot 1$, $\mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta)))$ has type $r = p$. Thus, generalizing $\mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta)))$ and performing a one-sided path induction on it replaces r with p and reduces the terms $\mathsf{idc}\big(1_{|-|_X \circ \mathbf{h}}, \mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta))), \varepsilon\big)$ and $\mathsf{idc}\big(1_{\mathbf{h}}, \mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta))), \gamma\big)$ to ε and γ themselves. Hence we are down to showing that the diagram below commutes:

$$
\begin{array}{ccc}
\left(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(p)\right)_*^E \; |\mathbf{h}(x)|_E & \overset{\textit{via } \varepsilon}{=\!=\!=\!=\!=\!=\!=} & (1)_*^E \; |\mathbf{h}(x)|_E \\
\Big| & & \Big| \\
\left(\mathsf{ap}_{|-|_X}(\mathsf{ap}_{\mathbf{h}}(p))\right)_*^E \; |\mathbf{h}(x)|_E & \underset{\textit{via } \gamma}{=\!=\!=\!=\!=\!=\!=} & \left(\mathsf{ap}_{|-|_X}(1)\right)_*^E \; |\mathbf{h}(x)|_E
\end{array}
$$

But this follows immediately from the definition of ε and we are done.

Step 2 We define the intended quasi-inverse $\mathsf{GQToTrFibAlg}$; fix algebra $(E, t_E, \mathsf{p}_E, \mathsf{c}_E, \mathsf{m}_E) : \mathsf{GQFibAlg}_{\mathcal{U}_k} \big(\mathsf{TrToGQAlg}\ \mathcal{X}\big)$. To construct a fibered truncation algebra, we can use the same underlying type family E, with t_E showing it is fiberwise a 1-type as desired. To obtain the last component, we need a function $W\text{-to-}E : \Pi_{w:W} E(|w|_X)$. For this we will first need a function $T\text{-to-}E : \Pi_{y:T} E(|\mathbf{h}(y)|_X)$. We proceed by induction, mapping $\mathsf{p}_T(a) \mapsto \mathsf{p}_E(a)$ for any $a : A$ and $\mathsf{c}_T(z) \mapsto \mathcal{D}^{-1}(\mathsf{c}_E(z))$ for any $a, b : A$, $z : R(a, b)$. This definition makes sense as $\mathsf{c}_E(z)$ provides us with a path from $\big(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))\big)_*^E \ \mathsf{p}_E(a)$ to $\mathsf{p}_E(b)$; this means that applying $\mathcal{D}^{-1}$ to it produces a path from $\mathsf{c}_T(z)_*^{E \circ |-|_X \circ \mathbf{h}} \ \mathsf{p}_E(a)$ to $\mathsf{p}_E(b)$ as desired.

The first computation rule gives us a family of paths $\beta_{T\text{-to-}E}(a) : T\text{-to-}E(\mathsf{p}_T(a)) = \mathsf{p}_E(a)$ for $a : A$, and the second computation rule implies the commutativity of the following diagram for any $a, b : A$, $z : R(a, b)$:

$$
\begin{array}{ccc}
\mathsf{c}_T(z)_*^{E\,\circ\,|-|_X\,\circ\,\mathbf{h}}\ T\text{-to-}E(\mathsf{p}_T(a)) & \overset{\mathsf{dap}_{T\text{-to-}E}(\mathsf{c}_T(z))}{\rule{3cm}{0.4pt}} & T\text{-to-}E(\mathsf{p}_T(b)) \\
\textit{via}\ \beta_{T\text{-to-}E}(a) \Big| & & \Big| \beta_{T\text{-to-}E}(b) \\
\mathsf{c}_T(z)_*^{E\,\circ\,|-|_X\,\circ\,\mathbf{h}}\ \mathsf{p}_E(a) & \underset{\mathcal{D}^{-1}(\mathsf{c}_E(z))}{\rule{3cm}{0.4pt}} & \mathsf{p}_E(b)
\end{array}
$$

To obtain W-to-E we again proceed by induction. For this we first need to define a function

$$e : \Pi_{x:T+D}\Pi_{t:\mathbf{0}\to W}\left(\Pi_{b:\mathbf{0}}E(|t\ b|_X)\right) \to E(|\mathsf{p}_W(x,t)|_X)$$

We define $e(x,t,-) :\equiv \mathbf{e}(x,t,\alpha_\star(t))$, where for any $x : T+D$, $t : \mathbf{0} \to W$, $\alpha : t = f_{\mathbf{0}}$, the term

$$\mathbf{e}(x,t,\alpha) : E(|\mathsf{p}_W(x,t)|_X)$$

is defined by one-sided path induction on α and the subsequent mapping

$$
\begin{aligned}
\mathsf{inl}(y) &\mapsto T\text{-to-}E(y) \\
\mathsf{inr}(d) &\mapsto \mathsf{c}_W((d,\mathsf{b}), f_{\mathbf{0}}, f_{\mathbf{0}})_*^{E\,\circ\,|-|_X}\ T\text{-to-}E(\mathbf{f}(d,\mathsf{b}))
\end{aligned}
$$

To complete the inductive definition we need to construct a function

$$
\begin{gathered}
q : \Pi_{c:D\times S}\Pi_{t:\mathbf{0}\to W}\Pi_{s:\mathbf{0}\to W}\Pi_{u:(\Pi b:\mathbf{0})E(|t\ b|_X)}\Pi_{v:(\Pi b:\mathbf{0})E(|s\ b|_X)} \\
\left(\mathsf{c}_W(c,t,s)_*^{E\,\circ\,|-|_X}\ e(\mathtt{l}(c),t,u) = e(\mathtt{r}(c),s,v)\right)
\end{gathered}
$$

For this it suffices to prove the following:

- **Goal:** For any $d : D$, $x : S$, there exists a path family

$$
\begin{aligned}
c_{\mathbf{f}}(d,x) : \mathsf{c}_W((d,x), f_{\mathbf{0}}, f_{\mathbf{0}})_*^{E\,\circ\,|-|_X}\ T\text{-to-}E(\mathbf{f}(d,x)) = \\
\mathsf{c}_W((d,\mathsf{b}), f_{\mathbf{0}}, f_{\mathbf{0}})_*^{E\,\circ\,|-|_X}\ T\text{-to-}E(\mathbf{f}(d,\mathsf{b}))
\end{aligned}
$$

We can then define $q(c,t,s,-,-) :\equiv \mathbf{q}(c,t,\alpha_\star(t),s,\alpha_\star(s))$, where for $c : D\times S$, $t : \mathbf{0} \to W$, $\alpha_t : t = f_{\mathbf{0}}$, $s : \mathbf{0} \to W$, $\alpha_s : s = f_{\mathbf{0}}$, the term

$$\mathbf{q}(c,t,\alpha_t,s,\alpha_s) : \mathsf{c}_W(c,t,s)_*^{E\,\circ\,|-|_X}\ \mathbf{e}(\mathtt{l}(c),t,\alpha_t) = \mathbf{e}(\mathtt{r}(c),s,\alpha_s)$$

is defined by one-sided path induction on α_t, α_s, and the subsequent mapping $(d,x) \mapsto c_{\mathbf{f}}(d,x)$. To prove the existence of the function $c_{\mathbf{f}}$ as specified above, we will make use of the following claim:

- **Claim:** For a type $Z : \mathcal{U}_i$, type family $Y : Z \to \mathcal{U}_k$, maps $r : S \to Z$, $f : \Pi_{x:S} Y(r(x))$, term $z_\star : Z$, and path $\alpha : r = \lambda_{(y:S)} z_\star$, if $\mathsf{dap}_f(\mathsf{lp})$ is equal to the path

$$\begin{array}{c} (\mathsf{lp})_*^{Y \circ r}\ f(\mathsf{b}) \\ \Big| \\ \big(\mathsf{ap}_r(\mathsf{lp})\big)_*^{Y}\ f(\mathsf{b}) \\ \Big|\ \textit{via}\ \mathsf{idc}\big(\alpha, 1_{\mathsf{lp}}, \mathsf{apc}(z_\star, \mathsf{lp})\big) \\ (1)_*^{Y}\ f(\mathsf{b}) \end{array}$$

$(\star)$

then there is a path family

$$c(x) : \Big({}^{=}\mathsf{E}^{\Pi}(\alpha, x)\Big)_*^{Y}\ f(x) = \Big({}^{=}\mathsf{E}^{\Pi}(\alpha, \mathsf{b})\Big)_*^{Y}\ f(\mathsf{b})$$

To prove this claim, we proceed by induction on α. The term $\mathsf{idc}\big(\alpha, 1_{\mathsf{lp}}, \mathsf{apc}(z_\star, \mathsf{lp})\big)$ then becomes just $\mathsf{apc}(z_\star, \mathsf{lp})$. Furthermore, it reduces the problem to that of finding a path family $c(x) : f(x) = f(x)$. As observed before, for this it suffices to show that $\mathsf{ap}_f(\mathsf{lp}) = 1$, which is well-typed since r is now a constant function on $z_\star$, which makes f a non-dependent function from S to $Y(z_\star)$. We now generalize this situation by considering an arbitrary $p : x_1 = x_2$ instead of lp. By an obvious path induction, $\mathsf{dap}_f(p)$ is equal to the path below:

$$\begin{array}{c} p_*^{Y \circ r}\ f(x_1) \\ \Big| \\ \big(\mathsf{ap}_r(p)\big)_*^{Y}\ f(x_1) \\ \Big|\ \textit{via}\ \mathsf{apc}(z_\star, p) \\ (1)_*^{Y}\ f(x_1) \\ \Big|\ \mathsf{ap}_f(p) \\ f(x_2) \end{array}$$

Performing the instantiation $p :\equiv \mathsf{lp}$, we see that by assumption, the term $\mathsf{dap}_f(\mathsf{lp})$ is in fact equal to the composition of the *first two* equalities in the above path, which means we must necessarily have $\mathsf{ap}_f(\mathsf{lp}) = 1$ as desired.

Having proved the claim, we now return to our original goal of constructing $c_{\mathbf{f}}$. We fix $d :\equiv (a, b, c, v, w) : D$ and instantiate the above claim by putting $Z :\equiv W$, $Y :\equiv E \circ |\cdot|_X$, $r :\equiv \mathbf{h} \circ \mathbf{f}(d)$, $f :\equiv T\text{-to-}E \circ \mathbf{f}(d)$, $z_\star :\equiv \mathbf{f}_\star(d)$, $\alpha :\equiv \eta_{\mathbf{f}}(d)$. Recalling the definition of $\eta_{\mathbf{f}}(d)$ and noting that ${}^{=}\mathsf{E}^{\Pi}$ and ${}^{\Pi}\mathsf{E}^{=}$ are quasi-inverses, we see that the conclusion of the claim guarantees the existence of our desired $c_{\mathbf{f}}$. It therefore remains to establish the hypothesis of the claim, that is, show that $\mathsf{dap}_{T\text{-to-}E \,\circ\, \mathbf{f}(d)}(\mathsf{lp})$ has the form specified in $(\star)$. This is precisely the conclusion of the following generalization: given terms

- $x_k : T$ and $y_k : E(|\mathbf{h}(x_k)|_X)$ for $k \in \{1, 2, 3\}$,
- $\gamma_k : T\text{-to-}E(x_k) = y_k$ for $k \in \{1, 2, 3\}$,
- $p : x_1 = x_3$ and $q_k : x_k = x_{k+1}$ for $k \in \{1, 2\}$,
- $\beta : \mathbf{f}(d, \mathsf{b}) = x_3$,
- $s_k : \left(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(q_k)\right)_*^E \, y_k = y_{k+1}$ for $k \in \{1, 2\}$,
- $\theta : \beta \cdot \left((q_1 \cdot q_2)^{-1} \cdot p\right) = \mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}) \cdot \beta$,

the term $\mathsf{dap}_{T\text{-to-}E \,\circ\, \mathbf{f}(d)}(\mathsf{lp})$ is equal to the path

$$\begin{array}{c}
(\mathsf{lp})_*^{E \,\circ\, |-|_X \circ\, \mathbf{h} \,\circ\, \mathbf{f}(d)} \; T\text{-to-}E(\mathbf{f}(d, \mathsf{b})) \\
\Big| \\
\left(\mathsf{ap}_{\mathbf{h} \,\circ\, \mathbf{f}(d)}(\mathsf{lp})\right)_*^{E \,\circ\, |-|_X} \; T\text{-to-}E(\mathbf{f}(a, \mathsf{b})) \\
\Big| \; \textit{via } \upsilon_{\mathbf{f}}(d) \\
(1)_*^{E \,\circ\, |-|_X} \; T\text{-to-}E(\mathbf{f}(d, \mathsf{b}))
\end{array}$$

provided the diagrams below commute for $k \in \{1, 2\}$:

$$\begin{array}{ccc}
p_*^{E \,\circ\, |-|_X \circ\, \mathbf{h}} \; T\text{-to-}E(x_1) & \xrightarrow{\;\mathsf{dap}_{T\text{-to-}E}(p)\;} & T\text{-to-}E(x_3) \\
\Big| \; \textit{via } \gamma_1 & & \Big| \; \gamma_3 \\
p_*^{E \,\circ\, |-|_X \circ\, \mathbf{h}} \; y_1 & \xrightarrow[\;\mathcal{D}^{-1}(\kappa)\;]{} & y_3
\end{array}$$

$$\begin{array}{ccc}
(q_k)_*^{E \,\circ\, |-|_X \circ\, \mathbf{h}} \; T\text{-to-}E(x_k) & \xrightarrow{\;\mathsf{dap}_{T\text{-to-}E}(q_k)\;} & T\text{-to-}E(x_{k+1}) \\
\Big| \; \textit{via } \gamma_k & & \Big| \; \gamma_{k+1} \\
(q_k)_*^{E \,\circ\, |-|_X \circ\, \mathbf{h}} \; y_k & \xrightarrow[\;\mathcal{D}^{-1}(s_k)\;]{} & y_{k+1}
\end{array}$$

Here κ denotes the following path:

$$\begin{array}{l}
\left(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(p)\right)^E_* \; y_1 \\
\quad \Big| \; \textit{via } \mathcal{H}_{\mathsf{c}}(\theta, \varepsilon_{\mathbf{f}}(d)) \\
\left(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(q_1) \cdot \mathsf{ap}_{|-|_X \circ \mathbf{h}}(q_2)\right)^E_* \; y_1 \\
\quad \Big| \; \mathcal{C}(s_1, s_2) \\
y_3
\end{array}$$

We instantiate this generalization by $\gamma_1 :\equiv \beta_{T\text{-to-}E}(a)$, $\gamma_2 :\equiv \beta_{T\text{-to-}E}(b)$, $\gamma_3 :\equiv \beta_{T\text{-to-}E}(c)$, $p :\equiv \mathsf{c}_T(\mathbf{c}(v,w))$, $q_1 :\equiv \mathsf{c}_T(v)$, $q_2 :\equiv \mathsf{c}_T(w)$, $\beta :\equiv \beta_{\mathbf{f}}(d)$, $s_1 :\equiv \mathsf{c}_E(v)$, $s_2 :\equiv \mathsf{c}_E(w)$, $\theta :\equiv \theta_{\mathbf{f}}(d)$. The commutativity of the diagrams in the hypotheses is implied by the second computation rule for T-to-E and $\mathsf{m}_E(v,w)$.

Finally, to prove this generalization we perform one-sided path induction on β, γ_1, γ_2, γ_3, q_1, q_2. This reduces the term $\mathcal{H}_{\mathsf{c}}(\theta, \varepsilon_{\mathbf{f}}(d))$ to $\mathsf{idc}\big(1_{|-|_X \circ \mathbf{h}}, \mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta))), \varepsilon_{\mathbf{f}}(d)\big)$, and the commutativity of the first diagram in the hypothesis then becomes equivalent to the condition $\mathsf{dap}_{T\text{-to-}E}(p) = \mathcal{D}^{-1}(\kappa)$, where κ is the path

$$\begin{array}{l}
\left(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(p)\right)^E_* \; T\text{-to-}E(\mathbf{f}(d, \mathsf{b})) \\
\quad \Big| \; \textit{via } \mathsf{idc}\big(1_{|-|_X \circ \mathbf{h}}, \mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta))), \varepsilon_{\mathbf{f}}(d)\big) \\
(1)^E_* \; T\text{-to-}E(\mathbf{f}(d, \mathsf{b})) \\
\quad \Big| \; \mathcal{C}(s_1, s_2) \\
T\text{-to-}E(\mathbf{f}(d, \mathsf{b}))
\end{array}$$

Furthermore, the commutativity of the remaining two diagrams in the hypothesis becomes equivalent to the conditions $s_1 = 1$, $s_2 = 1$. Performing one-sided path induction on these last two conditions then replaces s_1 and s_2 with reflexivities, which in turn reduces the term $\mathcal{C}(s_1, s_2)$ to reflexivity.

Also, θ now has type $1 \cdot (1 \cdot p) = \mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}) \cdot 1$, which means that the term $\mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta)))$ has type $p = \mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})$. Thus, generalizing $\mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta)))$ and performing a one-sided path induction on it replaces p with $\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})$ and reduces the term $\mathsf{idc}\big(1_{|-|_X \circ \mathbf{h}}, \mathcal{I}_l(\mathcal{I}_l(\mathcal{I}_r(\theta))), \varepsilon_{\mathbf{f}}(d)\big)$ to $\varepsilon_{\mathbf{f}}(d)$. Hence it suffices to establish the conclusion of the generalization under the hypothesis that $\mathsf{dap}_{T\text{-to-}E}(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}))$ equals the path

$$
\begin{array}{c}
\left(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})\right)_*^{E \,\circ\, |-|_X \circ\, \mathbf{h}}\ T\text{-to-}E(\mathbf{f}(d,\mathsf{b})) \\
\Big| \\
\left(\mathsf{ap}_{|-|_X \,\circ\, \mathbf{h}}\left(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})\right)\right)_*^{E}\ T\text{-to-}E(\mathbf{f}(d,\mathsf{b})) \\
\Big| \ \textit{via}\ \varepsilon_{\mathbf{f}}(d) \\
(1)_*^{E}\ T\text{-to-}E(\mathbf{f}(d,\mathsf{b}))
\end{array}
$$

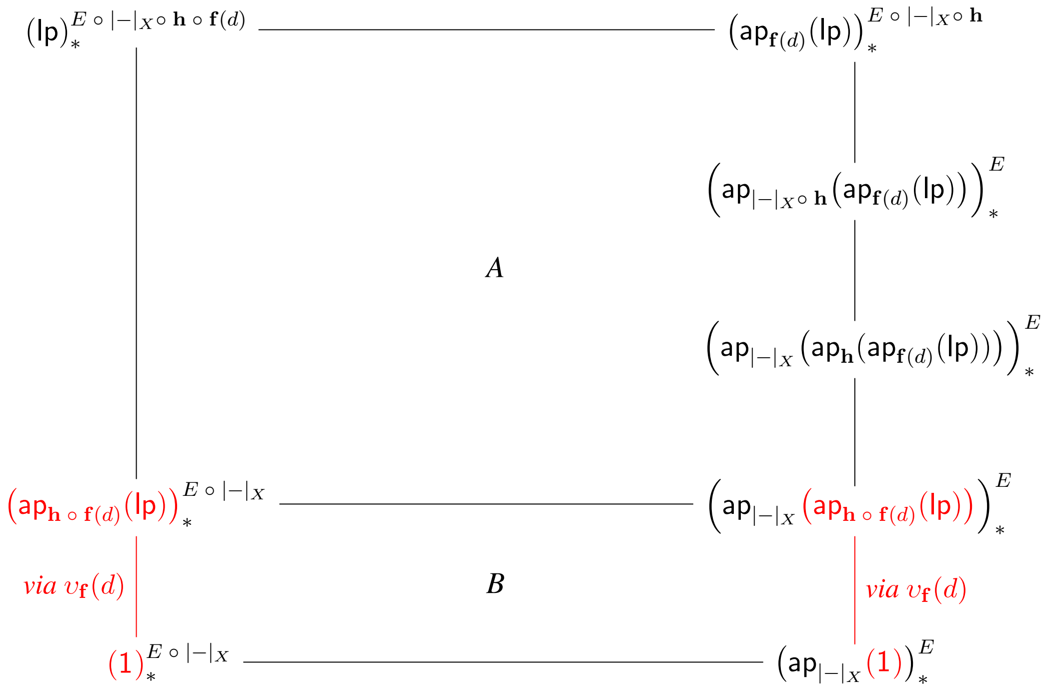
To do so, we first relate the two terms $\mathsf{dap}_{T\text{-to-}E \,\circ\, \mathbf{f}(d)}(\mathsf{lp})$ and $\mathsf{dap}_{T\text{-to-}E}(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp}))$ by observing that the former is equal to the path below by an easy path induction:

$$
\begin{array}{c}
(\mathsf{lp})_*^{E \,\circ\, |-|_X \circ\, \mathbf{h} \,\circ\, \mathbf{f}(d)}\ T\text{-to-}E(\mathbf{f}(d,\mathsf{b})) \\
\Big| \\
\left(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})\right)_*^{E \,\circ\, |-|_X \circ\, \mathbf{h}}\ T\text{-to-}E(\mathbf{f}(d,\mathsf{b})) \\
\Big| \ \mathsf{dap}_{T\text{-to-}E}\left(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})\right) \\
T\text{-to-}E(\mathbf{f}(d,\mathsf{b}))
\end{array}
$$

Hence it only remains to show that the following diagram commutes:

$$
\begin{array}{ccc}
(\mathsf{lp})_*^{E \,\circ\, |-|_X \circ\, \mathbf{h} \,\circ\, \mathbf{f}(d)} & \text{———} & \left(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})\right)_*^{E \,\circ\, |-|_X \circ\, \mathbf{h}} \\
\Big| & & \Big| \\
\left(\mathsf{ap}_{\mathbf{h} \,\circ\, \mathbf{f}(d)}(\mathsf{lp})\right)_*^{E \,\circ\, |-|_X} & & \left(\mathsf{ap}_{|-|_X \,\circ\, \mathbf{h}}\left(\mathsf{ap}_{\mathbf{f}(d)}(\mathsf{lp})\right)\right)_*^{E} \\
\textit{via}\ \upsilon_{\mathbf{f}}(d)\ \Big| & & \Big| \ \textit{via}\ \varepsilon_{\mathbf{f}}(d) \\
(1)_*^{E \,\circ\, |-|_X} & \text{———} & (1)_*^{E}
\end{array}
$$

Expanding the definition of $\varepsilon_{\mathbf{f}}(d)$, we see that it remains to show that the outer rectangle in the diagram below commutes:



But both of the inner rectangles *A* and *B* commute by an easy generalization and subsequent path induction, so we are done.

Although this step is finished, it will be useful to note that since W-to-E was defined by induction, the first computation rule gives us a family of paths

$$\beta_{W\text{-to-}E}(y) : W\text{-to-}E(\mathsf{h}(y)) = \mathbf{e}\big(\mathsf{inl}(y), f_{\mathbf{0}}, \alpha_\star(f_{\mathbf{0}})\big)$$

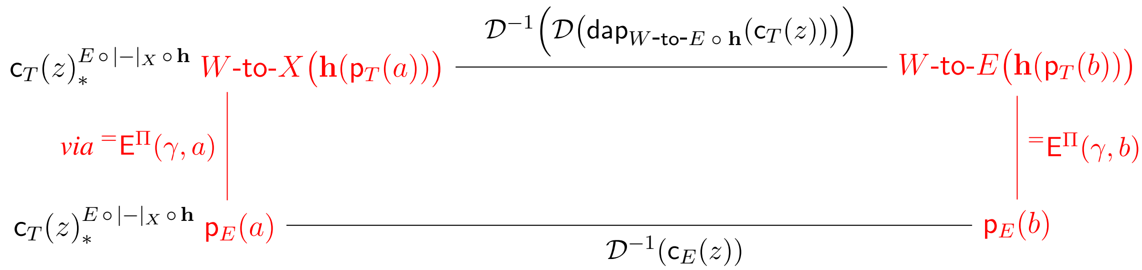
for any $y : T$. But of course we have $\alpha_\star(f_{\mathbf{0}}) = 1_{f_{\mathbf{0}}}$ so in fact we have a family of paths

$$\beta'_{W\text{-to-}E}(y) : W\text{-to-}E(\mathsf{h}(y)) = T\text{-to-}E(y)$$

for any $y : T$.

Step 3 We show that for a fibered groupoid quotient algebra $\mathcal{Y} : \mathsf{GQFibAlg}_{\mathcal{U}_k}\ \big(\mathsf{TrToGQAlg}\ \mathcal{X}\big)$, we have $\mathsf{TrToGQFibAlg}(\mathsf{GQToTrFibAlg}(\mathcal{Y})) = \mathcal{Y}$. Let such an algebra $(E, t_E, \mathsf{p}_E, \mathsf{c}_E, \mathsf{m}_E)$ be given. The first and second components of $\mathsf{TrToGQAlg}\big(\mathsf{GQToTrAlg}(E, t_E, \mathsf{p}_E, \mathsf{c}_E, \mathsf{m}_E)\big)$ are E and t_E themselves. The third and fourth components are the maps $a \mapsto W\text{-to-}E(\mathbf{h}(\mathsf{p}_T(a)))$ and $a, b, z \mapsto \mathcal{D}\big(\mathsf{dap}_{W\text{-to-}E \circ \mathbf{h}}(\mathsf{c}_T(z))\big)$ respectively.

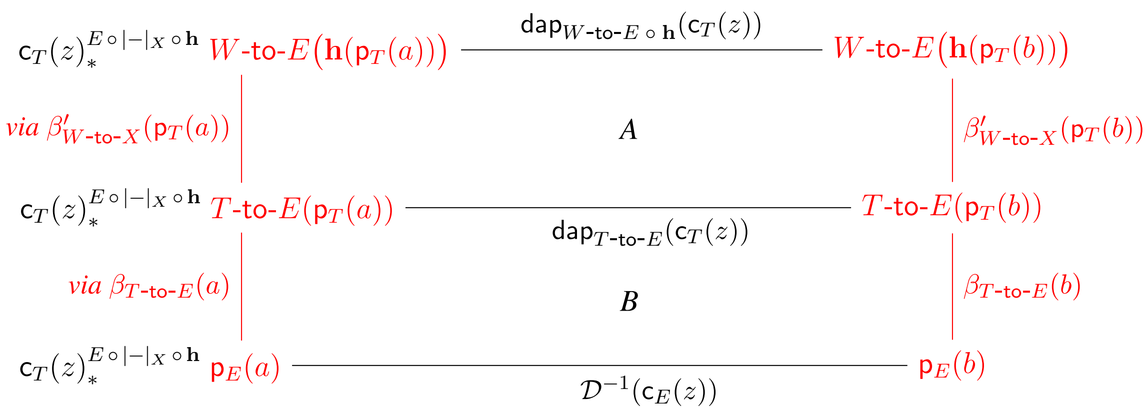
The type of the final component is a mere proposition, so all we need to show is that there is a path γ equating the third component with p_E, such that the fourth component transported along γ among the fibers of the type family $f \mapsto \Pi_{a,b:A}\Pi_{z:R(a,b)}\Big(\big(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))\big)_*^E\ f(a) = f(b)\Big)$ is equal to c_E. An easy generalization and path induction, with function extensionality, shows that the latter condition is equivalent to the assertion that the diagram below commutes for all a, b, z:



To construct γ, we put

$$\gamma :\equiv {}^{\Pi}\mathsf{E}^{=}\Big(a \mapsto \beta'_{W\text{-to-}E}(\mathsf{p}_T(a)) \cdot \beta_{T\text{-to-}E}(a)\Big)$$

It thus remains to show that the outer rectangle in the diagram below commutes:



But rectangle *A* commutes by an easy path induction and rectangle *B* commutes by the second computation rule for T-to-E, so we are done.

Step 4 Finally, we want to show that for any fibered truncation algebra $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}\ \mathcal{X}$, we have $\mathsf{GQToTrFibAlg}(\mathsf{TrToGQFibAlg}(\mathcal{Y})) = \mathcal{Y}$. Let such an algebra $(E, t_E, |\cdot|_E)$ be given. The first and second components of $\mathsf{GQToTrFibAlg}\big(\mathsf{TrToGQFibAlg}(E, t_E, |\cdot|_E)\big)$ are E and t_E themselves. The third component is the map W-to-E. We thus need to show that for any $w : W$, we have $W\text{-to-}E(w) = |w|_E$. Since the type $W\text{-to-}X(w) = |w|_E$ is a set for any $w : W$, by an earlier observation at the beginning of the proof of lemma 197 it suffices to show that for any $y : T$ we have $W\text{-to-}E(\mathbf{h}(y)) = |\mathbf{h}(y)|_E$. However, we recall that we have the path $\beta'_{W\text{-to-}E}(y) : W\text{-to-}E(\mathbf{h}(y)) = T\text{-to-}E(y)$. Hence it suffices to show that $T\text{-to-}E(y) = |\mathbf{h}(y)|_E$ for any $y : T$.

We proceed by induction, mapping $\mathsf{p}_T(a) \mapsto \beta_{T\text{-to-}E}(a)$ for any $a : A$. To map $\mathsf{c}_T(z)$ for $a, b : A$, $z : R(a, b)$, we need to show that $\beta_{T\text{-to-}E}(a)$ transported along $\mathsf{c}_T(z)$ among the fibers of the type family $x \mapsto T\text{-to-}E(x) = |\mathbf{h}(x)|_E$ is equal to $\beta_{T\text{-to-}E}(b)$. An easy generalization and path induction shows that this is equivalent to the assertion that the diagram below commutes:

$$
\begin{array}{ccc}
\mathsf{c}_T(z)_*^{E \circ |-|_X \circ \mathbf{h}}\ T\text{-to-}E(\mathsf{p}_T(a)) & \overset{\mathsf{dap}_{T\text{-to-}E}(\mathsf{c}_T(z))}{=\!=\!=\!=\!=} & T\text{-to-}E(\mathsf{p}_T(b)) \\
\textit{via } \beta_{T\text{-to-}E}(a) \Big\| & & \Big\| \beta_{T\text{-to-}E}(b) \\
\mathsf{c}_T(z)_*^{E \circ |-|_X \circ \mathbf{h}}\ |\mathbf{h}(\mathsf{p}_T(a))|_E & \underset{\mathsf{dap}_{|-|_E \circ \mathbf{h}}(\mathsf{c}_T(z))}{=\!=\!=\!=\!=} & |\mathbf{h}(\mathsf{p}_T(b))|_E
\end{array}
$$

But this is implied by the second computation rule for T-to-E. □

Lemma 199. $(\mathcal{H} + \mathsf{W})$ *For algebras* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(3, W)$ *and* $\mathcal{Y} : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$ *we have*

$$
\mathsf{TrFibMor}\ \mathcal{X}\ \mathcal{Y} \ \simeq\ \mathsf{GQFibMor}\ \Big(\mathsf{TrToGQAlg}\ \mathcal{X}\Big)\ \Big(\mathsf{TrToGQFibAlg}(\mathcal{X})\ \mathcal{Y}\Big)
$$

Proof. Fix an algebra $\mathcal{X} :\equiv (X, t_X, |\cdot|_X) : \mathsf{TrAlg}_{\mathcal{U}_j}(3, W)$. We recall that $\mathsf{TrToGQAlg}\ \mathcal{X}$ is the algebra $(X, t_X, \mathsf{p}_X, \mathsf{c}_X, \mathsf{m}_X)$, where

$$
\begin{aligned}
\mathsf{p}_X(a) &:\equiv |\mathbf{h}(\mathsf{p}_T(a))|_X && \text{for } a : A \\
\mathsf{c}_X(z) &:\equiv \mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z)) && \text{for } a, b : A,\ z : R(a, b)
\end{aligned}
$$

Fix an algebra $\mathcal{Y} :\equiv (E, t_E, |\cdot|_E) : \mathsf{TrFibAlg}_{\mathcal{U}_k}\ \mathcal{X}$. Then $\mathsf{TrToGQFibAlg}(\mathcal{X})\ \mathcal{Y}$ is the algebra $(E, t_E, \mathsf{p}_E, \mathsf{c}_E, \mathsf{m}_E)$, where

$$
\begin{aligned}
\mathsf{p}_E(a) &:\equiv |\mathbf{h}(\mathsf{p}_T(a))|_E && \text{for } a : A \\
\mathsf{c}_E(z) &:\equiv \mathcal{D}\big(\mathsf{dap}_{|-|_E \circ \mathbf{h}}(\mathsf{c}_T(z))\big) && \text{for } a, b : A,\ z : R(a, b)
\end{aligned}
$$

We now proceed in four steps:

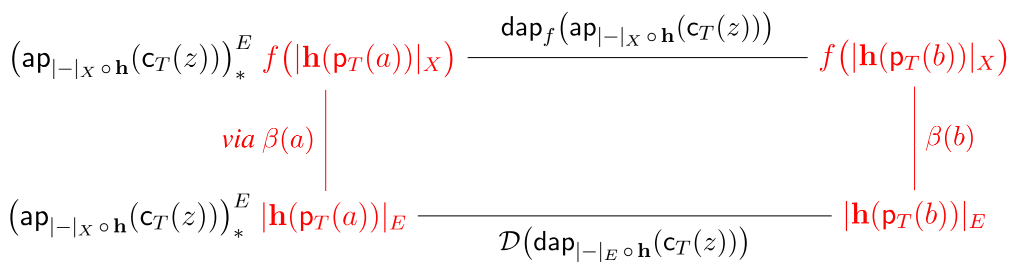
Step 1 First we define a function $\mathsf{TrToGQFibMor}$ going from left to right. For this, take a morphism $(f, \beta) : \mathsf{TrFibMor}\ \mathcal{X}\ \mathcal{Y}$. To construct the desired fibered group quotient morphism, we can use the same underlying map f. For the second component, we need a function $\beta_{./.} : \Pi_{a:A}\big(f(|\mathbf{h}(\mathsf{p}_T(a))|_X) = |\mathbf{h}(\mathsf{p}_T(a))|_E\big)$. The obvious choice is to define $\beta_{./.} :\equiv \beta(\mathbf{h}(\mathsf{p}_T(a)))$. To obtain the final component, we need to show that the following diagram commutes for all $a, b : A$, $z : R(a, b)$:

$$
\begin{array}{ccc}
\big(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))\big)_*^E\ f\big(|\mathbf{h}(\mathsf{p}_T(a))|_X\big) & \overset{\mathsf{dap}_f\left(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))\right)}{=\!=\!=\!=\!=} & f\big(|\mathbf{h}(\mathsf{p}_T(b))|_X\big) \\
\textit{via } \beta\big(\mathbf{h}(\mathsf{p}_T(a))\big) \Big\| & & \Big\| \beta\big(\mathbf{h}(\mathsf{p}_T(b))\big) \\
\big(\mathsf{ap}_{|-|_X \circ \mathbf{h}}(\mathsf{c}_T(z))\big)_*^E\ |\mathbf{h}(\mathsf{p}_T(a))|_E & \underset{\mathcal{D}\left(\mathsf{dap}_{|-|_E \circ \mathbf{h}}(\mathsf{c}_T(z))\right)}{=\!=\!=\!=\!=} & |\mathbf{h}(\mathsf{p}_T(b))|_E
\end{array}
$$

But this follows by a straightforward generalization and path induction.

Step 2 We define the intended quasi-inverse GQToTrFibMor going from right to left. For this, take a morphism (f, β, θ) : GQFibMor $\big(\mathsf{TrToGQAlg}\ \mathcal{X}\big)$ $\big(\mathsf{TrToGQFibAlg}(\mathcal{X})\ \mathcal{Y}\big)$. To construct a truncation morphism, we can use the same underlying map f. For the second component, we need a function $\beta_{|\cdot|} : \Pi_{w:W}(f(|w|_X) = |w|_E)$. As the type $f(|w|_X) = |w|_E$ is a set for any $w : W$, by an earlier observation at the beginning of the proof of lemma 197 it suffices to construct a function $\gamma_{|\cdot|} : \Pi_{y:T}(f(|\mathbf{h}(y)|_X) = |\mathbf{h}(y)|_E)$.

We construct $\gamma_{|\cdot|}$ by induction, mapping $\mathsf{p}_T(a) \mapsto \beta(a)$ for any $a : A$. To map $\mathsf{c}_T(z)$ for $a, b : A$, $z : R(a, b)$, we need to show that $\beta(a)$ transported along $\mathsf{c}_T(z)$ among the fibers of the type family $y \mapsto f(|\mathbf{h}(y)|_X) = |\mathbf{h}(y)|_E$ is equal to $\beta(b)$. An easy generalization and path induction shows that this is equivalent to the assertion that the diagram below commutes:



But this is implied right away by θ.

Although we are done with this step, it will be useful to note that by our definition of $\beta_{|\cdot|}$ and $\gamma_{|\cdot|}$, we have $\beta_{|\cdot|}(\mathbf{h}(y)) = \gamma_{|\cdot|}(y)$ for any $y : T$ and $\gamma_{|\cdot|}(\mathsf{p}_T(a)) = \beta(a)$ for any $a : A$.

Step 3 We show that for a morphism μ : GQFibMor $\big(\mathsf{TrToGQAlg}\ \mathcal{X}\big)$ $\big(\mathsf{TrToGQFibAlg}(\mathcal{X})\ \mathcal{Y}\big)$, we have TrToGQFibMor(GQToTrFibMor(μ)) $= \mu$. Let such a morphism (f, β, θ) be given. Then f itself is the first component of TrToGQFibMor$\big($GQToTrFibMor$(f, \beta, \theta)\big)$. The second component is the map $a \mapsto \beta_{|\cdot|}(\mathbf{h}(\mathsf{p}_T(a)))$. The type of the final component is a mere proposition, so all we need to show is that there is a path equating the third component with β. Using function extensionality, we only need to show that for any $a : A$ we have $\beta_{|\cdot|}(\mathbf{h}(\mathsf{p}_T(a))) = \beta(a)$. But as we noted at the end of the previous step, by construction we have

$$\beta_{|\cdot|}(\mathbf{h}(\mathsf{p}_T(a))) = \gamma_{|\cdot|}(\mathsf{p}_T(a)) = \beta(a)$$

so we are done.

Step 4 We show that we have GQToTrFibMor(TrToGQFibMor(μ)) $= \mu$ for any morphism μ : TrFibMor $\mathcal{X}$ $\mathcal{Y}$. Let such morphism (f, β) be given. Then f itself is the first component of GQToTrFibMor(TrToGQFibMor(f, β)). The second component is the map $\beta_{|\cdot|}$. Using function extensionality, we only need to show that for any $w : W$ we have $\beta_{|\cdot|} = \beta(w)$. Since the type $\beta_{|\cdot|}(w) = \beta(w)$ is a mere proposition for any $w : W$, and hence a set, by an earlier observation at the beginning of the proof of lemma 197 it suffices to show that for any $y : T$ we have $\beta_{|\cdot|}(\mathbf{h}(y)) = \beta(\mathbf{h}(y))$. However, we recall that we have $\beta_{|\cdot|}(\mathbf{h}(y)) = \gamma_{|\cdot|}(y)$. Hence it suffices to show that $\gamma_{|\cdot|}(y) = \beta(\mathbf{h}(y))$ for any $y : T$. But since the type $\gamma_{|\cdot|}(y) = \beta(\mathbf{h}(y))$ is a mere proposition for any $y : T$, it suffices to show that it is inhabited for $y :\equiv \mathsf{p}_T(a)$, *i.e.*, that for any

$a : A$ we have $\gamma_{|\cdot|}(\mathsf{p}_T(a)) = \beta(\mathbf{h}(\mathsf{p}_T(a)))$. But this follows right away from the construction of $\gamma_{|\cdot|}$. □

Lemma 200. $(\mathcal{H} + \mathsf{W})$ *For algebras* $\mathcal{X} : \mathsf{TrAlg}_{\mathcal{U}_j}(3, W)$ *and* $\mathcal{Y} : \mathsf{TrAlg}_{\mathcal{U}_k}(3, W)$ *we have*

$$\mathsf{TrMor}\ \mathcal{X}\ \mathcal{Y} \;\; \simeq \;\; \mathsf{GQFibMor}\left(\mathsf{TrToGQAlg}\ \mathcal{X}\right)\left(\mathsf{TrToGQAlg}\ \mathcal{Y}\right)$$

Proof. Exactly as in the fibered case. □

Corollary 201. $(\mathcal{H} + \mathsf{W})$ *For an algebra* $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A, \mathbf{G})$ *we have*

$$\begin{aligned} \mathsf{hasGQRec}_{\mathcal{U}_k}(\mathcal{X}) \;\; &\simeq \;\; \mathsf{hasTrRec}_{\mathcal{U}_k}\left(\mathsf{TrToGQAlg}^{-1}(\mathcal{X})\right) \\ \mathsf{hasGQInd}_{\mathcal{U}_k}(\mathcal{X}) \;\; &\simeq \;\; \mathsf{hasTrInd}_{\mathcal{U}_k}\left(\mathsf{TrToGQAlg}^{-1}(\mathcal{X})\right) \\ \mathsf{isGQHInit}_{\mathcal{U}_k}(\mathcal{X}) \;\; &\simeq \;\; \mathsf{isTrHInit}_{\mathcal{U}_k}\left(\mathsf{TrToGQAlg}^{-1}(\mathcal{X})\right) \end{aligned}$$

Corollary 202. $(\mathcal{H} + \mathsf{W})$ *For* $A : \mathcal{U}_i$, $\mathbf{G} : \mathsf{Grp}(A)$, *the following conditions on an algebra* $\mathcal{X} : \mathsf{GQAlg}_{\mathcal{U}_j}(A, \mathbf{G})$ *are equivalent:*

- $\mathcal{X}$ *satisfies the induction principle on the universe* $\mathcal{U}_k$
- $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_k$

for $k \geq j$. *In other words, we have*

$$\mathsf{hasGQInd}_{\mathcal{U}_k}(\mathcal{X}) \;\; \simeq \;\; \mathsf{isGQHInit}_{\mathcal{U}_k}(\mathcal{X})$$

provided $k \geq j$. *Moreover, the two types above are mere propositions.*

Corollary 203. $(\mathcal{H} + \mathsf{W} + \cdot/_1\cdot)$ *For* $A : \mathcal{U}_i$, $\mathbf{G} : \mathsf{Grp}(A)$, *the algebra*

$$\left(A/_1\mathbf{G}, \mathsf{point}_1, \mathsf{cell}_1\right) : \mathsf{GQAlg}_{\mathcal{U}_i}(A, \mathbf{G})$$

is homotopy-initial on any universe $\mathcal{U}_j$.

Analogously to set quotients, homotopy-initial groupoid quotient algebras enjoy the property of *effectiveness*, in the sense that the path structure on the groupoid quotient is "the same" as the structure specified by $\mathbf{G}$:

Lemma 204. $(\mathcal{H})$ *For* $A : \mathcal{U}_i$, $\mathbf{G} : \mathsf{Grp}(A)$, *and algebra* $\mathcal{X} :\equiv (D, t_D, \mathsf{p}, \mathsf{c}, \mathsf{m}) : \mathsf{GQAlg}_{\mathcal{U}_i}(A, \mathbf{G})$, *where* $\mathbf{G} :\equiv (R, t_R, \mathbf{r}, \mathbf{i}, \mathbf{c}, \ldots)$, *if* $\mathcal{X}$ *is homotopy-initial on the universe* $\mathcal{U}_{i+1}$, *then for any* $a, b : A$ *we have*

$$(\mathsf{p}(a) = \mathsf{p}(b)) \;\; \simeq \;\; R(a, b)$$

Proof. As in the case of set quotients, we employ the "encode-decode" method by Licata. We define a function $\mathsf{C} : D \to D \to \Sigma_{X:\mathcal{U}_i}\mathsf{isSet}(X)$, where $\mathsf{C}(x, y)$ is intended to explicitly describe the path type $x = y$, and then show that we indeed have $(x = y) \simeq \pi_1(\mathsf{C}(x, y))$. Since C will be defined in such a way that $\pi_1\big(\mathsf{C}(\mathsf{p}(a), \mathsf{p}(b))\big)$ is equivalent to $R(a, b)$, this will finish the proof. To obtain the desired equivalence, we construct maps

$$\mathsf{e}(x, y) : (x = y) \to \pi_1(\mathsf{C}(x, y))$$
$$\mathsf{d}(x, y) : \pi_1(\mathsf{C}(x, y)) \to (x = y)$$

and show that they compose to identities. We will also make use of a couple observations and notations:

- For any $a, b_1, b_2 : A$ and $z : R(b_1, b_2)$, we have the equivalence

  $$R(a, b_1) \simeq R(a, b_2)$$

  given by post-composition with z. Let $\mathsf{postceq}(a, z)$ denote the witness that this map is indeed an equivalence.
- For any $a_1, a_2, b : A$ and $z : R(a_1, a_2)$, we have the equivalence

  $$R(a_1, b) \simeq R(a_2, b)$$

  given by pre-composition with the inverse of z. Let $\mathsf{preceq}(b, z)$ denote the witness that this map is indeed an equivalence.
- For any $a, b : A$, we let $\mathbf{R}(a, b) :\equiv (R(a, b), t_R(a, b)) : \Sigma_{X:\mathcal{U}_i}\mathsf{isSet}(X)$.
- For any $\mathbf{X}, \mathbf{Y} : \Sigma_{X:\mathcal{U}_i}\mathsf{isSet}(X)$, we have the equivalence

  $$\mathsf{ap}_{\pi_1} : (\mathbf{X} = \mathbf{Y}) \to (\pi_1(\mathbf{X}) \simeq \pi_1(\mathbf{Y}))$$

  since the type $\mathsf{isSet}(X)$ is a mere proposition for any X. By the univalence axiom, we also have the equivalence

  $$^{=}\mathsf{E}^{\simeq} : (\pi_1(\mathbf{X}) = \pi_1(\mathbf{Y})) \to (\pi_1(\mathbf{X}) \simeq \pi_1(\mathbf{Y}))$$

  The composition thus yields an equivalence

  $$K : (\mathbf{X} = \mathbf{Y}) \to (\pi_1(\mathbf{X}) \simeq \pi_1(\mathbf{Y}))$$

  We let $K_1 :\equiv \pi_1 \circ K$.
- For any $\mathbf{X}, \mathbf{Y} : \Sigma_{X:\mathcal{U}_i}\mathsf{isSet}(X)$ and $\alpha_1, \alpha_2 : \mathbf{X} = \mathbf{Y}$, we have

  $$(\alpha_1 = \alpha_2) \simeq (K_1(\alpha_1) = K_1(\alpha_2))$$

  This follows as K is an equivalence and the type $\mathsf{isEq}(f)$ is a mere proposition for any f.
- For any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} : \Sigma_{X:\mathcal{U}_i}\mathsf{isSet}(X)$ and $\alpha_1 : \mathbf{X} = \mathbf{Y}$, $\alpha_2 : \mathbf{Y} = \mathbf{Z}$, we have

  $$K_1(\alpha_1 \cdot \alpha_2) = K_1(\alpha_2) \circ K_1(\alpha_1)$$

We now proceed in five steps:

Step 1 To define C, we proceed by recursion on the first argument. We are allowed to do this since the type $D \to \Sigma_{X:\mathcal{U}_i}\mathsf{isSet}(X)$ indeed belongs to $\mathcal{U}_{i+1}$ and is a 1-type (the latter follows since $\Sigma_{X:\mathcal{U}_i}\mathsf{isSet}(X)$ is itself a 1-type, as a simple exercise or an appeal to theorem 7.1.11 in [33] shows).

Now we need to define a function $F : A \to D \to \Sigma_{X:\mathcal{U}_i}\mathsf{isSet}(X)$. We do this by recursion on the second argument. So we fix $a : A$ and map $\mathsf{p}(b) \mapsto \mathbf{R}(a,b)$. To map $\mathsf{c}(z)$ for $b_1, b_2 : A$, $z : R(b_1, b_2)$, we need to construct a path $\gamma_F(a,z) : \mathbf{R}(a,b_1) = \mathbf{R}(a,b_2)$. Using the notation from above, we define

$$\gamma_F(a,z) :\equiv K^{-1}\big(\mathbf{c}(-,z), \mathsf{postceq}(a,z)\big)$$

To finish the definition of F, we need to show that for any $b_1, b_2, b_3 : A$ and $z_1 : R(b_1,b_2)$, $z_2 : R(b_2,b_3)$, we have

$$\gamma_F(a, \mathbf{c}(z_1,z_2)) = \gamma_F(a,z_1) \cdot \gamma_F(a,z_2)$$

As observed above, for this it suffices to show that

$$K_1\big(\gamma_F(a,\mathbf{c}(z_1,z_2))\big) = K_1\big(\gamma_F(a,z_1) \cdot \gamma_F(a,z_2)\big)$$

By yet another observation from above, it suffices to show that

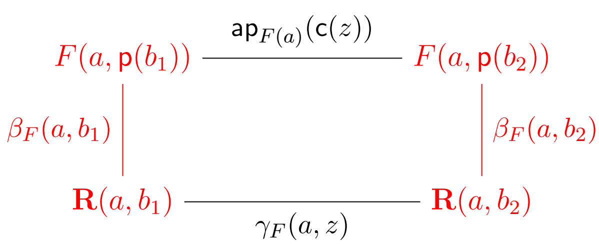
$$K_1\big(\gamma_F(a,\mathbf{c}(z_1,z_2))\big) = K_1(\gamma_F(a,z_2)) \circ K_1(\gamma_F(a,z_1))$$

Unfolding the definition of γ_F we see that this is equivalent to showing that

$$\mathbf{c}(-,\mathbf{c}(z_1,z_2)) = \mathbf{c}(-,z_2) \circ \mathbf{c}(-,z_1)$$

But this is obvious from function extensionality and the groupoid properties.

Although the definition of $F(a)$ is finished, we note that the computation rules give us a family of paths $\beta_F(a,b) : F(a,\mathsf{p}(b)) = \mathbf{R}(a,b)$ for $a, b : A$ such that the diagram below commutes for any $a, b_1, b_2 : A$ and $z : R(b_1,b_2)$:



We now continue with our definition of C. For any $a_1, a_2 : A$ and $w : R(a_1,a_2)$, we must construct a path from $F(a_1)$ to $F(a_2)$. By function extensionality, it suffices to construct a homotopy $\delta_{\mathsf{C}}(w) : F(a_1) \sim F(a_2)$. Our desired path between $F(a_1)$ and $F(a_2)$ will then be ${}^{\Pi}\mathsf{E}^{=}(x \mapsto \delta_{\mathsf{C}}(w))$. To construct $\delta_{\mathsf{C}}(w)$, we proceed by induction on its argument. It is clear that for any $x : D$, the type $F(a_1,x) = F(a_2,x)$ belongs to $\mathcal{U}_{i+1}$. We also need to show that it is a 1-type, which is true since it is in fact a set. To map $\mathsf{p}(b)$ for $b : A$, we need to construct a path equating $F(a_1,\mathsf{p}(b))$ with $F(a_2,\mathsf{p}(b))$. Appealing to $\beta_F(a_1,b)$ and $\beta_F(a_2,b)$, we see that for this it suffices to construct a path $\gamma_{\mathsf{C}}(w,b) : \mathbf{R}(a_1,b) = \mathbf{R}(a_2,b)$. We define

$$\gamma_{\mathsf{C}}(w,b) :\equiv K^{-1}\big(\mathbf{c}(\mathbf{i}(w),-), \mathsf{preceq}(b,w)\big)$$

Putting the above together, we map

$$\mathsf{p}(b) \mapsto \beta_F(a_1, b) \cdot \gamma_{\mathsf{C}}(w, b) \cdot \beta_F(a_2, b)^{-1}$$

To map $\mathsf{c}(z)$ for $b_1, b_2 : A$, $z : R(b_1, b_2)$, we must show that $\beta_F(a_1, b_1) \cdot \gamma_{\mathsf{C}}(w, b_1) \cdot \beta_F(a_2, b_1)^{-1}$ transported along $\mathsf{c}(z)$ among the fibers of the type family $x \mapsto F(a_1, x) = F(a_2, x)$ is equal to the path $\beta_F(a_1, b_2) \cdot \gamma_{\mathsf{C}}(w, b_2) \cdot \beta_F(a_2, b_2)^{-1}$. An easy generalization and path induction shows that this is equivalent to the assertion that the outer rectangle in the diagram below commutes:

$$\begin{array}{ccccc}
F(a_1, \mathsf{p}(b_1)) & \xrightarrow{\mathsf{ap}_{F(a_1)}(\mathsf{c}(z))} & F(a_1, \mathsf{p}(b_2)) \\
\beta_F(a_1, b_1) \Big| & A & \Big| \beta_F(a_1, b_1) \\
\mathbf{R}(a_1, b_1) & \xrightarrow{\gamma_F(a_1, z)} & \mathbf{R}(a_1, b_2) \\
\gamma_{\mathsf{C}}(w, b_1) \Big| & B & \Big| \gamma_{\mathsf{C}}(w, b_2) \\
\mathbf{R}(a_2, b_1) & \xrightarrow[\gamma_F(a_2, z)]{} & \mathbf{R}(a_2, b_2) \\
\beta_F(a_2, b_1)^{-1} \Big| & C & \Big| \beta_F(a_2, b_2)^{-1} \\
F(a_2, \mathsf{p}(b_1)) & \xrightarrow[\mathsf{ap}_{F(a_2)}(\mathsf{c}(z))]{} & F(a_2, \mathsf{p}(b_2))
\end{array}$$

Rectangles A and C commute by the construction of F, as observed earlier. It thus remains to show that rectangle B commutes, *i.e.*, that we have

$$\gamma_{\mathsf{C}}(w, b_1) \cdot \gamma_F(a_2, z) = \gamma_F(a_1, z) \cdot \gamma_{\mathsf{C}}(w, b_1)$$

For this it suffices to show that

$$K_1(\gamma_F(a_2, z)) \circ K_1(\gamma_{\mathsf{C}}(w, b_1)) = K_1(\gamma_{\mathsf{C}}(w, b_1)) \circ K_1(\gamma_F(a_1, z))$$

Using the definition of γ_F and γ_{C}, we see that this is equivalent to showing that

$$\mathbf{c}(-, z) \circ \mathbf{c}(\mathbf{i}(w), -) = \mathbf{c}(\mathbf{i}(w), -) \circ \mathbf{c}(-, z)$$

But this is obvious from function extensionality and the groupoid properties, which means that we are done with the mapping of $\mathsf{c}(z)$. To finish the definition of $\delta_{\mathsf{C}}(w)$, we should provide the appropriate mapping corresponding to m. But there is a propositionally unique way to do this since as noted before, the type $F(a_1, x) = F(a_2, x)$ is a set for each x. Hence the definition of $\delta_{\mathsf{C}}(w)$ is complete, and by the first computation rule we get

$$\delta_{\mathsf{C}}(w, \mathsf{p}(b)) = \beta_F(a_1, b) \cdot \gamma_{\mathsf{C}}(w, b) \cdot \beta_F(a_2, b)^{-1}$$

To finish the definition of C, we need to show that for any $a_1, a_2, a_3 : A$ and $w_1 : R(a_1, a_2)$, $w_2 : R(a_2, a_3)$, we have

$$ {}^{\Pi}\mathsf{E}^{=}\big(\delta_{\mathsf{C}}(\mathbf{c}(w_1, w_2))\big) = {}^{\Pi}\mathsf{E}^{=}(\delta_{\mathsf{C}}(w_1)) \cdot {}^{\Pi}\mathsf{E}^{=}(\delta_{\mathsf{C}}(w_2)) $$

This is equivalent to showing that for any $x : D$, we have

$$ \delta_{\mathsf{C}}(\mathbf{c}(w_1, w_2), x) = \delta_{\mathsf{C}}(w_1, x) \cdot \delta_{\mathsf{C}}(w_2, x) $$

We once again proceed by induction on $x : D$. However, the type above is a mere proposition so we only need to show that it is inhabited for the case $x :\equiv \mathsf{p}(b)$, *i.e.*, that for any $b : D$ we have

$$ \delta_{\mathsf{C}}\big(\mathbf{c}(w_1, w_2), \mathsf{p}(b)\big) = \delta_{\mathsf{C}}(w_1, \mathsf{p}(b)) \cdot \delta_{\mathsf{C}}(w_2, \mathsf{p}(b)) $$

By the construction of δ_{C}, this is equivalent to showing that

$$ \beta_F(a_1, b) \cdot \gamma_{\mathsf{C}}(\mathbf{c}(w_1, w_2), b) \cdot \beta_F(a_3, b)^{-1} = \\ \Big(\beta_F(a_1, b) \cdot \gamma_{\mathsf{C}}(w_1, b) \cdot \beta_F(a_2, b)^{-1}\Big) \cdot \Big(\beta_F(a_2, b) \cdot \gamma_{\mathsf{C}}(w_2, b) \cdot \beta_F(a_3, b)^{-1}\Big) $$

Clearly, it suffices to show that

$$ \gamma_{\mathsf{C}}(\mathbf{c}(w_1, w_2), b) = \gamma_{\mathsf{C}}(w_1, b) \cdot \gamma_{\mathsf{C}}(w_2, b) $$

This is equivalent to showing that

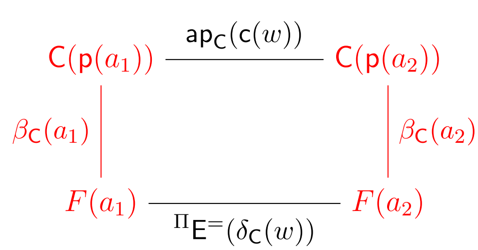
$$ K_1\big(\gamma_{\mathsf{C}}(\mathbf{c}(w_1, w_2), b)\big) = K_1(\gamma_{\mathsf{C}}(w_2, b)) \circ K_1(\gamma_{\mathsf{C}}(w_1, b)) $$

This is in turn equivalent to showing that

$$ \mathbf{c}\big(\mathbf{i}(\mathbf{c}(w_1, w_2)), -\big) = \mathbf{c}(\mathbf{i}(w_2), -) \circ \mathbf{c}(\mathbf{i}(w_1), -) $$

But this is obvious from function extensionality and the groupoid properties.

Although we are done with the definition of C, we note that the computation rules give us a family of paths $\beta_{\mathsf{C}}(a) : \mathsf{C}(\mathsf{p}(a)) = F(a)$ for $a : A$ such that the following diagram commutes for each $a_1, a_2 : A$, $w : R(a_1, a_2)$:



Putting together what we have so far, we see that for any $a, a', b, b' : A$ and $w : R(a, a')$, $z : R(b, b')$, all rectangles in the following diagram commute:

$$
\begin{array}{ccccc}
\mathsf{C}(\mathsf{p}(a),\mathsf{p}(b)) & \xrightarrow{{}^{=}\mathsf{E}^{\Pi}(\mathsf{ap}_{\mathsf{C}}(\mathsf{c}(w)),\mathsf{p}(b))} & \mathsf{C}(\mathsf{p}(a'),\mathsf{p}(b)) & \xrightarrow{\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a'))}(\mathsf{c}(z))} & \mathsf{C}(\mathsf{p}(a'),\mathsf{p}(b')) \\
\Big| \; {}^{=}\mathsf{E}^{\Pi}(\beta_{\mathsf{C}}(a),\mathsf{p}(b)) & A & \Big| \; {}^{=}\mathsf{E}^{\Pi}(\beta_{\mathsf{C}}(a'),\mathsf{p}(b)) & C & \Big| \; {}^{=}\mathsf{E}^{\Pi}(\beta_{\mathsf{C}}(a'),\mathsf{p}(b')) \\
F(a,\mathsf{p}(b)) & \xrightarrow[\delta_{\mathsf{C}}(w,\mathsf{p}(b))]{} & F(a',\mathsf{p}(b)) & \xrightarrow[\mathsf{ap}_{F(a')}(\mathsf{c}(z))]{} & F(a',\mathsf{p}(b')) \\
\Big| \; \beta_F(a,b) & B & \Big| \; \beta_F(a',b) & D & \Big| \; \beta_F(a',b') \\
\mathbf{R}(a,b) & \xrightarrow[\gamma_{\mathsf{C}}(w,b)]{} & \mathbf{R}(a',b) & \xrightarrow[\gamma_F(a',z)]{} & \mathbf{R}(a',b')
\end{array}
$$

Rectangle A commutes by the construction of C; B by the construction of $\delta_{\mathsf{C}}(w)$; C by path induction; and D by the construction of $F(a')$.

It will also be useful to give names to some of the edges in this diagram. We let $\varepsilon(a,b)$ be the path

$$
\begin{array}{c}
\mathsf{C}(\mathsf{p}(a),\mathsf{p}(b)) \\
\Big| \; {}^{=}\mathsf{E}^{\Pi}(\beta_{\mathsf{C}}(a),\mathsf{p}(b)) \\
F(a,\mathsf{p}(b)) \\
\Big| \; \beta_F(a,b) \\
\mathbf{R}(a,b)
\end{array}
$$

and $\eta(w,z)$ the path

$$
\mathsf{C}(\mathsf{p}(a),\mathsf{p}(b)) \xrightarrow{{}^{=}\mathsf{E}^{\Pi}(\mathsf{ap}_{\mathsf{C}}(\mathsf{c}(w)),\mathsf{p}(b))} \mathsf{C}(\mathsf{p}(a'),\mathsf{p}(b)) \xrightarrow{\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a'))}(\mathsf{c}(z))} \mathsf{C}(\mathsf{p}(a'),\mathsf{p}(b'))
$$

Step 2 To define $\mathsf{e}(x,y,u)$, we perform path induction on u. To construct $G(x) : \pi_1(\mathsf{C}(x,x))$, we proceed by induction on x, mapping $\mathsf{p}(a) \mapsto K_1(\varepsilon(a,a)^{-1})\ \mathbf{r}(a)$ for $a : A$. To map $\mathsf{c}(w)$ for $a_1, a_2 : A$ and $w : R(a_1,a_2)$, we must show that $K_1(\varepsilon(a_1,a_1)^{-1})\ \mathbf{r}(a_1)$ transported along $\mathsf{c}(w)$ among the fibers of the type family $x \mapsto \pi_1(\mathsf{C}(x,x))$ is equal to $K_1(\varepsilon(a_2,a_2)^{-1})\ \mathbf{r}(a_2)$. An easy generalization and path induction shows that this is equivalent to the assertion that

$$
K_1\big(\eta(\mathsf{c}(w),\mathsf{c}(w))\big)\ \big(K_1(\varepsilon(a_1,a_1)^{-1})\ \mathbf{r}(a_1)\big) = K_1(\varepsilon(a_2,a_2)^{-1})\ \mathbf{r}(a_2)
$$

The function $K_1\big(\eta(\mathsf{c}(w), \mathsf{c}(w))\big)$ goes from $\pi_1\big(\mathsf{C}(\mathsf{p}(a_1), \mathsf{p}(a_1))\big)$ to $\pi_1\big(\mathsf{C}(\mathsf{p}(a_2), \mathsf{p}(a_2))\big)$, so the above makes sense. By definition of γ_F, γ_{C}, and the groupoid properties, it is equivalent to showing that

$$K_1\big(\eta(\mathsf{c}(w), \mathsf{c}(w))\big)\ \big(K_1(\varepsilon(a_1, a_1)^{-1})\ \mathbf{r}(a_1)\big) = \\ K_1(\varepsilon(a_2, a_2)^{-1})\ \Big(K_1(\gamma_F(a_2, w))\ \big(K_1(\gamma_{\mathsf{C}}(w, a_1))\ \mathbf{r}(a_1)\big)\Big)$$

This in turn is equivalent to showing that

$$K_1\Big(\varepsilon(a_1, a_1)^{-1} \cdot \eta\big(\mathsf{c}(w), \mathsf{c}(w)\big)\Big)\ \mathbf{r}(a_1) = K_1\Big(\gamma_{\mathsf{C}}(w, a_1) \cdot \gamma_{\mathsf{C}}(w, a_1) \cdot \varepsilon(a_2, a_2)^{-1}\Big)\ \mathbf{r}(a_1)$$

For this it clearly suffices to show that

$$\varepsilon(a_1, a_1)^{-1} \cdot \eta\big(\mathsf{c}(w), \mathsf{c}(w)\big) = \gamma_{\mathsf{C}}(w, a_1) \cdot \gamma_{\mathsf{C}}(w, a_1) \cdot \varepsilon(a_2, a_2)^{-1}$$

But this is obvious from the commuting diagram noted at the end of the first step. The mapping of $\mathsf{c}(w)$ is finished, and since the type $\pi_1(\mathsf{C}(x, x))$ is a set for any $x : D$, it is not necessary to construct a mapping corresponding to m. The definition of e is thus complete. It will be useful to note that for any $a : A$, we have $G(\mathsf{p}(a)) = K_1(\varepsilon(a, a)^{-1})\ \mathbf{r}(a)$.

Step 3 We now show that for any $x : D$, the type $\Sigma_{y:D}\pi_1(\mathsf{C}(x, y))$ is contractible. We proceed by induction on x. Since the property of being contractible is a mere proposition, it suffices to prove it in the case $x :\equiv \mathsf{p}(a)$, *i.e.*, that for any $a : A$, the type $\Sigma_{y:D}\pi_1(\mathsf{C}(\mathsf{p}(a), y))$ is contractible. So fix $a : A$. The type in question is clearly inhabited by $\big(\mathsf{p}(a), G(\mathsf{p}(a))\big)$, so it suffices to show that for any $y : D$ and $u : \pi_1(\mathsf{C}(\mathsf{p}(a), y))$, we have $\big(\mathsf{p}(a), G(\mathsf{p}(a))\big) = (y, u)$. But it is easy to see that for any y and u we have

$$\begin{aligned} &\big(\mathsf{p}(a), G(\mathsf{p}(a))\big) = (y, u) \\ \simeq\ & \Sigma_{p:\mathsf{p}(a)=y}\Big(p_*^{x \mapsto \pi_1(\mathsf{C}(\mathsf{p}(a), x))}\ G(\mathsf{p}(a)) = u\Big) \\ \simeq\ & \Sigma_{p:\mathsf{p}(a)=y}\Big(K_1(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(p))\ G(\mathsf{p}(a)) = u\Big) \end{aligned}$$

Hence we aim to show that for any y and u, the last type in the above chain of equivalences is inhabited. We proceed by induction on y. We can do this since for any y, u, the type $\mathsf{p}(a) = y$ is a set and, furthermore, for any $p : \mathsf{p}(a) = y$ the type $K_1(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(p))\ G(\mathsf{p}(a)) = u$ is even a mere proposition (a fact we will use later). To map $\mathsf{p}(b)$ for $b : A$, we construct a function

$$H(b) : \Pi_{u:\pi_1(\mathsf{C}(\mathsf{p}(a), \mathsf{p}(b)))}\Sigma_{p:\mathsf{p}(a)=\mathsf{p}(b)}\Big(K_1(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(p))\ G(\mathsf{p}(a)) = u\Big)$$

To define $H(b)$, fix $u : \pi_1(\mathsf{C}(\mathsf{p}(a), \mathsf{p}(b)))$. The first component of our desired pair will be the path $\mathsf{c}(K_1(\varepsilon(a, b))\ u)$. To obtain the second component, we need to show that

$$K_1\Big(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}\big(\mathsf{c}(K_1(\varepsilon(a, b))\ u)\big)\Big)\ G(\mathsf{p}(a)) = u$$

By construction of G, the left-hand side is equal to

$$K_1\Big(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}\big(\mathsf{c}(K_1(\varepsilon(a,b))\ u)\big)\Big)\ \big(K_1(\varepsilon(a,a)^{-1})\ \mathbf{r}(a)\big)$$

This in turn is equal to

$$K_1\Big(\varepsilon(a,a)^{-1}\cdot\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}\big(\mathsf{c}(K_1(\varepsilon(a,b))\ u)\big)\Big)\ \mathbf{r}(a)$$

From the right part of the commuting diagram noted at the end of the first step we see that

$$\varepsilon(a,a)^{-1}\cdot\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}\big(\mathsf{c}(K_1(\varepsilon(a,b))\ u)\big)=\gamma_F\big(a,K_1(\varepsilon(a,b))\ u\big)\cdot\varepsilon(a,b)^{-1}$$

Hence the above is equal to

$$K_1\Big(\gamma_F\big(a,K_1(\varepsilon(a,b))\ u\big)\cdot\varepsilon(a,b)^{-1}\Big)\ \mathbf{r}(a)$$

This in turn is equal to

$$K_1(\varepsilon(a,b)^{-1})\ \Big(K_1\big(\gamma_F(a,K_1(\varepsilon(a,b))\ u)\big)\ \mathbf{r}(a)\Big)$$

By definition of γ_F and the groupoid properties, this is equal to

$$K_1(\varepsilon(a,b)^{-1})\ \big(K_1(\varepsilon(a,b))\ u\big)$$

But this is clearly equal to u so we are done. This finishes the definition of H and hence the mapping of $\mathsf{p}(b)$. We note that by the definition of H, we have $\pi_1(H(b,u))=\mathsf{c}(K_1(\varepsilon(a,b))\ u)$ for any $b:A$ and $u:\pi_1(\mathsf{C}(\mathsf{p}(a),\mathsf{p}(b)))$.

To map $\mathsf{c}(z)$ for $b_1,b_2:A$, $z:R(b_1,b_2)$, we must show that the map $H(b_1)$ transported along $\mathsf{c}(z)$ among the fibers of the type family $b\mapsto\Pi_{u:\pi_1(\mathsf{C}(\mathsf{p}(a),\mathsf{p}(b)))}\Sigma_{p:\mathsf{p}(a)=\mathsf{p}(b)}\ldots$ is equal to the map $H(b_2)$. Here we left out the body of the Σ-type because as noticed earlier, it is a mere proposition for any $b:A$, $u:\pi_1(\mathsf{C}(\mathsf{p}(a),\mathsf{p}(b)))$, and $p:\mathsf{p}(a)=\mathsf{p}(b)$. This observation together with path induction and function extensionality shows that in order to map $\mathsf{c}(z)$, it suffices to show that for any $u:\pi_1\big(\mathsf{C}(\mathsf{p}(a),\mathsf{p}(b_1))\big)$ we have

$$\pi_1(H(b_1,u))\cdot\mathsf{c}(z)=\pi_1\Big(H\big(b_2,K_1(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(\mathsf{c}(z)))\ u\big)\Big)$$

This makes sense because $K_1\big(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(\mathsf{c}(z))\big)\ u$ gives us a term of type $\pi_1\big(\mathsf{C}(\mathsf{p}(a),\mathsf{p}(b_2))\big)$. To show the above, we note that by our definition of H, it is equivalent to showing that

$$\mathsf{c}\big(K_1(\varepsilon(a,b_1))\ u\big)\cdot\mathsf{c}(z)=\mathsf{c}\Big(K_1(\varepsilon(a,b_2))\ \big(K_1(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(\mathsf{c}(z)))\ u\big)\Big)$$

By the groupoid properties, the left-hand side is equal to $\mathsf{c}\big(\mathbf{c}(K_1(\varepsilon(a,b_1))\ u,z)\big)$, so it suffices to show that

$$\mathbf{c}\big(K_1(\varepsilon(a,b_1))\ u,z\big)=K_1(\varepsilon(a,b_2))\ \Big(K_1(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(\mathsf{c}(z)))\ u\Big)$$

This is in turn equivalent to

$$\mathbf{c}\big(K_1(\varepsilon(a,b_1))\ u, z\big) = K_1\Big(\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(\mathsf{c}(z)) \cdot \varepsilon(a,b_2)\Big)\ u$$

As the right side of the commuting diagram at the end of the first step shows, we have

$$\mathsf{ap}_{\mathsf{C}(\mathsf{p}(a))}(\mathsf{c}(z)) \cdot \varepsilon(a,b_2) = \varepsilon(a,b_1) \cdot \gamma_F(a,z)$$

Hence it suffices to show that

$$\mathbf{c}\big(K_1(\varepsilon(a,b_1))\ u, z\big) = K_1\big(\varepsilon(a,b_1) \cdot \gamma_F(a,z)\big)\ u$$

This is equivalent to showing that

$$\mathbf{c}\big(K_1(\varepsilon(a,b_1))\ u, z\big) = K_1(\gamma_F(a,z))\ \big(K_1(\varepsilon(a,b_1))\ u\big)$$

By definition of γ_F, this is equivalent to showing

$$\mathbf{c}\big(K_1(\varepsilon(a,b_1))\ u, z\big) = \mathbf{c}\big(K_1(\varepsilon(a,b_1))\ u, z\big)$$

and we are done. This finishes the mapping of $\mathsf{c}(z)$, and as before, there is no need to supply a mapping corresponding to m. Hence, we have completed the proof that the type $\Sigma_{y:D}\pi_1(\mathsf{C}(x,y))$ is contractible for any $x : D$. In particular, this gives us a path family $\alpha(x,q) : q = (x, G(x))$ for any $x : D$ and $q : \Sigma_{y:D}\pi_1(\mathsf{C}(x,y))$.

Step 4 We define d by $\mathsf{d}(x,y,u) :\equiv \mathbf{d}\big(x, \alpha(x,(y,u))\big)$, where for $x : D$, $q : \Sigma_{y:D}\pi_1(\mathsf{C}(x,y))$, $\alpha_q : q = (x, G(x))$, the term

$$\mathbf{d}(x,\alpha_q) : x = \pi_1(q)$$

is defined by a one-sided path induction on α_q and the subsequent mapping $x \mapsto 1$. This finishes the definition of d.

It now remains to show that $\mathsf{e}(x,y)$ and $\mathsf{d}(x,y)$ are indeed quasi-inverse to each other for any x, y. In one direction, take $x, y : D$, $u : x = y$. To show that $\mathsf{d}(x,y,\mathsf{e}(x,y,u)) = u$, we proceed by path induction on u. This reduces the goal to showing that $\mathbf{d}\big(x, \alpha(x,(x,G(x)))\big) = 1$ for any $x : D$. But this follows at once from the fact that $\alpha(x,(x,G(x))) = 1_{(x,G(x))}$ by contractibility.

For the other direction, take $x, y : D$, $u : \pi_1(\mathsf{C}(x,y))$. The goal $\mathsf{e}(x,y,\mathsf{d}(x,y,u)) = u$ follows immediately from the generalization $\mathsf{e}\big(x, \pi_1(q), \mathbf{d}(x,\alpha_q)\big) = \pi_2(q)$ for any $x : D$, $q : \Sigma_{y:D}\pi_1(\mathsf{C}(x,y))$, $\alpha_q : q = (x, G(x))$. The generalization itself follows right away by one-sided path induction on α_q. □

5

Conclusion

We have investigated a class of higher inductive types with propositional computational behavior and shown that they can be equivalently characterized as homotopy-initial algebras. We have stated and proved this result for propositional truncations and for the so-called W-quotients, which subsume a number of other interesting cases - ordinary W-types, the unit circle $\mathbf{S}^1$, the interval type $\mathbf{I}$, all the higher spheres $\mathbf{S}^n$, all suspensions, and all type quotients. The characterization of these specific types as homotopy-initial algebras can be easily obtained as a corollary to our main theorem.

We have also established a characterization of truncations, set quotients, and groupoid quotients as homotopy-initial algebras. For set and groupoid quotients, we have shown that they can be recovered from W-quotients and truncations, the same way natural numbers or lists can be recovered from W-types. Furthermore, recent work by E. Rijke and F. van Doorn [36] shows that truncations can be reduced to type quotients (and hence to W-quotients). Thus, we conjecture that W-quotients play the same role in the higher dimensional setting as Martin-Löf's W-types do for ordinary inductive types: that of a simple, well-studied class of (higher) inductive types, which subsumes most of the other (higher) inductive types of interest as special cases.

This result would provide one possible answer to the earlier question of what a higher inductive type should be, and the characterization of higher inductive types as homotopy-initial algebras would follow from our main theorem. In this respect, the results in this thesis are related to the work of van Doorn, Rijke, and others, on reducing general higher inductive types to a combination of W-types and type quotients. We conjecture that W-quotients themselves arise as such a combination. While this decomposition of W-quotients into W-types and type quotients would not significantly simplify the development we presented here – establishing the characterization of type quotients as homotopy-initial algebras directly is almost as much work as doing it for the full W-quotients – it would provide further evidence for W-types and type quotients, either individually or combined into a W-quotient, as being the main building blocks for most (or even all?) other higher inductive types of interest.

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