Hodge structures and the topology of algebraic varieties

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Contents

Abstract

We review the major progress made since the 50's in our understanding of the topology of complex algebraic varieties. Most of the results we will discuss rely on Hodge theory, which provides by analytic tools the Hodge and Lefschetz decompositions, and the Hodge-Riemann relations. However, we will put emphasis on the algebraic arguments and definitions around the notion of Hodge structure. In another direction, we will emphasize the crucial importance of polarizations, which are missing in the general Kähler context. We will also discuss some results and problems related to algebraic cycles and motives.

0 Introduction

The study of the topology of complex manifolds starts with Riemann, whose work leads to the discovery of some remarkable features of integrals of differential forms on compact oriented Riemann surfaces, that is, compact 1-dimensional complex manifolds Σ. A key object is the space $H^{1,0}(\Sigma)$ of holomorphic 1-forms, which are locally of the form $g(z)dz$ where g is holomorphic and z is a local holomorphic coordinate. The Riemann-Roch formula implies that the space $H^{1,0}(\Sigma)$ has dimension g, where $2g = b_1(\Sigma)$. Furthermore, a local computation shows that

$$\int_\Sigma \alpha \wedge \beta = 0, \forall \alpha,\, \beta \in H^{1,0}(\Sigma), \tag{0.1}$$

$$\int_\Sigma i\alpha \wedge \overline{\alpha} > 0\ \forall 0 \neq \alpha \in H^{1,0}(\Sigma). \tag{0.2}$$

Noting that holomorphic 1-forms on Σ are closed, we deduce

Corollary 0.1. *(1) The map $H^{1,0}(\Sigma) \to H^1(\Sigma, \mathbb{C})$ which to a holomorphic 1-form associates its cohomology class is injective.*

(2) Furthermore, we have

$$H^1(\Sigma, \mathbb{C}) = H^{1,0}(\Sigma) \oplus \overline{H^{1,0}(\Sigma)}. \tag{0.3}$$

Proof. If α is exact, then $\int_\Sigma i\alpha \wedge \overline{\alpha} = 0$, hence (0.2) implies (1). The equations (0.1) and (0.2) together imply that the pull-back to the space $H^{1,0}(\Sigma) \oplus \overline{H^{1,0}(\Sigma)}$ of the pairing $\langle\, ,\, \rangle_\Sigma$ on $H^1(\Sigma, \mathbb{C})$ is nondegenerate, which implies that the natural map $H^{1,0}(\Sigma) \oplus \overline{H^{1,0}(\Sigma)} \to H^1(\Sigma, \mathbb{C})$ is injective. It is thus an isomorphism for dimension reasons. □

Remark 0.2. The decomposition (0.3) also says that the quotient $H^1(\Sigma, \mathbb{C})/H^{1,0}(\Sigma)$ is isomorphic to $H^1(\Sigma, \mathbb{R})$. We thus get from the data of the subspace $H^{1,0}(\Sigma) \subset H^1(\Sigma, \mathbb{C})$ a complex torus

$$J(\Sigma) = H^1(\Sigma, \mathbb{C})/(H^{1,0}(\Sigma) \oplus H^1(\Sigma, \mathbb{Z})) = \overline{H^{1,0}(\Sigma)}/H^1(\Sigma, \mathbb{Z}).$$

This torus is called the Jacobian of Σ. There are several ways of proving that this torus is in fact an algebraic variety. One can either use the Abel map or explicitly construct the Theta functions providing an embedding of the complex torus to projective space. Nowadays we can use the Kodaira embedding theorem (see Section 2.2) to show that a complex torus built on data as above, satisfying the Hodge-Riemann relations (0.1) and (0.2), is projective.

Considering higher dimensional complex compact manifolds, we see that the very simple arguments given above do not give enough information to understand the relation between complex geometry and topology. A few statements remain true: for example a nonzero holomorphic n-form η on a n-dimensional complex manifold X is closed and not exact because $\int_X \eta \wedge \overline{\eta} \neq 0$. It follows that holomorphic $n-1$-forms are closed. For surfaces, this is almost sufficient to prove that the Frölicher spectral sequence (see Section 1.1) degenerates at E_1, but the decomposition (0.3) is wrong for general compact complex surfaces, as the example of the Hopf surface H, quotient of $\mathbb{C}^2 \setminus \{0\}$ by the group $G \cong \mathbb{Z}$ generated by multiplication by λ, for some nonzero complex number λ such that $|\lambda| \neq 1$, shows. Indeed it has $b_1(H) \neq 0$ and $H^{1,0}(H) = 0$.

It is however a remarkable fact that higher dimensional *projective* manifolds carry some structure very similar to the one described above, namely the Hodge decomposition and sign relations of the intersection pairing, which in the case of surfaces take the very simple form that the intersection pairing restricted to the space $H^{1,1}$ of classes of real closed forms of type $(1,1)$ has signature $(1, h^{1,1} - 1)$ (the Hodge index theorem). In higher dimensions, formulating these sign conditions needs the introduction of the Lefschetz decomposition, which relies on Lefschetz work in algebraic geometry (see [53] and Section 2.2). The three celebrated theorems of Lefschetz are the most important results concerning the topology of complex algebraic manifolds obtained without using any modern sophisticated tool. They are the Lefschetz theorem on $(1,1)$-classes (Theorem 2.19), the Lefschetz theorem on hyperplane sections (Theorem 2.14), and the hard Lefschetz theorem (Corollary 2.13). The first one can be seen as an ancestor of the Hodge conjecture, as it deals with Hodge classes of degree 2. The original proof by Lefschetz is sketched in Griffiths' paper [28]. Nowadays it is given a formal proof based on sheaf-theoretic methods and the exponential exact sequence. The second one is a topological result comparing the topology of a smooth projective manifold and its hyperplane sections. It works with integral coefficients and can be given a Morse-theoretic proof (see [59]). The last one is a property of crucial importance satisfied by classes $c_1(L)$, where L is an ample holomorphic line bundle on a smooth projective complex variety or by a Kähler class on a compact Kähler manifold. It was not actually proved by Lefschetz, and Hodge gave an analytic proof of it based on Hodge theory. Deligne gave an arithmetic proof of this theorem in [20] using étale cohomology, but there is no algebraic geometry proof of this statement, for example in the setting of algebraic de Rham cohomology. The Lefschetz decomposition is a formal consequence of the hard Lefschetz theorem. As we will see in Section 2.2, the structure one gets by combining the Hodge and Lefschetz decompositions, namely a polarized Hodge structure on cohomology, exists only on projective manifolds. There are morphisms of Hodge structures associated to holomorphic maps or more generally correspondences between compact Kähler manifolds, but the category of polarized Hodge structures behaves much better. It is semi-simple, which is not the case for the category of Hodge structures. So, even if formally the natural context to do Hodge theory is the compact Kähler context, in fact the theory of Hodge structures seems to have its full power and applications only in the projective context.

The Kodaira embedding theorem (Theorem 2.33) is a major result obtained by combining sheaf-theoretic methods and Hodge theory, together with the Lefschetz theorem on $(1,1)$-classes. None of these results however uses in an essential way the notion of of (mixed, polarized) Hodge structures introduced by Griffiths and Deligne, that we will describe in Sections 1.5 and 2.2 and fully exploit the contents of the Hodge decomposition theorem in order to relate complex geometry and topology. Sheaf cohomology tells us that they are related by the Frölicher spectral sequence, and the Hodge decomposition theorem tells among other things that this spectral sequence degenerates at E_1. There are many consequences of the degeneration at E_1 of the Frölicher spectral sequence, as we will explain in Section 1.1.

The Hodge structures on the cohomology of a complex projective manifold have moduli and contain a lot of information on its geometry. Typically, the Hodge conjecture predicts which cohomology classes can be constructed from complex submanifolds. Although they are objects of a transcendental nature, they are predicted to contain a lot of information on the "motive" of a variety. To start with, the Hodge conjecture predicts that classes of cycles (combinations of varieties with rational coefficients) in a product $X \times Y$ are exactly the Hodge classes on $X \times Y$, or the morphisms of Hodge structures from the cohomology of X to the cohomology of Y. It is thus expected that the category of cohomological motives (see Sections 2.1 and 2.3) maps faithfully to the category of (polarized) Hodge structures, although a big problem in the theory comes from the fact that most Hodge structures do not come from the cohomology of algebraic varieties (or compact Kähler manifolds), as follows from Griffiths' transversality [25]. Another source for the extra structure on the cohomology of an algebraic manifold is the fact that it can be (at least with adequate coefficients), computed by different means, e.g. étale cohomology, algebraic de Rham cohomology (see Section 1.3), and of course Betti cohomology. Each of these realizations carries a specific information, e.g. étale cohomology carries the galois group action, while de Rham cohomology carries the Hodge filtration and is defined over the same field as the variety. The comparison isomorphisms then allow to put together these various structures. One should not believe however that this makes the cohomology theory of algebraic varieties understandable purely inside algebraic geometry. For example, Betti cohomology with integral coefficients has a transcendental nature.

We conclude this introduction with a general presentation of complex and Kähler geometries.

0.1 Complex manifolds

A complex manifold of dimension n is a real manifold of dimension $2n$ equipped with local holomorphic charts, namely homeomorphisms ϕ_U between open sets U of X and open sets of $\mathbb{C}^n$, such that the transition diffeomorphisms, given by the maps $\phi_{U'} \circ \phi_U^{-1}$ are holomorphic on $\phi_U(U \cap U') \subset \mathbb{C}^n$. The transition diffeomorphisms being holomorphic, they are in particular C^∞, and even real analytic, hence a complex manifold is real analytic. The homeomorphisms ϕ_U identify locally the real tangent bundle $T_{X,\mathbb{R}}$ with $T_{\mathbb{C}^n,\mathbb{R}}$ and the tangent bundle $T_{\mathbb{C}^n,\mathbb{R}}$ obviously has the structure of a complex vector bundle. The local structures of complex vector bundle on $T_{X,\mathbb{R}}$ (or almost complex structures) so constructed on open sets of X glue together, since the transition diffeomorphisms, being holomorphic, have complex linear differential. Hence a complex manifold has an almost complex structure, which is also described by an almost complex structure operator $I \in \operatorname{End} T_{X,\mathbb{R}}$ satisfying $I^2 = -Id$, and it is proved in [61] that an almost complex structure of class C^2 comes from a complex structure as above if and only if it satisfies the integrability condition saying that the Lie bracket of two complex vector fields of type $(1,0)$ is of type $(1,0)$. Here a complex vector field χ is said to be of type $(1,0)$ for I if $I\chi = i\chi$. In [80], Weil gives a simple argument proving this result in the real analytic case, where X and I are real analytic, as a consequence of Frobenius theorem on integrable distributions. The simplest examples of compact complex manifolds are projective space $\mathbb{CP}^n$ and complex tori $T = V/L$ where V is a complex vector space and L is a lattice in V. Projective space is a big source of further constructions of compact complex manifolds, as we can consider the projective manifolds, namely the complex submanifolds of $\mathbb{CP}^n$. It is known by work of Chow [14] that they are defined by algebraic equations, but there is no classification of smooth projective manifolds. The complete intersections in $\mathbb{CP}^N$ are those which are of codimension c and globally defined by exactly c homogeneous polynomials, seen as sections of line bundles on $\mathbb{CP}^N$. A celebrated conjecture by Hartshorne [35] states that a complex submanifold X of $\mathbb{CP}^N$ of dimension n and codimension c must be a complete intersection if $n > 2c$. It is completely open, but it is motivated by the nice observation due to Hartshorne that in this range, the rational cohomology of X satisfies many of the constraints imposed by the Lefschetz theorem on hyperplane sections to the cohomology of a complete intersection.

Another widely open problem is to decide which almost complex manifolds have a complex structure. In real dimension 4, using the Kodaira classification of complex surfaces [50], one can prove that there are almost complex compact fourfolds that do not admit a complex structure (see for example the work [71] by Van de Ven). In higher dimension, we do not know if the sphere $\mathbb{S}^6$, which has an almost complex structure, admits a complex structure (see [54] for a discussion).

0.2 Kähler manifolds

A Hermitian structure on a complex manifold is a Hermitian metric on its tangent bundle, equipped with its complex structure. In local holomorphic coordinates z_i, such a Hermitian metric is given by a Hermitian matrix (h_{ij}) of functions, and there is an associated real form

$$\omega = \frac{i}{2}\sum_{kl} h_{kl}dz_k \wedge d\overline{z_l},$$

called the associated Kähler form. This linear algebra construction works in the almost complex setting. The metric is said to be Kähler if the form ω is closed. This notion appears explicitly in the paper [40] but seems to have been already considered by Shouten and van Dantzig. As the Kähler form is obviously everywhere of maximal rank, it provides a symplectic structure on X. The set of Kähler form being obviously connected, a Kähler manifold thus has a natural deformation class of symplectic structures. In the reverse direction, Gromov and later Donaldson beautifully exploited the (not necessarily integrable) almost complex structures compatible with a given symplectic structure. Gromov used them in [32] to develop the theory of pseudoholomorphic curves, leading to the theory of Gromov-Witten invariants, and Donaldson used them in [23], in order to prove that a compact symplectic manifold contains many codimension 2 symplectic submanifolds.

There are various characterizations of Kähler metrics. The first one looks technical but it is very useful for computations, namely it says that the Hermitian metric

h on X is Kähler if and only if at any point $x \in X$, there are holomorphic coordinates such that the matrix of h in these coordinates is diagonal at first order, or equivalently

$$\omega = \frac{i}{2} \sum_k dz_k \wedge d\overline{z_k} + O(|z.|^2). \tag{0.4}$$

The second characterization is a consequence of the first, and it is very important in the theory of Calabi-Yau manifolds and Kähler-Einstein metrics, that is, Ricci flat Kähler metrics (see [82]). It says that the metric is Kähler if and only if the operator of almost complex structure I is parallel for the Levi-Civita connection of the metric $g = \mathrm{Re}\, h$. The "only if" easily follows from (0.4) because the Levi-Civita connection is computed using only the first order derivatives of the metric.

The importance of this notion for algebraic geometry is the fact that any smooth complex projective variety is Kähler. This follows from the fact that complex projective space $\mathbb{CP}^N$ is Kähler. It even admits explicit Kähler metrics like the Fubini-Study metric, which is invariant under the group $\mathbb{P}U(N+1)$ acting on $\mathbb{CP}^N$. The Kähler class of the Fubini-Study Kähler metric is a rational cohomology class on $\mathbb{CP}^N$, hence any complex projective manifold admits rational Kähler classes. This property characterizes complex projective manifolds by the celebrated Kodaira embedding theorem 2.33.

1 Differential forms on complex manifolds

1.1 Frölicher spectral sequence

On a complex manifold X, the cotangent bundle, seen as a complex vector vector bundle, as in fact the structure of a holomorphic vector bundle, which we denote Ω_X. In local holomorphic coordinates z_i, Ω_X is generated over $\mathcal{O}_X$ by the dz_i's. Using the local holomorphic charts to trivialize Ω_X, the transition matrices for Ω_X are the Jacobian matrices of the holomorphic change of coordinates. The holomorphic de Rham complex $\Omega_X^\bullet$ of a complex manifold X is defined by $\Omega_X^k := \bigwedge^k \Omega_X$, with differential $d : \Omega_X^k \to \Omega_X^{k+1}$ given by the exterior derivative. The holomorphic Poincaré lemma is quite easy to prove. It says that a holomorphic form of positive degree which is closed is locally an exact form $d\beta$ with β holomorphic, and a holomorphic form of degree 0 (that is, a holomorphic function) which is closed is locally constant. We summarize it in the form

Lemma 1.1. *The complex of sheaves $\Omega_X^\bullet$ on X is a resolution of the constant sheaf $\mathbb{C}$ on X.*

This resolution is not acyclic. One can obtain an acyclic resolution by choosing compatible acyclic resolutions of each sheaf Ω_X^p. For example the Dolbeault resolution will lead to the full de Rham complex, which is an acyclic resolution of the constant sheaf $\mathbb{C}$. We deduce from Lemma 1.1 a canonical isomorphism

$$H^k(X, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega_X^\bullet). \tag{1.5}$$

On the right, we have the hypercohomology of the holomorphic de Rham complex. The complex $\Omega_X^\bullet$ admits the Hodge filtration (or "filtration bête" or "naïve")

$$F^p\Omega_X^\bullet = \Omega_X^{\bullet \geq p}. \tag{1.6}$$

The corresponding spectral sequence (hypercohomology spectral sequence)

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet) = H^{p+q}(X, \mathbb{C})$$

associated to this filtration is called the Frölicher spectral sequence. Thus $H^k(X, \mathbb{C})$ has a decreasing filtration with graded pieces $E_\infty^{p,q}$, for $p+q=k$, which are subquotients of $E_1^{p,q}$.

Corollary 1.2. *One has*

$$b_k(X) \leq \sum_{p+q=k} h^{p,q}(X), \tag{1.7}$$

where $h^{p,q}(X) := \dim H^q(X, \Omega_X^p)$. Furthermore, the equality (1.7) holds for all k if and only if the Frölicher spectral sequence of X degenerates at E_1, that is, all the derivatives d_r vanish for $r \geq 1$ (or equivalently, $E_1^{p,q} = E_\infty^{p,q}$ for all p, q).

Note that $d_1 : H^q(X, \Omega_X^p) = E_1^{p,q} \to E_1^{p+1,q} = H^q(X, \Omega_X^{p+1})$ is induced by the exterior differential $d : \Omega_X^p \to \Omega_X^{p+1}$.

The degeneracy at E_1 of the Frölicher spectral sequence has some important consequences, particularly in deformation theory. Suppose that a compact complex manifold X is isomorphic to the central fiber $\mathcal{X}_0$ of a smooth proper holomorphic map

$$f : \mathcal{X} \to B,$$

where $(B, 0)$ is a pointed analytic space. By smoothness, and properness, f is a topological fibration (e's theorem), hence we have, for any $t \in B$, $b_k(\mathcal{X}_t) = b_k(\mathcal{X}_0) = b_k(X)$ at least if B is connected. The fibers $\mathcal{X}_t$ have their own Frölicher spectral sequence and satisfy the inquality (1.7). One can show by upper-semi-continuity of cohomology that for t close to 0, $h^{p,q}(\mathcal{X}_t) \leq h^{p,q}(\mathcal{X}_0)$. We thus get the chain of inequalities

$$b_k(\mathcal{X}_0) = b_k(\mathcal{X}_t) \leq \sum_{p+q=k} h^{p,q}(\mathcal{X}_t) \leq \sum_{p+q=k} h^{p,q}(\mathcal{X}_0). \tag{1.8}$$

We then conclude:

Theorem 1.3. *If the Frölicher spectral sequence of $X = \mathcal{X}_0$ degenerates at E_1, then*

(i) $h^{p,q}(\mathcal{X}_t) = h^{p,q}(\mathcal{X}_0)$ for t close to 0.

(ii) The Frölicher spectral sequence of $\mathcal{X}_t$ also degenerates at E_1 for t close to 0.

Indeed, in the chain of inequalities (1.8), the two extreme terms are equal hence the intermediate inequalities are equalities, which proves both statements using Corollary 1.2.

Writing this argument in a schematic form, Deligne proved an even better statement. Considering again a smooth proper family $f : \mathcal{X} \to B$, we have the relative holomorphic de Rham complex $\Omega_{\mathcal{X}/B}^\bullet$ which is (by analytic local triviality) a resolution of $\pi^{-1}\mathcal{O}_B$, and thus a relative Frölicher spectral sequence

$$E_1^{p,q} = R^q f_* \Omega_{\mathcal{X}/B}^p \Rightarrow R^k \pi_* \mathbb{C} \otimes \mathcal{O}_B. \tag{1.9}$$

The right hand side is locally free over B. On the other hand, it admits the Hodge filtration obtained as the abutment of the relative Frölicher spectral sequence. Note that, after restriction to $\mathcal{X}_0$, we get the Frölicher spectral sequence of $\mathcal{X}_0$. Assuming B is Artinian and replacing the dimension inequalities (1.7) by length inequalities for the sheaves $R^q f_* \Omega^p_{\mathcal{X}/B}$ an $R^k\pi_*\mathbb{C}\otimes\mathcal{O}_B$, one gets

Theorem 1.4. *(Deligne) Assume B is Artinian (or formal, or a germ). If the Frölicher spectral sequence of $X = \mathcal{X}_0$ degenerates at E_1, then*

(i) The sheaves $R^q f_* \Omega^p_{\mathcal{X}/B}$ are locally free on B and satisfy base change.

(ii) The relative Frölicher spectral sequence of $\mathcal{X}/B$ also degenerates at E_1.

Here the upper-semi-continuity of cohomology takes the form of the following inequality

$$l(R^q f_* \Omega^p_{\mathcal{X}/B}) \leq h^{p,q}(\mathcal{X}_0) l(\mathcal{O}_B)$$

with equality if and only if $R^q f_* \Omega^p_{\mathcal{X}/B}$ is locally free and satisfies base change, which means that the restriction map

$$(R^q f_* \Omega^p_{\mathcal{X}/B})_{|0} \to H^q(\mathcal{X}_0, \Omega^p_{\mathcal{X}_0})$$

is an isomorphism.

This theorem implies the following version of the Bogomolov-Tian-Todorov theorem [69].

Theorem 1.5. *Let X be a compact complex manifold with trivial canonical bundle. Assume the Frölicher spectral sequence of X degenerates at E_1. Then the deformations of X are unobstructed.*

The conclusion is equivalent to the fact that the first order deformations of X, or smooth morphisms $\mathcal{X}_1 \to B_1 := \operatorname{Spec}\mathbb{C}[t]/(t^2)$ of analytic spaces with fiber over 0 isomorphic to X extend to arbitrary high order n to a smooth morphism $\mathcal{X}_n \to B_n := \operatorname{Spec}\mathbb{C}[t]/[t^{n+1}]$ of analytic spaces. The original proof of the Bogomolov-Tian-Todorov theorem needs the stronger assumption that X is Kähler (see Section 1.4). The proof of Theorem 1.5 is done by induction on n. It relies on the very important T^1-lifiting principle proved in [62], which says the following. Associated to a smooth morphism $f_n : \mathcal{X}_n \to B_n := \operatorname{Spec}\mathbb{C}[t]/(t^{n+1})$ of analytic spaces, one has a relative Kodaira-Spencer class $\rho_n \in \Gamma(B_{n-1}, R^1 f_{n-1*} T_{\mathcal{X}_{n-1}/B_{n-1}}) = H^1(\mathcal{X}_{n-1}, T_{\mathcal{X}_{n-1}/B_{n-1}})$, which is defined as the extension class of the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}_{n-1}} \to \Omega_{\mathcal{X}_{n|\mathcal{X}_{n-1}}} \to \Omega_{\mathcal{X}_{n-1}/B_{n-1}} \to 0$$

of locally free sheaves on $\mathcal{X}_{n-1}$.

Theorem 1.6. *(T^1-lifting principle.) The morphism f_n extends to a smooth morphism $f_{n+1} : \mathcal{X}_{n+1} \to B_{n+1}$ if and only if the relative Kodaira-Spencer class ρ_n extends to a class $\rho_{n+1} \in \Gamma(B_n, R^1 f_{n*} T_{\mathcal{X}_n/B_n}) = H^1(\mathcal{X}_n, T_{\mathcal{X}_n/B_n})$.*

Suppose that X has trivial canonical bundle, that is, $\Omega^d_X := K_X \cong \mathcal{O}_X$, where $d := \dim X$, and that its Frölicher spectral sequence degenerates at E_1. The everywhere nonzero section $\eta \in H^0(X, K_X)$ extends to a section η_n of $K_{\mathcal{X}_n/B_n}$ by Theorem 1.4 (i), and, as η, η_n is everywhere nonzero, so it follows that the relative canonical bundle $K_{\mathcal{X}_n/B_n}$ is trivial. But then, by interior product, we get an isomorphism

$$T_{\mathcal{X}_n/B_n} \cong \Omega^{d-1}_{\mathcal{X}_n/B_n}$$

of holomorphic vector bundles on $\mathcal{X}_n$. The fact that the relative Kodaira-Spencer class extends to $\mathcal{X}_n$ then follows from Theorem 1.4 (i) applied to $\Omega^{d-1}_{\mathcal{X}_n/B_n}$.

1.2 Logarithmic de Rham complex

Let X be a complex manifold and $D \subset X$ be a reduced analytic hypersurface. D is said to be a normal crossing divisor if for any $x \in D$, there exist local holomorphic coordinates $z_1, \dots, z_n$ centered at x, such that D is defined by the equation $z_1 \dots z_k = 0$ near x. The number k of branches depends of course on x. The logarithmic de Rham complex associated to a normal crossing divisor D is denoted $\Omega_X^\bullet(\log D)$ and constructed as follows. We define the set of sections of the sheaf $\Omega_X^k(\log D)$ to be the set of holomorphic differentials η with pole order 1 along D, and whose differential $d\eta$ also has pole order 1 along D. It is clear from the definition that the exterior differential maps $\Omega_X^k(\log D)$ to $\Omega_X^{k+1}(\log D)$. The following is easily proved.

Proposition 1.7. *(i) In local holomorphic coordinates as above, $\Omega_X(\log D)$ is freely generated over $\mathcal{O}_X$ by $\frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}, dz_{k+1}, \dots, dz_n$. In particular, $\Omega_X(\log D)$ is locally free.*

(ii) One has $\Omega_X^k(\log D) = \bigwedge^k \Omega_X(\log D)$. In particular, $\Omega_X^k(\log D)$ is locally free.

Let $U := X \setminus D$ and let $j : U \hookrightarrow X$ be the open inclusion. If $x \in D$ with local holomorphic coordinates z_i as above, then for an adequate open neighborhood V_x of x in X, $U \cap V_x$ is biholomorphic to $\mathbb{D}^{*k} \times \mathbb{D}^{n-k}$, where $\mathbb{D}$ is the unit disc, hence has the same homotopy type as $(\mathbb{S}^1)^k$. The Cauchy formula shows that the classes of the closed logarithmic forms $\frac{dz_I}{z_I}$ restricted to $U \cap V_x$ provide a complex basis of $H^*((\mathbb{S}^1)^k, \mathbb{C})$. With a little more work, one concludes

Theorem 1.8. *The logarithmic de Rham complex is quasiisomorphic to $Rj_*\mathbb{C}$.*

Corollary 1.9. *There is a canonical isomorphism*

$$H^*(U, \mathbb{C}) = \mathbb{H}^*(X, Rj_*\mathbb{C}) \cong \mathbb{H}^*(X, \Omega_X^\bullet(\log D)). \tag{1.10}$$

Note that we also have the isomorphism (1.5) for U, which provides

$$H^*(U, \mathbb{C}) \cong \mathbb{H}^*(U, \Omega_U^\bullet)$$

and a Hodge filtration on $H^*(U, \mathbb{C})$ induced by the filtration (1.6) on $\Omega_U^\bullet$ and given by

$$F^p H^*(U, \mathbb{C}) = \mathrm{Im}\,(\mathbb{H}^*(U, \Omega_U^{\bullet \geq p}) \to \mathbb{H}^*(U, \Omega_U^\bullet)).$$

This filtration is not very interesting for the following reason. Assume that U is affine, so that the sheaves Ω_U^l are acyclic. Then the formula (1.10) gives in this case

$$H^k(U, \mathbb{C}) = \frac{H^0(U, \Omega_U^k)_{closed}}{H^0(U, \Omega_U^k)_{exact}}$$

and $H^k(U, \mathbb{C}) = F^k H^k(U, \mathbb{C})$. In fact, the Hodge filtration we will be considering on $H^*(U, \mathbb{C})$ for U quasiprojective, with projective compactification X with normal crossing boundary divisor $D = X \setminus U$, is induced by the filtration (1.6) on the logarithmic de Rham complex using (1.10).

$$F^p H^*(U, \mathbb{C}) = \mathrm{Im}\,(\mathbb{H}^*(X, \Omega_X(\log D)^{\bullet \geq p}) \to \mathbb{H}^*(X, \Omega_X^\bullet(\log D))). \tag{1.11}$$

Note that such a compactification always exists by Hironaka resolution of singularities.

Proposition 1.10. *The filtration defined by (1.11) does not depend on the choice of smooth projective compactification with normal crossing boundary.*

This follows from the fact that any two such compactifications are birational and dominated by a third, and from functoriality properties of the logarithmic de Rham complex.

1.3 Algebraic de Rham cohomology

Let X be an algebraic variety over $\mathbb{C}$. Then we can define the sheaf of Kähler differentials $\Omega_{X/\mathbb{C}}$. Its stalk at $x \in X$ is the $\mathcal{O}_{X,x}$-module of differentials of $\mathcal{O}_{X,x}$, generated by df, $f \in \mathcal{O}_{X,x}$, with relations given by the Leibniz rule $d(fg) = fdg + gdf$. Here $\mathcal{O}_X$ is the sheaf of algebraic functions on X, in the Zariski topology. We have the universal differential

$$d : \mathcal{O}_X \to \Omega_{X/\mathbb{C}}.$$

When X is smooth, $\Omega_{X/\mathbb{C}}$ is locally free and we have the algebraic de Rham complex

$$0 \to \mathcal{O}_X \stackrel{d}{\to} \Omega_{X/\mathbb{C}} \stackrel{d}{\to} \ldots \stackrel{d}{\to} \bigwedge^n \Omega_X \to 0,$$

where $n = \dim X$. The algebraic de Rham cohomology $H^k_{dR}(X/\mathbb{C})$ was introduced by Grothendieck [34] and is defined as $\mathbb{H}^k(X, \Omega^\bullet_{X/\mathbb{C}})$. We can now consider the corresponding complex manifold X_{an} (with sheaf of rings given by holomorphic functions in the Euclidean topology instead of algebraic functions in the Zariski topology). The sheaf of holomorphic differentials Ω_X is nothing but the analytization of the algebraic coherent sheaf $\Omega_{X/\mathbb{C}}$, and similarly for the whole de Rham complex. The following result is due to Grothendieck [34] (but some related remarks had been made by Serre in [65]).

Theorem 1.11. *If X is quasi-projective, one has*

$$H^k(X_{an}, \mathbb{C}) \cong H^k_{dR}(X/\mathbb{C}). \tag{1.12}$$

When X is projective, this follows from the comparison isomorphism (1.5) for X_{an} and the GAGA isomorphism (see [65])

$$H^k_{dR}(X/\mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_{X/\mathbb{C}}) \cong \mathbb{H}^k(X_{an}, \Omega^\bullet_{X_{an}}).$$

When X is not projective, one chooses a smooth projective compactification $\overline{X}$ such that $\overline{X} \setminus X$ is a normal crossing divisor D. One observes that the logarithmic de Rham complex has an algebraic version $\Omega^\bullet_{\overline{X}/\mathbb{C}}(\log D)$, whose analytization gives $\Omega^\bullet_{\overline{X}_{an}}(\log D)$. The GAGA theorem then provides an isomorphism

$$\mathbb{H}^k(\overline{X}, \Omega^\bullet_{\overline{X}/\mathbb{C}}(\log D)) \cong \mathbb{H}^k(\overline{X}_{an}, \Omega^\bullet_{\overline{X}_{an}}(\log D)),$$

and by (1.10), the last term is isomorphic to $\mathbb{H}^k(\overline{X}_{an}, \mathbb{C})$. Using the fact that $\mathbb{H}^k(X, \Omega^\bullet_{X/\mathbb{C}}) \cong \mathbb{H}^k(\overline{X}, j_*\Omega^\bullet_{X/\mathbb{C}})$, where j is the inclusion of X in $\overline{X}$, the proof is concluded by proving that the natural inclusion of complexes

$$\Omega^\bullet_{\overline{X}/\mathbb{C}}(\log D) \subset j_*\Omega^\bullet_{X/\mathbb{C}}$$

is a quasiisomorphism.

Remark 1.12. This last point is not proved in [34]. Grothendieck only uses the fact that $H^k(U,\mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_X(*D))$, where $\Omega^\bullet_X(*D) \subset j_*\Omega^\bullet_U$ is the complex of holomorphic differential forms on U with meromorphic growth along D. This is enough to prove Theorem 1.11 using GAGA.

The comparison isomorphism (1.12) has for consequence the fact that, if a smooth variety is defined over a subfield $K \subset \mathbb{C}$ (for example a number field), then its Betti cohomology $H^k(X_{an}, \mathbb{C})$ has a natural K-structure, that is, there is a K-vector space V such that $V \otimes_K \mathbb{C} = H^k(X_{an}, \mathbb{C})$. Indeed, it suffices to observe that the sheaves of Kähler differentials are defined over K, hence also the algebraic de Rham cohomology of X, so that we can take $V = H^k_{dR}(X_K/K)$. When $K = \mathbb{Q}$, the cohomology $H^k(X_{an}, \mathbb{C})$ thus has two $\mathbb{Q}$-structures, the Betti $\mathbb{Q}$-structure given by

$$H^k(X_{an}, \mathbb{C}) = H^k(X_{an}, \mathbb{Q}) \otimes \mathbb{C},$$

and the de Rham $\mathbb{Q}$-structure given by

$$H^k_{dR}(X/\mathbb{C}) = H^k_{dR}(X_{\mathbb{Q}}/\mathbb{Q}) \otimes \mathbb{C}.$$

Comparing these two structures is the contents of the arithmetic theory of periods. Consider the example of the cohomology of the algebraic variety $X = \mathbb{G}_m$ defined over $\mathbb{Q}$ with analytization $X_{an} = \mathbb{C}^*$, for which a $\mathbb{Q}$-generator of H^1_{dR} is given by the class α_{dR} of $\frac{dz}{z}$ while a $\mathbb{Q}$-generator of $H^1(X_{an}, \mathbb{Q})$ is given by a class α_{Betti} whose integral over the counterclokwise oriented unit circle is 1. The Cauchy formula shows that

$$\alpha_{dR} = (2i\pi)\alpha_{Betti}$$

in this case, so that a transcendental coefficient appears in any case. For even degree cohomology H^{2k}, it is more natural to compare the de Rham $\overline{\mathbb{Q}}$-structure divided by $(2i\pi)^k$ and the Betti $\overline{\mathbb{Q}}$-structure, as they coincide on classes of algebraic subvarieties (cycle classes) of X (still assumed to be defined over a number field). The Grothendieck period conjecture (see [2]) relates the transcendence degree of the arithmetic period matrix of X to the cycle classes on powers X^l of X.

1.4 The Hodge decomposition theorem

The almost complex structure I of a complex manifold X gives equivalently a decomposition of the complexified cotangent bundle $\Omega_{X,\mathbb{C}}$ into the $\pm i$-eigenspaces of I, which one writes as

$$\Omega_{X,\mathbb{C}} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X. \tag{1.13}$$

The differential forms of type $(1,0)$ are the $\mathbb{C}$-linear maps $T_{X,\mathbb{R}} \to \mathbb{C}$, where $T_{X,\mathbb{R}}$ is equipped with its structure of $\mathbb{C}$-vector space given by I, and the differential forms of type $(0,1)$ are the $\mathbb{C}$-antilinear ones. If X has a complex structure, the vector bundle $\Omega^{1,0}_X$ is generated in local holomorphic coordinates z_i by the dz_i and it has a holomorphic structure, namely it is the C^∞-bundle underlying the holomorphic bundle Ω_X of section 1.1. The decomposition (1.13) allows to define the $\overline{\partial}$-operator acting on differentiable functions $f : U \to \mathbb{C}$ defined over an open set $U \subset X$ by

$$\overline{\partial} f = (df)^{0,1},$$

where $(df)^{0,1} \in \mathcal{A}^{0,1}(U) = \Gamma(U, \Omega_U^{0,1})$ is the projection of $df \in \Gamma(\Omega_{U,\mathbb{C}})$. Defining the sheaf $\mathcal{A}^{p,q}$ of C^∞ forms of type (p,q) by the formula

$$\mathcal{A}^{p,q}(U) = \Gamma(U, \bigwedge^p \Omega_U^{1,0} \otimes \bigwedge^q \Omega_U^{1,0}),$$

(1.13) gives a decomposition of the set $\mathcal{A}^k(U)$ of complex differential forms on U as

$$\mathcal{A}^k(U) = \oplus_{p+q=k} \mathcal{A}^{p,q}(U). \tag{1.14}$$

Using (1.14), the operator $\overline{\partial}$ extends to an operator

$$\overline{\partial} : \mathcal{A}^{0,1}(U) \to \mathcal{A}^{0,2}(U)$$

where as before $\overline{\partial}\alpha$ is the $(0,2)$-component of $d\alpha$. One can show that the complex structure is integrable (that is, the Newlander-Nirenberg criterion [61] is satisfied) if and only if $\overline{\partial} \circ \overline{\partial} = 0$ on functions. When this is the case, the decomposition (1.14) has the properties that the differential $d\alpha$ of a form α of type $p,q)$, that is, $\alpha \in \mathcal{A}^{p,q}(U)$, is the sum of a form $\partial\alpha$ of type $(p+1,q)$ and a form $\overline{\partial}\alpha$ of type $(p,q+1)$. The operators $\overline{\partial} : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p,q+1}(U)$ so defined for any open set $U \subset X$ satisfy $\overline{\partial} \circ \overline{\partial} = 0$ hence give for fixed p a complex which can easily be shown to be the Dolbeault complex $\Omega_X \otimes \mathcal{A}^{0,q}$ of the holomorphic vector bundle Ω_X. Dolbeault theorem [22] says that the Dolbeault complex is a (acyclic) resolution of the sheaf of sections of Ω_X^p, so that one gets an isomorphism

$$H^q(X, \Omega_X^p) = \frac{\mathrm{Ker}\,(\mathcal{A}^{p,q}(X) \overset{\overline{\partial}}{\to} \mathcal{A}^{p,q+1}(X))}{\mathrm{Im}\,(\mathcal{A}^{p,q-1}(X) \overset{\overline{\partial}}{\to} \mathcal{A}^{p,q}(X))}. \tag{1.15}$$

Without more assumptions on X, relating the cohomology of d and $\overline{\partial}$ is not easy. The only general fact is that there is a natural map $H^k(X, \mathbb{C}) \to H^k(X, \mathcal{O}_X)$, induced by the morphism of sheaves $\mathbb{C} \to \mathcal{O}_X$, which to the de Rham class of a k-form associates the Dolbeault class of its $(0,k)$-component. The following major result of Hodge says that, to the contrary, when X is compact Kähler, the decomposition (1.14) descends to cohomology.

Theorem 1.13. *(Hodge [38].) For a complex manifold X, define $H^{p,q}(X) \subset H^{p+q}(X, \mathbb{C})$ as the set of de Rham cohomology classes of closed forms of type (p,q) on X. Then, if X is compact Kähler,*

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X), \tag{1.16}$$

$$H^{p,q}(X) \cong H^q(X, \Omega_X^p). \tag{1.17}$$

The isomorphism in (1.17) is not natural. Essentially it associates to the d-class of a closed (hence $\overline{\partial}$-closed) (p,q)-form α its $\overline{\partial}$-class, but one has to prove that the later depends only on the d-class of α, which is not a formal fact. Although Theorem 1.16 is very useful independently of its proof, it turns out that other results, for example the hard Lefschetz theorem cannot be deduced directly from it but need the whole power of Hodge theory. To summarize the main points of the proof, let us say that, choosing a Hermitian metric h on the compact complex manifold X, one gets a L^2-metric on the spaces of differential forms, or a L^2-hermitian metric on the spaces of differential forms of type (p,q). If $n=\dim X$, there is the Hodge $*$-operator

$$\mathcal{A}^k_X \to \mathcal{A}^{2n-k}_X,$$

which is of order 0, such that

$$||\alpha||_{L^2} = \int_X \alpha \wedge *\alpha.$$

One defines the first order adjoint operators d^*, $\overline{\partial}^*$, ∂^* by the formulas

$$d^* = -*d*,\ \overline{\partial}^* = -*\partial*,\ \partial^* = -*\overline{\partial}*.$$

They are formal adjoints in the sense that

$$\langle \alpha, d\beta\rangle_{L^2} = \langle d^*\alpha, \beta\rangle_{L^2} \tag{1.18}$$

and similarly for the other operators. One defines the Laplacians Δ_d, $\Delta_{\overline{\partial}}$, Δ_∂ by the formulas

$$\Delta_d = dd^* + d^*d,\ \Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial},\ \Delta_\partial = \partial\partial^* + \partial^*\partial. \tag{1.19}$$

Note that the Laplacians $\Delta_{\overline{\partial}}$ and Δ_∂ act on each space $A^{p,q}(X)$ of differential forms of type (p,q), which is a priori not the case for Δ_d. A differential form is harmonic (for d, ∂ or $\overline{\partial}$) if it is annihilated by the corresponding Laplacian. As one easily sees using formulas (1.18) and (1.19), a form α on a compact complex manifold is Δ_d-harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$, and similarly for the two other Laplacians.

The general theory of elliptic operators developed by Hodge says that one has orthogonal decompositions

$$A^k(X) = \operatorname{Im}\Delta_d \oplus \operatorname{Ker}\Delta_d \tag{1.20}$$

$$A^k(X) = \operatorname{Im} d \oplus \operatorname{Im} d^* \oplus \mathcal{H}^k(X), \tag{1.21}$$

hence an isomorphism

$$H^k(X,\mathbb{R}) \cong \mathcal{H}^k(X), \tag{1.22}$$

where $\mathcal{H}^k(X)$ is the space of real harmonic forms of degree k, and similarly for the other Laplacians, or with $\mathbb{C}$-coefficients. Up to now, we have not been using the Kähler assumption, and any Hermitian metric could be used. Theorem 1.13 is then a consequence of the following fact.

Theorem 1.14. *If the metric h is Kähler, one has $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$.*

Indeed, Theorem 1.14 implies that the Laplacian Δ_d preserves the decomposition (1.14), which implies that, if a form $\alpha = \sum_{p+q=k}\alpha^{p,q}$ is harmonic, then its components $\alpha^{p,q}$ are harmonic. We thus get a decomposition

$$\mathcal{H}^k = \oplus_{p+q=k}\mathcal{H}^{p,q}$$

where $\mathcal{H}^{p,q}$ is the set of Δ_d-harmonic (or Δ_∂-harmonic, or $\Delta_{\overline{\partial}}$-harmonic) forms on X, which implies (1.16). The statement (1.17) follows from the equality $\Delta_d = 2\Delta_{\overline{\partial}}$ which implies that $\mathcal{H}^{p,q}(X)$ is also the set of $\Delta_{\overline{\partial}}$-harmonic forms of type (p,q) and from the isomorphism (1.22) for the $\overline{\partial}$-operator, which gives

$$\mathcal{H}^{p,q} \cong H^q(X,\Omega^p_X).$$

Corollary 1.15. *The Frölicher spectral sequence of a compact Kähler manifold degenerates at E_1.*

Indeed, as we have seen in Section 1.1, this is a consequence of the equality $b_k(X) = \sum_{p+q=k} h^{p,q}(X)$, $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$.

Remark 1.16. The complex conjugacy acts on the space $H^k(X, \mathbb{C})$, with fixed part $H^k(X, \mathbb{R})$. It is clear that, with the definition above

$$\overline{H^{p,q}(X)} = H^{q,p}(X). \tag{1.23}$$

This phenomenon is called Hodge symmetry. It implies the restriction $h^{p,q}(X) = h^{q,p}(X)$ on the Hodge numbers $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$ of a compact Kähler manifold. This symmetry is strong enough to characterize compact Kähler surfaces since it implies in the case of surfaces that b_1 is even. Indeed, the Frölicher spectral sequence of a compact complex surface always degenerates at E_1, so that $b_1(S) = h^{1,0}(S) + h^{0,1}(S)$ holds without the Kähler assumption.

In the case of complex projective manifolds, the degeneracy at E_1 of the Frölicher spectral sequence has been given an algebraic proof by Deligne-Illusie [21] using reduction to nonzero characteristic. Note here that the degeneracy at E_1 of the analytic Frölicher spectral sequence is equivalent by GAGA [65] to the degeneracy at E_1 of the algebraic Frölicher spectral sequence (see Section 1.3). This is thus an algebraic statement.

One consequence of (the proof of) the Hodge theorem is the celebrated $\partial\overline{\partial}$-lemma.

Lemma 1.17. *Let X be a compact complex manifold. Let α be a closed form of type (p,q) on X which is d-closed and is ∂-closed or $\overline{\partial}$-closed, or d-closed. Then $\alpha = \partial\overline{\partial}\beta$ for some form of type $(p-1, q-1)$ on X.*

Note that each of the three conditions stated is equivalent by the arguments given above to the fact that the harmonic representative of α is 0 and, in particular, the three conditions are equivalent. The proof of Lemma 1.17 uses (1.20) and the Kähler identities, which are commutation relations between the operators ∂^* an $\overline{\partial}$, and are used in the proof of Theorem 1.14. The following important application of the $\partial\overline{\partial}$-lemma is the starting point in the proof of the Kodaira embedding theorem [49]. If L is a holomorphic line bundle on a complex manifold, and h_L is a Hermitian metric on L, the form $\omega_{h_L} = \frac{1}{2i\pi}\partial\overline{\partial}\log h_L$, computed as $\frac{1}{2i\pi}\partial\overline{\partial}\log h_L(\sigma)$ for some (in fact, any) local generating holomorphic sections σ of L, is a real closed form of type $(1,1)$ and it is a well-known fact of the theory of Chern classes that it is a de Rham representative of the second Chern class $c_1(L)$.

Proposition 1.18. *Let X be a compact Kähler manifold and L a holomorphic line bundle on X. Let α be a closed $(1,1)$-form which is a de Rham representative of the second Chern class $c_1(L)$. Then there exists a Hermitian metric h_L on L such that $\omega_{h_L} = \alpha$.*

Indeed, choose any Hermitian metric h'_L on L. Then the form $\omega_{h'_L} - \alpha$ is closed of type $(1,1)$ and its de Rham cohomoloy class is 0, so it is exact. The $\partial\overline{\partial}$-lemma says that it is $\frac{1}{2i\pi}\partial\overline{\partial}\log\phi$ for some real function ϕ. The Hermitian metric $e^{-\phi}h'_L$ satisfies the required properties.

1.5 Hodge structures and mixed Hodge structures

Definition 1.19. *An integral Hodge structure of weight k is a torsion free lattice L and a decomposition*

$$L_{\mathbb{C}} := L \otimes \mathbb{C} = \oplus_{p+q=k} L^{p,q} \tag{1.24}$$

into a direct sum of complex vector subspaces satisfying $\overline{L^{p,q}} = L^{q,p}$.

One can define as well rational Hodge structures. By the Hodge decomposition theorem 1.13, this structure appears on the integral cohomology modulo torsion of a compact Kähler manifold, using the isomorphism

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C}.$$

In this case, we have $H^{p,q}(X) = 0$ for $p < 0$ or $q < 0$ (Hodge structures satisfying this condition will be said effective), but one should not impose this as a general condition since it is useful to consider dual Hodge structures with weight of opposite sign. The Hodge decomposition is a data equivalent to the corresponding Hodge filtration

$$F^r L_{\mathbb{C}} = \oplus_{p \geq r} L^{p,q} \tag{1.25}$$

which has to be finite and to satisfy the condition

$$F^r L_{\mathbb{C}} \oplus \overline{F^{k+1-r} L_{\mathbb{C}}} = L_{\mathbb{C}} \tag{1.26}$$

for any r. Indeed, when (1.26) is satisfied, one defines $L^{p,q} = F^r L_{\mathbb{C}} \cap \overline{F^{k-r} L_{\mathbb{C}}}$ and one shows that it satisfies (1.25). For the Hodge structure on $H^k(X, \mathbb{Z})$, with X compact Kähler, one has

Lemma 1.20. *One has $F^r H^k(X, \mathbb{C}) = \mathrm{Im}\,(\mathbb{H}^k(X, \Omega_X^{\bullet \geq r}) \to \mathbb{H}^k(X, \Omega_X^{\bullet}))$.*

Indeed, let $V := \mathrm{Im}\,(\mathbb{H}^k(X, \Omega_X^{\bullet \geq r}) \to \mathbb{H}^k(X, \Omega_X^{\bullet}))$. The inclusion $F^r H^k(X, \mathbb{C}) \subset V$ follows from the definition of the spaces $H^{p,q}(X)$ in Theorem 1.13 and in the other direction, we can argue that, by a spectral sequence argument,

$$\dim V \leq \sum_{p \geq r, p+q=k} \dim H^q(X, \Omega_X^p).$$

As the right hand side is the dimension of $F^r H^k(X, \mathbb{C})$ by Theorem 1.13, they must be equal. **qed**

Corollary 1.21. *If X is a smooth projective variety defined over a subfield K of $\mathbb{C}$, the Hodge filtration on $H^k(X, \mathbb{C})$ is defined over K.*

This follows indeed from the algebraic de Rham cohomology arguments given in Section 1.3, and from the fact that the Serre isomorphism

$$\mathbb{H}^k(X_{an}, \Omega^{\bullet}_{X_{an}}) \cong \mathbb{H}^k(X, \Omega^{\bullet}_{X/\mathbb{C}})$$

is compatible with the filtrations induced by the naïve filtration on both sides.

The theory of Hodge structures is a very rich tool to study the topology of smooth projective varieties, but it is also crucial to be able to say something on nonprojective, for example quasi-projective, or singular varieties. One reason is the following: it is natural to study fibered smooth projective varieties, that is, morphisms $f : X \to B$. In most cases, the morphism f will have critical points, and will determine a topological fibration $X^0 \to B^0$ only over the Zariski open set $B^0$of regular values. The complement will consist of singular fibers. Usually the complement $B \setminus B^0$ (called the discriminant) is itself singular. Another example concerns the study of the coniveau of cohomology. Given a Betti cohomology class α on X, the coniveau of α is the maximal codimension of a closed algebraic subset $Y \subset X$ such that α is supported on Y, that is, vanishes on $X \setminus Y$. The support Y cannot in general taken to be smooth, because the topology of smooth subvarieties is too restricted (for example by the Lefschetz theorem on hyperplane sections in the case of hypersurfaces). Deligne [19] introduced the notion of mixed Hodge structure that turns out to exist on any cohomology group of any quasiprojective complex variety.

Definition 1.22. *([19], [63]) A mixed Hodge structure is the data of a lattice (or $\mathbb{Q}$-vector space) L, equipped with an increasing filtration W on L (the weight filtration), and a decreasing filtration F on $L_{\mathbb{C}}$ (the Hodge filtration) such that the filtration F induces a Hodge structure of weight k on $Gr_k^W L := W_k L/W_{k-1}L$.*

The indexing of weights is sometimes very hard to relate to the geometry so it might be preferable to choose a weight k_0 and to ask instead that the filtration F induces a Hodge structure of weight $k + k_0$ on $Gr_k^W L := W_k L/W_{k-1}L$. Hodge structures as in Definition 1.19 are called pure Hodge structures

Deligne's fundamental theorem states the existence of mixed Hodge structure on any cohomology group coming from algebraic geometry over $\mathbb{C}$. Its fundamental character and major applications, also developed in [19], come from the properties of the category of mixed Hodge structures that we will see in Section 2.1.

Theorem 1.23. *[19] The Betti cohomology groups $H^k_B(X, \mathbb{Q}) = H^k(X_{an}, \mathbb{Q})$ for X quasiprojective over $\mathbb{C}$, or relative cohomology groups $H^k_B(X, Y, \mathbb{Q})$, for $Y \subset X$ closed or open algebraic subvariety, carry natural mixed Hodge structures.*

These mixed Hodge structures are also functorial (see next Section). The easiest case is the case of a smooth quasiprojective variety X. By Hironaka resolution theorem [36], there exists a smooth projective variety $\overline{X}$ containing X as a Zariski open set, such that the boundary $\overline{X} \setminus X$ is a simple normal crossing divisor D. Denoting $j : X \hookrightarrow \overline{X}$ the inclusion map, one can then use the isomorphism (1.10) and compute $H^k(X, \mathbb{C})$ as $\mathbb{H}^k(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D))$. The two filtrations are then easy to define, at least on cohomology with $\mathbb{C}$-coefficients. The Hodge filtration F was already mentioned. It is induced by the naïve fitration $\Omega^{\bullet \geq r}_{\overline{X}}(\log D)$ on the complex $\Omega^\bullet_{\overline{X}}(\log D)$. The weight filtration is induced up to a shift by the filtration $W'_k \Omega^\bullet_{\overline{X}}(\log D) \subset \Omega^\bullet_{\overline{X}}(\log D)$ defined by

$$W'_k = \Omega^{\bullet \leq k}_{\overline{X}}(\log D) \wedge \Omega^{\bullet - k}_{\overline{X}}$$

is the subcomplex of sheaves of logarithmic forms with pole multiplicity at most k at any point. Thus W'_k ignores the singularities of the boundary divisor D where at least $k + 1$ branches intersect. It turns out that the filtration on cohomology induced by W' is in fact defined on cohomology with $\mathbb{Q}$-coefficients, as one can relate it directly to the Leray filtration associated to the map j. More precisely, if $W''_\bullet := W'_{-\bullet}$ to make it decreasing, the E_1-term of the W''-spectral sequence identifies with the E_2-term of the Leray specctral sequence. The fact that these two filtrations together endow the cohomology $H^k(X, \mathbb{Q})$ with a mixed Hodge structure uses the study of this filtration and its differentials. It turns out that it degenerates on E_2 (which corresponds to degeneracy on E_3 of the Leray spectral sequence). The differentials d_1 of the spectral sequence associated to the filtration W'' identify with Gysin morphisms induced by the inclusions $D_{(k)} \subset D_{(k-1)}$, where $D_{(k)}$ is the disjoint union of the intersection of k components of D. Hodge theory on the smooth projective varieties $D_{(k)}$ is needed in order to show that the differentials d_r vanish for $r \geq 2$.

2 Polarizations

2.1 The category of Hodge structures

Definition 2.1. *A morphism of integral (or rational) Hodge structures $(L, F^iL_{\mathbb{C}})$, $(L', F^iL'_{\mathbb{C}})$ of respective weights k, resp. $k + 2r$, is a morphism $\phi : L \to L'$ of lattices (or $\mathbb{Q}$-vector spaces) such that*

$$\phi_{\mathbb{C}}(F^iL_{\mathbb{C}}) \subset F^{i+r}L'_{\mathbb{C}}\ \forall i.$$

Equivalently, one has $\phi_{\mathbb{C}}(L^{p,q}) \subset L'^{p+r,q+r}$ for all (p, q) with $p + q = k$, and the morphism is said of type (r, r).

Definition 2.2. *A Hodge class $\alpha \in \mathrm{Hdg}(L)$, where $(L, F^iL_{\mathbb{C}})$ is a Hodge structure of even weight $2k$, is an element of $L \cap L^{k,k}$, where the intersection takes place in $L_{\mathbb{C}}$.*

Given two Hodge structures of weights k and k', we endow the lattice (or $\mathbb{Q}$-vector space) $\mathrm{Hom}\,(L, L')$ with the Hodge structure of weight $k' - k$ corresponding to the Hodge decomposition

$$\mathrm{Hom}\,(L, L')^{s,t} := \oplus \mathrm{Hom}\,(L^{p,q}, L'^{p+s,q+t}),\ s + t = k' - k.$$

The following lemma follows from the definition in a straightforward way.

Lemma 2.3. *A morphism $\phi \in \mathrm{Hom}\,(L, L')$ is a morphism of Hodge structures if and only if ϕ is a Hodge class.*

Remark 2.4. When L' is the trivial Hodge structure of weight 0, the definition above defines the dual of a Hodge structure.

The category of rational Hodge structures is not semisimple. This means that, given a Hodge substructure

$$L \subset L',$$

there does not exist in general a Hodge substructure $L'' \subset L'$ such that

$$L' \cong L \oplus L''$$

as Hodge structures. Let us construct an explicit example in weight 1. Let L be a rank 4 lattice and $L' \subset L$ be a sublattice of rank 2. Consider the effective Hodge structures of weight 1 on L such that L' is a Hodge substructure. The condition on $L^{1,0}$ is that $L'^{1,0} := L^{1,0} \cap L'_{\mathbb{C}}$ has dimension 1, thus defining a Hodge structure on L'. The parameter space for such an $L^{1,0}$ is an open set in a Schubert variety of $\mathrm{Grass}(2, L_{\mathbb{C}})$ and is easily seen to be of dimension 3. Next consider the condition that there is, at least with $\mathbb{Q}$-coefficients, a supplementary Hodge structure $L'' \subset L$. Fixing $L'' \subset L$ such that $L' \oplus L''$ has finite index in L, the condition on $L^{1,0}$ is that $L''^{1,0} := L^{1,0} \cap L''_{\mathbb{C}}$ also has dimension 1, so that

$$L^{1,0} = L'^{1,0} \oplus L''^{1,0}.$$

The parameters for $L'^{1,0}$ and $L''^{1,0}$ are open sets in $\mathbb{P}(L'_{\mathbb{C}})$ and $\mathbb{P}(L''_{\mathbb{C}})$ respectively, so the set of Hodge structures split as above by a fixed L'' has dimension 2. Thus the locus of split Hodge structures is the countable union (over all choices of L'') of loci of dimension 2, hence the very general Hodge structure L with Hodge substructure L' is not split.

Note first that the category of effective weight 1 integral Hodge structures is naturally equivalent to the category of complex tori while the category of effective weight 1 rational Hodge structures is naturally equivalent to the category of isogeny classes of complex tori. This correspondence is explained in the first case in Remark 0.2. In the rational case, we use the fact that choosing a lattice $L \subset L_{\mathbb{Q}} = H_1(T, \mathbb{Q})$ is equivalent to choosing a torus T' isogenous to T. With this translation, the existence of a non split Hodge substructure $L' \subset L$ of weight 1 is equivalent to the fact that one can have a surjective morphism

$$T \to T''$$

of compact complex tori of respective dimensions 2 and 1, such that T is not isogenous to $T'' \oplus T'$. If T is projective, this phenomenon is not possible, because one can then explicitly construct a section $T'' \to T$ up to isogeny. For example, one chooses a curve $C \subset T$ dominating T'' via a morphism r of degree d and then define as section up to isogeny the morphism of tori

$$T'' \to T,\ t \mapsto \mathrm{alb}_T(C_t - d0_T),$$

where C_t is the 0-cycle $r^{-1}(t)$ and 0_T is the origin of T. We will discuss in next section the notion of polarization for Hodge structures that generalizes the projectivity condition of the associated complex tori that can be used for effective weight 1 Hodge structures.

Hodge structures on the cohomology of compact Kähler manifolds have the following functoriality properties:

Proposition 2.5. *Let $\phi : X \to Y$ be a holomorphic map, where X and Y are compact Kähler of respective dimensions n and m. Then, for any k*

$$\phi^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is a morphism of Hodge structures.

The Gysin morphism

$$\phi_* : H^k(X, \mathbb{Z}) \to H^{k-2(n-m)}(Y, \mathbb{Z})$$

is a morphism of Hodge structures.

To be perfectly consistent, we should consider cohomology modulo torsion in this statement. The first statement is obvious since using the definition of the $H^{p,q}$ spaces given in Theorem 1.13, we just have to observe that the pull-back of a closed form of type (p,q) is a closed form of type (p,q). The second statement follows from the first by duality, since one checks that the Hodge structures on $H^{2n-k}(X,\mathbb{Q})$, resp. $H^{2n-k}(Y,\mathbb{Q})$ are dual in the sense described in Remark 2.4 to the Hodge structures on $H^k(X,\mathbb{Q})$, resp. $H^{k-2(n-m)}(Y,\mathbb{Q})$. (This is in fact true only up to a shift of bidegree, that is called a Tate twist.) One then uses the fact that the Gysin morphism identifies via Poincaré duality to the transpose of the morphism ϕ^*.

Definition 2.6. *A morphism of mixed Hodge structures (L,W,F) and (L',W',F') is a morphism $\phi : L \to L'$ of $\mathbb{Q}$-vector spaces such that*

$$\phi(W_rL) \subset W'_rL',\ \phi_{\mathbb{C}}(F^pL_{\mathbb{C}}) \subset F^pL'_{\mathbb{C}}.$$

Such a morphism induces for each r a morphism $\overline{\phi}_r : \mathrm{Gr}_r^W L \to \mathrm{Gr}_{r'}^W L$ of Hodge structures of weight r. A variant of this definition allows a shift by a bidegree (k,k), where for each r,

$$\phi(W_rL) \subset W'_{r+2k}L',\ \phi_{\mathbb{C}}(F^pL_{\mathbb{C}}) \subset F^{p+k}L'_{\mathbb{C}}.$$

Although this is a purely formal result, the following theorem due to Deligne [19] has striking consequences (see Section 2.3.1).

Theorem 2.7. *A morphism of Hodge structures is strict for both filtrations, that is*

$$\mathrm{Im}\,\phi \cap W'_rL' = \phi(W_rL),\ \mathrm{Im}\,\phi_{\mathbb{C}} \cap F^pL'_{\mathbb{C}} = \phi_{\mathbb{C}}(F^pL_{\mathbb{C}}).$$

This theorem is an easy consequence of the following formal but important result.

Theorem 2.8. *Let (L,W,F) be a mixed Hodge structure. Then there is a decomposition*

$$L_{\mathbb{C}} = \oplus_{p,q} I^{p,q} \tag{2.27}$$

of $L_{\mathbb{C}}$ into complex vector subspaces, such that

$$W_kL_{\mathbb{C}} = \oplus_{p+q\leq k} I^{p,q} \tag{2.28}$$

$$F^rL_{\mathbb{C}} = \oplus_{p\geq r} I^{p,q}. \tag{2.29}$$

Furthermore, one can construct such a decomposition to be functorial.

Note that, in particular, $I^{p,q} \subset W_{p+q}L_{\mathbb{C}} \cap F^pL_{\mathbb{C}}$ and via the projection map $W_{p+q}L_{\mathbb{C}} \to Gr^W_{p+q}L_{\mathbb{C}}$, $I^{p,q}$ is naturally isomorphic to $H^{p,q}(Gr^W_{p+q}L_{\mathbb{C}})$. However, unlike the case of a pure Hodge structure, the $I^{p,q}$ of decomposition (2.27) cannot in general be imposed to satisfy both the Hodge symmetry property (1.23) and the properties (2.28), (2.29). Theorem 2.8 implies Theorem 2.7 as follows. Let $\alpha \in W'_kL' \cap \mathrm{Im}\,\phi$ so that $\alpha = \phi(\beta)$. Write $\beta = \sum_{p,q} \beta^{p,q}$, using the decomposition (2.27) for L. Then $\phi(\beta) = \sum_{p,q} \phi(\beta^{p,q})$ with $\phi(\beta^{p,q}) \in I'^{p,q}$. Using the decomposition (2.27) for L' and $\alpha \in W_kL'$, we deduce that $\phi(\beta^{p,q}) = 0$ for $p+q > k$, and thus $\alpha = \phi(\beta')$, where $\beta' = \sum_{p+q\leq k} \beta^{p,q}$. As $\beta' \in W_kL$, this proves the result for the weight filtration W and the proof for the Hodge filtration F is the same. **qed**

2.2 Polarizations and the Hodge-Riemann bilinear relations

Definition 2.9. *A polarization on a rational Hodge structure $(H, F^pH_{\mathbb{C}})$ of weight k is a perfect pairing $\langle\, ,\, \rangle$ on H, which is symmetric if k is even and skew-symmetric if k is odd, and has the following properties:*

(i) (First Hodge-Riemann bilinear relations.) One has $\langle H^{p,q}, H^{p',q'}\rangle = 0$ if $(p', q') \neq (k-p, k-q)$.

(ii) (Second Hodge-Riemann bilinear relations.) The sesquilinear intersection pairing $h(\alpha) := i^k\langle \alpha, \overline{\alpha}\rangle$ is definite on $H^{p,q}$, of sign $(-1)^p$ (up to a global sign).

It is sometimes better to formulate (i) in the equivalent form $\langle F^pH_{\mathbb{C}}, F^{k+1-p}H_{\mathbb{C}}\rangle = 0$, which makes clear that the condition is holomorphic in the Hodge filtration F. These rules may seem too complicated to be useful but this is what one gets from geometry and the best possible generalization of the relations (0.1), (0.2) of the introduction, which were valid in the case of degree 1 cohomology. The category of polarized Hodge structures (where the morphisms are morphisms of rational Hodge structures as in Definition 2.1) is semi-simple, as shows the following

Lemma 2.10. *Let H be a polarized Hodge structure, and $H' \subset H$ be a Hodge structure. Then there exists a Hodge substructure $H'' \subset H$ such that*

$$H \cong H' \oplus H'' \tag{2.30}$$

(as vector spaces, hence as Hodge structures).

Proof. Choose a polarization $\langle\, ,\, \rangle$ on H. We define H'' as the orthogonal of H' w.r.t. $\langle\, ,\, \rangle$. Using property (i) in Definition 2.9, we see that H'' is also a Hodge substructure of H. In order to prove (2.30) we just have to show that $\langle\, ,\, \rangle_{|H'}$ is nondegenerate. Of course this can be checked after complexification, and after replacing $\langle\, ,\, \rangle_{|H'}$ by the sesquilinear pairing $h(\alpha, \beta) = i^k\langle \alpha, \overline{\beta}\rangle$. The Hodge decomposition of $H_{\mathbb{C}}$ is orthogonal for this pairing, and $H'_{\mathbb{C}}$ is the direct sum of its components $H'^{p,q}$. It thus suffices to show that $h_{|H'^{p,q}}$ is nondegenerate, and this is obvious since by (ii), $h_{|H^{p,q}}$ is definite nondegenerate. □

The following result is fundamental.

Theorem 2.11. *The Hodge structures $H^k(X, \mathbb{Q})$, for X smooth projective, admit polarizations.*

Let us sketch the proof, as this will make clear the nature and the importance of the problem formulated under the name of *standard conjectures* (see [47]). We consider X as a compact Kähler manifold (see Section 0.2) with Kähler form ω. Let $n = \dim X$. It is an easy algebraic fact that for $k \leq n$, the cup-product map acting on differential forms

$$\omega^{n-k}\wedge : \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}} \tag{2.31}$$

is an isomorphism of real vector bundles. One deduces a decomposition (Lefschetz decomposition on differential forms).

Corollary 2.12. *For $k \leq n = \dim X$, define $\Omega^k_{X,\mathbb{R},\mathrm{prim}}$ as*

$$\mathrm{Ker}\,(\omega^{n-k+1}\wedge : \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k+2}_{X,\mathbb{R}}).$$

Then $\Omega^k_{X,\mathbb{R}} = \oplus_{k-2r\geq 0}\omega^r \wedge \Omega^{k-2r}_{X,\mathbb{R},\mathrm{prim}}$.

It is not very hard to prove that the Laplacian commutes with the operator $\omega^{n-k}\wedge$. Hence we conclude that the operator $\omega^{n-k}\wedge$ acts on harmonic form and induces an injective linear map

$$L^{n-k} : \mathcal{H}^k \to \mathcal{H}^{2n-k} \tag{2.32}$$

which is nothing than the map (called the Lefschetz operator)

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R}) \tag{2.33}$$

of cup-product by $[\omega]^{n-k}$, after applying the identifications

$$\mathcal{H}^k \cong H^k(X, \mathbb{R}),\ \mathcal{H}^{2n-k} \cong H^{2n-k}(X, \mathbb{R})$$

of (1.22). As the two spaces in (2.33) have the same dimension, we conclude

Corollary 2.13. *(Hard Lefschetz theorem.) The operator L^{n-k} in (2.33) is an isomorphism.*

When the class ω is the class of a hyperplane section of a smooth projective manifold of dimension n, it is interesting to compare the hard Lefschetz theorem with the Lefschetz theorem on hyperplane sections, saying the following.

Theorem 2.14. *For any smooth complete intersection $j : Y \hookrightarrow X$ of $n-k$ hyperplane sections H_i of X, the restriction map $j^* : H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$ is an isomorphism for $i < k$ and is injective for $i = k$.*

Consider for simplicity the case $k = n-1$. The operator L of cup-product with the hyperplane class is equal to the composite map

$$j_* \circ j^* : H^{n-1}(X, \mathbb{Z}) \stackrel{j^*}{\to} H^{n-1}(Y, \mathbb{Z}) \stackrel{j_*}{\to} H^{n+1}(X, \mathbb{Z}).$$

Theorem 2.14 says that j^* is injective, so that by Poincaré duality, j_* is surjective. However the hard Lefschetz theorem is the stronger statement that

$$j_* \circ j^* : H^{n-1}(X, \mathbb{Q}) \to H^{n+1}(X, \mathbb{Q})$$

is an isomorphism, and this is equivalent to the fact that the Poincaré intersection pairing $\langle\, ,\, \rangle_Y$ on $H^{n-1}(Y, \mathbb{Q})$ remains nondegenerate on

$$\mathrm{Im}\,(H^{n-1}(X, \mathbb{Q}) \to H^{n-1}(Y, \mathbb{Q})).$$

There exist different proofs of Theorem 2.14. Note that, by induction, it clearly suffices to do the case of hypersurfaces (where $n-k=1$). One proof is by Morse theory and it is due to Andreotti and Frankel [3] (see also [59]) who prove that a smooth complex affine manifold of dimension n has the homotopy type of a CW complex of real dimension $\leq n$ and then apply Poincaré duality. The first part of the argument can be replaced by Serre vanishing on affine varieties. There is also an algebraic proof (working only with $\mathbb{C}$-coefficients) using the Akizuki-Kodaira-Nakano vanishing theorem (see [51]). In contrast, no algebraic proof of the hard Lefschetz theorem, using algebraic de Rham cohomology, is known.

Using the hard Lefschetz isomorphism (in all degrees) on harmonic differential forms, we also conclude that the space of harmonic forms $\mathcal{H}^k$ is stable under the

Lefschetz decomposition appearing in Corollary 2.12. Defining, for any $k' \leq n$, $H^{k'}(X,\mathbb{R})_{\text{prim}}$ as

$$\operatorname{Ker}(L^{n-k'+1} : H^{k'}(X,\mathbb{R}) \to H^{2n-k'+2}(X,\mathbb{R}))$$

we see from the above arguments that each space $H^{k-2r}(X,\mathbb{R})_{\text{prim}}$ is represented by harmonic forms annihilated by $\omega^{n-k+2r+1}$, that is, primitive forms, and that the *Lefschetz decomposition* on harmonic forms induces the Lefschetz decomposition on cohomology

$$H^k(X,\mathbb{R}) \cong \oplus_{k-2r\geq 0} L^r H^{k-2r}(X,\mathbb{R})_{\text{prim}}, \tag{2.34}$$

formally deduced from the hard Lefschetz isomorphism.

We now explain how to construct a polarization on the Hodge structure $H^k(X,\mathbb{Q})$. Recall from section 0.2 that on a complex projective manifold, one can choose a rational Kähler class $[\omega]$. It follows that the operator L of cup-product with $[\omega]$ acts on rational cohomology and that the Lefschetz decomposition (2.34) is defined on cohomology with rational coefficients. Next we use the rationality of the class $[\omega]$ to define the Lefschetz intersection pairing on cohomology

$$\langle \alpha, \beta\rangle_{\text{Lef}} = \langle \alpha, L^{n-k}\beta\rangle, \tag{2.35}$$

for α, $\beta \in H^k(X,\mathbb{Q})$. Using the fact that the class $[\omega]$ has type $(1,1)$, one sees immediately that this pairing satisfies the first Hodge-Riemann bilinear relations (i), but unfortunately, it does not satisfy the second ones. In fact, we have

Theorem 2.15. *The Lefschetz intersection pairing induces a polarization on each Hodge structure $L^rH^{k-2r}(X,\mathbb{Q})$ appearing in the Lefschetz decomposition.*

Unfortunately, the global sign appearing in Definition 2.9 (ii) changes with r, so the statement above is not true for the whole of $H^k(X,\mathbb{Q})$. The recipe to remedy this is the following. We observe that the Lefschetz decomposition (2.34) is orthogonal for $\langle\, ,\rangle_{\text{Lef}}$. We define the polarization h to be $(-1)^r\langle\, ,\rangle_{\text{Lef}}$ on the r-th piece of the Lefschetz decomposition, and we impose them to be mutually orthogonal for h. Theorem 2.15 says that, up to a global sign, we get this way a polarization of the Hodge structure.

For the proof of Theorem 2.15, we have to check the sign condition (ii) in Definition 2.9. It follows from the fact that, as already mentioned, we can work at the level of forms using harmonic representatives. Furthermore, on primitive forms α of type (p,q), $p+q=k$, we have a relation of the form

$$*\alpha = \pm L^{n-k}\alpha,$$

where the sign depends only on p, q, n. It follows that on each piece of the Hodge-Lefschetz decomposition, the Hermitian Lefschetz intersection pairing $\langle \alpha, \overline{\beta}\rangle_{\text{Lef}}$ equals up to a sign the L^2 Hermitian intersection pairing.

2.3 Motives and algebraic cycles

2.3.1 The Hodge and generalized Hodge conjectures

Let X be a smooth projective variety and $\alpha \in H^k(X,\mathbb{Q})$ be a cohomology class.

Definition 2.16. *The class α is of coniveau $\geq c$ if there exists a closed algebraic subset $Y \subset X$ of codimension $\geq c$ such that $\alpha_{|U} = 0$, where $U := X \setminus Y$.*

An easy statement is the following.

Lemma 2.17. *If α is of coniveau $\geq c$, α is of Hodge coniveau $\geq c$, that is,*

$$\alpha^{p,q} = 0 \text{ for } p < c \text{ or } q < c. \tag{2.36}$$

Proof. The class α is supported on Y, that is, it is the Poincaré dual of a homology class $T \in H_{2n-k}(Y, \mathbb{Q})$. We can assume that T is not supported on $\operatorname{Sing} Y$ (otherwise we replace Y by $\operatorname{Sing} Y$). The vanishings (2.36) are equivalent to $\int_T \beta = 0$ for $\beta \in H^{p',q'}(X)$, with $p' + q' = 2n - k$, $p' > n - c$ or $q' > n - c$. But then β vanish on Y_{reg}, since it is smooth of complex dimension $\leq n - c$, hence on T. (For the proof to be rigorous, one should introduce here stratification of singular analytic spaces). □

The extreme example is the case of Hodge classes (cf. Definition 2.2), which are rational cohomology classes of degree $2k$ and Hodge coniveau c. If $Y \subset X$ is an algebraic subvariety of codimension c, it has a cohomology class $[Y] \in H^{2c}(X, \mathbb{Z})$, which by Lemma 2.17 is a (integral) Hodge class. The class $[Y]$ can be defined using stratification, or using a desingularization $\tilde{j} : \widetilde{Y} \to Y \to X$ and defining $[Y] = \tilde{j}_*(1_{\tilde{Y}})$. When Y is irreducible, the class $[Y]$ can be shown to generate the kernel of the restriction map $H^{2c}(X, \mathbb{Z}) \to H^{2c}(X \setminus Y, \mathbb{Z})$. The Hodge conjecture is the following statement.

Conjecture 2.18. *The $\mathbb{Q}$-vector space* $\operatorname{Hdg}^{2c}(X, \mathbb{Q})$ *of rational Hodge classes is generated by classes $[Y]$as above.*

The only nontrivial case of the Hodge conjecture that is known in full generality is the case of degree 2, which is the Lefschetz theorem on $(1,1)$-classes. It has the specifity that it works as well with integral Hodge classes.

Theorem 2.19. *Let $\alpha \in \operatorname{Hdg}^2(X, \mathbb{Z})$, where X is smooth projective. Then $\alpha = [D]$ for some divisor $D = \sum_i n_i D_i$,* $\operatorname{codim} D_i = 1$ *of X.*

Griffiths explained in [28] the Lefschetz idea of using normal functions to prove this result for surfaces. The modern proof is very short, very elegant, but also very mysterious. One ingredient is the Serre GAGA theorem [65], which shows that it suffices to find a *holomorphic* line bundle H on X, such that

$$c_1(L) = \alpha.$$

Indeed, such a holomorphic line bundle is algebraic and for any rational section σ of H, the divisor D of σ satisfies $[D] = c_1(H)$ by Lelong formula. The end of the argument works on any complex manifold; it rests on the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^* \to 1,$$

where $\mathcal{O}_X$ is the sheaf of holomorphic functions. Taking the associated cohomology long exact sequence, one finds that classes $c_1(H) \in H^2(X, \mathbb{Z})$ are exactly those mapping to 0 in $H^2(X, \mathcal{O}_X)$, and when X is projective or compact Kähler, they are the integral Hodge classes.

We know examples of compact Kähler manifiolds not containing any closed analytic subset of codimension 1 but having nonzero degree 2 Hodge classes. Take for example a general complex torus of dimension 2 having a degree 2 Hodge class represented by a $(1,1)$-form which is nondegenerate and indefinite. The proof given above shows that in this case, the Hodge class is nevertheless the first Chern class of a holomorphic line bundle. This suggests that in the more general setting of compact Kähler manifolds, Hodge classes could be generated by Chern classes of holomorphic vector bundles or coherent sheaves. This has been disproved in [78], where it is shown that on a very general 4-dimensional Weil torus T (a torus admitting an endomorphism ϕ such that $\phi^2 = -Id$, with some condition on the eigenvalues of ϕ), the Weil Hodge classes (which are of Hodge classes of degree 4) cannot be generated by Chern classes of coherent sheaves. Indeed, it is proved that, on such a torus, any coherent sheaf $\mathcal{F}$ satisfies $c_1(\mathcal{F}) = c_2(\mathcal{F}) = 0$. The proof uses the extension to reflexive coherent sheaves, due to Bando and Siu, of the Uhlenbeck-Yau theorem [70] on the existence of Hermite-Einstein metrics on stable holomorphic vector bundles.

Hodge asked, inspired by Lemma 2.17 if more generally a rational cohomology class of Hodge coniveau c (but any degree $\geq 2c$) must be of coniveau $\geq c$, and Grothendieck disproved this statement in [33]. One reason why this is not true is the following result, in fact due to Deligne.

Theorem 2.20. *The set $N^cH^k(X,\mathbb{Q})$ of cohomology classes of coniveau $\geq c$ is a Hodge substructure of $H^k(X,\mathbb{Q})$, contained in $\oplus_{p\geq c, q\geq c}H^{p,q}(X)$ (that is, of Hodge coniveau $\geq c$).*

Proof. It suffices to show that the set of cohomology classes on X vanishing on $X\setminus Y$, where $Y \subset X$ is a closed algebraic subset of codimension c, is a Hodge substructure of $H^k(X,\mathbb{Q})$, (contained in $\oplus_{p\geq c, q\geq c}H^{p,q}(X)$ by Lemma 2.17). This set is the image of $H_{2n-k}(Y,\mathbb{Q})$ in $H^k(X,\mathbb{Q})$ via the composition

$$H_{2n-k}(Y,\mathbb{Q}) \xrightarrow{j} H_{2n-k}(X,\mathbb{Q}) \cong H^k(X,\mathbb{Q})$$

where j is the inclusion of Y in X, and the last map is Poincaré duality on X. The morphism j_* above is (up to a shift of (n,n)) a morphism of mixed Hodge structures. As the Hodge structure on $H^k(X,\mathbb{Q})$ is pure of weight k, the image is $W_kH^k(X,\mathbb{Q}) \cap \operatorname{Im} j_*$. By Theorem 2.7, one thus has

$$\operatorname{Im} j_* = j_*(W_{k-2n}H_{2n-k}(Y,\mathbb{Q})).$$

Looking at the construction of the weight filtration on $H_{2n-k}(Y,\mathbb{Q})$, one has

$$W_{k-2n}H_{2n-k}(Y,\mathbb{Q}) = \operatorname{Im}(\tau_* : H_{2n-k}(\widetilde{Y},\mathbb{Q}) \to H_{2n-k}(Y,\mathbb{Q}))$$

(and this is the minimal weight piece of $H_{2n-k}(Y,\mathbb{Q})$). Here $\tau : \widetilde{Y} \to Y$ is any desingularization. Thus we conclude that

$$\operatorname{Im} j_* = \operatorname{Im}((j\circ\tau)_* : H_{2n-k}(\widetilde{Y},\mathbb{Q}) \to H^k(X,\mathbb{Q})),$$

which concludes the proof since, $\widetilde{Y}$ being smooth, $(j\circ\tau)_*$ is a morphism of Hodge structures. $\square$

This statement allows to give counterexamples to the statement expected by Hodge (although this is not the way Grothendieck argues). Consider the family of hypersurfaces of degree 5 in $\mathbb{P}^4$. They have $h^{3,0}(X) = 1$. Choose a nonzero degree 3 rational cohomology class α on X, and consider the locus of deformations X_t of X for which $\alpha_t^{0,3} = 0$, where $\alpha_t^{0,3}$ is the $(0,3)$-component of the Hodge decomposition of α on X_t. This condition implies $\alpha_t^{3,0} = 0$ by Hodge symmetry, because α is rational, and it imposes one holomorphic equation to the parameter t, defining a hypersurface $\mathcal{D}$ in the parameter space for X. We can now show that along any hypersurface $\mathcal{D}$ in this parameter space, the general element X_t for $t \in \mathcal{D}$ does not contain any nontrivial Hodge substructure contained in $H^{2,1}(X_t) \oplus H^{1,2}(X_t)$. It then follows from Theorem 2.20 that the class α is not of coniveau ≥ 1 on X_t for a general $t \in \mathcal{D}$, while by construction, it is of Hodge coniveau 1.

The generalized Hodge conjecture corrects the expectation of Hodge taking into account Theorem 2.20 and gives the following generalization of the Hodge conjecture 2.18.

Conjecture 2.21. *Let $L \subset H^k(X, \mathbb{Q})$ be a Hodge substructure of Hodge coniveau $\geq c$, that is, $L^{p,q} = 0$ for $p < c$ or $q < c$. Then $L \subset N^c H^k(X, \mathbb{Q})$.*

This conjecture is widely open. The Hodge conjecture is the particular case where one considers a Hodge substructure of $H^{2c}(X, \mathbb{Q})$ of Hodge coniveau c. It is much easier to exhibit concrete unsolved instances of the generalized Hodge conjecture than the Hodge conjecture (the already mentioned Weil construction of Weil Hodge classes on Weil tori is rather sophisticated). Consider smooth hypersurfaces X of degree d in $\mathbb{P}^n$. By the Lefschetz theorem on hyperplane sections (Theorem 2.14), only the degree $n-1$ cohomology of X carries a nontrivial Hodge structure. Work of Griffiths [29] implies that the Hodge coniveau of $H^{n-1}(X, \mathbb{Q})$ is ≥ 2 if $n \geq 2d$. For n large enough, it is not known that the cohomology of a smooth hypersurface of degree d in $\mathbb{P}^n$ with $n \geq 2d$ has coniveau ≥ 2. For $d = 3$, $n = 6$, or $d = 4$, $n = 8$, one can check this is true.

2.3.2 Motives

Let X be an algebraic variety over a field K. We define the group of k-cycles $\mathcal{Z}_k(X)$ as the free abelian group generated by letters Z, for each closed irreducible algebraic subset $Z \subset X$ of dimension k. When the field is not algebraically closed, there is a subtlety in the fact that being irreducible over K is not equivalent to being irreducible over the algebraic closure $\overline{K}$. As we will work over $\mathbb{C}$, this is not very important here, but the great interest of algebraic cycles is that they are sensitive to the field of definition. When X is smooth of dimension n, we denote $\mathcal{Z}^k(X) =: \mathcal{Z}_{n-k}(X)$. There is no good intersection theory on the space of cycles, but after putting an adequate equivalence relation $\equiv$ on $\mathcal{Z}^k(X)$ for which a "Chow moving lemma" holds, one can define when X is smooth, a graded ring structure on $\mathcal{Z}^\bullet(X)/\equiv$ (see [24]). The smallest equivalence relation satisfying the Chow moving lemma and functoriality properties (stability under proper push-forward and flat pull-back) is rational equivalence, where one defines the subgroup $\mathcal{Z}^k(X)_{\mathrm{rat}}$ of cycles rationally equivalent to 0 as the subgroup generated by $n_*(\operatorname{div}\phi)$, for all subvarieties $W \subset X$ of codimension $k-1$ with normalization $n : W' \to W$, and all rational functions $\phi \in K(W)^* = K(W')^*$. We denote the quotient $\mathcal{Z}^k(X)/\mathcal{Z}^k(X)_{\mathrm{rat}}$

by $\mathrm{CH}^k(X)$. Having the Fulton refined intersection product (or by Chow moving lemma), we have a pull-back $\phi^* : \mathrm{CH}^k(Y) \to \mathrm{CH}^k(X)$ for any morphism between smooth algebraic varieties. Assuming now we are over $\mathbb{C}$, the cycle class map

$$Z \mapsto [Z_{an}] \in H^{2k}(X, \mathbb{Z}),$$

defined on $\mathcal{Z}^k(X)$, factors through $\mathrm{CH}^k(X)$ and is compatible with the functorialities (pull-back, proper push-forward) and intersection product. We can define much weaker equivalence relations, like algebraic equivalence, homological equivalence, and, when X is projective, numerical equivalence. The subgroup $\mathcal{Z}_{\mathrm{hom}}$ defining homological equivalence is the group of cycles homologous to 0, and the subgroup $\mathcal{Z}_{\mathrm{num}}$ defining numerical equivalence is the group of cycles $Z \in \mathcal{Z}^k(X)$ such that $\deg Z \cdot Z' = 0$ for any $Z' \in \mathcal{Z}^{n-k}(X)$. Here $Z \cdot Z'$ is a 0-cycle of X and its degree is the number of points counted with their multiplicities. The intersection pairing so defined can also be computed in cohomology as $\int_X [Z_{\mathrm{an}}] \cup [Z'_{\mathrm{an}}]$. The following conjecture, for smooth projective complex varieties, will be discussed in next section.

Conjecture 2.22. *Let X be smooth projective. On cycles of X with rational coefficients, numerical and homological equivalence coincide.*

Remark 2.23. As we use the Betti cycle class, the conjecture seems to assume working over $\mathbb{C}$. In fact this is not true, as one can construct a cycle class in algebraic de Rham cohomology (see Section 1.3), which over $\mathbb{C}$ differs from the Betti cycle class with complex coefficients by a coefficient of $(2i\pi)^k$ for codimension k cycles (here we use the comparison isomorphism of Theorem 1.11).

Remark 2.24. Suppose X is defined over a subfield $K \subset \mathbb{C}$. For any subvariety $Z \subset X$ of codimension k, there exists a Hilbert scheme, or Chow variety W defined over K and parameterizing the set of subvarieties of X with the same numerical invariants as Z. This variety being defined over K, its geometric irreducible components are defined over $\overline{K}$. The subvariety Z is parameterized by a point z in such a component W', and all complex points in this component parameterize subvarieties $Z' \subset X$ which are deformation equivalent to Z, hence have the same cycle class. As W' is defined over $\overline{K}$, it has points over $\overline{K}$. We thus conclude that any cycle class on X is the class of a cycle defined over $\overline{K}$, hence, seen as a algebraic de Rham cohomology class, is defined over $\overline{K}$ (up to a coefficient of $(2i\pi)^k$).

The interest of studying cycles modulo numerical equivalence is that these groups (or $\mathbb{Q}$-vector spaces) are defined inside algebraic geometry, and independently of any cohomology theory. They provide $\mathbb{Q}$-vector spaces, and not $\mathbb{Q}_\ell$-vector spaces as étale cohomology would do, while algebraic de Rham cohomology would produce K-vector spaces, K being the field of definition.

Definition 2.25. *A correspondence between two varieties X and Y is a cycle in $X \times Y$ modulo a given adequate equivalence relation. We denote by* $\mathrm{Corr}(X, Y)$ *the group of correspondences.*

Of course, the nature of the correspondence will depend on the equivalence relation we consider on cycles.We can thus speak of homological, numerical, Chow correspondences. It is also useful to work with $\mathbb{Q}$-cycles. Assume now that X, Y, T

are three smooth varieties, with Y projective. Then we can compose correspondences $Z \in \mathrm{Corr}(X,Y)$, $Z' \in \mathrm{Corr}(Y,T$ by the rule

$$Z' \circ Z = \mathrm{pr}_{XT*}(\mathrm{pr}_{XY}^* Z \cdot \mathrm{pr}_{YT}^* Z'). \tag{2.37}$$

When Z, Z' are respectively the graphs of morphisms $f : X \to Y$, $g : Y \to T$, $Z' \circ Z$ is the graph of the composed morphism $g \circ f$. Motives introduced by Grothendieck use correspondences in place of morphisms. They also enlarge the category of varieties by adding projectors:

Definition 2.26. *A (smooth, projective, effective) motive is a pair (X,Z) where X is smooth projective and $Z \in \mathrm{Corr}(X,X)$ is a projector : $Z \circ Z = Z$.*

Example 2.27. *The full motive of X is the pair (X, Δ_X), where Δ_X is the diagonal of X. If $x \in X$, we can also consider the motives*

$$(X, X \times x),\ (X, x \times X),\ (X, \Delta_X - X \times x - x \times X).$$

Correspondences act in a compatible way on cycles (modulo the given equivalence relation) and on cohomology. If $\gamma \in H^{2k}(X \times Y, \mathbb{Q})$, the action of γ on $H^\bullet(X,\mathbb{Q})$ (resp. $H^\bullet(Y,\mathbb{Q})$) is given by

$$\gamma_*(\alpha) = \mathrm{pr}_{Y*}(\gamma \cup \mathrm{pr}_X^* \alpha) \in H^{\bullet+2k-2n}(Y,\mathbb{Q}),$$

$$\gamma^*(\alpha) = \mathrm{pr}_{X*}(\gamma \cup \mathrm{pr}_Y^* \alpha) \in H^{\bullet+2k-2m}(X,\mathbb{Q}),$$

where $n = \dim X$, $m = \dim Y$. The Hodge structure on $H^*(X \times Y, \mathbb{Q})$ identifies canonically, via the Künneth decomposition and Poincaré duality on X, with the Hodge structure on $\mathrm{Hom}\,(H^*(X,\mathbb{Q}), H^*(Y,\mathbb{Q}))$ as defined in Section 2.1. Lemma 2.3 then tells us that the Hodge classes on $X \times Y$ identify to the morphisms of Hodge structures inside $\mathrm{Hom}\,(H^*(X,\mathbb{Q}), H^*(Y,\mathbb{Q}))$. If $\dim X = \dim Y$, the Hodge classes of degree $2n$ on $X \times Y$ identify to the degree preserving morphisms of Hodge structures in $\mathrm{Hom}\,(H^*(X,\mathbb{Q}), H^*(Y,\mathbb{Q}))$. (Here, by Hodge structures, we mean direct sums of Hodge structures, allowing different weights.) There is a faithful functor from the category of cohomological motives to the category of Hodge structures, which to a pair (X,π) associates the Hodge structure $\mathrm{Im}\,\pi_* \subset H^*(X,\mathbb{Q})$. By the arguments above, the Hodge conjecture 2.18 predicts that it is fully faithful.

2.3.3 Standard conjectures

The standard conjectures (see [47]) are specific cases of the Hodge conjecture, applied to natural Hodge classes on squares $X \times X$ of projective manifolds X, and designed to produce self-correspondences of X allowing to split the motive of X, as introduced in the previous section, according to the Lefschetz decomposition on cohomology (2.34). We will focus on one main conjecture and its many implications, namely the Lefschetz standard conjecture. Recall that if X is smooth projective of dimension n with polarizing class $[\omega] = c_1(H) \in H^2(X,\mathbb{Q})$, H an ample line bundle on X, one has, for any $k \le n$, the hard Lefschetz isomorphism

$$L^{n-k} : H^k(X,\mathbb{Q}) \stackrel{\cong}{\to} H^{2n-k}(X,\mathbb{Q}) \tag{2.38}$$

of Corollary 2.13, where L is the operator of cup-product with the class $[\omega]$. The morphism L^{n-k} is a morphism of Hodge structures, and it is induced by the class of the cycle $Z_{n-k} = \delta_{X*}(H^{n-k}) \subset X \times X$, where H^{n-k} is the k-cycle of X represented by the intersection of $n-k$ members of $|H|$, and $\delta_X : X \to X \times X$ is the diagonal embedding. The inverse

$$(L^{n-k})^{-1} : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q}) \tag{2.39}$$

is also a morphism of Hodge structures. According to the discussion in the previous section, the following conjecture is thus implied by the Hodge conjecture.

Conjecture 2.28. *(Lefschetz standard conjecture.) There exists a cycle*

$$Z_{\mathrm{Lef},k} \in \mathrm{CH}^k(X \times X)_{\mathbb{Q}},$$

such that

$$[Z_{\mathrm{Lef},k}]^* \circ L^{n-k} = Id_{H^k(X,\mathbb{Q})},\ L^{n-k} \circ [Z_{\mathrm{Lef},k}]_* = Id_{H^{2n-k}(X,\mathbb{Q})}. \tag{2.40}$$

Note that the second equality follows from the first, by dualizing via by Poincaré duality. There are good reasons to state the Lefschetz standard conjecture separately from the Hodge conjecture. The Lefschetz standard conjecture could be true even if the Hodge conjecture is wrong. Indeed, the Hodge classes appearing in the standard conjectures are very special. We will discuss this point in Section 4. The second point is that the Hodge conjecture is specific to Betti cohomology while the statement of the Lefschetz standard conjecture can be made in other cohomology theories, like étale cohomology or algebraic de Rham cohomology. The Lefschetz standard conjecture is known in full generality for degree 1 cohomology. In this case, the cycle $Z_{\mathrm{Lef},1}$ is obtained up to a coefficient by pulling-back to $X \times X$ the Poincaré (or universal) divisor $\mathcal{P}$ on $\mathrm{Pic}^0(X) \times X$, via the morphism $(j, Id_X) : X \times X \to \mathrm{Pic}^0(X) \times X$, where the morphism $j : X \to \mathrm{Pic}^0(X)$ is the composition of the Albanese morphism $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ and the isogeny $\mathrm{Alb}(X) \to \mathrm{Pic}^0(X)$ induced by the choice of ample line bundle H.

The main general consequence of Conjecture 2.28 is the following (cf. Conjecture 2.22).

Theorem 2.29. *(see [55]) If X satisfies the Lefschetz standard conjecture in all even degrees, homological and numerical equivalence coincide on algebraic cycles of X with $\mathbb{Q}$-coefficients. Equivalently, the pairing between $H^{2k}(X, \mathbb{Q})_{\mathrm{alg}}$ and $H^{2n-2k}(X, \mathbb{Q})_{\mathrm{alg}}$ is perfect.*

Here we denote by $H^{2k}(X, \mathbb{Q})_{\mathrm{alg}} \subset H^{2k}(X, \mathbb{Q})$ the $\mathbb{Q}$-vector subspace generated by classes of codimension k closed algebraic subsets of X. Conjecturally, it is equal to $\mathrm{Hdg}^{2k}(X, \mathbb{Q})$.

Proof. The equivalence between the two statements is clear. Indeed, if Z is homologous to 0, then it is numerically equivalent to 0, since the pairing $\deg(Z \cdot Z')$ for any cycle of codimension $n-k$ can be computed as $\int_X [Z] \cup [Z']$, hence is 0 if $[Z] = 0$. In the other direction, if $Z \in \mathrm{CH}^k(X)_{\mathbb{Q}}$ is numerically equivalent to 0, the above formula shows that $[Z]$ pairs to 0 with any cycle class of degree $2n-2k$, so if the pairing is perfect, one must have $[Z] = 0$. Let us now explain the proof of the second statement. We assume $2k \leq n$ and X satisfies the Lefschetz standard conjecture in even degrees. We claim that the pairing $\langle\, ,\rangle_{\text{Lef}}$ on $H^{2k}(X,\mathbb{Q})_{\text{alg}}$ given by

$$\langle [Z],[Z'] \rangle_{\text{Lef}} = \int_X [Z] \cup L^{n-2k}[Z']$$

is perfect. This will conclude the proof because $L^{n-2k}[Z'] = [H^{n-2k}\cdot Z']$ is algebraic and, by formula (2.40), the Lefschetz standard conjecture for degree $2k$ implies that the Lefschetz operator induces an isomorphism

$$L^{n-2k} : H^{2k}(X,\mathbb{Q})_{\text{alg}} \cong H^{2n-2k}(X,\mathbb{Q})_{\text{alg}}$$

since it is injective and (2.40) provides a left inverse. To prove the claim, observe that, assuming the Lefschetz standard conjecture in even degrees $\leq 2k$, we also have a restricted hard Lefschetz isomorphism

$$L^{n-2k+2r} : H^{2k-2r}(X,\mathbb{Q})_{\text{alg}} \cong H^{2n-2k+2r}(X,\mathbb{Q})_{\text{alg}}$$

for any $r \leq k$. It follows that $H^{2k}(X,\mathbb{Q})_{\text{alg}}$ is stable under the Lefschetz decomposition (2.34). Thus we have

$$H^{2k}(X,\mathbb{Q})_{\text{alg}} = \oplus_{r\leq k} L^r H^{2k-2r}(X,\mathbb{Q})_{\text{alg,prim}}. \tag{2.41}$$

The decomposition (2.41) is orthogonal for $\langle\, ,\rangle_{\text{Lef}}$ so it suffices to show that the restriction of $\langle\, ,\rangle_{\text{Lef}}$ on each term $L^r H^{2k-2r}(X,\mathbb{Q})_{\text{alg,prim}}$ is nondegenerate. This follows from the second Hodge-Riemann relations (see Theorem 2.15), since one has

$$L^r H^{2k-2r}(X,\mathbb{Q})_{\text{alg,prim}} \subset L^r H^{k-r,k-r}(X)_{\text{prim}},$$

and $\langle\, ,\rangle_{\text{Lef}}$ is definite nondegenerate on $L^r H^{k-r,k-r}(X)_{\text{prim}}$. $\square$

Formal consequences of Theorem 2.29 are as follows:

Corollary 2.30. *Let $Y \subset X$ be smooth complex projective varieties, both satisfying the standard Lefschetz conjecture. Denote by j the inclusion of Y in X. Let $c = \operatorname{codim} Y \subset X$.*

(i) Let $Z \in \mathrm{CH}^k(Y)_{\mathbb{Q}}$ and assume that there exists $\beta \in H^{2k}(X,\mathbb{Q})$ such that $\beta_{|Y} = [Z]$ in $H^{2k}(Y,\mathbb{Q})$. Then there exists $Z' \in \mathrm{CH}^k(X)_{\mathbb{Q}}$ such that

$$[Z']_{|Y} = [Z].$$

(ii) Let $Z \in \mathrm{CH}^k(X)_{\mathbb{Q}}$ and assume that there exists $\beta \in H^{2k-2c}(Y,\mathbb{Q})$ such that $j_*\beta = [Z]$ in $H^{2k}(X,\mathbb{Q})$. Then there exists $Z' \in \mathrm{CH}^{k-c}(Y)_{\mathbb{Q}}$ such that

$$j_*[Z'] = [Z].$$

Proof. We only prove (i), as the argument for (ii) is exactly the same. Let $n = \dim X$. The class β defines by Poincaré pairing a linear form β^* on $H^{2n-2k}(X,\mathbb{Q})_{alg}$, given by

$$\beta^*([W]) = \int_X \beta \cup [W].$$

By Theorem 2.29, assuming the Lefschetz standard conjecture for X, there exists a cycle $Z' \in \mathrm{CH}^k(X)_{\mathbb{Q}}$ such that

$$\beta^*([W]) = \int_X [Z'] \cup [W]$$

for any $W \in \mathrm{CH}^{n-k}(X)_{\mathbb{Q}}$. We claim that $[Z']_{|Y} = [Z]$. By Theorem 2.29 now applied to Y which by assumption also satisfies the Lefschetz standard conjecture, it suffices to prove that for any $W \in \mathrm{CH}^{n-c-k}(Y)_{\mathbb{Q}}$,

$$\int_Y [W] \cup [Z']_{|Y} = \int_Y [W] \cup [Z]. \tag{2.42}$$

We have

$$\int_Y [W] \cup [Z']_{|Y} = \int_X j_*[W] \cup [Z'] = \beta^*(j_*[W]) = \int_X j_*[W] \cup \beta$$

$$\int_Y [W] \cup \beta_{|Y} = \int_Y [W] \cup [Z],$$

which proves (2.42). □

Remark 2.31. We stated Corollary 2.30 only for inclusion morphisms j, but it would work as well for any morphism, and even any correspondence between smooth complex projective varieties.

The Lefschetz standard conjecture concerns Hodge classes on the self-product $X \times X$ of a smooth complex projective variety X. There are other Hodge classes on $X \times X$, which are much easier to construct, namely the Künneth components of the diagonal. As we mentioned, any endomorphism of Hodge structure $H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ can be seen as a Hodge class on $X \times X$. If the endomorphism preserves the degree, then the Hodge class is of degree $2n$, $n = \dim X$. The Künneth components of the diagonal are the projectors

$$\delta_k = Id_{H^k(X,\mathbb{Q}} \in \mathrm{End}\, H^*(X, \mathbb{Q})$$

onto the degree k cohomology of X. One has $\sum_k \delta_k = Id_{H^*(X,\mathbb{Q})}$ hence

$$\sum_k \delta_k = [\Delta_X],$$

where the right hand side is the class of the diagonal of X, hence is an algebraic class. However it is not known if each δ_k individually is algebraic, a problem which is referred to as the Künneth standard conjecture.

Proposition 2.32. *([47]) The Lefschetz standard conjecture for X implies the Künneth standard conjecture for X.*

Proof. One argues inductively on k. Choose a polarizing class $h = c_1(H) \in H^2(X, \mathbb{Q})_{alg}$ on X, and suppose the Lefschetz standard conjecture holds for X and the δ_i are algebraic on $X \times X$ for $i < k$. It follows that the class

$$\delta_{\geq k} := \sum_{i \geq k} \delta_i = [\Delta_X] - \sum_{i<k} \delta_i$$

is algebraic and acts as the projector on $\oplus_{l\geq k}H^l(X,\mathbb{Q})$. For $k \leq n$, let $T_k = \delta_{X*}h^{n-k} \in H^{2n+2k}(X\times X,\mathbb{Q})_{\rm alg}$ be the class of a complete intersection of $n-k$ members of $|H|$ supported on the diagonal of X, and $Z_k \in H^{2k}(X\times X,\mathbb{Q})_{\rm alg}$ be a Lefschetz class, such that

$$[Z_k\circ T_k]_* = Id : H^k(X,\mathbb{Q}) \to H^k(X,\mathbb{Q}). \tag{2.43}$$

The cycle class $[Z_k\circ T_k\circ \delta_{\geq k}] \in H^{2n}(X\times X,\mathbb{Q})_{\rm alg}$ acts as 0 on $H^l(X,\mathbb{Q})$ for $l<k$ because $\delta_{\geq k}$ does; it acts as the identity on $H^k(X,\mathbb{Q})$ and as 0 on $H^l(X,\mathbb{Q})$ for $l>2k$ because T_k acts as 0 on $H^l(X,\mathbb{Q})$ for $l>2k$ (indeed, $\dim T_k = k$). We have to understand what happens on $H^l(X,\mathbb{Q})$ for $k<l<2k$ but in fact, as $[Z_k\circ T_k\circ \delta_{\geq k}]_*$ factors through the action of $[T_k]$ in this range, that is, through the Lefschetz operator h^{n-k}, it is only the action of $[Z_k]$ on $h^{n-k}H^l(X,\mathbb{Q}) \subset H^{l+2n-2k}X,\mathbb{Q})$, with $l+2n-2k > 2n-k$ that we have to consider. Using the Lefschetz cycles for $l'<k$, we can thus correct $[Z_k\circ T_k\circ\delta_{\geq k}]$ by algebraic classes on $X\times X$ so that the resulting class acts trivially on $H^{>k}(X,\mathbb{Q})$, hence is the Künneth projector δ_k. □

2.4 On the topology of Kähler and projective manifolds

We are going to discuss in this section a completely different application of polarizations on Hodge structures, showing that their existence produces some topological restrictions on complex projective manifolds, that general compact Kähler manifolds do not necessary satisfy. To start with, the Kodaira embedding theorem says the following.

Theorem 2.33. *A compact Kähler manifold X is projective if and only if it admits a Kähler form ω whose cohomology class is rational, i.e. $[\omega]\in H^2(X,\mathbb{Q})\subset H^2(X,\mathbb{R})$.*

The proof beautifully uses the sheaf-theoretic language combined with Hodge theory. The first step consists in using the Lefschetz $(1,1)$-theorem to conclude that (after passing to a multiple), $\omega = c_1(H)$, for some holomorphic line bunlde H on X. The $\partial\overline{\partial}$-lemma 1.17 then guarantees that ω is the Chern form of a Hermitian metric h on H (see Proposition 1.18). The second part consists in proving that the line bundle H is ample, that is a power $H^{\otimes N}$ is very ample. A first reduction step introduces blow-ups $\tau : \widetilde{X}\to X$ of X at points x, y, thus introducing exceptional divisors E_x, E_y in $\widetilde{X}$. The goal of this operation is the following: in order to prove that holomorphic sections of $H^{\otimes N}$ separate the points x, y, it suffices to show the vanishing $H^1(X, H^{\otimes N}\otimes\mathcal{M}_x\otimes\mathcal{M}_y)=0$, where $\mathcal{M}_x\subset\mathcal{O}_X$ is the subsheaf of holomorphic functions vanishing at x. This vanishing is equivalent to the vanishing

$$H^1(\widetilde{X},\tau^*H^{\otimes N}(-E_x-E_y))=0, \tag{2.44}$$

where $\mathcal{O}_{\widetilde{X}}(-E_x-E_Y)$ is the sheaf of holomorphic functions on $\widetilde{X}$ vanishing along the hypersurfaces E_x, E_y. This sheaf is the sheaf of sections of a holomorphic line bundle on $\widetilde{X}$, to which Kodaira applies his celebrated vanishing theorem

Theorem 2.34. *(Kodaira vanishing theorem) Let Y be a smooth compact complex manifold, and M a holomorphic line bundle on Y admitting a Hermitian metric h whose Chern form ω_h is positive. Then*

$$H^i(Y,K_Y\otimes M)=0 \tag{2.45}$$

for $i>0$.

The proof of the vanishing theorem rests on Hodge theory, that is the study of harmonic forms, and Kähler identities with the inclusion of a curvature term.

The Kodaira vanishing theorems has been improved later on in various ways. First of all, the statement itself has been improved, with weakened positivity properties, by Kawamata [44] and Viehweg [73] independently. Second, by the introduction due to Nadel [60] of the multiplier ideals, which give vanishing theorems with coefficients in ideals sheaves which are not locally free. In the above line of argument, the Hermitian metric on the original line bundle is allowed to have singularities. The multiplier ideals appear as the necessary corrections to make certain singular forms to be L^2.

Theorem 2.33 has the following corollary

Corollary 2.35. *Let X be a compact Kähler manifold such that $H^2(X, \mathcal{O}_X) = 0$. Then X is projective.*

Proof. The Hodge decomposition theorem says, when $H^2(X, \mathcal{O}_X) = 0$, that $H^2(X, \mathbb{R}) = H^{1,1}(X)_{\mathbb{R}}$, where $H^{1,1}(X)_{\mathbb{R}}$ is the set of de Rham cohomology classes of closed real forms of type $(1,1)$. Among these forms, the Kähler condition is a positivity condition which is open. There is thus a non-empty open cone in $H^2(X, \mathbb{R})$ consisting of Kähler classes. As $H^2(X, \mathbb{Q})$ is dense in $H^2(X, \mathbb{R})$, this cone contains rational cohomology classes. □

When $H^2(X, \mathcal{O}_X) = 0$, there are examples of Kähler nonprojective manifolds, for example a general complex torus of dimension ≥ 2 has no topologically nontrivial holomorphic line bundle, but Theorem 2.33 allows in many cases to prove that, given a deformation family, that is, a smooth proper holomorphic map $\pi : \mathcal{X} \to B$, the set of points $t \in B$ such that the fibre $\mathcal{X}_t$ is projective is dense in B, being locally the countable union of closed analytic spaces. More precisely, we can assume B is contratible, so that by Ehresman's fibration theorem, the family $\mathcal{X}$ is topologically a product $\mathcal{X}_0 \times B$ and any cohomology class $\alpha \in H^2(\mathcal{X}_0, \mathbb{Q})$ extends to a class $\alpha \in H^2(\mathcal{X}, \mathbb{Q})$, with restrictions $\alpha_t \in H^2(\mathcal{X}_t, \mathbb{Q})$. Then we get for each $\alpha \in H^2(\mathcal{X}_0, \mathbb{Q})$ a locally closed analytic space $B_\alpha \subset B$ defined by

$$B_\alpha^K = \{t \in B,\, \alpha_t \text{ is Kaehler on } \mathcal{X}_t\}.$$

The locus B_α^K is locally closed, because the Kähler condition is open on the sets of classes of closed $(1,1)$-forms, so B_α^K is open in the set

$$B_\alpha = \{t \in B,\, \alpha_t \text{ is of type } (1,1) \text{ on } \mathcal{X}_t\}$$

which is closed analytic in B. Kodaira theorem 2.33 says that the union of the loci B_α parameterizes the projective fibers in the family $\mathcal{X} \to B$. This analysis allows to show that compact Kähler manifolds with trivial or torsion canonical bundle admit algebraic approximations, meaning that the points parameterizing projective fibres are dense in their Kuranishi families. The following result, originally proved by Kodaira using his classification of surfaces, has been reproved by Buchdahl [11] by such a deformation argument, and even an infinitesimal argument.

Theorem 2.36. *(Kodaira [50], [11]) Compact Kähler surfaces admit algebraic approximations.*

This theorem implies in particular

Corollary 2.37. *A compact Kähler surface is homeomorphic to a projective complex surface.*

This last result is completely wrong in higher dimensions.

Theorem 2.38. *([75]) Starting from dimension 4, there exist compact Kähler manifolds X whose cohomology algebra is not isomorphic to the cohomology algebra of a complex projective manifold.*

The case of dimension 3 left by Theorems 2.36 and 2.38 has been solved by Lin [56], who proved that Theorem 2.36 also holds in dimension 3. Our original argument used the cohomology ring, but with the help of Deligne, it turned out that the cohomology algebra with rational coefficients provides enough obstructions. It was even proved in [76] that, starting from dimension 8, there exist compact Kähler manifolds X with the property that for any compact complex manifold $\widetilde{X}$ bimeromorphic to X, the cohomology algebra of $\widetilde{X}$ is not isomorphic to the cohomology algebra of a complex projective manifold. These compact Kähler manifolds are not homeomorphic to any complex projective manifold, and neither any of their smooth bimeromorphic models.

Let us explain the ingredients in the proof of 2.38. The key point is the observation that the Hodge structures on the various cohomology groups $H^k(X, \mathbb{Q})$ are constrained by their compatibility with the algebra structure on cohomology. This is summarized in the following definition. Here we will call a cohomology algebra (say with $\mathbb{Q}$-coefficients) any finite dimensional graded and graded-commutative $\mathbb{Q}$-algebra satisfying $A^{2n} = \mathbb{Q}$ for some integer n, and the condition that $A^i \otimes A^{2n-i} \to A^{2n}$ is a perfect pairing.

Definition 2.39. *A Hodge structure on a cohomology algebra A is the data of a Hodge structure of weight k on A^k for each k, such that the multiplication map $A^k \otimes A^l \to A^{k+l}$ is a morphism of Hodge structures for each k, l.*

If X is compact Kähler, its cohomology algebra carries a Hodge structure, hence this criterion cannot be used in itself to distinguish topologically projective manifolds from general compact Kähler manifolds. Note however that it is very restrictive, and this provides new ways of constructing compact symplectic non-Kähler manifolds (classical examples can be found in [68], [50]). For example, in [77], the following result is proved:

Theorem 2.40. *Let M be a real oriented compact manifold of dimension $2m$, whose cohomology algebra is generated in degrees ≤ 2. Let $E \to M$ be a complex vector bundle on M such that $c_1(E) = 0$. Then the manifold N constructed as the complex projective bundle $\mathbb{P}(E)$ admits a Hodge structure on its cohomology algebra if and only if M admits a Hodge structure on its cohomology algebra, such that the Chern classes $c_i(E) \in H^{2i}(M, \mathbb{Q})$ are Hodge classes.*

Using this result, one can construct symplectic non-Kähler manifolds as complex projective bundles over complex tori. In fact, most complex projective bundles over complex tori are topologically non-Kähler because, using Theorem 2.40, they do not admit a Hodge structure on their cohomology algebra.

Coming back to the proof of Theorem 2.38, the key point is to use the fact that the cohomology algebra of a projective complex manifold admits a polarized Hodge structure, namely a Hodge structure in the sense of Definition 2.39 and a degree 2 Hodge class ω satisfying the hard Lefschetz property and the Hodge-Riemann bilinear relations of Definition 2.9. Our strategy was to show that, for some well chosen examples X, any Hodge structure on the cohomology algebra of X is not polarizable. The first examples in [75] were constructed starting from the observation that certain automorphisms ϕ prevent a complex torus T to be algebraic. More precisely, as described in the introduction, T corresponds to a weight 1 Hodge structure on $H^1(T,\mathbb{Z})$, and ϕ^* acts on the lattice $H^1(T,\mathbb{Z})$ as an automorphism of Hodge structures. The presence of such an automorphism ϕ^* may prevent the Hodge structure to be polarizable, for example if the action of ϕ^* on $H^2(T,\mathbb{Q}) = \bigwedge^2 H^1(T,\mathbb{Q})$ is irreducible, since this implies that $\mathrm{Hdg}^2(T) = 0$ when $\dim T \geq 2$. Indeed, $\mathrm{Hdg}^2(T)$ cannot be equal to the whole of $H^2(T,\mathbb{Q})$ if $H^{2,0}(T) \neq 0$, that is, once $\dim T \geq 2$. On the other hand, $\mathrm{Hdg}^2(T)$ is preserved by ϕ^*, so if the action of ϕ^* is irreducible, $\mathrm{Hdg}^2(T)$ must be 0. We thus started from such a pair (T,ϕ) and constructed the compact Kähler manifold X as follows: inside $T\times T$, blow-up $T\times x$, $x\times T$, the diagonal Δ_T and the graph of ϕ. The resulting compact Kähler manifold satisfies the property stated in Theorem 2.38, that is

Theorem 2.41. *The manifold X so constructed does not have the cohomology algebra of a smooth projective manifold.*

If we consider the cohomology ring, instead of the cohomology algebra, then one can argue as follows: Note that, because X is bimeromorphic to a complex torus $T\times T$ of dimension $2n$, with $n \geq 2$, the natural map

$$\bigwedge^{4n} H^1(T\times T,\mathbb{Z}) \to H^{4n}(X,\mathbb{Z}) \tag{2.46}$$

given by cup-product is an isomorphism. This property is a characterization, among compact Kähler manifolds, of manifolds bimeromorphic to complex tori. Indeed, any compact Kähler manifold admits an Albanese map, which is a holomorphic map

$$\mathrm{alb}_Y : Y \to \mathrm{Alb}\, Y = H^{1,0}(Y)^*/H_1(Y,\mathbb{Z})$$

given by integrating holomorphic 1-forms along paths. The map alb_Y induces by construction an isomorphism

$$\mathrm{alb}_Y^* : H^1(\mathrm{Alb}\, Y,\mathbb{Z}) \cong H^1(Y,\mathbb{Z})$$

so that (2.46) identifies with the pull-back map

$$\mathrm{alb}_Y^* : H^{4n}(\mathrm{Alb}\, Y,\mathbb{Z}) \cong H^{4n}(Y,\mathbb{Z})$$

which is an isomorphism if and only if the map alb_Y is of degree 1, that is bimeromorphic. Having this, suppose Y is a compact Kähler manifold with a cohomology ring isomorphic to the one of X, say

$$i : H^*(X,\mathbb{Z}) \cong H^*(Y,\mathbb{Z}).$$

Then $\mathrm{alb}_Y : Y \to \mathrm{Alb}(Y)$ is also bimeromorphic by the characterization above and thus the classes in $\mathrm{Ker}\,\mathrm{alb}_{Y*} \subset H^2(Y,\mathbb{Z})$ are divisor classes, and they are images

$i(d)$ of the similarly defined divisor classes $d \in \operatorname{Ker} \mathrm{alb}_{X*} \subset H^2(X, \mathbb{Z})$ on X. The group of these divisor classes has rank 4 and contains exactly (up to a multiple) 4 generators d_j having the property that

$$d_j : H^1(X, \mathbb{Z}) \to H^3(X, \mathbb{Z})$$

is not injective. These divisor classes are exactly (up to a multiple) the classes of the four exceptional divisors and their kernels are Hodge substructures $L_j \subset H^1(X, \mathbb{Z}) = H^1(T, \mathbb{Z}) \times H^1(T, \mathbb{Z})$, defined as

$$H^1(T, \mathbb{Z}) \times 0,\ 0 \times H^1(T, \mathbb{Z}),\ \Gamma_{-1},\ \Gamma_{-\phi},$$

where $\Gamma_{-1} \subset H^1(T, \mathbb{Z}) \times H^1(T, \mathbb{Z})$ is the graph of $-Id$ and $\Gamma_{-\phi} \subset H^1(T, \mathbb{Z}) \times H^1(T, \mathbb{Z})$ is the graph of $-\phi^*$. The images under i of the L_j are Hodge substructures of $H^1(Y, \mathbb{Z})$, being the kernels of the cup-product maps $i(d_j)\cup : H^1(Y, \mathbb{Z}) \to H^3(Y, \mathbb{Z})$. One then uses these four Hodge substructures to conclude that the Hodge structure on $H^1(Y, \mathbb{Z})$ is, as a Hodge structure, a direct sum $H \oplus H$, where H carries an endomorphism of Hodge structures conjugate to ϕ^*. But we already know that the presence of an endomorphism in this conjugacy class prevents the weight 1 Hodge structure on H to be polarized and Y cannot be projective. This concludes the proof in the case of the cohomology ring. **qed**

In the case of the rational cohomology algebra, there is an alternative argument, due to Deligne, which is purely algebraic and replaces the geometric part above involving the Albanese map. It relies on the following.

Lemma 2.42. *Let A^* be a cohomology algebra (say with $\mathbb{Q}$-coefficients) equipped with a Hodge structure in the sense of Definition 2.39. Let $W \subset A^k$ be a closed algebraic subset which is defined by homogeneous equations involving only the algebra structure on A^*. Let $W' \subset W$ be an irreducible component of W which is defined over $\mathbb{Q}$. Then the $\mathbb{Q}$-vector subspace $\langle W' \rangle \subset A^k$ is a Hodge substructure of A^k.*

This lemma applies to algebraic subsets defined by rank conditions, e.g. fixing l and s,

$$W = \{\alpha \in A^k,\ \alpha : A^l \to A^{k+l} \text{ has rank } \leq s\}, \tag{2.47}$$

or more simply, defined by a simple algebraic equation in A^*, e.g. fixing l,

$$W = \{\alpha \in A^k,\ \alpha^l = 0 \text{ in } A^{kl}\}. \tag{2.48}$$

Example (2.47) applies to the previous proof and directly shows that the classes $i(d_j)$ must be divisor classes (equivalently, degree 2 Hodge classes), even if we know only the cohomology algebra with $\mathbb{Q}$-coefficients. This proves Theorem 2.41 starting from the cohomology algebra instead of the cohomology ring.

Lemma 2.42 combined with Example (2.48) is the main ingredient in the proof of Theorem 2.40. Consider the projective bundle $p : N = \mathbb{P}(E) \to M$. It has the Hopf line bundle H with first Chern class $h = c_1(H) \in H^2(N, \mathbb{Q})$ and we have

$$H^*(N, \mathbb{Q}) = \oplus_{i=0}^{r-1} h^i p^* H^{k-2i}(N, \mathbb{Q}), \tag{2.49}$$

the cohomology algebra structure being determined by the equation

$$h^r = \sum_{i=0}^{r-1} \pm h^i p^* c_{r-i}(E). \tag{2.50}$$

Using the fact that the cohomology algebra of M is generated in degree ≤ 2, it is clear that Theorem 2.40 is implied by the fact that $h \in H^2(N, \mathbb{Q})$ is a Hodge class of degree 2 and $H^2(M, \mathbb{Q}) \subset H^2(N, \mathbb{Q})$ is a Hodge substructure. Indeed, this implies that $H^k(M, \mathbb{Q}) \subset H^k(N, \mathbb{Q})$ is a Hodge substructure for all k, and (2.50) then implies that $c_i(E) \in H^{2i}(M, \mathbb{Q})$ are Hodge classes. In order to prove that $p^* H^2(M, \mathbb{Q}) \subset H^2(N, \mathbb{Q})$ is a Hodge substructure, we can apply Lemma 2.42. Indeed, elements of $p^* H^2(M, \mathbb{Q})$ satisfy $\alpha^{m+1} = 0$ in $H^{2m+2}(N, \mathbb{Q})$ and one can show they form an irreducible component of the closed algebraic subset of $H^2(N, \mathbb{Q})$ defined by this condition. To prove that the class h is a Hodge class, we observe that, since $c_1(E) = 0$, it satisfies the equations

$$h^r p^* H^{2m-2}(M, \mathbb{Q}) = 0,$$

and is an isolated solution of this equation. Hence a variant of Lemma 2.42 implies that h is a Hodge class. **qed**

3 The topology of families

3.1 The Leray spectral sequence and Deligne's decomposition theorem

Let $f : X \to Y$ be a smooth proper C^∞ map and A an abelian group. Then by Ehresmann's fibration theorem, f is a local fibration, so that the sheaves $R^k f_* A$ are locally constant on Y. The Leray spectral sequence of f has E_2-term

$$E_2^{p,q} = H^p(Y, R^k f_* A) \Rightarrow H^{p+q}(X, A).$$

It is usually very complicated with nonzero differentials of high degree, as the example of sphere bundles show: for a $r-1$-sphere bundle $S \to Y$, $S \subset E$, where E is a real oriented vector bundle of rank r, the only nonzero cohomology sheaves are $R^0 f_* \mathbb{Z} = \mathbb{Z}$, $R^{r-1} f_* \mathbb{Z} = \mathbb{Z}$ and thus the nontrivial differentials are

$$d_r : H^p(Y, \mathbb{Z}) \to H^{p+r}(Y, \mathbb{Z}).$$

For $p = 0$, $d_r(1) \in H^r(Y, \mathbb{Z})$ is well-known to give the Euler class of E, which is nonzero in general.

At least with $\mathbb{Q}$-coefficients, this situation cannot happen for projective morphisms, or more generally proper morphisms $X \to Y$ with Kähler fibers, carrying a degree 2 cohomology class α whose restriction α_t to the fiber X_t of f is a Kähler class, for any $t \in Y$.

Theorem 3.1. *([9], [19]) Let $f : X \to Y$, $\alpha \in H^2(X, \mathbb{R})$ be as above. Then the Leray spectral sequence of f with $\mathbb{Q}$-coefficients degenerates at E_2.*

Proof. It suffices to prove the result with $\mathbb{R}$-coefficients. Choosing a closed representative of α, we see that the maps of cup-product with the powers α^r act on the Leray spectral sequence. The only fact we will need is the hard Lefschetz property satisfied by the class α_t. In particular the result has nothing to do with the complex structure of the fibers. One proves by induction on r that $d_r = 0$ for $r \geq 2$. Let us just prove that

$$d_2 : H^p(Y, R^q f_*\mathbb{R}) \to H^{p+2}(Y, R^{q-1} f_*\mathbb{R})$$

is 0. First of all, we reduce the problem to the case where $q \leq n$ because for $q > n$, we can apply the Lefschetz isomorphism

$$\alpha^{q-n} \cup : R^{2n-q} f_*\mathbb{R} \cong R^q f_*\mathbb{R}$$

with $2n - q < n$, which commutes with the Leray differentials. For $q \leq n$, we have the Lefschetz decomposition of the local system $R^q f_*\mathbb{R}$

$$R^q f_*\mathbb{R} = \oplus_{q-2r \geq 0} \alpha^r \cup (R^{q-2r} f_*\mathbb{R})_{\mathrm{prim}},$$

and using the induction hypothesis, we just have to prove the vanishing of d_2 on the primitive part

$$(R^q f_*\mathbb{R})_{\mathrm{prim}} := \mathrm{Ker}\,(\alpha^{n-q+1} \cup : R^q f_*\mathbb{R} \to R^{2n-q+2} f_*\mathbb{R}).$$

The vanishing of d_2 on $R^q f_*\mathbb{R})_{\mathrm{prim}}$ now follows from the following commutative diagram

$$\begin{array}{ccc} H^p(Y, (R^q f_*\mathbb{R})_{\mathrm{prim}}) & \stackrel{d_2}{\to} & H^{p+2}(Y, R^{q-1} f_*\mathbb{R}) \\ \alpha^{n-q+1} \downarrow & & \alpha^{n-q+1} \downarrow \\ H^p(Y, (R^{2n-q+2} f_*\mathbb{R})_{\mathrm{prim}}) & \stackrel{d_2}{\to} & H^{p+2}(Y, R^{2n-q+1} f_*\mathbb{R}) \end{array},$$

where the first vertical arrow is by definition 0, while the second vertical arrow is the Lefschetz isomorphism in degree $q - 1$. $\square$

With integral coefficients, the degeneracy at E_2 of the Leray spectral sequence for smooth projective morphisms is completely wrong. For example, the Brauer-Severi varieties $p : P \to Y$ are projective bundles which are locally trivial in the analytic or étale topology, but not in the Zariski topology. Assume the fibers P_t are isomorphic to $\mathbb{P}^r$. Then P is a projectivized vector bundle

$$P \cong \mathbb{P}(E)$$

if and only if there exists a holomorphic line bundle H on P whose restriction to the fibers is the line bundle $\mathcal{O}_{P_t}(1)$. Indeed, if H exists, P is canonically isomorphic to $\mathbb{P}((R^0 p_* H)^*)$. We have $R^2 p_*\mathbb{Z} = \mathbb{Z}$ and, assuming $H^2(Y, \mathcal{O}_Y) = 0$, the existence of H is equivalent to the surjectivity of the restriction map

$$H^2(P, \mathbb{Z}) \to H^0(Y, R^2 p_*\mathbb{Z}),$$

appearing in the Leray spectral sequence. The existence of nontrivial Brauer-Severi varieties over bases Y with $H^2(Y, \mathcal{O}_Y) = 0$ thus entirely depends on the nondegeneracy at E_2 of the Leray spectral sequence with integral coefficients.

The Deligne decomposition theorem for projective smooth morphisms is the following stronger version of Theorem 3.1.

Theorem 3.2. *[17] Let $f : X \to Y$ be a C^∞ map satisfying the same assumptions as in Theorem 3.1. Then, in the bounded derived category $\mathcal{D}_Y$ of (locally constant) sheaves of $\mathbb{Q}$-vector spaces on Y, there is a decomposition*

$$Rf_*\mathbb{Q} = \oplus R^q f_*\mathbb{Q}[-q].$$

Proof. We observe that the argument used for the proof of Theorem 3.1 would work as well with the constant sheaf on X replaced by any local system V of $\mathbb{Q}$-vector spaces pulled-back from Y. In particular we can take $V = f^{-1}((R^q f_*\mathbb{Q})^*)$, and we know that the Leray spectral sequence of (V, f) degenerates at E_2. There is a canonical section

$$s_q \in H^0(Y, R^q f_* V),$$

since $R^q f_* V \cong R^q f_*\mathbb{Q} \otimes (R^q f_*\mathbb{Q})^*$. The degeneracy at E_2 of the Leray spectral sequence now gives a class $\alpha_q \in H^q(X, V)$, which maps to $s_q \in H^0(Y, R^q f_* V)$. Next,

$$H^q(X, V) = \mathbb{H}^q(Y, Rf_* V) = H^q(Y, (R^q f_*\mathbb{Q})^* \otimes Rf_*\mathbb{Q}) = \mathrm{Hom}_{\mathcal{D}_Y}(R^q f_*\mathbb{Q}[-q], Rf_*\mathbb{Q}),$$

so the α_q give morphisms $R^q f_*\mathbb{Q}[-q] \to Rf_*\mathbb{Q}$ inducing an isomorphism on degree q cohomology. The α_q together provide the desired decomposition. □

3.2 The global invariant cycle theorem

Let X, Y be smooth projective complex varieties with Y connected, and let $f : X \to Y$ be a surjective morphism. There is a dense Zariski open set Y^0 of Y (the open set of regular values) such that, denoting $X^0 := f^{-1}(Y^0)$, the restricted map

$$f : X^0 \to Y^0$$

is a smooth morphism, which is proper, hence a local fibration. Note that, since Y is connected and $R^k f_*\mathbb{Q}$ is a local system with stalk $H^k(X_{0_Y}, \mathbb{Q})$ at 0_Y, if $0_Y \in Y$ is a base-point, the space $H^0(Y^0, R^k f_*\mathbb{Q})$ identifies via the restriction map

$$H^0(Y^0, R^k f_*\mathbb{Q}) \to H^k(X_{0_Y}, \mathbb{Q})$$

with the invariant subspace

$$H^k(X_{0_Y}, \mathbb{Q})_{\mathrm{inv}} \subset H^k(X_{0_Y}, \mathbb{Q})$$

under the monodromy representation

$$\rho : \pi_1(Y^0, 0_Y) \to \mathrm{End}\, H^k(X_{0_Y}, \mathbb{Q}). \tag{3.51}$$

Theorem 3.1 tells us that the natural map

$$H^k(X^0, \mathbb{Q}) \to H^0(Y^0, R^k f_*\mathbb{Q})$$

is surjective, or equivalently that

$$H^k(X_{0_Y}, \mathbb{Q})_{\mathrm{inv}} = \mathrm{Im}\,(r^0_{0_Y} : H^k(X^0, \mathbb{Q}) \to H^k(X_{0_Y}, \mathbb{Q})), \tag{3.52}$$

where $r^0_{0_Y}$ is the restriction map from X^0 to X_{0_Y}. The following crucial result is due to Deligne [19].

Theorem 3.3. *One has*

$$H^k(X_{0_Y},\mathbb{Q})_{\mathrm{inv}} = \mathrm{Im}\,(r_{0_Y} : H^k(X,\mathbb{Q}) \to H^k(X_{0_Y},\mathbb{Q})),$$

where r_{0_Y} is the restriction map from X to X_{0_Y}.

Corollary 3.4. *$H^k(X_{0_Y},\mathbb{Q})_{\mathrm{inv}} \subset H^k(X_{0_Y},\mathbb{Q})$ is a Hodge substructure of $H^k(X_{0_Y},\mathbb{Q})$ whose isomorphism class does not depend on the reference point.*

The corollary follows from the theorem because the restriction map $r_{0_Y} : H^k(X,\mathbb{Q}) \to H^k(X_{0_Y},\mathbb{Q})$ is a morphism of Hodge structures. Hence its image is a Hodge substructure. It is a constant Hodge substructure because the morphism is locally constant. If $\alpha = r_{0_Y}(\beta^{p,q})$ is of type (p,q), then $r_t(\beta^{p,q}) = \alpha_t$ is also of type (p,q), hence the Hodge decomposition on the image is constant. Note that Corollary 3.4 can also formulated as follows.

Corollary 3.5. *Let $\alpha \in H^k(X_{0_Y},\mathbb{Q})_{\mathrm{inv}}$. Write $\alpha = \sum_{p,q}\alpha^{p,q}$ in the Hodge decomposition of $H^k(X_{0_Y},\mathbb{C})$. Then each $\alpha^{p,q}$ belongs to $H^k(X_{0_Y},\mathbb{C})_{\mathrm{inv}}$.*

Theorem 3.3 follows from (3.52) and Theorem 2.7, using the fact that the restriction map $r^0_{0_Y} : H^k(X^0,\mathbb{Q}) \to H^k(X_{0_Y},\mathbb{Q})$ is a morphism of mixed Hodge structures, where on the right the Hodge structure is pure of weight k. Theorem 2.7 thus says that $\mathrm{Im}\, r^0_{0_Y} = r^0_{0_Y}(W_kH^k(X^0,\mathbb{Q}))$. On the left, the minimal weight is k and

$$W_kH^k(X^0,\mathbb{Q}) = \mathrm{Im}\,(H^k(X,\mathbb{Q}) \to H^k(X^0,\mathbb{Q})).$$

It follows that

$$\mathrm{Im}\, r^0_{0_Y} = \mathrm{Im}\, r_{0_Y},$$

which implies the result since we already know by (3.52) that $\mathrm{Im}\, r^0_{0_Y} = H^k(X_{0_Y},\mathbb{Q})_{\mathrm{inv}}$. **qed**

A first important consequence is the rigidity theorem which says that a variation of Hodge structures is determined by the underlying local system and its value at one point.

Corollary 3.6. *Let $f : X \to Y$, $g : X' \to Y$ be two smooth projective morphisms, where Y is smooth quasi-projective and connected. Assume the local systems $R^kf_*\mathbb{Q}$ and $R^kf'_*\mathbb{Q}$ are isomorphic and that the isomorphism*

$$i : R^kf_*\mathbb{Q} \cong R^kf'_*\mathbb{Q}$$

is at some point $y \in Y$ an isomorphism of Hodge structures $H^k(X_y,\mathbb{Q}) \cong H^k(X'_y,\mathbb{Q})$. Then i induces an isomorphism of Hodge structures at any point of Y.

Indeed, we already explained that the isomorphism i_y can be seen via Künneth decomposition as a cohomology class in the product $X_y \times X'_y$. By assumption this class is monodromy invariant, and it is a Hodge class at the point y. Hence it is a Hodge class everywhere on Y. **qed**

Another corollary is the following.

Corollary 3.7. *The situation and notation being as in Theorem 3.3, the set*

$$H^k(X_{0_Y},\mathbb{Q})_{\mathrm{fin}} \subset H^k(X_{0_Y},\mathbb{Q})$$

of elements $\alpha \in H^k(X_{0_Y},\mathbb{Q})$ such that the orbit of α under the monodromy action (3.51) is finite, is a Hodge substructure of $H^k(X_{0_Y},\mathbb{Q})$.

Indeed, it suffices to prove the statement for the set of elements

$$H^k(X_{0_Y}, \mathbb{Q})^{G'} \subset H^k(X_{0_Y}, \mathbb{Q})$$

invariant under any fixed finite index subgroup $G' \subset \pi_1(Y^0, 0_Y)$. Such a finite index subgroup produces a finite étale cover Y' of Y^0 which is algebraic and a base-changed family $X^0_{Y'} \to Y'$, for which $H^k(X_{0_Y}, \mathbb{Q})^{G'}$ is now monodromy invariant, and to which Corollary 3.4 applies.

In the opposite direction, the following result, which is again a consequence of the Hodge-Riemann relations, exhibits elements of $H^k(X_{0_Y}, \mathbb{Q})_{\mathrm{fin}}$. We assume now $k = 2l$ is even.

Theorem 3.8. *Let $X \to Y$ be as above, and let $t \in Y$ be a very general point. Then the $\mathbb{Q}$-vector space $\mathrm{Hdg}^{2l}(X_t, \mathbb{Q})$ of Hodge classes on X_t is stable under the monodromy action*

$$\pi_1(Y^0, t) \to \mathrm{Aut}\, H^{2l}(X_t, \mathbb{Q})$$

and we have

$$\mathrm{Hdg}^{2l}(X_t, \mathbb{Q}) \subset H^{2l}(X_t, \mathbb{Q})_{\mathrm{fin}}. \tag{3.53}$$

Here by "very general", we mean that t can be chosen outside a countable union of Zariski closed subsets of Y.

Proof of Theorem 3.8. As it is stated, the theorem needs hard results about Hodge loci (see Section 3.6), from which we extract the following statement: There exists a countable union $\mathcal{H}$ of closed proper algebraic subsets $Y_i \subset Y^0$, such that for any $t \in Y^0 \setminus \mathcal{H}$, and any Hodge class

$$\alpha \in \mathrm{Hdg}^{2l}(X_t, \mathbb{Q}),$$

the class α remains Hodge in a neighborhood of t. Here we take a contractible neighborhood $U \subset Y^0$ of t, so that the family $X^0 \to Y^0$ is topologically trivial and the cohomology groups $H^{2l}(X_t, \mathbb{Q})$, $H^{2l}(X_{t'}, \mathbb{Q})$ are canonically isomorphic for any $t' \in U$; the class α can thus be transported to a class $\alpha_{t'} \in H^{2l}(X_{t'}, \mathbb{Q})$, and our statement is that $\alpha_{t'}$ is still a Hodge class on $X_{t'}$. To conclude that for any path $\gamma : [0,1] \to Y^0$ from t to another point $t' \in Y$, the class α remains Hodge on $X_{\gamma(s)}$ for any $s \in [0,1]$, we use then the analytic continuation principle. This proves that α remains Hodge under parallel transport along paths, hence a fortiori under parallel transport along loops. This implies that the classes $\gamma(\alpha)$ are Hodge on X_t for any $\gamma \in \pi_1(Y^0, t)$. Thus $\mathrm{Hdg}^{2l}(X_t, \mathbb{Q})$ is stable under the monodromy action.

In order to prove that the monodromy acting on $\mathrm{Hdg}^{2l}(X_t, \mathbb{Q})$ is finite, we use the fact that the morphism $X \to Y$ is projective, so that there is an integral degree 2 cohomology class $l = c_1(L)$ whose restriction to the fibers X_t is a Kähler class l_t which induces the Lefschetz isomorphism (see (2.33))

$$l^{d-2l} : H^{2l}(X_t, \mathbb{Q}) \cong H^{2d-2l}(X_t, \mathbb{Q})$$

where $d = \dim(X/Y)$ and assuming $2l \leq d$. The Lefschetz isomorphisms preserve Hodge classes since they are isomorphisms of Hodge structures, hence they provide restricted isomorphisms

$$l^{d-2l} : \mathrm{Hdg}^{2l}(X_t, \mathbb{Q}) \cong \mathrm{Hdg}^{2d-2l}(X_t, \mathbb{Q}). \tag{3.54}$$

When $2l \geq d$, we can use the Lefschetz isomorphism (3.54) to deduce finite monodromy on $\mathrm{Hdg}^{2l}(X_t, \mathbb{Q})$ from finite monodromy on $\mathrm{Hdg}^{2d-2l}(X_t, \mathbb{Q})$. Assuming $2l \leq d$, the isomorphisms (3.54) imply that $\mathrm{Hdg}^{2l}(X_t, \mathbb{Q})$ is stable under the Lefschetz decomposition which provides

$$\mathrm{Hdg}^{2l}(X_t, \mathbb{Q}) \cong \oplus_{2l-2r \geq 0} l^r \mathrm{Hdg}^{2l-2r}(X_t, \mathbb{Q})_{\mathrm{prim}}. \tag{3.55}$$

The end of the proof is the following. Recall the Lefschetz intersection pairing $\langle\,,\,\rangle_{\mathrm{Lef}}$ on $\mathrm{Hdg}^{2l}(X_t, \mathbb{Q})$ defined on $H^{2l}(X_t, \mathbb{Q})$ by

$$\langle \alpha, \beta \rangle_{\mathrm{Lef}} = \langle l^{n-2l}\alpha, \beta \rangle_{X_t}.$$

The Lefschetz decomposition is orthogonal for $\langle\,,\,\rangle_{\mathrm{Lef}}$, and furthermore the second Hodge-Riemann relations (2.15) say that the pairing $\langle\,,\,\rangle'_{\mathrm{Lef}}$ which is equal to $(-1)^r \langle\,,\,\rangle_{\mathrm{Lef}}$ on $l^r \mathrm{Hdg}^{2l-2r}(X_t, \mathbb{Q})_{\mathrm{prim}}$, and for which the Lefschetz decomposition is also orthogonal, is definite.

The proof is now finished because the monodromy action preserves l, hence $\langle\,,\,\rangle_{\mathrm{Lef}}$ and the Lefschetz decomposition on $\mathrm{Hdg}^{2l}(X_t, \mathbb{Q})$, so it also preserves $\langle\,,\,\rangle'_{\mathrm{Lef}}$. Moreover, there is an integral structure on the $\mathbb{Q}$-vector spaces considered, and it is preserved by the monodromy action. So we conclude that the monodromy action on $\mathrm{Hdg}^{2l}(X_t, \mathbb{Q})$ factors through the orthogonal group $O(L, \langle\,,\,\rangle)$ of a lattice equipped with a definite intersection form, and such a group is finite. □

A consequence of these results is the following statement concerning the *variational Hodge conjecture.*

Proposition 3.9. *Let $\mathcal{X} \to B$ be a smooth projective morphism, where B is connected smooth quasi-projective. Let $0 \in B$, and $Z_0 \in \mathrm{CH}^k(\mathcal{X}_t)$ be a cycle with class $\alpha \in \mathrm{Hdg}^{2k}(\mathcal{X}_0, \mathbb{Q})$. Assume the class α remains Hodge on $\mathcal{X}_t$ for t in a neighborhood of 0. Then assuming the Lefschetz standard conjecture, the class α_t remains algebraic on $\mathcal{X}_t$ for any $t \in B$.*

In fact, as the proof will show, we only need the Lefschetz standard conjecture for both $\overline{\widetilde{\mathcal{X}}}$ and the fiber $\mathcal{X}_0$, where $\overline{\widetilde{\mathcal{X}}}$ is a variety deduced from $\mathcal{X}$ by an étale base change $\widetilde{B} \to B$ and by taking a projective compactification.

Proof. By analytic continuation, the class α_t remains Hodge on $\mathcal{X}_t$ for any $t \in B$. By the finiteness theorem 3.8, the orbit of the class α under the monodromy action is finite, so after passing to an étale cover $\widetilde{B}$ of B, we can assume that the class α is invariant under monodromy. By Theorem 3.3, there exists a class $\beta \in H^{2k}(\overline{\widetilde{\mathcal{X}}}, \mathbb{Q})$ such that (denoting also 0 any point of $\widetilde{B}$ over $0 \in B$) $\beta_{|\widetilde{\mathcal{X}}_0} = \alpha$. As we assumed the Lefschetz standard conjecture for both $\overline{\widetilde{\mathcal{X}}}$ and $\mathcal{X}_0 = \widetilde{\mathcal{X}}_0$, Corollary 2.30 shows that there exists a cycle $\mathcal{Z}$ with $\mathbb{Q}$-coefficients on $\overline{\widetilde{\mathcal{X}}}$ such that, denoting $\mathcal{Z}_t = \mathcal{Z}_{|\mathcal{X}_t}$, $[\mathcal{Z}_0] = \alpha$, hence $[\mathcal{Z}_t] = \alpha_t$ for any $t \in \widetilde{B}$, which proves the result. □

3.3 Variational aspects; Griffiths' transversality

So far, we have been discussing the formal properties of Hodge structures and their polarizations. In the next sections, we are going to concentrate on how they vary, and how the study of the so-called "period map" leads to further results on the topology of families. Most of the results here are due to Griffiths, except in the weight 1 case, where the polarized Hodge structures correspond to abelian varieties, and a lot of work had been done previously using the theory of automorphic forms (see [6]).

Let X be a compact Kähler or complex projective manifold and k be an integer. The Hodge filtration $F^iH^k(X,\mathbb{C}) \subset H^k(X,\mathbb{C})$ (see (1.25)) determines a point in the flag manifold $\mathrm{Fl}_{b_{i,k}}(H^k(X,\mathbb{C}))$ parameterizing the filtrations on $H^k(X,\mathbb{C})$ with successive dimensions $b_{i,k} := \dim F^iH^k(X,\mathbb{C})$, $i = k, \ldots, 0$. Recall from Section 1.1 (combined with Corollary 1.15) that, under a small deformation $(X_t)_{t\in B}$ of the complex structure of X, the dimensions $b_{i,k}(X_t)$ remain constant, and it follows by the Hodge theory of harmonic representative that the subspace

$$F^iH^k(X_t,\mathbb{C}) \subset H^k(X_t,\mathbb{C}) \cong H^k(X,\mathbb{C})$$

vary in a C^∞ way with t.

Griffiths [25] proved much more. We concentrate on a fixed i, since the flag manifold $\mathrm{Fl}_{b_{i,k}}(H^k(X,\mathbb{C}))$ is contained in the product $\prod_{i=0}^k \mathrm{Grass}(b_{i,k}, H^k(X,\mathbb{C}))$. It is indeed the closed algebraic subvariety

$$Z = \{(W_1,\ldots,W_k) \in \prod_{i=0}^k \mathrm{Grass}(b_{i,k}, H^k(X,\mathbb{C})),\, W_i \subset W_{i-1},\, \forall i \geq 1\}.$$

The period map

$$\mathcal{P}_{i,k} : B \to \mathrm{Grass}(b_{i,k}, H^k(X,\mathbb{C}))$$

maps t to the point $[F^iH^k(X_t,\mathbb{C}))] \in \mathrm{Grass}(b_{i,k}, H^k(X,\mathbb{C}))$. It is a priori of class C^∞ in t. Let us now assume that the family of deformations of X is holomorphic in the sense that it comes from a family given by an analytic variety equipped with a smooth proper holomorphic map

$$\pi : \mathcal{X} \to B \tag{3.56}$$

and a topological trivialization

$$\mathcal{X} \cong B \times \mathcal{X}_0$$

over B, with $\mathcal{X}_0 \cong X$ as a complex manifold. Let W be a complex vector space of dimension b_k and $b_{i,k} \leq \dim W$ be an integer. The Grassmannian $\mathrm{Grass}(b_{i,k}, W)$ is a projective complex manifold. It has a cell decomposition with affine cells isomorphic to $\mathbb{A}^{b_{i,k}(b_k - b_{i,k})}$ constructed as follows. Let $V \subset W$ be a complex vector space of dimension $b_{i,k}$, defining a point $[V] \in \mathrm{Grass}(b_{i,k}, W)$. Choose a decomposition

$$W = V \oplus V',$$

with $\dim V' = b_k - b_{i,k}$. Then, an open set of deformations of the vector subspace $V \subset W$ parameterizes vector subspaces V_t of W of dimension $b_{i,k}$ which are transverse to V', and such V_t is exactly parameterized by an element h_t of $\mathrm{Hom}\,(V,V')$. (To h_t one associates the graph of h_t in $V \oplus V'$, and to $V_t \subset V \oplus V'$ transverse to V', one associates $h_t = \mathrm{pr}_2 \circ (\mathrm{pr}_{1|V_t})^{-1}$.) This construction gives local holomorphic coordinates for $\mathrm{Grass}(b_{i,k}, W)$. This is used to show the following

Lemma 3.10. *The tangent space to* $\mathrm{Grass}(b_{i,k}, W)$ *at* $[V]$ *is canonically isomorphic to* $\mathrm{Hom}\,(V, W/V)$.

The canonical isomorphism given above is constructed as follows. Let $[V] \in \mathrm{Grass}(b_{i,k}, W)$. Choose a basis α_j of V. In a neighborhood of $[V]$ in $\mathrm{Grass}(b_{i,k}, W)$, one can choose holomorphic sections $\tilde{\alpha}_j$ of the universal subbundle

$$\mathcal{V} \subset \mathrm{Grass}(b_{i,k}, W) \times W$$

such that $\tilde{\alpha}_i([V]) = \alpha_i$. For any $u \in T_{\mathrm{Grass}(b_{i,k},W),[V]}$, we get a linear map

$$h_u : V \to W/V,$$

defined by

$$h_u(\alpha_i) = d_u\tilde{\alpha}_j \bmod V.$$

One checks using the Leibniz formula that h_u does not depend on the choice of the sections $\tilde{\alpha}_j$ and one checks using the coordinates described above that this construction gives a linear isomorphism $u \mapsto h_u$, proving Lemma 3.10. Coming back to the case $W = H^k(X, \mathbb{C})$, $b_{i,k} = \dim F^iH^k(X, \mathbb{C})$, one has

Theorem 3.11. *(Griffiths) (i) The period map is holomorphic.*

(ii) The period map satisfies the transversality condition

$$\begin{aligned} d\mathcal{P}(T_{B,0}) &\subset \mathrm{Hom}\,(F^iH^k(X, \mathbb{C}), F^{i-1}H^k(X, \mathbb{C})/F^iH^k(X, \mathbb{C})) \\ &\subset \mathrm{Hom}\,(F^iH^k(X, \mathbb{C}), H^k(X, \mathbb{C})/F^iH^k(X, \mathbb{C})). \end{aligned} \tag{3.57}$$

It is interesting to note that the transversality property is automatic in the case of the variation of Hodge structure on degree 1 cohomology, because the varying subspace is F^1H^1 and F^0H^1 is everything. Griffiths' proof of Theorem 3.11 is by explicit computation relying on the Cartan-Lie formula, using the recipe described above and differentiating families of cohomology classes to compute the differential of the period map. The operation of differentiating families of cohomology classes in the fibers of a fibration is called the Gauss-Manin connection. More generally, for any local system $\mathbb{H}$ of $\mathbb{R}$-vector spaces on B, there is a connection ∇ on the associated holomorphic vector bundle $\mathcal{H} := \mathbb{H} \otimes \mathcal{O}_B$, which is characterized by the fact that sections of $\mathbb{H}$ are the ∇-flat sections of $\mathcal{H}$. The connection ∇ is flat, that is has 0-curvature, and conversely by Frobenius theorem, a connection with 0-curvature gives rise to a local system of flat sections (hence to a representation of the fundamental group of the base). This correspondence is called the Riemann-Hilbert correspondence. In our context, the Gauss-Manin connection on the vector bundle $\mathcal{H} = R^k\pi_*\mathbb{C} \otimes \mathcal{O}_B$ is extremely interesting because, as we will see below, it is an algebraic data, and even more, it is algebraic defined over K if π is a morphism of quasiprojective varieties defined over K. By contrast, the associated local system is a highly transcendental data. The reason is that solutions of an algebraic linear differential equation are usually transcendental, like the function $\log z$ which is the solution of the linear equation $f' = \frac{1}{z}$.

The proof given by Katz-Oda of Griffiths transversality (Theorem 3.11(ii)) relies on the following description of ∇ (see [43]). Consider the smooth holomorphic map $\pi : \mathcal{X} \to B$. There is the relative holomorphic de Rham complex

$$\Omega^\bullet_{\mathcal{X}/B}$$

(see Section 1.1) which restricts on each fiber $\mathcal{X}_t$ to the holomorphic de Rham complex $\Omega^\bullet_{\mathcal{X}_t}$, and which provides, using the local analytic triviality of the map π (even if B is singular), a resolution of the sheaf $\pi^{-1}\mathcal{O}_B$. It follows that

$$R^k\pi_*\mathbb{C}\otimes\mathcal{O}_B \cong \mathbb{R}^k\pi_*\Omega^\bullet_{\mathcal{X}/B}. \tag{3.58}$$

The right hand side has the Hodge filtration by analytic coherent sheaves

$$F^p\mathbb{R}^k\pi_*\Omega^\bullet_{\mathcal{X}/B} = \mathbb{R}^k\pi_*\Omega^{\bullet\geq p}_{\mathcal{X}/B}$$

and it follows from the base change property explained in Section 1.1 that this filtration is a filtration by locally free subsheaves which induces the Hodge filtration on each fiber. This proves part (i) of Theorem 3.11.

Let us assume for simplicity that $\mathcal{X}$ is smooth. The absolute de Rham complex of $\mathcal{X}$ carries the following filtration

$$L^i\Omega^\bullet_{\mathcal{X}} := \pi^*\Omega^i_B \wedge \Omega^{\bullet-i}_{\mathcal{X}}.$$

We have by construction an isomorphism of complexes

$$\Omega^\bullet_{\mathcal{X}}/L^1\Omega^\bullet_{\mathcal{X}} \cong \Omega^\bullet_{\mathcal{X}/B}$$

and an exact sequence

$$0 \to \pi^*\Omega_B \otimes \Omega^{\bullet-1}_{\mathcal{X}/B} \to \Omega^\bullet_{\mathcal{X}}/L^2\Omega^\bullet_{\mathcal{X}} \to \Omega^\bullet_{\mathcal{X}/B} \to 0. \tag{3.59}$$

Applying $\mathbb{R}\pi_*$, we get a connecting map (which is not $\mathcal{O}_B$-linear because the differential in the complex $\Omega^\bullet_{\mathcal{X}}/L^2\Omega^\bullet_{\mathcal{X}}$ is not $\mathcal{O}_B$-linear)

$$\delta : R^k\pi_*\Omega^\bullet_{\mathcal{X}/B} \to R^k\pi_*\Omega^\bullet_{\mathcal{X}/B}\otimes\Omega_B.$$

Theorem 3.12. *Via the isomorphisms (3.58), ∇ identifies to the Gauss-Manin connection.*

This theorem immediately implies the transversality property. Indeed, the exact sequence (3.60) is compatible with the Hodge (or naïve) filtrations

$$F^p\Omega^\bullet_{\mathcal{X}} = \Omega^{\bullet\geq p}_{\mathcal{X}},\ F^p\Omega^\bullet_{\mathcal{X}/B} = \Omega^{\bullet\geq p}_{\mathcal{X}/B},$$

inducing an exact sequence

$$0 \to \pi^*\Omega_B \otimes \Omega^{\bullet-1\geq p-1}_{\mathcal{X}/B} \to \Omega^{\bullet\geq p}_{\mathcal{X}}/L^2\Omega^{\bullet\geq p}_{\mathcal{X}} \to \Omega^{\bullet\geq p}_{\mathcal{X}/B} \to 0, \tag{3.60}$$

so that the map δ induces as well for each p

$$\nabla : F^p\mathcal{H}^k \to F^{p-1}\mathcal{H}^k\otimes\Omega_B,$$

proving transversality. The great advantage of this construction is that it works as well, using the relative version of Theorem 1.11, on the level of algebraic (relative) de Rham complexes and over the same definition field as the family $\mathcal{X}\to B$, providing a natural algebraic structure for the Hodge bundles $F^p\mathcal{H}^k$ and the Gauss-Manin connection.

3.4 Polarized period domains

If we work with a smooth projective morphism

$$\pi : \mathcal{X} \to B,$$

choosing a relatively ample holomorphic line bundle $\mathcal{L}$ on $\mathcal{X}$, and denoting $l := c_1(\mathcal{L}) \in H^2(\mathcal{X}, \mathbb{Z})$, the class l provides a Lefschetz decomposition on the local systems $R^k\pi_*\mathbb{Q}$, which for each $t \in B$ gives a decomposition of $H^k(\mathcal{X}_t, \mathbb{Q})$ into a direct sum of Hodge structures. Note also that there is an integral structure on each term (namely primitive cohomology groups), although the Lefschetz decomposition itself works only with $\mathbb{Q}$-coefficients. We can thus restrict to study the variation of Hodge structure on each primitive cohomology group $H^k(\mathcal{X}_t, \mathbb{Z})_{\text{prim}}$ and each of them gives a polarized variation of Hodge structures, where the polarization is given (up to sign) by the integral locally constant nondegenerate intersection pairing $\langle\, ,\rangle_{\text{Lef}}$ of (2.35).

Choose a reference point $0 \in B$. Fix k, the numbers $b_{p,k,\text{prim}} := \dim F^pH^k(\mathcal{X}_0, \mathbb{C})_{\text{prim}}$ for $p+q = k$, and a lattice $(L, (\, ,\,))$ isomorphic to $(H^k(\mathcal{X}_0, \mathbb{Z})_{\text{prim}}, \langle\, ,\rangle_{\text{Lef}})$. The polarized period domain $\mathcal{D}$ is the locally closed subspace of the flag manifold $\text{Fl}(b_{p,k}, L_{\mathbb{C}})$ parameterizing decreasing filtrations

$$\ldots \subset F^pL_{\mathbb{C}} \subset F^{p-1}L_{\mathbb{C}} \subset \ldots \subset L_{\mathbb{C}}$$

on $L_{\mathbb{C}}$, with $\dim F^pL_{\mathbb{C}} = b_{p,k}$ and satisfying the Hodge-Riemann relations 2.9, (i) and (ii).

Note that there is no need to ask for the opposite filtration property, that is

$$F^pL_{\mathbb{C}} \oplus \overline{F^{k-p+1}L_{\mathbb{C}}} = L_{\mathbb{C}}$$

because it is automatic in this case. Indeed, a class in $F^pL_{\mathbb{C}} \cap \overline{F^{k-p+1}L_{\mathbb{C}}}$ is in the kernel of $(\, ,\,)$ by 2.9, (i), hence is 0 since the pairing is nondegenerate.

Coming back to our family $\pi : \mathcal{X} \to B$, if we restrict to simply connected open sets $U \subset B$ containing the reference point $0 \in U$ (e.g. a neighborhood of a path γ from t to 0 in B), we have by topological trivialization a canonical isomorphism

$$(H^k(\mathcal{X}_t, \mathbb{Z})_{\text{prim}}, \langle\, ,\rangle_{\text{Lef}}) \cong (H^k(\mathcal{X}_0, \mathbb{Z})_{\text{prim}}, \langle\, ,\rangle_{\text{Lef}}) \cong (L, (\, ,\,)), \tag{3.61}$$

and the Hodge filtration on $H^k(\mathcal{X}_t, \mathbb{Z})_{\text{prim}}$ provides via this isomorphism a polarized Hodge structure (or filtration) on L, that is, a point of $\mathcal{D}$. This defines the polarized local period map

$$\mathcal{P} : U \to \mathcal{D}.$$

This map is only locally defined because it depends on the choice of isomorphism (3.61) which itself depends on the choice of U, or of γ. A different choice γ' produces a loop $\gamma \circ \gamma'^{-1}$ which acts on $(L, (\, ,\,))$ via the monodromy representation

$$\rho : \pi_1(B, 0) \to \text{Aut}\,(H^k(\mathcal{X}_0, \mathbb{Z})_{\text{prim}}, \langle\, ,\rangle_{\text{Lef}}).$$

The group $\Gamma := \text{Aut}\,(L, (\, ,\,))$ acts in an obvious way on $\mathcal{D}$ and the considerations above show that the global period map is well defined modulo this action, that is, with value in $\Gamma \setminus \mathcal{D}$. Here we see immediately a major difference between the unpolarized case and the polarized case. In the unpolarized case, the group $\text{Aut}\, L$

acts on the set of Hodge structures on L with given Hodge numbers, but this action is far from being properly discontinuous; to the contrary, it tends to be ergodic (see [72]). The stabilizer groups are not necessarily finite in the unpolarized case: for example, in weight 1, assuming the rank of the lattice L is $2p$ with $p \geq 2$, we have weight 1 Hodge structures on L wich are the direct sum L_1^p, of p copies of a weight 1 Hodge structure of rank 2. The automorphisms group of the Hodge structure L_1^p contains $\mathrm{Gl}(p, \mathbb{Z})$, which is infinite when $p \geq 2$. To the contrary, in the polarized case we have:

Lemma 3.13. *The group G of automorphisms of an integral weight k polarized Hodge structure $(L, (\ ,\ ), F^pL_{\mathbb{C}})$ is finite.*

Proof. As the group is contained in $\mathrm{Aut}\, L_{\mathbb{Z}}$, it suffices to show that G is contained in a compact subgroup of $\mathrm{Aut}\, L_{\mathbb{C}}$. The Hodge decomposition and the Hermitian form $h_{\mathbb{C}}(\alpha, \beta) = i^k(\alpha, \overline{\beta})$ on $L_{\mathbb{C}}$, for which, by definition of a polarization, the $L^{p,q}$ are mutually orthogonal, provide a modified Hermitian form $h'_{\mathbb{C}}$ on $L_{\mathbb{C}}$ which is defined as $(-1)^p h_{\mathbb{C}}$ on $L^{p,q}$, the spaces $L^{p,q}$ being also mutually orthogonal for $h'_{\mathbb{C}}$. By the second Hodge-Riemann bilinear relations, $h'_{\mathbb{C}}$ is definite. Furthermore the group G is contained in $\mathrm{Aut}\,(L_{\mathbb{C}}, h'_{\mathbb{C}})$ since it preserves both $h_{\mathbb{C}}$ and the Hodge decomposition which is used to construct $h'_{\mathbb{C}}$. As $\mathrm{Aut}\,(L_{\mathbb{C}}, h'_{\mathbb{C}})$ is compact, this concludes the proof. ◻

It is not hard to modify this proof to prove

Proposition 3.14. *The action of $\Gamma = \mathrm{Aut}\,(L, (\ ,\ ))$ is properly discontinuous on the polarized period domain $\mathcal{D}$.*

The polarized period domain is defined, inside the flag manifold $\mathrm{Fl}\,(L_{\mathbb{C}}, b_{p,k})$ by algebraic equations and by open conditions. The algebraic equations correspond to the first Hodge-Riemann bilinear relations (Definition 2.9 (i)), while the open conditions are the sign conditions given by the second Hodge-Riemann bilinear relations (Definition 2.9 (ii)). Although the polarized period domain is smaller than the set of unpolarized Hodge structures, the transversality property of the period map imposes in general (but not always) further restrictions. To see this, consider the case of effective weight 2 Hodge structures. A polarized effective weight 2 Hodge structure $(L, (\ ,\ ), F^pL_{\mathbb{C}})$ is determined by the subpace $L^{2,0} \subset L_{\mathbb{C}}$, which has to be totally isotropic for $(\ ,\ )$ by 2.9 (i) (and such that $\mathrm{Re}\, L^{2,0} \subset L_{\mathbb{R}}$ is positive definite for $(\ ,\ )$ by 2.9 (ii)). Indeed, one then defines $F^1L_{\mathbb{C}}$ as $(L^{2,0})^{\perp}$ and then (as in (1.26) one must have

$$L^{0,2} = \overline{L^{2,0}},\ L^{1,1} = F^1L_{\mathbb{C}} \cap \overline{F^1L_{\mathbb{C}}}.$$

With these definitions, the Hodge structure one gets is indeed polarized. Thus in our case, the polarized period domain identifies with the positive open subset of the isotropic Grassmannian $G(h^{2,0}, L_{\mathbb{C}})_{\mathrm{isot}}$. Its tangent space a point $[L^{2,0}]$ is the set of $\phi : L^{2,0} \to L_{\mathbb{C}}/L^{2,0}$ such that

$$(v, \phi(w)) = -(\phi(v), w) \tag{3.62}$$

for any $v,\ w \in L^{2,0}$. The transversality condition restricts the tangent space to be contained in $\mathrm{Hom}\,(L^{2,0}, F^1L_{\mathbb{C}}/L^{2,0})$, and indeed, any $\phi \in \mathrm{Hom}\,(L^{2,0}, F^1L_{\mathbb{C}}/L^{2,0})$ satisfies (3.62) since $(L^{2,0}, F^1L_{\mathbb{C}}) = 0$. However the set of ϕ satisfying (3.62) is larger than $\mathrm{Hom}\,(L^{2,0}, F^1L_{\mathbb{C}}/L^{2,0})$ once $h^{2,0} > 1$. When $h^{2,0} = 1$, the transversality is automatically implied by the polarization condition.

In this example, we find that, due to transversality and the existence of countably many families parameterizing all smooth projective complex varieties, a general polarized Hodge structure of weight 2 with $h^{2,0} > 1$ is not the Hodge structure on the H^2 of a projective complex manifold, and neither a direct summand in it. To the contrary, the theory of hyper-Kähler manifolds [39] gives many examples of families of projective manifolds whose period points exhaust a period domain for weight 2 polarized Hodge structures with $h^{2,0} = 1$.

3.4.1 Curvature properties

Consider first a variation of effective Hodge structures of weight 1, given by a (integral, rational or real) local system H^1 over a complex manifold B and a holomorphic vector subbundle

$$\mathcal{H}^{1,0} \subset \mathcal{H}^1 = H^1 \otimes \mathcal{O}_B.$$

The vector bundle $\mathcal{H}^1$ is flat, equipped with the Gauss-Manin connexion ∇, so the subbundle $\mathcal{H}^{1,0}$ should not have any positivity properties. To the contrary, it is positive when the variation of Hodge structure is polarized. Let $\langle\, ,\, \rangle$ be the skew-symmetric intersection pairing on H^1 giving the polarization. The sesquilinear intersection pairing

$$h_1(\alpha, \beta) = i\langle \alpha, \beta \rangle$$

on H^1 or $\mathcal{H}^1$ is not Hermitian positive definite, as it has signature (n, n), where $2n = \mathrm{rank}\, H^1$. By definition of a polarization, h_1 restricts to a Hermitian metric h on $\mathcal{H}^{1,0}$, so that we can compute the curvature of the Chern connection of h. It is nonnegative and 0 if and only if the variation of Hodge structure is constant. This phenomenon is well explained in [27]. The key point in the computation is the alternance in the signs of the Hermitian intersection pairings induced by h_1 on $\mathcal{H}^{1,0}$ and $\mathcal{H}^{0,1}$ (one is positive, the other is negative, according to (0.1), (0.2)). The statement generalize to any weight as follows

Theorem 3.15. *For a variation of Hodge structures of weight k, the Hodge bundle $\mathcal{H}^{k,0}$ are positive semidefinite.*

This is particularly troubling in the case where the base B is a projective manifold. Although this rarely happens, we have, at least abstractly, plenty of examples obtained by observing that the moduli space $\mathcal{M}_g$ parameterizing smooth (say automorphisms free) curves of genus $g \geq 4$ admits a projective compactification with boundary of codimension ≥ 2. Thus any curve B avoiding the boundary in this compactified moduli space will produce a family of smooth genus g curves parameterized by a projective base B. This observation was used by Kodaira [48] to construct interesting surfaces. There is of course no contradiction between flatness of $\mathcal{H}^1$ and the existence of the positively curve subbundle $\mathcal{H}^{1,0}$. If B is a curve, an indecomposable holomorphic vector bundle $\mathcal{F}$ on B admits a flat connection if and only if it satisfies $c_1(\mathcal{F}) = 0$. This result is due to Weil [81]. The presence of the positive subbundle $\mathcal{H}^{1,0} \subset \mathcal{H}^1$ simply says that the holomorphic vector bundle $\mathcal{H}^1$ is unstable.

3.4.2 The horizontal distribution

Let $\mathcal{D}$ be a period domain, that is, $\mathcal{D}$ is, in the unpolarized case, an open subset of a flag manifold $\mathrm{Fl}(b_{k,p}, H_{\mathbb{C}})$ for given numbers $b_{k,k} \leq b_{k,k-1} \ldots \leq b_{k,0} = \dim H_{\mathbb{C}}$ and in the polarized case, $H_{\mathbb{C}}$ is endowed with a $(-1)^k$-symmetric pairing $\langle\, ,\, \rangle$ and $\mathcal{D}$ is an open subset of the isotropic flag manifold $\mathrm{Fl}^{\mathrm{isot}}(b_{k,p}, H_{\mathbb{C}})$ for given numbers $b_{k,k} \leq b_{k,k-1} \ldots \leq b_{k,0} = \dim H_{\mathbb{C}}$, where the isotropy conditions are $F^p H_{\mathbb{C}} \subset F^{k-p+1} H_{\mathbb{C}}^{\perp}$ for any p.

Forgetting the polarization conditions, the tangent space to the flag manifold at a point $[F^\bullet H_{\mathbb{C}}]$ parameterizing a filtration with given dimensions $b_{k,p}$ is $T_{\mathcal{D},[F^\bullet H_{\mathbb{C}}]} = W \subset \prod_p \mathrm{Hom}\,(F^p H_{\mathbb{C}}, H_{\mathbb{C}}/F^p H_{\mathbb{C}})$, where

$$W = \{(\phi_p),\ \phi_{p+1,p} = \phi_{p|F^{p+1}H_{\mathbb{C}}}\ \forall p\}.$$

Here $\phi_{p+1,p} : F^{p+1}H_{\mathbb{C}} \to H_{\mathbb{C}}/F^p H_{\mathbb{C}}$ is the composition of $\phi_{p+1} : F^{p+1}H_{\mathbb{C}} \to H_{\mathbb{C}}/F^{p+1}H_{\mathbb{C}}$ and the projection $H_{\mathbb{C}}/F^{p+1}H_{\mathbb{C}} \to H_{\mathbb{C}}/F^p H_{\mathbb{C}}$.

The Griffiths transversality condition (Theorem 3.11) says that the local period map $\mathcal{P} : U \to \mathcal{D}$ associated with a family of complex projective or compact Kähler manifolds with trivialized local system has the image of its differential contained in a subspace $W^{\mathrm{hor}} \subset W$ defining the horizontal distribution

$$T_{\mathcal{D}}^{\mathrm{hor}} \subset T_{\mathcal{D}}.$$

The subspace W^{hor} is the set of $(\phi_p) \in W$ such that $\mathrm{Im}\,\phi_p \subset F^{p-1}H_{\mathbb{C}}$ and it naturally identifies with $\oplus_p \mathrm{Hom}\,(H^{p,k-p}, H^{p-1,k-p+1})$, where $H^{p,k-p} = F^p H_{\mathbb{C}}/F^{p+1}H_{\mathbb{C}}$.

A difficulty of the subject is the fact that the horizontal distribution so defined is not integrable. To see this, let us mention the following lemma (see [12]). Let B be a smooth complex manifold and $\mathcal{P} : B \to \mathcal{D}$ be a holomorphic map satisfying the transversality condition, that is, $\mathrm{Im}\, d\mathcal{P} \subset T_{\mathcal{D}}^{\mathrm{hor}}$. For any $b \in B$, and any $u \in T_{B,b}$, denote by $(\phi_p(u)) \in \oplus_p \mathrm{Hom}\,(H^{p,k-p}, H^{p-1,k-p+1})$ the element $d\mathcal{P}(u)$.

Lemma 3.16. *For any u, $v \in T_{B,b}$, one has*

$$\phi_{p-1}(v) \circ \phi_p(u) = \phi_{p-1}(u) \circ \phi_p(v) : H^{p,k-p} \to H^{p-2,k-p+2}.$$

This result, which easily follows from the symmetry of double derivatives, shows that the distribution $T_{\mathcal{D}}^{\mathrm{hor}}$ has a nontrivial curvature. In [12], the authors prove that for most hypersurfaces in projective space, the corresponding period map provides maximal solutions of the transversality equation for the polarized period map.

The lack of integrability makes a little painful to speak of the curvature of the period domain in the horizontal directions. Nevertheless the holomorphic sectional curvature of any holomorphic distribution makes sense, since it is computed by restricting the metric to holomorphic discs tangent to the distribution, and taking the usual Ricci curvature of the restricted metric. It can be used in the same way as the general holomorphic sectional curvature in order to prove hyperbolicity results (see next section). A natural Hermitian metric exists on the polarized period domain $\mathcal{D}$, and is defined up to a scalar. It is induced by the Hodge metric on the $\mathcal{H}^{p,q}$-bundles. Griffiths proves the following result.

Theorem 3.17. *The holomorphic sectional curvature K of the horizontal subbundle $T_{\mathcal{D}}^{\mathrm{hor}}$ is bounded above by a negative constant. There exists $A > 0$ such that for any $0 \neq \xi \in T_{\mathcal{D}}^{\mathrm{hor}}$, $K(\xi) \leq -A$.*

3.4.3 The monodromy theorem

Consider a polarized variation of Hodge structures over the punctured disc Δ^* of radius $1+\epsilon$, so we can take the reference point $1 \in \Delta^*$. We thus have a local system H of $\mathbb{Z}$-modules on Δ^*, which is entirely characterized by its monodromy operator

$$T = \rho(\gamma_1) : H_1 \to H_1,$$

where γ_1 is a counterclockwise loop around the origin based at 1, equipped with a T-invariant pairing $\langle\, , \rangle$ and a holomorphic filtration $F^i\mathcal{H}$ on $\mathcal{H} := H \otimes \mathcal{O}_{\Delta^*}$, which satisfies two conditions

(1) The filtration $F^\bullet\mathcal{H}_{\mathbb{C},t}$ induces a Hodge structure on $\mathcal{H}_t \cong H_t \otimes \mathbb{C}$ which is polarized by $\langle\, , \rangle$.

(2) The filtration $F^\bullet\mathcal{H}_t$ has the transversality property

$$\nabla F^i\mathcal{H} \subset F^{i-1}\mathcal{H} \otimes \Omega_{\Delta^*}$$

with respect to the Gauss-Manin connection ∇ of $\mathcal{H}$.

The following result is of major importance. It has several proofs (see [41], [30]), especially in the algebraic geometry context, where the variation of Hodge structures comes from an algebraic family of projective varieties.

Theorem 3.18. *The monodromy operator T is quasiunipotent of order $\leq k+1$.*

This means that the eigenvalues of ρ are roots of unity, that is, $Id - \rho^N$ is nilpotent for some N, and that $(Id - T^N)^{k+1} = 0$. The proof of the quasiunipotency that is sketched by Griffiths in [27] and attributed to Borel goes as follows. Let $\mathbb{H}$ be the upper-half plane, which is a uniformization of the punctured disc Δ^*. When we pull-back our variation of Hodge structures to $\mathbb{H}$, the local system being trivialized gives a period map

$$\mathcal{P} : \mathbb{H} \to \mathcal{D}$$

where $\mathcal{D}$ is the polarized period domain associated to the data $(H_1, \langle\, , \rangle, b_{p,k} = \dim F^pH_{t,\mathbb{C}}$. The upper-half space $\mathbb{H}$ admits the Poincaré metric, and, using Theorem 3.17 and the Ahlfors-Schwarz lemma, Griffiths shows that, up to a coefficient, the period map is distance decreasing, where the distance on $\mathcal{D}$ is the Hodge metric mentioned previously. The argument is now the following. Let $x \in \mathbb{H}$ be a point. Then x, $x+1$, $x+2, \ldots$ are all over the same point $\overline{z} \in \Delta^*$ and we have by definition of monodromy $\mathcal{P}(x+1) = T \cdot \mathcal{P}(x)$. The distance decreasing property mentioned above gives

$$d_{\mathcal{D}}(\mathcal{P}(x+1), \mathcal{P}(x)) \leq d_{\mathbb{H}}(x, x+1),$$

and the right hand side tends to 0 when $\operatorname{Im} x$ tends to 0. Write $\mathcal{P}(x) = g_x \cdot \mathcal{P}(x_0)$, where x_0 is a fixed point in $\mathbb{H}$ and $g_x \in \operatorname{Aut}(H_{1,\mathbb{R}}, \langle\, , \rangle)$. Then

$$d_{\mathcal{D}}(\mathcal{P}(x+1), \mathcal{P}(x)) = d_{\mathcal{D}}(T \cdot \mathcal{P}(x), \mathcal{P}(x)) = d_{\mathcal{D}}(g_x^{-1} \circ T \circ g_x \cdot \mathcal{P}(x_0), \mathcal{P}(x_0)).$$

Hence we conclude that

$$\lim_{\operatorname{Im} x \to 0} d_{\mathcal{D}}(g_x^{-1} \circ T \circ g_x \cdot \mathcal{P}(x_0), \mathcal{P}(x_0)) = 0.$$

In other words, the conjugates $g_x^{-1} \circ T \circ g_x$ are as close as one wants to the stabilizer of $\mathcal{P}(x_0)$ in $\operatorname{Aut}(H_{1,\mathbb{R}}, \langle\, , \rangle)$, that we already proved to be a compact group. It easily follows that T has its eigenvalues of modulus 1. To conclude, one can apply Kronecker's lemma, since T acts preserving a lattice, hence its eigenvalues are algebraic integers. It follows that the eigenvalues are roots of unity. **qed**

Following [63, 11.2.4], one can be even more precise in the geometric setting, using semistable reduction. Assume one has a projective morphism $f : X \to \Delta$, which is smooth over Δ^*. Then, assuming X irreducible, one can arrange by Hironaka resolution theorem, that the central fiber $X_0 = X^* \setminus \Delta^*$ is a simple normal crossing divisor. One cannot however always arrange that the schematic central fiber X_0 (which equals Y set-theoretically) is a reduced normal crossing divisor. This can be achieved only after base-change by the semi-stable reduction theorem (see [45]). An important improvement of Theorem 3.18 is the following (see [63, 11.2.4])

Theorem 3.19. *If $f : X \to \Delta$ is a projective morphism with reduced normal crossing central fiber X_0, the monodromy T on $H^k(X_1, \mathbb{Q})$ is unipotent for all k.*

Another application of the Ahlfors-Schwarz lemma and curvature computations of the horizontal distribution is the following result by Griffiths [31].

Theorem 3.20. *Let $(H, \langle\, , \rangle, F^p\mathcal{H})$ be a polarized variation of Hodge structure on the punctured disc Δ^*. Then, if the monodromy T is trivial, the period map $\mathcal{P} : \Delta^* \to \mathcal{D}$ extends over 0.*

3.5 Degenerations and limit mixed Hodge structures

In order to understand the topology and motive of a family parameterized by a quasi-projective basis, it is essential to understand what happens at infinity. We mentioned that one can construct families of smooth projective varieties with projective base. In dimension 2, one can construct many of them as follows: start with a smooth projective threefold X and choose a Lefschetz pencil $(X_t)_{t\in\mathbb{P}^1}$ of hyperplane sections of X. This means that the base locus $Z := X_0 \cap X_\infty$ is smooth and the singular points of fibers are ordinary double points. The Lefschetz pencil provides, after blow-up of Z, a morphism $\phi : \widetilde{X} \to \mathbb{P}^1$ and the local monodromies around the critical values are, thanks to the Picard-Lefschetz formula [79, 3.2.1], of order 2. Choose a proper morphism $C \to \mathbb{P}^1$, where C is a smooth curve, which is fully ramified with ramification of order 2 over each critical value of ϕ. Then the base-changed family $\widetilde{X}_C \to C$ admits a small resolution of singularities at each of its singular points, which produces a smooth fibration $\widetilde{\widetilde{X}}_C \to C$. This construction, which provides a simultaneous resolution of singularities of the singular fibers, is due to Atiyah [4]. In higher even dimension, this construction provides families for which the local monodromies are trivial (but for which there is in general no simultaneous resolution of singularities). For families of varieties of odd dimension, the local monodromy, even for an innocent singularity like an ordinary double point, is infinite, so we need to go farther than Theorem 3.20 and this is exactly what is done in the crucial series of papers [64], [15], [67].

We consider a projective morphism $f : X \to \Delta$, smooth over Δ^*, with a central fiber X_0 that we can assume after base change to be a reduced normal crossing divisor. Denote by $f' : X^* \to \Delta^*$ the restriction of f to $X^* = f^{-1}(\Delta^*)$. As observed in Section 3.3, the vector bundles $\mathcal{H}^k = R^k f'_*\mathbb{C} \otimes \mathcal{O}_{\Delta^*}$ on Δ^* are holomorphic vector bundles with a natural meromorphic structure, which means that there is a natural meromorphic equivalence class of extensions of these bundles to Δ. This follows from the construction of $\mathcal{H}^k$ using holomorphic de Rham cohomology:

$$\mathcal{H}^k = R^k f'_*(\Omega^\bullet_{X^*/\Delta^*}). \tag{3.63}$$

The regularity theorem says that the Gauss-Manin connection has regular singular points, which means that there is a canonical extension $\mathcal{H}_e$ of $\mathcal{H}$ to Δ, in the same bimeromorphic equivalence class of extensions, for which the Gauss-Manin connection extends to a connection with pole order 1 at 0.

$$\nabla_e : \mathcal{H}_e \to \mathcal{H}_e \otimes \Omega_\Delta(\log 0).$$

Such an extension is provided by Steenbrink [67] and Katz-Oda [43], see also Theorem 3.12. Namely, Steenbrink introduces the relative logarithmic de Rham complex

$$\Omega^\bullet_{X/\Delta}(\log X_0) := \Omega^\bullet_X(\log X_0)/(f^*\Omega_\Delta(\log 0) \wedge \Omega^{\bullet-1}_X(\log X_0)). \tag{3.64}$$

Theorem 3.21. *[67] The sheaves $R^k f_*(\Omega^\bullet_{X/\Delta}(\log X_0))$ are locally free and satisfy base change.*

Having this result, it is clear using Theorem 3.12 over Δ_* that $R^k f_*(\Omega^\bullet_{X/\Delta}(\log X_0))$ is the canonical extension $\mathcal{H}^k_e$, for which the Gauss-Manin extension extends with logarithmic pole. Steenbrink also proves that the bundles

$$F^p R^k f_*(\Omega^\bullet_{X/\Delta}(\log X_0)) := R^k f_*(\Omega^{\bullet \geq p}_{X/\Delta}(\log X_0))$$

are subbundles of $R^k f_*(\Omega^\bullet_{X/\Delta}(\log X_0))$, thus extending the Hodge filtration on $\mathcal{H}^k$ to a Hodge filtration on $\mathcal{H}^k_e$.

By Theorem 3.19, the monodromy T is unipotent, so we can define its logarithm $N = \log T$ as a polynomial in $Id - T$. This is thus a matrix with $\mathbb{Q}$-coefficients. The local system H^k on Δ^* pulls-back to the upper-half plane with coordinate $q = \log z$ to a constant local system, and we thus have the period map

$$\mathcal{P} : \mathbb{H} \to \mathcal{D},$$

where $\mathcal{D}$ is the polarized period domain for H^k equipped with the adequate intersection form and with the given Hodge numbers. We have $\mathcal{P}(q+1) = T \cdot \mathcal{P}(q)$, so that

$$\psi(q) = \exp(-qN) \cdot \mathcal{P}(q)$$

depends only on $z = \exp(z)$ which provides a holomorphic map

$$\Psi : \Delta^* \to \mathcal{D}\check{}.$$

Here the space $\mathcal{D}\check{}$ is called the compact dual of $\mathcal{D}$. Typically, the polarized period domain $\mathcal{D}$ is defined inside a flag manifold, first by algebraic equations related to the first Hodge-Riemann bilinear relations (2.9(i)), and second by certain inequalities related to the second Hodge-Riemann relations (2.9(ii)). The compact dual $\mathcal{D}\check{}$ is only defined by the first Hodge-Riemann relations. It is thus an algebraic variety.

Let $\widetilde{X} = X \times_{\Delta^*} \mathbb{H}$ and $H_{\lim} := H^k(\widetilde{X}, \mathbb{Q})$. Schmid's nilpotent orbit theorem says the following.

Theorem 3.22. *[64] (i) The limit* $\lim_{z\to 0}\Psi(z)$ *exists. The filtration so defined on* $\mathcal{H}_{\lim}\otimes\mathbb{C}$ *is the Hodge filtration of a mixed Hodge structure on* $\mathcal{H}_{\lim}$*.*

(ii) The weight filtration of this mixed Hodge structure is the monodromy weight filtration.

(iii) The nilpotent orbit $z\mapsto \exp(qN)\psi_0$ *has the property that it defines a polarized Hodge structure on* $H_{\lim}$ *for* $\operatorname{Im} q$ *large enough. Furthermore, it is an excellent approximation of the original period map.*

3.6 Hodge loci

Let $\phi : \mathcal{X} \to B$ be a smooth projective morphism between smooth complex quasi-projective varieties. For any $b \in B$, denote by $\mathcal{X}_b$ the fiber $\phi^{-1}(b)$. Let $\alpha \in \mathrm{Hdg}^{2k}(\mathcal{X}_{b_0})$ be a Hodge class. The Hodge locus of α can be first defined locally in a simply connected open neighborhood $U \subset B$ of b_0, on which the local system $R^{2k}\phi_*\mathbb{Z}$ with stalk $H^{2k}(\mathcal{X}_b,\mathbb{Z})$ at b is trivial. The class α thus provides a section $\tilde{\alpha}$ of $R^{2k}\phi_*\mathbb{Z}$ on U, transporting the class α to a class α_b for any $b \in U$.

We define the Hodge locus U_α of α in U as the set of points $b \in U$, such that the class $\alpha_b \in H^{2k}(\mathcal{X}_b,\mathbb{Z})$ is a Hodge class. The following result is a consequence of Theorem 3.11.

Theorem 3.23. *The Hodge locus* U_α *is closed analytic, of codimension* $\leq h^{k-1,k+1} := \dim H^{k-1,k+1}(\mathcal{X}_b)$*.*

Indeed, the local analytic equations are obtained as follows : the locally constant section $\tilde{\alpha}$ provides as well a holomorphic section of the holomorphic bundle $\mathcal{H}^{2k}$ on U, and it projects to a holomorphic section $\overline{\tilde{\sigma}}$ of the quotient bundle $\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$. The Hodge locus U_α identifies with the vanishing locus of $\overline{\tilde{\sigma}}$, because a rational (or real) cohomology class is of type (k,k) if and only if it belongs to F^kH^{2k}, as follows from Hodge symmetry. The second statement needs Griffiths transversality which implies that the $\sum_{p+q=k,\,p<k} h^{p,q} = \operatorname{rank} \mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$ local equations described above can be reduced, at least generically, to $h^{k,k}$ equations.

The global structure and definition of the Hodge locus is made complicated by the presence of monodromy, although we can of course use analytic continuation. However in the paper [10], Cattani, Deligne and Kaplan introduced the following convenient viewpoint on the Hodge locus, by introducing the locus of Hodge classes. We observe that in the definition above, the Hodge class α_b lies in $F^kH^{2k}(\mathcal{X}_b) \cap H^{2k}(\mathcal{X}_b,\mathbb{Z})_{\mathrm{tf}}$ (where "tf" stands for "torsion free part"). It follows that, denoting F^kH^{2k} the total space of the holomorphic (in fact algebraic) vector bundle $F^k\mathcal{H}^{2k} := R^{2k}\phi_*\Omega^{\bullet\geq k}_{\mathcal{X}/B}$, we have a natural section of this bundle over U_α

$$U_\alpha \to F^kH^{2k}$$

$$b \mapsto \alpha_b \in F^kH^{2k}(\mathcal{X}_b).$$

The image of this lift (or rather the union over all α of such subsets) has now a global definition for which no local trivialization is needed, namely

Definition 3.24. *The locus of Hodge class for the family* $\phi : \mathcal{X} \to B$ *is the set of* $(b,\alpha_b) \in F^kH^{2k}$*, with* $\alpha_b \in F^kH^{2k}(\mathcal{X}_b) \cap H^{2k}(\mathcal{X}_b,\mathbb{Z})_{\mathrm{tf}}$*.*

In other words, in the first approach, we first choose an integral class and imposed to it the condition of being of Hodge type $(2k, 0) + \ldots + (k, k)$, while in the second approach, we choose a class of Hodge type $(2k, 0) + \ldots + (k, k)$ and impose to it the condition of being integral. What we gained is that the locus of Hodge classes is globally defined and is naturally a closed (for the Euclidean topology) subset of a complex algebraic variety. Furthermore this construction allows to globalize the definition of the Hodge loci U_α by considering the images in B of the connected components of the locus of Hodge classes. The following result heavily uses the nilpotent orbit theorem 3.22. It has been recently reproved in [8] using definability theory (but their proof also uses the Schmid asymptotic analysis) and it is probably the most important general result on the Hodge conjecture.

Theorem 3.25. *(Cattani, Deligne, Kaplan 1995) The locus of Hodge classes is a countable union of closed algebraic subsets of $F^k H^{2k}$.*

That this is indeed the structure predicted by the Hodge conjecture for the Hodge locus of α was observed by Weil and follows from the existence of relative Hilbert schemes (or Chow varieties) which are projective over B and parameterize subschemes (or effective cycles) $Z_t \subset \mathcal{X}_t$ of a given cohomology class. Using these relative Hilbert schemes M_i, we can construct a countable union of varieties M_{ij} projective over B, defined by $M_{ij} = M_i \times_B M_j$ parameterizing all cycles $\mathcal{Z}_t = \mathcal{Z}_t^+ - \mathcal{Z}_t^-$ in the fibers $\mathcal{X}_t$. For any point $t \in B$, if the class α_t on $\mathcal{X}_t$ is algebraic, α_t is the class of a cycle $\mathcal{Z}_t^+ - \mathcal{Z}_t^-$ parameterized by a point in the fiber of at least one of these varieties M_{ij}. Hence the locus of Hodge classes is the union of the images of M_{ij} in $F^k H^{2k}$ under the relative algebraic de Rham cycle class.

4 More structure and more open problems

4.1 Coefficients and comparisons

The cohomology of algebraic varieties over a field K of characteristic zero can be computed by different means, however with different coefficients. Each of these tools, combined with comparison theorems, provides extra structure on cohomology. The most elementary and traditional way is to imbed the field K in $\mathbb{C}$, so that the variety X defined over K can be seen as a variety defined over $\mathbb{C}$, which itself has an analytic variant X_{an}, a complex manifold whose underlying topological space has Betti cohomology groups. We denote

$$H^k_B(X, \mathbb{Z}) = H^k(X_{an}, \mathbb{Z}).$$

This notation needs however a caveat. Indeed, the homotopy type of the underlying topological space of X_{an} depends on the choice of embedding $K \subset \mathbb{C}$. This is a very subtle phenomenon discovered by Serre [66]. For another embedding $\tau : K \to \mathbb{C}$, we denote by X^τ the complex variety deduced from X/K using τ, and we say that X^τ and X are conjugate varieties. Serre exhibited examples where X^τ_{an} and X_{an} do not have the same fundamental group. Charles constructed in [13] examples where X^τ_{an} and X_{an} do not have isomorphic cohomology algebras. When K is a finitely generated field over an algebraically closed field k (for example $\overline{\mathbb{Q}}$), it is the function field of an irreducible variety B defined over k, and we can spread the variety X/K into a morphism $\mathcal{X} \to B$, which we can assume (even without properness assumptions) to be a topological fibration over a Zariski open set B^0 of B, with complement also defined over k. All the complex points of B^0, which correspond to fields embeddings $K \hookrightarrow \mathbb{C}$ exactly when they are not contained in a proper subvariety of B defined over k, have in this case the same topology. When k is not algebraically closed, the above argument does not work because the variety B which is irreducible over k may be not irreducible, and in fact not connected, over $\mathbb{C}$. The above argument shows more generally that the phenomenon of conjugate not homeomorphic complex varieties is of an arithmetic nature and concerns varieties defined over a number field.

The two other main theories are the algebraic de Rham cohomology (see Section 1.3), which works for smooth X, and étale cohomology.

Étale cohomology $H^k_{et}(X, \mathbb{Z}_\ell(i))$ is defined for a variety X over a field k as the projective limit over n of the étale cohomology groups $H^k_{et}(X_{\overline{k}}, \mu_{l^n}^{\otimes i})$, where μ_{l^n} is the étale sheaf (i.e. sheaves for the étale topology, or rather site) of l^n-roots of unity on X. If the field k is algebraically closed (or contains all roots of unity), the étale sheaf μ_{ℓ^n} is isomorphic to the constant sheaf $\mathbb{Z}/\ell^n$. In general however, the étale sheaves $\mu_{l^n}^{\otimes i}$ depend on the twist i. There is a Galois group action of $\mathrm{Gal}(\overline{k}/k)$ on $H^k_{et}(X_{\overline{k}}, \mu_{l^n}^{\otimes i})$ and the Galois group action depends on the twist i. This Galois group action is essential in the arithmetic context, when X is defined over a number field.

Over $\mathbb{C}$, the easiest comparison theorem between étale and Betti cohomology theory is the Artin comparison theorem (see [58]).

Theorem 4.1. *Let X be an algebraic variety over $\mathbb{C}$. Then there is a canonical isomorphism*

$$H^k_B(X_{\mathrm{an}}, \mathbb{Z}/\ell^n) \cong H^k_{et}(X, \mathbb{Z}/\ell^n). \tag{4.65}$$

By taking projective limits, we get as well an isomorphism

$$H^k_B(X_{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{Q}_\ell = H^k_B(X_{\mathrm{an}}, \mathbb{Q}_\ell) \cong H^k_{et}(X, \mathbb{Q}_\ell), \tag{4.66}$$

where we pass here to $\mathbb{Q}_\ell$-coefficients because the change of coefficients that is used in the first equality is more complicated with $\mathbb{Z}_\ell$-coefficients.

We already commented on the comparison theorem between algebraic de Rham cohomology and Betti cohomology of a smooth variety defined over $\mathbb{C}$. When the variety is defined over a subfield $K \subset \mathbb{C}$, algebraic de Rham cohomology has a version with coefficients in K, and the comparison gives an isomorphism after tensoring with $\mathbb{C}$:

$$H^k_B(X_{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{C} = H^k_B(X_{\mathrm{an}}, \mathbb{C}) \cong H^k(X, \Omega^\bullet_{X/\mathbb{C}}) = H^k(X, \Omega^\bullet_{X/K}) \otimes \mathbb{C}. \tag{4.67}$$

We wish now to discuss the constraints of an arithmetic nature on cycle classes provided (even for cycles on complex algebraic manifolds) by the existence of a cycle class in these various theories and the comparison theorems. When X is smooth, there is a cycle class in the three theories. The étale and Betti cycle classes correspond under the isomorphism (4.66), while the de Rham and Betti cycle classes compare as follows

$$[Z]_{dR} = (2i\pi)^c [Z]_{\mathrm{Betti}} \tag{4.68}$$

for any cycle Z of codimension c in a smooth variety X defined over a field $K \subset \mathbb{C}$.

Let us make the following

Definition 4.2. *[18] A Hodge class $\alpha \in \mathrm{Hdg}^{2c}(X_{\mathrm{an}}, \mathbb{Q})$ is absolute Hodge if for any automorphism τ of the field $\mathbb{C}$, the class*

$$\frac{1}{(2i\pi)^c}((2i\pi)^c\alpha)^\tau \in F^cH^{2c}(X^\tau_{\mathrm{an}}, \mathbb{C}) \tag{4.69}$$

is also a Hodge class, that is, belongs to $H^{2c}(X^\tau_{\mathrm{an}}, \mathbb{Q})$.

In (4.69), we use the transportation via τ of a class $\beta \in F^cH^{2c}(X_{\mathrm{an}}, \mathbb{C})$ to a class $\beta^\tau \in F^cH^{2c}(X_{\mathrm{an}}, \mathbb{C})$ using the comparison isomorphism

$$F^cH^{2c}(X_{\mathrm{an}}, \mathbb{C}) \cong \mathbb{H}^{2c}(X, \Omega^{\bullet \geq c}_{X/\mathbb{C}}).$$

Thinking to X as defined in projective space by algebraic equations with coefficients α_i, and letting τ act on these coefficients, we get the algebraic variety X^τ in projective space, whose analytization will provide the complex manifold X^τ_{an}. It is not hard to construct a τ-linear isomorphism

$$\mathbb{H}^{2c}(X, \Omega^{\bullet \geq c}_{X/\mathbb{C}}) \cong \mathbb{H}^{2c}(X^\tau, \Omega^{\bullet \geq c}_{X^\tau/\mathbb{C}}) \tag{4.70}$$

which, combined with the Betti-de Rham comparison

$$\mathbb{H}^{2c}(X^\tau, \Omega^{\bullet \geq c}_{X^\tau/\mathbb{C}}) \cong F^cH^{2c}(X_{\mathrm{an}}, \mathbb{C})$$

gives our transportation $\beta \mapsto \beta^\tau$.

Let $Z \in \mathrm{CH}^c(X)$ be a cycle of codimension c. Then for any field automorphism τ of $\mathbb{C}$, transporting via (4.70) the class $[Z]_{dR} \in \mathbb{H}^{2k}(X, \Omega^{\bullet \geq c}_{X/\mathbb{C}})$ to an algebraic de Rham class in $\mathbb{H}^{2k}(X_\tau, \Omega^{\bullet \geq c}_{X_\tau/\mathbb{C}})$ provides the de Rham cohomology class of the conjugate subvariety Z^τ of X^τ. It follows that the Hodge class

$$[Z]_{\mathrm{Betti}} = \frac{1}{(2i\pi)^c}[Z]_{dR} \in \mathrm{Hdg}^{2k}(X_{\mathrm{an}}, \mathbb{Q}),$$

is absolute Hodge, since its transportations as in (4.69) give the Betti Hodge class of Z_τ in X_τ. In other words, the Betti cycle classes are absolute Hodge classes and thus, an intermediate step towards the Hodge conjecture is

Conjecture 4.3. *Hodge classes on smooth projective complex varieties are absolute Hodge.*

Example 4.4. The Hodge classes on products $X \times X$ appearing in the standard conjectures are absolute Hodge, because they are constructed from cycle classes, which are absolute Hodge, by formal operations.

Coming back to the étale setting, there is a cycle class $[Z]_{et}$ for codimension c cycles of smooth projective varieties defined over any field K and it takes value in $H^{2c}_{et}(X_{\overline{K}}, \mathbb{Q}_\ell(c))$. We already mentioned that the Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on $H^{2c}_{et}(X_{\overline{K}}, \mathbb{Q}_\ell(c))$. Let Z be a cycle of codimension c on $X_{\overline{K}}$. Then there is an algebraic extension $K \subset K'$, and a cycle Z' defined over K' which is algebraically equivalent to Z, hence has the same cycle class. Indeed, the Hilbert scheme parameterizing subvarieties of X of codimension c and given degree is defined over K, so its irreducible components, which parameterize deformation classes of subvarieties of X, are defined over algebraic extensions of K. It follows that the étale cycle class $[Z]_{et} \in H^{2c}_{et}(X_{\overline{K}}, \mathbb{Q}_\ell(c))$ is invariant under the finite index subgroup $\mathrm{Gal}(\overline{K}/K') \subset \mathrm{Gal}(\overline{K}/K)$. The Tate conjecture is that this is a characterization of cycle classes.

Conjecture 4.5. *Assume K is a number field (or finitely generated over a number field). Then the image of the cycle class map*

$$\mathrm{CH}^c(X_{\overline{K}}) \otimes \mathbb{Q}_\ell \to H^{2c}_{et}(X_{\overline{K}}, \mathbb{Q}_\ell(c))$$

is equal to the set of Tate classes, that is, classes which are stabilized by a finite index subgroup $\mathrm{Gal}(\overline{K}/K') \subset \mathrm{Gal}(\overline{K}/K)$.

This conjecture is widely open, even for divisor classes.

4.2 Bloch-Ogus theory

The various cohomology theories mentioned in the previous section involve not only different sheaves of coefficients but also different topologies. Bloch-Ogus theory [7] directly addresses the change of topology. Let us focus on the Euclidean and Zariski topologies. Let X be an algebraic variety (in particular, it is irreducible and we can speak of its function field). If X is defined over $\mathbb{C}$, we can consider two topologies on $X(\mathbb{C})$, namely the Euclidean (or analytic) topology and the Zariski topology. We will denote X_{an}, resp. X_{Zar}, the topological space $X(\mathbb{C})$ equipped with the Euclidean topology, resp. the Zariski topology. As Zariski open sets are open for the Euclidean topology, the identity of $X(\mathbb{C})$ is a continuous map

$$f : X_{an} \to X_{Zar}.$$

Given any abelian group A, the Bloch-Ogus spectral sequence is the Leray spectral sequence of f, abutting to the cohomology $H^i_B(X, A) := H^i(X_{an}, A)$. It starts with

$$E_2^{p,q}(A) = H^p(X_{Zar}, \mathcal{H}^q(A)), \tag{4.71}$$

where $\mathcal{H}^q(A)$ is the sheaf on X_{Zar} associated with the presheaf $U \mapsto H^q_B(U, A)$. The Betti cohomology groups $H^n_B(X, A) = H^n(X_{an}, A)$ thus have a filtration, namely the Leray filtration for which $Gr^p_L H^{p+q}_B(X_{an}, A) = E_\infty^{p,q}$, the latter group being a subquotient of $E_2^{p,q}$.

A fundamental result of Bloch-Ogus is the Gersten-Quillen resolution for the sheaves $\mathcal{H}^q(A)$. It is constructed as follows: For any variety Y, we denote by $H^i(\mathbb{C}(Y), A)$ the direct limit over all dense Zariski open sets $U \subset Y$ of the groups $H^i_B(U, A)$:

$$H^i(\mathbb{C}(Y), A) := \lim_{\substack{\rightarrow \\ \emptyset \neq U \subset Y, \mathrm{open}}} H^i_B(U, A). \tag{4.72}$$

Let now Z be a normal irreducible closed algebraic subset of X, and let Z' be an irreducible reduced divisor of Z. At the generic point of Z', both Z' and Z are smooth. There is thus a residue map $\partial : H^i(\mathbb{C}(Z), A) \to H^{i-1}(\mathbb{C}(Z'), A)$. It is defined as the limit over all pairs of dense Zariski open sets $V \subset Z_{reg}$, $U \subset Z'_{reg}$ such that $U \subset V \cap Z'_{reg}$, of the residue maps

$$Res_{Z,Z'} : H^i((V \setminus V \cap Z')_{an}, A) \to H^{i-1}(U_{an}, A).$$

If now $Z' \subset Z$ is a divisor, with Z not necessarily normal along Z', we can introduce the normalization $n : \widetilde{Z} \to Z$ with restriction $n' : Z'' \to Z'$, where $Z'' = n^{-1}(Z')$, and then define $\partial : H^i(\mathbb{C}(Z), A) \to H^{i-1}(\mathbb{C}(Z'), A)$ as the composite

$$H^i(\mathbb{C}(Z), A) \cong H^i(\mathbb{C}(\widetilde{Z}), A) \stackrel{\partial}{\to} H^{i-1}(\mathbb{C}(Z''), A) \stackrel{n'_*}{\to} H^{i-1}(\mathbb{C}(Z'), A). \tag{4.73}$$

In (4.73), the pushforward morphism

$$n'_* : H^{i-1}(\mathbb{C}(Z''), A) \to H^{i-1}(\mathbb{C}(Z'), A)$$

is defined by restricting to pairs of Zariski open sets $U \subset Z''_{reg}$, $V \subset Z'_{reg}$ such that n' restricts to a proper (in fact, finite) morphism $U \to V$. More precisely, as Z'' is not necessarily irreducible, we should in the above definition write $Z'' = \cup_j Z''_j$ as a union of irreducible components, and take the sum over j of the morphisms (4.73) defined for each Z''_j.

For each subvariety $j : Z \hookrightarrow X$, we consider the group $H^i(\mathbb{C}(Z), A)$ as a constant sheaf supported on Z and we get the corresponding sheaf $j_* H^i(\mathbb{C}(Z), A)$ on X_{Zar}. Finally, we observe that we have a natural sheaf morphism

$$\mathcal{H}^i(A) \to H^i(\mathbb{C}(X), A)$$

where we recall that the second object is a constant sheaf on X_{Zar}. This sheaf morphism is simply induced by the natural maps $H^i(U_{an}, A) \to H^i(\mathbb{C}(X), A)$ for any Zariski open set $U \subset X$, given by (4.72). The residue maps have the following property: Let D_1, $D_2 \subset Y$ be two smooth divisors in a smooth variety, let Z be a smooth reduced irreducible component of $D_1 \cap D_2$ and let $\alpha \in H^i_B(U, A)$, where $U : Y \setminus (D_1 \cup D_2)$. Then

$$\mathrm{Res}_Z(\mathrm{Res}_{D_1}(\alpha)) = -\mathrm{Res}_Z(\mathrm{Res}_{D_2}(\alpha)), \tag{4.74}$$

where on the left Z is seen as a divisor in D_1, and on the right it is seen as a divisor in D_2. Considering the case where $Y \subset X$ is the regular locus of any subvariety of codimension k of X, D, $D' \subset Y$ are of codimension $k+1$, and $Z \subset D \cap D' \subset Y$ is of codimension $k+2$ in X, we conclude from (4.74) that for any i, the two sheaf maps

$$\partial : \oplus_{\mathrm{codim}\, Y = k} H^i(\mathbb{C}(Y), A) \to \oplus_{\mathrm{codim}\, D = k+1} H^{i-1}(\mathbb{C}(D), A)$$

and

$$\partial : \oplus_{\mathrm{codim}\, D = k+1} H^{i-1}(\mathbb{C}(D), A) \to \oplus_{\mathrm{codim}\, Z = k+2} H^{i-2}(\mathbb{C}(Z), A)$$

satisfy $\partial \circ \partial = 0$.

Theorem 4.6. *(Bloch-Ogus, [7]) Let X be smooth. The complex*

$$\begin{aligned} 0 \to \mathcal{H}^i(A) \to H^i(\mathbb{C}(X), A) \to \oplus_{\mathrm{codim}\, D = 1} H^{i-1}(\mathbb{C}(D), A) \to \\ \ldots \to \oplus_{\mathrm{codim}\, Z = i} H^0(\mathbb{C}(Z), A) \to 0 \end{aligned} \tag{4.75}$$

is an acyclic resolution of $\mathcal{H}^i(A)$.

It is clear that this resolution is acyclic. Indeed, all these sheaves appearing in the resolution are acyclic, being constant sheaves for the Zariski topology on algebraic subvarieties of X. Note that the codimension i subvarieties Z of X appearing above are all irreducible, so that $H^0(\mathbb{C}(Z), A) = A$ and the global sections of the last sheaf appearing in this resolution is the group $\mathcal{Z}_i(X) \otimes A$ of codimension i cycles with coefficients in A.

Theorem 4.6 says first that the sheaf map $\mathcal{H}^i(A) \to H^i(\mathbb{C}(X), A)$ is injective, which is by no means obvious. The meaning of this assertion is that if a class

$\alpha \in H^i_B(U, A)$ vanishes on a dense Zariski open set $V \subset U$, then U can be covered by Zariski open sets V_i such that $\alpha_{|V_i} = 0$. This is a moving lemma for the support of cohomology.

Theorem 4.6 has the following consequence, simplifying the Bloch-Ogus spectral sequence (4.71).

Corollary 4.7. *One has $E_2^{p,q} = H^p(X_{\mathrm{Zar}}, \mathcal{H}^q(A)) = 0$ for $p > q$.*

Indeed, as (4.75) is an acyclic resolution of $\mathcal{H}^q(A)$, the complex of global sections of (4.75) has degree p cohomology equal to $H^p(X_{Zar}, \mathcal{H}^q(A))$. As the resolution has length q, the degree p cohomology vanishes for $p > q$. **qed**

4.2.1 Finite coefficients

The Bloch-Ogus theory can be combined with the analogous resolution for K-theory sheaves. The original Gersten-Quillen resolution worked for Quillen K-theory sheaves, but Kerz [46] recently established a Gersten-Quillen resolution for Milnor K-theory sheaves which are much more concrete and easy to define. Define the Milnor K-theory groups of a field K (or a ring R) as follows

$$K_i^M(K) = (K^*)^{\otimes i}/I,$$

where I is the ideal generated by $x \otimes (1-x)$ for $x \in K^*$, $1-x \in K^*$. In particular, we have $K_1^M(K) = K^*$. Fix an integer n prime to the characteristic of K. The exact sequence of Galois modules

$$0 \to \mu_n \to \overline{K}^* \to \overline{K}^* \to 1,$$

where $\mu_n \subset \overline{K}^*$ is the group of n-th roots of unity, gives a map

$$\partial : K^*/n \to H^1(K, \mu_n) := H^1(G_K, \mu_n), \tag{4.76}$$

where $G_K = \mathrm{Gal}\,(\overline{K}/K)$, which is known by Hilbert's Theorem 90 to be an isomorphism (this is equivalent to the vanishing $H^1(G_K, \overline{K}^*) = 0$). More generally, one has a morphism (called the Galois symbol or norm residue map)

$$\partial_i : K_i^M(K)/n \to H^i(G_K, \mu_n^{\otimes i}) \tag{4.77}$$

which to $(x_1, \ldots, x_i)$ associates $\partial x_1 \cup \ldots \cup \partial x_i$. The following fundamental result generalizing the isomorphism (4.76) is the Bloch-Kato conjecture solved by Voevodsky [74].

Theorem 4.8. *The map ∂_i is an isomorphism for any i and n prime to* $\mathrm{char}\, K$.

This result was known for $i = 2$ as the Merkur'ev-Suslin theorem [57].

We now work over $\mathbb{C}$. We observe that Galois cohomology of the function field $L = \mathbb{C}(Y)$ of a smooth algebraic variety Y over $\mathbb{C}$ can be expressed, using the Artin comparison theorem as

$$H^i(G_L, \mathbb{Z}/n) = \lim_{\substack{\to \\ \emptyset \neq U \subset Y, \text{open}}} H^i_{et}(U, \mathbb{Z}/n) = \lim_{\substack{\to \\ \emptyset \neq U \subset Y, \text{open}}} H^i_B(U, \mathbb{Z}/n) =: H^i(\mathbb{C}(Y), \mathbb{Z}/n).$$

Combining Theorem 4.8 with Kerz resolution for the Milnor K-theory sheaves on one hand, and the Bloch-Ogus Gersten-Quillen resolution on the other hand, one then gets

Corollary 4.9. *Let X be smooth over $\mathbb{C}$. Then the sheafified norm residue map gives sheaf isomorphisms*

$$\partial_i : \mathcal{K}_i^M/n \to \mathcal{H}^i(\mathbb{Z}/n).$$

Here the sheaves $\mathcal{K}_i^M$ are associated to the presheaves $U \mapsto K_i^M(\Gamma(U, \mathcal{O}_U))$.

4.2.2 Integral coefficients

The Hodge conjecture is not true with integral coefficients. There are counterexamples due to Atiyah and Hirzeburch [5] who exhibit nontrivial topological obstructions for an integral cohomology class on a differentiable manifold to be the class of submanifold with a complex structure on the normal bundle. There are also examples due to Kollár [52] which are much more mysterious, as they are not of a topological nature. Kollár proves for example that for a very general hypersurface X of degree divisible by p^n in $\mathbb{P}^{n+1}$, all the curves $C \subset X$ are of degree divisible by p. When $n \geq 3$, by the Lefschetz theorem on hyperplane sections, $H_2(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, with generator α of degree 1 (with respect to the hyperplane class $c_1(\mathcal{O}_X(1))$). The class $d\alpha$ is the class of a plane section of X, so $d\alpha$ is algebraic. Kollár's theorem shows that α itself is not algebraic in general. However, when X contains a line (a degree 1 curve), its class is the class α, which is then algebraic. This example shows that the variational form of the Hodge conjecture mentioned in Section 3.2 is not true for integral Hodge classes.

The defect of the Hodge conjecture with integral coefficients provides in some cases obstructions to stable rationality for a complex projective manifold X. Here X is said to be rational if it is birational to $\mathbb{P}^n$ and stably rational if $X \times \mathbb{P}^r$ is rational for some r.

Proposition 4.10. *If X is stably rational, integral Hodge classes of degree 4 and degree $2n - 2$ on X are classes of cycles with integral coefficients.*

In [16], the case of degree 4 has been related to another sort of stable birational invariants provided by Bloch-Ogus theory, namely unramified cohomology. With the notation as in the previous section, one defines unramified cohomology $H^i_{nr}(X, A)$ as $H^0(X_{Zar}, \mathcal{H}^i(A))$.

The following results are proved in [16], as a consequence of the Bloch-Kato conjecture (Theorem 4.8) and its consequence Corollary 4.9.

Theorem 4.11. *If X is a smooth algebraic variety over $\mathbb{C}$, the sheaves $\mathcal{H}^i(\mathbb{Z})$ have no torsion.*

This means that a torsion integral cohomology class α on a smooth complex variety X vanishes on the open sets U_i of a Zariski open covering. A formal consequence is

Corollary 4.12. *For each i, N there is an exact sequence of sheaves on X_{Zar}*

$$0 \to \mathcal{H}^i(\mathbb{Z}) \stackrel{N}{\to} \mathcal{H}^i(\mathbb{Z}) \to \mathcal{H}^i(\mathbb{Z}/N) \to 0. \tag{4.78}$$

The next result is a consequence of Theorem 4.11.

Theorem 4.13. *For any smooth algebraic variety over $\mathbb{C}$, there is an exact sequence*

$$0 \to H^3_{nr}(X,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}\,(H^4(X,\mathbb{Z})/\mathrm{Im}\,c) \to 0, \qquad (4.79)$$

where $c : \mathrm{CH}^2(X)/\mathrm{alg} \to H^4(X,\mathbb{Z})$ is the cycle class map.

The key point is the identification, due to Bloch-Ogus [7], of the cycle map c to the natural map

$$H^2(X_{\mathrm{Zar}}, \mathcal{H}^2(\mathbb{Z})) \to H^4(X,\mathbb{Z})$$

appearing in the Bloch-Ogus spectral sequence. The existence of this map is due to the vanishing of Corollary 4.7, which says that there are no nonzero differentials starting from $H^2(X_{\mathrm{Zar}}, \mathcal{H}^2(\mathbb{Z})) = E_2^{2,2}$ which thus admits $E_\infty^{2,2}$ as a quotient, and that, also by Corollary 4.7, $E_\infty^{2,2} \subset H^4(X,\mathbb{Z})$ as the deepest term in the Leray filtration. It follows that

$$\mathrm{Tors}\,(H^4(X,\mathbb{Z})/\mathrm{Im}\,c) = \mathrm{Tors}\,(H^4(X,\mathbb{Z})/E_\infty^{2,2}).$$

The Bloch-Ogus spectral sequence has two other terms in degree 4 at E_2, namely

$$E_2^{1,3} = H^1(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z})),\ E_2^{0,4} = H^0(X_{\mathrm{Zar}}, \mathcal{H}^4(\mathbb{Z})).$$

We have $H^0(X_{\mathrm{Zar}}, \mathcal{H}^4(\mathbb{Z})) \subset E_\infty^{0,4}$ and $H^1(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z})) \cong E_\infty^{1,3}$, again as a consequence of the vanishing of Corollary 4.7. It follow that there is an exact sequence

$$0 \to H^1(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z})) \to H^4(X,\mathbb{Z})/\mathrm{Im}\,c \to H^0(X_{\mathrm{Zar}}, \mathcal{H}^4(\mathbb{Z}))$$

where the last term is torsion free by Theorem 4.11. It follows that

$$\mathrm{Tors}\,(H^4(X,\mathbb{Z})/E_\infty^{2,2}) = \mathrm{Tors}\,(H^1(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z}))).$$

To compute the right hand side, we apply the exact sequence (4.78) in degree 3, and this provides the exact sequence (4.79). **qed**

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