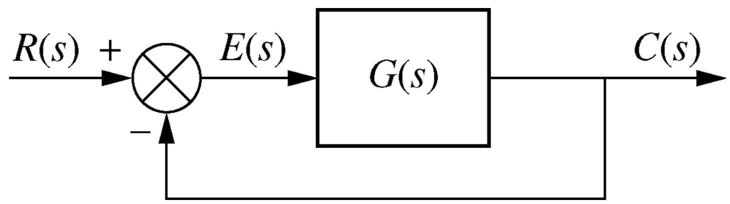
Homework 6 - Solutions

Note: This homework is worth a total of 45 points.

1. Steady State Error Analysis of a unity feedback system (5 points)
For the unity feedback system shown below, where

$$G(s) = \frac{5000}{s(s+75)}$$



(a) What is the expected percent overshoot for a unit step input?

(b) What is the settling time for a unit step input?

(c) What is the steady-state error for an input of $5u(t)$?

(d) What is the steady-state error for an input of $5tu(t)$?

(e) What is the steady-state error for an input of $5t^2u(t)$?

Solution: The closed-loop transfer function is given by:

$$T(s) = \frac{5000}{s^2 + 75s + 5000}$$

(a) From $T(s)$, we can check that $\omega_n = \sqrt{5000}$ and $2\zeta\omega_n = 75$. Thus, $\zeta = 0.53$ and %OS is given by,

$$\%\text{OS} = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 14.01\%$$

(b) $T_s = \dfrac{4}{\zeta\omega_n} = 0.107$ sec.

(c) Error is given by:

$$E(s) = R(s) - C(s) = R(s) - \frac{R(s)C(s)}{R(s)} = R(s)(1 - T(s))$$

$$E(s) = R(s)\left(1 - \frac{5000}{s^2 + 75s + 5000}\right)$$

Now,

$$e_{ss} = e(\infty) = \lim_{s\to 0} sE(s) = s\frac{5}{s}\left(1 - \frac{5000}{s^2 + 75s + 5000}\right) = 0$$

(d) Using $E(s)$ from above, the steady state error for a ramp input is given by:

$$e_{ss} = e(\infty) = \lim_{s\to 0} sE(s) = s\frac{5}{s^2}\left(1 - \frac{5000}{s^2 + 75s + 5000}\right) = 0.075$$

(e) Using $E(s)$ from above, the steady state error for a parabolic input is given by:

$$e_{ss} = e(\infty) = \lim_{s\to 0} sE(s) = s\frac{5}{s^3}\left(1 - \frac{5000}{s^2+75s+5000}\right) = \infty$$

2. Steady State Error Analysis of a unity feedback system (3 points)
Consider the unity feedback system shown in Problem 1. If $G(s)$ is given as follows, find the value of α to yield a $K_v = 25000$.

$$G(s) = \frac{100500(s+5)(s+14)(s+23)}{s(s+27)(s+\alpha)(s+33)}$$

Solution: We know that $K_v = \lim_{s\to 0} sG(s)$, where $G(s)$ is the open-loop transfer function. Thus,

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{100500(s+5)(s+14)(s+23)}{s(s+27)(s+\alpha)(s+33)} = \frac{100500(5)(14)(23)}{27(\alpha)(33)} = 25000$$

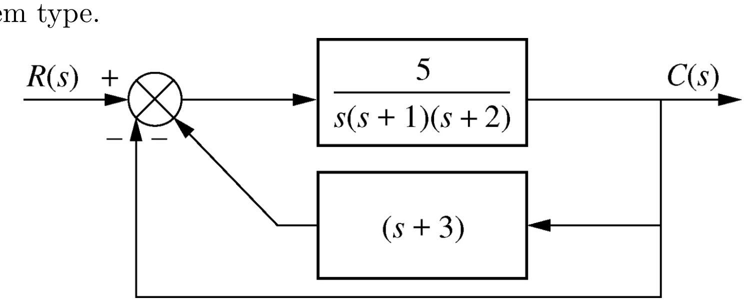
$$\therefore \alpha = 7.26$$

3. Steady State Error Analysis (5 points)
For the system shown below,

(a) Find K_p, K_v and K_a.

(b) Find the steady-state error for an input of $50u(t)$, $50tu(t)$ and $50t^2u(t)$.

(c) State the system type.



Solution: Let $G_e(s)$ be the equivalent open-loop transfer function, then

$$G_e(s) = \frac{\dfrac{5}{s(s+1)(s+2)}}{1 + \dfrac{5(s+3)}{s(s+1)(s+2)}} = \frac{5}{s^3+3s^2+7s+15}$$

(a) Now, K_p, K_v and K_a can be calculated as follows

$$K_p = \lim_{s\to 0} G_e(s) = 0.33$$

$$K_v = \lim_{s\to 0} sG_e(s) = 0$$

$$K_a = \lim_{s\to 0} s^2G_e(s) = 0$$

(b) Using the values of K_p, K_v and K_a, we can find the steady-state errors as follows

For $r(t) = 50u(t)$, $e_{ss} = e(\infty) = \dfrac{50}{1+K_p} = 37.5$

For $r(t) = 50tu(t)$, $e_{ss} = e(\infty) = \dfrac{50}{K_v} = \infty$

For $r(t) = 50t^2u(t)$, $e_{ss} = e(\infty) = \dfrac{50}{K_a} = \infty$

(c) Since there are no pure integrations in the open-loop transfer function, the system is **Type 0**.

4. Steady State Error Analysis (5 points)

Given the system shown below, find the following

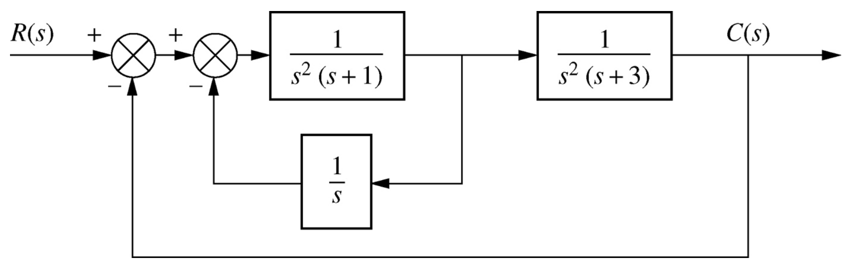
(a) The closed-loop transfer function

(b) The system type

(c) The steady-state error for an input of $5u(t)$

(d) The steady-state error for an input of $5tu(t)$

(e) Discuss the validity of your answers to Parts c and d.



Solutions:

(a) For the inner loop:

$$G_1(s) = \frac{\frac{1}{s^2(s+1)}}{1 + \frac{1}{s^3(s+1)}} = \frac{s}{s^4+s^3+1}$$

The equivalent open-loop transfer function $G_e(s)$ is then given by

$$G_e(s) = \frac{1}{s^2(s+3)}G_1(s) = \frac{1}{s(s^5+4s^4+3s^3+s+3)}$$

$$T(s) = \frac{G_e(s)}{1+G_e(s)} = \frac{1}{s^6+4s^5+3s^4+s^2+3s+1}$$

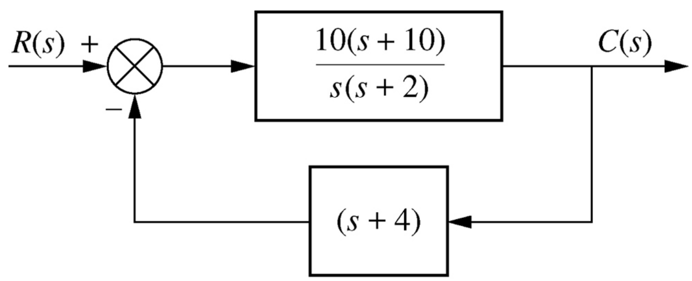
(b) From inspection of $G_e(s)$, we can say that $K_p = \infty, K_v =$ constant and $K_a = 0$. Thus the system is **Type 1**.

(c) Since the system is type 1, for $r(t) = 5u(t)$, $e_{ss} = e(\infty) = 0$.

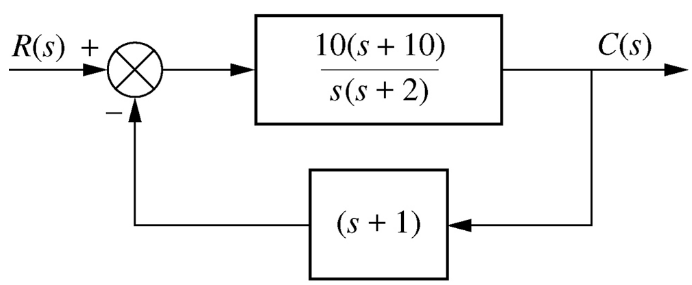
(d) From $G_e(s)$, $K_v = \lim_{s\to 0} sG_e(s) = 1/3$. Therefore, $e_{ss} = 5/K_v = 15$.

(e) Poles of closed-loop transfer function $T(s) : -3.02, -1.32, 0.34 \pm j0.77, -0.35$. Since we have a pair of complex poles in the RHP, the closed-loop system is unstable. Thus, our solutions for parts (c) and (d) are **meaningless**.

5. Steady State Error Analysis (10 points)
For each of the systems shown below, find the following
   (a) The system type
   (b) The appropriate static error constant
   (c) The input waveform to yield a constant error
   (d) The steady-state error for a unit input of the waveform found in Part c
   (e) The steady-state value of the actuating signal.



System 1



System 2

Solutions:
System 1
Let $G(s)$ be the feedforward transfer function and $H(s)$ be the feedback transfer function. Then, the equivalent open-loop transfer function with unity feedback loop, $G_e(s)$ is given by:

$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{10(s+10)}{11s^2 + 132s + 300}$$

(a) Since there are no pure integrators in $G_e(s)$, the system is **Type 0**.

(b) K_p in type 0 systems is constant.

$$K_p = \lim_{s\to 0} G_e(s) = 0.33$$

(c) Type 0 systems require a **step input** for a constant error.

(d) For $r(t) = u(t), e_{ss} = e(\infty) = \dfrac{1}{1+K_p} = 0.75$

(e) Let $e_a(t)$ be the actuation signal. Then,

$$e_a(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{1}{1+\dfrac{10(s+10)(s+4)}{s(s+2)}} = 0$$

System 2

Let $G(s)$ be the feedforward transfer function and $H(s)$ be the feedback transfer function. Then, the equivalent open-loop transfer function with unity feedback loop, $G_e(s)$ is given by:

$$G_e(s) = \frac{G(s)}{1+G(s)H(s)-G(s)} = \frac{10(s+10)}{s(11s+102)}$$

(a) Since there is one pure integrator in $G_e(s)$, the system is **Type 1**.

(b) K_v in type 1 systems is constant.

$$K_v = \lim_{s\to 0} sG_e(s) = 0.98$$

(c) Type 0 systems require a **ramp input** for a constant error.

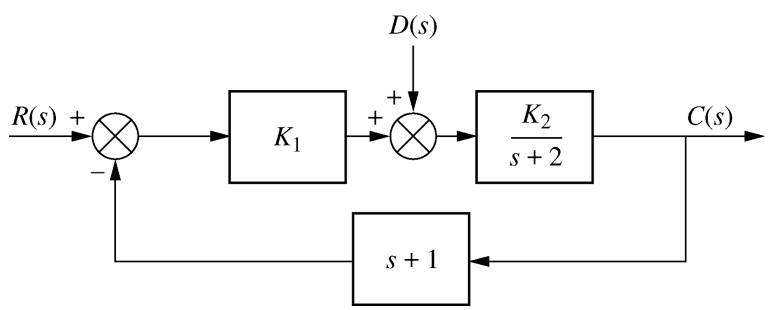
(d) For $r(t) = tu(t), e_{ss} = e(\infty) = \dfrac{1}{K_v} = 1.02$

(e) Let $e_a(t)$ be the actuation signal. Then,

$$e_a(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s\to 0} \frac{s\dfrac{1}{s^2}}{1+\dfrac{10(s+10)(s+1)}{s(s+2)}} = \frac{1}{50}$$

6. Sensitivity Of Steady State Error (3 points)

For the system shown below, find the sensitivity of the steady-state error for changes in K_1 and in K_2, when $K_1 = 100$ and $K_2 = 0.1$. Assume step inputs for both the input and the disturbance.



Solutions: Using Equation (7.70) from the textbook, we can write the steady-state error as follows:

$$e(\infty) = 1 - \lim_{s\to 0} \frac{\dfrac{K_1K_2}{s+2}}{1 + \dfrac{K_1K_2(s+1)}{s+2}} - \lim_{s\to 0} \frac{\dfrac{K_2}{s+2}}{1 + \dfrac{K_1K_2(s+1)}{s+2}} = \frac{2-K_2}{2+K_1K_2}$$

Sensitivity to K_1:

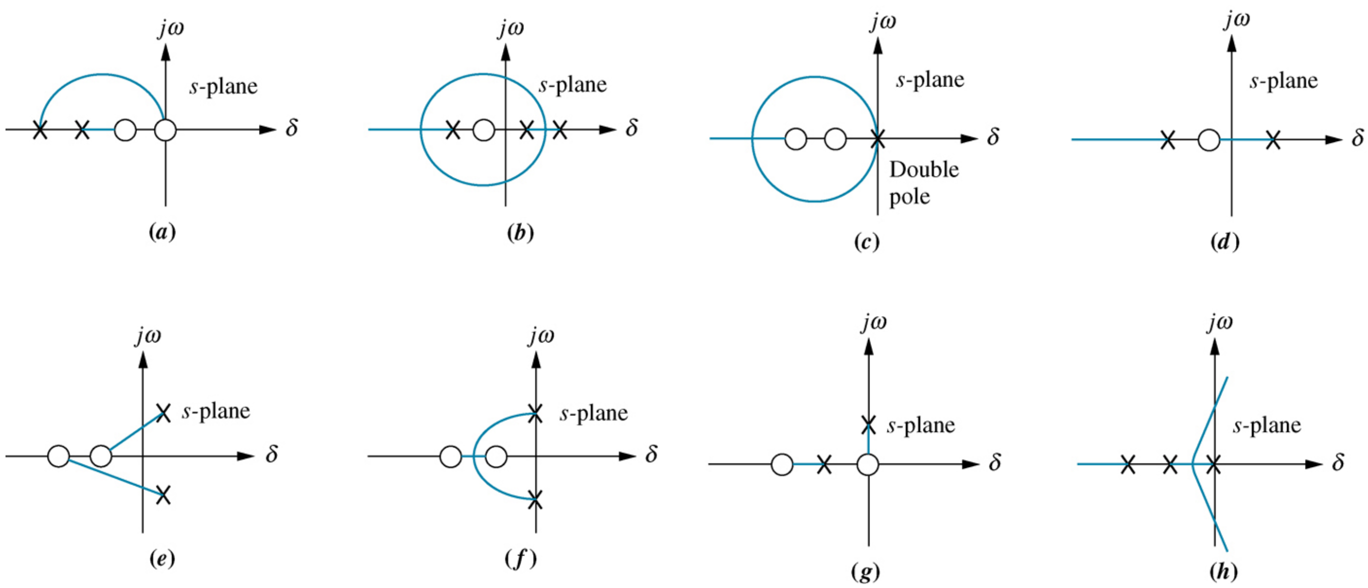
$$S_{e:K_1} = \frac{K_1}{e}\frac{\partial e}{\partial K_1} = -\frac{K_1K_2}{2+K_1K_2} = -\frac{(100)(0.1)}{2+(100)(0.1)} = -0.833$$

Sensitivity to K_2:

$$S_{e:K_2} = \frac{K_2}{e}\frac{\partial e}{\partial K_2} = \frac{2K_2(1+K_1)}{(K_2-2)(2+K_1K_2)} = \frac{2(0.1)(1+100)}{(0.1-2)(2+10)} = -0.89$$

7. Root Loci Inspection (8 points)

For each of the root loci shown below, tell whether or not the sketch can be a root locus. If the sketch cannot be a root locus, explain why. Give **all** reasons.



Solutions:

(a) No. Root Locus is always symmetric about the real axis.

(b) No. Root Locus is always to the left of an odd number of poles or zeros on the real axis.

(c) No. Root Locus is always to the left of an odd number of poles or zeros on the real axis.

(d) Yes.

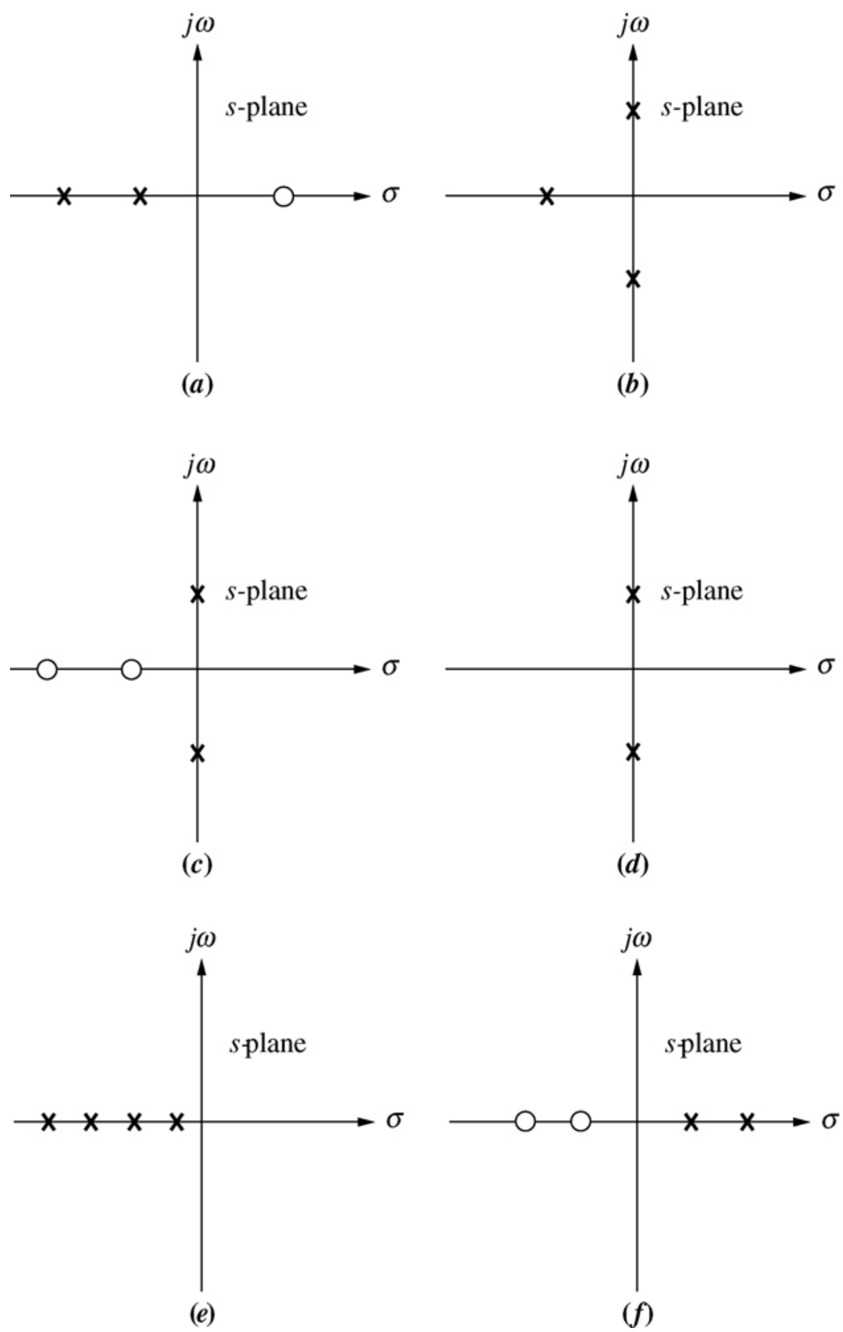
(e) No. Root Locus is always symmetric about the real axis. Root Locus is always to the left of an odd number of poles or zeros on the real axis.

(f) Yes.

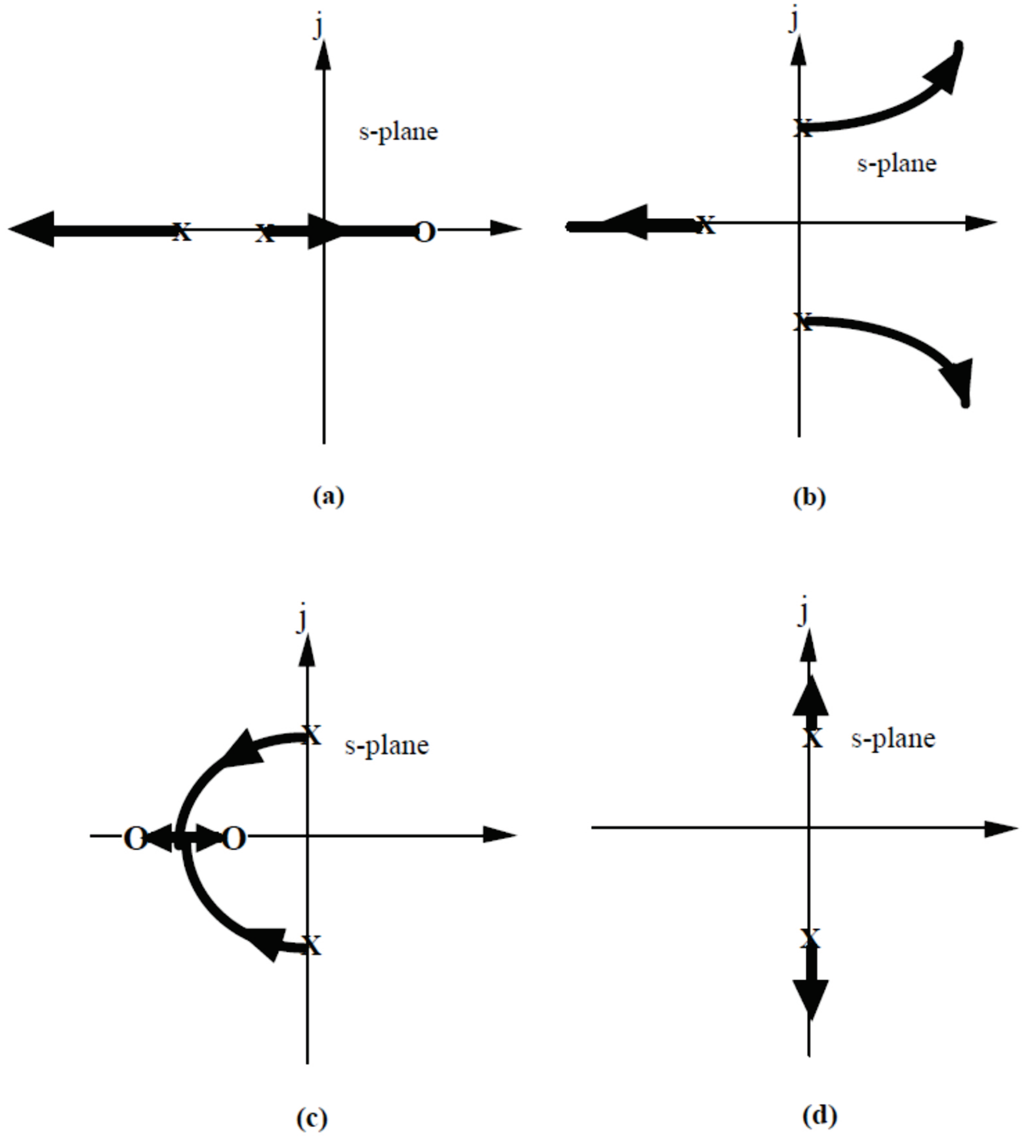
(g) No. Root Locus is always symmetric about the real axis.

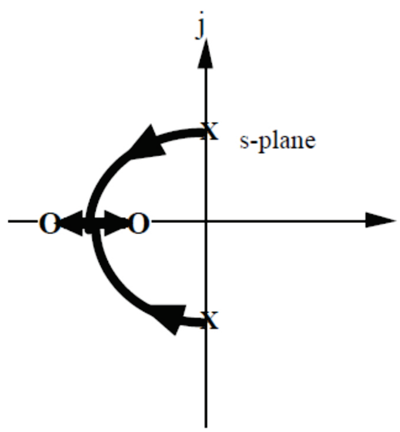
(h) Yes.

8. Sketching Root Loci (6 points)
Sketch the general shape of the root locus for each of the open-loop pole-zero plots shown below. Please print out this page and attach it with your solutions to other problems.

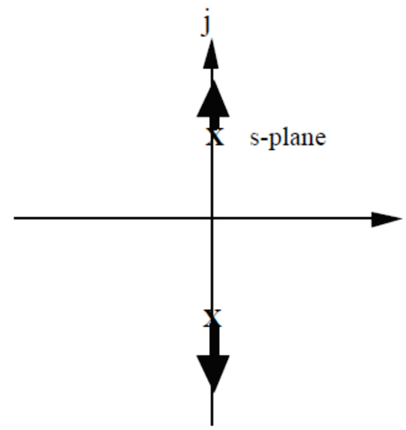


Solutions:

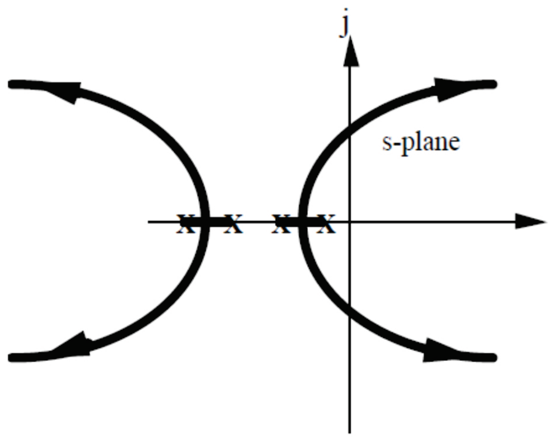




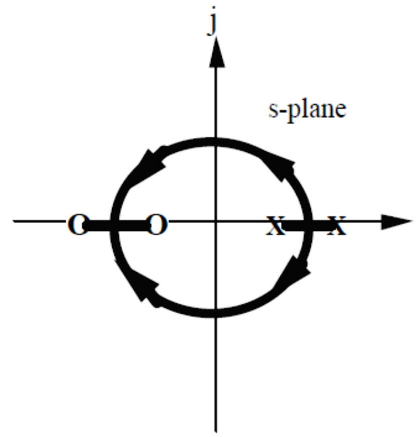
(c)



(d)



(e)



(f)