Homework for
Complex Analysis

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Most exercises are from
Functions of One Complex Variable I (2nd Edition) by Conway.
For example, “5.3.10” means exercise 10 from
section 3 of chapter 5 in Conway.
Beware: Some solutions may be incorrect!

Exercise 1. Let $1 \leq p < \infty$. Show that a closed, bounded subset $S \subseteq \ell^p(\mathbb{N})$ is compact if and only if it is equisummable in the sense that for every $\epsilon > 0$ there exists an index N for which $\sum_{k=N}^{\infty} |x_k|^p < \epsilon$ for all $x = \{x_n\} \in S$.

Proof. ($\Rightarrow$) Let $\epsilon > 0$ and cover S with the collection $\{B(x,\epsilon)\}_{x \in S}$. Then there exists $x^1, \dots, x^k$ so that $S \subseteq \bigcup_{i=1}^{k} B(x^i, \epsilon)$. Since $x^1, \dots, x^n \in \ell^p(\mathbb{N})$, then for all i we have

$$||x^i||_p^p = \sum_{n=1}^{\infty} |x_n^i|^p < \infty,$$

So for all i there exits N_i so that $\sum_{n=N_i}^{\infty} |x_n^i|^p < \epsilon$. Define $N := \max_i\{N_i\}$. Now let $y = \{y_n\} \in S$. Then $y \in B(x^i, \epsilon)$ for some i and then by the triangle inequality,

$$\left(\sum_{n=N}^{\infty} |y_n|^p\right)^{1/p} \leq \left(\sum_{n=N}^{\infty} |y_n|^p\right)^{1/p} + \left(\sum_{n=N}^{\infty} |x_n^i|^p\right)^{1/p} < \epsilon + \epsilon^p$$

and so $\sum_{n=N}^{\infty} |y_n|^p < (\epsilon + \epsilon^p)^p$.

($\Leftarrow$) Let x^ℓ be a point in S, and let x_m^ℓ denote its m-th term. Now, let $\{x^{\boldsymbol{n}}\} = \{x^1, x^2, \dots\}$ be a sequence in S. For $\epsilon > 0$, since S is equisummable, we have an N so that for all x^j in the sequence $\{x^{\boldsymbol{n}}\}$,

$$\sum_{k=N}^{\infty} |x_k^j|^p < \epsilon,$$

which gives that for any particular term x_a^j in x^j for $a \geq N$,

$$|x_a^j| \leq \sum_{k=N}^{\infty} |x_k^j|^p < \epsilon.$$

In other words, each term x_a^j of the sequence x^j is bounded. So, when we consider the collection of a-th terms over all j, $\{x_a^1, x_a^2, \dots\}$ we have a bounded sequence! For convenience, suppose $N = 1$.

Now, consider the collection of "first terms" $\{x_1^1, x_1^2, x_1^3, \dots\}$. By the argument above, this collection (sequence) is bounded, and therefore has a convergent subsequence, $\{x_1^{s(1,n)}\}_{n=1}^{\infty} \to a_1$. Now consider the collection of second terms $\{x_2^{s(1,n)}\}_{n=1}^{\infty}$. Again, this sequence is bounded and therefore has a convergent subsequence, $\{x_2^{s(2,n)}\} \to a_2$ where the indices $s(2,n) \subseteq s(1,n)$. Continuing in this way, once we have the j-th subsequence for the j-th terms constructed, $\{x_j^{s(j,n)}\}_{n=1}^{\infty} \to a_j$, we consider the collection $\{x_{j+1}^{s(j,n)}\}_{n=1}^{\infty}$, which is bounded and therefore has a convergent subsequence $\{x_{j+1}^{s(j+1,n)}\} \to a_{j+1}$ where $s(j+1,n) \subseteq s(j,n)$. Setting $n_k := s(k,k)$, we get for all j

$$\lim_{k \to \infty} x_j^{n_k} = a_j,$$

and so the subsequence $\{x_1^n, x_2^n, \dots\}$ of $\{x^{\boldsymbol{n}}\}$ converges pointwise to the sequence $\{a_j\}_{j=1}^{\infty}$.

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Exercise 2. Show that $z_1, z_2, z_3 \in \mathbb{C}$ are colinear if and only if $\text{Im}(z_1\bar{z_2} + z_2\bar{z_3} + z_3\bar{z_1}) = 0$.

Proof. We assume from the outset that z_1, z_2, and z_3 are distinct.

($\Rightarrow$) Suppose $z_1, z_2, z_3 \in \mathbb{C}$ are colinear. Then, there exists a $t \in \mathbb{R}$ such that $z_3 = z_1 t + (1-t)z_2$. Let $z_j = a_j + ib_j$ for $j = 1, 2, 3$. Then

$$a_3 = ta - 1 + a_2 - ta_2 \quad \text{and} \quad b_3 = tb_1 + b_2 - tb_2.$$

For $k \neq j$, we have $\text{Im}(z_k\overline{z_j}) = a_j b_k - a_k b_j$. So

$$\begin{aligned}\text{Im}(z_1\bar{z_2} + z_2\bar{z_3} + z_3\bar{z_1}) &= [a_2b_1 - a_1b_2] + [(ta_1 + a_2 - ta_2)b_2 - a_2(tb_1 + b_2 - tb_2)] \\ &\quad + [a_1(tb_1 + b_2 - tb_2) - (ta_1 + a_2 - ta_2)b_1] \\ &= a_1(-b_2 + tb_2 + tb_1 + b_2 - tb_2 - tb_1) \\ &\quad + a_2(b_1 + b_2 - tb_2 - tb_1 - b_2 + tb_2 - b_1 + tb_1) \\ &= 0\end{aligned}$$

($\Leftarrow$) We have

$$0 = a_1(b_3 - b_2) + a_2(b_1 - b_3) + a_3(b_2 - b_1). \tag{$*$}$$

Notice that if $a_1 = a_2$, then ($*$) gives $a_1(b_2 - b_1) = a_3(b_2 - b_1)$ and so $a_1 = a_2 = a_3$ and the vertical line through a_1 contains z_1, z_2, z_3. Similarly, if $b_1 = b_2$ then $b_1 = b_2 = b_3$ and the horizontal line through b_1 contains z_1, z_2, z_3. So, assume $a_1 \neq a_2$ and $b_1, \neq b_2$.

By ($*$), we get

$$\frac{a_3 - a_2}{a_1 - a_2} = \frac{b_3 - b_2}{b_1 - b_2}. \tag{$**$}$$

We claim that if t equals the real number in ($**$), then $z_3 = z_1 t + (1-t)z_2$ and we are done. Note that

$$1 - t = \frac{a_1 - a_3}{a_1 - a_2} = \frac{b_1 - b_3}{b_1 - b_2}.$$

Then

$$\begin{aligned}z_1 t + (1-t)z_2 &= \left[a_1\left(\frac{a_3 - a_2}{a_1 - a_2}\right) + ib_1\left(\frac{b_3 - b_2}{b_1 - b_2}\right)\right] + \left[a_2\left(\frac{a_1 - a_3}{a_1 - a_2}\right) + ib_2\left(\frac{b_1 - b_3}{b_1 - b_2}\right)\right] \\ &= \left[\frac{a_1a_3 - a_1a_2 + a_1a_2 - a_2a_3}{a_1 - a_2}\right] + i\left[\frac{b_1b_3 - b_1b_2 + b_1b_2 - b_2b_3}{b_1 - b_2}\right] \\ &= \left[\frac{(a_1 - a_2)a_3}{a_1 - a_2}\right] + i\left[\frac{(b_1 - b_2)b_3}{b_1 - b_2}\right] \\ &= z_3.\end{aligned}$$

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§1.4, #4 Use the binomial equation

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

and compare the real and imaginary parts of each side of de Moivre's formula to obtain the formulas:

$$\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \dots$$
$$\sin n\theta = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \dots$$

Proof. Let $z = a + ib = r \operatorname{cis} \theta$. Then by De Moivre's formula, $z^n = (a+ib)^n = r^n \operatorname{cis} n\theta$. Using the binomial equation and the fact that $a = r\cos\theta$ and $b = r \sin\theta$,

$$\begin{aligned}
r(\cos n\theta + i \sin n\theta) = (a+ib)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (ib)^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (r\cos\theta)^{n-k} i^k (r \sin\theta)^k \\
&= r \sum_{k=0}^{n} \binom{n}{k} i^k \cos^{n-k}\theta \sin^k \theta \\
&= r \Bigg[\cos^n \theta + \binom{n}{1} i \cos^{n-1} \sin\theta - \binom{n}{2} \cos^{n-1} \sin^2\theta \\
&\qquad + \dots + \binom{n}{n-1} i^{n-1} \cos\theta \sin^{n-1}\theta + i^n \sin^n \theta \Bigg] \\
&= r \Bigg\{ \Bigg[\cos^n \theta - \binom{n}{2} \cos^{n-1} \sin^2 \theta + \binom{n}{4} \cos^{n-4} \sin^4 \theta - \dots \Bigg] \\
&\qquad + i \Bigg[\binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \dots \Bigg] . \Bigg\}
\end{aligned}$$

Comparing the real and imaginary parts, we obtain the desired formulas. ☕

§1.4, #7 If $z \in \mathbb{C}$ and $\operatorname{Re}(z^n) \geq 0$ for every positive integer n, show that z is a positive real number.

Proof. Let $z = r \operatorname{cis} \theta$. By De Moivre's formula, $z^n = r \operatorname{cis} \theta$. We have $0 \leq \operatorname{Re}(z^n) = r^n \cos n\theta$ for all n. This implies $r^n \geq 0$ and $\cos n\theta \geq 0$. Working modulo 2π, the latter gives $\frac{-\pi}{2} \leq n\theta \leq \frac{\pi}{2}$ and so $\frac{-\pi}{(2n)} \leq \theta \leq \frac{\pi}{(2n)}$. Since this is true for all n, when $n \to \infty$, $\theta \to 0$. So, $\theta = 0$, which means $z = r \operatorname{cis} \theta = r \geq 0$. ☕

§3.1, #6 Find the radius of convergence for each of the following power series:

$$(a)\sum_{n=0}^{\infty} a^n z^n, a \in \mathbb{C}; \quad (b)\sum_{n=0}^{\infty} a^{n^2} z^n, a \in \mathbb{C}; \quad (c)\sum_{n=0}^{\infty} k^n z^n, k \in \mathbb{Z}; \quad (d)\sum_{n=0}^{\infty} z^{n!}.$$

Proof. (a) $\limsup |a_n|^{1/n} = \limsup |a| = |a| \implies R = 1/|a|$ is $|a| \neq 0$ and $\mathbb{R} = \infty$ if $|a| = 0$.

(b) For fixed n,

$$\sup_{k \geq n}\left\{\left|a^{k^2}\right|^{1/k}\right\} = \sup_{k\geq n}\{|a|^k\} = \begin{cases} 0 & \text{if } |a| = 0 \\ 1 & \text{if } |a| = 1 \\ |a|^n & \text{if } 0 < |a| < 1 \\ \infty & \text{if } |a| > 1, \end{cases}$$

which gives

$$\frac{1}{R} = \limsup\{|a|^k\} = \begin{cases} 0 & \text{if } |a| = 0 \\ 1 & \text{if } |a| = 1 \\ 0 & \text{if } 0 < |a| < 1 \\ \infty & \text{if } |a| > 1, \end{cases} \implies R = \begin{cases} \infty & \text{if } |a| = 0 \\ 1 & \text{if } |a| = 1 \\ \infty & \text{if } 0 < |a| < 1 \\ 0 & \text{if } |a| > 1, \end{cases} = \begin{cases} \infty & \text{if } |a| < 1 \\ 1 & \text{if } |a| = 1 \\ 0 & \text{if } |a| > 1. \end{cases}$$

(c) We have $\sup_{\ell \geq n}\{|k^\ell|^{1/\ell}\} = |k|$ and so $\limsup\{|k|\} = |k|$ which implies $R = 1/|k|$.

(d)

$$\begin{aligned}\sum_{n=0}^{\infty} z^{n!} &= z^{0!} + z^{1!} + z^{2!} + z^{3!} + z^{4!} + \dots \\ &= z^1 + z^1 + z^2 + z^6 + z^2 4 + \dots \\ &= (0)z^0 + 2(z) + 1(z^2) + (0)z^3 + (0)z^4 + (0)z^5 + (1)z^6 + \dots \\ &= \sum_{n=1}^{\infty} a_n z^n\end{aligned}$$

where

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 1 & \text{if } n = k! \text{ for some } k \in \mathbb{N}_{\geq 1} \\ 0 & \text{otherwise .} \end{cases}$$

Then for $n > 1$,

$$\sup_{k\geq n}\{|a_k|^{1/k}\} = \sup_{k \geq n}\{1^{1/k}\} = 1 \implies \limsup\{|a_k|^{1/k}\} = 1 \implies R = 1.$$

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§3.1, #7 Show that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss convergence for $z = 1, -1$, and i

Proof. Define

$$a_n := \begin{cases} \dfrac{(-1)^k}{k} & \text{if } n = k(k+1) \text{ f or some } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{2}{-1+\sqrt{1+4n}} \cdot (-1)^{\frac{-1+\sqrt{1+4n}}{2}} & \text{if } n = k(k+1) \text{ f or some } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise.} \end{cases}$$

Then the series $\sum_{n=0}^{\infty} a_n z^n$ is equivalent to the one in question. Notice that for nonzero a_n,

$$|a_n|^{1/n} = \left| \frac{2}{-1+\sqrt{1+4n}} \cdot (-1)^{\frac{-1+\sqrt{1+4n}}{2}} \right|^{1/n} = \left(\frac{2}{|-1+\sqrt{1+4n}|} \right)^{1/n}$$

Now

$$\lim_{n\to\infty} \ln \left(\left(\frac{2}{-1+\sqrt{1+4n}} \right)^{1/n} \right) = \lim_{n\to\infty} \frac{\ln\left(\frac{2}{-1+\sqrt{1+4n}}\right)}{n}$$
$$= \lim_{n\to\infty} \frac{\ln(2)}{n} - \lim_{n\to\infty} \frac{\ln(-1+\sqrt{1+4n})}{n}$$
$$= 0,$$

and so

$$\lim_{n\to\infty} |a_n|^{1/n} = e^{\lim_{n\to\infty} \ln \left(\left(\frac{2}{-1+\sqrt{1+4n}} \right)^{1/n} \right)} = e^0 = 1.$$

Thus the radius of convergence is 1! If $z = 1$, then we have the alternating series, which converges. If $z = -1$ the power $n(n+1)$ will always be even, so we get the same series as the case $z = 1$. When $z = i$ we again note that we have even powers on i, and so $i^{n(n+1)}$ is either 1 or -1. ☕

Exercise 3.2.2*. Prove that if b_n, a_n are real an positive and $0 < b = \lim b_n$, $a = \limsup a_n$, then $ab = \limsup(a_n b_n)$. Does this remain true if the requirement of positivity is dropped?

Proof. Fix $n \in \mathbb{N}$ and let $k \geq n$. Then

$$a_k \leq \sup_{\ell \geq n}\{a_\ell\} \quad \text{and} \quad b_k \leq \sup_{\ell \geq n}\{b_\ell\}.$$

So

$$a_k b_k \leq \sup_{\ell \geq n}\{a_\ell\} \sup_{\ell \geq n}\{b_\ell\},$$

and so

$$\sup_{k \geq n}\{a_k b_k\} \leq \sup_{\ell \geq n}\{a_\ell\} \sup_{\ell \geq n}\{b_\ell\},$$

which gives

$$\limsup(a_n b_n) \leq \limsup(a_n) \limsup(b_n) = ab.$$

Conversely, we can pick a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ which converges to a. Then $\{a_{n_k} b_{n_k}\}$ converges to ab, and since $\limsup(a_n b_n)$ is the *largest* subsequential limit of $\{a_n b_n\}$, then $ab \leq \limsup(a_n b_n)$.

Exercise 3.2.3. Show that $\lim n^{1/n} = 1$.

Proof. Since the linear function n grows faster than $\ln n$, we get

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0, \implies \lim_{n\to\infty} n^{1/n} = e^{\lim \frac{\ln n}{n}} = e^0 = 1.$$

Exercise 3.2.4*. Show that $(\cos z)' = -\sin z$ and $(\sin z)' = \cos z$.

Proof. Using the power series of $\cos z$ and $\sin z$ and applying Proposition 2.5,

$$\begin{aligned}
(\cos z)' = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} z^{2n}\right)' = \sum_{n=1}^{\infty} (2n) \frac{(-1)^n}{2n!} z^{2n-1} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} z^{2n-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2(n+1)-1)!} z^{2(n+1)-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(-1)}{(2n+1))!} z^{2n+1} \\
&= -\sin z
\end{aligned}$$

$$\begin{aligned}
(\sin z)' = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} z^{2n-1}\right)' = \sum_{n=1}^{\infty} (2n-1) \frac{(-1)^{n-1}}{(2n-1)!} z^{2n-2} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-2)!} z^{2n-2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{(n+1)-1}}{(2(n+1)-2)!} z^{2(n+1)-2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} z^{2n} \\
&= \cos z.
\end{aligned}$$

Exercise 3.2.10. Let G and Ω be open in $\mathbb{C}$ and suppose f and h are functions defined on G, $g : \Omega \to \mathbb{C}$ and suppose that $f(G) \subset \Omega$. Suppose that g and h are analytic, $g'(\omega) \neq 0$ for any ω, that f is continuous, h is one-to-one, and that they satisfy $h(z) = g(f(z))$ for $z \in G$. Show that f is analytic. Give a formula for $f'(z)$.

Proof. We have

$$\lim_{\omega \to z} \frac{f(\omega) - f(z)}{\omega - z} = \lim_{\omega \to z} \frac{f(\omega) - f(z)}{g(f(\omega)) - g(f(z))} \frac{g(f(z)) - g(f(\omega))}{z - \omega} = \frac{1}{(h'(z))} \cdot g'(f(z)),$$

and thus f is differentiable. Since h and g are analytic, so is f. Moreover,

$$f'(z) = \frac{1}{(h'(z))} \cdot g'(f(z)).$$

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Exercise 3.2.11. Suppose that $f : G \to \mathbb{C}$ is a branch of the logarithm and that n is an integer. Prove that $z^n = e^{nf(z)}$ for all $z \in G$.

Proof. Since $e^{f(z)} = z$, then $e^{nf(z)} = (e^{f(z)})^n = z^n$ for all $z \in G$. ☕

Exercise 3.2.13*. Let $G = \mathbb{C} - \{z \in \mathbb{R} \mid z \leq 0\}$ and let n be a positive integer. Find all analytic functions $f : G \to \mathbb{C}$ such that $z = (f(z))^n$ for all $z \in G$.

Proof. Notice that for a function f to satisfy $z = (f(z))^n$, that would imply $z^{1/n} = f(z)$. Then all analytic functions f with the required property are of the form

$$f(z) = e^{\log f(z)} = e^{\log z^{1/n}} = e^{1/n \log z} = e^{1/n(|z| + i(\arg(z) + 2\pi k))} = e^{1/n|z|} e^{(i \arg(z) + 2\pi k)/(n)}$$

for $k \in \mathbb{Z}$. These functions are analytic since they are the product of the composition of analytic functions. For each $k \in \mathbb{Z}$, define

$$f_k(z) = e^{1/n|z|} e^{i \arg(z)/n} e^{2\pi k/n}.$$

Then if $k \equiv m \mod n$, then $f_k = f_m$ since $e^{2\pi k/n} = e^{2\pi m/n}$. So, we have exactly n analytic functions satisfying the given property. ☕

Exercise 3.2.18*. Let $f : G \to \mathbb{C}$ and $g : G \to \mathbb{C}$ be branches of z^a and z^b, respectively. Show that fg is a branch of z^{a+b} and f/g is a branch of z^{a-b}. Suppose that $f(G) \subset G$ and $g(G) \subset G$ and prove that both $f \circ g$ and $g \circ f$ are branches of z^{ab}.

Proof. Let $f(z) = e^{a(p(z))}$ and $g(z) = e^{b(q(z))}$ where $p(z), q(z)$ are branches of the logarithm. As such, there exists $k \in \mathbb{Z}$ so that $p(z) = q(z) + 2\pi ik$. So

$$f(z)g(z) = e^{a(q(z)+2\pi ik)+bq(z)} = e^{aq(z)+a2\pi ik+bq(z)} = e^{(a+b)q(z)}(e^{2\pi i})^{ak} = e^{(a+b)q(z)},$$

and thus fg is a branch of z^{a+b}. Similarly,

$$f(z)/g(z) = e^{a(q(z)+2\pi ik)-bq(z)} = e^{aq(z)+a2\pi ik-bq(z)} = e^{(a-b)q(z)}(e^{2\pi i})^{ak} = e^{(a-b)q(z)}.$$

Let $\text{Log}(z)$ be the principal branch of the logarithm. Then there exist $m, n \in \mathbb{Z}$ so that $p(z) = \text{Log}(z) + 2\pi im$ and $q(z) = \text{Log}(z) + 2\pi in$. Then

$$\begin{aligned}(f \circ g)(z) = e^{ap\left(e^{b(q(z))}\right)} &= e^{a\,\text{Log}\left(e^{b(q(z))}\right)+2\pi im} \\ &= e^{a\,\text{Log}\left(e^{b(q(z))}\right)}e^{2\pi im} \\ &= e^{ab\ q(z)},\end{aligned}$$

and similarly,

$$\begin{aligned}(g \circ f)(z) = e^{bq\left(e^{a(p(z))}\right)} &= e^{b\,\text{Log}\left(e^{a(p(z))}\right)+2\pi in} \\ &= e^{b\,\text{Log}\left(e^{a(p(z))}\right)}e^{2\pi in} \\ &= e^{ba\ p(z)}.\end{aligned}$$

Thus $f \circ g$ and $g \circ f$ are branches of z^{ab}. ☕

Exercise 3.2.21*. Prove that there is no branch of the logarithm defined on $G = \mathbb{C} - \{z \in \mathbb{R} \mid z \leq 0\}$. (Hint: Suppose such a branch exists and compare this with the principal branch.)

Proof. Suppose such a branch exists and call it $f(x)$. If $h(z) = \ln(|z|) + i\theta, -\pi < \theta < \pi$ is the principal branch, then there is a $k \in \mathbb{Z}$ so that $f(z) = h(z) + 2\pi ik$. Let $\theta_n = \pi - 1/n$ and $\theta_k = -\pi + 1/n$. Then $\{\theta_n\} \to \pi$ and $\{\theta_k\} \to -\pi$, and so

$$\{e^{i\theta_n}\} \to -1 \quad \text{and} \quad \{e^{i\theta_k}\} \to -1.$$

Since f is a branch of the logarithm, it must be continuous. So,

$$\lim_{n\to\infty} f\left(e^{i\theta_n}\right) = f(-1) = \lim_{n\to\infty} f\left(e^{i\theta_k}\right).$$

So,

$$\begin{aligned}\lim_{n\to\infty} f\left(e^{i\theta_n}\right) &= \lim_{n\to\infty} h\left(e^{i\theta_n}\right) + 2\pi ik \\ &= \lim_{n\to\infty}[\ln(1) + i\theta_n] + 2\pi ik \\ &= \lim_{n\to\infty} i(\pi - 1/n) + 2\pi ik \\ &= i\pi + 2\pi ik.\end{aligned}$$

However, by a similar computation, we get $\lim_{n\to\infty} f\left(e^{i\theta_k}\right) = -i\pi + 2\pi ik$. Together, these imply $-i\pi = i\pi$, which means $-i = 0$. Contradiction! ☕

Exercise 3.3.1. Find the image of $\{z : \operatorname{Re} z < 0, |\operatorname{Im} z| < \pi\}$ under the exponential function.

Proof. Let $z = a + ib$ be an element of the set described. Since the exponential is never 0, $e^z \neq 0$. Because $a < 0$, then $|e^z| = e^a < 1$ since the real exponential is increasing and $e^a < e^0 = 1$. Hence e^z will be strictly contained in the unit disk. Since $|\arg(e^z)| = |\operatorname{Im} z| = |b| < \pi$, then $e^z \notin \mathbb{R}^-$. Therefore the image of $\{z : \operatorname{Re} z < 0, |\operatorname{Im} z| < \pi\}$ under the exponential function is the open unit disk minus the nonpositive real axis. That is,

$$\{w \in \mathbb{C} - \{0\} : |w| < 1, w \notin \mathbb{R}^-\}.$$

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Exercise 3.3.7. If $Tz = \dfrac{az+b}{cz+d}$, find z_2, z_3, z_4 (in terms of a, b, c, d) such that $Tz = (z, z_2, z_3, z_4)$.

Proof. We want that $Tz_2 = 1, Tz_3 = 0$, and $Tz_4 = \infty$. So

$$\begin{aligned} 1 &= \frac{az_2+b}{cz_2+d} \implies z_2 = \frac{d-b}{a-c} \\ 0 &= \frac{az_3+b}{cz_3+d} \implies z_3 = -b/a \\ \infty &= \frac{az_4+b}{cz_4+d} \implies z_4 = -d/c \end{aligned}$$

If $a = c$, then $z_2 = \infty$. If $a = 0$, then $z_3 = \infty$. If $c = 0$, then $z_4 = \infty$. ☕

Exercise 3.3.9. If $Tz = \dfrac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$, where Γ is the unit circle $\{z : |z| = 1\}$.

Proof. Let $z \in \Gamma$ and suppose $T(z) \in \Gamma$, i.e., $T(z)\overline{T(z)} = 1$. Then

$$1 = \frac{az+b}{cz+d}\frac{\overline{az+b}}{\overline{cz+d}}.$$

Simplifying this expression, we get

$$0 = z\overline{z}(a\overline{a} - c\overline{c}) + z(a\overline{b} - c\overline{d}) + \overline{z}(b\overline{a} + d\overline{c}) + b\overline{b} - d\overline{d},$$

Now since $z\overline{z} = 1$,

$$z\overline{z} - 1 = z\overline{z}(a\overline{a} - c\overline{c}) + z(a\overline{b} - c\overline{d}) + \overline{z}(b\overline{a} + d\overline{c}) + b\overline{b} - d\overline{d}.$$

Comparing coefficients, we get the following conditions:

$$a\overline{b} - c\overline{d} = 0 \quad \text{and} \quad |a|^2 + |b|^2 = |c|^2 + |d|^2 \tag{$*$}$$

Hence if we want $T(\Gamma) = \Gamma$, then it is necessary that these conditions hold.

Conversely, suppose the equations in $(*)$ hold. Since T sends a circle to a circle, and a circle is determined by three points (as is T), we show $|T(1)|^2 = |T(-1)|^2 = |T(i)|^2 = 1$. Then we can conclude $T(\Gamma) = \Gamma$. So,

$$\begin{aligned}1 = |T(1)|^2 &\iff |a+b|^2 = |c+d|^2 \\ &\iff |a|^2 + 2\operatorname{Re}(a\bar{b}) + |b|^2 = |c|^2 + 2\operatorname{Re}(c\bar{d}) + |d|^2\end{aligned}$$

$$\begin{aligned}1 = |T(-1)|^2 &\iff |-a+b|^2 = |-c+d|^2 \\ &\iff |-a|^2 + 2\operatorname{Re}(-a\bar{b}) + |b|^2 = |-c|^2 + 2\operatorname{Re}(-c\bar{d}) + |d|^2 \\ &\iff |a|^2 - 2\operatorname{Re}(a\bar{b}) + |b|^2 = |c|^2 - 2\operatorname{Re}(c\bar{d}) + |d|^2\end{aligned}$$

Combining these, we get $\operatorname{Re}(a\bar{b}) = \operatorname{Re}(c\bar{d})$. Then,

$$\begin{aligned}1 = |T(i)|^2 &\iff |ai+b|^2 = |ci+d|^2 \\ &\iff |ai|^2 + 2\operatorname{Re}(ai\bar{b}) + |b|^2 = |c|^2 + 2\operatorname{Re}(ci\bar{d}) + |d|^2 \\ &\iff \operatorname{Re}(ia\bar{b}) = \operatorname{Re}(ic\bar{d}) && \text{(by } (*)) \\ &\iff -\operatorname{Im}(a\bar{b}) = -\operatorname{Im}(c\bar{d}) \\ &\iff \operatorname{Im}(a\bar{b}) = \operatorname{Im}(c\bar{d}).\end{aligned}$$

Therefore,

$$\operatorname{Re}(a\bar{b}) = \operatorname{Re}(c\bar{d}) \quad \text{and} \quad \operatorname{Im}(a\bar{b}) = \operatorname{Im}(c\bar{d})$$

gives $a\bar{b} = c\bar{d}$. Now, applying this to the first computation $1 = |T(1)|^2$, we get that

$$|a|^2 + |b|^2 = |c|^2 + |d|^2.$$

☕

Exercise 3.3.17. Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is constant.

Proof. Let $z_0 \in G$ so that $f'(z_0)$ and $B \subseteq G$ be an open ball around z_0. Pick two points $z_1, z_2 \in B$ so that if

$$\ell_1(t) = (z_0 + (z_1 - z_0)t) \quad \text{and} \quad \ell_2(t) = (z_0 + (z_2 - z_0)t)$$

are equations of the lines joining z_0 with z_1 and z_0 with z_2, respectively, then

$$\arg(z_1 - z_0) - \arg(z_2 - z_0) = \arg \ell_1'(0) - \arg \ell_2'(0) = \frac{\pi}{2}.$$

Since f is analytic, it is angle preserving, and so

$$\frac{\pi}{2} = \arg f'(\ell_1(0))\ell_1'(0) - \arg f'(\ell_2(0))\ell_2'(0).$$

However, since f maps the paths ℓ_1 and ℓ_2 in the circle, the vectors $f'(\ell_1(0))$ and $f'(\ell_2(0))$ must either be identical or pointing in opposite directions. In other words, the vectors will have angle 0 or angle π between them. Thus, $f' = 0$ on G, i.e., f is constant. ☕

Exercise 3.3.27. Prove that the group $\mathcal{M}$ of Möbius transformations is a simple group.

Proof. Define a group homomorphism

$$\varphi : GL_2(\mathbb{C}) \to \mathcal{M} \quad \text{by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}.$$

Then φ is certainly surjective with

$$\ker \varphi = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}.$$

Then by the first isomorphism theorem, $GL_2(\mathbb{C}) / \ker \varphi \cong \mathcal{M}$. Moreover, $GL_2(\mathbb{C}) / \ker \varphi \cong PSL_2(\mathbb{C})$, which is simple[1], and hence so is $\mathcal{M}$. ☕

[1] According to Frauke Bleher, this is difficult to prove, and requires a good amount of advanced algebra.

Exercise 4.1.9. Define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = e^{int}$ where $n \in \mathbb{Z}$. Show that $\int_\gamma \frac{1}{z} dz = 2\pi in$.

Solution:

$$\int_\gamma \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{int}} d\gamma = \int_0^{2\pi} \frac{ine^{int}}{e^{int}} dt = \int_0^{2\pi} in \; dt = 2\pi in.$$

Exercise 4.1.20. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$ and find $\int_\gamma (z^2 - 1)^{-1} dz$.

Solution:
If $f(z) = 1/(z^2 - 1)$, then

$$f(z) = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right] \quad \text{and} \quad f(\gamma(t)) = \frac{1}{2}\left[\frac{1}{e^{it}} - \frac{1}{2+e^{it}}\right].$$

Letting $Log(z)$ be the principal log defined on the set $G = \mathbb{C} - \{z \in \mathbb{R} \mid z \leq 0\}$,

$$\int_0^{2\pi} f(\gamma(t))\gamma'(t) dt = \frac{1}{2}\left[\int_0^{2\pi} i \; dt - \int_0^{2\pi} \frac{ie^{it}}{2+e^{it}} dt\right] = \frac{1}{2}\left[2\pi i - Log(2+e^{it})|_0^{2\pi}\right] = \pi i.$$

Exercise 4.1.23. Let γ be a closed rectifiable curve in an open set G and $a \notin G$. Show that for $n \geq 2$, $\int_\gamma (z-a)^{-n} dz = 0$.

Proof. Since $\gamma : [a, b] \to \mathbb{C}$ is closed, $\gamma(a) = \gamma(b)$. Letting $f(z) = (z-a)^{-n}$, since $a \notin G$, then f and

$$F(z) = \frac{(z-a)^{-n+1}}{-n+1}$$

are defined and continuous on G, and $F' = f$. So $\int_\gamma f(z) dz = F(\beta) - F(\alpha) = 0$. ☕

Exercise 4.2.4. (a) Prove Abel's Theorem: Let $\sum a_n(z-a)^n$ have radius of convergence 1 and suppose that $\sum a_n$ converges to A. Prove that

$$\lim_{r\to 1^-} \sum a_n r^n = A.$$

(b) Use Abel's Theorem to prove that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots$.

Proof. *** The following proof belongs to Alexander Bates***
We may assume that $a = 0$ and $A = 0$. Define $s_k = \sum_{n=0}^{k} a_n$ and $s_{-1} := 0$. Notice that $a_n = s_n - s_{n-1}$. Furthermore, $\lim_{k\to\infty} s_k = \lim_{k\to\infty}\sum_{n=0}^{k} a_n = \sum_{n=0}^{\infty} a_n = A = 0$. Define $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Letting $z \in \{z \in \mathbb{R} \mid 0 < z < 1\}$, we have:

$$\begin{aligned}
f(z) = \sum_{n=0}^{\infty} z^n a_n = \lim_{k\to\infty}\sum_{n=0}^{k} z^n a_n &= \lim_{k\to\infty}\sum_{n=1}^{k} z^n (s_n - s_{n-1}) \\
&= \lim_{k\to\infty}\left(z^{k+1}s_k - z^0 s_{-1} - \sum_{n=1}^{k} s_n(z^{n+1} - z^n)\right) \\
&= \lim_{k\to\infty}\left(z^{k+1}s_k - \sum_{n=1}^{k} s_n(z^{n+1} - z^n)\right) \\
&= \lim_{k\to\infty} z^{k+1}s_k - \lim_{k\to\infty}\sum_{n=1}^{k} s_n(z^{n+1} - z^n) \\
&= \lim_{k\to\infty}\sum_{n=1}^{k} s_n(z^n - z^{n+1}) \\
&= (1-z)\lim_{k\to\infty}\sum_{n=1}^{k} s_n z^n = (1-z)\sum_{n=1}^{\infty} s_n z^n.
\end{aligned}$$

Letting $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $|s_n| < \epsilon/2$ for all $n \geq N$. For real r with $0 < r < 1$,

$$\begin{aligned}
|f(r)| &\leq (1-r)\left(\left|\sum_{n=1}^{N-1} s_n r^n\right| + \sum_{n=N}^{\infty} |s_n||r^n|\right) \\
&< (1-r)\left(\left|\sum_{n=1}^{N-1} s_n r^n\right| + \sum_{n=N}^{\infty} \frac{\epsilon}{2} r^n\right) \\
&= (1-r)\left(\left|\sum_{n=1}^{N-1} s_n r^n\right| + \frac{\epsilon}{2}\frac{r^N}{1-r}\right) \\
&\leq (1-r)\left|\sum_{n=1}^{N-1} s_n r^n\right| + \frac{\epsilon}{2}.
\end{aligned}$$

If $\left|\sum_{n=1}^{N-1} s_n r^n\right| < \epsilon/2$ then we are done. Otherwise, pick $r \in \mathbb{R}$ with $0 < r < 1$ so that $1 - r < \frac{\epsilon}{2\left|\sum_{n=1}^{N-1} s_n r^n\right|}$. Then $|f(r)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence, $\lim_{r\to 1^-} f(r) = 0$.

We have $\log(1+z) = \sum_{n=0}^{\infty} a_n(z+1-1)^n = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = \frac{1}{n!}f^{(n)}(1) = (-1)^n\frac{1}{n+1}$. That is, $\log(1+z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{n+1}}{n+1}$. This series has radius of convergence 1 and the sum $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n+1}$ is convergent. By part (a), the conclusion follows. ☕

Exercise 4.2.7. Use the results of this section to evaluate the following integrals:

(c) $\displaystyle\int_\gamma \frac{\sin z}{z^3}dz, \quad \gamma(t) = e^{it}, \quad 0 \le t \le 2\pi.$

(d) $\displaystyle\int_\gamma \frac{\log z}{z^n}dz, \quad \gamma(t) = 1 + \tfrac{1}{2}e^{it}, \quad 0 \le t \le 2\pi \text{ and } n \ge 0.$

Solution:
Letting $f(z) = \sin z$, we have

$$0 = -\sin 0 = f''(0) = \frac{2!}{2\pi i}\int_\gamma \frac{f(z)}{(z-0)^{2+1}} = \frac{1}{\pi i}\int_\gamma \frac{\sin z}{z^3}dz.$$

In the disk $B(1;1/3)$, γ is a closed rectifiable curve, and $\log z/z^n$ is analytic and hence has a primitive. So by Proposition 2.15 the integral in part (d) is 0.

Exercise 4.2.9. Evaluate the following integrals:

(c) $\displaystyle\int_\gamma \frac{dz}{z^2+1}, \quad \gamma(t) = 2e^{it}, \quad 0 \le t \le 2\pi.$

(d) $\displaystyle\int_\gamma \frac{\sin z}{z}dz, \quad \gamma(t) = e^{it}, \quad 0 \le t \le 2\pi.$

Solution:
Let $f(z) = 1$. Then f is analytic on $\mathbb{C}$ with $\overline{B}(0,2) \subset \mathbb{C}$. Hence by Proposition 2.6

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$$

for $|z-0| < 2$. Since $|i| = |-i| = 1 < 2$, we have

$$\int_\gamma \frac{dz}{z^2+1} = -\frac{2}{i}\int_\gamma \frac{1}{z-i}dz + \frac{2}{i}\int_\gamma \frac{1}{z+i}dz = -\frac{2}{i}\cdot(2\pi i)f(i) + \frac{2}{i}\cdot(2\pi i)f(-i) = 0.$$

Now, letting $g(w) = \sin w$, we have

$$0 = \sin 0 = g(0) = \frac{1}{2\pi i}\int_\gamma \frac{g(w)}{(w-0)}dw = \frac{1}{2\pi i}\int_\gamma \frac{\sin w}{w}dw.$$

Exercise 4.3.1. Let f be an entire function and suppose there is a constant M, and $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.

Proof. Since f is continuous and $\overline{B}(0; R)$ is compact, then there exists $C > 0$ such that $|f| < C$ on $\overline{B}(0; R)$. Choose $r > R$ so that $C < Mr^n$, and let $R < |z| < r$. Then

$$|f(z)| \leq M|z|^n < Mr^n,$$

and hence $|f| < Mr^n$ on $B(0; r)$. For any $k > n$, we have by Cauchy's Estimate

$$\left|f^{(k)}(0)\right| \leq \frac{k!Mr^n}{r^k} = \frac{k!M}{r^{k-n}}.$$

Letting $r \to \infty$ gives that $f^k(0) = 0$ for all $k > n$. Since f is entire, we can write $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and for all $k > n$,

$$a_k = \frac{1}{k!} f^{(k)}(0) = 0.$$

Hence $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. ☕

Exercise 4.3.8. Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all $z \in G$. Show that either $f \equiv 0$ or $g \equiv 0$.

Proof. Suppose without loss of generality that $g \not\equiv 0$ on G. So there exists $a \in G$ such that $g(a) \neq 0$. Let $R > 0$ so that $B(a; R) \subset G$. The function $h(z) := f(z)g(z) = 0$ is analytic in $B(a; R)$, and so we can write $0 = h(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$. This implies that $a_k = 0$ for all $k \geq 1$.

Fix $n \in \mathbb{N}$. We show by induction that $f^{(n)}(a) = 0$. We have $f(a)g(a) = 0$ by hypothesis which gives $f(a) = 0$. Moreover,

$$0 = a_1 = h'(a) = (fg)'(a) = f'(a)g(a) + f(a)g'(a) = f'(a)g(a) = 0 \implies f'(a) = 0.$$

Now suppose for induction that $f^{(k)}(a) = 0$ for all $k \in \{0, \ldots, n-1\}$. Then

$$\begin{aligned} 0 = a_n = \frac{1}{n!} h^{(n)}(a) &= \frac{1}{n!} (fg)^{(n)}(a) \\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} f^{(n-\ell)}(a) g^{(\ell)}(a) \\ &= f^{(n)}(a) g(a), \end{aligned}$$

and hence $f^{(n)}(a) = 0$. Since this is true for all $n \in \mathbb{N}$, we have by Theorem 3.7 that $f \equiv 0$ on G. ☕

Exercise 4.3.9. Let $U : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $U(z) \geq 0$ for all $z \in \mathbb{C}$; prove that U is constant.

Proof. Since U is harmonic, it has a harmonic conjugate and hence U is the real part of an analytic function f. Define $g(z) = e^{-f(z)}$. So g is analytic on all of $\mathbb{C}$ hence entire. Then

$$|g(z)| = |e^{-f(z)}| = e^{\operatorname{Re} -f(z)} = e^{-\operatorname{Re} f(z)} = e^{-U(z)} \leq 1,$$

and so g is a constant function by Louiville's Theorem. It follows that U is also constant. ☕

Exercise 4.4.3. Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : |z| \geq R\}$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, show that $\int_\gamma \frac{p'(z)}{p(z)} dz = 2\pi i n$.

Proof. We can write $p(z) = c(z - \alpha_1) \cdots (z - \alpha_n)$ where $\alpha_1, \ldots, \alpha_n$ are the (not necessarily distinct) roots of $p(z)$ and $c \in \mathbb{C}$. Then

$$p'(z) = c \sum_{i=1}^{n} (z - \alpha_1) \cdots (z - \alpha_{i-1})(z - \alpha_{i+1}) \cdots (z - \alpha_n).$$

So

$$\begin{aligned}
\int_\gamma \frac{p'(z)}{p(z)} dz &= \int_\gamma \sum_{i=1}^{n} \frac{c(z - \alpha_1) \cdots (z - \alpha_{i-1})(z - \alpha_{i+1}) \cdots (z - \alpha_n)}{c(z - \alpha_1) \cdots (z - \alpha_n)} \\
&= \sum_{i=1}^{n} \int_\gamma \frac{(z - \alpha_1) \cdots (z - \alpha_{i-1})(z - \alpha_{i+1}) \cdots (z - \alpha_n)}{(z - \alpha_1) \cdots (z - \alpha_n)} \\
&= \sum_{i=1}^{n} \int_\gamma \frac{1}{z - \alpha_i} \\
&= \sum_{i=1}^{n} n(\gamma; a_i) 2\pi i \\
&= 2\pi i n.
\end{aligned}$$

☕

Exercise 4.5.6. Let f be analytic on $D = B(0; 1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show $|f'(0)| \leq 1$.

Proof. By Cauchy's Estimate, $|f'(0)| \leq \frac{1! \cdot 1}{1^1} = 1$. ☕

Exercise 4.5.8. Let G be a region and suppose $f_n : G \to \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f : G \to \mathbb{C}$. Show that f is analytic.

Proof. Since $\{f_n\} \to f$ uniformly, then f is continuous. Let $a \in G$, and let $r > 0$ be such that $D := \overline{B}(a; r) \subset G$. Let T be a triangular path in D. Then $\int_T f_n(z) dz = 0$.

Since $\{f_n\} \to f$ uniformly, then $0 = \lim \int_T f_n = \int_T \lim f_n = \int_T f$. So by Morera's Theorem, f is analytic on D, and in particular at $a \in D$. Since a was arbitrary, f is analytic on G. ☕

Exercise 4.6.5. Evaluate the integral $\int_\gamma \frac{dz}{z^2+1}$ where $\gamma(\theta) = 2|\cos 2\theta|e^{i\theta}$ for $0 \le \theta \le 2\pi$.

Solution:
We have

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+1} &= \frac{1}{2i}\int_\gamma \frac{1}{z-i}dz - \frac{1}{2i}\int_\gamma \frac{1}{z+i}dz \\ &= \frac{\pi}{2\pi i}\int_\gamma \frac{1}{z-i}dz - \frac{\pi}{2\pi i}\int_\gamma \frac{1}{z+i}dz \\ &= \pi(n(\gamma; i) - n(\gamma; -i)).\end{aligned}$$

So it suffices to find $n(\gamma; i)$ and $n(\gamma; -i)$. By a quick sketch of the curve γ, it is easily seen that $n(\gamma; i) = n(\gamma; -i) = 1$ and hence the integral in question is 0.

Exercise 4.6.7. Let $f(z) = \frac{1}{[(z-\frac{1}{2}-i)\cdot(z-1-\frac{3}{2}i)\cdot(z-1-\frac{i}{2})\cdot(z-\frac{3}{2}-i)]}$ and let γ be the polygon $[0, 2, 2+2i, 2i, 0]$. Find $\int_\gamma f$.

Solution:
Define triangular paths:

$$\gamma_1 = [0, 2, i+i, 0], \quad \gamma_2 = [0, i = i, 2i, 0], \quad \gamma_3 = [2, i+i, 2+2i, 2], \quad \gamma_4 = [2+2i, i+i, 2, 2+2i].$$

Then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. For convenience, define the points

$$p_1 = 1 + \frac{1}{2}i, \quad p_2 = \frac{1}{2} + i, \quad p_3 = 1 + \frac{3}{2}i, \quad p_4 = \frac{3}{2} + i.$$

For each $i \in \{1, 2, 3, 4\}$ let G_i be a simply connected open set containing γ_i but not containing the points p_j for $j \in \{1, 2, 3, 4\} - \{i\}$. Then define functions $f_i : G_i \to \mathbb{C}$ by $f_i = (z - p_i)f$. Then each f_i is analytic in G_i with $n(\gamma_i; a) = 0$ for all $a \in \mathbb{C} - G_i$ and so

$$n(\gamma_i; p_i)f_i(p_i) = \frac{1}{2\pi i}\int_{\gamma_i} \frac{f_i(z)}{z-p_i}dz = \frac{1}{2\pi i}\int_{\gamma_i} f(z)dz.$$

We have $f_1(p_1) = 2/i, f_2(p_2) = -2, f_3(p_3) = -2/i$, and $f_4(p_4) = 2$. Note that $n(\gamma_i; p_i) = 1$. Then

$$\int_\gamma f(z)dz = \sum_{i=1}^4 \int_{\gamma_i} f(z)dz = \sum_{i=1}^4 \int_{\gamma_i} \frac{f_i(z)}{z-p_i}dz = 2\pi i \sum_{i=1}^4 n(\gamma_i; p_i)f_i(p_i) = 2\pi i \sum_{i=1}^4 f_i(p_i) = 0.$$

Exercise 4.7.3. Let f be analytic in $B(a;R)$ and suppose that $f(a)=0$. Show that a is a zero multiplicity m if and only if $f^{(m-1)}(a)=\cdots=f(a)=0$ and $f^{(m)}(a)\neq 0$.

Proof. ($\Rightarrow$) We can write $f(z)=(z-a)^m g(z)$ for some analytic function g of which a is not a zero. Then by the general Leibniz rule,

$$f^{(k)}(z)=((z-a)^m g(z))^{(k)}=\sum_{i=1}^{k}\binom{k}{i}((z-a)^m)(k-i)g^{(k)}(z)$$

For $0\leq \ell\leq m-1$, $((z-a)^m)^{(\ell)}$ will have a factor of $z-a$ since $\ell<m$. In particular, if $\ell=k-i$ for $0\leq k\leq m-1$ and $0\leq i\leq k$, we see that $f^{(k)}(z)$ will have a factor of $z-a$. Hence $f^{(k)}(a)=0$ for all $0\leq k\leq m-1$.

($\Leftarrow$) Let $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$ in $B(a;R)$. Then $0=\frac{f^{(k)}(a)}{k!}=a_k$ for all $0\leq k\leq m-1$, and hence

$$f(z)=\sum_{n=m}^{\infty}a_n(z-a)^n=(z-a)^m\sum_{n=0}^{\infty}a_{n+m}(z-a)^{n+m}.$$

Letting $g(z):=\sum_{n=0}^{\infty}a_{n+m}(z-a)^{n+m}$, we have $f(z)=(z-a)^m g(z)$. Moreover, $g(a)=a_m\neq 0$ since $f^{(m)}(a)\neq 0$, and hence a is a zero of multiplicity m. ☕

Exercise 4.7.4. Suppose that $f:G\to\mathbb{C}$ is analytic and one-to-one; show that $f'(z)\neq 0$ for any z in G.

Proof. By the corollary to the Open Mapping Theorem, $f^{-1}:f(G)\to G$ is analytic. Suppose $f'(z)=0$ for some $z\in G$ and $f(z)=\omega$. Then $(f^{-1})'(\omega)$ is undefined and therefore not analytic since $(f^{-1})'(\omega)=\dfrac{1}{f'(z)}$, a contradiction. ☕

Exercise 1. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{e^{a+ix}}{(a+ix)^b} dx$, where $a > 1$ and $b > 0$.

Exercise 2. Let Π be the open right half plane. Suppose that f is analytic on Π and satisfies the following: (i) $|f(z)| < 1$ for all $z \in \Pi$; and (ii) there exists $-\pi/2 < \alpha < \pi/2$ such that $\dfrac{\log(|f(re^{i\theta})|)}{r} \to \infty$, as $r \to \infty$. Show that $f = 0$.

Exercise 3. : Let G be a region and let $f_n : G \to \mathbb{C}$ be analytic functions such that f_n has no zero in G. If f_n converges to f uniformly on the compact subsets of G then show that either $f = 0$ or f has no zero in G.

Proof. Assume that $f \not\equiv 0$, and suppose $f(a) = 0$ for some $a \in G$. Since the zeroes of an analytic function are isolated, there exists $r > 0$ such that f does not vanish in $\overline{B}(0, r) \subseteq G$. Let $\epsilon = \min\{|f(z)| \ : \ |z - a| = r\} > 0$. Since $\{f_n\}$ is uniformly convergent to f on compact subsets of G, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$|f_n(z) - f(z)| < \epsilon \leq |f(z)| \text{ for all } |z - a| = r.$$

By Rouche's Theorem, $0 = Z_{f_n} = Z_f > 0$, a contradiction. So f has no zeroes in G. ☕

Exercise 5.1.6. If $f : G \to \mathbb{C}$ is analytic except for poles show that the poles of f cannot have a limit point in G.

Proof. We assume that f is not constant, otherwise the statement is false. If f has a pole at $z = a$ then $\lim_{z\to a} |f(z)| = \infty$, and so $\lim_{z\to a} 1/|f(z)| = 0$. Since $1/f$ is analytic, it is in particular continuous and hence $1/f(a) = 0$. Hence the poles of f are precisely the zeroes of $1/f$. If the poles of f has a limit point in G, then the zeroes of $1/f$ have a limit point in G. Hence $1/f \equiv 0$, and so $f(z) = \infty$ for all $z \in G$, a contradiction. ☕

Exercise 5.1.13. Let $R > 0$ and $G = \{z : |z| > R\}$; a funtion $f : G \to \mathbb{C}$ ia a *removable singularity, a pole, or an essential singularity at infinity* if $f(z^{-1})$ has, respectively, a removable singularity, a pole, or an essential singularity at $z = 0$. If f has a pole at ∞ then the order of the pole is the order of the pole of $f(z^{-1})$ at $z = 0$.

(a) Prove that an entire function has a removable singularity at infinity iff it is a constant.

Proof. ($\Rightarrow$) Let $f(z) = \sum_{n\geq 0} a_n z^n$ be entire with a removable singularity at infinity. Then $f(1/z)$ has a removable singularity at 0, and so

$$0 = \lim_{z\to 0} z f(1/z) = \lim_{z\to 0} \sum_{n\geq 0} \frac{a_n}{z^{n-1}} = \sum_{n\geq 0} \lim_{z\to 0} \frac{a_n}{z^{n-1}} \qquad (\mathfrak{D})$$

Since the sum on the right hand side exists (and equals 0), each summand must be finite, i.e., each limit $\lim_{z\to 0} \frac{a_n}{z^{n-1}}$ exists. In particular, when $n \geq 2$, $\lim_{z\to 0} \frac{a_n}{z^{n-1}} = \infty$, unless $a_n = 0$. Hence $a_n = 0$ for all $n \geq 2$. Then $(\mathfrak{D})$ becomes

$$0 = \lim_{z\to 0} z f(1/z) = a_0 z + a_1 = a_1,$$

which gives $f(z) = a_0$.

($\Leftarrow$) If $f(z) = c$, then $\lim_{z\to 0} f(1/z)z = \lim_{z\to 0} cz = 0$ and hence $f(z)$ has a removable singularity at infinity. ☕

(b) Prove that an entire function has a pole at infinity of order m iff it is a polynomial of degree m.

Proof. ($\Rightarrow$) Suppose $f(z) = \sum_{n\geq 0} a_n z^n$ is entire with a pole at infinity of order m. Then $f(1/z)$ has a pole of order m at $z = 0$ and hence $f(1/z)z^m$ has a removable singularity at 0. So

$$0 = \lim_{z\to 0} z^{m+1} f(1/z) = \lim_{z\to 0} z^{m+1} \sum_{n\geq 0} \frac{a_n}{z^n} = \sum_{n\geq 0} \lim_{z\to 0} \frac{a_n}{z^{n-(m+1)}}. \qquad (ʘ)$$

As before, each summand must be finite, i.e., each limit $\lim_{z\to 0} \frac{a_n}{z^{n-(m+1)}}$ exists. In particular, when $n \geq m+1$, $\lim_{z\to 0} \frac{a_n}{z^{n-(m+1)}} = \infty$, unless $a_n = 0$. Hence $a_n = 0$ for all $n \geq m+2$. Then (ʘ) becomes

$$0 = \lim_{z\to 0} z^{m+1} f(1/z) = \lim_{z\to 0} (a_0 z^{m+1} + a_1 z^m + \dots a_m z + a_m + 1),$$

which gives $f(z) = a_m z^m + \cdots + a_1 z + a_0$.

($\Leftarrow$) Suppose $f(z) = a_m z^m + \cdots + a_1 z + a_0$ for $a_m \neq 0$. Then

$$f(1/z) = a_m z^{-m} + \cdots + a_1 z^{-1} + a_0.$$

Since $f(1/z)$ has a pole at 0, the above is the Laurent Expansion in $\text{Ann}(0; 0, R)$ for some $R > 0$. We see then that $a_{m-1} = a_m \neq 0$ and $a_n = 0$ for all $n \leq -(m+1)$. Hence $f(1/z)$ has a pole of order m at 0 by Proposition 1.18(b). ☕

(c) Characterize those rational functions which have a removable singularity at infinity.

Proof. *** The following two proofs belong to Curits Balz ***
We can write a rational function as $r(z) = p(z)/q(z) = a(z) + p_1(z)/q(z)$ where $a(z)$ is a polynomial and $\deg(p_1) < \deg(q)$. If $r(z)$ has a removable singularity at infinity, then $r(z)$ is bounded at infinity. So $a(z) + p_1(z)/q(z)$ must also be bounded at infinity. By the degrees of $p_1(z)$ and $q(z)$, we see that $p_1(z)/q(z)$ will be bounded at infinity, thus $a(z)$ will be bounded at infinity. But by part (a), we get that $a(z)$ must be constant, say $a(z) = c$, and so $r(z) = p(z)/q(z) = c + p_1(z)/q(z)$. So $p(z) - aq(z) + p_1(z)$ must be a polynomial with degree less than or equal to the degree of $q(z)$. ☕

(d) Characterize those rational functions which have a pole of order m at infinity.

Proof. As in part (c), write $r(z) = p(z)/q(z) = a(z) + p_1(z)/q(z)$. By the degree requirements, $p_1(z)/q(z)$ has a removable singularity at infinity, so we must have $a(z)$ has a pole of order m at infinity. Thus $a(z)$ is a polynomial of degree m. So the degree of $p(z)$ must be m greater than the degree of $q(z)$ when $r(z) = p(z)/q(z)$. ☕

Exercise 5.1.17. Let f be analytic in the region $G = \text{Ann}(a; 0, R)$. Show that if

$$\int\int_G |f(x+iy)|^2 dxdy < \infty$$

then f has a removable singularity at $z = a$. Suppose that $p > 0$ and

$$\int\int_G |f(x+iy)|^p dxdy < \infty;$$

what can be said about the nature of the singularity at $z = a$?

Proof. Without loss of generality, assume $a = 0$. Since $f(z)$ is analytic in G, we can write $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ for all $z \in G$. Using the parametrization $\gamma(r, \theta) = re^{i\theta}$ for $r \in (0, R]$ and $\theta \in (0, 2\pi]$ for G, we have

$$\begin{aligned}
\infty > \int\int_G |f(x+iy)|^2 dxdy &= \int_0^R \int_0^{2\pi} \left(\sum_{n\in\mathbb{Z}} a_n r^n e^{in\theta}\right)\left(\overline{\sum_{m\in\mathbb{Z}} a_m r^m e^{im\theta}}\right) r d\theta dr \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \int_0^R \int_0^{2\pi} a_n \overline{a_m} r^{n+m+1} e^{i\theta(n-m)} d\theta dr \\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_n \overline{a_m} \int_0^R r^{n+m+1} \underbrace{\left(\int_0^{2\pi} e^{i\theta(n-m)} d\theta\right)}_{=0 \text{ when } n\neq m} dr \\
&= \sum_{n\in\mathbb{Z}} |a_n|^2 2\pi \underbrace{\int_0^R r^{2n+1} dr}_{\text{goes to } \infty \text{ if } n\leq -1} .
\end{aligned}$$

The last integral goes to ∞ if $n \leq -1$. Since we know the integral is finite, we must have $a_n = 0$ for all $n \leq 1$. Hence $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$, and hence f has a removable singularity at $z = a$. ☕

Exercise 5.2.2. Verify the following equations:

(b) $\displaystyle\int_0^\infty \frac{(\log x)^3}{1+x^2}dx = 0$

(c) $\displaystyle\int_0^\infty \frac{\cos ax}{(1+x^2)^2}dx = \frac{\pi(a+1)e^{-a}}{4}$ if $a > 0$.

Solution:

Define $f(z) = \frac{e^{iaz}}{(1+z^2)^2}$ and let γ be the closed path which is the boundary of the upper half disk of radius $R > 1$, traversed in the counterclockwise direction. The poles of $f(z)$ are i, $-i$ and $n(\gamma, i) = 1, n(\gamma, -i) = 0$. Let $g(z) = (z-i)^2 f(z)$. Then $\operatorname{Res}(f; i) = g'(i) = \frac{e^{-a}(a-1)}{4i}$. Then by the Residue Theorem,

$$\int_\gamma f = 2\pi i \operatorname{Res}(f; i) = \frac{\pi(a+1)e^{-a}}{2}.$$

Then

$$\begin{aligned}
\frac{\pi(a+1)e^{-a}}{2} = \int_\gamma f &= \int_{-R}^R \frac{e^{iax}}{(1+x^2)^2} + Ri\int_0^\pi \frac{e^{iaRe^{ix}}}{(1+Re^{2ix})^2} \\
&= \int_{-R}^R \frac{\cos ax}{(1+x^2)^2} + Ri\int_0^\pi \frac{e^{iaRe^{ix}}}{(1+Re^{2ix})^2} \\
&= \int_{-R}^0 \frac{\cos ax}{(1+x^2)^2} + \int_0^R \frac{\cos ax}{(1+x^2)^2} + Ri\int_0^\pi \frac{e^{iaRe^{ix}}}{(1+Re^{2ix})^2} \\
&= 2\int_0^R \frac{\cos ax}{(1+x^2)^2} + Ri\int_0^\pi \frac{e^{iaRe^{ix}}}{(1+Re^{2ix})^2},
\end{aligned}$$

where the last equality follows since $\cos x$ is an even function. Now,

$$\begin{aligned}
\left| Ri\int_0^\pi \frac{e^{iaRe^{ix}}}{(1+Re^{2ix})^2}\right| \le R\int_0^\pi \frac{\left|e^{iaRe^{ix}}\right|}{|1+2Re^{2ix}+R^2e^{4ix}|} &\le R\int_0^\pi \frac{1}{1+2Re^{2ix}+R^2e^{4ix}} \\
&\le \frac{\pi R}{1+2Re^{2ix}+R^2e^{4ix}} \xrightarrow{R\to\infty} 0.
\end{aligned}$$

Hence $\frac{\pi(a+1)e^{-a}}{4} = \int_0^\infty \frac{\cos ax}{(1+x^2)^2}$.

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin a\pi}$ if $0 < a < 1$.

Solution:

Define $f(z) = \frac{e^{az}}{1+e^z}$. Let $R > 0$ and let γ be the rectangular region $[-R, R, R+2\pi]$. Then each edge of γ can be parametrized by

$$\begin{aligned}
&\gamma_1(t) = R + it, \quad t \in [0, 2\pi] \\
&\gamma_2(t) = 2\pi i - t, \quad t \in [-R, R] \\
&\gamma_3(t) = -R + i(2\pi - t), \quad t \in [0, 2\pi] \\
&\gamma_4(t) = t, \quad t \in [-R, R].
\end{aligned}$$

The poles of $f(z)$ are $\{z = \pi i + 2\pi i k \mid k \in \mathbb{Z}\}$. Notice that πi is the only pole of $f(z)$ such that $n(\gamma; \pi i) \neq 0$. Then using L'Hôpital's rule

$$\text{Res}(f; \pi i) = \lim_{z\to\pi i} (z - \pi i f(z) = \lim_{z\to\pi i} \frac{(z-\pi i)ae^{az} + e^{az}}{e^z} = e^{a\pi i}e^{-\pi i} = -e^{a\pi i}.$$

By the Residue Theorem

$$\begin{aligned}
2\pi i e^{\pi(a-1)} &= \int_\gamma f(z) \\
&= i\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{R+it}} - \int_{-R}^{R} \frac{e^{a(2\pi i+t)}}{1+e^{(2\pi i+t)}} - i\int_0^{2\pi} \frac{e^{a(-R+it)}}{1+e^{(-R+it)}} + \int_{-R}^{R} \frac{e^{at}}{1+e^t}.
\end{aligned}$$

We want to show that the first and third integrals above go to 0 as R goes to ∞. For the first integral, since $|1 + e^{R+it}| \geq |e^R - 1|$, we have

$$\left| i\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{R+it}} \right| \leq \int_0^{2\pi} \frac{e^{aR}}{|1+e^{R+it}|} \leq \int_0^{2\pi} \frac{e^{aR}}{|e^R - 1|}.$$

Then

$$\lim_{R\to\infty} \frac{e^{aR}}{e^R - 1} = \lim_{R\to\infty} \frac{e^{R(a-1)}}{1 - 1/e^R} = 0 \qquad \text{(since } a < 1\text{)}$$

Since $|1 + e^{-R+it}| \geq |e^{-R} - 1|$, then for the third integral, we have

$$\left| i\int_0^{2\pi} \frac{e^{a(-R+it)}}{1+e^{-R+it}} \right| \leq \int_0^{2\pi} \frac{e^{-aR}}{|1+e^{-R+it}|}, \quad \text{and} \quad \lim_{R\to\infty} \frac{1}{e^{aR}(e^{-R}-1)} = 0.$$

Then we have

$$2\pi i e^{\pi(a-1)} = -\int_{-\infty}^{\infty} \frac{e^{a(2\pi i+t)}}{1+e^{(2\pi i+t)}} + \int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t} = (1 - e^{a2\pi i}) \int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t},$$

which gives

$$\int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t} = \frac{2\pi i(-e^{a\pi i})}{1 - e^{a2\pi i}} = \frac{2\pi i}{e^{a\pi i} - e^{-a\pi i}} = \frac{\pi}{\sin(a\pi)}.$$

(h) $\displaystyle\int_0^{2\pi} \log \sin^2 2\theta d\theta = 4\int_0^{\pi} \log \sin\theta d\theta = -4\pi \log 2.$

Exercise 5.2.6. Let γ be the rectangular path

$$[n+1/2+ni, -n-1/2+ni, -n-1/2-ni, n+1/2-ni, n+1/2+ni]$$

and evaluate the integral $\int_\gamma \pi(z+a)^{-2}\cot \pi z dz$ for $a \notin \mathbb{Z}$. Show that $\lim_{n\to\infty}\int_\gamma \pi(z+a)^{-2}\cos \pi z dz = 0$ and, by using the first part, deduce that

$$\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}$$

(*Hint:* Use the fact that for $z = x+iy$, $|\cos z|^2 = \cos^2 x + \sinh^y$ and $|\sin z|^2 = \sin^2 x + \sinh^2 y$ to show that $|\cot \pi z| \le 2$ for z on γ if n is sufficiently large.)

Proof. *** The following proof belongs to Curits Balz ***
Let $f(z) = \frac{1}{(z+a)^2}$. We want to find $\int_\gamma \cot \pi z f(z)$. Define $g(z) := \pi \cot \pi z f(z)$. By the residue theorem, since $\pi \cot \pi z$ has simple poles when $z \in \mathbb{Z}$, we get

$$\int_\gamma g(z) = 2\pi i \left(\sum_{n\in\mathbb{Z}} \mathrm{Res}(g;n) + \mathrm{Res}(g,-a) \right).$$

At each integer n, the residue of $g(z)$ is

$$\mathrm{Res}(g;n) = \lim_{z\to n}(z-n)\pi\cot \pi z f(z) = \lim_{z\to n}\frac{z-n}{\sin \pi z}\lim_{z\to n}\pi \cot \pi z f(z) = f(n).$$

We need to show that $\int_\gamma \pi \cot \pi z f(z) = 0$ as $n\to\infty$, so we show $\cot \pi z$ is bounded on γ.

- For $z = n+1/2+iy, -1/2 \le y \le 1/2$,
  $|\cot(\pi z)| = |\cot(\pi(N+1/2_i y))| = |\cot(\pi/2 + i\pi y)| = |\tanh \pi y| \le \tanh \pi/2$
- For $z = -n-1/2+iy, -1/2 \le y \le 1/2$
  $|\cot(\pi z)| = |\cot(\pi(-N-1/2_i y))| = |\cot(\pi/2 - i\pi y)| = |\tanh \pi y| \le \tanh \pi/2$
- For $y > 1/2$,
  $$|\cot(\pi z)| = \left|\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\right| = \left|\frac{e^{-\pi y}+e^{\pi y}}{e^{\pi y}-e^{-\pi y}}\right| = \frac{1+e^{2\pi y}}{1-e^{-2\pi y}} \le \frac{1+e^{-\pi}}{1-e^{-\pi}} = \coth \pi/2$$
- For $y < -1/2$,
  $|\cot(\pi z)| = |\cot(\pi(-N-1/2_i y))| = |\cot(\pi/2 - i\pi y)| = |\tanh \pi y| \le \tanh \pi/2.$

We also have $|f(z) \le \frac{1}{z+a|^2}$. So

$$\lim_{n\to\infty}\left|\int_\gamma \pi\cot \pi z f(z)\right| \le \lim_{n\to\infty}\int_g a\pi|\cot(\pi z)||f(z)| \le \lim_{n\to\infty}\frac{\pi}{n^2}(8n+4)\coth \pi/2 = 0$$

where $8n+4 = V(\gamma)$. This gives $\sum_{n\in\mathbb{Z}} f(n) = \mathrm{Res}(g;-a)$. But

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in Z}\frac{1}{(a+n)^2} \quad \text{and} \quad \mathrm{Res}(g;-a) = \lim_{z\to -a}\frac{(z+a)^2\pi\cot \pi z}{(z+a)^2} = -\pi^2\csc^2 \pi a.$$

So $\dfrac{\pi^2}{\sin^2 \pi a} = \displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^2}$.

Exercise 5.3.5. Let f be meromorphic on the region G and not constant; show that neither the poles nor the zeros of f have a limit point in G.

Proof. That the poles of f do not have a limit point in in G was proved in Exercise 5.1.6. If the zeroes of f have a limit point in G, then the poles of the meromorphic function $1/f$ has a limit point in G, contradicting Exercise 5.1.6. ☕

Exercise 5.3.10. Let f be analytic in a neighborhood of $D = \overline{B}(0; 1)$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. If $|f(z)| \leq 1$ for $|z| = 1$, what can you say?

Proof. Define $g(z) = f(z) - z$ and let $h(z) = z$. Then on ∂D, we have

$$|g(z) + h(z)| = |f(z)| < 1 \leq |g(z)| + 1 = |g(z)| + |h(z)|.$$

By Rouche's Theorem, $Z_g = Z_h$ (where Z_f denotes the number of zeroes of f). Since $Z_g = Z_h = 1$, then there is a unique $z_0 \in D$ such that $0 = g(z_0) = f(z_0) - z_0$, i.e., $f(z_0) = z_0$. ☕

Exercise 6.2.3. Suppose $f : \mathbb{D} \to \mathbb{C}$ satisfies $\operatorname{Re} f(z) \geq 0$ for all z in $\mathbb{D}$ and suppose that f is analytic and not constant.

(a) Show that $\operatorname{Re} f(z) > 0$ for all $z \in \mathbb{D}$.

Proof. Let $\Pi = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ be the open right-half plane. By the open mapping theorem, $f(\mathbb{D})$ is open. Therefore since $f(\mathbb{D}) \subseteq \overline{\Pi}$, we must have $f(\mathbb{D}) \subseteq \Pi$. ☕

(b) By using an appropriate Möbius transformation, apply Schwartz's Lemma to prove that if $f(0) = 1$ then

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

Proof. Define a Möbius transformation $g(z) = \dfrac{z-1}{z+1}$. Then $g(\Pi) \subseteq \mathbb{D}$ because if $\operatorname{Re} z > 0$, then

$$\left|\frac{z-1}{z+1}\right|^2 = \frac{z-1}{z+1} \cdot \frac{\overline{z}-1}{\overline{z}+1} = \frac{|z|^2 - 2\operatorname{Re} z + 1}{|z|^2 + 2\operatorname{Re} z + 1} < 1.$$

Consider $g \circ f : \mathbb{D} \to \Pi \to \mathbb{D}$. Since $g(f(0)) = 0$ and $|(g \circ f)(z)| < 1$, we can apply Swartz's Lemma to obtain the inequality $|(g \circ f)(z)| \leq |z|$ for all $z \in \mathbb{D}$. This yields

$$|f(z) - 1| \leq |z||f(z) + 1| = |zf(z) + z| \leq |z||f(z)| + |z| \qquad (🦇)$$

By the reverse triangle inequality, we have $|f(z)| - 1 \leq ||f(z)| - 1| \leq |f(z) - 1|$. So (🦇) becomes

$$\begin{aligned} |f(z)| - 1 &\leq |z||f(z)| + |z| \\ |f(z)|(1 - |z|) &\leq 1 + |z| \\ |f(z)| &\leq \frac{1 + |z|}{1 - |z|}. \end{aligned}$$

☕

(c) Show that if $f(0) = 1$, f also satisfies

$$|f(z)| \geq \frac{1 - |z|}{1 + |z|}.$$

(*Hint:* Use part (a)).

Proof. Since $\operatorname{Re} f(z) > 0$ on $\mathbb{D}$, then $1/f(z)$ is analytic on $\mathbb{D}$. Let $h(z) = \frac{1-z}{1+z}$. Then $(h \circ 1/f)(0) = 0$ and $|(h \circ 1/f)(z)| \leq 1$. So by Swartz's Lemma, $|(h \circ 1/f)(z)| \leq |z|$. Using the reverse triangle inequality as in part (b), we get

$$\frac{1}{|f(z)|} - 1 \leq \left|1 - \frac{1}{f(z)}\right| \leq |z| \left|1 + \frac{1}{f(z)}\right| = \left|z + \frac{z}{f(z)}\right| \leq |z| + \frac{|z|}{|f(z)|},$$

which gives

$$\frac{1}{|f(z)|}(1 - |z|) \leq 1 + |z| \implies |f(z)| \geq \frac{1 - |z|}{1 + |z|}.$$

☕

Exercise 1. : Suppose $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $B = \{z \in \mathbb{C} : 4 < |z| < 5\}$. Is there a one-to-one analytic function from A to B? Justify your answer.

Proof. Suppose there exists a one-to-one onto analytic function from A onto B. Then f can be extended to an analytic function $\tilde{f} : \tilde{A} \to B$ where $\tilde{A} = \{z \in \mathbb{C} : 0 \leq |z| < 1\}$. Let $\tilde{f}(0) = b$. By the Open Mapping Theorem, a neighborhood of 0 must get mapped to an open set in B, i.e., b must lie in the interior of B.

Since f is onto, there exists $a \in A$ such that $\tilde{f}(a) = b$ (as $\tilde{f}|_A = f$). Now, let C and D be disjoint neighborhoods of 0 and a, respectively. Then $E := \tilde{f}(C) \cap \tilde{f}(D)$ is open since C and D and $\tilde{f}$ are open. But then $f^{-1}(E) \cap C$ and $f^{-1}(E) \cap D$ are are two disjoint open sets in A which get mapped onto the same set E, contradicting the injectivity of f on A. Hence such a function cannot exist. ☕

Exercise 2. How many zeros does the function $z^8 + e^{-2016\pi z}$ have in the region $\mathrm{Re}(z) > 0$?

Functions of One Complex Variable, Conway - Exercises

Exercise 7.1.6. (Dini's Theorem) Consider $C(G,\mathbb{R})$ and suppose that $\{f_n\}$ is a sequence in $C(G,\mathbb{R})$ which is monotonically increasing and $\lim f_n(z) = f(z)$ for each $z \in G$ where $f \in C(G,\mathbb{R})$. Show that $f_n \to f$.

Proof. We need to show that $f_n \to f$ in $(C(G,\mathbb{R}),\rho)$. This is equivalent to showing that $f_n \to f$ uniformly on compact subsets of G by Proposition 1.10 (b) of this section. So, let $K \subset G$ be a compact subset of G.

Let $\epsilon > 0$. Define $g_n = f - f_n$ and $E_n = \{z \in K \mid |f(z) - f_n(z)| < \epsilon\}$ for all n. Then $\{g_n\}$ is a collection of continuous and decreasing functions (since the f_n are increasing). So, E_n is open since $E_n = g_n^{-1}(-1,\epsilon)$. Notice that $E_n \subseteq E_{n+1}$ for all n because if $z \in K$ satisfies $|f(z) - f_n(z)| < \epsilon$, then $|f(z) - f_{n+1}(z)| \leq |f(z) - f_n(z)| < \epsilon$.

Since $f_n \to f$ pointwise, if $z \in K$, there exists $n \in \mathbb{N}$ such that $z \in E_n$. Hence $\{E_n\}$ is an open cover for K, and since K is compact, there exists $E_{n_1},\dots,E_{n_k}$ which cover K. By reordering if necessary, we assume $n_k > n_j$ for all $1 \leq j \leq k-1$. Hence $E_{n_k} \supseteq E_{n_i}$ for all j and so $K_i = \bigcup_{j=1}^k E_{n_j} = E_{n_k}$. Hence if $z \in K$ and $n \geq n_k$, then $z \in E_n$, i.e., $|f(z) - f_n(z)| < \epsilon$. Therefore, $f_n \to f$ uniformly on K. ☕

Exercise 7.2.1. Let $f, f_1, f_2, \dots$ be elements of $H(G)$ and show that $f_n \to f$ iff for each closed rectifiable curve γ in G, $f_n(z) \to f(z)$ for $z \in \{\gamma\}$.

Proof. ($\Rightarrow$) If $f_n \to f$ uniformly on G, then certainly $f_n \to f$ uniformly on the (compact) subset $\{\gamma\} \subset G$.

($\Leftarrow$) Let $a \in G$ and let $r > 0$ be such that $\overline{B}(a;2r) \subset G$. Let $\gamma(t) = a + 2re^{it}$, $t \in [0,2\pi]$. Then for any $z \in B(a;r)$ and $w \in \{\gamma\}$, we have $|w - z| > r$. So for $z \in B(a;r)$ we have by Cauchy's Theorem

$$\begin{aligned}|f(z) - f_n(z)| \leq \frac{1}{2\pi}\int_\gamma \frac{|f(w) - f_n(w)|}{|w-z|}dw &< \frac{1}{2\pi}\int_\gamma \frac{|f(w) - f_n(w)|}{r}dw \\ &\leq \frac{1}{2\pi}(2\pi 2r)\frac{1}{r}\sup_{w\in\{\gamma\}}\{|f(w) - f_n(w)|\} \\ &= 2\sup_{w\in\{\gamma\}}\{|f(w) - f_n(w)|\}.\end{aligned}$$

Since $f_n \to f$ on $\{\gamma\}$, then $2\sup_{w\in\{\gamma\}}\{|f(w)-f_n(w)|\} \to 0$ as $n \to \infty$. So $f_n \to f$ uniformly on $B(a;r)$.

Now if $K \subset G$ is compact, we can cover K with finitely many balls $\{B(k_i;r_i)\}_{i=1}^m$ where $k_i \in K$ and $r_i > 0$ is such that $\overline{B}(k_i, 2r_i) \subset G$. Then by the above argument, $f_n \to f$ uniformly on each ball $B(k_i;r_i)$. If $\epsilon > 0$, for each $B(k_i;r_i)$, there exists $N_i \in N$ such that for all $n \geq N_i$, $|f - f_n| < \epsilon$ on $B(k_i;r_i)$. Letting $N = \max\{N_1,\dots,N_m\}$, we get that for all $n \geq N$, $|f - f_n| < \epsilon$ on K. Hence $f_n \to f$ uniformly on any compact subset of G, and so $f_n \to f$ on G. ☕

Exercise 7.2.13.

(a) Show that if f is analytic on an open set containing the disk $\overline{B}(a;R)$ then

$$|f(a)|^2 \leq \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R |f(a+re^{i\theta})|^2 r dr d\theta \tag{$*$}$$

Proof. Let $0<r<R$ and $\gamma(t)=a+re^{i\theta}$, $t\in[0,2\pi]$. By Cauchy's Theorem,

$$\begin{aligned}|f(a)|^2 = |f^2(a)| &\leq \frac{1}{2\pi}\int_\gamma \frac{|f^2(z)|}{z-a}|dz| \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(a+re^{i\theta})|^2}{r}|ire^{i\theta}|d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}|f(a+re^{i\theta})|^2 d\theta\end{aligned}$$

Multiplying both sides by r and integrating from 0 to R with respect to r,

$$\begin{aligned}|f(a)|^2\frac{R^2}{2} = |f(a)|^2\int_0^R r dr &\leq \frac{1}{2\pi}\int_0^R\left(\int_0^{2\pi}|f(a+re^{i\theta})|^2 d\theta\right) r dr \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^R |f(a+re^{i\theta})|^2 r dr d\theta,\end{aligned}$$

which gives $(*)$. ☕

(b) Let G be a region and let M be a fixed positive constant. Let $\mathcal{F}$ be the family of all functions f in $H(G)$ such that $\int\int_G |f(z)|^2 dxdy \leq M$. Show that $\mathcal{F}$ is normal.

Proof. We show $\mathcal{F}$ is locally bounded and hence normal. Let $K\subset G$ be compact. If $a\in K$, then by part (a), and our assumption that $\int\int_G |f(z)|^2 dxdy \leq M$, we get $|f(a)| \leq \frac{\sqrt{M}}{\sqrt{\pi}R}$. Hence $\mathcal{F}$ is locally bounded. ☕