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Homological algebra and Yang–Mills theory

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Abstract

The antifield-BRST formalism and the various cohomologies associated with it are surveyed and illustrated in the context of Yang–Mills gauge theory. In particular, the central role played by the Koszul–Tate resolution and its relation to the characteristic cohomology are stressed.

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1. Introduction

Since the discovery by Becchi, Rouet and Stora [8] and Tyutin [31] of the remarkable symmetry that now bears their names, it has been realized that many fundamental questions of local field theory can be reformulated as cohomological ones. This is true both at the classical and at the quantum levels.

Reformulating physical problems as cohomological issues is more than just aesthetically appealing since one can then analyse them by using the powerful machinery of homological algebra. For instance, the homological point of view has enabled us in [22] to streamline the demonstration of a fundamental theorem of pertubative quantum Yang–Mills theory [26,13] and, more recently [6,2], to settle definitely a long-standing conjecture due to Kluberg-Stern and Zuber [28,36] on the structure of renormalized gauge invariant operators.

There exists various formulations of the BRST symmetry. We shall follow in this paper the approach that is now known as the "antifield formalism". A detailed exposition of the antifield-BRST construction with an emphasis on its cohomological aspects can be found in [25].

The distinguishing feature of the antifield formalism is the introduction of new variables, the "antifields". These variables serve a definite purpose, namely, they generate

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the "Koszul–Tate" differential complex, which provides a (homological) resolution of the algebra of on-shell functions [1]. The Koszul–Tate complex plays a crucial role for at least two reasons. First, its acyclicity properties guarantee the existence of the BRST differential for an arbitrary gauge system, no matter how complicated the structure of its gauge algebra is. Second, the Koszul–Tate differential is a central tool in the calculation of the BRST cohomology.

The aim of this paper is to illustrate this second aspect of the Koszul–Tate differential in the physically important case of the Yang–Mills theory. It is sometimes stated that the antifield formalism is an unnecessary complication for Yang–Mills models, whose gauge transformations are irreducible and close off-shell. While this is certainly true for the definition of the BRST symmetry itself, the full calculation of the BRST cohomology is greatly simplified (and actually has been carried out only) by following the homological ideas underlying the antifield theory. On these grounds, the Yang–Mills models provide a nice illustration of the power of the homological techniques applied to local field theory.

The antifield construction of the BRST differential finds its roots in the papers [27,34] and is due to Batalin and Vilkovisky [7] (see also [32]). The introduction of the antifields in the Yang–Mills case was actually performed earlier by Zinn-Justin and Becchi, Rouet and Stora in [35,8], where the antifields are called "sources for the BRST variations of the fields". The key importance of the Koszul–Tate complex in the antifield formalism has been uncovered and stressed in [17,20], where various cohomologies occurring in BRST theory are also defined and related with one another. Our approach to the antifield formalism follows the lines of our earlier work on the Hamiltonian formulation of the BRST theory [19,24,16], where identical algebraic features hold (see also [14,11,29]). A unified and systematic exposition may be found in [25].

2. Yang–Mills action

Like any local field theory with a gauge freedom, the Yang–Mills system is characterized by its dynamics (partial differential equations deriving from a variational principle) and by its symmetries. These two aspects are of course connected since the symmetries leave, by definition, the action invariant. The different cohomologies of interest to the analysis of a gauge system are either associated with its dynamics (characteristic cohomology, Koszul–Tate cohomology), to its symmetries (Lie algebra cohomology in appropriate representation spaces) or to both (BRST cohomology).

The Yang–Mills action is given by

$$S[A^a_\mu] = -\frac{1}{4}\int F^a_{\mu\nu}F^{b\mu\nu}\delta_{ab}\, d^n x, \tag{1}$$

[1] By definition, the algebra of on-shell functions is the algebra of functions defined on the stationary surface, i.e., on the surface where the equations of motion hold.

where A^a_μ is the Yang–Mills connection. The field strengths read explicitly[2]

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - C^a_{bc} A^b_\mu A^c_\nu. \tag{2}$$

Here, C^a_{bc} are the structure constants of the Lie algebra $\mathcal{G}$ of the gauge group G and fulfill $\delta_{ab} C^b_{cd} + \delta_{bc} C^b_{ad} = 0$ (δ_{ab} is invariant). We assume that G is compact and is thus the direct product of a semi-simple compact Lie group H by abelian factors,

$$G = U(1)^k \times H. \tag{3}$$

Couplings to matter will not be considered here for simplicity but can be handled along the same lines [2]. The equations of motion following from the action (1) are the standard Yang–Mills equations,

$$D_\mu F^{a\mu\nu} = 0, \tag{4}$$

where D_μ denotes the covariant derivative.

Let us now turn to the gauge symmetries of Yang–Mills theory. These are given by

$$\delta_\varepsilon A^a_\mu = \partial_\mu \varepsilon^a - C^a_{bc} A^a_\mu \varepsilon^c. \tag{5}$$

The invariance of the Yang–Mills action under the gauge transformations (5) is a direct consequence of the Noether identities

$$D_\mu \left(\frac{\delta \mathcal{L}}{\delta A^a_\mu} \right) \equiv D_\mu D_\nu F^{a\mu\nu} \equiv 0. \tag{6}$$

3. Algebra of local forms

The differentials that we shall introduce are defined in the algebra of local forms. A local form is by definition a form on spacetime (assumed to be R^n for simplicity), which depends also on the fields A^a_μ and a finite number of their derivatives. This can be formalized by using the jet space language. We refer the reader to [1] for a presentation of jet bundle theory adapted to our purposes and for further references. Let E be the bundle $E = R^n \times F$ over spacetime of Lie algebra-valued exterior forms. Coordinates on the fibers are given by A^a_μ. A section of E is a field configuration $A^a_\mu(x)$. The kth order jet bundle over E is denoted by $J^k(E)$. It is coordinatized by x^μ, the components A^a_μ of the vector potential and their successive derivatives up to order k. The infinite jet bundle over E is denoted by $J^\infty(E)$. The exterior algebra of differential forms $\Omega(J^\infty(E))$, together with its exterior derivative, splits according to vertical and horizontal degrees, $\Omega(J^\infty(E)) = \bigoplus_{r,s} \Omega^{r,s}(J^\infty(E))$ where r is the horizontal degree and s is the vertical degree. The algebra of local forms is by definition the algebra $\Omega^{*,0}(J^\infty(E))$ of purely

[2] Although the analysis below can be carried out in a coordinate free way, we shall be quite explicit about indices. In terms of forms, Eqs. (1) and (2) can be respectively written as $S = -\frac{1}{4}\int F^a \wedge F^{*b}\delta_{ab}$ and $F = dA - A \wedge A$, with $A = A^a_\mu dx^\mu T_a$. The T_a's are the generators of the adjoint representation of the Lie algebra.

horizontal forms, i.e., of elements of $\Omega(J^\infty(E))$ with zero vertical degree. A generic element ω of $\Omega^{*,0}(J^\infty(E))$ can be written as

$$\omega = \sum_{0\leq p\leq n} \omega^{(p)} \tag{7}$$

with

$$\omega^{(p)} = \sum \frac{1}{p!}\omega_{\mu_1\dots\mu_p}\, dx^{\mu_1}\dots dx^{\mu_p} \tag{8}$$

(we do not write the exterior product $\wedge$ explicitly). The coefficients $\omega_{\mu_1\dots\mu_p}$ are functions of the spacetime coordinates, of the fields A^a_μ and of a finite number of their derivatives. We shall consider in the sequel only polynomial functions of the A^a_μ and their derivatives. Thus, the algebra $\mathcal{E}$ of local forms can be identified with the tensor product

$$\mathcal{E} = \Omega(R^n)\otimes \mathcal{A}, \tag{9}$$

where $\Omega(R^n)$ is the algebra of exterior differential forms on R^n and where $\mathcal{A}$ is the algebra of "local functions", i.e., of (polynomial) functions of the fields A^a_μ and of a finite number of their derivatives.

We shall introduce later further fields, which are the ghost fields and the antifields. This means that the fibers F of E will be replaced by new fibers denoted by $\bar{F}$ and coordinatized not only by A^a_μ but also by the new fields. The concept of local forms and local functions will be modified accordingly, namely, the coefficients $\omega_{\mu_1\dots\mu_p}$ in (8) will depend also (polynomially) on the ghosts, the antifields and a finite number of their derivatives.

In the algebra of local forms, one can define the differential d as follows,

$$d\omega = \sum_p d\omega^{(p)}, \tag{10}$$

$$d\omega^{(p)} = \sum \frac{1}{p!}(d\omega_{\mu_1\dots\mu_p})\, dx^{\mu_1}\dots dx^{\mu_p}, \tag{11}$$

$$d\omega_{\mu_1\dots\mu_p} = dx^\mu\, \partial_\mu \omega_{\mu_1\dots\mu_p}. \tag{12}$$

It coincides with the horizontal d of the jet space formulation. To describe the cohomology of d, it is convenient to introduce a further concept. One says that an n-form $\omega \equiv a\, dx^0\dots dx^{n-1}$ is variationally closed if and only if the variational derivatives of a with respect to all fields (and antifields if any) identically vanish,

$$\omega \equiv a\, dx^0\dots dx^{n-1} \text{ is variationally closed} \quad\Longleftrightarrow\quad \frac{\delta a}{\delta \Phi^A}\equiv 0. \tag{13}$$

Here, Φ^A stands for the A^a_μ, as well as the ghosts and the antifields if these are present. Furthermore, $\delta a/\delta\Phi^A$ is the variational derivative of a with respect to Φ^A,

$$\frac{\delta a}{\delta\Phi^A} \equiv \frac{\partial a}{\partial \Phi^A} - \partial_\mu \frac{\partial a}{\partial(\partial_\mu \Phi^A)} + \partial_\mu\partial_\nu \frac{\partial a}{\partial(\partial_\mu\partial_\nu\Phi^A)} - \cdots. \tag{14}$$

Two n-forms are said to be equivalent if their difference is variationally closed.

Theorem 1. (Algebraic Poincaré Lemma) *The cohomology of d is given by*

$$H^0(d) = R, \tag{15}$$

$$H^k(d) = 0, \tag{16}$$

$$H^n(d) = \{\text{equivalent } n\text{-forms}\}. \tag{17}$$

Proof. This is a classical result from the calculus of variations, see e.g. [1]. □

We shall also need the cohomology of d in the algebra of invariant (local) forms. A local form depending on A^a_μ and its derivatives is gauge invariant iff its coefficients involve only the field strength components and their covariant derivatives contracted with tensors invariant for the adjoint representation of the gauge group G (invariant polynomials in $F^a_{\mu\nu}$ and $D_\rho F^a_{\mu\nu}$, $D_\sigma D_\rho F^a_{\mu\nu}$, etc). The exterior derivative of an invariant local form is also an invariant local form. Accordingly, one may investigate the invariant cohomology of d, i.e., the cohomology of d in the algebra of invariant local forms. We shall call somewhat loosely "characteristic classes" the invariant polynomials in the curvature two forms $F^a = \frac{1}{2}F^a_{\mu\nu}\, dx^\mu\, dx^\nu$, e.g. $\delta_{ab}F^aF^b$ is a characteristic class. The characteristic classes are closed and thus exact by the algebraic Poincaré lemma. However, they cannot be written as the d of an invariant local form. Indeed, one can prove:

Theorem 2. (Covariant Poincaré Lemma) *The invariant cohomology of d in form degree $k < n$ is given by the characteristic classes. In degree n, it contains both the characteristic classes of degree n (if any) and the equivalence classes of invariant n-forms that differ by a variationally closed one.*

Proof. See [9,15]. □

There is an analogous theorem in Riemannian geometry, see [18]. The covariant Poincaré lemma provides a very nice algebraic characterization of the characteristic classes.

4. Characteristic cohomology

The next cohomology that we shall discuss is related to the equations of motion. It is the characteristic cohomology [12]. To illustrate the idea, we start with the familiar concept of conserved currents. A conserved current j may be defined as an $(n-1)$-form that is closed when the equations of motion hold. We say that j is closed "on-shell" and we write

$$dj \approx 0. \tag{18}$$

One easily constructs solutions of (18) by taking simply

$$j \approx dk, \tag{19}$$

where k is an $(n-2)$-form. These solutions are called trivial. The characteristic cohomology in form degree $n-1$ is the quotient space of the cocycles (18) by the coboundaries (19).

More generally, the characteristic cohomology in form degree k is defined as the space of equivalence classes of cocycles a solutions of

$$da \approx 0 \tag{20}$$

modulo the coboundaries c defined by

$$c \approx db. \tag{21}$$

That is, the characteristic cohomology consists of the non-trivial k-forms that are closed on-shell.

If the gauge group G has abelian factors, the characteristic cohomology in form degree $n-2$ does not vanish and contains the (equivalence classes) of the cocycles $^*F^A$, where A ranges over the abelian factors and where $^*F^A$ is the $(n-)2$-form dual of the abelian field strength 2-form F^A. Indeed, the Maxwell equations for the abelian gauge fields read

$$d(^*F^A) \approx 0, \tag{22}$$

and clearly $^*F^A$ is not equal on-shell to the exterior derivative of a *local* form. We shall indicate below that there are no other non-trivial cocycles in form degree $n-2$. Furthermore, general arguments establish the following theorem:

Theorem 3. (Vanishing theorem for the characteristic cohomology in degree $\leq n-3$) *The characteristic cohomology for Yang-Mills theory in form degree $k < n-2$ vanishes, except, of course, in degree* 0, *where the characteristic cohomology is isomorphic to the constants.*

Proof. See [12,3]. □

Thus, the characteristic cohomology for Yang-Mills theory is completely known up to form degree $n-2$ inclusive. In form degree $n-1$, one has the conserved currents. These probably reduce to the currents associated with the Poincaré transformations (and in four dimensions, with the conformal symmetries as well) [30]. Restrictions on the form of the conserved currents are given in [4].

5. Koszul-Tate complex

The second differential associated with the equations of motion is the Koszul-Tate differential δ. The equations of motion (4), together with their successive derivatives up to order $k-2$,

$$D_\nu F^{a\mu\nu} = 0, \tag{23}$$

$$\partial_{\lambda_1\ldots\lambda_i} D_\nu F^{a\mu\nu} = 0, \quad i = 1, \ldots, k-2, \tag{24}$$

determine a surface Σ_k in $J^k(E)$ $(k = 2, 3, \ldots)$. They form what is known as the "prolonged system" and are subject to the Noether identities (6) together with their derivatives up to order $k-3$,

$$D_\mu D_\nu F^{a\mu\nu} = 0, \tag{25}$$

$$\partial_\lambda D_\mu D_\nu F^{a\mu\nu} = 0, \quad \text{etc.} \tag{26}$$

The Koszul–Tate differential implements the equations of motion in cohomology. Technically, this means that it provides for each value of k a resolution of the algebra $C^\infty(\Sigma_k)$ of functions on Σ_k. It is defined on the generators of

$$C^\infty(J^k(E)) \otimes \Lambda\Big[A^{*\mu}_a, \partial_\lambda A^{*\mu}_a, \ldots, \partial_{\lambda_1\lambda_2\ldots\lambda_{k-2}} A^{*\mu}_a\Big] \otimes R\Big[C^*_a, \partial_\lambda C^*_a, \partial_{\lambda_1\lambda_2\ldots\lambda_{k-3}} C^*_a\Big]$$

as follows

$$\delta f = 0 \quad \text{for all } f \in C^\infty(J^k(E)), \tag{27}$$

$$\delta(\partial_{\lambda_1}\partial_{\lambda_2}\ldots\partial_{\lambda_i} A^{*\mu}_a) = \partial_{\lambda_1}\partial_{\lambda_2}\ldots\partial_{\lambda_i} D_\nu F^{\mu\nu}_a, \tag{28}$$

$$\delta(\partial_{\lambda_1}\partial_{\lambda_2}\ldots\partial_{\lambda_j} C^*_a) = \partial_{\lambda_1}\partial_{\lambda_2}\ldots\partial_{\lambda_j} D_\mu A^{*\mu}_a \tag{29}$$

and extended to the full algebra as an odd derivation. The $A^{*\mu}_a$ and C^*_a are the "antifields"; they are respectively odd and even and have the following gradings

$$antigh(A^{*\mu}_a) = 1, \tag{30}$$

$$antigh(C^*_a) = 2 \tag{31}$$

(of course $antigh(A^a_\mu) = 0$). The $A^{*\mu}_a$ are actually the usual Koszul generators. There is no need for further antifields of degree 3 or higher because the Noether relations among the equations of motion are independent. One has $\delta^2 = 0$ because of the Noether identities (6). Thus, δ is a differential of antighost number -1. Furthermore, δ commutes with ∂_μ, which implies $\delta d + d\delta = 0$.

The cohomology of the Koszul–Tate differential δ is given by the following theorem:

Theorem 4. (Cohomology of Koszul–Tate differential δ) *For each k, the Koszul–Tate complex (K_k, δ) of order k, with*

$$K_k \equiv C^\infty(J^k(E)) \otimes \Lambda\Big[A^{*\mu}_a, \ldots, \partial_{\lambda_1\lambda_2\ldots\lambda_{k-2}} A^{*\mu}_a\Big] \otimes R\Big[C^*_a, \ldots, \partial_{\lambda_1\lambda_2\ldots\lambda_{k-3}} C^*_a\Big] \tag{32}$$

provides a resolution of the algebra $C^\infty(\Sigma_k)$ of functions defined on the stationary surface. That is,

$$H_0(\delta) = C^\infty(\Sigma_k), \tag{33}$$

$$H_j(\delta) = 0, \quad j > 0. \tag{34}$$

Proof. See [16,17,20,21]. □

It follows from this theorem that the equation $f \approx 0$, where f has antighost number 0, is equivalent to $f = \delta m$ for some m of antighost number 1. One can replace $C^\infty(J^k(E))$ in K_k by $\Omega^{*,0}(J^\infty(E))$, i.e., one can take the tensor product of K_k with $\Lambda[dx^\mu]$. The theorem remains then true provided one replaces in (33) the algebra $C^\infty(\Sigma_k)$ by $C^\infty(\Sigma_k) \otimes \Lambda[dx^\mu]$. Also, one may consider – as we do here – polynomial functions on the jet spaces without changing the conclusions since the equations of motion themselves are polynomial.

Another cohomology of interest is the cohomology $H^j_i(\delta|d)$ of δ modulo d. Here, j is the form degree while i is the antighost number. The cohomology $H^*_*(\delta|d)$ is defined by the cocycle condition

$$\delta a + db = 0 \tag{35}$$

with the coboundary condition

$$a \text{ is a coboundary for } \delta \text{ modulo } d \quad \Leftrightarrow \quad a = \delta c + de. \tag{36}$$

This cohomology arises when studying the cohomology of δ in the space of local functionals, i.e., in the space of integrated local n-forms $\int \omega^{(n)}$ [21]. Contrary to $H(\delta)$, it may not vanish for antighost number $i \neq 0$ [21]. In fact, in the Yang–Mills case, one has:

Theorem 5. (Cohomology of δ modulo d, $n > 2$) *The only non-vanishing cohomological groups $H^j_i(\delta|d)$, $i > 0$, $j > 0$ are $H^n_2(\delta|d)$, $H^n_1(\delta|d)$ and $H^{n-1}_1(\delta|d)$. The group $H^n_1(\delta|d)$ is isomorphic to the set of equivalence classes of symmetries, where two symmetries (i.e., two transformations of the A^a_μ leaving the Yang–Mills action invariant) are identified if their difference reduces to a gauge transformation on-shell. Furthermore, $H^n_2(\delta|d)$ and $H^{n-1}_1(\delta|d)$ are isomorphic. A complete list of independent representatives of the classes of $H^n_2(\delta|d)$ is given by*

$$C^*_A \, dx^0 dx^1 \ldots dx^{n-1}, \tag{37}$$

where A ranges over the abelian factors of the gauge group.

Proof. See [3]. □

The isomorphism of $H^n_2(\delta|d)$ and $H^{n-1}_1(\delta|d)$ follows from standard descent equation techniques and from the vanishing theorems for $H(\delta)$ and $H(d)$. For a semi-simple gauge group, $H^n_2(\delta|d)$ and $H^{n-1}_1(\delta|d)$ vanish. Only $H^n_1(\delta|d)$ is non-trivial.

6. A new look at the characteristic cohomology

The characteristic cohomology can be reformulated as a "mod" cohomology with the help of the Koszul-Tate differential. Indeed, the cocycle condition (20) can be rewritten as

$$da + \delta m = 0 \tag{38}$$

for some form m of antighost number 1. Similarly, the coboundary condition (21) is equivalent to

$$c = db + \delta n. \tag{39}$$

for some n of antighost number 1. Thus, the characteristic cohomology is just the cohomology $H_0^j(d|\delta)$ of d modulo δ. We shall in the sequel refer to $H_i^j(d|\delta)$ as the characteristic cohomology even when $i \neq 0$.

The vanishing theorems for $H(\delta)$ and $H(d)$ easily lead to the following isomorphism between the characteristic cohomology and $H_i^j(\delta|d)$.

Theorem 6. (Isomorphism between the characteristic cohomology and $H_i^j(\delta|d)$) *One has*

$$H_j^i(\delta|d) \simeq H_{j-1}^{i-1}(d|\delta) \quad (i > 0,\ j > 0,\ (i,j) \neq (1,1)), \tag{40}$$

$$H_1^1(\delta|d) \simeq \frac{H_0^0(d|\delta)}{R}. \tag{41}$$

Proof. See [3]. □

In particular, the isomorphism $H_0^{n-1}(d|\delta) \simeq H_1^n(\delta|d)$ is just a cohomological reformulation of the Noether theorem since $H_0^{n-1}(d|\delta)$ contains the conserved currents and $H_1^n(\delta|d)$ the symmetries of the action. Moreover, the elements in $H_2^n(\delta|d)$ corresponding to ${}^*F_A \in H_0^{n-2}(d|\delta)$ through the isomorphisms $H_0^{n-2}(d|\delta) \simeq H_1^{n-1}(\delta|d) \simeq H_2^n(\delta|d)$ are just the n-forms $C_A^* \, dx^0 \ldots dx^{n-1}$ of Eq. (37). It is actually by using the isomorphism theorem and computing directly $H_2^n(\delta|d)$ that we have been able to work out the characteristic cohomology in degree $n-2$ for Yang-Mills theory in [3].

7. Lie algebra cohomology

We now introduce a differential which, contrary to the previous ones, is related to the gauge transformations rather than to the equations of motion.

The gauge transformations (5) act on the bundle $E \equiv J^0(E)$. They provide a nonlinear realization of the Lie algebra $\mathcal{G}_1$ parametrized by ε^a and $\partial_\mu \varepsilon^a$, regarded as independent parameters. The commutator of two elements $(\varepsilon^a, \partial_\mu \varepsilon^a)$ and $(\eta^a, \partial_\mu \eta^a)$ is parametrized by $(\zeta^a, \partial_\mu \zeta^a)$, with

$$\zeta^a = C^a_{bc}\varepsilon^b\eta^c. \tag{42}$$

The Lie algebra $\mathcal{G} \equiv \mathcal{G}_0$ of the gauge group is the subalgebra with $\partial_\mu\varepsilon^a = 0$ (constant gauge transformations). The action is non-linear because $\delta_\varepsilon A^a_\mu$ does not depend linearly on A^a_μ in (5): if one multiplies A^a_μ by 5, say, $\delta_\varepsilon A^a_\mu$ is not multiplied by 5 because of the inhomogeneous term $\partial_\mu\varepsilon^a$.

One can extend the action of the gauge transformations (5) to any finite order jet bundle $J^k(E)$ by simply taking their successive derivatives with respect to the coordinates k times ("prolongation of the symmetry" in jet bundle terminology). For instance, in $J^1(E)$, the gauge transformations are given by (5) and

$$\delta_\varepsilon\left(\partial_\mu A^a_\nu\right) \equiv \partial_\mu\delta_\varepsilon A^a_\nu = \partial_{\mu\nu}\varepsilon^a - C^a_{bc}\partial_\mu A^b_\nu\varepsilon^c - C^a_{bc}A^b_\nu\partial_\mu\varepsilon^c. \tag{43}$$

They are parametrized by $(\varepsilon^a, \partial_\mu\varepsilon^a, \partial_\nu\partial_\mu\varepsilon^a)$ and form a Lie algebra $\mathcal{G}_2$ of which $\mathcal{G}_0$ is again the subalgebra with gauge parameters having vanishing derivatives. The commutator of two elements $(\varepsilon^a, \partial_\mu\varepsilon^a, \partial_\nu\partial_\mu\varepsilon^a)$ and $(\eta^a, \partial_\mu\eta^a, \partial_\nu\partial_\mu\eta^a)$ of $\mathcal{G}_2$ is given by $(\zeta^a, \partial_\mu\zeta^a, \partial_\nu\partial_\mu\zeta^a)$ with ζ^a given again by (42). We shall denote by $\mathcal{G}_{k+1}$ the Lie algebra acting in $J^k(E)$. One can even go all the way to the infinite jet space $J^\infty(E)$. The corresponding Lie algebra of gauge transformations is parametrized by ε^a and all its derivatives, and is denoted by $\mathcal{G}_\infty$.

The Lie algebra $\mathcal{G}_0$ of the gauge group is more than just a subalgebra of $\mathcal{G}_\infty$. It is also the quotient algebra of $\mathcal{G}_\infty$ by the infinite-dimensional ideal $\mathcal{G}'_\infty$ of transformations with $\varepsilon^a = 0$ (but $\partial_\mu\varepsilon^a \neq 0$, $\partial_\nu\partial_\mu\varepsilon^a \neq 0$, etc.).

$$\mathcal{G}_0 = \frac{\mathcal{G}_\infty}{\mathcal{G}'_\infty}. \tag{44}$$

One calls tensor representations of $\mathcal{G}_\infty$ the linear representations of $\mathcal{G}_\infty$ in which the subalgebra $\mathcal{G}'_\infty$ is mapped to zero. These are in bijective correspondence with the linear representations of $\mathcal{G}_0$. The curvature components $F^a_{\mu\nu}$ and their successive covariant derivatives, for fixed values of the spacetime indices, all transform in the tensor representation of $\mathcal{G}_\infty$ corresponding to the adjoint representation of $\mathcal{G}_0$.

Because $\mathcal{G}_\infty$ acts on $J^\infty(E)$ and thus also on functions on $J^\infty(E)$, one can introduce in the usual manner the coboundary operator γ for the Lie algebra cohomology in the module of functions on $J^\infty(E)$ through the formula

$$\gamma\partial_{\mu_1\mu_2\ldots\mu_k}A^a_\mu = \partial_{\mu_1\mu_2\ldots\mu_k}D_\mu C^a, \tag{45}$$

$$\gamma\partial_{\mu_1\mu_2\ldots\mu_k}C^a = \partial_{\mu_1\mu_2\ldots\mu_k}\left(\frac{1}{2}C^a_{bc}C^bC^c\right). \tag{46}$$

One extends γ to the algebra $C^\infty(J^\infty(E))\otimes\Lambda[C^a, \partial_\mu C^a, \partial_{\mu\nu}C^a, \ldots]$ as an odd derivation. The odd generators C^a are known as the ghosts and are assigned pure ghost number equal to 1, whereas A^a_μ has pure ghost number equal to zero. Accordingly, γ has pure ghost number equal to 1. It is immediate to verify that $\gamma\partial_\mu = \partial_\mu\gamma$. Since the action of γ, although non-linear, is nevertheless polynomial, one may restrict $C^\infty(J^\infty(E))$ to polynomial functions.

One may then extend the action of γ to the antifields by demanding $\gamma\delta + \delta\gamma = 0$ and $\delta\partial_{\mu_1\mu_2\ldots\mu_k}C^a = 0$. Because the left-hand sides $D_\nu F_a^{\mu\nu}$ of the Yang–Mills equations of motion transform tensorially in the coadjoint representation of $\mathcal{G}_0$, this requirement is equivalent to stating that the antifields (and their covariant derivatives) transform also in the tensorial representation of $\mathcal{G}_\infty$ corresponding to the coadjoint representation of $\mathcal{G}_0$. Consequently,

$$\gamma A_a^{*\mu} = C_{ab}^c C^b A_c^{*\mu}, \tag{47}$$

$$\gamma C_a^* = C_{ab}^c C^b C_c^*. \tag{48}$$

The action of γ on the derivatives of the antifields is obtained by the rule $\gamma\partial_\mu = \partial_\mu\gamma$. To complete the definition of the differential γ in the (polynomial) algebra $\bar{\mathcal{E}}$ of local forms involving the vector potentials, the antifields, the ghosts and their derivatives, one sets $\gamma dx^\mu = 0$, which implies $\gamma d + d\gamma = 0$. The antifields are of course assigned pure ghost number equal to zero, while the ghosts have antighost number zero (see e.g. [2] for more information).

The cohomology of γ in the algebra $\bar{\mathcal{E}}$ is the Lie algebra cohomology of the infinite-dimensional Lie algebra $\mathcal{G}_\infty$ in the module of the polynomials in the fields A_μ^a, the antifields and their derivatives. To compute it more explicitly, it is necessary to understand the role played by the non-linear term $\partial_\mu C^a$ appearing in the transformation law of A_μ^a. To that end, let us first consider the case of a single abelian field A_μ. One then has

$$\gamma\partial_{\mu_1\mu_2\ldots\mu_k}A_\mu = \partial_{\mu\mu_1\mu_2\ldots\mu_k}C, \tag{49}$$

γ(everything else) $= 0$. One sees that the symmetrized derivatives of A_μ and the derivatives of the ghost form contractible pairs and drop from the cohomology. Thus, the cohomology of the infinite-dimensional Lie algebra $\mathcal{G}_\infty$ acting on the $\mathcal{G}_\infty$-module of the polynomials in the A_μ's, the antifields and their derivatives actually reduces to the cohomology of the finite-dimensional Lie algebra $\mathcal{G}_0$ acting on the $\mathcal{G}_0$-module of the polynomials in the $F_{\mu\nu}$'s, the antifields and their derivatives (which actually all transform according to the trivial representation of $\mathcal{G}_0$).

The same property is true in the general case: the derivatives of the ghosts are killed in cohomology by the symmetrized derivatives of the vector potential. Let $\mathcal{C}$ be the algebra of local polynomial forms that involve only the field strength components $F_{\mu\nu}^a$, the antifields $A_a^{*\mu}$, the antifields C_a^* and their successive covariant derivatives $D_{\lambda_1\ldots\lambda_i}F_{\mu\nu}^a$, $D_{\lambda_1\ldots\lambda_i}A_a^{*\mu}$, $D_{\lambda_1\ldots\lambda_i}C_a^*$ $(i = 1,2,3,\ldots)$. We shall denote all these variables, which transform tensorially, by z^Δ. The algebra $\mathcal{C}$ is clearly the representation space of a *tensor* representation of $\mathcal{G}_\infty$. Thus, contrary to $\bar{\mathcal{E}}$, it provides also a representation of the finite-dimensional Lie algebra $\mathcal{G}_0$. The theorem that generalizes the situation found in the abelian case is:

Theorem 7. (Isomorphism between $H(\gamma,\bar{\mathcal{E}})$ and $H(\mathcal{G}_0,\mathcal{C})$) *The cohomology of the differential γ in $\bar{\mathcal{E}}$, which is identical to the Lie algebra cohomology $H(\mathcal{G}_\infty,\bar{\mathcal{E}})$ of* *the infinite-dimensional Lie algebra $\mathcal{G}_\infty$ in the $\mathcal{G}_\infty$-module $\bar{\mathcal{E}}$, is isomorphic to the Lie algebra cohomology $H(\mathcal{G}_0, C)$ of the finite-dimensional Lie algebra $\mathcal{G}_0$ in the $\mathcal{G}_0$-module C,*

$$H(\gamma, \bar{\mathcal{E}}) \equiv H(\mathcal{G}_\infty, \bar{\mathcal{E}}) \simeq H(\mathcal{G}_0, C). \tag{50}$$

Proof. See [15,9,22]. □

Since the algebra $\mathcal{G}_0$ is finite-dimensional, one can use standard theorems on Lie algebra cohomology to find

Theorem 8. (Cohomology of γ) *Up to γ-exact terms, the general solution of the cocycle condition $\gamma a = 0$ in $\mathcal{E}$ reads*

$$a = \sum P_J(z^\Delta)\omega^J(C^a), \tag{51}$$

where the sum is over a basis $\{\omega^J\}$ of the Lie algebra cohomology $H(\mathcal{G}_0) \equiv H(\mathcal{G}_0, C)$ of $\mathcal{G}_0$ and where the P_J are invariant polynomials in the z^Δ (which may also involve the coordinates x^μ and the dx^μ).

In the case of an abelian gauge group of dimension one, there is just one ω^J, namely, the abelian ghost C itself. For $SU(m)$, the algebra $H(\mathcal{G}_0)$ is generated by the "primitive forms" $\operatorname{tr} C^3$, $\operatorname{tr} C^5$ up to $\operatorname{tr} C^{2m-1}$. Here we have set $C \equiv C^a T_a$, where the T_a are the generators of the adjoint representation. Finally, we point out that the cohomology $H(\gamma|d)$ of γ modulo d has also been investigated in the literature [9,15,2], but we shall not report the results here.

8. BRST cohomology

The BRST differential takes into account both the dynamics and the gauge symmetries. For a theory with a gauge algebra that closes off-shell, it is simply the sum of δ and γ,

$$s = \delta + \gamma. \tag{52}$$

It increases by one unit the (total) ghost number, defined to be the difference between the pure ghost number and the antighost number. The BRST cohomology can be shown on general grounds to be equal to

$$H^k(s) \simeq H^k(\gamma, H_0(\delta)), \tag{53}$$

where k stands on the left-hand side for the total ghost number while it stands on the right hand side for the pure ghost number. This follows from a simple spectral sequence argument explained for example in this particular context in [25].

In practice, however, one needs a more precise characterization of the representatives of the BRST cohomology. This is given by the following theorem.

Theorem 9. (Joglekar and Lee Theorem) *In each equivalence class of the BRST cohomology, one can find a representative that does not depend on the antifields and that is thus annihilated by* γ. *In particular, if* a *is a BRST cocycle of ghost number zero, then one has*

$$sa = 0, \; gh(a) = 0 \quad \Leftrightarrow \quad a = \bar{a} + sb, \tag{54}$$

where $\bar{a}$ *is an invariant polynomial in the field strength components and their covariant derivatives.*

Proof. See [26,22]. □

This theorem plays a key role in the analysis of the renormalization of local gauge invariant operators [13].

9. Wess–Zumino consistency condition

Another cohomology that is also of fundamental physical importance is the cohomology $H(s|d)$ of s modulo d. The corresponding cocycle condition reads

$$sa + db = 0 \tag{55}$$

and is known as the Wess–Zumino consistency condition [33]. Trivial solutions of (55) are of the form $a = sm + dn$. The Wess–Zumino consistency condition arose first in the context of anomalies, where it constrains severely the form of the candidate anomalies, but it is also quite important in analyzing the counterterms that are needed in the renormalization of integrated gauge invariant operators and, classically, in determining the form of the consistent deformations of the action.

The general solution of the Wess–Zumino consistency condition for Yang–Mills theory has been worked out in [2] and involves all the cohomologies described previously. There is indeed a close connection between $H(s|d)$, $H(\delta|d)$ and $H(\gamma|d)$, given again by a spectral sequence argument [25,3]. We shall not give here the results of our general investigation but shall merely quote two theorems of direct physical interest. These theorems are valid in *four spacetime dimensions* and for a *semi-simple* gauge group (no abelian factors).

Theorem 10. (On the Kluberg-Stern and Zuber conjecture) *Up to trivial terms, the general solution of the Wess–Zumino consistency condition with ghost number zero is exhausted by the invariant polynomials in the field strength components and their covariant derivatives.*

Proof. See [6,2]. □

Theorem 11. (Adler–Bardeen anomaly) *The only solutions of the Wess–Zumino consistency condition with ghost number one are given by the multiples of the Adler–Bardeen anomaly*

$$a = \mathrm{tr}\left\{C\left[dAdA + \frac{1}{2}\left(AdAA - A^2dA - dAA^2\right)\right]\right\} \tag{56}$$

(*up to trivial terms*). *Here, A is the 1-form* $A^a_\mu dx^\mu T_a$.

Proof. See [6,2]. □

These theorems are not true anymore in other spacetime dimensions (in odd dimensions, one has the Chern–Simons forms that solve (55) at ghost number zero), or in the presence of abelian factors. We refer to [2] for more information.

10. Conclusions

In this paper, we have illustrated the usefulness and power of cohomological concepts in local field theory. Even though its gauge structure is simple (off-shell closure, irreducibility), the Yang–Mills system provides a great diversity of cohomologies of direct physical significance. These cohomologies are the basic ingredients of the antifield formalism and allow one to describe the general solution of the crucial Wess–Zumino consistency condition with antifields included. The approach to Yang–Mills gauge models outlined here can be characterized as the "cohomological approach to Yang–Mills theory".

Following analogous ideas, a similar analysis of the Wess–Zumino consistency condition has been carried out successfully for other theories with a gauge freedom, including Einstein gravity [5], two-dimensional gravity [10] as well as theories involving p-form gauge fields [23].

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