Name Answer Key .

Teacher/Block (circle): Kelly/H Olsen/C Olsen/F Verner/G

Honors Advanced Math
Final Exam 2009

Lexington High School
Mathematics Department

This is a 90-minute exam, but you will be allowed to work for up to 120 minutes.

The exam has 3 parts. Directions for each part appear below.

In total, there are 58 points that you can earn. A letter grade scale will be set by the course faculty after the tests have been graded.

Part A. Short Problems

7 questions, 2 points each, 14 points total

You must write your answers in the answer boxes.

If your answer is correct, you will receive full credit. Showing work is not required.

If your answer is incorrect, you may receive half credit if you have shown some correct work.

A good pace on this part would be to spend 2-4 minutes per problem.

Part B. Medium Problems

5 problems, 4 points each, 20 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend 4-6 minutes per problem.

Part C. Long Problems

3 problems, 8 points each, 24 points total

Write a complete, clearly explained solution to each problem. Partial credit will be given.

A good pace on this part would be to spend 8-12 minutes per problem.

Part A. Short Problems

7 problems, 2 points each, 14 points total

1. Algebraically, solve the equation $2\cos(3x-1)=\sqrt{3}$. Find all solutions, and give their exact values, not decimal approximations.

$$2\cos(3x-1)=\sqrt{3}$$
$$\cos(3x-1)=\frac{\sqrt{3}}{2}$$
$$(3x-1)=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}+2\pi n,\quad \frac{11\pi}{6}+2\pi n$$
$$x=\frac{\pi}{18}+\frac{2\pi n}{3}+\frac{1}{3},\quad \frac{11\pi}{18}+\frac{2\pi n}{3}+\frac{1}{3}$$

Answer to question 1:

$$x=\frac{\pi+6+12\pi n}{18},\quad \frac{11\pi+6+12\pi n}{18}$$

2. Consider the infinite **geometric series** $4x + 12x^3 + 36x^5 + 108x^7 + \cdots$. Find the interval of x values for which the sum of this series is finite. For those x values, find the sum of the series.

Geometric series so $t_1 = 4x$ and $r = 3x^2$ with $-1 < r < 1$

Answers to question 2:

interval of x values: $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

sum of the series: $S=\frac{t_1}{1-r}=\frac{4x}{1-3x^2}$

3. Consider the ellipse given by the equations $\left(\frac{x-6}{3}\right)^2+\left(\frac{y-8}{4}\right)^2=1$. Write parametric equations for this ellipse.

Answer to question 3:

$x=$ $3\cos(t)+6$ $y=$ $4\sin(t)+8$

4. Kryptonite and adamantium are (fictitious) elements that exhibit radioactive decay.
Suppose there are 2500 mg of kryptonite at time $t = 0$ days, and that kryptonite has a half-life of 10 days. Also suppose that there are 3000 mg of adamantium at $t = 4$ days and 750 mg of adamantium at $t = 12$ days.
At what time is the amount of kryptonite equal to the amount of adamantium?
(You may give your answer as a decimal approximation accurate to the nearest 0.01.)

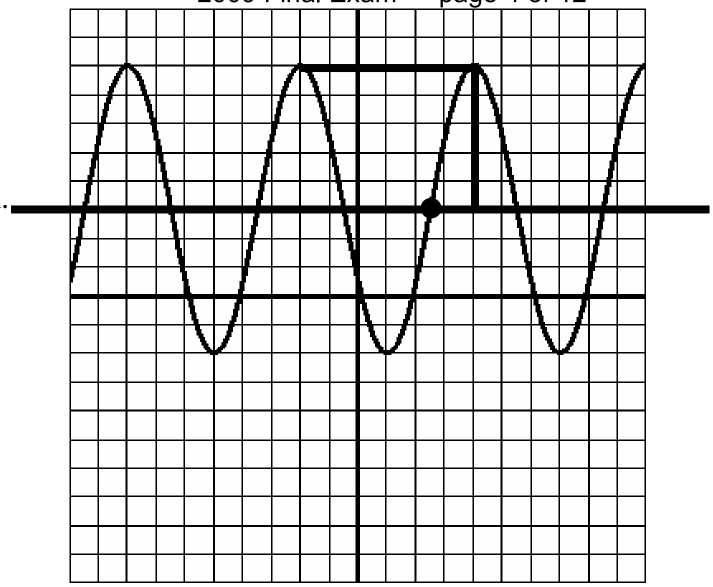
Find *K* (kryptonite equation)	**Find *D* (aDamantium equation)**	**Find *t***
$K=ab^t$	$D=ab^t$	$2500\cdot 2^{-\frac{t}{10}}=6000\cdot 2^{-\frac{t}{4}}$
$2500=ab^0$	$3000=ab^4$	$2^{-\frac{t}{10}+\frac{t}{4}}=\frac{6000}{2500}=\frac{12}{5}=2^{\frac{3t}{20}}$
$0.5a=ab^{10}$	$750=ab^{12}$	$\frac{3t}{20}=\frac{\ln(2.4)}{\ln(2)}$
$a=2500$	$b^8=750/3000=0.25$	$t=\frac{20\cdot\ln(2.4)}{3\cdot\ln(2)}$
$b=\sqrt[10]{\frac{1}{2}}=2^{-0.1}$	$b=2^{\frac{-2}{8}}=2^{-0.25}$	
$K=2500\cdot 2^{-\frac{t}{10}}$	$3000=a\left(2^{-0.25}\right)^4=\frac{a}{2}$	
	$D=6000\cdot 2^{-\frac{t}{4}}$	

Answer to question 4:

$t = 8.42$ days

5. The graph of a sinusoidal function $f(x)$ on the window [–10, 10] by [–10, 10] is shown. All maxima and minima have whole-number coordinates.

Write a formula **involving sine** for this function.



Axis of the Wave = 3

Amplitude = 5

Period = 6

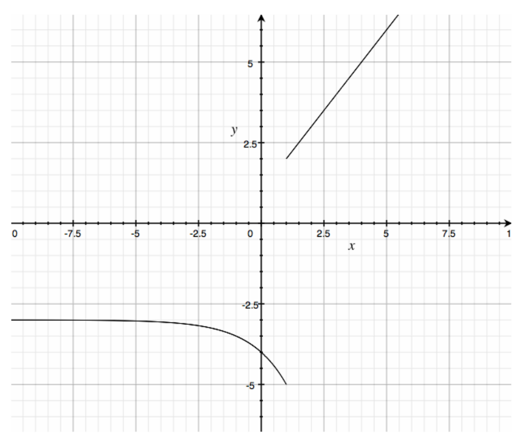
Start of sine at 2.5

Answer to question 5:

$$f(x) = 5\sin\left(\frac{2\pi}{6}(t-2.5)\right)+3$$

6. Find the domain and range for the piecewise function $f(x)=\begin{cases} \dfrac{x^2-2x-3}{x-3} & \text{if} \quad x \geq 1, \\ -2^x-3 & \text{if} \quad x < 1. \end{cases}$

$$\frac{x^2-2x-3}{x-3} = \frac{(x-3)(x+1)}{(x-3)} \approx (x+1)$$



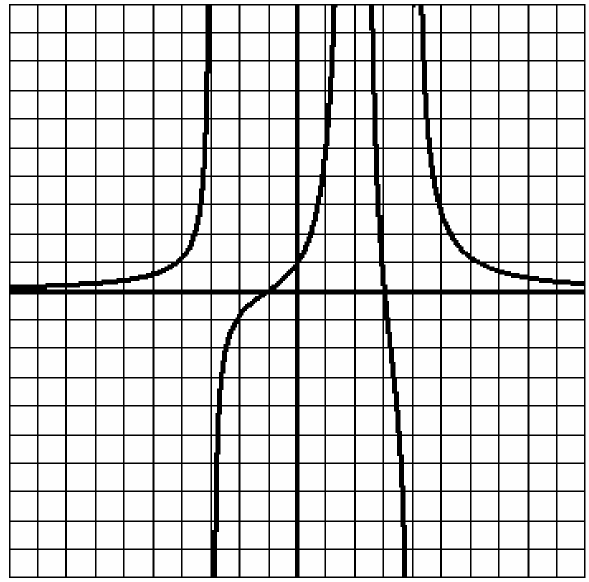
Answers to question 6:

domain: $x \neq 3$

range: $(-5, -3) \cup [2, 4) \cup (4, \infty)$

7. The graph of a rational function $r(x)$ on the window [–10, 10] by [–10, 10] is shown. Although not visible, the graph has a one hole, at the point (–1, 0).

Write a possible function formula for $r(x)$.



Hole at $x = -1$
so $(x + 1)$ in numerator and denominator

Zeros at $x = -1, 3$
so $(x + 1)(x - 3)$ in numerator

Asymptotes at $x = -3, 2, 4$
so $(x + 3)(x - 2)(x - 4)$ in denominator

Same direction on both sides of $x = 2$
so multiplicity of 2

y – intercept at $y = 1$ so adjust leading coefficient

$$r(0) = \frac{a(x-3)(x+1)^2}{(x+3)(x+1)(x-2)^2(x-4)} = 1 \quad \text{so } a = 16$$

Answer to question 7:

$$r(x) = \frac{16(x-3)(x+1)^2}{(x+3)(x+1)(x-2)^2(x-4)}$$

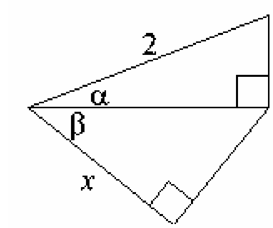
Part B. Medium Problems

5 problems, 4 points each, 20 points total

8. Suppose b, N, and M are positive numbers. Prove $\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$. You may assume the properties of exponents and you may assume that logarithmic and exponential functions are inverses of each other. You may **not** assume log properties such as $\log_b M + \log_b N = \log_b(MN)$.

$$b^m = M \iff m = \log_b(M)$$
$$b^n = N \iff n = \log_b(N)$$
$$\frac{b^m}{b^n} = b^{m-n} = \frac{M}{N} \iff m - n = \log_b\left(\frac{M}{N}\right)$$
$$\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right)$$

9. a. In the diagram at the right, find an expression for the length x in terms of trigonometric functions of α and/or β.



$$\cos(\alpha) = \frac{y}{2} \Rightarrow y = 2\cos(\alpha)$$

$$\cos(\beta) = \frac{x}{y} \Rightarrow x = y\cos(\beta) \Rightarrow x = 2\cos(\alpha)\cos(\beta)$$

b. Suppose $\alpha + \beta = \frac{\pi}{3}$. Show that $x = 1 + 2\sin(\alpha)\sin(\beta)$.

$$\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \cos(\alpha+\beta)$$
$$\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \sin(\alpha)\sin(\beta)$$
$$x = 2\cos(\alpha)\cos(\beta) = 2\big(\cos(\alpha+\beta) + \sin(\alpha)\sin(\beta)\big)$$
$$x = 2\left(\cos\left(\frac{\pi}{3}\right) + \sin(\alpha)\sin(\beta)\right) = 2\left(\frac{1}{2} + \sin(\alpha)\sin(\beta)\right) = 1 + 2\sin(\alpha)\sin(\beta)$$

10. Consider the function $f(x)=\sin(2x-6)+(2x-6)^3+7$.

a. Find a sequence of two basic transformations (translations, dilations, rotations, or reflections) that transforms the graph of $f(x)=\sin(2x-6)+(2x-6)^3+7$ into the graph of an odd function. Let $g(x)$ stand for the function after the first transformation, and let $h(x)$ be the final function that should be odd. Identify the two transformations and give formulas for $g(x)$ and $h(x)$.

first transformation: move function left 3 .

function after first transformation: $g(x) = f(x+3) = \sin(2x) + (2x)^3 + 7$.

second transformation: move function down 7 .

odd function after second transformation: $h(x) = g(x) - 7 = \sin(2x) + (2x)^3$.

b. Algebraically prove that function $h(x)$ from above is an odd function. Make sure your proof is clear and well organized.

$$
\begin{aligned}
h(-x) &= \sin(2(-x)) + (2(-x))^3 \\
&= \sin(-2x) + (-1)^3(2x)^3 \\
&= -\sin(2x) - (2x)^3 \\
&= -\left(\sin(2x) + (2x)^3\right) \\
&= -h(x)
\end{aligned}
$$

11. a. Find the **complex conjugate** of the polar complex number $\sqrt{2}\,\text{cis}\,(\frac{5\pi}{4})$,
where cis(θ) denotes cos(θ) + i sin(θ). Express your answer in the form $a + bi$.

$$\begin{aligned}\bar{z} &= \sqrt{2}\cos\left(\frac{5\pi}{4}\right) - i\sqrt{2}\sin\left(\frac{5\pi}{4}\right)\\ &= \sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) - i\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right)\\ &= -1+i\end{aligned}$$

b. Suppose $p(x)$ is polynomial function with the following properties:

- $p(x)$ has degree 4. $\Rightarrow$ **four factors**
- $p(x)$ has real coefficients. $\Rightarrow$ **imaginary zeros in conjugate pairs**
- When the real numbers are used as inputs, the range of $p(x)$ is $(-\infty,0]$.$\Rightarrow$ **all negative**
- $p(1) = -39$, $p(-2) = 0$, and $p\left(\sqrt{2}\,\text{cis}\,(\frac{5\pi}{4})\right) = 0$. $\Rightarrow$ **zeros and extra point for *a***

Find a function formula for $p(x)$. You may leave your answer in factored form.

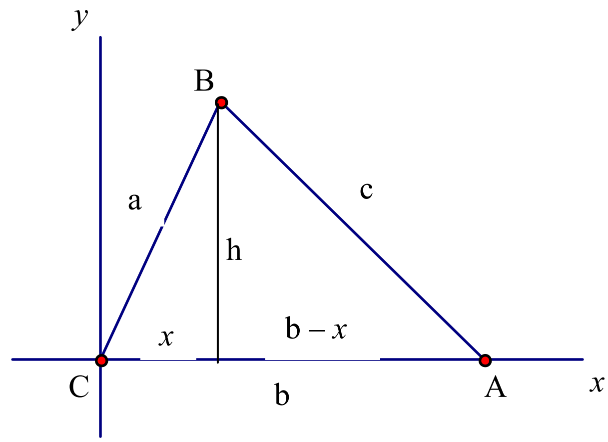
$$\begin{aligned}P(x) &= a(x+2)^2(x-(-1+i))(x-(-1-i))\\ &= a(x+2)^2((x+1)-i)((x+1)+i)\\ &= a(x+2)^2\left((x+1)^2+1\right)\end{aligned}$$

$$\begin{aligned}P(1) &= a(3)^2\left(2^2+1\right)\\ -39 &= a(9)(5)\\ a &= -\frac{13}{15}\end{aligned}$$

$$P(x) = -\frac{13}{15}(x+2)^2\left((x+1)^2+1\right)$$

12. In triangle ΔABC, let a stand for the length of the side opposite angle A, let b stand for the length of the side opposite angle B, and let c stand for the length of the side opposite angle C.

Prove the **Law of Cosines**, $c^2 = a^2 + b^2 - 2\,ab \cos \mathrm{C}$, in the case where angle C is an acute angle. You may use any formulas from this course or a previous course that are not dependent on, or equivalent to, the Law of Cosines.



Using notation added to the graph:

$x^2 + h^2 = a^2$

$(b - x)^2 + h^2 = c^2$

$b^2 - 2bx + x^2 + h^2 = c^2$

$b^2 - 2bx + a^2 = c^2$

$c^2 = a^2 + b^2 - 2bx \qquad \cos C = \dfrac{x}{a} \Rightarrow x = a \cos C$

$c^2 = a^2 + b^2 - 2ba \cos C$

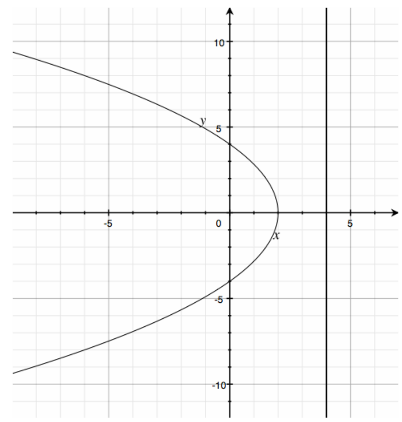
$c^2 = a^2 + b^2 - 2ab \cos C$

Part C. Long Problems

3 problems, 8 points each, 24 points total

13. A parabola has its focus at the origin, and its directrix is the line $x = 4$.

a. Find an *x*-*y* equation (i.e., rectangular equation or Cartesian equation) for the parabola. Sketching the parabola first may help.



$$(x-2) = \frac{1}{4(-2)}(y-0)^2$$

$$\boxed{(x-2) = -\frac{1}{8}y^2}$$

b. Find a polar equation for the parabola.

$k = 4, \quad e = 1$

$$\boxed{r = \frac{4}{1+\cos\theta}}$$

c. Now suppose the parabola is rotated 30º counterclockwise about the origin. Find an *x*-*y* equation for the rotated parabola. *Note:* Rotation by angle θ counterclockwise about the origin has the transformation equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. You do not need to simplify the resulting expression.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 30^\circ & \sin 30^\circ \\ -\sin 30^\circ & \cos 30^\circ \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' \\ -\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' \end{bmatrix}$$

plug these versions of x and y in the equation in part ***a*** and multiply through by 2.

$$\boxed{\sqrt{3}x' + y' - 4 = -\frac{1}{16}\left(-x' + \sqrt{3}y'\right)^2}$$

d. Find a polar equation for the parabola after the 30° counterclockwise rotation.
Hint: Take your equation from part **b**, and replace θ with an expression involving θ.

$$\boxed{r = \frac{4}{1+\cos(\theta - 30^\circ)}}$$

14. The equations of three planes are given below:

$$\begin{aligned} &\text{Plane } A: \quad x + y - 5z = -1 \\ &\text{Plane } B: \quad 2x + y - 8z = -3 \\ &\text{Plane } C: \quad -3x + 2y + 5z = 8 \end{aligned}$$

a. The intersection of these three planes is a line. Find the vector equation of the line where the planes intersect.

$$\begin{bmatrix} 1 & 1 & -5 & -1 \\ 2 & 1 & -8 & -3 \\ -3 & 2 & 5 & 8 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x &= 3z - 2 \\ y &= 2z + 1 \end{aligned}$$

$$\boxed{(x, y, z) = (-2, 1, 0) + t\langle 3, 2, 1\rangle}$$

b. Find the angle formed by planes A and B. Give an answer that is accurate to the nearest 0.01 radian or 0.01 degree.

$$\overrightarrow{v_A} = \langle 1,\ 1,\ -5\rangle \quad \left|\overrightarrow{v_A}\right| = \sqrt{1^2 + 1^2 + (-5)^2} = \sqrt{27}$$

$$\overrightarrow{v_B} = \langle 2,\ 1,\ -8\rangle \quad \left|\overrightarrow{v_B}\right| = \sqrt{2^2 + 1^2 + (-8)^2} = \sqrt{69}$$

angle between these vectors $\beta = \cos^{-1}\left(\dfrac{2+1+40}{\sqrt{27}\cdot\sqrt{69}}\right) = 0.0868$ radians

angle between planes $\alpha = \pi - \beta = 3.05$ radians $= 175.03°$

c. Let **u** = <1, 1, –5>, **v** = <2, 1, –8>, and **w** = <–3, 2, 5>. Show that these three vectors are coplanar. *Hint:* One valid method would be to show that **w** is a linear combination of **u** and **v**.

Show **w** = a**u** + b**v**

$$\left.\begin{aligned} a + 2b &= -3 \\ a + b &= 2 \\ -5a - 8b &= 5 \end{aligned}\right\} \quad \begin{aligned} b &= -5 \\ a &= 7 \end{aligned} \qquad \mathbf{w} = 7\mathbf{u} - 5\mathbf{v}$$

d. Explain why the plane containing **u**, **v**, and **w** must be perpendicular to the intersection line found in part **a**. You may give either a geometric or an algebraic explanation.

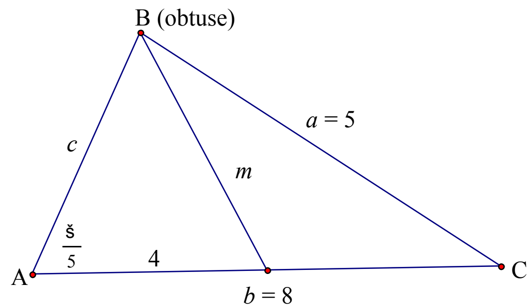
Since vectors **u**, **v**, and **w** are perpendicular to planes A, B, and C, respectively, they are perpendicular to every line in the planes. This includes the line found in part ***a.*** As such, the plane containing **u**, **v**, and **w** must be perpendicular to the line found in part ***a***.

15. In triangle ΔABC, let a stand for the length of the side opposite angle A, let b stand for the length of the side opposite angle B, and let c stand for the length of the side opposite angle C.

Here is some given information about the triangle: $\angle A = \frac{\pi}{5}$, $\angle B$ is obtuse, $a = 5$, and $b = 8$.

Answers to all parts may be given as decimal approximations accurate to the nearest 0.01.

a. Find side length c. *Hint:* You may wish to solve the triangle or find other measurements first.



$$\frac{\sin A}{a} = \frac{\sin\left(\frac{\pi}{5}\right)}{5} = \frac{\sin B}{8}$$

$$\sin^{-1}\left(\frac{8\sin\left(\frac{\pi}{5}\right)}{5}\right) = 1.224 \text{ (acute angle)}$$

$$\angle B = \pi - 1.224 = 1.9176 \text{ radians}$$

$$\angle C = \pi - \tfrac{\pi}{5} - 1.9176 = 0.5956 \text{ radians}$$

$$c^2 = a^2 + b^2 - 2ab\cos C = 5^2 + 8^2 - 2(5)(8)\cos(0.5956)$$

$$c = 4.77$$

b. Find the area of triangle ΔABC.

$$A = \tfrac{1}{2}ab \cdot \sin C = 11.22$$

c. Find the length of the median segment from point B to the midpoint of side AC.

$$m^2 = 4^2 + 4.77^2 - 2(4)(4.77)\cos\left(\tfrac{\pi}{5}\right) = 2.81$$

d. Find the radius of the circle that passes through points A, B, and C.

Proved in homework (Brown section 9-3 problem 23) that diameter of circumscribed circle $d = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 8.506$ so radius $= 4.25$