15-859T: A Theorist’s Toolkit, 2013

Lecture 4

How to Be a Mathematician

(or Theoretical Computer Scientist)

(This title chosen pretentiously for humorous effect.)

Part I: How to present mathematics

Part II: How to do mathematics

Part I: LaTeX

Q: What is mathematics?

A1: "Mathematics is the abstract study of topics such as quantity (numbers), structure, space, and change." – Wikipedia

A2: "Mathematics is about figuring the logical consequences of ideas we have made up – according to [the] notion of logical consequence that we have made up." – Alexander Woo

A3: "Mathematics is what mathematicians do." – Poincaré(?)

A4: Mathematics is the branch of science written in **LaTeX**.

LaTeX workflow

Select a good text editor that understands you are writing in LaTeX.

Should have:

- syntax highlighting
- hotkey to compile/display
- “synchronization”/“roundtripping”
- text/reference autocompletion
- block commenting

E.g.: WinEdt, BaKoMa, TeXnicCenter, AUCTeX, Kile, TeXshop, etc.

Use PDFLatex; nobody uses .dvi or .ps these days.

LaTeX workflow

Create a stub .tex file,
a lifetime .sty file,
and a lifetime .bib file.

LaTeX — stub .tex file example

```
\documentclass[11pt]{article}
\usepackage{odonnell}

\begin{document}

\title{}
\author{Ryan O'Donnell\thanks{odonnell@cs.cmu.edu}}
\date{\today}

\maketitle

%\begin{abstract}
%\end{abstract}


%\section{}


%\bibliographystyle{alpha}
%\bibliography{odonnell}

\end{document}
```

refers to odonnell.sty, my lifetime .sty file

refers to odonnell.bib, my lifetime .bib file

LaTeX — lifetime .sty file

```
\usepackage{fixltx2e,amsmath,amssymb,amsthm,amsfonts,bbm,graphicx,fullpage}
\usepackage[colorlinks,citecolor=blue,bookmarks=true]{hyperref}

\theoremstyle{plain}
\newtheorem{theorem}{Theorem}[section]
\newtheorem{lemma}[theorem]{Lemma}
\newtheorem{corollary}[theorem]{Corollary}
\newtheorem{proposition}[theorem]{Proposition}

\theoremstyle{definition}
\newtheorem{definition}[theorem]{Definition}
\newtheorem{remark}[theorem]{Remark}


% for ``commenting out'' chunks of text
\newcommand{\ignore}[1]{}

% for notes on the text
\newcommand{\ryansays}[1]{{\color{red}{\tiny [Ryan: #1]}}}

% macros
\newcommand{\R}{{\mathbbm R}}
\newcommand{\eps}{\epsilon}
% ...
```

LaTeX — lifetime .bib file

Start yours today!

Use .bib file management software:

e.g., JabRef, BibDesk

```
@STRING{ann = {Annual}}
@STRING{proc = {Proceedings of the}}
@STRING{focs = {IEEE Symposium on Foundations of Computer Science}}
@STRING{focs12 = proc # { 53rd } # ann # { } # focs}
@STRING{focs13 = proc # { 54th } # ann # { } # focs}
@STRING{stoc = {ACM Symposium on Theory of Computing}}
@STRING{stoc12 = proc # { 44th } # ann # { } # stoc}
@STRING{stoc13 = proc # { 45th } # ann # { } # stoc}

@ARTICLE{AGHP92,
   author = {Alon, Noga and Goldreich, Oded and H{\aa}stad, Johan and Peralta, Ren{\'e}},
    title = {Simple constructions of almost {$k$}-wise independent random variables},
  journal = {Random Structures \& Algorithms},
   volume = {3},
     year = {1992},
   number = {3},
    pages = {289--304},
      doi = {10.1002/rsa.3240030308},
      url = {http://dx.doi.org/10.1002/rsa.3240030308},
}

@ONLINE{CLRS13,
      author = {Chan, Siu On and Lee, James and Raghavendra, Prasad and Steurer, David},
       title = {Approximate constraint satisfaction requires large {LP} relaxations},
     version = {1},
        date = {2013-09-03},
        note = {arXiv:1309.0563},
  eprinttype = {arXiv},
 eprintclass = {cs.CC},
      eprint = {1309.0563}
}

@INCOLLECTION{MR12,
    author = {Mossel, Elchanan and R{\'a}cz, Mikl{\'o}s},
     title = {A quantitative {G}ibbard--{S}atterthwaite theorem without neutrality},
 booktitle = stoc12,
     pages = {1041--1060},
 publisher = {ACM},
      year = {2012},
       doi = {10.1145/2213977.2214071},
       url = {http://dx.doi.org/10.1145/2213977.2214071},
}
```

LaTeX — lifetime .bib file

Please make your .bib entries high quality!

References

[AKK+08] Sanjeev Arora, Subhash Khot, Alexandra Kolla, David Steurer, Madhur Tulsiani, and Nisheeth K. Vishnoi, *Unique games on expanding constraint graphs are easy: extended abstract*, STOC (Richard E. Ladner and Cynthia Dwork, eds.), ACM, 2008, pp. 21–28.

[Alo86] Noga Alon, *Eigenvalues and expanders*, Combinatorica **6** (1986), no. 2, 83–96.

[AM85] N. Alon and V. D. Milman, λ_1, *isoperimetric inequalities for graphs, and superconcentrators*, Journal of Combinatorial Theory. Series B **38** (1985), 73–88.

[AR98] Yonatan Aumann and Yuval Rabani, *An* $O(\log k)$ *approximate min-cut max-flow theorem and approximation algorithm*, SIAM Journal on Computing **27** (1998), no. 1, 291–301.

[ARV04] Sanjeev Arora, Satish Rao, and Umesh Vazirani, *Expander flows, geometric embeddings and graph partitioning*, Proceedings of the thirty-sixth annual ACM Symposium on Theory of Computing (STOC-04) (New York), ACM Press, June 13–15 2004, pp. 222–231.

[Che70] J. Cheeger, *A lower bound on smallest eigenvalue of a laplacian*, Problems in Analysis (Papers dedicated to Salomon Bochner) (1970), 195–199.

[DKSV06] Nikhil R. Devanur, Subhash Khot, Rishi Saket, and Nisheeth K. Vishnoi, *Integrality gaps for sparsest cut and minimum linear arrangement problems*, STOC, 2006, pp. 537–546.

[FS02] Feige and Schechtman, *On the optimality of the random hyperplane rounding technique for MAX CUT*, RSA: Random Structures Algorithms **20** (2002).

[GMPT07] Konstantinos Georgiou, Avner Magen, Toniann Pitassi, and Iannis Tourlakis, *Integrality gaps of 2 - o(1) for vertex cover SDPs in the lovész-schrijver hierarchy*, FOCS, IEEE Computer Society, 2007, pp. 702–712.

[Kho02] Subhash Khot, *On the power of unique 2-prover 1-round games*, STOC, 2002, pp. 767–775.

← This is gross. You may as well have a spelling mistake in your title.

LaTeX — lifetime .bib file

Please make your .bib entries high quality!

- Have a standard for citing proceedings

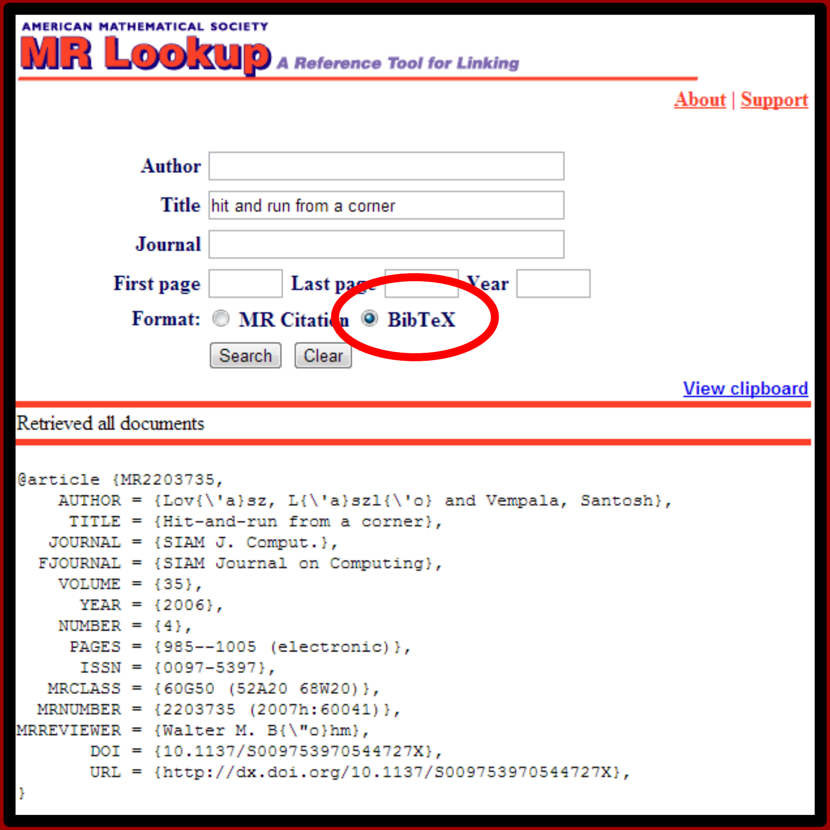
  (FOCS, STOC, etc.) and arXiv and ECCC.
- Get capitalization correct: {B}races needed
- Put in people’s first and last names—with the diacritcs!
- Use math mode for math parts of titles

LaTeX — lifetime .bib file

Where to get .bib entries:

1. Always try **www.ams.org/mrlookup** first.

   (The **best** entries, and you **don’t** need ams.org access.)

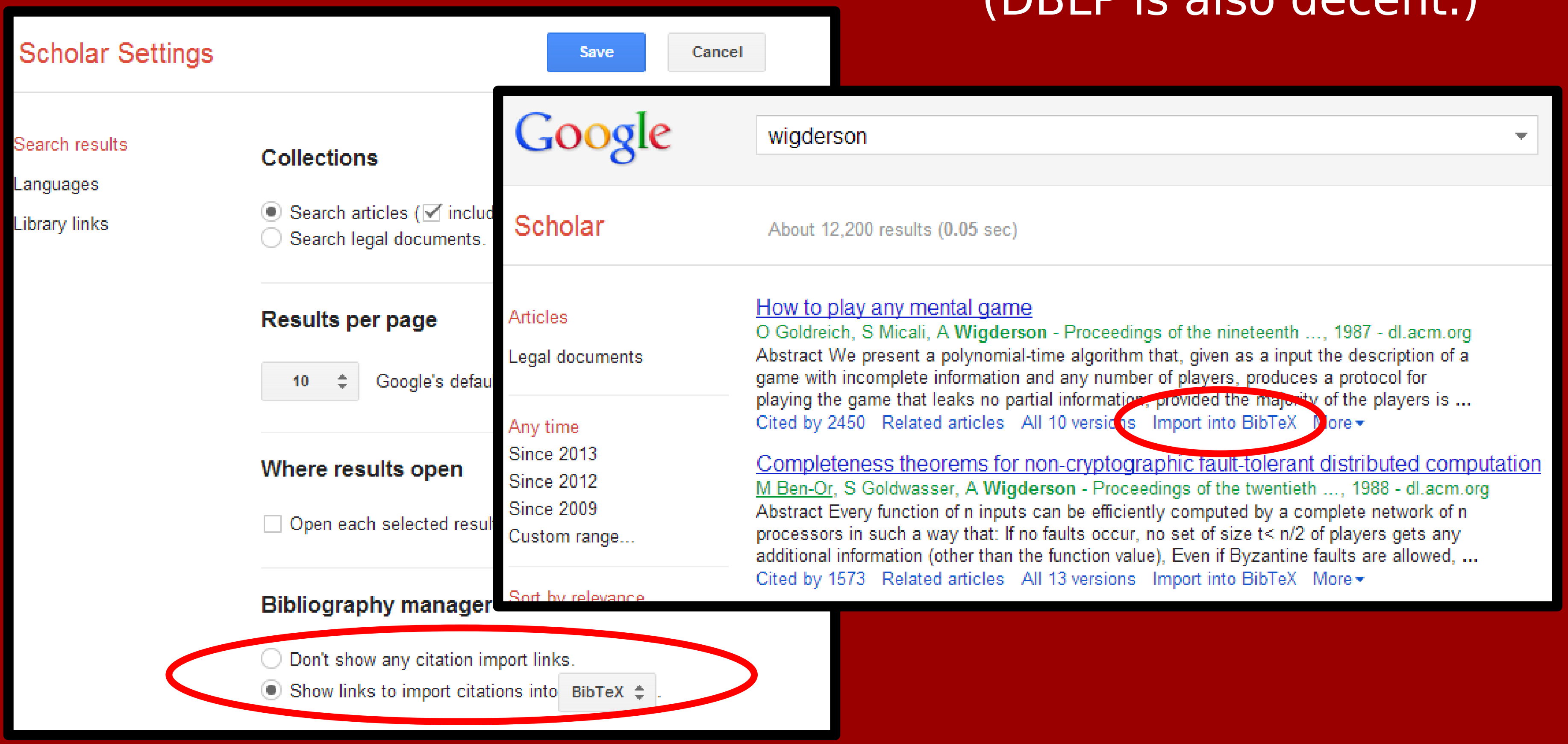


LaTeX — lifetime .bib file

Where to get .bib entries:

2. Failing that, **scholar.google.com**.

(DBLP is also decent.)



LaTeX — version control

Version control software will save you heartache and help you collaborate with others.

At the very least, get **Dropbox** (or an equivalent) to put all of your LaTeX in.

The next step up is Subversion (SVN). Try it.

LaTeX — my top peeves

DON'T	DO
`$ < U, V > $`	`$ \langle U, V \rangle $`
`"quotes"`	``` ``quotes'' ```

`$$ f(x) = x^2 $$`	`\[`
	`    f(x) = x^2`
	`\]`

`$ log(1+x) $`	`$ \log(1+x) $`

`\[`
`  (\frac{ax+b}{cy})^2`
`\]`
	`\[`
	`  \left(\frac{ax+b}{cy}\right)^2`
	`\]`

LaTeX — my top peeves

DON'T

```
\begin{eqnarray}
  y &=& (x+1)^2 \\
    &=& x^2+2x+1
\end{eqnarray}
```

```
If A is a matrix, then
```

```
Lemma \ref{lem:big} is
due to Blum \cite{Blu99}
```

```
assuming- as we do – that
the Birch-Swinnerton-Dyer
Conjecture holds
```

DO

```
\begin{align}
  y &= (x+1)^2 \\
    &= x^2+2x+1
\end{align}
```

```
If $A$ is a matrix, then
```

```
Lemma~\ref{lem:big} is
due to Blum~\cite{Blu99}
```

```
assuming---as we do---that
the Birch--Swinnerton-Dyer
Conjecture holds
```

LaTeX — my top peeves

DON'T

```
one party, e.g. Alice, is

\[
  a-b=2 ~~~~~~~~ a+b=4
\]

we execute $ALG(x)$

\begin{proof}
\[
  x=1 \implies x^2=1.
\]
\end{proof}
```

DO

```
one party, e.g.\ Alice, is

\[
  a-b=2 \qquad a+b=4
\]

we execute $\textnormal{ALG}(x)$

\begin{proof}
\[
  x=1 \implies x^2=1. \qedhere
\]
\end{proof}
```

LaTeX — my top peeves

I could go on. When in doubt, look up the correct thing to do at **tex.stackexchange.com** !

Writing mathematics well

This is a challenging, lifelong skill.

If I had to give two pieces of advice…

1. This is math, so it has to be 100% correct.

that said,

2. Take pity on your poor reader; help them out.

LaTeX — drawing

DON'T BE LAZY: include figures to help the reader.

```
\usepackage{graphicx}
...
\includegraphics{mypicture.png}
```

Was that so hard?

Works with `.jpg`, `.png`, `.pdf`, `.eps`

To draw figs: *Inkscape*, *TikZ*, *Processing*...
but ∃ learning curve.

Recommendation: draw figs with your presentation software (PowerPoint, Keynote, ...), since you have to learn it anyway...

Presentation software

If you write a paper, you'll have to make a talk.

To make a talk, you'll need PowerPoint/Keynote/Beamer.

Any of these is fine, but you'll still suffer the "drawing figures" challenge with Beamer.

It's not "hip", but **become a hacker** in one of these.

Learn to integrate **beautiful math equations**:

PowerPoint:	IguanaTex (or Office 2010 Eq'n Editor)
Keynote:	LaTeXiT (I'm told)
Beamer:	Automatic

Presenting math well

I like Kayvon Fatahalian’s tips, which I just found recently:

http://www.cs.cmu.edu/~kayvonf/misc/cleartalktips.pdf

Part I: How to present mathematics

Part II: How to do mathematics

Finding papers

Use **Google Scholar**.

- Use CMU credentials (VPN) to get journal issues online.
- Alternative: check the author's home page.
- Books/some older journal articles can be found in the actual physical science library, in Wean.
- For books, first use Google Books / Amazon's "Read Inside" feature to try to find what you want.
- All else fails: Interlibrary loan is not too slow (`https://illiad.library.cmu.edu/illiad/illiad.dll`)

Finding papers

Use **Google Scholar**.

If you look at a paper, even briefly:

1. Check its “cited by” link on Google Scholar
2. **Save a local copy.**

Beginning today, maintain a giant folder of saved papers.

Use a consistent naming convention.

E.g., `nisan-wigderson-log-rank-conj.pdf`

This will save you 100’s of hours, lifetime.

How to find papers to read

Papers citing / cited by the paper you're reading.

Proceedings of recent FOCS/STOC/SODA/CCC.

Google Scholar Alerts. (Surprisingly good.)

Recent posts to ECCC (`http://eccc.hpi-web.de/`)
or arXiv (`http://arxiv.org/corr/home`)

Stay au courant

Read TCS blogs: **http://feedworld.net/toc**

Watch videos:

http://video.ias.edu/csdm

http://www.youtube.com/user/SimonsInstitute

http://research.microsoft.com/

(search for "[researcher name/topic] video")

http://intractability.princeton.edu/videos/

Conference talks; e.g., STOC 2013:

http://dl.acm.org/citation.cfm?id=2213977

Streetfighting Mathematics

(title stolen from Sanjoy Mahajan)

Q: What is the next number in the series?

1, 2, 5, 20, 125, 1070, ???

A: Just look it up at oeis.org

(Online Encyclopedia of Integer Sequences)

1, 2, 5, 20, 125, 1070 [Search] Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,2,5,20,125,1070**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A076795 Partial sums of (2n-1)!!. +20 3

0, **1, 2, 5, 20, 125, 1070**, 11465, 146600, 2173625, 36633050, 691362125, 14440672700, 330674815925, 8236528396550, 221694575073425, 6411977928702800, 198310761891213425, 6530970632654064050 (list; graph; refs; listen; history; text; internal format)

OFFSET	0,3
LINKS	Vincenzo Librandi, Table of n, a(n) for n = 0..300
FORMULA	E.g.f.: exp(x)* Integral_{t=0, x} exp(-t)/sqrt(1-2*t) dt. a(n)=a(n-1)(2n-2)-a(n-2)(2n-3) . a(n) ~ 1/sqrt(2)/n * 2^n * (n/e)^n. G.f.: A(x)=x/(1-x)*(1 + x/(U(0)-x)), where U(k)= (2*k+1)*x + 1 - (2*k+3)*x/U(k+1); (continued fraction Euler's 1st kind, 1-step). - Sergei N. Gladkovskii, Jun 27 2012 G.f.: x/(1-x)/Q(0), where Q(k)= 1 - x*(k+1)/Q(k+1); (continued fraction). - Sergei N. Gladkovskii, May 19 2013 G.f.: G(0)*x/(1-x), where G(k)= 1 - x*(k+1)/(x*(k+1) - 1/G(k+1)); (continued fraction). - Sergei N. Gladkovskii, Aug 04 2013
MATHEMATICA	Join[{0}, Accumulate[Table[(2n-1)!!, {n, 0, 20}]]] (* Harvey P. Dale, Jan 27 2013 *)
PROG	(PARI) a(n)=if(n<0, 0, sum(k=0, n-1, (2*k)!/k!/2^k))
CROSSREFS	Cf. A001147.
KEYWORD	nonn
AUTHOR	Michael Somos, Nov 16 2002

Q: What are the

Stirling numbers of the second kind ?

What is the explicit formula for them?

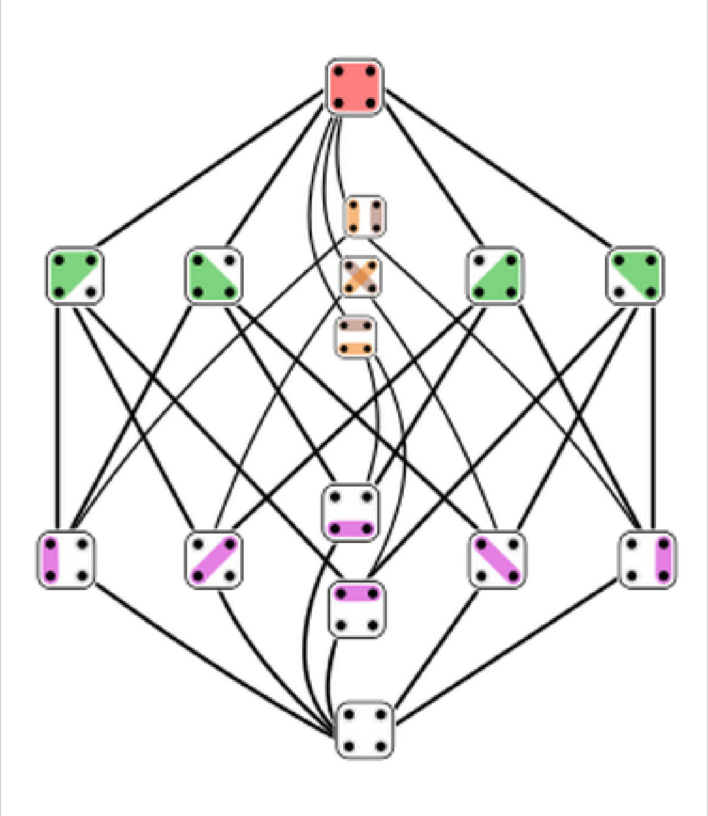
A: Look it up on Wikipedia

Stirling numbers of the second kind

From Wikipedia, the free encyclopedia

In mathematics, particularly in combinatorics, a **Stirling number of the second kind** is the number of ways to partition a set of *n* objects into *k* non-empty subsets and is denoted by $S(n,k)$ or $\left\{ {n \atop k} \right\}$.[1] Stirling numbers of the second kind occur in the field of mathematics called combinatorics and the study of partitions.

Stirling numbers of the second kind are one of two kinds of Stirling numbers, the other kind being called Stirling numbers of the first kind. Mutually inverse (finite or infinite) triangular matrices can be formed by arranging the Stirling numbers of the first respectively second kind according to the parameters *n*, *k*.



The 15 partitions of a 4-element set ordered in a Hasse diagram

Contents [show]

Definition [edit source | edit beta]

The Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ count the number of ways to partition a set of *n* labelled objects into *k* nonempty unlabelled subsets. Equivalently, they count the number of different equivalence relations with precisely *k* equivalence classes that can be defined on an *n* element set. Obviously,

$$\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1.$$

Q: What is 0.6019072301 97?

(This question based on a true story.)

A: Look it up at Inverse Symbolic Calculator inverse symbolic calculator

Standard lookup results for **0.601907230197**

Best guess: BesK(1,1)

BesK(1,1)-exp(-Pi)^GAM(1/12)	6019072301972343
BesK(1,1)	6019072301972345
BesselK(1,1)	
BesK(1,1)+exp(-Pi)^GAM(1/12)	6019072301972347

The original ISC

Q: What is the Bessel K function?

A: Look it up on Wikipedia

An anecdote

Ryan Williams had an awesome CMU PhD thesis.

I read the first draft. Its #1 theorem was:

Theorem: Any alg. for SAT using $n^{o(1)}$ space requires time $\tilde{\Omega}(n^c)$, where c is the largest root of $c^3-c^2-2c+1=0$; i.e., $c \approx 1.801$.

I had my computer calculate a few more digits:

$$c \approx 1.801937736.$$

Plugged it into Inverse Symbolic Calculator...

isc+ inverse symbolic calculator

Standard lookup results for **1.801937736**

Best guess: 1+2*x-x^2-x^3

rootofx^3-x^2-2*x+1;	1801937735804838
F(2/7,5/7;1/2;3/4)	
-3-3*x+6*x^2-6*x^3+3*x^5	
1+2*x-x^2-x^3	
cos(Pi*1/7)/cos(Pi*1/3)	
cos(Pi*1/7)+cos(Pi*1/7)	

An anecdote

I let him know, and now his famous theorem reads:

Theorem: Any alg. for SAT using $n^{o(1)}$ space requires time $\tilde{\Omega}(n^{2\cos(\pi/7)})$.

Q: Let $K, L \subseteq \mathbf{R}^n$ be closed, bounded, convex sets with smooth boundary.
Does $K \cup L$ have piecewise-smooth boundary?

A: Well, I didn’t know, but it’s the kind of question where you just **know** that some expert in analysis knows the answer.

Ask on mathoverflow.net.

If K and L are compact convex sets with smooth boundary, does their union have piecewise-smooth boundary?

2

Clarification: by "piecewise", I mean a *finite* number of pieces.

I'm sure this must be true, but my search for a citation was in vain (although I did learn the new term "polyconvex").

Thanks!

integral-geometry dg.differential-geometry analytic-geometry convexity smoothness

share edit close delete flag

asked Nov 10 '10 at 16:57
Ryan O'Donnell
2,269 ● 8 ● 19

add comment

start a bounty

Tagged
dg.differential-geometry × 2347
convexity × 194
smoothness × 34
analytic-geometry × 16
integral-geometry × 13

Asked 2 Years Ago
Viewed 293 Times
Active 2 Years Ago

Community Bulletin
meta Who gets the reputation?
meta Best of MathOverflow
meta Long-term archiving of MathOverflow
meta Two tags for partially ordered sets

2 Answers

active oldest votes

9

I don't think this is true. Suppose one of the sets is essentially $\{(x,y)\,:\,y\geq x^2\}$ in the plane (cut off in some smooth way at the top, to make it compact). And suppose the other one is the same except that the parabolic lower boundary has been replaced by the graph of something like $y=x^2+e^{-1/x^2}\sin(1/x)$ In other words add a fierce oscillation but damped so strongly that the region above the curve is still convex (i.e., d^2y/dx^2 remains positive). (I haven't done the arithmetic to make sure my e^{-1/x^2} damping is sufficient; if it isn't, then replace it by a more vigorous damping.) The union of the two convex sets will have infinitely many corners, where $\sin(1/x)$is 0.

share edit flag

answered Nov 10 '10 at 17:16
Andreas Blass
35.4k ● 3 ● 62 ● 126

Stackexchange sites

Mathoverflow.net:

For **research-level** questions about math.

CSTheory.stackexchange.com:

For **research-level** questions about TCS.

math.stackexchange.com:

For help with math questions at any level.

(Do not post your homework here!!!)

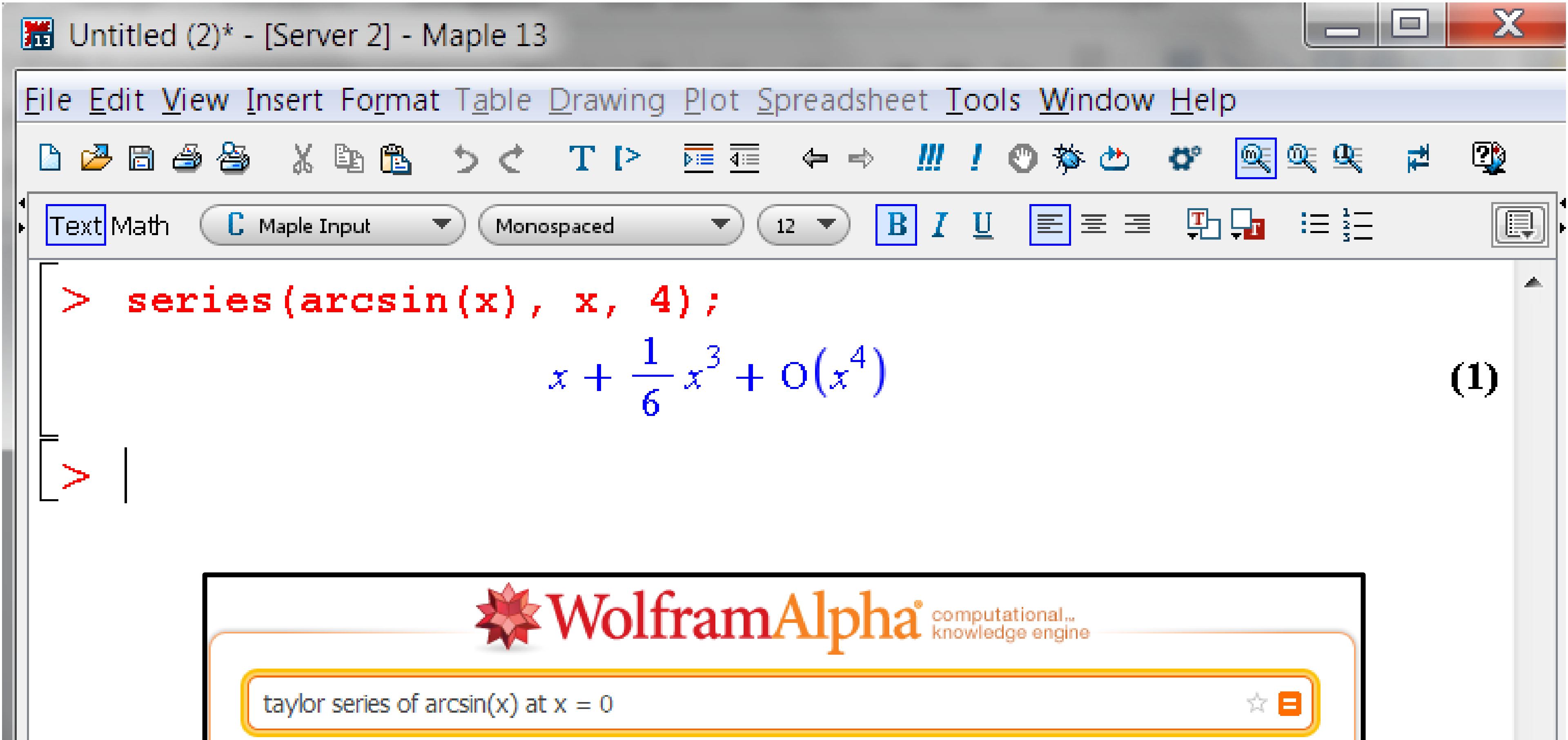
tex.stackexchange.com:

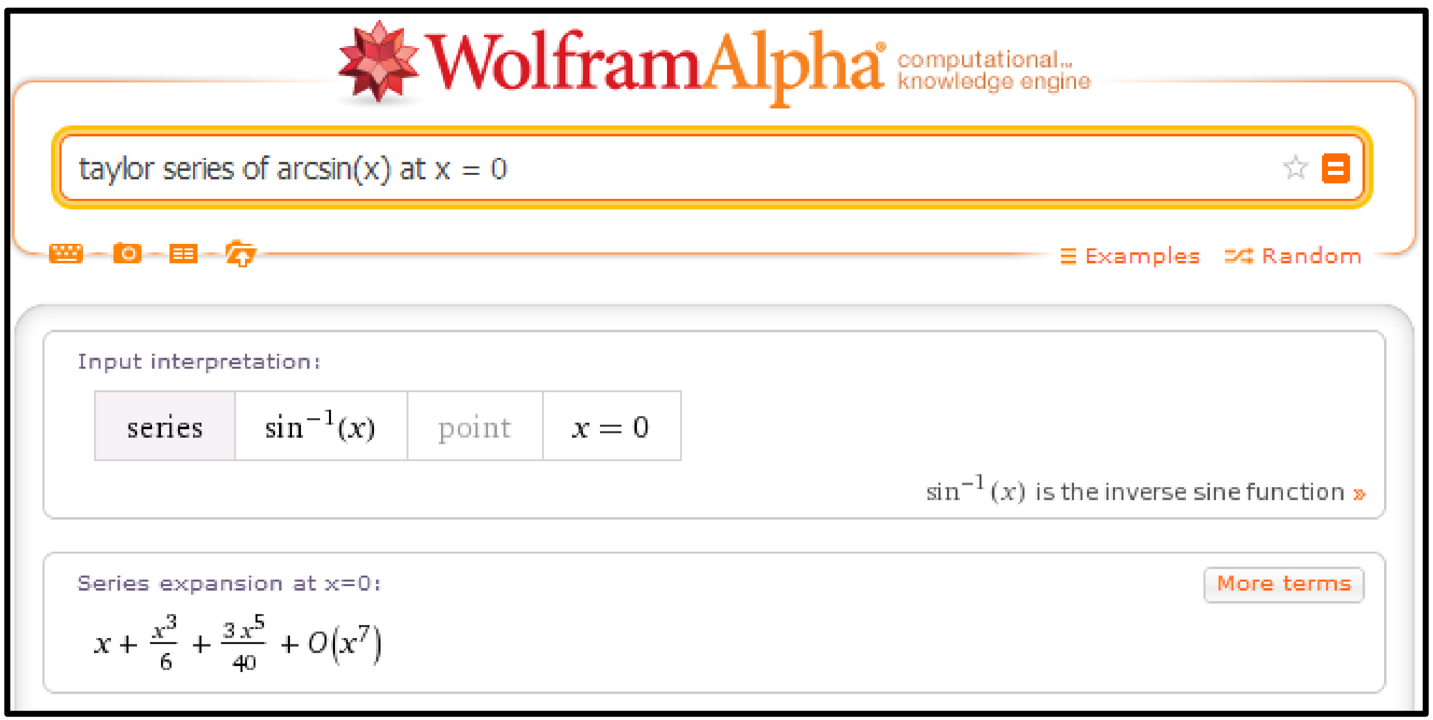
For any questions about LaTeX.

Q: What's the 4^{th}-order Taylor series for arcsin(x)?

A: Ask Maple/Mathematica/Sage.

(The first two are equally awesome.
Sage is free, and is based around python.
Maple/Mathematica freely accessible at CMU.
For quick things, use wolframalpha.com.)


Untitled (2)* - [Server 2] - Maple 13
File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help
Text Math
Maple Input
Monospaced
12
> series(arcsin(x), x, 4);
$x+\frac{1}{6}x^3+O(x^4)$
(1)
>


WolframAlpha computational.. knowledge engine
taylor series of arcsin(x) at x = 0
Examples
Random
Input interpretation:
series
$\sin^{-1}(x)$
point
$x = 0$
$\sin^{-1}(x)$ is the inverse sine function »
Series expansion at x=0:
More terms
$x+\frac{x^3}{6}+\frac{3x^5}{40}+O(x^7)$

What else are Maple/Mathematica good for?

A: Everything. Use liberally.

- Plotting functions
- Testing numerical conjectures
- Solving linear progs (symbolically, too)
- Simplifying complicated expressions
- Generating random numbers
- Integrating (symbolically/numerically)
- Finding roots of equations
- Empirically checking inequalities
- Maximizing/minimizing expressions
- Outputting LaTeX of expressions
- Inverting matrices (symbolically, too)
- Testing primality/irreducibility
- Finding eigenvalues, SVDs
- Writing code
- Gröbner bases
- Finite field arithmetic
- Solving differential equations
- Explicit computations
- Visualizing graphs
- Solving systems of equations
- Quadratic programming
- Curve fitting
- Asymptotics and Taylor series
- Differentiating

What else are Maple/Mathematica good for?

A: Everything. Use liberally.

Basically, if it’s a math problem, and
you think someone in history ever thought of
using a computer to do it,
then Maple/Mathematica can do it.

PS: You should also learn Matlab.
Often better for numerical things.

Streetfighting Mathematics

an example

Q: Suppose $p(x)$ is a polynomial of degree $\leq k$ which is bounded in $[-1,+1]$ for $x \in [-1,+1]$.

What is the largest $p'(0)$ can be?

Remark: This question actually comes up from time to time in analysis of boolean functions.

You can *probably* solve it with judicious Googling.

Also appropriate for math.stackexchange.com, **if** you put in a reasonable effort first.

Let's solve it using streetfighting mathematics.

Q: Suppose p(x) is a polynomial of degree ≤ k which is bounded in [−1,+1] for x ∊ [−1,+1].

What is the largest p′(0) can be?

Let's think about k = 3, say, so

$$p(x) = a + bx + cx^2 + dx^3$$

For each value of x, e.g. x = .2, we have a **constraint**:

$$-1 \leq a + .2b + .04c + .008d \leq +1$$

We want to maximize **b**

Q: Suppose p(x

which is bour

What is the

We have infinitely many constraints, but probably not much changes if we just take some random 5000 of them.

Let's think about k = 3, say, so

$$p(x) = a + bx + cx^2 + dx^3$$

For each value of x, e.g. $x = .2$,

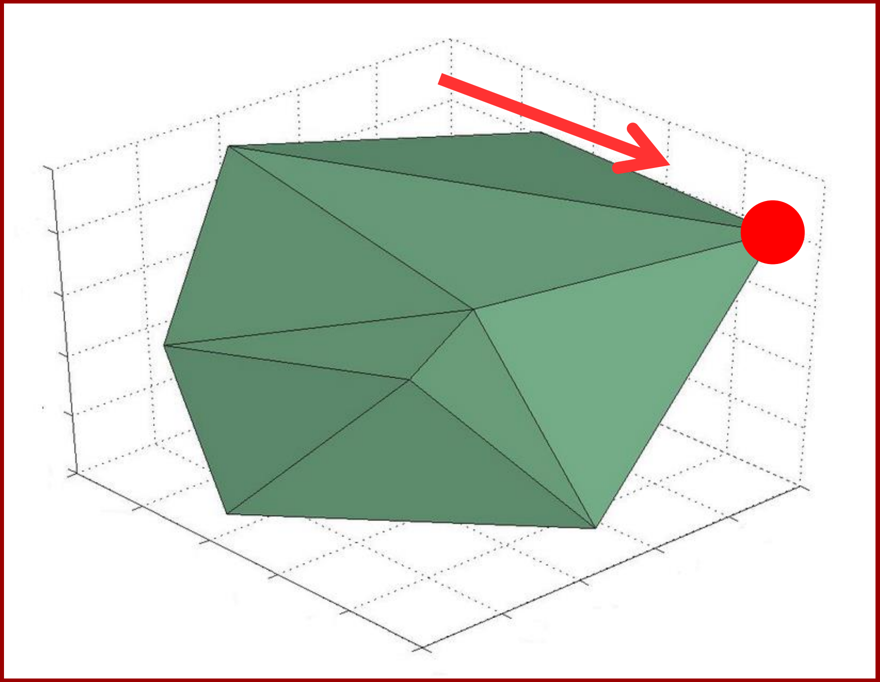
we have a **constraint**:

$$-1 \le a + .2b + .04c + .008d \le +1$$

We want to maximize **b**

So say we have 10,000 linear inequalities
over the variables a, b, c, d;
they form some polytope in $\mathbf{R}^4$.

We want to maximize b.



This is a “**Linear Program**”. Maple can solve it.

```
> with(Optimization): with(RandomTools):
> deg := 3; numPts := 5000:
> P := x -> add(c[i]*x^i, i = 0..deg):
> ConstrPts := [seq(2*Generate(float(range = 0..1, method = uniform))-1, i=1..numPts)]:
> Constrs := map(y -> (P(y) >= -1, P(y) <= 1), ConstrPts):
> Soln := LPSolve(c[1], Constrs, maximize):
> BestDeriv := Soln[1];
> BestPoly := subs(op(Soln[2]), P(x));
> plot(BestPoly, x = -1..1);
```

$$deg := 3 \quad (1)$$

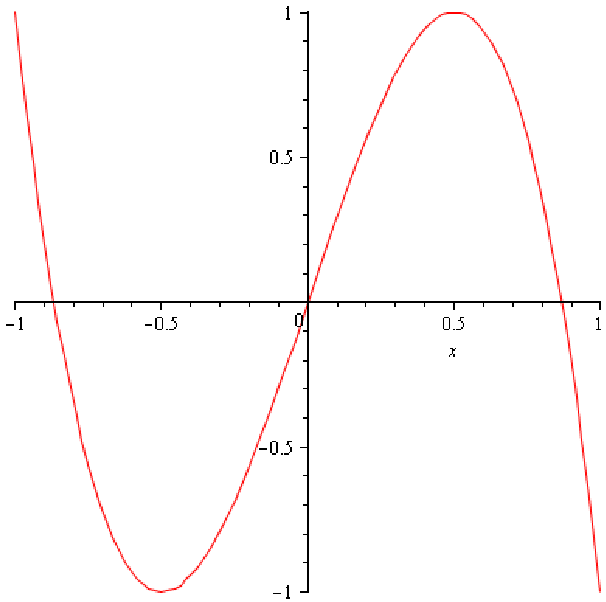
```
> P := x -> add(c[i]*x^i, i = 0..deg):
> ConstrPts := [seq(2*Generate(float(range = 0..1, method = uniform))-1, i=1..numPts)]:
> Constrs := map(y -> (P(y) >= -1, P(y) <= 1), ConstrPts):
> Soln := LPSolve(c[1], Constrs, maximize):
> BestDeriv := Soln[1];
```

$$BestDeriv := 3.00056057358036 \quad (2)$$

```
> BestPoly := subs(op(Soln[2]), P(x));
```

$$BestPoly := -0.000426544117962384708 + 3.00056057358035799\, x + 0.00170592447557217143\, x^2 - 4.00223867084451790\, x^3 \quad (3)$$

```
> plot(BestPoly, x = -1..1);
```



deg = 3: looks like maximizer is p(x) = **3**x−4x^3

$$deg := 1 \quad (1)$$

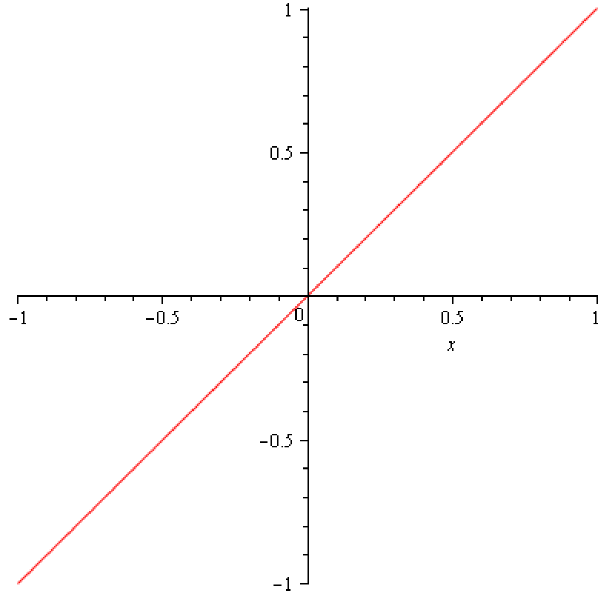
```
> P := x -> add(c[i]*x^i, i = 0..deg):
> ConstrPts := [seq(2*Generate(float(range = 0..1, method = uniform))-1, i=1..numPts)]:
> Constrs := map(y -> (P(y) >= -1, P(y) <= 1), ConstrPts):
> Soln := LPSolve(c[1], Constrs, maximize):
> BestDeriv := Soln[1];
```

$$BestDeriv := 1.00041785758203 \quad (2)$$

```
> BestPoly := subs(op(Soln[2]), P(x));
```

$$BestPoly := 0.000239437959177035980 + 1.00041785758202950\,x \quad (3)$$

```
> plot(BestPoly, x = -1..1);
```



deg = 1: p(x) = **1**x

$$deg := 2 \tag{1}$$

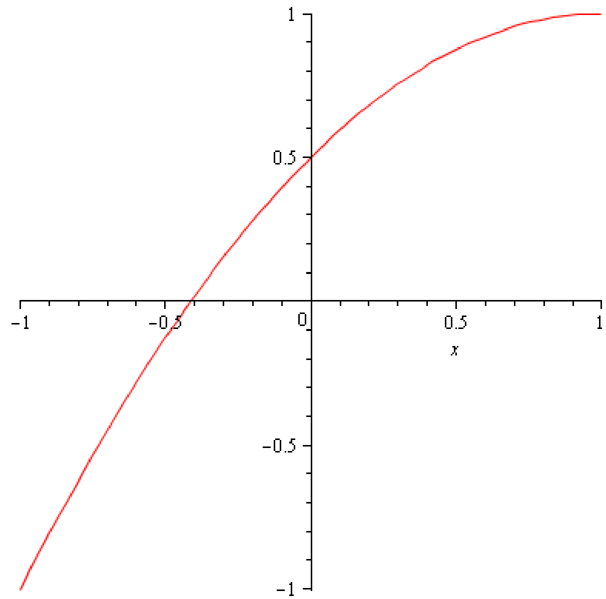
```
> P := x -> add(c[i]*x^i, i = 0..deg):
> ConstrPts := [seq(2*Generate(float(range = 0..1, method = uniform))-1, i=1..numPts)]:
> Constrs := map(y -> (P(y) >= -1, P(y) <= 1), ConstrPts):
> Soln := LPSolve(c[1], Constrs, maximize):
> BestDeriv := Soln[1];
```

$$BestDeriv := 1.00154659899863 \tag{2}$$

```
> BestPoly := subs(op(Soln[2]), P(x));
```

$$BestPoly := 0.500012915154317117 + 1.00154659899863407\, x - 0.501560741214689410\, x^2 \tag{3}$$

```
> plot(BestPoly, x = -1..1);
```



deg = 2: p(x) = .5+**1**x−.5x^2

$$deg := 3 \quad (1)$$

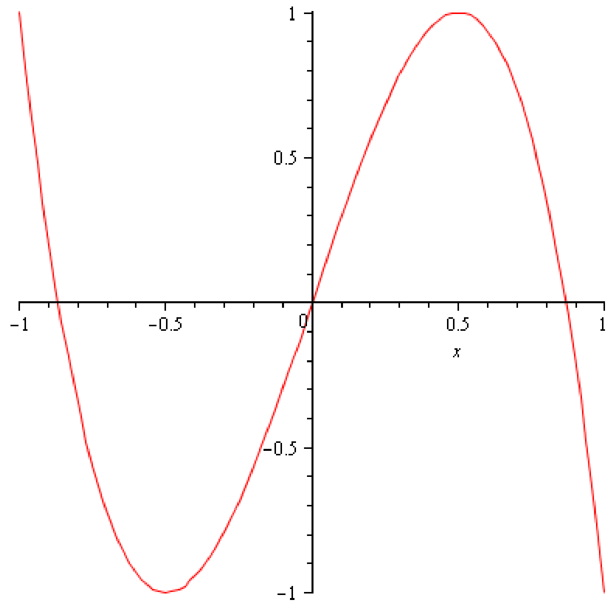
```
> P := x -> add(c[i]*x^i, i = 0..deg):
> ConstrPts := [seq(2*Generate(float(range = 0..1, method = uniform))-1, i=1..numPts)]:
> Constrs := map(y -> (P(y) >= -1, P(y) <= 1), ConstrPts):
> Soln := LPSolve(c[1], Constrs, maximize):
> BestDeriv := Soln[1];
```

$$BestDeriv := 3.00056057358036 \quad (2)$$

```
> BestPoly := subs(op(Soln[2]), P(x));
```

$$BestPoly := -0.000426544117962384708 + 3.00056057358035799\, x + 0.00170592447557217143\, x^2 - 4.00223867084451790\, x^3 \quad (3)$$

```
> plot(BestPoly, x = -1..1);
```



deg = 3: $p(x) = \mathbf{3}x - 4x^3$

$$deg := 4 \quad (1)$$

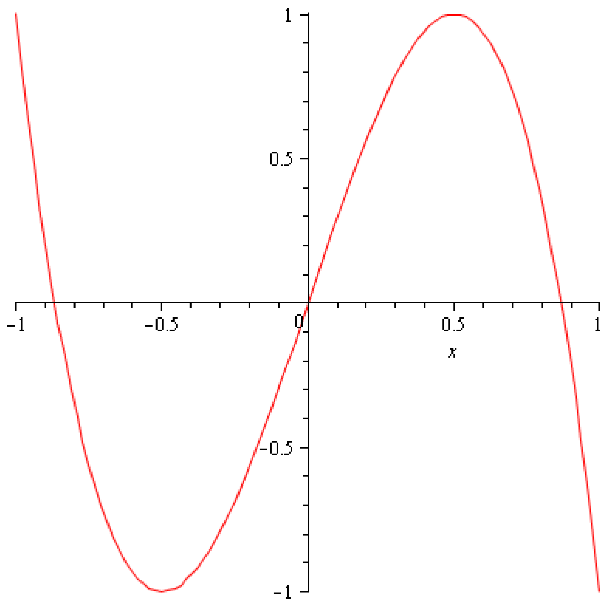
```
> P := x -> add(c[i]*x^i, i = 0..deg):
> ConstrPts := [seq(2*Generate(float(range = 0..1, method = uniform))-1, i=1..numPts)]:
> Constrs := map(y -> (P(y) >= -1, P(y) <= 1), ConstrPts):
> Soln := LPSolve(c[1], Constrs, maximize):
> BestDeriv := Soln[1];
```

$$BestDeriv := 3.00028125652306 \quad (2)$$

```
> BestPoly := subs(op(Soln[2]), P(x));
```

$$BestPoly := 0.0000121482101641112857 + 3.00028125652306343\, x - 0.000326369013471309970\, x^2 - 4.00112504781797629\, x^3 + 0.00111128812486230256\, x^4 \quad (3)$$

```
> plot(BestPoly, x = -1..1);
```



deg = 4: p(x) = **3**x−4x^3 again (!)

$$deg := 5 \quad (1)$$

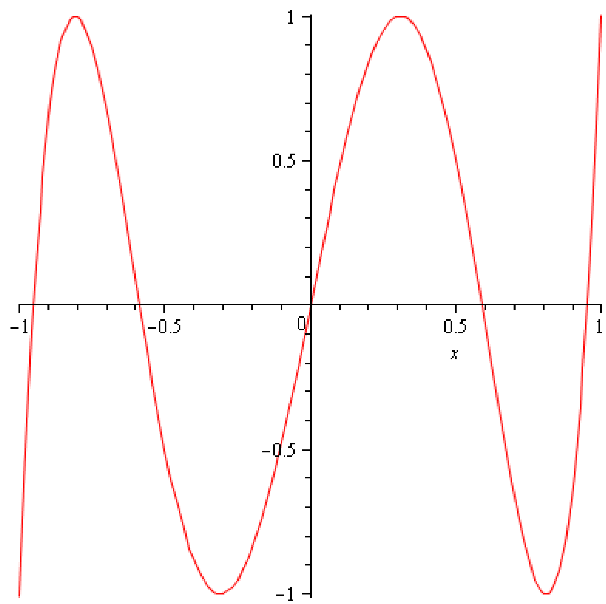
```
> P := x -> add(c[i]*x^i, i = 0..deg):
> ConstrPts := [seq(2*Generate(float(range = 0..1, method = uniform))-1, i=1..numPts)]:
> Constrs := map(y -> (P(y) >= -1, P(y) <= 1), ConstrPts):
> Soln := LPSolve(c[1], Constrs, maximize):
> BestDeriv := Soln[1];
```

$$BestDeriv := 5.00120532339474 \quad (2)$$

```
> BestPoly := subs(op(Soln[2]), P(x));
```

$$BestPoly := -0.00106623559767311328 + 5.00120532339473822\, x + 0.0127965087812789788\, x^2 - 20.0144708379618130\, x^3 - 0.0170841289591253838\, x^4 + 16.0192856795011309\, x^5 \quad (3)$$

```
> plot(BestPoly, x = -1..1);
```



deg = 5: p(x) = **5**x−20x^3+16x^5

$$deg := 6 \tag{1}$$

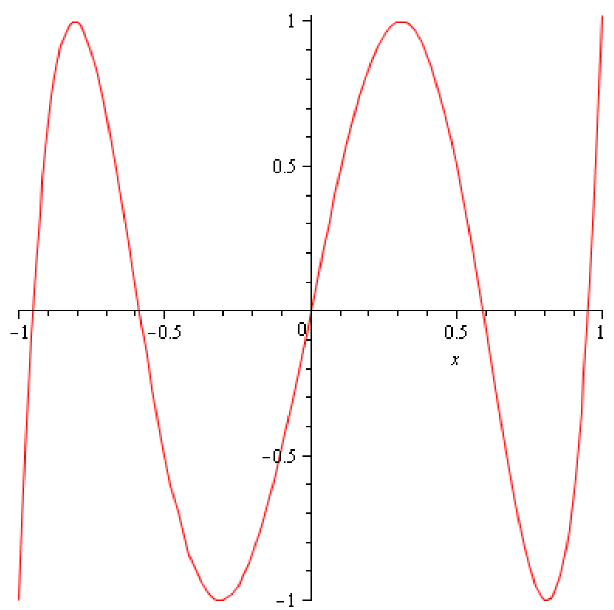
```
> P := x -> add(c[i]*x^i, i = 0..deg):
> ConstrPts := [seq(2*Generate(float(range = 0..1, method = uniform))-1, i=1..numPts)]:
> Constrs := map(y -> (P(y) >= -1, P(y) <= 1), ConstrPts):
> Soln := LPSolve(c[1], Constrs, maximize):
> BestDeriv := Soln[1];
```

$$BestDeriv := 5.00176146095726 \tag{2}$$

```
> BestPoly := subs(op(Soln[2]), P(x));
```

$$BestPoly := 0.000368034916638261978 + 5.00176146095725738\,x - 0.00321856542933066266\,x^2 - 20.0210030986764060\,x^3 - 0.00827062048655928528\,x^4 + 16.0279861868147115\,x^5 + 0.0188502914009799974\,x^6 \tag{3}$$

```
> plot(BestPoly, x = -1..1);
```



deg = 6: p(x) = **5**x−20x^3+16x^5 again

Summary:

Except for weird anomaly at degree 2, looks like degree 2k optimizer is the same as the degree 2k−1 optimizer.

(In fact, that's true; can you see why?)

So let's focus on odd degree.

$p_1(x) = \mathbf{1}x$

$p_3(x) = \mathbf{3}x - 4x^3$

$p_5(x) = \mathbf{5}x - 20x^3 + 16x^5$

Largest p′(0) seems to equal degree, but now what?

Summary:

Except for weird anoma

looks like degree 2k opt

as the degree 2k−1 opt

(In fac

Try typing these coefficients into oeis.org

So let's focus on odd degree.

$p_1(x) = \mathbf{1}x$

$p_3(x) = \mathbf{3}x - 4x^3$

$p_5(x) = \mathbf{5}x - 20x^3 + 16x^5$

Largest p′(0) seems to equal degree, but now what?

1,3,-4,5,-20,16 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,3,-4,5,-20,16**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A084930 Triangle of coefficients of Chebyshev polynomials T_{2n+1} (x). +10 3

1, -3, 4, 5, -20, 16, -7, 56, -112, 64, 9, -120, 432, -576, 256, -11, 220, -1232, 2816, -2816, 1024, 13, -364, 2912, -9984, 16640, -13312, 4096, -15, 560, -6048, 28800, -70400, 92160, -61440, 16384, 17, -816, 11424, -71808, 239360, -452608, 487424, -278528, 65536, -19, 1140, -20064, 160512, -695552 (list; table; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Comment from Herb Conn, HCR 83, Box 93, Custer, SD 57730, Jan 28 2005: "Letting x = 2 Cos 2A, we have the following trigonometric identities:
"Sin 3A = 3*Sin A - 4*Sin^3 A
"Sin 5A = 5*Sin A - 20*Sin^3 A + 16*Sin^5 A
"Sin 7A = 7*Sin A - 56*Sin^3 A + 112*Sin^5 A - 64*Sin^7 A
"Sin 9A = 9*Sin A - 120*Sin^3 A + 432*Sin^5 A - 576*Sin^7 A + 256*Sin^9 A, etc."
Cayley (1876) states "Write sin u = x, then we have sin u = x, [...] sin 3u = 3x - 4x^3, [...] sin 5u = 5x - 20x^3 + 16 x^5, [...]". Since T_n(cos(u)) = cos(nu) for all integer n, sin(u) = cos(u-pi/2), and sin(u+k pi) = (-1)^k sin(u) it follows that T_n(sin(u)) = (-1)^((n-1)/2) sin(nu) for all odd integer n. - Michael Somos, Jun 19 2012

REFERENCES A. Cayley, On an Expression for 1 +- sin(2p+1)u in Terms of sin u, Messenger of Mathematics, 5 (1876), pp. 7-8 = Mathematical Papers Vol. 10, n. 630, pp. 1-2.
Theodore J. Rivlin, Chebyshev polynomials: from approximation theory to algebra and number theory, 2. ed., Wiley, New York, 1990. p. 37, eq. (1.96) and p. 4, eq.(1.10).

LINKS Table of n, a(n) for n=0..49.
M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions, National Bureau of Standards, Applied Math. Series 55, Tenth Printing, 1972 [alternative scanned copy].
M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions, National Bureau of Standards Applied Math. Series 55, Tenth Printing, 1972, p. 795.
Index entries for sequences related to Chebyshev polynomials.

FORMULA Alternate rows of A008310.

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Chebyshev polynomials

From Wikipedia, the free encyclopedia

Not to be confused with discrete Chebyshev polynomials.

It has been suggested that *Dickson polynomial* be merged into this article. (Discuss) *Proposed since September 2011.*

In mathematics the **Chebyshev polynomials**, named after Pafnuty Chebyshev,[1] are a sequence of orthogonal polynomials which are related to de Moivre's formula and which can be defined recursively. One usually distinguishes between **Chebyshev polynomials of the first kind** which are denoted T_n and **Chebyshev polynomials of the second kind** which are denoted U_n. The letter T is used because of the alternative transliterations of the name *Chebyshev* as *Tchebycheff*, *Tchebyshev* (French) or *Tschebyschow* (German).

The Chebyshev polynomials T_n or U_n are polynomials of degree n and the sequence of Chebyshev polynomials of either kind composes a polynomial sequence.

Chebyshev polynomials are polynomials with the largest possible leading coefficient, but subject to the condition that their absolute value is bounded on the interval by 1. They are also the extremal polynomials for many other properties.[2]

Chebyshev polynomials are important in approximation theory because the roots of the Chebyshev polynomials of the first kind, which are also called Chebyshev nodes, are used as nodes in polynomial interpolation. The resulting interpolation polynomial minimizes the problem of Runge's phenomenon and provides an approximation that is close to the polynomial of best approximation to a continuous function under the maximum norm. This approximation leads directly to the method of Clenshaw–Curtis quadrature.

In the study of differential equations they arise as the solution to the Chebyshev differential equations

$$(1-x^2)\,y'' - x\,y' + n^2\,y = 0$$

Notes [edit source | edit beta]

1. ^ Chebyshev polynomials were first presented in: P. L. Chebyshev (1854) "Théorie des mécanismes connus sous le nom de parallélogrammes," *Mémoires des Savants étrangers presentés à l'Académie de Saint-Pétersbourg*, vol. 7, pages 539–586.
2. ^ Rivlin, Theodore J. **The Chebyshev polynomials.** Pure and Applied Mathematics. *Wiley-Interscience [John Wiley & Sons]*, New York-London-Sydney,1974. Chapter 2, "Extremal Properties", pp. 56--123.
3. ^ Jeroen Demeyer Diophantine Sets over Polynomial Rings and Hilbert's Tenth Problem for Function Fields, Ph.D. theses (2007), p.70.
4. ^ a b c Boyd, John P. (2001). *Chebyshev and Fourier Spectral Methods* (second ed.). Dover. ISBN 0-486-41183-4.
5. ^ Chebyshev Interpolation: An Interactive Tour

References [edit source | edit beta]

- Abramowitz, Milton; Stegun, Irene A., eds. (1965), "Chapter 22", *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover, p. 773, ISBN 978-0486612720, MR 0167642.
- Dette, Holger (1995), A Note on Some Peculiar Nonlinear Extremal Phenomena of the Chebyshev Polynomials, *Proceedings of the Edinburgh Mathematical Society* **38**, 343-355
- Eremenko, A.; Lempert, L. (1994), An Extremal Problem For Polynomials, *Proceedings of the American Mathematical Society*, Volume 122, Number 1, 191-193
- Koornwinder, Tom H.; Wong, Roderick S. C.; Koekoek, Roelof; Swarttouw, René F. (2010), "Orthogonal Polynomials", in Olver, Frank W. J.; Lozier, Daniel M.; Boisvert, Ronald F.; Clark, Charles W., *NIST Handbook of Mathematical Functions*, Cambridge University Press, ISBN 978-0521192255, MR2723248
- Remes, Eugene, On an Extremal Property of Chebyshev Polynomials
- Suetin, P.K. (2001), "C/c021940", in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4

External links [edit source | edit beta]

- Weisstein, Eric W., "Chebyshev Polynomial of the First Kind", *MathWorld*.
- Module for Chebyshev Polynomials by John H. Mathews
- Chebyshev Interpolation: An Interactive Tour, includes illustrative Java applet.
- Numerical Computing with Functions: The Chebfun Project

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Rivlin Chebyshev Polynomials Theorem 2.20

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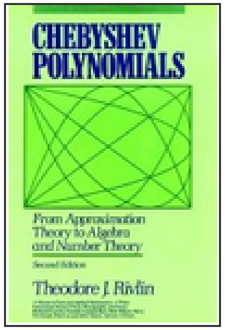
Chebyshev polynomials: from approximation theory to algebra and number theory



Theodore J. Rivlin

0 Reviews

J. Wiley, Jul 4, 1990 - Mathematics - 249 pages

This Secodnd Edition continues the fine tradition of its predecessor by surveying the most important propertie
New to this edition are approximately 80 exercises and a chapter which introduces some elementary algebrai
Additional coverage focuses on extremal and iterative properties.

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18 pages matching **Rivlin Chebyshev Polynomials Theorem 2.20** in this book

Page 107

CHEBYSHEV POLYNOMIALS ARE EXTREMAL 107

Page 108

Theorem 2.20. Let F be a linear functional on $\mathscr{P}_n$ such that (i) neither $\pm T$ is an extremal for F, (ii) $p \in \mathscr{P}_n$, $p \neq 0$, having n distinct zeros in I, implies $Fp \neq 0$. Then

$$|Fp| \leq |FT_n|, \qquad p \in C_n, \qquad (2.36)$$

with equality holding if and only if $p = \pm T$

Page 114

3 Bounding the approximate degree of the majority function

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We will require the following well-known fact, which follows from, e.g., [Riv90, Theorem 2.20].

Proposition 3.1. *Let $p(x)$ be a polynomial of degree at most k which satisfies $|p(x)| \leq 1$ whenever $|x| \leq 1$. Then $|p'(0)| \leq 2\lceil \frac{k}{2} \rceil - 1 \leq k$.*

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References

[Riv90] Theodore J. Rivlin. *Chebyshev Polynomials.* John Wiley & Sons, New York, second edition, 1990.

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