How to Verify Trigonometric Identities

An identity is an equation that will always remain true, regardless of the value of the variables. To verify an identity is to simplify one side to show that it equals the other side. It includes the manipulation of trigonometric expressions through algebraic operations. This handout will explain how to verify trigonometric identities, including the use of fundamental identities such as quotient identities, reciprocal identities, and Pythagorean identities. To review trigonometric functions and their identities, please refer to the Common Trigonometric Angle Measurements handout.

Fundamental Identities

The fundamental identities will be the foundation for which most trigonometric identities will be verified. These are the identities that are commonly utilized and manipulated when verifying identities.

Reciprocal Identities		
$\sin x = \frac{1}{\csc x}$	$\cos x = \frac{1}{\sec x}$	$\tan x = \frac{1}{\cot x}$
$\csc x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$	$\cot x = \frac{1}{\tan x}$

Quotient Identities		
$\cot x = \frac{\cos x}{\sin x}$	$\tan x = \frac{\sin x}{\cos x}$	

Pythagorean Identities		
$\cos^2 x + \sin^2 x = 1$	$1 + \tan^2 x = \sec^2 x$	$1 + \cot^2 x = \csc^2 x$

Guidelines for Validating Identities

These are some suggested tips to follow when verifying identities, in order of precedence.

1. Manipulate the side of the equation that is more complex. For example, terms that are being multiplied/divided are more complex than terms that are being added/subtracted. Do not attempt to move terms from one side of the equation to the other.
2. When manipulating, try to simplify expressions in terms of sine and cosine.

3. Multiply by the conjugate when there is a binomial, such as $1 - \cos x$ in the numerator or denominator, or both. It reduces the number of terms in the expression.
4. Rewrite a single fraction as two separate fractions when there are two terms in the numerator. For example: $\frac{1+\cos x}{\sin x} \rightarrow \frac{1}{\sin x} + \frac{\cos x}{\sin x}$

The next four examples will show the general step-by-step process for verifying identities.

Example 1: $\sin x \sec x = \tan x$

Begin with the more complex side to simplify. In this case, work with the left side because two terms are provided while the right side only has one term. First, the equation will be converted into terms of $\sin x$ and $\cos x$. According to one of the reciprocal identities, $\sec x = \frac{1}{\cos x}$.

$$\frac{\sin x}{1} \bullet \frac{1}{\cos x} = \tan x$$

When multiplying across the top and the bottom, the result is $\frac{\sin x}{\cos x}$.

$$\frac{\sin x}{\cos x} = \tan x$$

Now, refer to the quotient identities and note that $\frac{\sin x}{\cos x} = \tan x$. Since both sides are equal, the proof is now complete.

$$\tan x = \tan x$$

Example 2: $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$

In this example, it does not matter what side to start with because both are equally complex. This example will demonstrate the steps when starting from the left side of the identity. Because there is a binomial in the denominator, multiply the numerator and denominator by the conjugate of $1 - \cos x$.

$$\frac{\sin x}{1 - \cos x} \bullet \frac{(1 + \cos x)}{(1 + \cos x)} = \frac{1 + \cos x}{\sin x}$$

After combining expressions, the denominator simplifies to $1 - \cos^2 x$ in the denominator.

$$\frac{\sin x + \sin x \cos x}{1 - \cos^2 x} = \frac{1 + \cos x}{\sin x}$$

The Pythagorean identity $\cos^2 x + \sin^2 x = 1$ can be applied here to change $1 - \cos^2 x$ to $\sin^2 x$. Having a $\sin^2 x$ in the denominator will allow a $\sin x$ to be canceled out in in the numerator and denominator in a future step.

$$\frac{\sin x + \sin x \cos x}{\sin^2 x} = \frac{1 + \cos x}{\sin x}$$

Factor out a $\sin x$ from both terms in the numerator.

$$\frac{\sin x(1 + \cos x)}{\sin^2 x} = \frac{1 + \cos x}{\sin x}$$

Since the $\sin x$ is common to the numerator and denominator, they can be canceled out. As a result, only one $\sin x$ will be remaining in the denominator. This will complete the verification.

$$\frac{1 + \cos x}{\sin x} = \frac{1 + \cos x}{\sin x}$$

Example 3: $\mathbf{\frac{\cot^2 x}{\csc x} = \csc x - \sin}$

Manipulate the left side of the equation rather than the right side because there are more opportunities for progression when dividing, as opposed to subtracting. Next, apply one of the Pythagorean identities: $\csc^2 x = 1 + \cot^2 x$. By doing so, it will allow two terms to be created on the left side.

$$\frac{\csc^2 x - 1}{\csc x} = \csc x - \sin x.$$

Separate the fraction into two fractions. This is not always needed, but here it allows the expansion of the expression. There are now two terms on each side.

$$\frac{\csc^2 x}{\csc x} - \frac{1}{\csc x} = \csc x - \sin x$$

Now, refer to the trig identities to see that $\frac{1}{\csc x} = \sin x$.

$$\frac{\csc^2 x}{\csc x} - \sin x = \csc x - \sin x$$

One $\csc x$ in the numerator will cancel the $\csc x$ in the denominator; thus, the left-hand side equals the right-hand side.

$$\csc x - \sin x = \csc x - \sin x$$

Example 4: $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$

In this case, each side has the same number of terms. Thus, one side of the equation is not any more complicated than the other. However, one side has multiplication while the other has addition. According to the guidelines, start on the side that has multiplication. Next, apply one of the Pythagorean identities. Replace $\sec^2 x$ with $1 + \tan^2 x$ because the desired outcome is having two terms for that expression.

$$(1 + \tan^2 x)\csc^2 x = \sec^2 x + \csc^2 x$$

Distribute the $\csc^2 x$ to 1 and $\tan^2 x$.

$$\csc^2 x + \csc^2 x \tan^2 x = \sec^2 x + \csc^2 x$$

The first $\csc^2 x$ can be left alone because there is a single $\csc^2 x$ on the right side of the identity. Next, change $\csc^2 x \tan^2 x$ to $\sec^2 x$. When trying to combine trigonometric terms that are multiplied, rewrite them in terms of $\sin x$ and $\cos x$. In this case, rewrite $\csc^2 x$ as $\frac{1}{\sin^2 x}$ via the reciprocal identities.

$$\csc^2 x + \left(\frac{1}{\sin^2 x}\right)(\tan^2 x) = \sec^2 x + \csc^2 x$$

Next, $\tan^2 x$ can be rewritten as $\frac{\sin^2 x}{\cos^2 x}$ via the quotient identities.

$$\csc^2 x + \left(\frac{1}{\sin^2 x}\right)\left(\frac{\sin^2 x}{\cos^2 x}\right) = \sec^2 x + \csc^2 x$$

When the two terms are multiplied from above, the two $\sin^2 x$ cancel each other out leaving behind $\frac{1}{\cos^2 x}$.

$$\csc^2 x + \frac{1}{\cos^2 x} = \sec^2 x + \csc^2 x$$

Lastly, to wrap up the proof, remember that one of the reciprocal identities states that $\sec x = \frac{1}{\cos x}$.

$$\csc^2 x + \sec^2 x = \sec^2 x + \csc^2 x$$

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Practice

1. $\tan x \csc x \cos x = 1$
2. $\frac{1-\cos x}{\sin x} = \csc x - \cot x$
3. $\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x} = 2\sec x$

Practice Solutions

1. $\tan x \csc x \cos x = 1$

$\left(\frac{\sin x}{\cos x}\right)(\csc x)(\cos x) = 1$

$\left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\sin x}\right)(\cos x) = 1$

$\frac{\sin x \cos x}{\cos x \sin x} = 1$

$\frac{\sin x}{\sin x} = 1$

$1 = 1$

2. $\frac{1-\cos x}{\sin x} = \csc x - \cot x$

$\frac{1}{\sin x} - \frac{\cos x}{\sin x} = \csc x - \cot x$

$\csc x - \frac{\cos x}{\sin x} = \csc x - \cot x$

$\csc x - \cot x = \csc x - \cot x$

3. $\frac{\cos x}{1-\sin x}+\frac{1-\sin x}{\cos x}=2\sec x$

$$\frac{\cos x}{1-\sin x}\bullet\frac{(\cos x)}{(\cos x)}+\frac{1-\sin x}{\cos x}\bullet\frac{(1-\sin x)}{(1-\sin x)}=2\sec x$$

$$\frac{(\cos x)(\cos x)}{(\cos x)(1-\sin x)}+\frac{(1-\sin x)(1-\sin x)}{(\cos x)(1-\sin x)}=2\sec x$$

$$\frac{(\cos x)(\cos x)+(1-\sin x)(1-\sin x)}{(\cos x)(1-\sin x)}=2\sec x$$

$$\frac{\cos^2 x+1-2\sin x+\sin^2 x}{(\cos x)(1-\sin x)}=2\sec x$$

$$\frac{1+1-2\sin x}{(\cos x)(1-\sin x)}=2\sec x$$

$$\frac{2-2\sin x}{(\cos x)(1-\sin x)}=2\sec x$$

$$\frac{2(1-\sin x)}{(\cos x)(1-\sin x)}=2\sec x$$

$$\frac{2}{\cos x}=2\sec x$$

$$2\sec x=2\sec x$$

Examples adapted from the Precalculus 6th edition textbook by Pearson.