How We Use This Book

How do you spend time in class?

Our dominant class modes are group work and teacher-led discussions with the whole class. We don't use the books purely as workbooks. Some classes can work through several problems on their own with periodic check-ins, but just as often the teacher will choose one problem as a way of introducing a topic, anticipating the details that a class might have trouble with.

We give students a significant amount of time to work on each problem, and the type of hints we give vary with the needs of the class. We value the conversation more than we value getting through many problems quickly.

Just because a formula, theorem, or principle does not appear in our textbook, does not mean that we do not intend to discuss the topic in class. Many problems are intended to prompt student discussion and questions to lead to the introduction of important concepts, and indeed this is how we spend much of class time. Were students to see the result already written in the textbook, they would lose the opportunity to construct their own questions and make conjectures.

We section our classes (roughly by interest and ability) so that every student can be in a class where they can keep up with the discussion, and so that students can make discoveries together.

Our classes are (usually) 90-minute blocks, held on alternate school days.

Why is there a "Habits of Mind" lesson at the beginning of every chapter? What's the difference between "Development," "Practice," "Problems," etc?

The Habits of Mind lesson is a collection of problems designed to teach a particular problem-solving strategy, usually done concurrently with the other lessons in the chapter. These problems generally have a playful feel. A given homework assignment might contain one habits problem. Teachers strive to make connections between the habits of mind students are using in the current habits lesson and the ways they approach problems in the content lessons.

Within a content lesson, students develop the ideas behind the topic of the lesson and do practice problems to make sure they have the basics down, and then progress to the "Problems" section. "Problems" are all related to the topic at hand, but often admit of multiple solution methods, involve connections to other areas of mathematics, and require thought beyond copying some model students have seen before. For more information on the structure of a lesson see http://parkmath.org/curriculum/.

The book doesn't contain many formulas or example problems. How are students expected to know how to do basic problems?

We want students to be the authors of their own mathematics. We expect students to take notes in class and keep track of important results themselves, with the idea that there is value in figuring out how to put into words what they have discovered. Each teacher has a different way of handling this. Some teachers have their class keep a "toolkit," a collection of facts and examples that the class has agreed are important. In any case, a large part of the teacher's role is to point out what parts of the assigned problems and discussion are important to remember for later.

Depending on the needs of the class, we might provide extra practice before tackling the "problems" section of the lesson.

Some problems seem very difficult, or at least difficult for students to figure out without direct instruction. Can students really do the problems in this book?

Students are not expected to solve all of their homework problems. Rather, the harder problems serve as ways to get the discussion going in class the next day.

Some problems are too hard to be homework problems; we assign them as group problems in class.

We don't assign every problem in a lesson. We choose problems to assign based on the interest and capability of each class.

… but yes, for the most part, students can handle the problems in this book, either on their own, in groups, or through discussion with their classmates and teacher. Once students get used to each problem requiring some thought, they come to see investigating new problems as normal. Practice with the habits of mind helps them to make significant headway on problems that at first seem very difficult.

How do you test on this material?

The format of our tests is fairly traditional. Our tests differ from traditional tests in that they are not particularly repetitive or computation-based. As is the case with the problems we assign, we include some problems that may not look exactly like problems that students have seen before. We want to assess students' ability to think flexibly about the mathematics they are studying.

The "right" way to assess students' progress in mathematics is an ongoing conversation in our department. We don't feel that we've hit upon the perfect method.

What is pacing like?

Each chapter of our book consists of one or two habits of mind lessons and three to six content lessons. We spend about two to four weeks on each content lesson, with habits of mind problems assigned concurrently. Depending on the pace of each class, we might get through three or four chapters in a year.

We don't always test at the end of every lesson. More often we test on one and a half or two lessons at a time, and we sometimes work in more than one lesson at once.

We skip some lessons with certain classes, either because they are too involved or because students would have seen the material before.

Park School Mathematics Curriculum Book 6: Geometry and Proof

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HABITS

look for patterns: to look for patterns amongst a set of numbers or figures

tinker: to play around with numbers, figures, or other mathematical expressions in order to learn something more about them or the situation; experiment

describe: to describe clearly a problem, a process, a series of steps to a solution; modulate the language (its complexity or formalness) depending on the audience

visualize: to draw, or represent in some fashion, a diagram in order to help understand a problem; to interpret or vary a given diagram

represent symbolically: to use algebra to solve problems efficiently and to have more confidence in one's answer, and also so as to communicate solutions more persuasively, to acquire deeper understanding of problems, and to investigate the possibility of multiple solutions.

prove: to desire that a statement be proved to you or by you; to engage in dialogue aimed at clarifying an argument; to establish a deductive proof; to use indirect reasoning or a counter-example as a way of constructing an argument

check for plausibility: to routinely check the reasonableness of any statement in a problem or its proposed solution, regardless of whether it seems true or false on initial impression; to be particularly skeptical of results that seem contradictory or implausible, whether the source be peer, teacher, evening news, book, newspaper, internet or some other; and to look at special and limiting cases to see if a formula or an argument makes sense in some easily examined specific situations.

OF MIND

take things apart: to break a large or complex problem into smaller chunks or cases, achieve some understanding of these parts or cases, and rebuild the original problem; to focus on one part of a problem (or definition or concept) in order to understand the larger problem

conjecture: to generalize from specific examples; to extend or combine ideas in order to form new ones

change or simplify the problem: to change some variables or unknowns to numbers; to change the value of a constant to make the problem easier; change one of the conditions of the problem; to reduce or increase the number of conditions; to specialize the problem; make the problem more general

work backwards: to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

re-examine the problem: to look at a problem slowly and carefully, closely examining it and thinking about the meaning and implications of each term, phrase, number and piece of information given before trying to answer the question posed.

change representations: to look at a problem from a different perspective by representing it using mathematical concepts that are not directly suggested by the problem; to invent an equivalent problem, about a seemingly different situation, to which the present problem can be reduced; to use a different field (mathematics or other) from the present problem's field in order to learn more about its structure

create: to invent mathematics both for utilitarian purposes (such as in constructing an algorithm) and for fun (such as in a mathematical game); to posit a series of premises (axioms) and see what can be logically derived from them

TAKE THINGS APART CONJECTURE EXAMINE A SIMILAR PROBLEM USE INVERSE THINKING DETERMINE RELEVANCE USE MULTIPLE POINTS OF VIEW CREATE LOOK FOR PATTERNS TINKER DESCRIBE VISUALIZE PROVE USE PLAUSIBILITY TAKE THINGS

HABITS OF MIND

WORK BACKWARDS

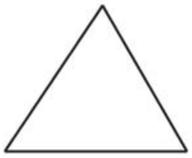
to reverse a process as a way of trying to understand it or as a way of learning something new; to work a problem backwards as a way of solving

CONJECTURE

to generalize from specific examples; to extend or combine ideas in order to form new ones

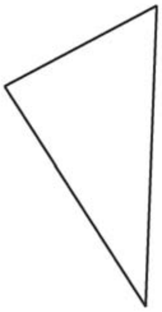
1 For each triangle, is it possible to draw a circle such that all three vertices are on the circle? (This is called *circumscribing* the triangle by a circle.)

a.



b.

c.



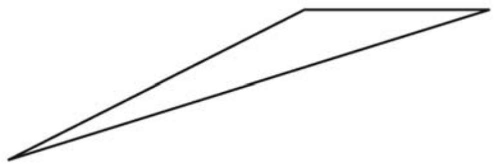
conjecture

Aside from this problem being an interesting puzzle in itself, you might find yourself wondering: what was the *point* of the problem? This is always a good question. It's also one that, often, you can answer on your own.

After solving a problem, ask yourself: *what's the next question?* In almost every problem, the ideas you have that allow you to solve that specific problem can connect to other, larger mathematical truths or ideas.

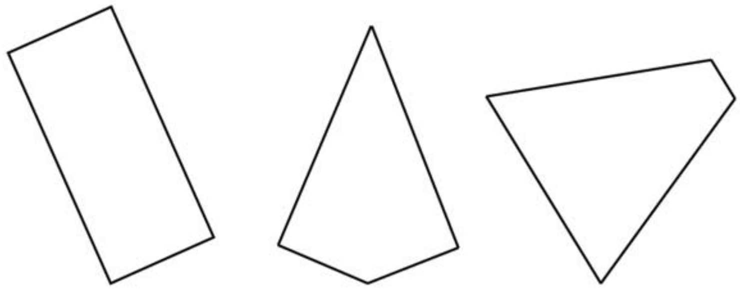
After you were able to solve the problem for the three triangles above, did it occur to you to ask whether or not this could be generalized? If not, consider this question now. Do you think it's always possible to circumscribe a triangle by a circle? Asking a question like this will often lead you to make a **conjecture**: that is, a new idea that goes beyond the original problem. In this case, your conjecture might be that *all triangles can be circumscribed by a circle*.

Of course, when you first make such a conjecture, it is purely hypothetical, and you can't be totally sure if it's true until you prove it. Before trying to prove your conjecture, though, it can be helpful to test it out a bit first. An obvious way is to try more examples. Now, if we drew a bunch of triangles that looked like the ones in the original problem, we'd run the risk of deceiving ourselves: perhaps there's something special about triangles like these that makes them work? A more strategic approach is to *try extreme cases*. In other words: try it on weird examples — the weirder, the better. That way, if it works, we can have much more confidence that your conjecture is worth trying to prove. Take a look at this triangle:



Will our conjecture hold up in this case? Try it and see.

2 For each quadrilateral, is it possible to draw a circle such that all four vertices are on the circle?



Before making a conjecture in response to Problem 2, did you make up your own examples as well, including shapes that aren't "regular," like rectangles? If so, you might have made a different conjecture than if you assumed that the three quadrilaterals in Problem 2 were representative of all shapes. Keep in mind that conjectures don't have to be as broad as possible; if you find exceptions to a conjecture that you originally thought was true, you might be able to make a different conjecture. For example, it is worth checking out whether all rectangles can be circumscribed.

For the rest of the problems in this lesson, answering each question as it is literally asked is only, say, a third of the battle — perhaps less. Once you've answered the question, you should step back from the situation and look for other questions you're now in a position to ask. Then make up simple examples, look for patterns, and otherwise do work to come up with a conjecture.

3 Find three positive integers that cannot be written as the sum of two primes.

4 In how many ways is it possible to write the number 23 as a sum of powers of two, using each power at most once? How about 24?

5 Alice the ant walks along the dashed curve on her search for a breadcrumb. Then she finds one, and takes the solid path back home. Which is longer: the path to the breadcrumb, or the way home?

6 The number 34 can be written as a sum of square numbers: 9+9+16. Write each of the following as a sum of squares, using the fewest number of squares possible.

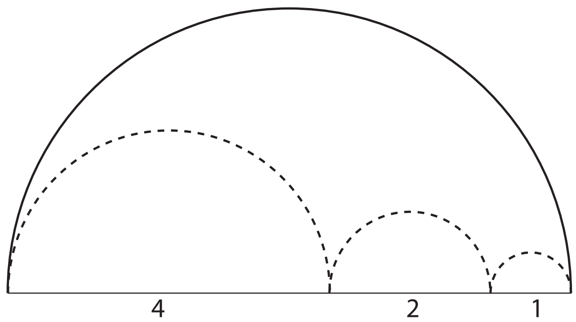
a. 41

b. 37

c. 70

d. 11

e. 1009



7 At Hillsbury high, they have seven class periods per day. Their schedule rotates, so that one day the order of the class periods is ABCDEFG, the next day it is BCDEFGA, then CDEFGAB, etc. If Monday is an ABCDEFG day, and school is held five days a week without any breaks for holidays, how many days will it be before there is another Monday ABCDEFG day?

8 Draw a circle. Then draw a line segment from one point on the circle to another. It divides the circle into two pieces, right? How many pieces can you get when you add a second line? How about a third line?

9 You may know that a Pythagorean triple is a set of integers that, together, could be the sides of a right triangle. Check that 3,4,5 and 5,12,13 are Pythagorean triples. Now find some more.

10 Color in the states in this US map so that no two states that are touching are the same color (But it's OK if states that only touch at corners share a color). What's the fewest number of colors it takes?

U.S. MAP: WWW.50STATES.COM

11 The symbol | means "divides." $3|12$ because 3 goes into 12 four times — it divides 12 evenly. But 3 does not divide 13. (Note: for this problem, restrict your consideration to positive integers only.)

a. Consider the following claim: If $(a \cdot b)|c$, then $a|b$ or $a|c$.

b. And this one: If $(a \cdot b)|c$, then $a|c$ and $b|c$.

c. And this one: If $a|b$, then $a|(b \cdot c)$.

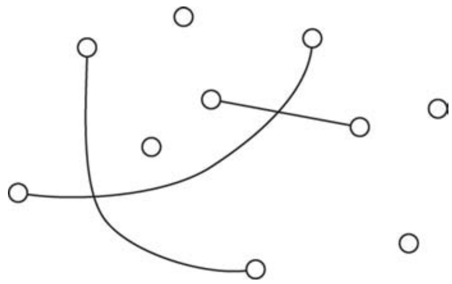
d. And this one: If $a|b$ and $a|c$, then $a|(b+c)$.

12 If p is a prime number, is $2^p - 1$ always prime? How about the converse of this statement?
(*Copyright www.mathleague.com*)

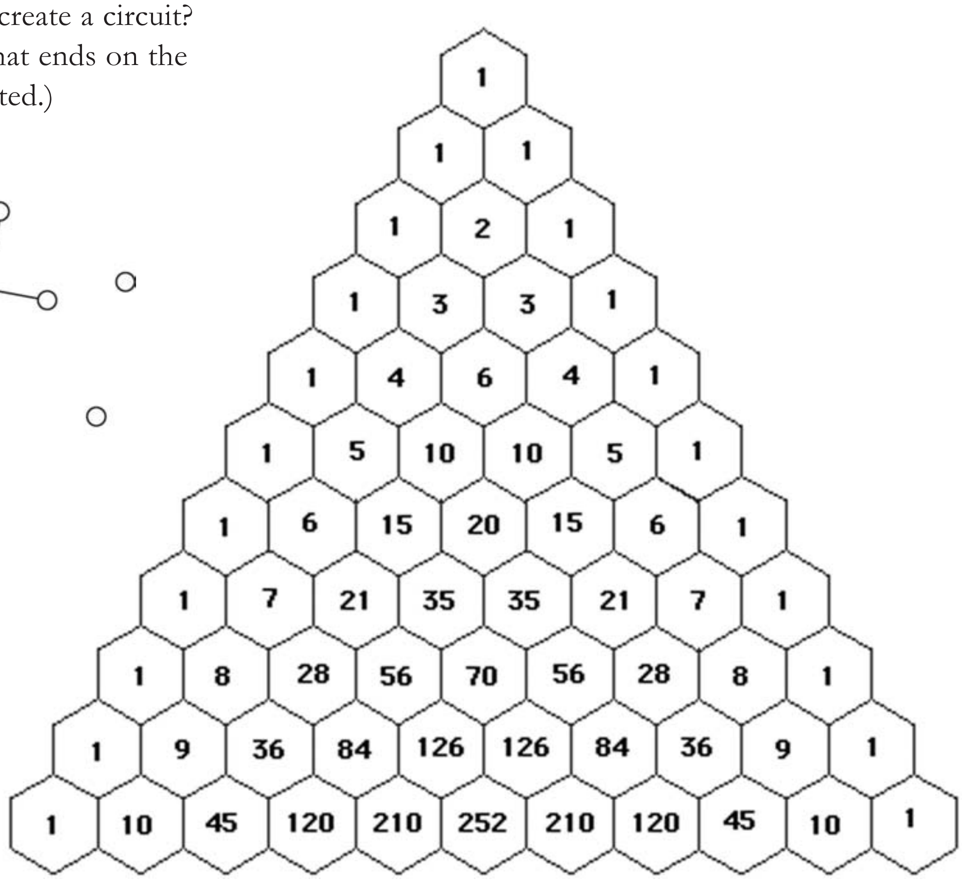
13 (See Problem 8 in the *Work Backwards* lesson) If you tried to create your own dark and stormy night problem with three apartments and choose the number of inhabitants to be 18, the problem would not work. Check this out. What numbers of inhabitants *do* work?

conjecture

14 Consider the incomplete graph below. Adding edges one at a time, what is the largest number of edges you can add before you're forced to create a circuit? (A *circuit* is just a path that ends on the same node where it started.)



15 Below are the first eleven rows of Pascal's Triangle: *(The rest of this question intentionally left blank.)*



1 Hannah had some eggs to sell, and Sierra, Matan and Kojo wanted to buy them. Hannah sells half the eggs plus half an egg to Sierra, then sells half the remaining eggs plus half an egg to Matan, and finally sells half the remaining eggs plus half an egg to Kojo. At the end of the three sales, Hannah is out of eggs. And Hannah never had to break an egg! How many eggs did Hannah begin with? Is there more than one solution?

This problem describes a chain of events in which first Sierra, then Matan, and finally Kojo buy eggs from Hannah. If you try to follow the chain through, starting with the number of eggs that Sierra buys, you will have to do a lot of guesswork to find the answer. However, you will be able to eliminate much of the guesswork from this problem if you **work backwards**. If you start instead by thinking about the number of eggs that Kojo buys, you will realize that there is only one way Hannah could sell him half of some number of eggs plus half an egg, and still be left with no eggs at the end. You can then follow the chain backwards to figure out how many eggs Hannah has at each step of the way.

The solution to the next problem is less obviously a case of working backwards:

2 A number ends in a 2, and the rest of its digits add up to 16. Prove that it is divisible by 6.

work backwards

When you are proving something, it is important to figure out before you start what it is that you need to prove. In this case, we need to show that the number is divisible by 6. However, that in itself is a tricky thing to show because, unlike numbers like 3 or 11, there is no nice rule that tells you whether a number is divisible by 6. Instead, you might ask yourself, "what are some things I can show that *would* make the number divisible by 6?" Now you have backed up one step in the proof — you've changed your goal to proving something else, perhaps easier to prove. The working backward here involves working backward from the stated goal to give yourself a better goal.

In the problems that follow, keep in mind both ways of working backward. Some problems will require a "starting at the end" approach, as in problem #1. Others will simply require that you pick a goal at the start of the problem and understand clearly the sub-goals needed to achieve it, as in #2. Still others may inspire a "working backward and forward" approach, where you will profit from starting at the beginning of a problem as well as working backwards from a goal, trying to arrange it so that your work will connect in the middle.

3 Using any particular number only once, can you choose numbers from the following set of numbers that will sum to 173?

{31, 7, 2, 4, 9, 19, 11, 32, 21, 15, 10, 17}

4 You and a friend are going to play a game where you take turns laying down bricks. On each turn, you can add between 1 and 10 bricks (inclusive), and whoever adds the 100th brick wins. It's your turn first: how many bricks should you put down in order to guarantee a win?

5 Use algebra to prove that $n^2 + 3n + 2$ is even for every integer n.

6 In the word chains game, you change one letter of a word at a time, preserving the order of the letters. Each intermediate step must spell a real English word.

a. Change MARE into COLT.
b. Change PASS into BUCK.
c. Change BASS into MIST.
d. Change WORD into BALD.

7 When you leave for school at 8 AM, the two coat hangers in your mostly-empty closet begin to multiply. They multiply in such a way that the number of coat hangers doubles every minute. You arrive home at 5 PM, the exact moment when the closet fills to capacity. Estimate as best you can the moment at which the closet was half full.

8 **A Dark and Stormy Night... and a Leak**
(compliments of ***Car Talk:*** *see transcript at http://www.cartalk.com/content/puzzler/transcripts/200511/index.html*)

RAY: It was a dark and stormy night. The 24 inhabitants of a small three-story apartment building began to worry. The rain got heavier and soon the roof began to leak.

The people living in the top story sought refuge in the apartment one floor below, i.e., the second floor. When they did, the second floor people said, "We can't admit all of you, we can just accept the same number of people that we already have living here."

So some people moved from the top floor to the middle floor and the rest stayed behind.

TOM: Let me guess, the leak came down to the second floor.

RAY: Soon the rain got heavier. Those on the second floor began to get wet and sought refuge on the first floor. They were told the same thing. We can only accept the same number of people that we have already. And so some moved, and some stayed behind.

The next morning, when the rescue workers arrived, an equal number of people emerged from each of the three floors.

The question is how many people started out on each floor?

9 Try to solve the dark and stormy night problem when the number of inhabitants of the building is:

a. 12

b. 36

10 Last week, Leo was given a list of transformations to apply to a polygon that he had constructed. The list, given in order, is

Translate two to the right.
Stretch vertically by a factor of three.
Translate four down.
Reflect over the y-axis.

work backwards

Compress horizontally (centered around the origin) by a factor of $\frac{1}{2}$.

Leo knows that one of the vertices of his polygon landed on the point $(1.5, -5)$, but he doesn't remember which one. Help him find the original vertex.

11 Prove that $\left(a^2+b^2\right)\left(c^2+d^2\right)$ is equivalent to $\left(ac+bd\right)^2+\left(ad-bc\right)^2$.

12 Each of the 4 cards shown at the right has a letter on one side and a digit on the other side. Now read the *test sentence* below:

Whenever there is a vowel on one side of a card, there is an even number on the other side of that card.

Identify every card that you ***must*** turn over to determine if the *test sentence* above is true or false for this set of 4 cards.

A	F	2	3

(Copyright marylandmathleague.com)

13 Here are some more word chains. (See http://thinks.com/puzzles/doublets.htm for these and more.)

a. Change FISH into LAKE.

b. Change ELM into OAK. (Some people can do this in six moves.)

c. Change ONE into TWO.

d. Change FOUR into FIVE.

e. Change SLEEP into DREAM.

f. Change LINENS into SHEETS.

g. Change FLUTE into CELLO.

14 (Adapted from *Gödel, Escher, Bach)* Here are the rules of a game:

*If a list of letters ends in A, you can add a B at the end (ABAA becomes ABAAB)

*You can double a list of letters (ABAA becomes ABAAABAA)

*AAA can be replaced by B (ABAAAB becomes ABBB)

*BB can disappear (ABAABBA becomes ABAAA)

a. Turn A into ABBA by following one rule at a time.

b. Turn A into BAAB.

c. Can you turn A into B?

15 Skeletor is about to give you four large prime numbers, and the only way he'll decide not to destroy the planet is if you respond within five seconds with a number not divisible by any of the four primes. All you have is a pocket calculator. What should your plan be?

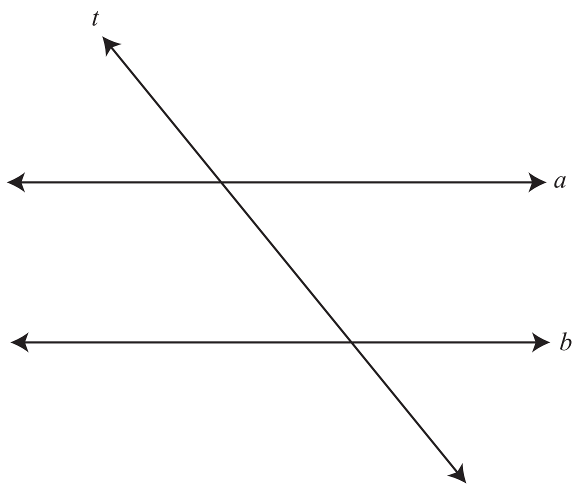
LESSON 1: PARALLEL LINES AND FIRST PRINCIPLES

Introduction

You have already seen how coordinate geometry works, with the focus on slope and distance. Now you will be considering an earlier geometry, where there are no coordinates, so lines don't have particular slopes, nor do we determine numerical distances between points. It's just our reasoning and imagination that enables us to discover valuable relationships among the figures we draw.

In this lesson, you will start to use this reasoning and imagination to help uncover the structure of geometry. Some things you know about geometry are simple and you may be happy to take them on faith. Others are less obvious, but can be explained by a careful argument. Still others may remain mysterious for now.

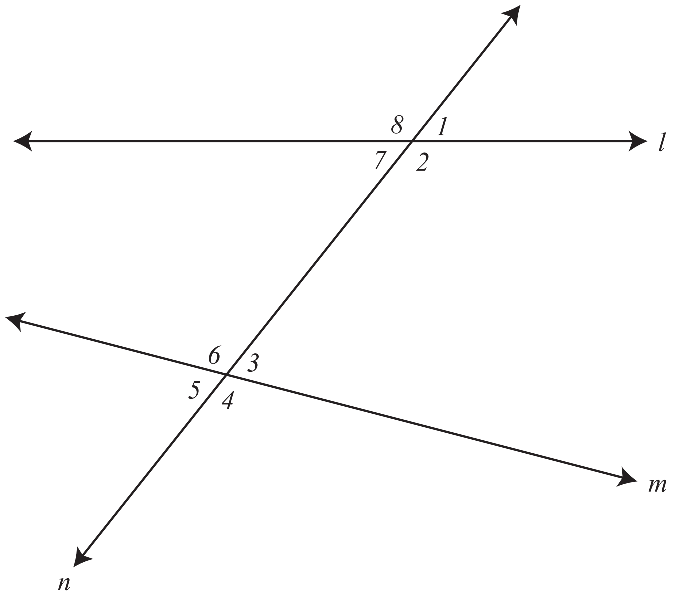
To give you some tools, let's start with the figure below, which you have probably seen in various contexts. The figure shows parallel lines *a* and *b* and line *t* crossing them. The line *t* is called a **transversal**, as it "moves" across the other pair of lines.



1 In the figure above, label all the acute angles you believe to be equal in measure with the letter x. Then label all the obtuse angles you believe to be equal in measure with the letter y. Are there any angles that you haven't labeled?

2 What is the relationship between the value of x and the value of y? Make an argument that doesn't depend on knowing the specific value of x or of y.

Some of these pairs of angles come up often enough so as to deserve special mention. In the diagram on the next page angles 1 and 3 are **corresponding angles**. Angles 3 and 7 are **alternate interior angles**. Angles 2 and 3 are **same-side interior angles**. Notice that lines l and m need not be parallel, yet the names still apply.



3 Name another pair of angles in the above figure that are

a. Corresponding.

b. Alternate interior.

c. Same-side interior.

4 In the figure showing lines *l* and *m* not parallel, you'll notice that there are no longer so many pairs of equal angles. But there is a fact that carries over from the previous figure. What is it, and how do you know for sure?

5 If you have two parallel lines crossed by a transversal, do you think that the pairs of corresponding angles will always be equal? How about the pairs of alternate interior angles? Try to find an exception.

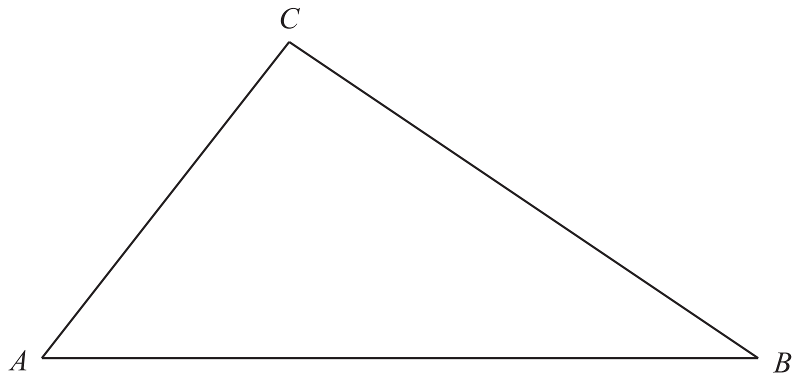
6 Draw another two parallel lines crossed by a transversal. The same-side interior angles probably don't look equal. Is there anything else you can say about them? If you're not sure, take some measurements.

Development

The results you found/reviewed in the introduction are basic results in geometry. You may have learned them before, and you may consider them so obvious that you wonder why anyone would point them out. The reason is that we can use those simple facts to convince ourselves of some results that are not so simple!

For instance, you have probably been raised to believe that there are 180 degrees in a triangle, even though that fact is not visually obvious. But the few results we've stated so far in this chapter can serve to convince us once and for all.

Below is a triangle, meant to be general enough to stand for any triangle. It would be nice to use some of your findings so far to prove that the sum of its angles is 180 degrees. However, there are no parallel lines in this picture, so there doesn't appear to be much we can do.



7 Add something to the diagram so that you do have some parallel lines.

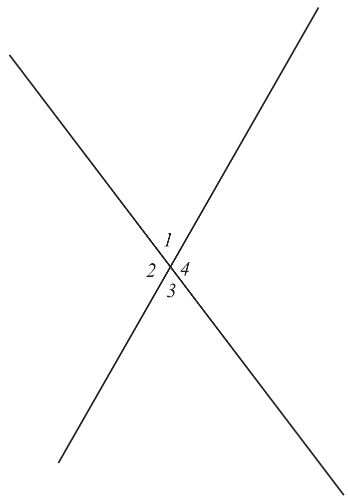
8 Now that you have some parallel lines, label as many angles as you can that you know to be equal.

9 Now finish the proof that the angles of a triangle add up to 180 degrees.

You have probably known for a long time that the angles of a triangle add up to 180 degrees — in that sense, you did not learn any new facts by doing this proof. However, you may not have realized how strongly the result depends on alternate interior angles (or whichever angle pair you decided to use) being equal when lines are parallel. In fact, there are non-Euclidean geometries in which the alternate interior result is not true, where the alternate interior angles result is not true — and, sure enough, in those geometries, the angles of a triangle can actually add up to more or less than 180 degrees!

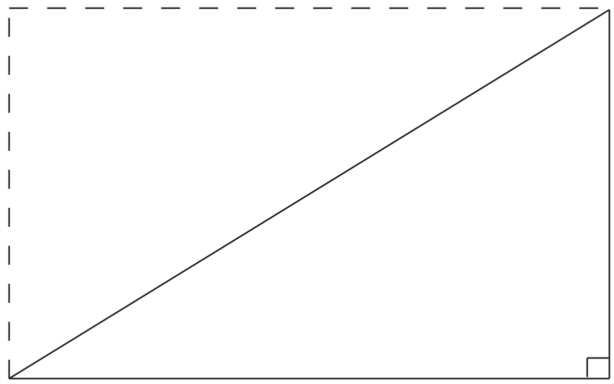
The ancient Greeks appreciated how important it was to uncover the structure of geometry, and very often looked for proofs of statements they were pretty sure were true using statements they already knew to be true. They tried to reduce the number of statements they accepted without proof to the smallest number possible. The most famous Greek geometer, Euclid, was able to reduce his number of assumptions to five mathematical statements and five "common notions" — like "two things equal to the same thing are equal to each other." He was able to prove a huge amount of geometry from these simple assumptions.

10 In the spirit of Euclid, try to prove that angles 1 and 3 below are equal. What facts are you relying on?

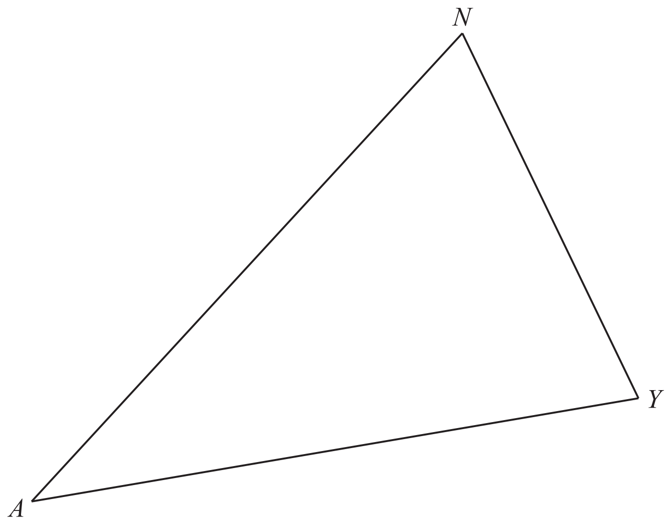


The rest of this Development consists of examples of some other argument styles that do not involve coordinates.

Another fact you have known for a long time is that the area of a triangle can be found by the formula $\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height}$. You might also have seen the following picture, designed to show why the formula is true for right triangles.



But alas, not all triangles are right, and we'd still like to be able to use this familiar formula with any triangle. So below is triangle *ANY*.



11 Add something to the diagram so that you can use what you know about the area of rectangles and right triangles to find its area. Is your formula equivalent to

$$\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height}?$$

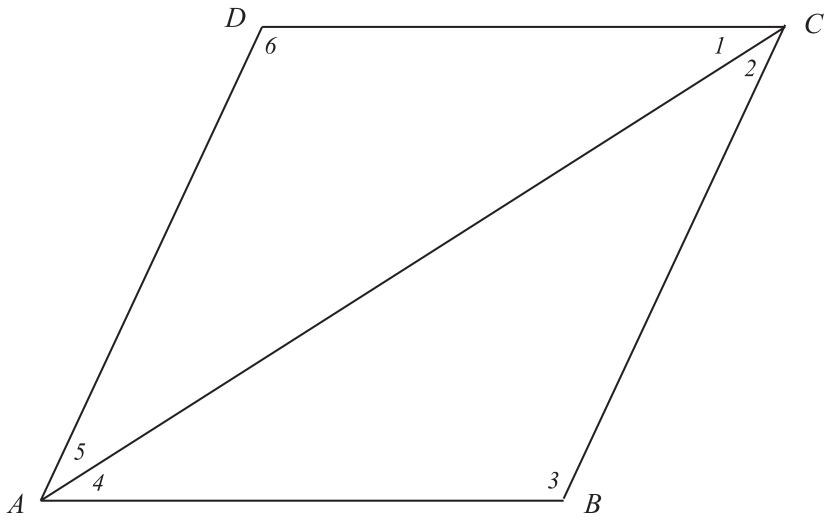
12 And last, but not least, a Physical Challenge:

a. Using only a compass and a tool for drawing straight lines, construct an equilateral triangle. Your tool for drawing straight lines may not be a ruler, or anything else that can take measurements.

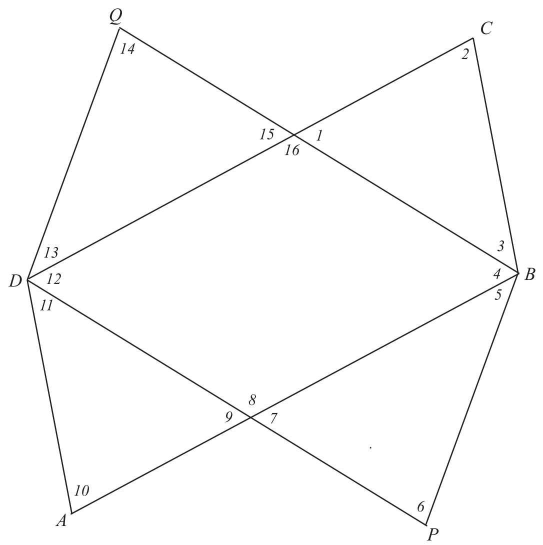
b. Come up with an argument that would convince a friend that the triangle you constructed really has to be equilateral and, if they did the same thing, their triangle would be equilateral, too.

Practice

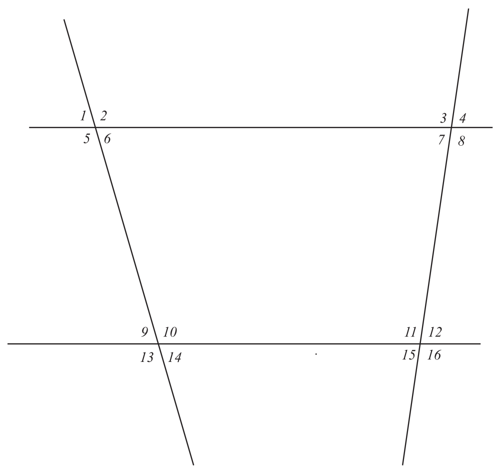
13 Given that $\overline{AB}$ is parallel to $\overline{CD}$, which pairs of angles must be equal?



14 In the figure below, ABCD and PBQD are parallelograms. Which of the numbered angles must be the same size as the angle numbered 1?
(Copyright Phillips Exeter Academy)



15 Given the following figure, where only the horizontal segments are parallel, name three angle pairs that are equal and three angle pairs you can conclude are **supplementary** — that is, sum to 180°.

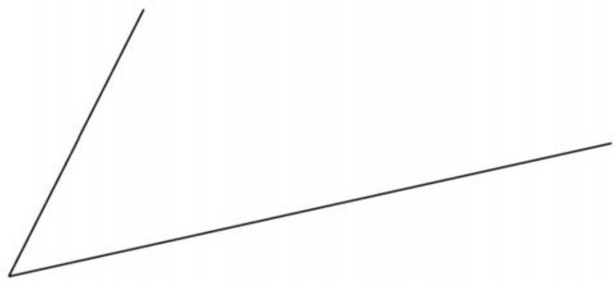


16 Assuming that alternate interior angles are equal when lines are parallel, prove that corresponding angles must be as well.

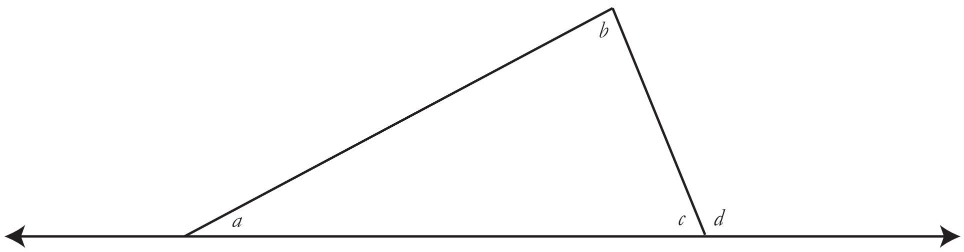
17 In problem 11, triangle *ANY* was meant to stand for any triangle, but notice that triangle ANY was drawn to be acute. Try the argument again with an obtuse triangle, and see if you can still obtain the formula

$$\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height}.$$

18 The two line segments below are "half" of a parallelogram. Trace it into your notebook. Then use a compass and straightedge to draw in the remaining two line segments. How do you know for sure that what you've got is a parallelogram?



19 In the figure below, the angle d is an *exterior angle* of this triangle. Determine experimentally if there is any relationship between d and the measure of any angles of the triangle. Does your observation apply even for an obtuse triangle (a triangle where one angle is greater than 90 degrees)?



20 Prove the result you found in the previous problem.

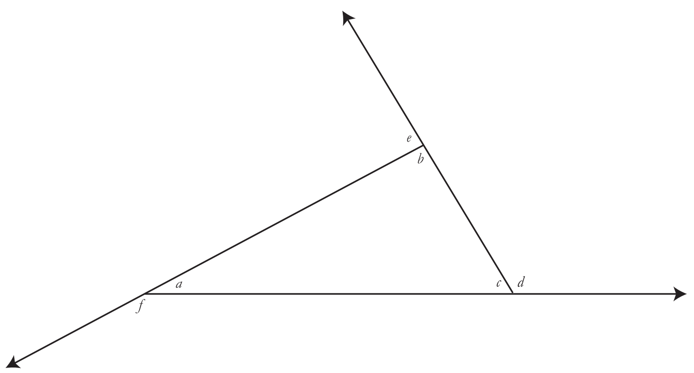
Problems

21 Now that you've proved that the sum of the measures of the interior angles of any triangle is 180°, experiment with the sum of the measures of the interior angles of any quadrilateral. How about other polygons? Prove what you find.

22 How many regular polygons have interior angles that are integers? How can you be sure your answer is correct?

23 On the bus, Rafael asked his three friends what they had figured out for the sum of the interior angles of a 500-sided polygon. Lemuel said he got 8640. Julia got 89640, and Marla got 899640. Rafael didn't have any of his books, pencil, or paper, but he was still able to determine who was likely to be right. Can you?

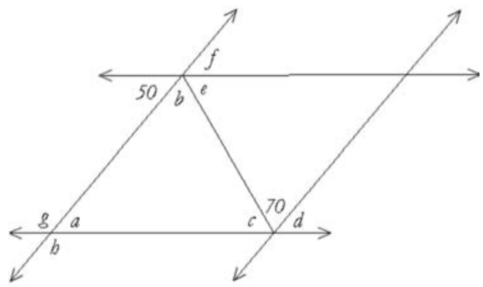
24 You know the sum of the interior angles of a triangle, but how about the sum of the exterior angles of any triangle? (see the figure below — the exterior angles are d, e, and f.) Take measurements, if necessary, and then try to prove what you found.



25 What do you suppose would be the sum of the exterior angles of a quadrilateral? Try it out on some different quadrilaterals. See if you can prove it.

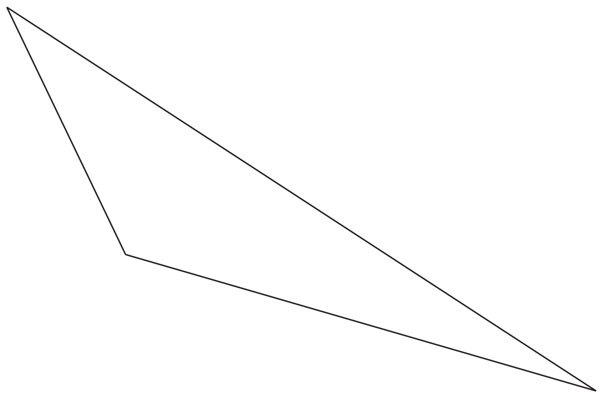
26 Draw any triangle *ANY*. Recall that a **median** is a line segment from a vertex of a triangle to the midpoint of the opposite side. A median will split a triangle into two smaller triangles. Try this with a few different triangles. Try to draw a triangle so that the left triangle created by the median visibly has a larger area than the right.

27 Given the following figure, where both pairs of lines that appear parallel are assumed parallel, find all the angles represented by letters.

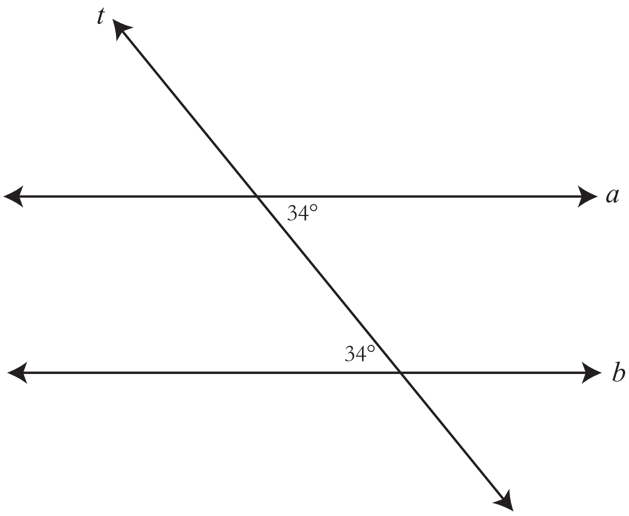


28 Try to prove or disprove the statement "If two lines intersect the same line, they must intersect each other."

29 Here is a triangle. Using only a compass and straightedge, make an exact copy of the triangle in your notebook. No tracing allowed.

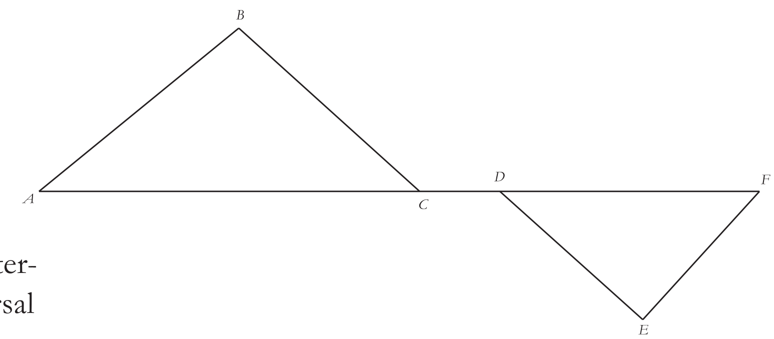


30 You know from the beginning of the chapter that, if lines are parallel, then alternate interior angles created by a transversal are equal. So, does this imply that in the figure below, must lines *a* and *b* be parallel?

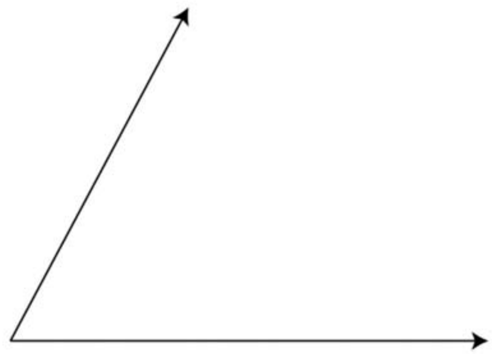


31 Though it seems obvious, one of Euclid's theorems was "If two lines are parallel to a third line, they are parallel to each other." Prove this statement. Remember the tools you have to prove that lines are parallel.

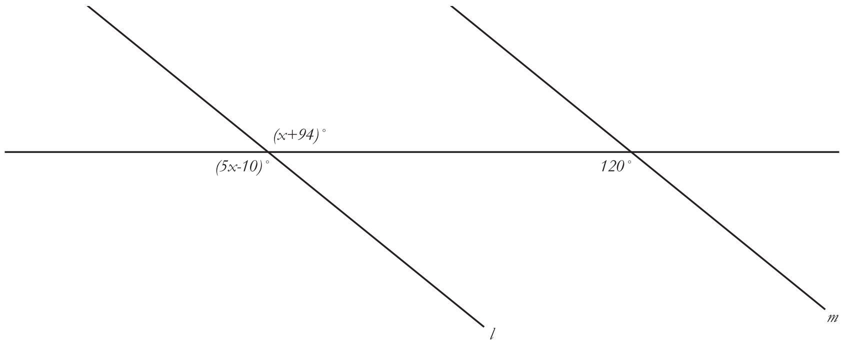
32 In the figure below, $m\angle BCA = m\angle FDE$ (which means the measure of angle BCA equals the measure of angle FDE). Also, the points A, C, D, and F all lie on a straight line. Is this enough evidence to claim that $\overline{BC}$ must be parallel to $\overline{DE}$? Explain.



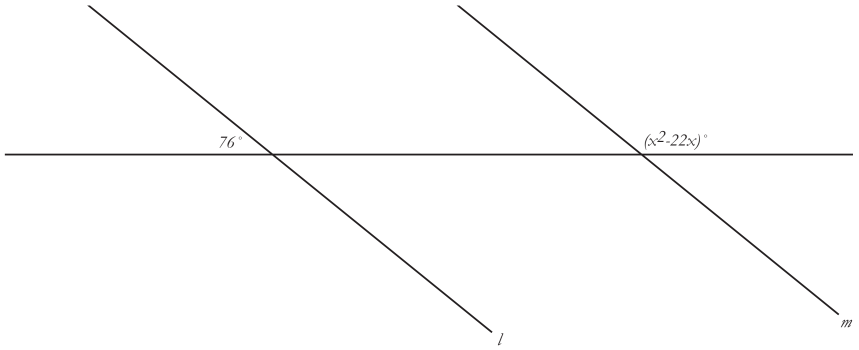
33 Here is an angle. Using only a compass and straightedge, find a way to copy the angle into your notebook. (Again, no tracing allowed.)



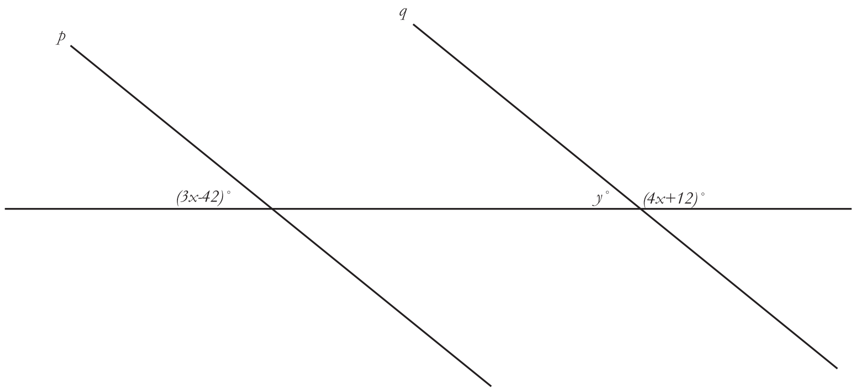
34 Would you claim that l and m are parallel in the following figure? Explain.



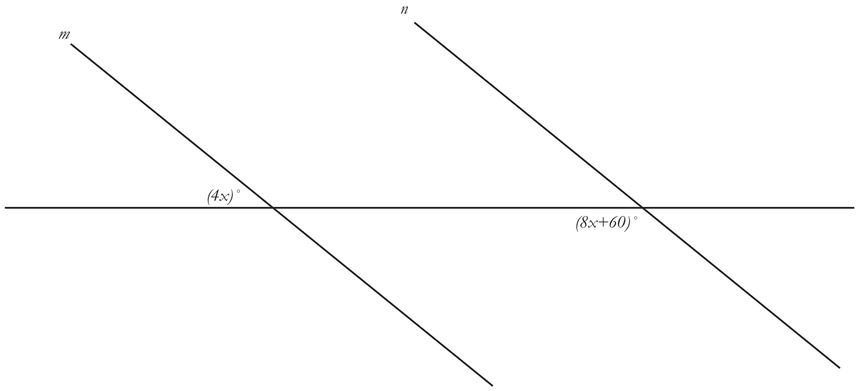
35 In the figure below, what would x need to equal if you could claim correctly that l and m are parallel? Explain.



36 If p and q are parallel, find y.

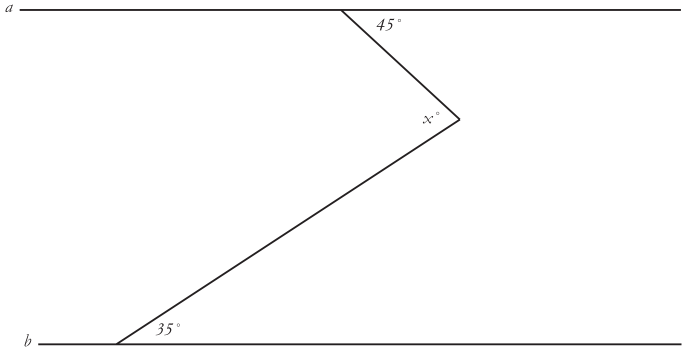


37 If m is not parallel to n, what can't x equal?

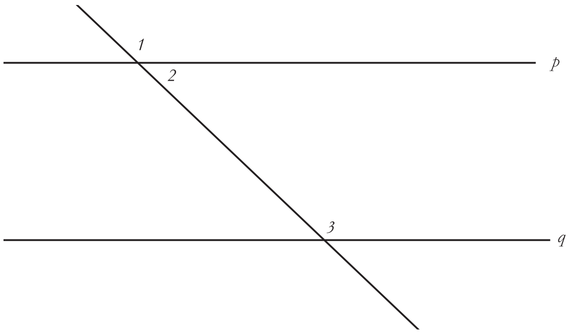


38 The shorter side of a rectangle is 10 units. If the diagonal is two units more than the longer side of the rectangle, what is the area of the triangles created by the diagonal?

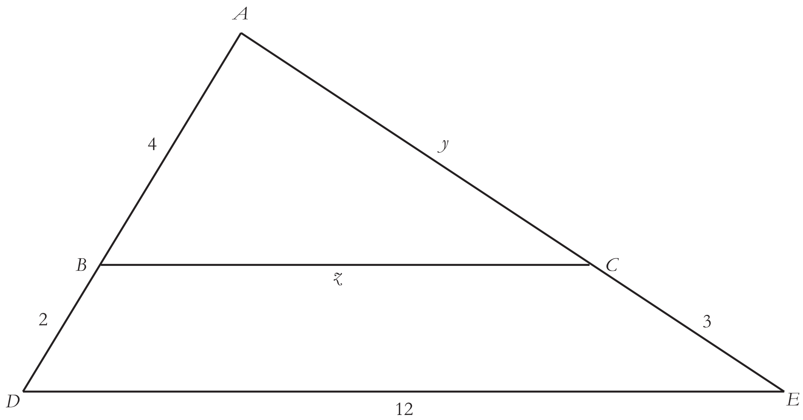
39 If a is parallel to b, find the measure of x. (Hint: this is a problem for which geometric tinkering will prove useful.)



40 Assume line p is parallel to line q. You are given that $m\angle 1 = 2x + 6y$, $m\angle 2 = x + 15$, and $m\angle 3 = 10y + 40$. Find $m\angle 2$.



41 Given that $\overline{BC}$ is parallel to $\overline{DE}$ (see to the right), what can you conclude about triangles ABC and ADE? What would be the length of $\overline{AC}$? $\overline{BC}$?



42 Don't use a calculator for this problem.

a. Simplify $\sqrt{\dfrac{3}{4}}$

b. Simplify $y^2-(\dfrac{1}{2}y)^2$

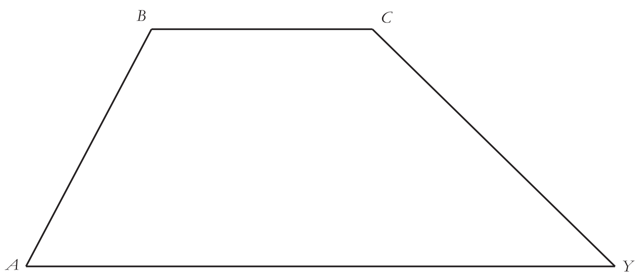
c. Factor x^2+3x+2

d. Reduce the fraction $\dfrac{x^2-8x+16}{x^2-11x+28}$

e. What value of x satisfies $2^5+2^5+2^5+2^5=2^x$?

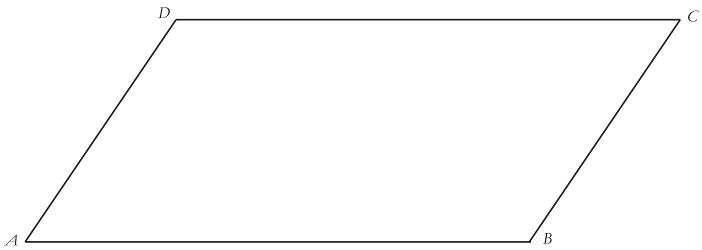
There are a number of other two-dimensional forms that we can find the areas of now that we know the area formulas for rectangles and triangles. Here are some.

Draw a triangle *ANY* and a line parallel to $\overline{AY}$ so that it cuts through the sides $\overline{AN}$ and $\overline{NY}$. Let $\overline{BC}$ represent the parallel line intersecting the triangle. Now erase what's above $\overline{BC}$. You're looking at what is called a trapezoid — a four-sided figure with one pair of parallel sides.



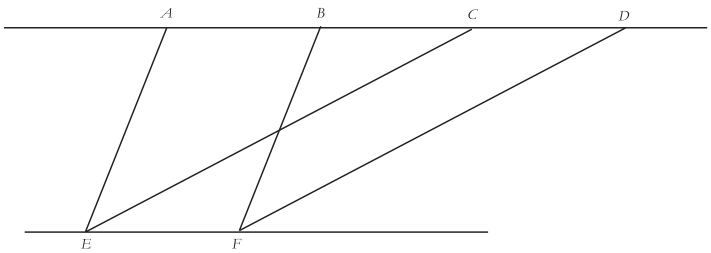
43 Knowing what you do about the areas of triangles and rectangles, come up with a formula for the area of a trapezoid. You may want to label *BC*, the top base, as b_1 and *AY*, the bottom base, as b_2. What other dimensions of the trapezoid might be useful?

If we now consider any parallelogram *ABCD*, not just those with right angles, we can consider whether there is a unique formula for the area of any and all parallelograms, including rectangles and squares.



44 Partition the parallelogram (divide it up) into shapes that you already know the area of. By summing the areas of these shapes, find an area formula that should work for any parallelogram.

45 Given the image below where it is assumed the horizontal segments are parallel, what can you conclude, if anything, regarding the areas of parallelograms *ABFE* and *CDFE*?



46 If the lengths of the sides of a parallelogram are doubled, what seems plausible regarding the area of the new parallelogram? Can you make an argument that your conjecture is the case?

47 A certain trapezoid has bases of length 10 and 24, sides of length 13 and 15, and a height of 12. If you were to make an enlarged, scale copy of the trapezoid whose area was 4 times that of the original, then what would the measurements of the larger trapezoid be?

48 Let *ABCD* be a square, and *EFGH* another square whose area is twice that of square *ABCD*. Prove that the ratio of *AB* to *AC* is equal to the ratio of *AB* to *EF*.

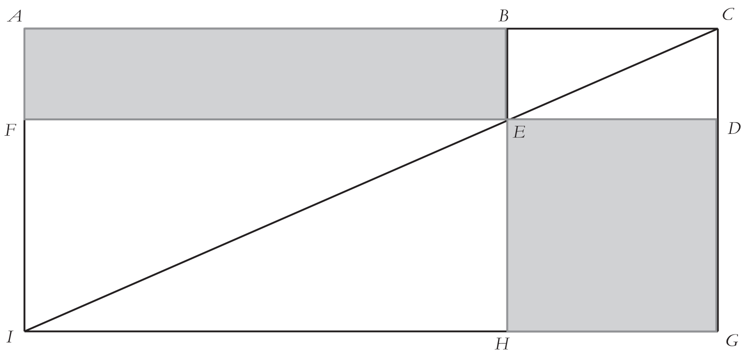
The ancients who studied form naturally included others besides the Greeks. There were the Egyptians who designed and built pyramids and came up with a very interesting formula for the area of a circle (see if you can find it on Google and see what an impressive fit it provides). The Babylonians also studied form, and they came up with a formula for the area of a quadrilateral.

49 The Babylonians' formula for the area of a quadrilateral is $\text{Area} = \dfrac{(a+b)(c+d)}{4}$, where a and c are the lengths of one pair of opposite sides of a quadrilateral, and b and d are the other. Does their formula always, sometimes, or never work? (Try this with quadrilaterals you actually know the area of.)

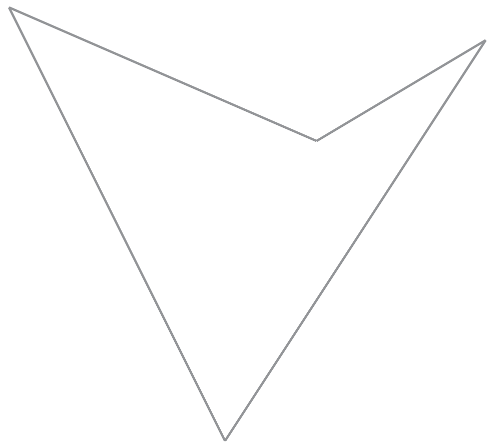
Exploring in Depth

50 Earlier in this lesson, you probably decided that, if you have lines crossed by a transversal in such a way that the alternate interior angles created are equal, then those lines must be parallel. Here's a suggestion for arguing that your conjecture must be the case. Suppose that you had alternate interior angles that were equal, but the lines were NOT parallel. Do some geometric tinkering to derive some interesting things.

51 If *BCDE*, *FEHI*, and ACGI are rectangles, can you logically conclude that quadrilaterals *ABEF* and *EDGH* have the same area?



52 The polygons you have studied in this unit have mostly been **convex**. When you pick any two points inside a convex polygon and connect them with a straight line segment, the line segment always stays within the polygon. If there are two points where the line connecting them must go outside the polygon, then the polygon is called **nonconvex**.

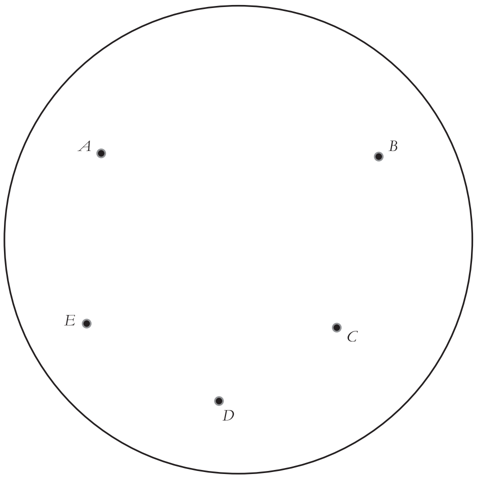


a. Show that the polygon above is nonconvex.

b. Draw in the exterior angles of this polygon. See Problem 24 if you need a reminder.

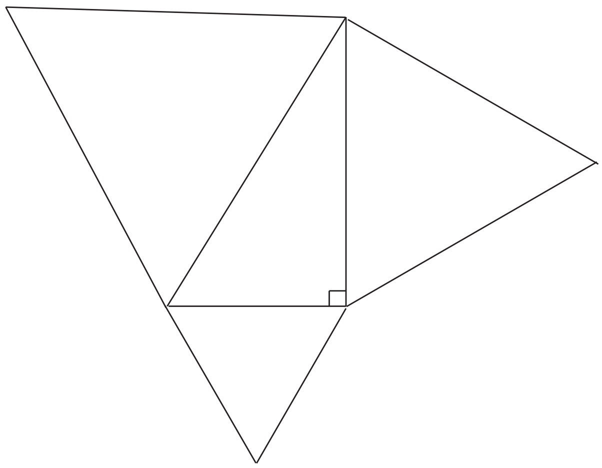
c. Do the exterior angles of a non-convex polygon still add up to 360 degrees? Experiment.

53 First remind yourself of the result in problem #31. Then consider the following image. As you have seen in Euclidean geometry parallel lines are infinite lines in the same plane that don't intersect. Given the circle as the entire universe, $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ are said to be parallel, as are $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$, as neither pair has any intersection points within the circle. However, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are not parallel. How does this finding fit with your finding in problem 31?



54 Draw a good-sized equilateral triangle and pick any point inside the triangle. From that point draw the three perpendiculars to the three sides of the triangle. Let the lengths of the three perpendiculars be a, b, and c. If d = height or altitude of the triangle, show that $a+b+c=d$. (Quite a surprise?)

55 Draw a right triangle. On each of the three sides of the triangle draw equilateral triangles where the side of the original triangle is also a side of the new triangle. Each of those equilateral triangles has an area. Does it seem that the sum of the areas of the equilateral triangles on the legs of the right triangle is equal to the area of the equilateral triangle on the hypotenuse? Can you prove this? (Hint: what do you know about 30°-60°-90° triangles?)

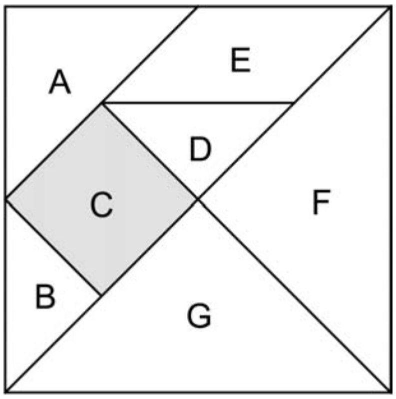


56 Draw any line segment on a piece of paper. Now come up with a compass-and-straightedge construction to *bisect* the segment — cut it exactly in half.

LESSON 2: CONGRUENT TRIANGLES

Introduction

1 The figure below, called a tangram puzzle and originating in China, consists of seven pieces — five isosceles right triangles, a square, and a parallelogram.



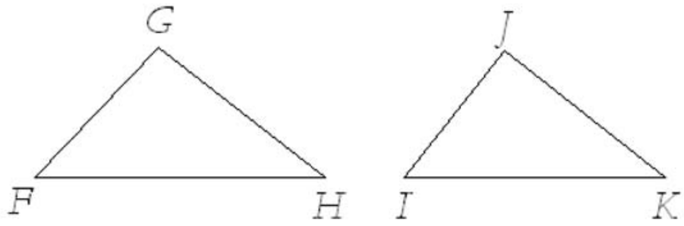
a. If the area of the shaded square C is 9 cm^2, find the dimensions of the other pieces.

b. Are any pairs of triangles identical in shape and size? Explain your answer.

Development

When two triangles are similar with scale factor one, we say that they are **congruent**. More precisely, if $\triangle ABC \sim \triangle DEF$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF} = 1$ (or equivalently, $AB = DE$, $BC = EF$, and $AC = DF$), we say that $\triangle ABC$ and $\triangle DEF$ are congruent and write $\triangle ABC \cong \triangle DEF$.

Carly is building a house for her dog, and she found two triangular trusses, $\triangle FGH$ and $\triangle IJK$ shown below, that could be used to support the roof of the doghouse. However, these triangular frames would be too heavy for her to put them together easily if she tried to check whether or not they overlap. But she needs to make sure that the edges of the trusses are congruent triangles; otherwise, she couldn't use them as frames for the roof of the doghouse.



As you can see, each truss has six main features, three angles and three sides. Before making a decision, Carly wants to know the minimum amount of measurements of each truss that she must take to ensure the trusses are identical in shape and size.

2 Suppose that Carly only wants to take one measurement.

a. If she found that one angle of one triangular truss is the same measure as one angle of the other (Case A), could she conclude that the two triangular trusses are congruent?

b. How about if she found that one side of one triangular truss is the same measure as one side of the other (Case S)?

3 Now suppose that Carly takes two measurements.

a. If she found that two angles of one triangular truss are the same measure as two angles of the other (Case AA), could she then conclude that the trusses are congruent?

b. How about if two sides of one triangular truss are the same measures as two sides of the other (Case SS)?

c. Finally, what if one side and one angle of one triangular truss are the same measures as one side and one angle of the other (Case SA), for example that $FG = IJ$ and $m\angle F = m\angle I$, or that $FG = IJ$ and $m\angle H = m\angle K$, or that $FG = IJ$ and $m\angle F = m\angle K$?

4 Now, if she found that the three angles of one truss are the same measure as the three angles of the other (Case AAA), or that the three sides of one truss are the same measure as the three sides of the other (Case SSS), might she conclude in each case that the two triangular trusses are congruent? Explain your answer.

5 What about if she found that three out of the six features of one truss — other than the ones considered in Exercise 4 — are the same measure as three features of the other truss? For example, if she found that two angles and the side between these angles in a truss are the same measure as two angles and the side between them in the other truss, such as if $m\angle F = m\angle I$, $FH = IK$, and $m\angle H = m\angle K$ (Case ASA), would this be enough information to conclude that the two triangular trusses are congruent? Or what if she found that two angles and a side other than the one between these two angles in a truss are the same measure as two angles and a side other than the one between them in the other truss, such as if $FH = IK$, $m\angle H = m\angle K$, and $m\angle G = m\angle J$ (Case SAA)?

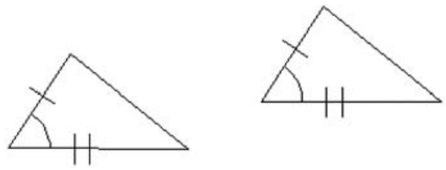
6 Describe all the possibilities where three features of one truss are the same measure as three features of the other truss using the case notation we have been using: Case ASA, or Case SAA etc. Explain your answer.

7 Use the ideas in the previous problems to discuss with the members of your group a general answer to the following question: What are the minimum conditions under which we may assume that two triangles are congruent? Identify these conditions by case, for example, Case AA, or Case ASA.

If two segments $\overline{AB}$ and $\overline{CD}$ are the same length, that is, if $AB = CD$, we say that these segments are **congruent**, and this can be written as $\overline{AB} \cong \overline{CD}$. Similarly, if two angles $\angle P$ and $\angle Q$ are the same measure — $m\angle P = m\angle Q$ — we say that these angles are **congruent**. This can be written as $\angle P \cong \angle Q$.

8 Matt's group found only one case with minimal conditions, Case SAS, under which two triangles are congruent. This group expressed and illustrated this case as follow.

Case SAS (side-angle-side): If two sides and the included angle (angle between them) in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.



Write and illustrate as Matt's group did all the cases with minimal conditions under which you consider that two triangles are congruent.

9 As shown in the figure, airports A and C are 600 km apart. The ground controllers at airport A monitor planes within a 400-km radius of the airport. A plane takes off from airport C in the direction W 30° N, as indicated in the figure.

David predicts that the plane will be within range of airport A ground control at just one point. Jackson, on the other hand, affirms that the plane will be within range of airport A ground control for quite some time, between a point when it first comes within range, and a point when it is last within range. Dan states that David and Jackson are both wrong, and that the plane will never be within range of airport A ground control.

Who is right? Explain your answer.

10 Claire claimed that SSA is a case of triangle congruence. She described this case as follows:

Case SSA (side-side-angle)

If two sides and an angle not included by these sides in one triangle are congruent to two sides and an angle not included by these sides in another triangle, then the two triangles are congruent.

Discuss with the members of your group whether or not this is a case of triangle congruence. If you agree with Claire, explain why you do. If you don't agree with her, give a counterexample.

11 For each of the following statements, determine whether it is true or false. If true, explain why; if false, give a counterexample.

a. If two triangles are congruent, they are similar.

b. If two triangles are similar, they are congruent.

12 For each figure below, determine the validity of each of the two statements that follow. Explain your answers. In Figure 1, the side length of each pentagon is 2 units.

a. The two figures are congruent.

b. The two figures are similar.

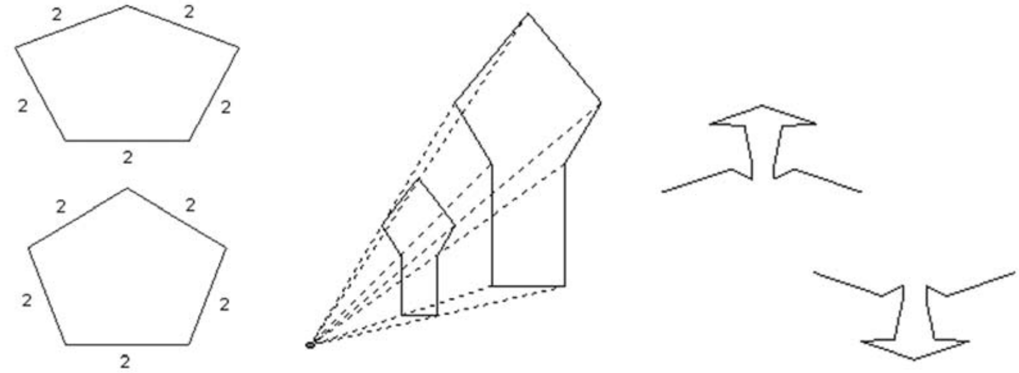


Figure 1 Figure 2 Figure 3

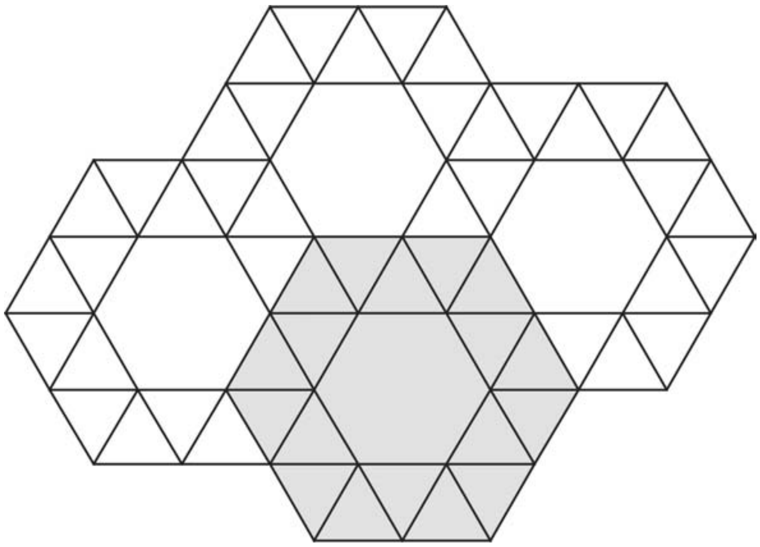
13 Discuss with the members of your group what may be an appropriate answer to each of the following questions.

a. What does it mean for two arbitrary shapes to be similar?

b. What does it mean for two arbitrary shapes to be congruent?

Practice

14 The figure below shows a tessellation. The smallest triangles in this tessellation are both equilateral and congruent. Look carefully at this tessellation and answer the questions that follow.



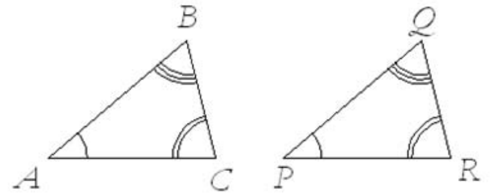
a. How many triangles are similar but not congruent to each of the smallest ones?

b. How many hexagons are congruent to the shaded one?

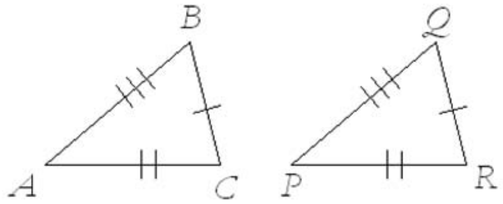
c. How many hexagons are similar to the shaded one?

15 Below, you are given several pairs of triangles with some information about them. In each case, determine whether or not the information given allows us to affirm that the two triangles are either congruent (C), or similar but not necessarily congruent (S), or not similar (NS), or there is insufficient information to decide (I). Justify your answers. Angles and segments marked equally have the same magnitude. However, do not assume that segments or angles have the same measurements just because it appears so.

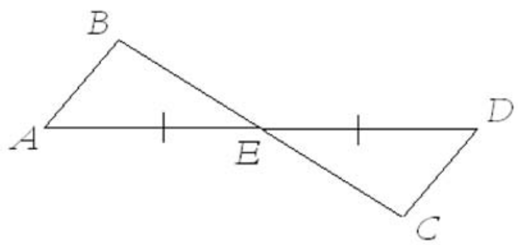
a. ______



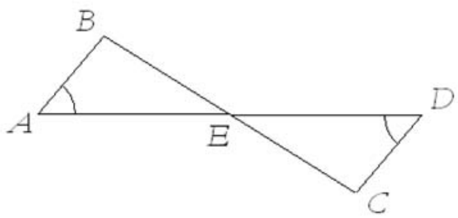
b. ______



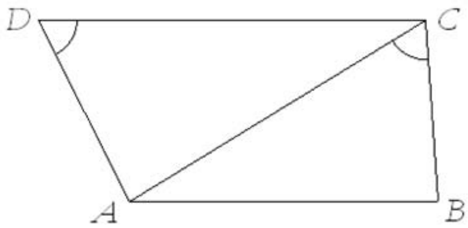
c. ______ $\overline{BC}$ and $\overline{AD}$ intersect at point E, and $\overline{AB} \parallel \overline{CD}$.



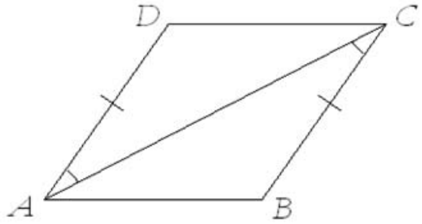
d. ______ $\overline{AD}$ and $\overline{BC}$ intersect at point E.



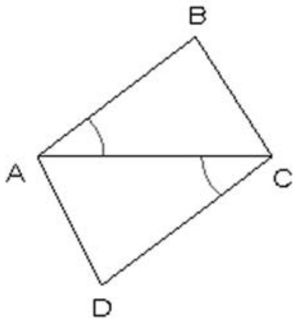
e. ______ $\overline{AB} \parallel \overline{CD}$



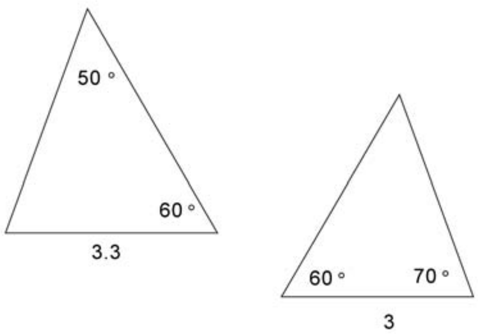
f. ______



g. ______

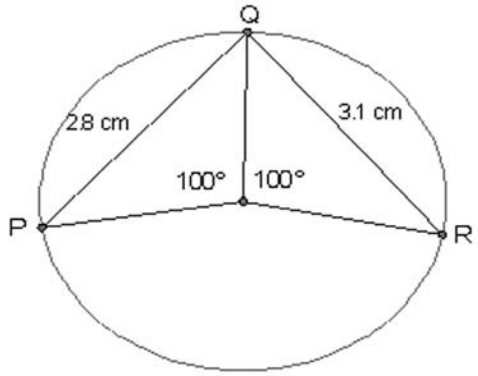


h. ______

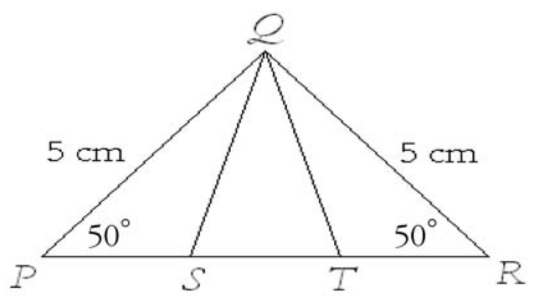


16 In each exercise below, you are given pieces of information about two figures. In the first two exercises the figures are shown, and you must determine whether the information is consistent (non-contradictory). In the third exercise, determine whether or not the statement is necessarily true. In each of these exercises, if you think that something is wrong, explain why.

a. Points P, Q, and R below lie on the circle centered at O. $m\angle POQ = m\angle QOR = 100^\circ$, $PQ = 2.8$ cm and $QR = 3.1$ cm.



b. In the figure below, $PQ = RQ = 5$ cm and $m\angle P = m\angle R = 50^\circ$. Furthermore, points S and T divide $\overline{PR}$ into three segments of equal length. $QS = 3.5$ cm and $QT = 3.8$ cm.

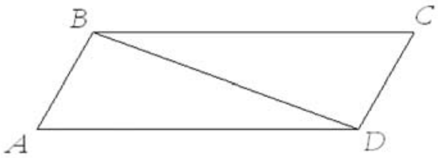


c. If $AB = PQ = 3$ cm, $AC = PR = 2$ cm, $m\angle B = m\angle Q = 34^\circ$, then necessarily $\triangle ABC \cong \triangle PQR$.

Further Development

17 A parallelogram is a quadrilateral whose opposite sides are parallel. $ABCD$ below is a parallelogram.

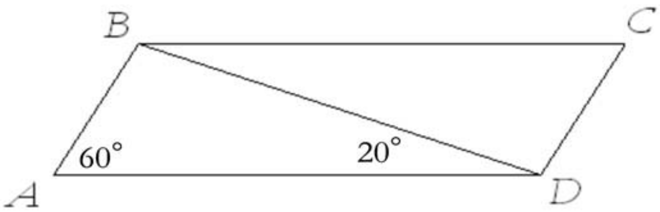
a. How do the two triangles determined by the diagonal $\overline{BD}$ compare? Explain.



b. In general, what can you conjecture about the two triangles determined by a diagonal of a parallelogram? Could you prove your conjecture? If so, prove it!

18 Silvia is a sharp mathematics student who frequently uses "mathematical" arguments to prove weird statements. This time, she claimed to be able to use congruent triangles to prove that $60^\circ = 100^\circ$. This was Silvia's argument:

In the figure below diagonal $\overline{BD}$ divides parallelogram $ABCD$ into two congruent triangles.

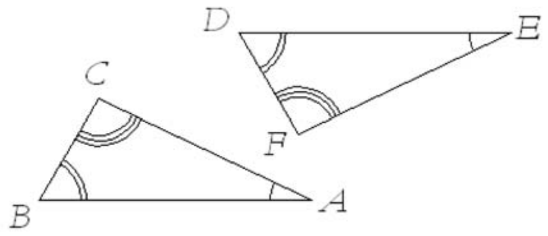


Furthermore, $m\angle ABD = 100^\circ$ and $m\angle BDC = m\angle ABD$ because $\overline{AB}$ and $\overline{DC}$ are parallel. Thus, $m\angle BDC = 100^\circ$. Now, since $\triangle ABD \cong \triangle DBC$, then $m\angle A = m\angle BDC$, that is, $60^\circ = 100^\circ$.

How would you explain to Silvia that her argument (not just her conclusion) is wrong?

As we have stated earlier, we say that $\triangle ABC$ and $\triangle EDF$ are congruent and write $\triangle ABC \cong \triangle EDF$ if $\triangle ABC \sim \triangle EDF$ and the scale factor is 1. Therefore, as illustrated in the figure below,

If $\triangle ABC \cong \triangle EDF$, then $m\angle A = m\angle E$, $m\angle B = m\angle D$, $m\angle C = m\angle F$, $AB = ED$, $BC = DF$, and $AC = EF$.

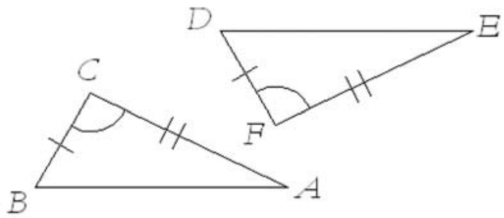


The converse of the statement in the previous box is also true; that is, if the six equalities in the box above hold, then $\triangle ABC \cong \triangle EDF$. Why?

In two congruent triangles, angles with the same angle measurement are called **corresponding angles**, and sides opposite corresponding angles are called **corresponding sides**. As a matter of fact, in the figure above, angles $\angle A$ and $\angle E$ are corresponding angles, and $\overline{BC}$ and $\overline{DF}$ are corresponding sides. Thus, The set of (six) equalities in the box above can be expressed by saying that *Corresponding Parts of Congruent Triangles are Congruent* (CPCTC).

Given misunderstandings such as that of Silvia's "proof" in Problem 18, it is important to show explicitly what pair of angles have the same measurement when describing congruent triangles. A natural way of doing this is by listing their corresponding angles in the same order.

19 In the figure below, angles or segments marked equally have the same measurements.



How many, if any, of the following statements are true? Explain your answer.

a. $\triangle CBA \cong \triangle EDF$

b. $\triangle CBA \cong \triangle DFE$

c. $\triangle BAC \cong \triangle DEF$

d. $\triangle CBA \cong \triangle FED$

e. $\triangle CBA \cong \triangle DEF$

f. $\triangle CBA \cong \triangle EFD$

g. $\triangle CBA \cong \triangle FDE$

20 In each case below, draw a triangle with the given side lengths. In each case, make any relevant comments about your work.

a. 5 cm, 6 cm, and 8 cm

b. 4 cm, 7 cm, and 11 cm

c. 3 cm, 5 cm, and 10 cm

d. Looking at parts a, b and c above, what would you conjecture?

21 Some ant scientists as well as some ornithologists believe that ants and birds have an intuitive sense of energy saving.

a. Two ants move from point A to point T, starting at the same time and with the same constant speed. The first goes directly from A to T, but the second goes first to point N which does not lie on the line segment joining A and T. Which ant will get sooner to point T? Why?

b. A bird needs to go from point *B* to point *R*; however, the bird has two options: either going directly from point *B* to point *R*, or going first to a point *I* which does not lie on the segment joining *B* and *R*. Which path do you think the bird should take? Why?

22 You are sitting at a vertex of a triangular region *DES* and you can move only along the edges of this region.

a. If you want to move from D to E, what path would minimize your travel in this region? Why?

b. What path would be the shortest if you what to go from D to S? Why?

c. What path would be the shortest if you what to go from E to *S*? Why?

d. Each of the inequalities you may have found in your answers to parts a, b, and c above is known as a **triangle inequality**. What do you think should be the general statement of the Triangle Inequality Conjecture? (Consider the three sides of the triangle.)

e. You can express the Triangle Inequality Conjecture from part d by referring only to the side lengths rather than to the labels of the vertices or sides of the triangle. Do so if you have not already.

23 Draw a triangle with three different side lengths. Answer the following questions in terms of angles with greater or smaller magnitudes, not in terms of their labels.

a. Which angle is opposite the longest side?

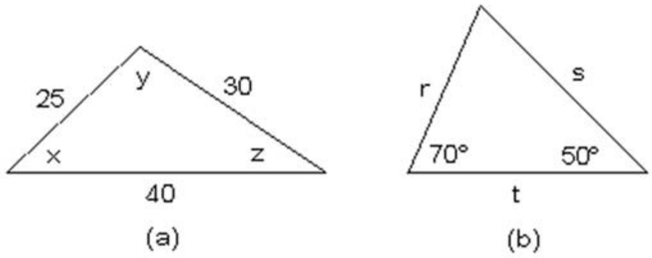
b. Which angle is opposite the shortest side?

c. Draw another triangle with three different side lengths. Are your observations in parts a and b above true in this triangle too?

d. Based on your previous observations, conjecture what may be true in general.

Practice

24 Based on the information given, arrange the letters in each of these figures in order from large to small. Explain.



25 In each case below, determine whether it is possible to draw a triangle with the given side lengths. Write *yes* if so, and *no* otherwise. Explain, giving at least one reason for each of your answers.

a. 4 m, 5 m, and 8 m

b. 5 cm, 6 cm, and 13 cm

c. 6 ft, 7 ft, and 12 ft

d. 3 cm, 4 cm, and 7 cm

26 If $\triangle CBA \cong \triangle FDE$, $m\angle A = 90^\circ$, $m\angle B = 30^\circ$, and $BA = 6$ cm, find DE and $m\angle F$.

Problems

27 A triangle has integer length sides, two of them having lengths 5 and 8. What is its largest possible perimeter?

28 Given that $\triangle PQR \cong \triangle STU$, $m\angle Q = 60^\circ$, $m\angle U = 30^\circ$, $QR = 5$ cm, and $TS = 2.5$ cm, find the perimeter of $\triangle STU$.

Recall that the vertices in a triangle are labeled using capital letters, while each of its sides is labeled using the corresponding lower-case letter of the opposite vertex.

29 Given that $\triangle PQR \sim \triangle MLN$, $m\angle P = 65^\circ$, and $m\angle Q = 75^\circ$, arrange the numbers l, m, and n in order from largest to smallest.

30 The 10 students of a math class were asked to draw a triangle PRO with the following dimensions: $PR = 4$ cm, $RO = 3$ cm, and $m\angle P = 40^\circ$. David affirms that the probability that all the students drew a triangle congruent to his own triangle is 1. Do you agree with David? If you don't agree, explain why and discuss with the members of your group what the probability should be.

31 Two triangles, $\triangle ABC$ and $\triangle DEF$, have the following side lengths and angle measures: $AC = 3$, $AB = 2$, $m\angle A = m\angle F = 60^\circ$, $FE = 6$, and $FD = 4$. In the bigger triangle, another angle is 79.1°. Furthermore, in one of the two triangles the length of one of its sides is 2.65. Determine the missing side lengths and angle measures of the two triangles.

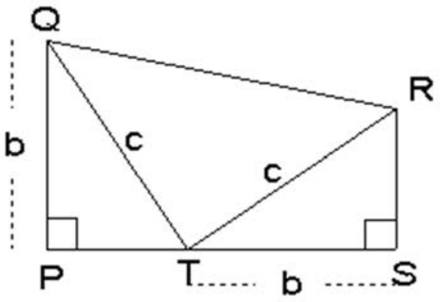
32 Lisa, a bright student in Mrs. Krabappel's class, proposed what she called the "HL Right Triangle Congruence Case." This was her statement.

HL Right Triangle Congruence Case

If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and one leg of another right triangle, then the two triangles are congruent.

Do you agree with Lisa that this is a case of triangle congruence for right triangles? Explain your answer.

33 In the figure below, trapezoid $PQRS$ is formed by two right triangles and an isosceles triangle, with the dimensions indicated in the figure.



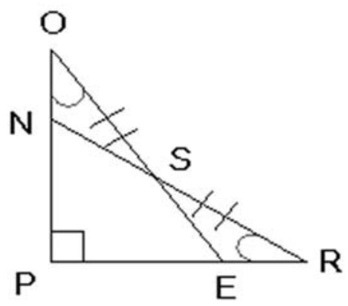
a. How do the two right triangles explicitly shown in this figure compare?

b. Is $\angle QTR$ a right angle? Explain your answer.

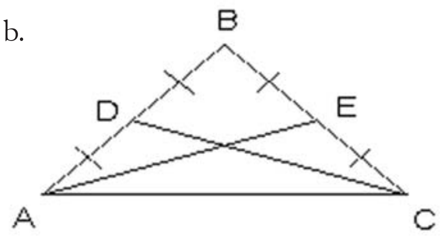
c. Determine the area of this trapezoid in terms of b and c.

34 In each of the first three exercises below, list all pairs of congruent triangles. In the fourth one, answer the question being asked. In each exercise, state explicitly ALL THE REASONS for your answers.

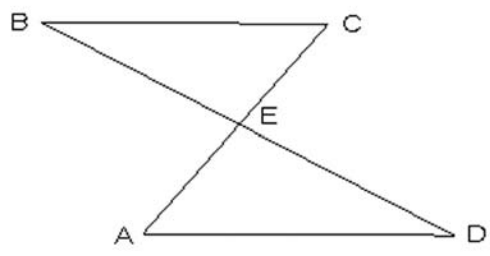
a. $PO = PR$, $m\angle O = m\angle R$, and $OS = SR$.



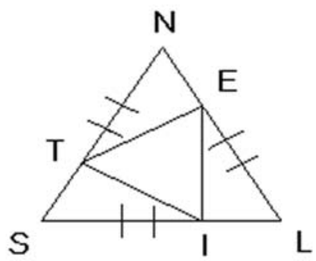
b.



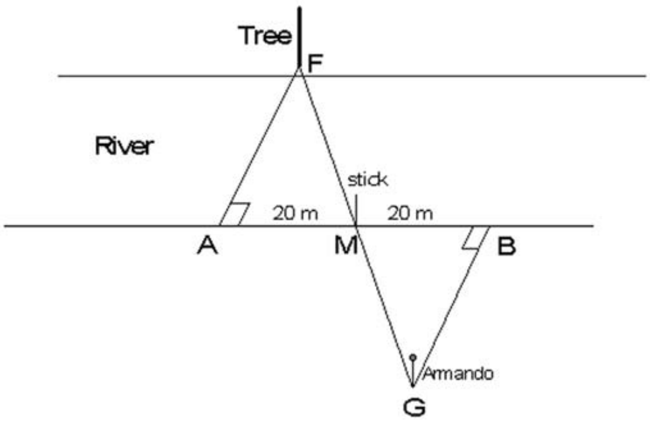
c. If $\overline{BC} \parallel \overline{AD}$ and $\overline{BD}$ and $\overline{AC}$ intersect at E.



d. If $\triangle SLN$ is equilateral, is $\triangle TIE$ equilateral? Why?



35 Armando wants to estimate the width of a river. As indicated in the figure below, standing at point A directly across from a tree located on the other side of the river, Armando walked 20 meters to M, placed a stick at M, and then walked 20 meters to B. Armando then turned 90° and walked to point G, where Armando, the stick, and the tree were lined up. Here is the diagram of Armando's walk drawn in perspective:

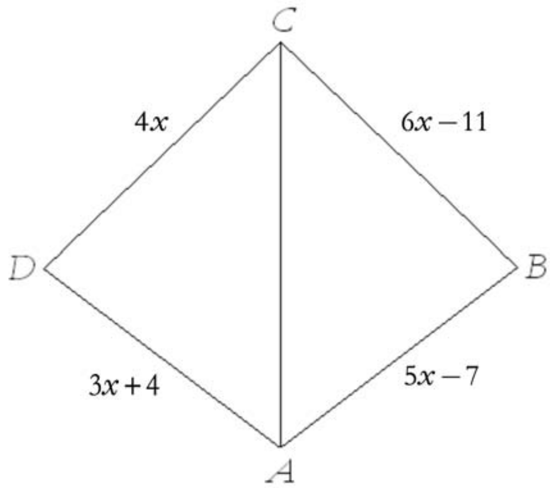


a. What line segments have lengths equal to the width of the river? Thoroughly explain your answer.

b. Is this a method that can really be used to estimate the width of the river? Explain.

c. How relevant to the final result is the 20 meter-long segment? That is, if Armando walked from A to M a distance other than 20 m, would it change the final estimation of the width of the river? Explain.

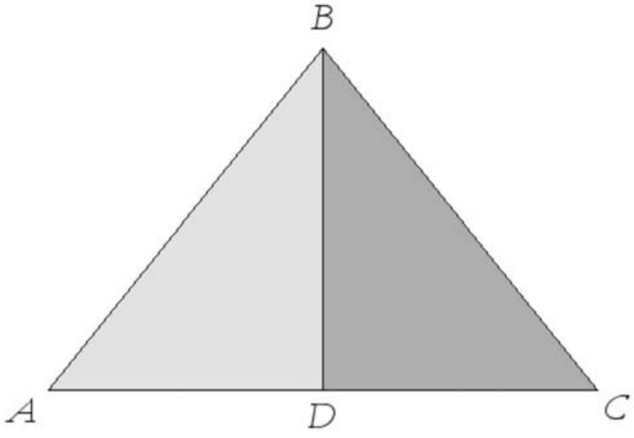
36 The perimeter of quadrilateral $ABCD$ below is 85.



a. Find the value of x.

b. Is $\triangle ADC \cong \triangle ABC$? Explain.

37 In the figure below, $m\angle ABD = (3x-9)^\circ$, $m\angle CBD = (2x+7)^\circ$, $AB = \frac{x}{2}+5$, $BC = x-3$, and $\overline{BD}$ bisects $\angle ABC$ (that is, $\overline{BD}$ divides $\angle ABC$ into two angles of equal measure). However, don't assume that $\overline{BD}$ is perpendicular to $\overline{AC}$, just because it appears so.



a. Find the magnitudes of $\angle ABD$ and $\angle CBD$, and the lengths of AB and BC.

b. Triangle $\triangle ABD$ is to be covered with yellow paper and triangle $\triangle BDC$ with blue paper. How do you compare the amount of paper needed to cover each one of these triangles? Thoroughly explain your answer.

c. If the side lengths of these triangles are in centimeters, and the yellow paper is \$0.50 per square centimeter and the blue paper is \$0.45 per square centimeter, how much money must be spent to cover these triangles with yellow and blue paper as indicated above?

38 **Isosceles Triangle Theorem**

Prove the following statements.

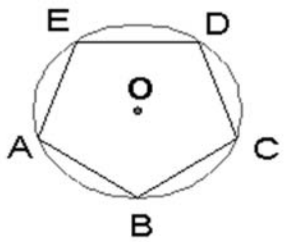
a. If $\triangle ABC$ is a triangle and $AB = BC$, then $m\angle A = m\angle C$.

Hint: Try breaking the triangle into two triangles that you can compare.

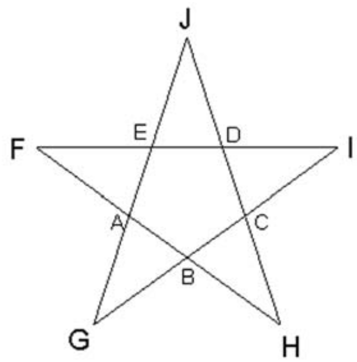
b. If $\triangle ABC$ is a triangle and $m\angle A = m\angle C$, then $AB = BC$.

39 In the figure below, regular pentagon *ABCDE* is inscribed in a circle centered at O.

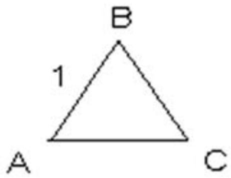
a. Find the measurement of $\angle A$, $\angle B$, $\angle C$, $\angle D$, and $\angle E$. Explain.



b. Each side of the pentagon has been extended to form a star. Find the sum, $m\angle F + m\angle G + m\angle H + m\angle I + m\angle J$. Explain.

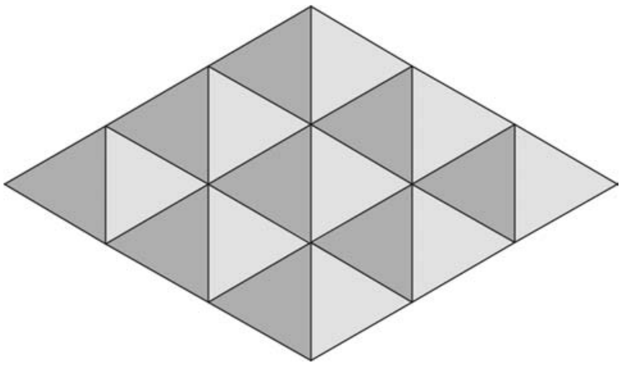


40 The equilateral triangle below has a side length of 1 unit.



a. Determine accurately, not approximately, the area of this triangle.

b. A **tessellation** or **tiling** is an arrangement of closed shapes that completely covers the plane without overlapping and without leaving gaps. A tessellation is called **pure** if it uses only one shape. The honeycomb of the bee is a pure tessellation using regular hexagons. Below, there is a tessellation of a rhombus-shaped region formed using the equilateral triangle in Part a.



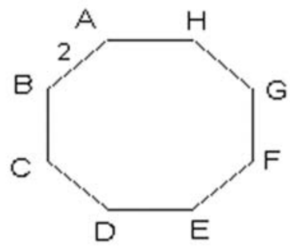
Complete this table for tessellations of the rhombus-shaped regions whose side lengths are indicated, formed using the equilateral triangles (tiles) in Part a.

Side length	# of tiles	Area of the region (in units)
2		
3		
4		
5		
n		

c. If the area of one of these rhombus-shaped regions is $32\sqrt{3}$, how many tiles would be needed to tessellate this region?

41 In the figure below, regular octagon is *ABCDEFGH* has a side length of 2 units. Determine its area.

Hint: You may want to consider this octagon as one inscribed in a circle centered at the center of the octagon. The use of trigonometric ratios may be appropriate as well.



42 Don't use a calculator for this problem.

a. Evaluate $\dfrac{7}{5} \div \dfrac{1}{10}$

b. Evaluate $\dfrac{a^5b^2}{c^2d^3} \div \dfrac{a^3b^3}{c^7d}$

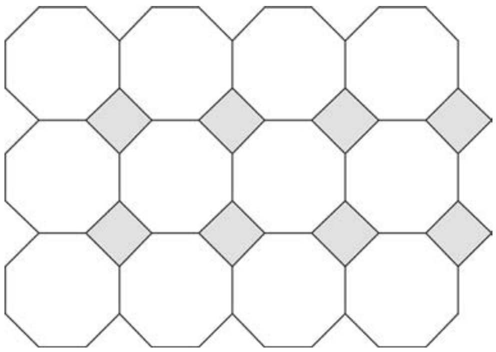
c. Simplify $\sqrt{\dfrac{27}{16}}$

d. Find an equation of the line that passes through the points $\left(5, \dfrac{26}{3}\right)$ and $(9,10)$.

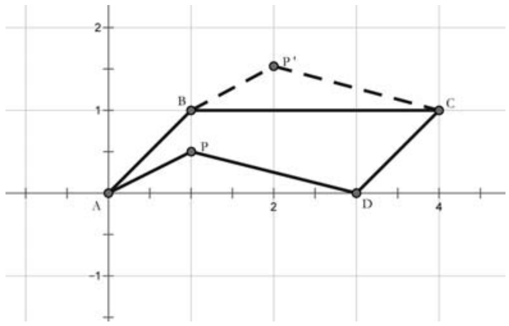
e. Solve the system of equations

$$10x + 12y = 20$$
$$5x - 3y = 8$$

43 Tessellations formed with more than one type of shape are called semi-pure tessellations. In the **semi-pure tessellation** below, formed using regular octagons and squares, the area of the shaded region is 128 square units. Determine the total area of this tessellation.

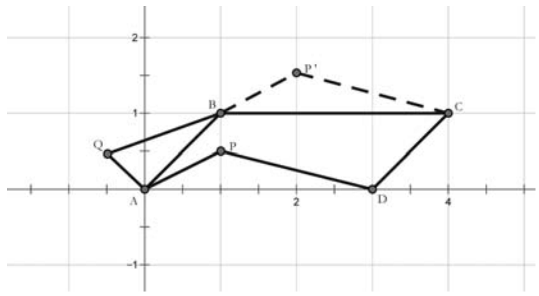


44 Point $P(1,0.5)$ is inside the parallelogram *ABCD* shown below. The two-straight-line-segment curve *APD* is translated up along the segment $\overline{AB}$. The image of this translation is the dashed curve *BP'C*, as shown.

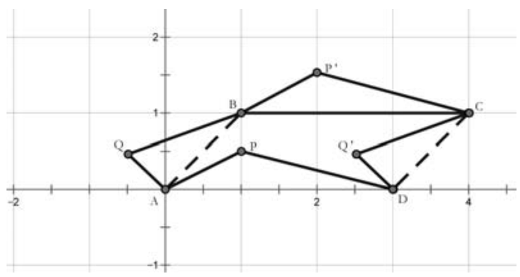


a. Find the area of $\triangle BP'C$.

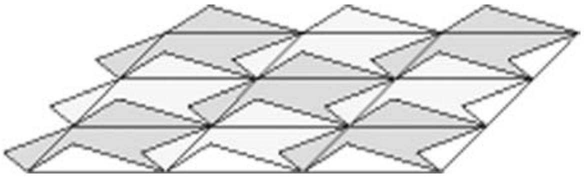
b. Point $Q(-0.5,0.5)$ is outside the parallelogram *ABCD*. Translate the two-straight-line-segment curve *AQB* 3 units horizontally. Call *Q'* the image of point *Q* by this translation.



c. Curves *APD* and *AQB* along with their translated images are shown below. Calculate the area of the fish-shaped polygon *APDQ'CP'BQ* thus formed.



d. When on a 3 by 3 parallelogram grid the fish-shape above is translated horizontally 3 units and vertically along the direction of the segment AB each time, we obtain the following translation tessellation. Consecutives fish-shapes have been colored with different colors.



If the tessellation extended out onto a 10 by 10 parallelogram grid, what would be the area of the darker region of the tessellation?

e. What would be the probability of selecting a darker fish in this tessellation if it were extended onto an 11 by 11 parallelogram grid?

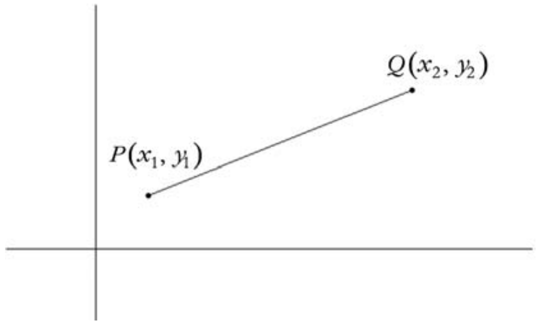
You may have already conjectured how to determine the coordinates of the midpoint of a segment, given the coordinates of its endpoints. However, with your current knowledge on congruent triangles, you may be better prepared to prove this conjecture, if you have not already done so.

45 Midpoint of a Segment

a. Determine the coordinates of the midpoint, A, of $\overline{CD}$, where $C(2,1)$ and $D(8,3)$. Explain with a mathematical argument, as opposed as a simple conjecture, how you can get to the answer to this question.

b. Determine the coordinates of the midpoint, E, of $\overline{FG}$, where $F(2,-1)$ and $G(-6,5)$. Explain how you got to your answer.

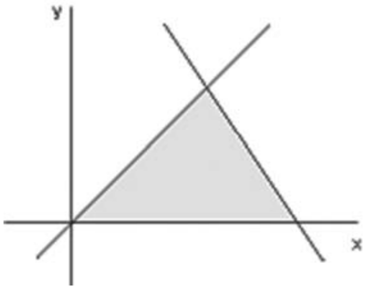
c. Explain with a mathematical argument, as opposed to a simple conjecture, how you can determine the coordinates of the midpoint, M, of $\overline{PQ}$, where $P(x_1,y_1)$ and $Q(x_2,y_2)$. You may want to use a diagram like the one shown below.



d. Would the result you may have found in Part c change if P or Q were located anywhere else in the plane? Explain.

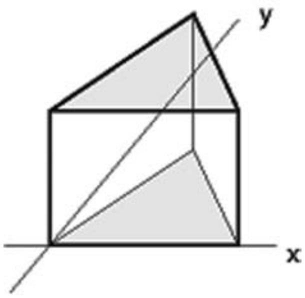
Exploring in Depth

46 The vertices of a triangle are determined by the intersections of the x-axis, the line $y=x$, and the line $3x+2y=15$ as shown in the figure.



a. Determine the area of this triangle.

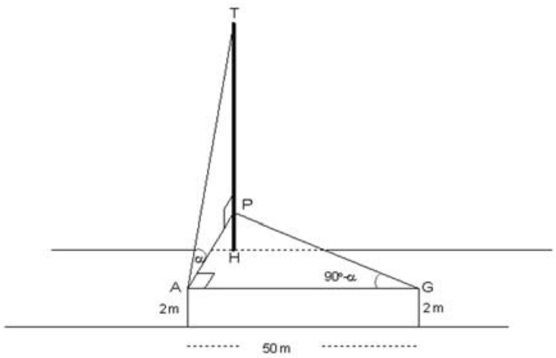
b. The triangle in Part a is used as the base of a fish tank, which has the shape of a prism whose sides are rectangles perpendicular to the base.



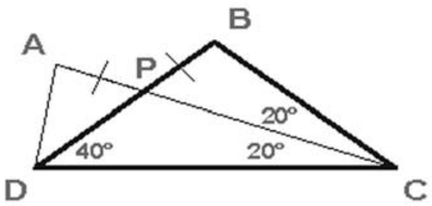
The height of the prism—the distance between its bases, the top and bottom faces—is 3 units. If this fish tank is to be made of glass, determine the total area of the piece of glass needed to make this fish tank, assuming its thickness is small enough as to be ignored.

c. What is the amount of water needed to completely fill this fish tank?

47 In the vertical plane of the tridimensional figure below, $m\angle TPA = 90^\circ$ and $m\angle TAP = \alpha$. In the horizontal plane, $m\angle PAG = 90^\circ$, $m\angle AGP = 90^\circ - \alpha$, and this horizontal plane is 2 meters above the ground. As shown, and points A and G are 50 meters apart. If it takes a black ant 17 minutes to go from point H to point P and then to point A, moving at a constant speed of 5 cm per second, how long would it take the ant, moving at the same constant speed, to go from point A to the top of the pole $\overline{HT}$ along the wire $\overline{AT}$?



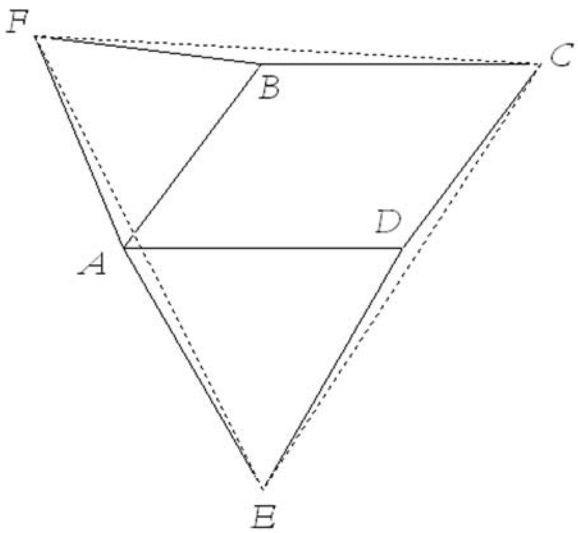
48 The following was Question 2-6 on the November 20, 2007 Maryland Mathematics League Contest.



The degree-measure of each base angle of an isosceles triangle $\triangle DBC$, as indicated in the figure, is 40°. $\overline{AC}$ bisects $\angle C$. Point P lies on $\overline{BD}$ so that $PA = PB$. What is the measurement of $\angle A$?

Hint: Reflect triangle $\triangle PBC$ over $\overline{PC}$.

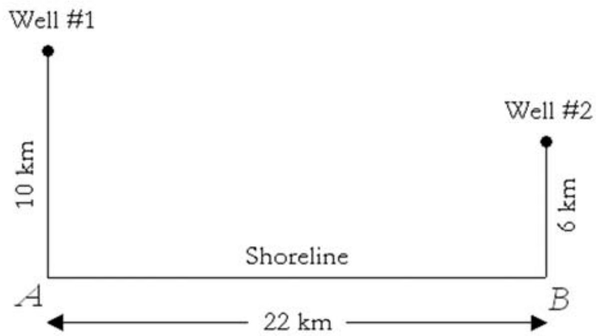
49 In the following figure, $ABCD$ is a parallelogram with equilateral triangles $\triangle ABF$ and $\triangle ADE$ drawn on sides $\overline{AB}$ and $\overline{AD}$ respectively.



Prove that $\triangle FCE$ is equilateral.

You may have seen the following problem earlier. However, this problem can be formally solved using your current knowledge of congruent triangles.

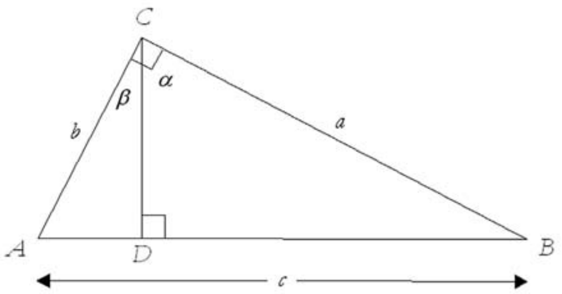
50 Ecopetrol, an oil company, has drilled two high-capacity wells in the Gulf of Maracaibo 10 km and 6 km from the shore, as shown in the diagram below. The 22 km of shoreline is nearly straight, and the company wants to build a refinery on shore between the two wells. Since pipe and labor cost money, the company wants to find the location that will serve both wells and uses the least amount of pipe when it is laid in lines from each well to the refinery.



a. Do you think the refinery should be closer to A, to B, or at the midpoint? Explain your reasoning.

b. What is your best estimate for the location of the refinery? How do you decide on the location?

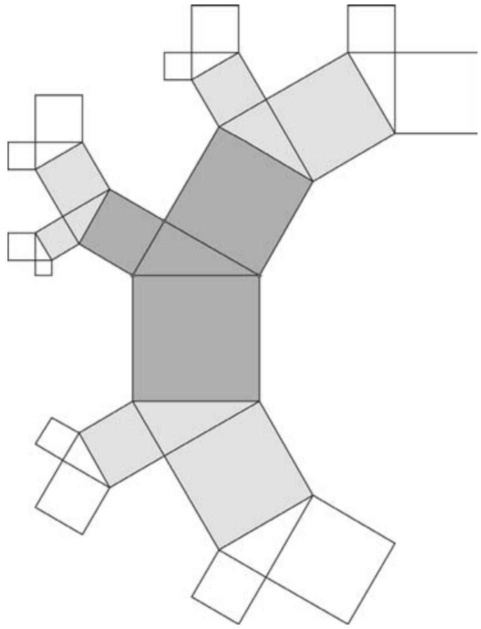
51 In the following figure, $\triangle ABC$ is a right triangle and $\overline{CD}$ is the altitude to the hypotenuse.



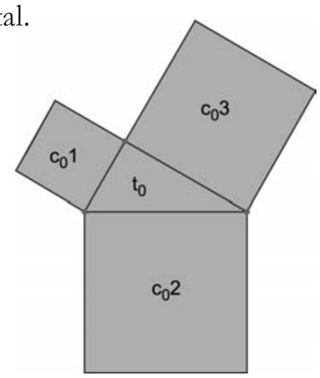
a. Find another pair of angles equal in measure to alpha (α) and beta (β).

b. Show that $\overline{CD}$ divides right triangle $\triangle ABC$ into two triangles, each of which is similar to $\triangle ABC$.

The features of a right triangle described in the previous problem along with those described by the Pythagorean theorem can be used to form a **fractal**—an irregular or fragmented geometric shape that can be repeatedly subdivided into parts, each of which is a smaller copy of the whole. The fractal below starts with a right triangle with squares on each side. Then triangles similar to the first one are built onto the squares. Then squares are built onto the new triangles, and so on.

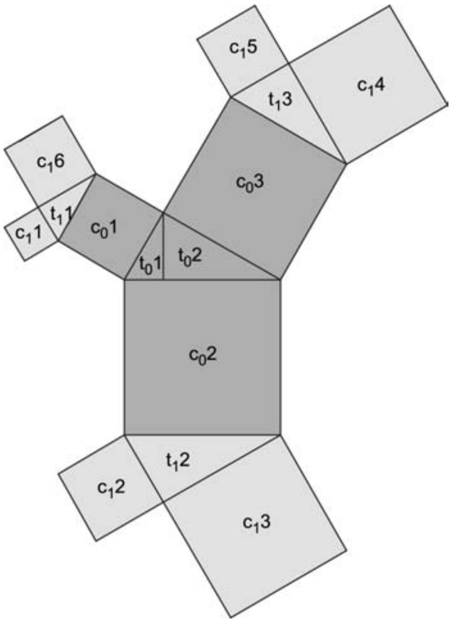


52 In the piece of the fractal above, consider your original sketch to be a single right triangle, t_0, with squares $c_0 1$, $c_0 2$, and $c_0 3$ built on each side, as indicated in the figure below. Call this sketch Stage 0 of your fractal.



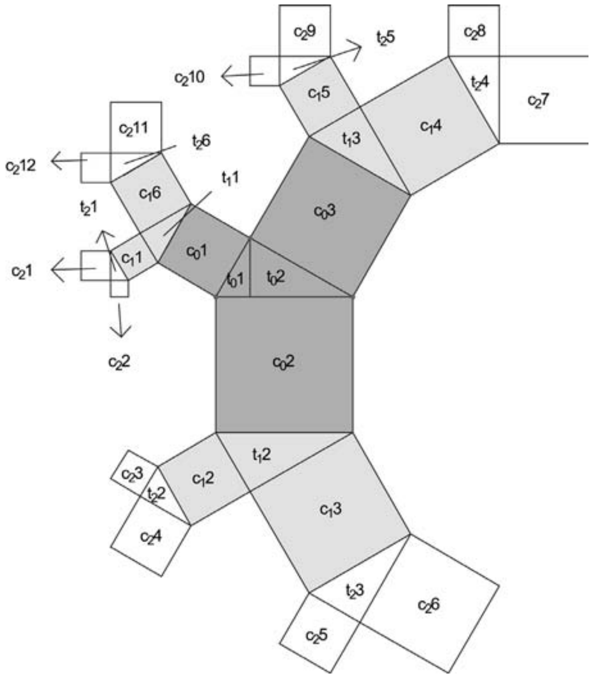
a. How are the areas of the squares $c_0 1$, $c_0 2$, and $c_0 3$ related? Explain.

b. At Stage 1, you add three triangles, $t_1 1$, $t_1 2$, and $t_1 3$ and six squares, $c_1 1$, $c_1 2$, $c_1 3$, $c_1 4$, $c_1 5$, and $c_1 6$ to your construction. The original right triangle at Stage 0, t_0, is divided into two triangles, $t_0 1$ and $t_0 2$, by the altitude to the hypotenuse. How are these triangles $t_0 1$ and $t_0 2$ related to the triangles $t_1 1$, $t_1 2$, and $t_1 3$ added at Stage 1? Explain.



c. How much area do you add to this fractal between Stage 0 and Stage 1? You may assume that the initial triangle, t_0, has dimensions 3, 4, and 5. (In reality, you don't need to measure any areas to answer this question.)

d. At Stage 2, you add six triangles, $t_2 1$, $t_2 2$, $t_2 3$, $t_2 4$, $t_2 5$, and $t_2 6$, and twelve squares, $c_2 1$, $c_2 2$, …, and $c_2 12$, to your construction. How are these triangles added at Stage 2 related to the triangles $t_1 1$, $t_1 2$, and $t_1 3$ added at Stage 1? Explain your answer.



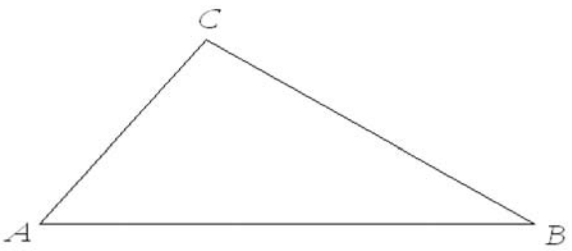
e. How much area do you add to this fractal between Stage 1 and Stage 2? (Again, you can answer this question without actually measuring any areas.)

f. A true fractal exists only after an infinite number of stages. How much area is added at any new stage?

In the following problem you will have the opportunity to tinker and conjecture about the best spot to build an airport which can serve three neighboring cities.

53 *The Airport Conjecture.*

Three cities are located at the vertices of $\triangle ABC$. The distance from city A to city B is 90 miles, from city B to city C 80 miles, and from city C to city A 60 miles. The figure below has been drawn to scale.



The city council members of these cities have decided to build together an airport which can be used by the three cities on a regular basis. Tinker and discuss with the members of your group about the best place for the airport to be built.

54 In Figure 1 below, a point D is randomly chosen in the interior of $\triangle ABC$. The distances from D to vertices A, B, and C are respectively x, y, and z as indicated in Figure 1.

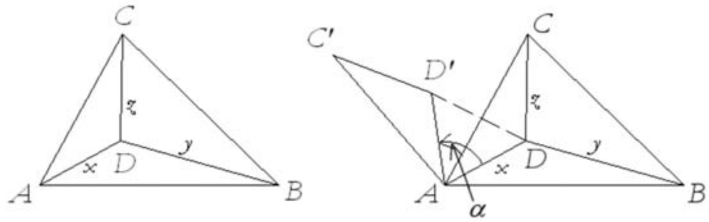


Figure 1 Figure 2

In Figure 2, $\triangle ADC$ has been rotated by an angle α about vertex A. The rotated $\triangle AD'C'$ is shown. D' and C' are the images of D and C respectively under this rotation.

a. In terms of x, y, or z, express the lengths of $\overline{AD'}$ and $\overline{C'D'}$. Explain.

b. How are $\angle AD'D$ and $\angle ADD'$ related? That is, which one is bigger, if either? Explain your answer.

c. If $\alpha = 60^\circ$, express in terms of x, y, or z, the sum $C'D' + D'D + DB$.

d. Would your answers to any of the previous questions be different if point D were located anywhere else in the interior of $\triangle ABC$? Explain.

55 Both in Figure 3 and Figure 4 below, we have the same triangle of the previous problem, but point D has been chosen to be at a different place in the interior of this triangle. Furthermore, $\triangle ADC$ has been rotated 60° about vertex A.

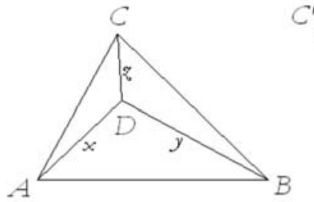


Figure 3

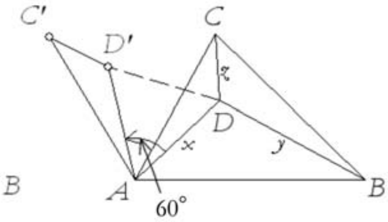


Figure 4

How should the position of points C', D', D, and B be related in order for $C'D' + D'D + DB$ to be as small as possible, while the rotation of $\triangle ADC$ about A remains 60°? Explain.

56 The figure below shows another selection of point D. As indicated in the figure, $\triangle ADC$ has been rotated an angle $\alpha = 60^\circ$ about vertex A, and points C', D', D, and B lie on a straight line.

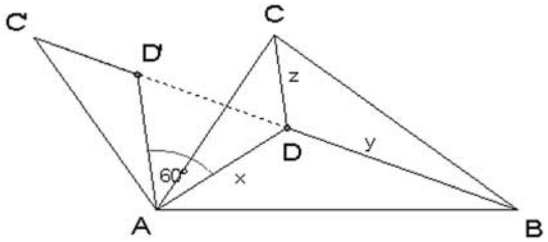


Figure 5

a. Determine $m\angle ADD'$ and $m\angle ADB$.

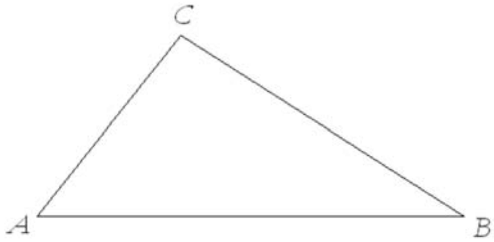
b. Express each of the distances $C'D'$, $D'D$, and DB in terms of x, y, or z.

c. Make a conjecture about the magnitude of $\angle ADB$ that would minimize $x + y + z$. Explain your answer.

d. Is your answer to Part c just a conjecture, or can you actually prove it?

57 *The Airport Conjecture* Revisited.

Three cities are located at the vertices of $\triangle ABC$. The distance from city A to city B is 90 miles, from city B to city C 80 miles, and from city C to city A 60 miles. The figure below has been drawn to scale.



The city council members of these cities have decided to build together an airport which can be used by the three cities on a regular basis.

Are you now better prepared to make a conjecture about the best place for the airport to be built? If so, make and explain your conjecture.

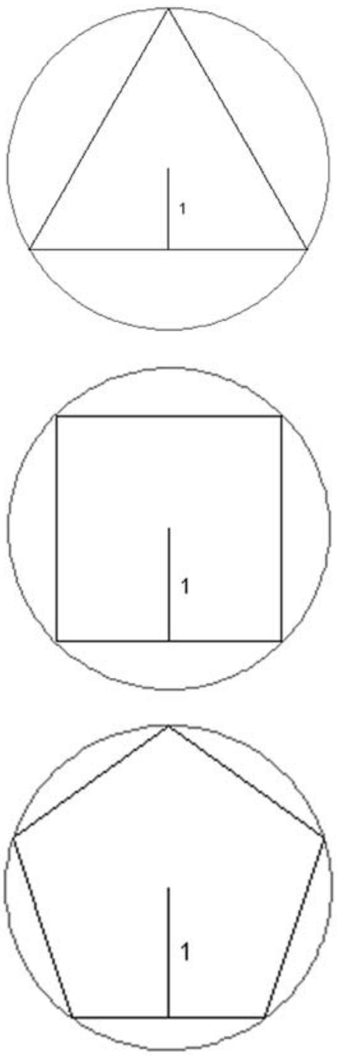
58 Can any triangle be divided into four congruent triangles by joining the midpoints of its sides? Explain your answer.

59 Can any triangle be divided into as many congruent triangles as a positive power of four, 4^n, where $n = 1, 2, 3, ...$ Thoroughly explain your answer.

60 In the graphs below, for each polygon the length of the segment drawn from the center of the circle to one of its sides is one.

a. Fill out the following table about the area and perimeter of regular polygons of n sides.

n	3	4	5	6	n
Area					
Perimeter					



b. Is the area of the polygons approaching a number as the number of sides, n, gets bigger and bigger? If so, what is that number?

c. Is the perimeter of the polygons approaching a number as the number of sides, n, gets bigger and bigger? If so, what is that number?

d. What happens to the shape of the polygons as the number of sides, n, gets bigger and bigger?

e. What would your answer to Parts a, b, and c be if the distance from the center of the circumscribed circle to one of the sides of the polygon were any positive number r?

LESSON 3: GEOMETRIC PROOF

Introduction

C'est Mathematique! is a French expression meaning that something is so certain, it is definite. Indeed, the 19th-Century French mathematician Pierre-Simon Laplace imagined a demon that could predict every future event because it knew the exact position of every particle in the universe, and the mathematical laws that governed the motion of those particles.

Though no such demon exists, we can see why Laplace might have imagined one. Our mathematical knowledge does seem more certain than some other facts we believe to be true about politics or human nature. For instance, maybe you think that passing laws to ban handguns will reduce homicides. Then you read a study that says that actually these bans don't have that effect. You might change your mind about whether banning handguns is a good idea… until you read another study that says that the bans *do* reduce accidental death among children. You'll always have to remain open to changing your opinion based on new information.

Likewise, in the history of science, people once thought that the atom was the smallest particle that existed (the word "atom" comes from a Greek word meaning "indivisible"). Now we know that atoms are made up of protons and neutrons, and in recent decades scientists have discovered quarks. In the scientific community, people are constantly revising and improving accepted theories.

In mathematics, on the other hand, we still accept the results of ancient texts. Archimedes, Pythagoras, and Euclid are credited with many of the results that we still use today. You have probably already **proven** things mathematically in such a way that you're sure they must be true. In this lesson we'll look at what it means for a proof to be absolutely airtight — with no possibility you'll want to revise your opinion later.

Development

Let's think about the following two statements:

The diagonals of a parallelogram bisect each other.
The diagonals of a parallelogram bisect the angles from which they're drawn.

1 Make some sketches to determine whether you think each of these statements is true or false. Recall that "bisect" means "cut exactly in half."

When we do proofs in this section, we're going to focus on making them airtight. We'll use the following standard. Any time we make a statement as part of a proof, we need to be able to give a reason for that statement. That reason, in turn, needs to be a statement about geometry that we *already* know to be true.

Here are some statements about geometry you already know to be true:

In an isosceles triangle, the angles opposite the congruent sides are congruent.

When parallel lines are cut by a transversal, same-side interior angles add up to 180 degrees.

Vertical angles are equal.

If two sides and the angle between them in one triangle are congruent to two sides and the angle between them in another triangle, then the triangles are congruent. (SAS congruence theorem)

If the three sides of one triangle are all related to the sides of another triangle by the same proportion, then the triangles are similar. (SSS similarity theorem)

In a parallelogram, opposite sides are congruent.

2 Make a list of statements about geometry that you know to be true, including the six statements above. To get ideas, try paging through your notes from the past few weeks, or through the lessons earlier in this textbook. Which results seem worth remembering? Also think about some basic shapes you know, like quadrilaterals, and write down some facts you know to be true about each one.

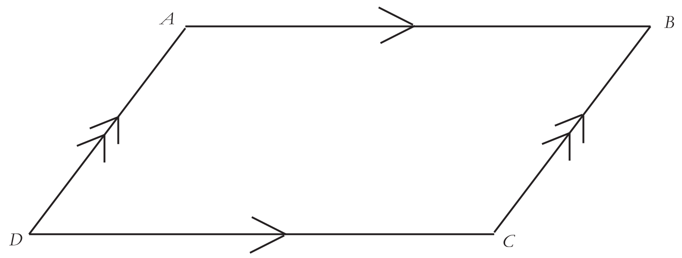
As you may recall from the first lesson of this chapter, the Greek mathematician Euclid did something very similar to what you are about to do. He boiled down his geometric knowledge to a few assumptions and then proved the rest of what he knew about geometry using those few things. He called the facts he assumed without proof "postulates." Any fact that he had proved was a "theorem."

3 Consolidate the list you made in the previous problem with the rest of the class. Discuss which items on it are postulates and which are theorems. Now separate a blank page of your notebook into two columns. Label one column "postulates" and the other "theorems." Write the items in the class list in the appropriate column. Leave yourself lots of room in both columns to add to the list, probably over multiple pages.

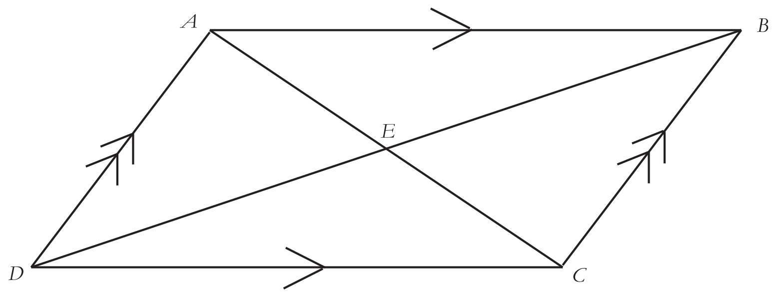
Hang on to this list. You'll be referring to it for the remainder of this chapter.

Now let's return to the problem of proving the statements from the beginning of this lesson. If you did your job carefully, you'll have found that the first statement was true — the diagonals of a parallelogram bisect each other.

Before writing our airtight proof of this fact, it's important to know where to start. Since we're proving something about a parallelogram, it's all right to assume that we *have* a parallelogram. So let's draw one…



… and label the vertices so that we can talk about them later. So, specifically, $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$. Those are two things we know for sure. It's also important to be sure of what we want to show about this diagram. Let's draw in some diagonals and call their intersection point E.



Then what we want to show is that $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$.

The comic book hero G.I. Joe used to say "knowing is half the battle." In the case of proof, knowing what information you have to start with and where you want to end up with is half the battle. Really!

What we need now is a chain of reasoning, beginning with the facts that $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$ and ending with the statement that $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$. In this chain of reasoning, every step needs to be justified with some reason we *know for sure*.

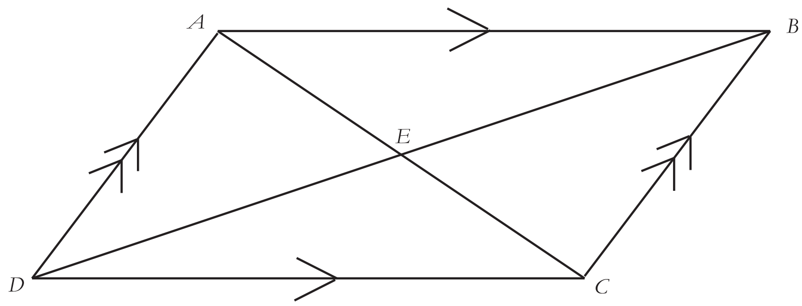
4 Try this yourself before reading the proof on the next page.

When thinking about what will help us to write this proof, let's think backwards a little bit. We want to show that certain lengths are the same. Look back at your list of postulates. What tools do you have for showing that lengths are the same? Not the results about parallel lines. Not vertical angles or linear pairs. Possibly some statements you have about parallelograms or rhombuses, but in this case the lengths we're interested in are diagonals rather than sides of those shapes. That really only leaves one thing that could help — congruent triangles.

If we can show that certain triangles are congruent, then we can conclude that the lengths making them up are also congruent. That should be our strategy in writing this proof.

5 Find some congruent triangles in this diagram that you think would help. Do you know for sure they're congruent?

Depending on the way you answered question #5, you may have an airtight proof already. Let's write one out as an example of a chain of reasoning in which all the steps are justified.



Given: $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$.

Goal: PROVE that $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$.

$\angle DAE \cong \angle BCE$ because they are alternate interior angles of the parallel lines $\overline{AD}$ and $\overline{BC}$.

$\angle ADE \cong \angle CBE$ for the same reason.

$\overline{AD} \cong \overline{BC}$ because opposite sides of a parallelogram are congruent.

Therefore, $\triangle AED \cong \triangle CEB$ by the ASA congruence theorem.

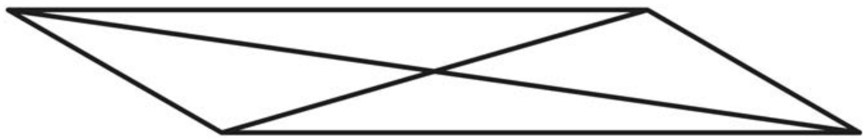
$\overline{AE} \cong \overline{EC}$ because they are corresponding parts of congruent triangles.

$\overline{BE} \cong \overline{ED}$ for the same reason.

Are we done? YES! We've accomplished our goal, showing that the diagonals cut each other in half.

That proof might have been written with a little more care than you are used to. Your class will have to decide exactly how much writing is expected in a proof. But notice that every step in the argument comes from your list of postulates, so that anyone who accepts the postulates can't disagree with your conclusion — there is no "wiggle room" in this proof.

Meanwhile, you should have found that it is *not* true that the diagonals of a parallelogram bisect the angles. You may remember from last year that an example designed to show that something isn't true is called a **counterexample**. Here's what a counterexample to this statement might look like:



You can see from this picture that it doesn't look true that the angles are bisected, and so it isn't worth your time to look for a proof.

In order to start the proof that the diagonals of a parallelogram bisect each other, we had to know exactly what the word "parallelogram" meant. That's how we knew to start with the statements "$\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$". Whenever we write a proof, we'll have to be very precise about what we mean by the objects we're talking about. For instance, last year you worked with the definition of a square: a quadrilateral with four congruent sides and four right angles.

6 Come up with a definition for each shape. Then compare notes with your class if you haven't already done so.

a. A rectangle

b. A parallelogram

c. A rhombus

d. A trapezoid

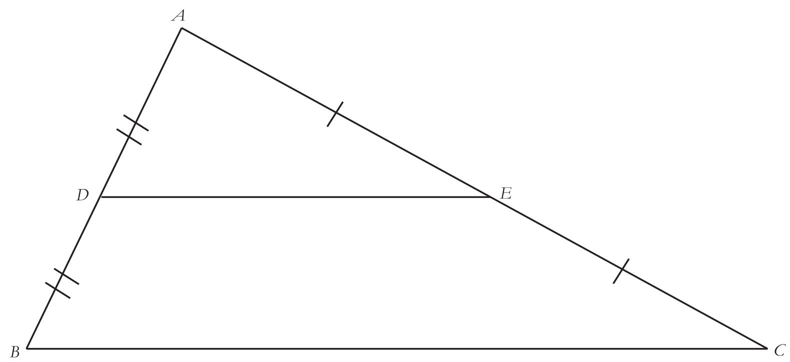
e. A kite

f. A circle

7 Now begin a section in your notebook labeled "definitions," again leaving space for you to add definitions as you consider new shapes. Start with the definitions you came up with in Problem 6.

8 Considering the shapes in the previous question might remind you of some *more* things you know to be true about geometry. Add them to your list of postulates and theorems (which should they be?), and confer again with your classmates.

Much as you can use congruent triangles to prove things about diagrams, you can also use similar triangles.



9 The diagram has been marked and labeled for you.

a. Based on the diagram, write what is given.

b. Can you remember the two things you showed about $\overline{DE}$ and $\overline{BC}$ last year? If not, try drawing this diagram with triangles of a few different shapes, and take some measurements in order to make a conjecture.

c. With an eye towards proving these results, prove that there are similar triangles in this picture (The three similarity theorems are AA, SSS, and SAS). What is the scale factor?

d. Finish proving what you wrote in part b about $\overline{BC}$ and $\overline{DE}$.

You've proven a couple of new things so far in this lesson. The interesting thing is, not only do you know those things, but you have proofs of them. Write the new things you know under the column labeled "theorems" on your list. Whenever you prove something new that you think may come in handy later, or that you just think is interesting or important, add it to this theorem list.

By the way, is there anything under "postulates" that you could prove and make a theorem?

Practice

10 In this problem, you will prove that the diagonals of a rectangle are congruent.

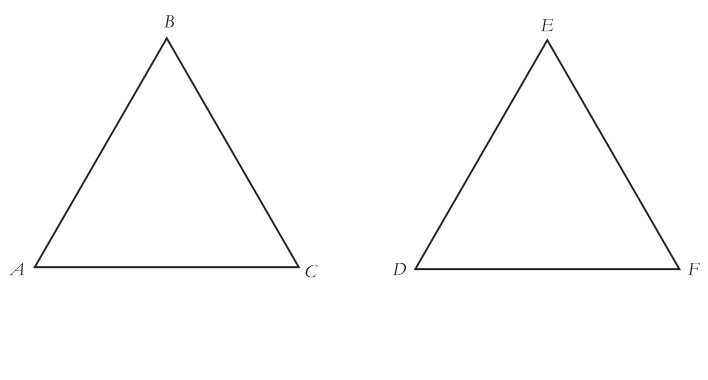
a. Draw a rectangle that can stand for any rectangle and label the vertices.

b. In terms of your diagram, write what is given and what is to be proved.

c. Finish the proof.

In the following five problems, you should mark up the diagram with the starting information before you attempt a proof.

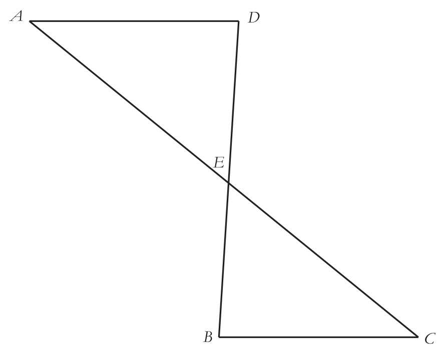
11 Given: $\overline{AB} \cong \overline{EF}$, $\angle A \cong \angle E$, $\angle C \cong \angle D$
Prove: $\overline{BC} \cong \overline{DF}$



12 Given: $\overline{AE} \cong \overline{CE}$, $\overline{BE} \cong \overline{ED}$

Prove: $\overline{AD} \parallel \overline{BC}$

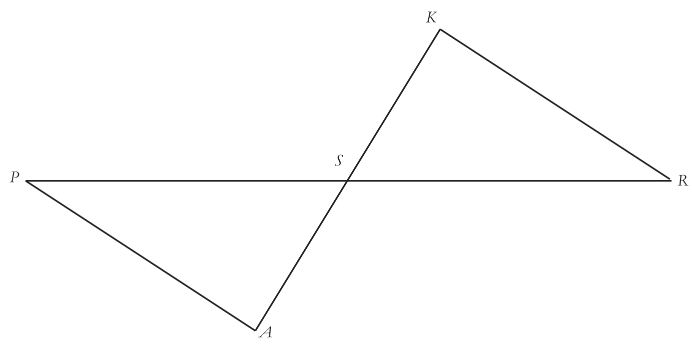
(In this problem and in the remaining problems in this lesson, assume that lines that appear to be straight in a figure really are straight.)



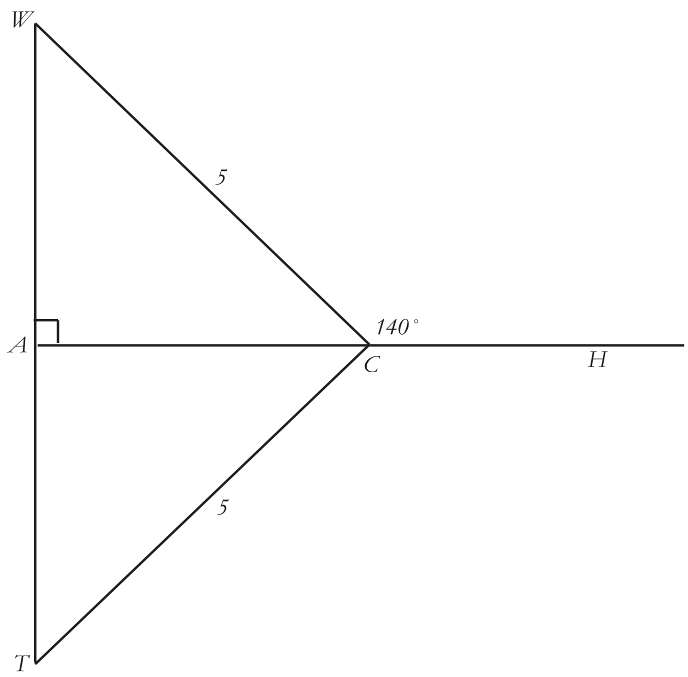
13 There is a joke about a mathematician asked to describe an algorithm for boiling an egg. She responds, "you take a pot, fill it with water, put in an egg, put the pot on the burner, turn the burner on, and wait 20 minutes." "All right," you say, but what if the pot is already sitting on the burner, filled with water?" "Then," she says, "you take the pot off the burner and empty the water, and then you've reduced the problem to the previous case."

a. Suppose, in the diagram below, that $\overline{PA} \cong \overline{RK}$, $\angle A \cong \angle K$
Prove: The length of $\overline{KA}$ is twice the length of $\overline{KS}$.

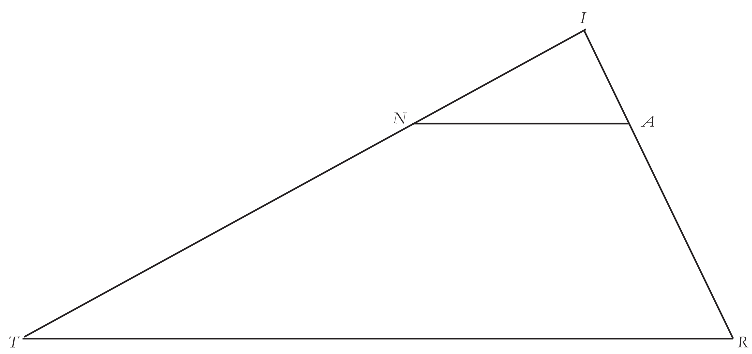
b. Now suppose instead that $\overline{PA} \cong \overline{RK}$, $\overline{PA} \parallel \overline{KR}$
Prove: The length of $\overline{KA}$ is twice the length of $\overline{KS}$.



14 Given: the angles and sides as marked; W, A, and T are points on a line segment.
Prove: $\angle T = 50^\circ$



15 Given: $TI = 3NI$, $RI = 3AI$
Prove: $\overline{TR} \parallel \overline{NA}$ and $TR = 3NA$



16 Which of these statements appear true, and which are false? If they are true, you don't need to prove them, but if you think they are false you should provide a counterexample.

a. If one isosceles triangle has side lengths of 5 cm and 5 cm, and so does another, then those triangles are congruent.

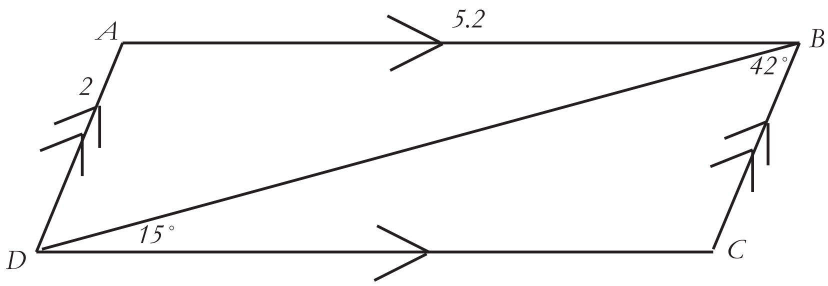
b. In an obtuse triangle, label the longest side c and the other two sides a and b. Then $a^2 + b^2 < c^2$.

c. The three medians of a triangle always cross in a single point.

d. If someone tells you three positive integer lengths, you will always be able to draw a triangle that has those lengths.

For each figure, determine as many missing sides and angles as you can. (You don't need to write a formal proof of your answers.) You will not be able to find *all* of the sides and angles. Avoid using trigonometry.

17



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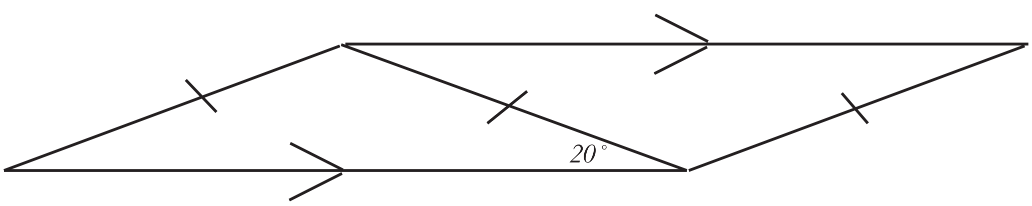
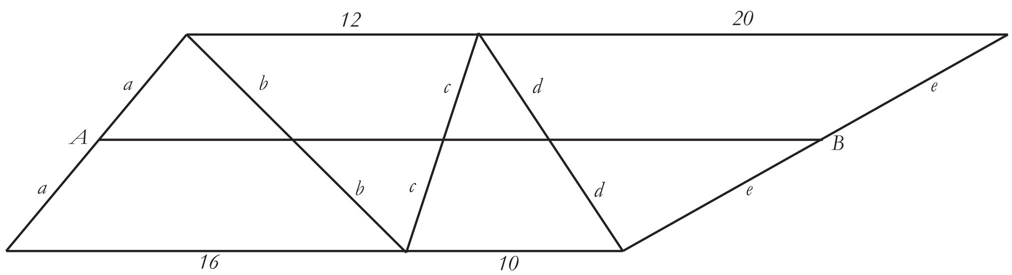
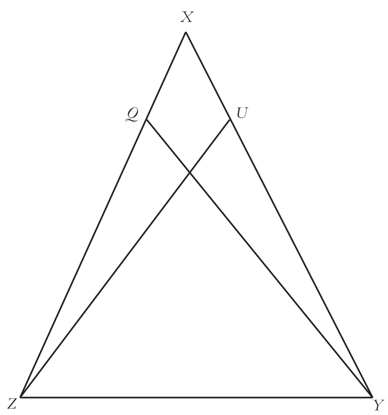


19 Find the length of the line segment $\overline{AB}$.



Going Further: Longer Proofs; Pitfalls When Writing Proofs

Sometimes the only way you can see that a statement must be true is to talk yourself through a series of steps in proof-like form. It would be hard to come to know the statement to be proven below, in any other way. In addition, the statement's proof deals with some complications that you will sometimes come across.



Given: $\overline{XQ} \cong \overline{XU}$ and $\angle XQY \cong \angle XUZ$.
Prove: $\angle UZY \cong \angle QYZ$.

"Working backwards and forwards" will be very important in the planning stages of this proof. At first glance, it seems that we should try to prove triangles *QYZ* and *UZY* congruent, since those are the triangles that contain the two angles we want to prove congruent.

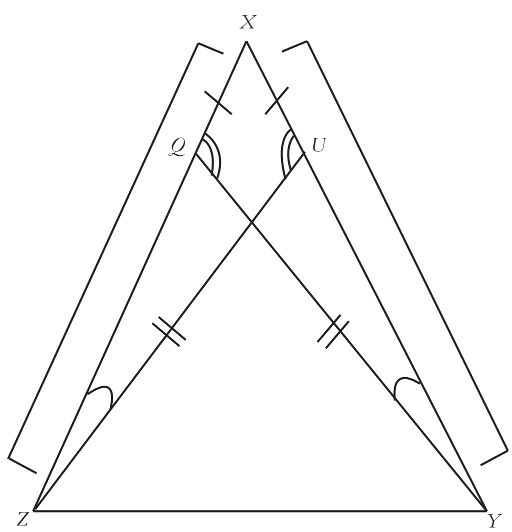
20 Find as much evidence as you can that $\triangle QYZ \cong \triangle UZY$.

In the previous problem you should have found a (shared) side and an angle, which by themselves are not enough to prove the triangles congruent. So we need to find another strategy. It may be that, using some other pair of congruent triangles in the figure, we can find that third piece of information needed to prove $\triangle QYZ \cong \triangle UZY$. Or, perhaps the solution is something completely different. Instead of doing all of our planning backwards, let's return to thinking forwards and just look for any pair of triangles that might be congruent.

21 Name some other pairs of triangles that might be congruent in the figure. Try to find enough evidence for their congruence.

The triangles that we can actually prove congruent may have been hard to spot. As a hint, they share angle X. If you haven't already, prove that these two triangles are congruent.

Now that we have congruent triangles, we can mark all of their sides and angles congruent, as follows:



22 Now we can return to the question of how to prove $\angle UZY \cong \angle QYZ$. Using the information we've derived, argue that these angles are congruent.

One way to complete the proof is to notice that, since we now know $\triangle XYZ$ is isosceles, $\angle XZY \cong \angle XYZ$. Since these angles are both composed of two parts, and the parts $\angle XZU$ and $\angle XYQ$ are congruent to each other, that means the remaining parts must be equal as well. In other words, $\angle UZY \cong \angle QYZ$! Though this kind of part/whole reasoning doesn't appeal to theorems or postulates that are most likely on your list, it is a common type of geometrical reasoning. Your class can decide what kind of language it wants to use for these types of arguments.

After all that, we have used congruent triangles to figure out why $\angle UZY \cong \angle QYZ$. However, much of the argument was probably done orally, in combination with a marked-up diagram. It is now time to

23 Write a formal proof of the fact that $\angle UZY \cong \angle QYZ$.

In the proof you just wrote, you took pains to make sure that each statement was correctly justified. However, it is easy to be sloppy when writing proofs. Two examples of proofs follow that illustrate some mistakes that are easy to make.

Consider the proof below showing that, if a quadrilateral has two pairs of congruent, opposite angles, then the quadrilateral is a parallelogram.

First, draw a quadrilateral that can stand for any quadrilateral with congruent, opposite angles. Label the vertices $ABCD$, where $\angle A \cong \angle C$ and $\angle B \cong \angle D$. Also, draw the diagonal AC, as you can do for any quadrilateral.

Given: Quadrilateral ABCD, $\angle A \cong \angle C$, $\angle B \cong \angle D$.

Prove: $\overline{AB} \parallel \overline{CD}$, $\overline{BC} \parallel \overline{AD}$.

$\angle B \cong \angle D$, as given.

$\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$, because opposite sides of a parallelogram are congruent.

Therefore, $\triangle ABC \cong \triangle CDA$ by the SAS congruence theorem.

Then $\angle DAC \cong \angle BCA$ since they are corresponding parts of congruent triangles.

That makes $\overline{BC} \parallel \overline{AD}$ since alternate interior angles are congruent.

Similarly, $\angle BAC \cong \angle DCA$, again since they are corresponding parts of congruent triangles.

So $\overline{AB} \parallel \overline{CD}$.

Therefore, $ABCD$ is a parallelogram.

24 This proof may seem to work quite well. However, it contains a major error that invalidates the proof entirely. What is this error?

25 This type of error is sometimes called *circular reasoning*. Why do you suppose this phrase is used?

26 Do you still think it's true that, if a quadrilateral has two pairs of congruent, opposite angles, it must be a parallelogram?

Finally, here is an example of a proof that just can't be right: if a triangle is isosceles, then it must be equilateral. You have seen lots of isosceles triangles that are not equilateral, so you know that this proof must contain a mistake! The challenge is to find out what that mistake is. Remember that, if you agree with every step of a proof, you have no choice but to accept its conclusion.

Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$.

Prove: $\overline{AC} \cong \overline{BC}$ (which would also make it congruent to $\overline{AB}$).

Draw the median from C and label the point where it hits $\overline{AB}$ as D.

$\overline{AD} \cong \overline{DB}$ because $\overline{CD}$ is a median.

Also, $\angle CDA \cong \angle CDB$ because, as a median, $\overline{CD}$ splits $\overline{AB}$ into two right angles.

$\overline{CD} \cong \overline{CD}$ because any line segment is congruent to itself.

Therefore, by SAS congruence, $\triangle CAD \cong \triangle CBD$.

Since they are corresponding parts of congruent triangles, $\overline{AC} \cong \overline{BC}$.

Therefore, all three sides of $\triangle ABC$ are congruent, making it equilateral.

27 Find the flaw in the proof above.

Practice

28 Which of the following proofs are "airtight"?

a. Prove that, in a quadrilateral, if a pair of opposite sides is both congruent and parallel, then the quadrilateral is a parallelogram.

Proof: Draw segments $\overline{AB}$ and $\overline{CD}$ so as to be congruent and parallel. Connect points A and C. Then connect points B and D. Because $\overline{AB}$ and $\overline{CD}$ are congruent and parallel, the line segments $\overline{AC}$ and $\overline{BD}$ have to slope in the same direction. Therefore, $\overline{AC} \parallel \overline{BD}$, and so ABCD is a parallelogram.

b. Prove that the diagonals of a rhombus bisect each other.

Proof: We've already proven, and put on our theorem list, that the diagonals of a parallelogram bisect each other. A rhombus is a type of parallelogram. Therefore, any statement that applies to a parallelogram also applies to a rhombus, and so the diagonals of a rhombus bisect each other.

c. Prove that the diagonals of a parallelogram bisect each other. (We've already proved this, but here is an alternate version of a proof.)

Proof: Draw parallelogram $ABCD$, and add diagonals that cross at E. We're going to show that triangles AED and CEB are congruent. The two pairs of angles needed for congruence are $\angle DAE \cong \angle ACB$ and $\angle ADB \cong \angle DBC$. Those pairs really are congruent because they are both pairs of alternate interior angles created by parallel lines and a transversal. Also, sides $\overline{AD}$ and $\overline{BC}$ are congruent because they are opposite sides of a parallelogram. So now we have enough to show triangles AED and CEB are congruent, by ASA. Therefore, the diagonals do bisect each other: $\overline{DE} \cong \overline{EB}$ and $\overline{AE} \cong \overline{EC}$ as they are corresponding parts of congruent triangles.

29 The following are situations in which you might want to add an extra line to a diagram to help you with a proof. For each situation, say whether there could be multiple lines that fit the description, exactly one line that would fit the description, or whether a line that fits the description might not exist at all.

a. Given line AB and point C not on it, draw a line segment from C to AB.

b. Given line AB and point C not on it, draw a perpendicular line segment from C to AB.

c. Given line AB and point C not on it, draw a line through C parallel to AB.

d. Given parallelogram $ABCD$, draw diagonal AC so that it bisects angles A and C.

e. Given triangle ABC, draw a line segment from points D and E, the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively, so that $\overline{DE}$ is parallel to and half the length of $\overline{AC}$.

f. Given triangle ABC, draw the perpendicular bisector of side $\overline{AB}$ from C.

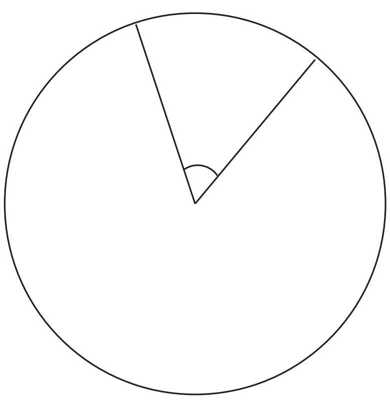
Problems

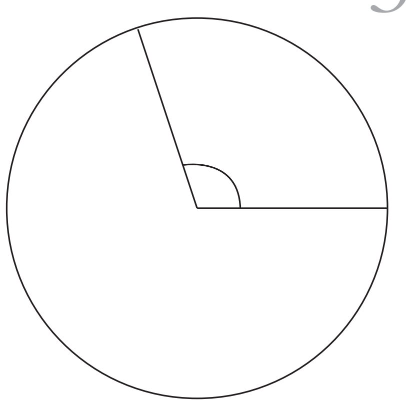
30 At the beginning of a lesson, you found that the diagonals of a parallelogram do not always bisect the angles from which they are drawn. How about for a rhombus? Prove it or find a counterexample.

31 For which of the following shapes — square, rectangle, parallelogram, trapezoid, kite, rhombus — are the diagonals always perpendicular to each other? Investigate with sketches.

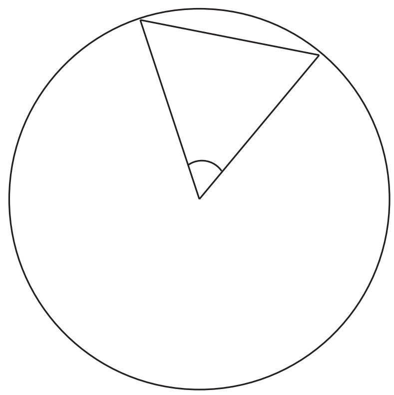
32 For the shapes in the previous problem where the answer is "yes," prove your response. Before beginning each proof, carefully state the given and what is to be proved. Hint: how can you possibly show that an angle is 90 degrees if you don't know the measures of any other angles in the figure?

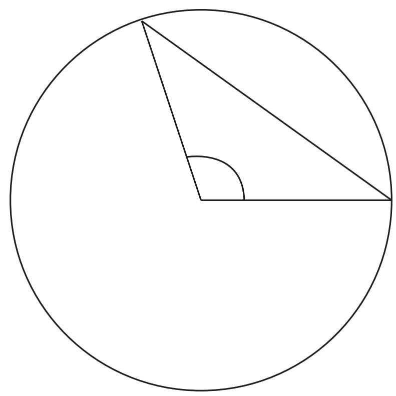
The angles below are called **central angles** of each circle. (The vertex of each angle is on the center of the circle.)





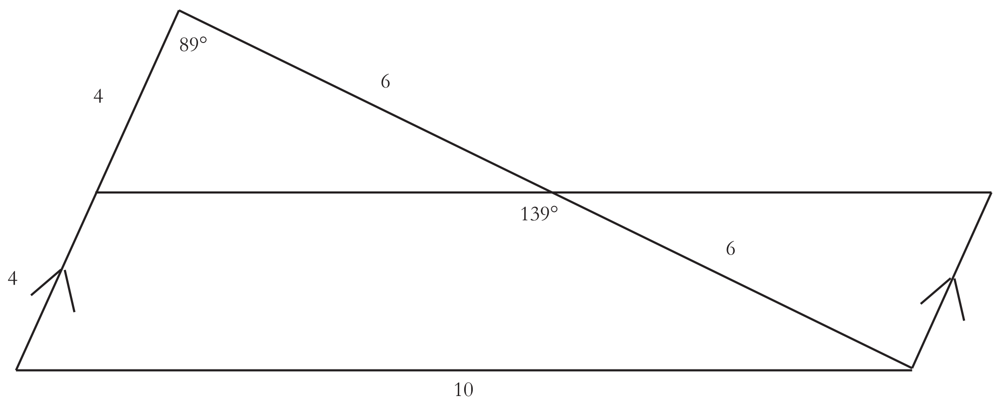
If you use a straight line segment to connect the points where the angle meets the circle, you get a *chord* of the circle.



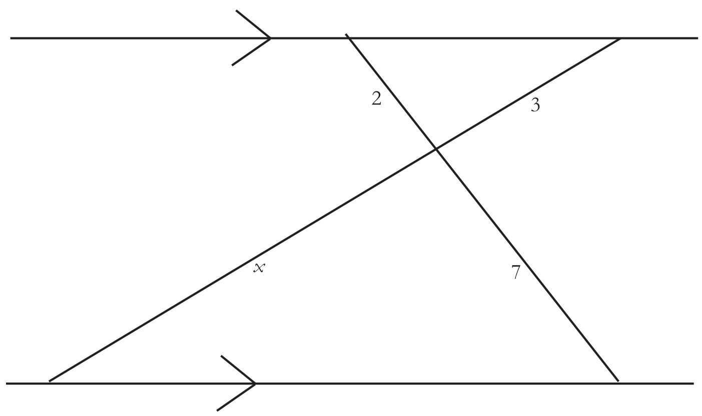


33 Prove that if a circle contains two central angles of the same size, then their corresponding chords are congruent.

34 Determine as many missing lengths and angles as you can. For each angle or length you find, give the reason you used to figure it out.

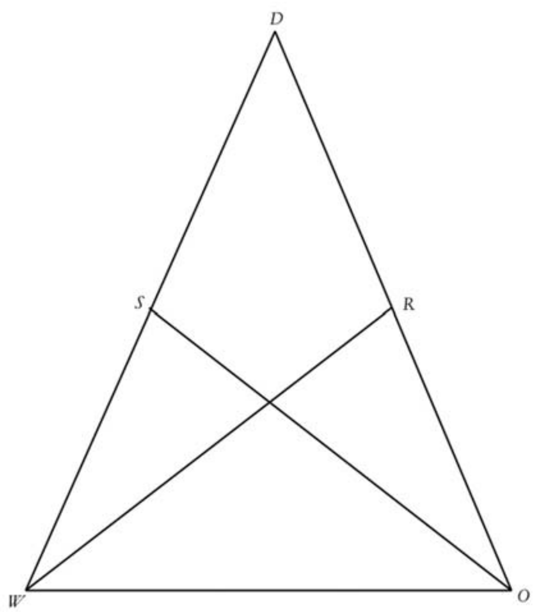


35 Find x. Write out the reasoning you used.



36 If you make an "X" shape in between two parallel lines, as in the figure for the previous problem, will the resulting triangles always be similar? Either prove it, or draw an "X" shape that creates non-similar triangles.

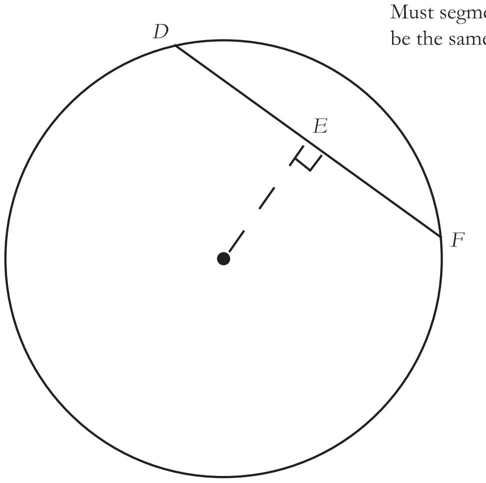
37 Given: S is the midpoint of $\overline{WD}$, R is the midpoint of $\overline{DO}$. Also, $\overline{WD} \cong \overline{DO}$.
Prove: $\overline{WR} \cong \overline{SO}$



38 Emmy and Sophie are standing in the center of the free throw circle on a basketball court. They want to measure their distance from the special free-throw line they put in for Emmy's little brother — it's a chord of the circle positioned close to the hoop.

Emmy says, "It's easy — we just measure along a perpendicular path from where we are standing to the special free-throw line. After all, that's how you find the distance from a point to a line.

Sophie says, "You're right. And I think the line we measure along will not only be perpendicular to the special free-throw line, but it will *bisect* the line as well."



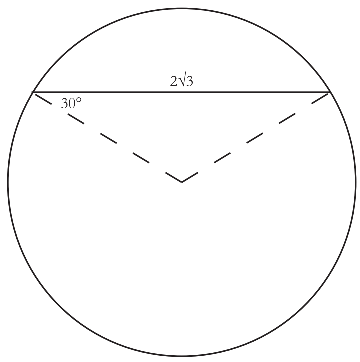
Must segments *DE* and *EF* be the same length?

a. Is Sophie right? Prove it or find a counterexample.

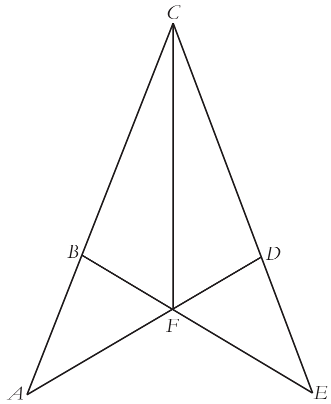
b. Suppose that Emmy and Sophie realize that they don't have a ruler, but they remember that the radius of the circle they're standing in is 10 feet, and the length of the special free-throw line is 14 feet. Help them calculate their distance from the line.

39 What's the distance from the center of a 41-foot radius circle to an 80-foot chord?

40 Someone saw this circle with chord of length $2\sqrt{3}$ drawn in. They connected the endpoints of the chord to the center of the circle. Find as many lengths and angles in this picture as you can. Also find the measure and lengths of each of the 2 arcs that the chord separates the circle into.



41 Given: $\overline{CB} \cong \overline{CD}$ and $\overline{BA} \cong \overline{DE}$
Prove: $\angle BCF \cong \angle DCF$



42 Don't use a calculator for this problem.

a. Simplify $\dfrac{x^{-3}y^{-4}}{x^{2}y^{-6}}$

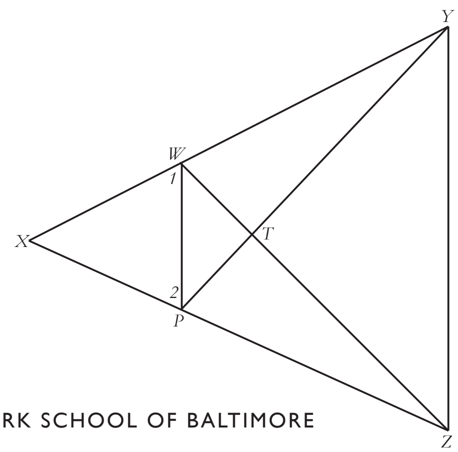
b. Simplify $x^2 + {\dfrac{3}{4}x}^{2}$

c. Reduce $\dfrac{3x^2}{6xy - 9x^3}$

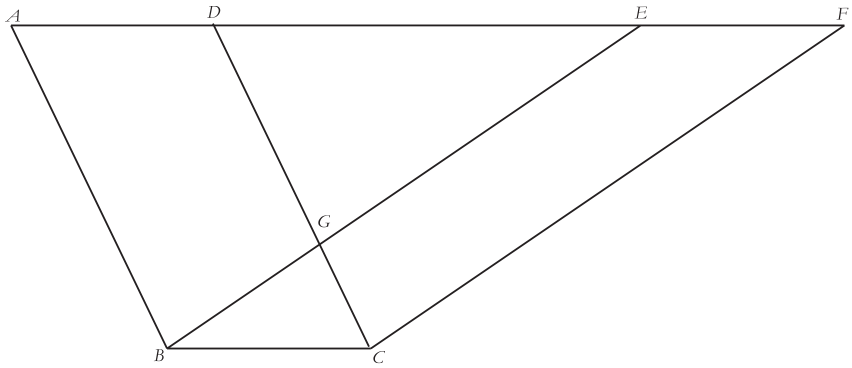
d. Reduce $\dfrac{x^2 - 1}{x^2 - 2x + 1}$

e. Solve for x if $x^2 + 1728x = 1729$

43 Given: $\overline{PT} \cong \overline{WT}$ and $\angle 1 \cong \angle 2$
Prove that $\triangle ZYT$ is isosceles



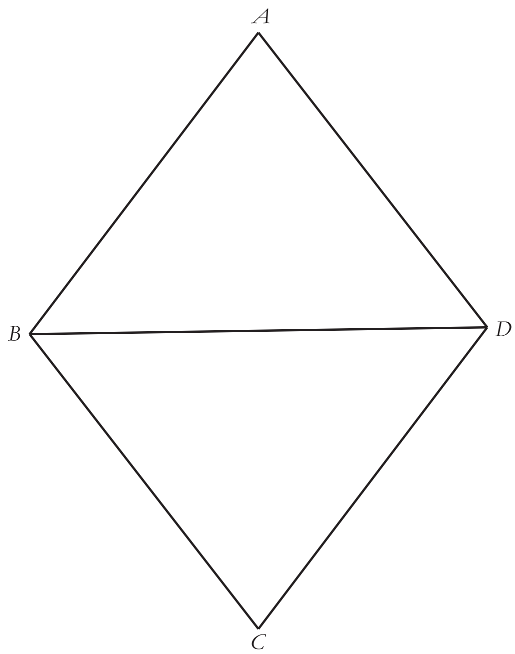
44 Given: ADCB and EFCB are parallelograms. Prove: $\triangle AEB \cong \triangle DFC$.



45 Given: $\overline{AB} \cong \overline{CB}$, $\overline{AD} \cong \overline{DC}$

a. Prove that the two triangles are congruent.

b. Can you then show that $\overline{AD} \cong \overline{BC}$?



46 Let ABC be any triangle. Choose point P somewhere on side AB of the triangle, and draw a line through P parallel to $\overline{CB}$. Where this line intersects $\overline{AC}$, label the point Q.

a. Prove that $\frac{AP}{AB} = \frac{AQ}{AC}$.

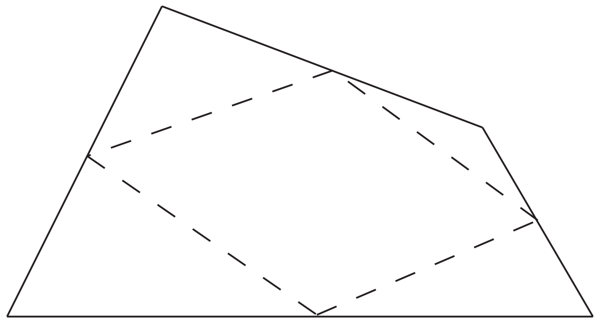
b. Prove that $\frac{AP}{PB} = \frac{AQ}{QC}$.

47 Gina did the previous problem and wonders if it's possible that $AP = x - 2$, $AB = x - 1$, $AQ = x - 3$, and $AC = x$. What should you tell her? Provide two explanations:

a. The first explanation should involve solving an algebraic equation.

b. The second explanation should involve considering a proportion.

48 The diagram below was formed by starting with any old quadrilateral, then connecting the midpoints of each side.



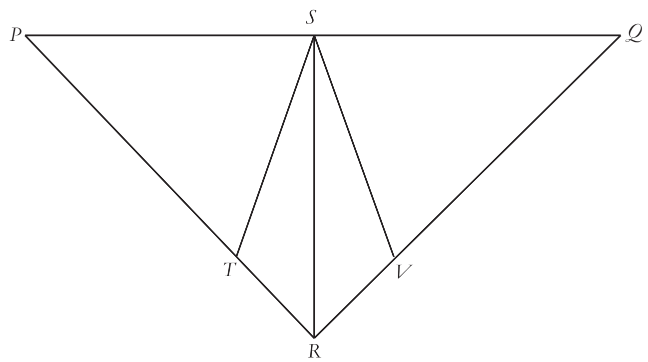
a. Try this several times on your own, with many different quadrilaterals. Form a conjecture about what always happens.

b. Prove your conjecture. (Hint: this is a problem where adding extra lines is helpful.)

Exploring in Depth

49 Given: $\overline{RS}$ is the perpendicular bisector of $\overline{PQ}$, and $\angle PST \cong \angle QSV$

Prove: $\overline{ST} \cong \overline{SV}$



50 Draw triangle ABC, and choose X to be a point on $\overline{AB}$. Draw $\overline{CX}$. Now add two lines to the picture that are parallel to $\overline{CX}$: one through A and the other through B. Y is the point where the first added line intersects line CB. (You may need to extend $\overline{CB}$ to find this point.) Z is the point where the second added line intersects line AC.

Now you have the line segments $\overline{AY}$, $\overline{CX}$, and $\overline{BZ}$. This problem will lead you towards proving a relationship between the lengths of those three segments.

a. Find two pairs of similar triangles in this figure, each pair involving the segment $\overline{CX}$.

b. Your two pairs of similar triangles allow you to write two proportions. Using these proportions, and facts such as $AB = AX + XB$, find an equation that relates $\overline{AY}$, $\overline{CX}$, and $\overline{BZ}$.

c. Prove that your answer is equivalent to $\frac{1}{AY} + \frac{1}{BZ} = \frac{1}{CX}$.

LESSON4:
LOGIC AND GEOMETRY

Introduction

Lewis Carroll, of *Alice in Wonderland* fame, made the following set of statements in his less famous book, *Symbolic Logic*."

No potatoes of mine, that are new, have been boiled.
All my potatoes in this dish are fit to eat.
No unboiled potatoes of mine are fit to eat.

1 What, if anything, can you conclude from these three statements?

Development

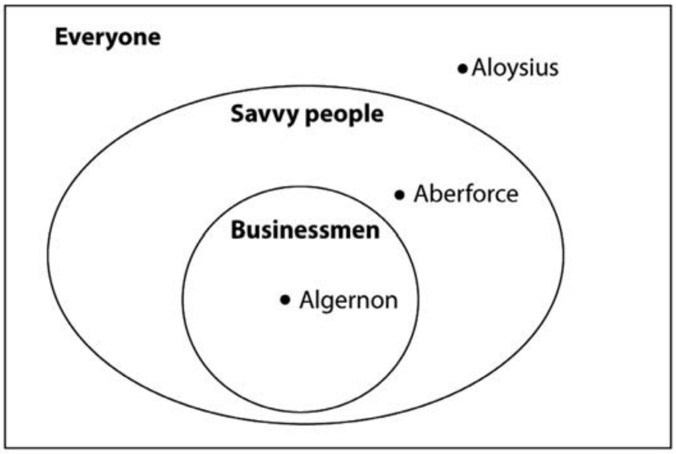
It's a well-known fact that all squares are rectangles, but not all rectangles are squares. We can express these statements in the form below.

If a figure is a square, then it is a rectangle. (true)
If a figure is a rectangle, then it is a square. (false)

You may recall that these statements are called **converses** of each other, as is true of any pair of statements of the form

If A then B.
If B then A.

You may have seen Venn diagrams at some point in your life. They are handy when thinking about logic. For instance, the statement "All businessmen are savvy" (or "If a person is a businessman, then he is savvy") can be represented by the following diagram:



In this example, Algernon is a savvy businessman, Aberforce is savvy but not a businessman, and Aloysius is neither savvy nor a businessman.

2 Draw a similar diagram illustrating the statement "All savvy people are businessmen."

a. How would you write the statement in "if/then" form?

b. Do the diagrams make it clear that the statements "All businessmen are savvy" and "All savvy people are businessmen" mean different things?

You've seen that if/then statements have converses, and that just because the if/then statement is true doesn't necessarily mean the converse is. Here are two other types of statements you can get from an if/then statement.

Take the statement "if A then B".
"If not-A then not-B" is called the **inverse.**
"If not-B then not-A" is called the **contrapositive.**

3 Pick three "test" statements of the form "if A then B."
Then use them to decide if the inverse and/or the contrapositive must be true when the original statement is.
If you're not sure, try drawing Venn diagrams.

Some statements on your list of theorems and postulates can be written in if/then form as well, and it is worth investigating if their converses are true. For example, you probably have the following statement:

Opposite sides of a parallelogram are congruent.

4 Write a pair of if/then statements expressing this statement and its converse.

5 If you were going to prove the converse, what would be the given? What would be the result that you are trying to prove?

6 Go ahead and prove the statement "if a quadrilateral has both pairs of opposite sides congruent, then it is a parallelogram." (Hint: this is a problem where adding something extra to the diagram is useful.)

Another way of saying what you just proved is that having both pairs of opposite sides congruent is *sufficient* for a quadrilateral to be a parallelogram. In other words, if you have a quadrilateral and you know its opposite sides are congruent, that's enough — you don't need to know anything else about it to know that it's a parallelogram.

On the other hand, the original statement on your list, "Opposite sides of a parallelogram are congruent," provides a statement that is true about all parallelograms. This kind of statement is called a *necessary* condition. For if you've got a quadrilateral without congruent opposite sides, you know you definitely don't have a parallelogram. You're lacking one of the necessary criteria.

Conditions aren't always both necessary and sufficient. For example, having four right angles is necessary for a quadrilateral to be a square, but having four right angles is not sufficient for a quadrilateral to be a square. The quadrilateral might be only a rectangle and still have four right angles.

Also, conditions can be sufficient without being necessary. For example, a closed shape's having three 60-degree angles is sufficient for the shape to be a triangle, but it is certainly not necessary that a triangle have three 60-degree angles.

What's more, necessary and sufficient conditions don't have to be about shapes. For example, at Park, playing soccer for all four years is sufficient to fill your athletic requirement. Also, being a senior is necessary for taking Senior Studio — you can't take Senior Studio without being a senior.

7 Is having four congruent sides necessary, sufficient, or both for being a square?

8 Is having four congruent sides necessary, sufficient, or both for being a rhombus?

9 Is playing soccer all four years a necessary condition for Park students to satisfy their athletic requirement? Is being a senior at Park sufficient for taking Senior Studio?

Practice

10 Poet Li-Young Lee says "every wise child is a sad child." According to him, is being a wise child sufficient for being a sad child?

11 Is being seventeen sufficient for having a provisional driver's license in Maryland?

12 Is being an even number necessary for being divisible by eight?

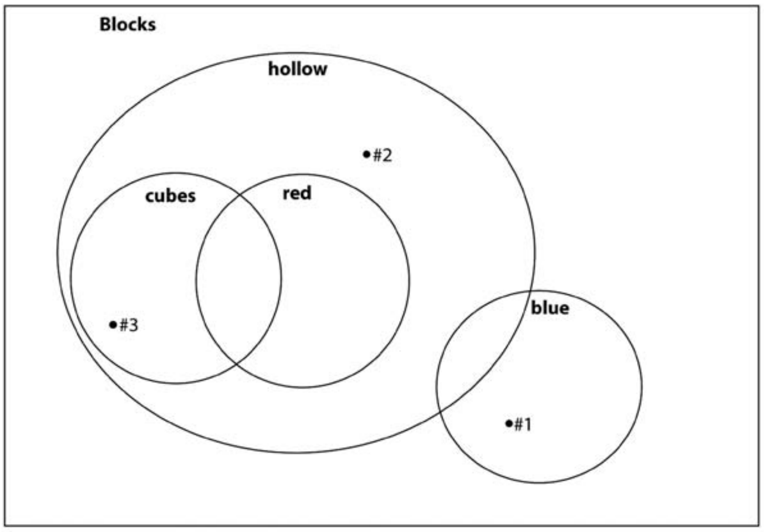
13 Is being a pair of numbers' LCM…

a. …sufficient for being divisible by each of those numbers?

b. …necessary for being divisible by each of those numbers?

14 Is being divisible by 108 a sufficient condition for a number to be divisible by 8? By 6? by 54?

15 You're organizing your little sister's blocks. While doing so, you make the following Venn diagram to classify them. Say whether each statement is true.



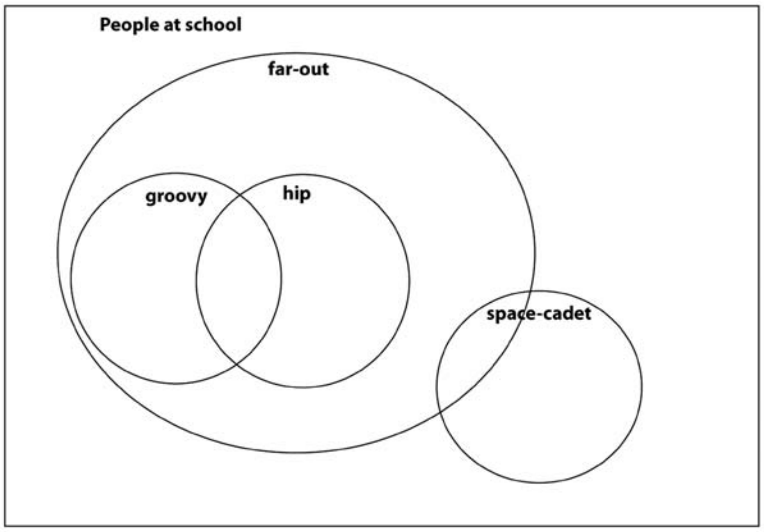
a. If a block is blue, then it is hollow.

b. If a block is red, then it is hollow.

c. If a block is hollow, then it is red.

d. Three of your sister's blocks are shown in the Venn diagram. Describe each block according to the characteristics described in the picture.

16 Bill and Ted come up with the following classification system of the students attending their high school.



According to their scheme,

a. Is being hip necessary for being far-out?

b. Is being groovy sufficient for being far-out?

c. Is being a space-cadet necessary for being far-out?

d. Is being far-out necessary for being hip?

17 Draw a diagram (perhaps a Venn diagram or a “tree” diagram) that clearly indicates which quadrilaterals are types of other quadrilaterals (for instance, a square is a type of rectangle).

Problems

18 Draw a Venn Diagram and use it to see what you can conclude from the following Lewis Carroll puzzle.

- All babies are illogical.
- Nobody is despised who can manage a crocodile.
- Illogical persons are despised.

19 The statement "no circle is a parallelogram" can't be represented in a Venn diagram as a circle within a circle.

a. Write "no circle is a parallelogram" as an if/then statement.

b. Draw a Venn diagram that you think captures the meaning of the statement.

c. Write the converse, inverse, and contrapositive of that statement. Does the same diagram work for the contrapositive?

20 In the last lesson you proved that if you have two central angles of the same size in a circle, then their corresponding chords are congruent.

a. Is the converse of this statement true?

b. Either prove the converse or provide a counterexample.

21 In the last lesson you proved that if you draw a line from the center of a circle perpendicular to a chord, the line would bisect the chord.

a. Write the converse of this statement in a form that you might be able to prove.

b. Is the converse true?

c. Either prove it or provide a counterexample.

22 Euclid's fifth postulate is famous for being more complicated than his others. The postulate is similar to the statement

If, when two lines are cut by a transversal, the alternate interior angles are not equal, then the lines will meet.

Euclid also proved a theorem which states

If, when two lines are cut by a transversal, the alternate interior angles are equal, then the two lines are parallel.

a. How are these two statements related to one another? (Converse? Inverse? Contrapositive?)

b. Did Euclid need to prove the second statement separately, or would it have been enough to appeal to the first statement?

23 Dr. Gordon says, "Say you have a statement like 'if p then q'. If this statement is true, then p is sufficient for q and q is necessary for p."

a. Test this condition out by trying some examples of statements (like, "If you live in Baltimore, then you live in Maryland.")

b. Explain, in language that an eighth-grader could understand, why Dr. Gordon's claim is true.

24 In the 2008 presidential campaign, some people were worried about whether Obama could still be a good president even though he lacked military experience. At the same time, General Wesley Clark pointed out that, though McCain had impressive military experience, that experience didn't necessarily prepare him well for the presidency.

a. Are Obama's critics arguing that military service is necessary to become a good president, or sufficient for being a good president?

b. Is Wesley Clark arguing that military service is not necessary to become a good president, or that it is not sufficient?

25 Decide if each of the following is a sufficient condition for a quadrilateral to be a parallelogram. You can think of these as "tests" to decide if the shape is really a parallelogram.

a. Both pairs of opposite angles are congruent.

b. When you draw a diagonal, a pair of alternate interior angles is congruent.

c. One pair of opposite sides and one pair of opposite angles are congruent.

d. Two pairs of adjacent (next to each other) angles are supplementary.

e. One pair of opposite sides is both parallel and congruent.

26 Two of the statements in the previous problem are true. Write a proof of each of them. (If you're stuck on a proof, **work backwards**; what do you need to show in order to show that something is a parallelogram? What tools do you have to show those things?)

27 Draw a Venn diagram to illustrate the following statements:

a. If you can make it through *War and Peace*, you're a superstar.

b. If you're not a superstar, you can't make it through *War and Peace*.

c. In either diagram, can you have not read *War and Peace* and yet still be a superstar?

28 Being an ack is a necessary condition for being a kook. Being an ook is a sufficient condition for being an ack. There are 22 acks, 7 ooks, and 10 kooks. Of the acks, 8 are neither ooks nor kooks. Now, supposing you pick a random ack, what is the probability that it will be an ook, but not a kook?

29 Investigate the following:

a. Draw a bunch of quadrilaterals where the diagonals are perpendicular and where one diagonal bisects the other (but not necessarily the other way around). It may help to start with the diagonals, then draw in the rest of the shape around them. What shape do these always turn out to be?

b. Prove that all quadrilaterals with perpendicular diagonals, one bisecting the other, are this shape.

c. Did you just find a necessary condition for a quadrilateral to be this shape, a sufficient condition, or both?

30 Name some necessary conditions for a shape to be a rhombus.

31 Are the conditions you named in the previous problem on your theorem list already? If the class agrees, add them to your theorem list.

32 Suppose that someone asked you to prove that, if a quadrilateral is not a parallelogram, at least one pair of opposite sides must not be congruent. Explain why you wouldn't have to do a lot of work, even though this statement is not on your theorem list.

33 In 9th grade, you learned about a made-up rule, $a \triangle b$, which takes the first number, adds the second number to it, adds the first number to the sum, then takes that whole answer and multiplies it by the second number. For example, $7 \triangle 2$ is 32.

a. Also in 9th grade, a mysterious fellow named John claimed: "To get an odd number for your answer from $a \triangle b$, you need to input odd numbers for a and b. Did John get it right?

b. Did he identify a necessary or a sufficient condition for getting an odd number, or was his condition both necessary and sufficient?

34 The symbol "&" from the 9th grade means that you take the first number and then add the product of the numbers. About this symbol, John claimed: "To get an odd number for your answer from $a \,\&\, b$, you need to input an odd number for a and an even number for b."

a. Did John get this one right?

b. Did he identify a necessary or a sufficient condition for getting an odd number, or was his condition both necessary and sufficient?

35 *Rational numbers* are those numbers that can be expressed as a fraction (where both numerator and denominator are whole numbers) or a repeating decimal (like 4/3, .256256256..., and 8.000000...). Many *roots* (such as square roots, cube roots, and higher roots) are not rational. Some roots of integers work out to be integers themselves, like $\sqrt[4]{16} = 2$. However, roots of integers that don't themselves work out to be integers are never rational; instead, they can be expressed as non-repeating decimals. Examples of these are $\sqrt{2}$ and $\sqrt[3]{7}$.

Since all numbers can be expressed as repeating or non-repeating decimals, can we conclude that all numbers are either rational or the root of an integer?

36 Recall from your study of quadratic equations that, so long as $a \neq 0$, the equation $ax^2 + bx + c = 0$ has as its solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

a. What is the converse of this statement? Is it true?

b. If the converse is true, then prove it. If it is not true, find a counterexample.

37 Update your theorem list with some of the sufficient conditions you have proven in this lesson.

Exploring in Depth

Many arguments have this type of structure:

- If a Park Student is in fall semester of 10^{th} grade, then they are taking a "Writing about…" class.
- Gwendolyn is in fall semester of 10^{th} grade.
- Therefore, Gwendolyn is taking a "Writing about" class.

Aristotle, a Greek philosopher who lived close to the time of Euclid, called this a **syllogism.** A syllogism has the form

- If A then B
- A
- Therefore, B.

In this example, the statements "if A then B" and "A" are called **premises** and "therefore, B" is called the **conclusion**. Note that if an argument really works, if you agree with the premises you should also agree with the conclusion. (And if you don't, that's a sign that something is wrong with the argument!)

Another form of argument can lead you to form an "if/then" statement as the conclusion. For example,

- If a Park Student is in fall semester of 10^{th} grade, then they are taking a "Writing about…" class.
- If a student is taking a "Writing about" class, then they have to write a self-reflection.
- Therefore, if a Park student is in fall semester of 10^{th} grade, then they have to write a self-reflection.

This type of argument has the form

- If A then B
- If B then C
- Therefore, if A then C.

Note the important thing that allows you to conclude the final step is that "B" occurs as the "then" clause of the first statement and the "if" clause of the second statement.

Boiling arguments down to their logical structure can be handy, especially when you start using the equivalence of the contrapositive. For example, recall the Lewis Carroll puzzle of problem #18:

a) All babies are illogical.
b) Nobody is despised who can manage a crocodile.
c) Illogical persons are despised.

Suppose you have managed to rewrite the puzzle this way:

(from a and c) If you are a baby, then you are despised
(from b) If you can manage a crocodile, then you are not despised.

Right now the premises are written as

If A then B.
If C then not-B.

It's difficult to know what we can conclude just now, but if we replace the second premise with its contrapositive, "if B then not-C," we can make things match up in the right way:

- If you are a baby, then you are despised.
- If you are despised, then you cannot manage a crocodile.

This allows us to easily see that we can conclude

- If you are a baby, then you cannot manage a crocodile.

…which is the solution to the puzzle.

For exercises 38-41 and 43-46, say what, if anything, you can conclude from the premises given.

38 If you are on Park's basketball team, then you are over six feet tall.
If you are over six feet tall, you can't go on the Pirate ride at Six Flags.

39 If you live in Baltimore, you should go see an Orioles game.
If you don't like baseball, then you shouldn't go see an Orioles game.

40 If it snows a lot, they'll cancel school.
If they don't cancel school, I'll scream.

41 If Tom committed the murder, he was in the library at 10 last night.
If Tom wasn't in the library at 10 last night, then he was at home.

42 Don't use a calculator for this problem.

a. Simplify $\sqrt{4a^2}$

b. Factor $x^2 + 3x - 28$

c. Find x if $\dfrac{17x}{3x+2} = 5$

d. Expand $(x+2)^3$

e. Simplify $\left(\dfrac{w^5 x^{-2} y}{(wx^2)^2 y^{-1}}\right)^2$

43 If Jean committed the murder, she was in the library at 10 last night.
If Jean bought groceries on her way home, then she wasn't in the library at 10 last night.

44 All brilliant mathematicians have a heart of gold.
If you study for hours, then you don't have a heart of gold

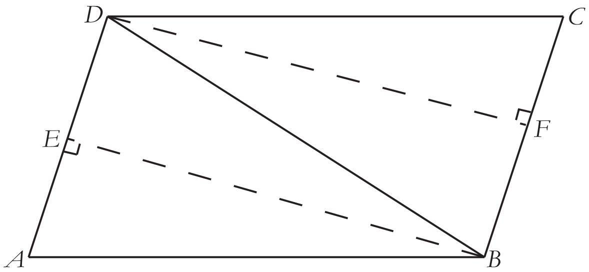
45 If you don't file a form, you can't get good service.
All people who get good service are content.

46 This puzzle comes from Lewis Carroll, author of *Alice in Wonderland*.
(note: "affected" means "written in a snobby, insincere way")
(other note: assume that all poems are either ancient or modern)

No affected poetry is popular among people of real taste.
No modern poetry is free from affectation.
No ancient poem is on the subject of soap-bubbles.
All your poems are on the subject of soap-bubbles.

47 Part c of problem #25 might have seemed true. ("One pair of opposite sides and one pair of opposite angles is sufficient for a quadrilateral to be a parallelogram.") What goes wrong when you try to attempt a proof? Does this suggest a method for finding a counterexample?

48 In fact, P. Halsey published the following "proof" that one pair of opposite sides and angles in a quadrilateral was sufficient for it to be a parallelogram, challenging his readers to find the flaw. Can you?



(source: Jim Loy's puzzle pages: http://www.jimloy.com/geometry/quad.htm)

Given: $\overline{CD} \cong \overline{AB}$, $\angle A \cong \angle C$.
Prove: $ABCD$ is a parallelogram.

Draw perpendiculars from points B and D that intersect the opposite sides of the quadrilateral at E and F, respectively. Then $DCF \cong BAE$ by AAS. Since the triangles are congruent, $\overline{AE} \cong \overline{CF}$.

Also, from the triangle congruency we know $\overline{BE} \cong \overline{FD}$ as well. This means that we can now show $DEB \cong BFD$ by Hypotenuse-Leg. From this, we can conclude $\overline{DE} \cong \overline{FB}$.

Now, the lengths of $\overline{DE}$ and $\overline{EA}$ add up to the length of $\overline{DA}$, and likewise the lengths of $\overline{BF}$ and $\overline{FC}$ add up to the length of $\overline{BC}$. Since $\overline{DA}$ and $\overline{BC}$ are made up of the same-sized "pieces", they must themselves be congruent. Therefore, $ABCD$ is a parallelogram.

LESSON 5: BUILDING UP MORE THEOREMS

Introduction

In this chapter, you have proven many properties of triangles and quadrilaterals. You've known about some of these properties for a long time; others may have been new to you. You have also set about creating a structure for the geometry you know, choosing certain facts to consider as basic and showing how other facts depend on those. At this point, you are well beyond the basics, and you have probably already begun to make conjectures about ideas you haven't thought about before.

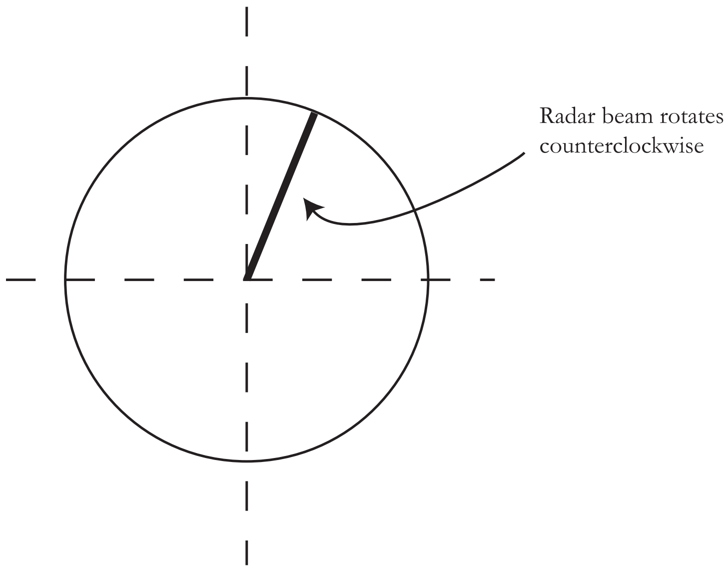
1 In your notebook, use a compass to draw a circle. Then draw two chords of two different lengths that cross somewhere inside the circle. Label the endpoints of one chord A and B, and the endpoints of another chord C and D. The chords cross at point E. Now measure segments $\overline{AE}$, $\overline{EB}$, $\overline{CE}$, and $\overline{ED}$. Try calculating $AE \cdot EB$ and $CE \cdot ED$. What do you find?

In this section, most ideas that come up will probably be new, and many, like the result above, will be unexpected. You will be in a good position to make some conjectures and to prove many of the conjectures you make.

After looking at parallel lines, triangles, and quadrilaterals, we'll now turn our attention to circles. Many of the theorems you've already established will prove useful in investigating circles. Circles themselves often prove useful in analyzing other shapes, as you've probably realized when doing compass-and-straightedge constructions. In the development, most of your time will be spent learning the substantial vocabulary that has to do with circles. Then you'll learn and prove a few basic properties of circles. In the problems section, you will have freer rein to conjecture and prove more things about circles, and about geometry in general.

Development

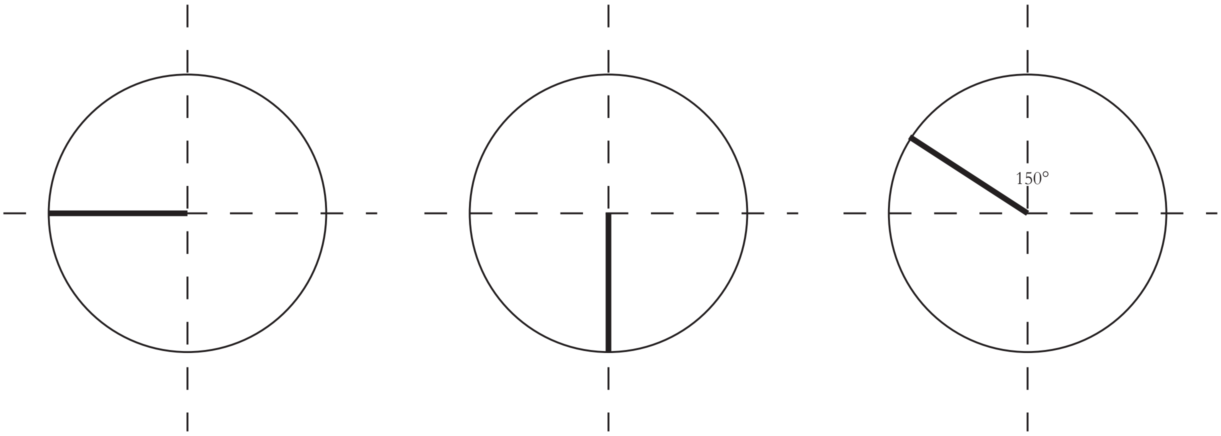
Your submarine has a rotating radar beam that is used to get a picture of the surrounding area. Your radar screen looks like this,



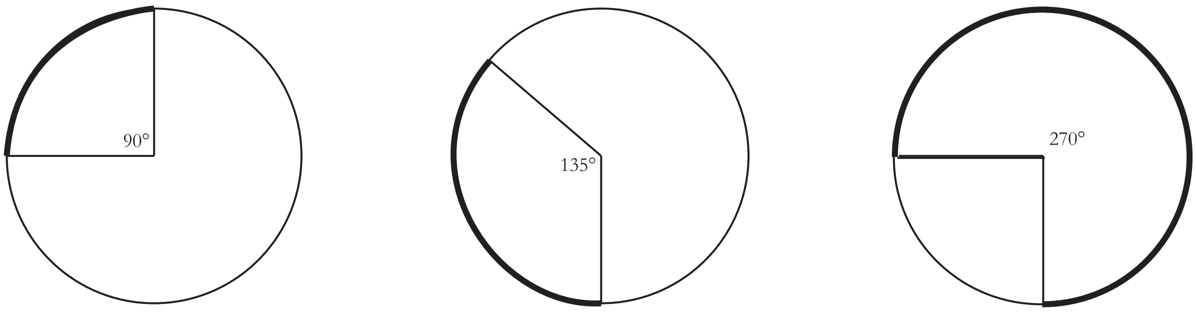
with the bold line segment showing the motion of the radar beam. As the beam rotates, the screen updates based on what the beam finds.

2 Why do you think the radar screen is in the shape of a circle, rather than, say, a rectangle?

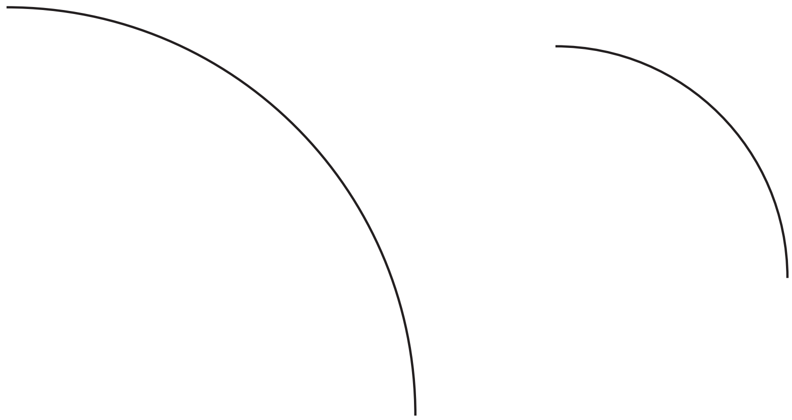
3 Say that you know that the circle the screen makes has a circumference of 360 millimeters. What's the length of the edge of the screen that has been updated since the beam was pointing **due east**? Answer this question for each picture below.



Recall that the 150° angle in the rightmost picture above is called a **central angle.** As you can see, there's a direct relationship between a central angle and the arc it **intercepts**, even if the circumference of the circle is not 360 millimeters.



Because of this relationship, people also talk about the *measure* of an arc. The measures of the arcs above are 90°, 135°, and 270°. So both of the arcs below have measure 90°, even though their lengths are very different.

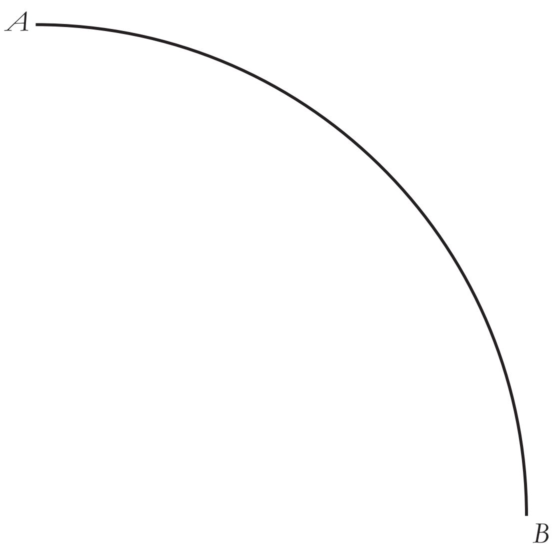


4 What's the measure of the arc that borders the portion of the radar screen between east and northwest?

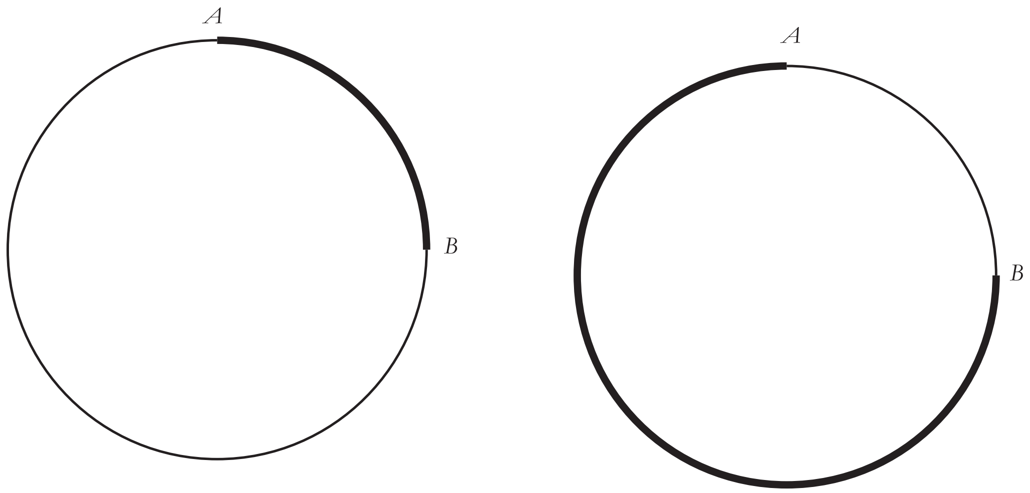
5 What's the measure of the arc that borders the portion of the radar screen between north and south-southeast?

6 If an arc of a circle has measure 47°, what's the measure of the arc you'd need to form the rest of the circle?

To refer to an arc, you use the two points where the arc begins and ends. Here's arc AB, or, better yet, $\overset{\frown}{AB}$.



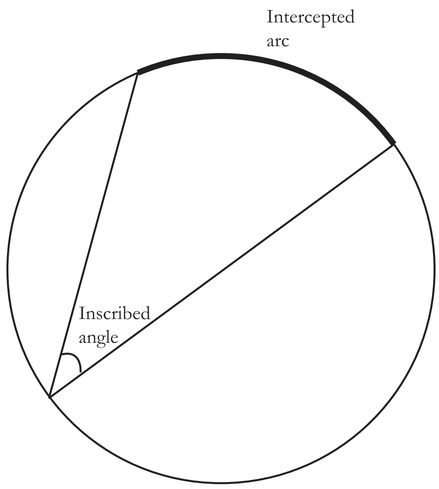
Of course, when we're talking about arcs that are parts of circles, naming arcs this way has a little problem…



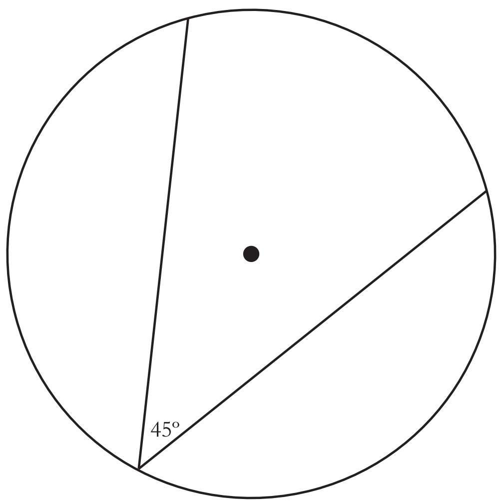
…because how would we know whether we meant the arc that looks like it's about 90°, or the one that looks like it's about 270°? When you don't specify, it's assumed that you mean the smaller, or *minor* arc.

7 Draw a circle in your notebook. Pick points Q and Z anywhere on the circle. Use measuring tools (and perhaps a bit of ingenuity) to find the measure of arc QZ. Then describe a general strategy for finding arc measure.

Another type of angle in a circle is called an *inscribed angle*. An inscribed angle has its vertex on the circumference of a circle, not at the center. Inscribed angles can intercept arcs, too.



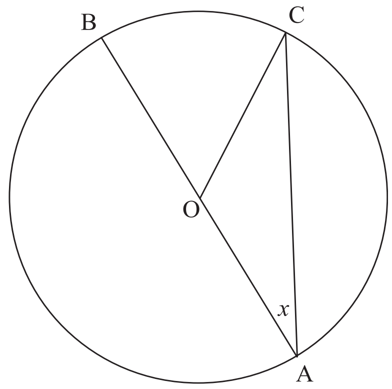
8 Here's a 45° inscribed angle. Does it look like it also intercepts a 45° arc?



9 There is a conjecture to be found here. Draw some more circles with inscribed angles, and find it.

Now that you have a conjecture, it's time to look for a proof. The following problems will help you.

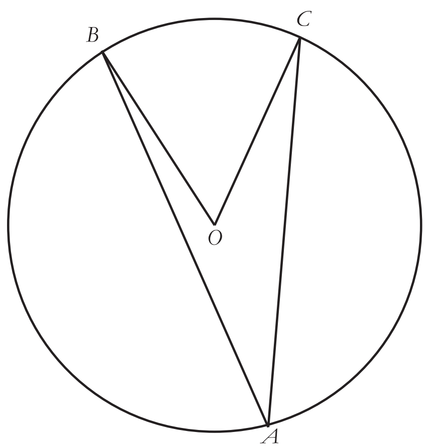
10 Below is a picture of an inscribed angle *BAC*, which intercepts the same arc as central angle *BOC*.



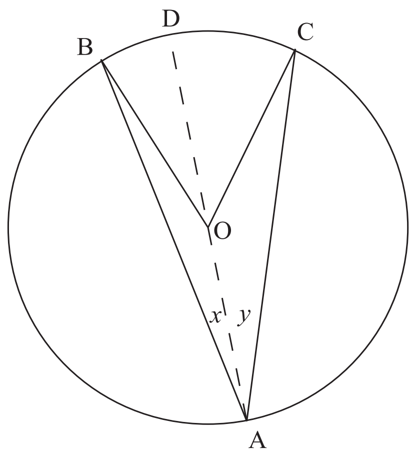
a. Using geometric tinkering, find the measure of $\angle BOC$ in terms of x.

b. The proof that you've done does not work for the diagram in problem 8. What goes wrong?

Here is another diagram, one that looks more like the situation in problem 8.



Since we already know something about when one of the sides of an angle is a diameter, let's draw in the diameter and label the angles this way:



11 Using this diagram, prove that the central angle is still twice the inscribed angle.

12 Between problems 10 and 11, will your proofs work for all inscribed angles? If so, great. If not, draw a picture of an inscribed angle, along with its associated central angle, for which the proofs will not work.

Practice

13 For this problem, think of an analog clock (a clock with hands).

a. Find the angle determined by the hands of a clock when it is 4 o'clock. What's the measure of the arc formed by the edge of the clock between the 12 and the 4?

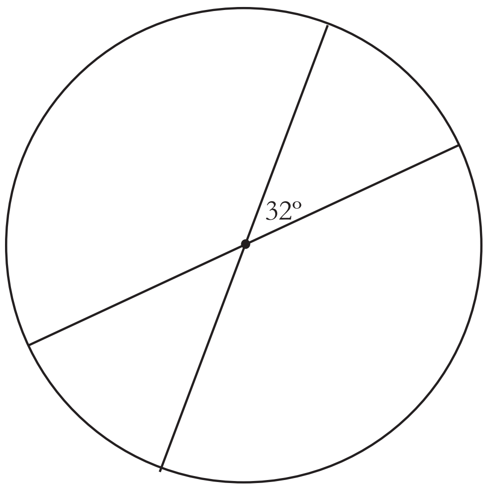
b. Now imagine that you draw a line from the 12 to the 8 and then from the 8 to the 4. What's the angle formed at the 8?

c. What's the measure of the arc formed by the edge of a clock between the 7 and the 11?

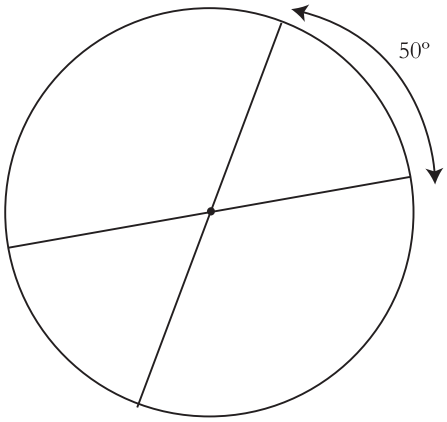
d. Create an inscribed angle by starting at the 7, going to any time on the clock not between 7 and 11 o'clock, and then going to the 11. How will this compare with your angle in part c?

14 For each figure, find as many angles and arc measures as you can. A dot indicates the center of the circle.

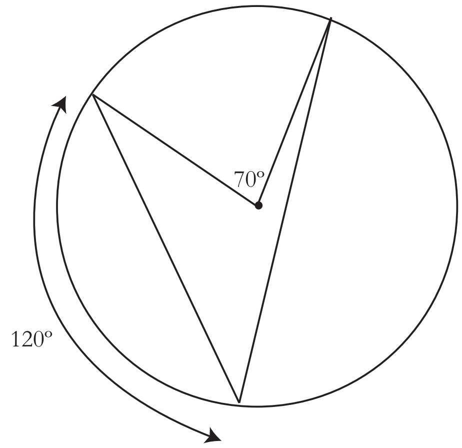
a.



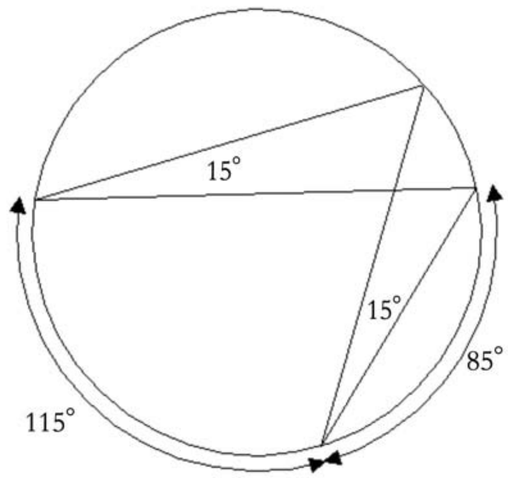
b.



c.

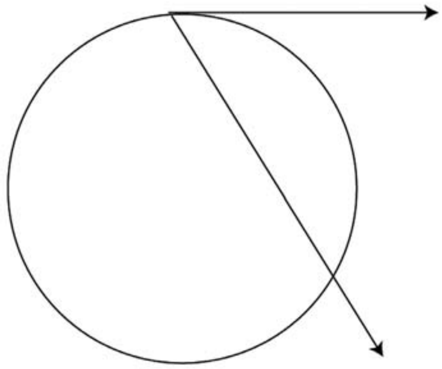


d.

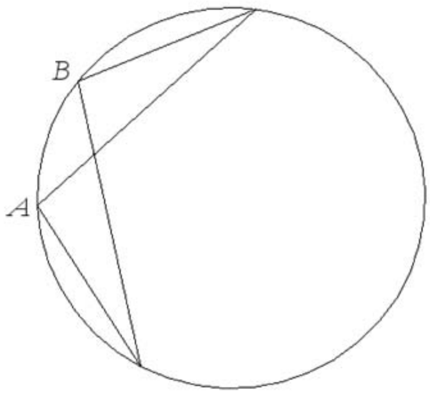


15 What can you say about two inscribed angles that intercept the same arc? Why?

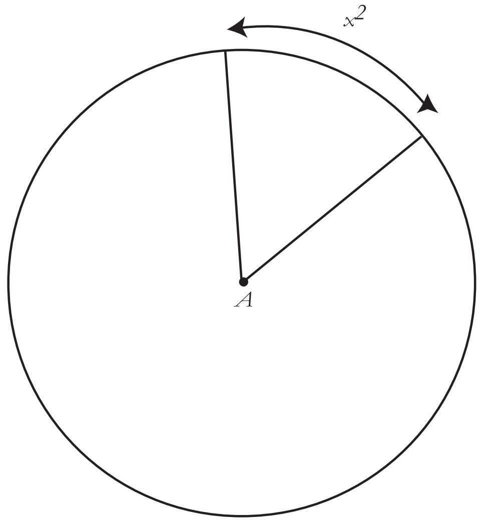
16 Below is an angle formed not by two chords, but by a chord and a tangent segment. Note that this angle still can be said to intercept an arc. Make a conjecture about the measure of a chord-tangent angle and the measure of its intercepted arc.



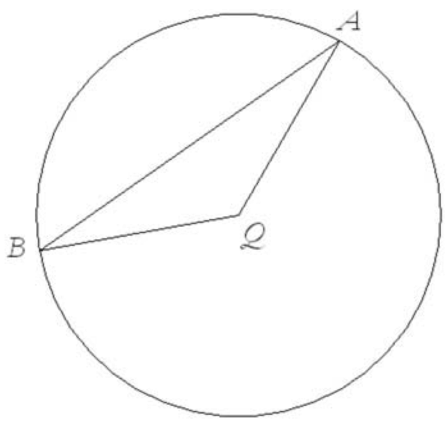
17 In the diagram below, $m\angle A = \left(x^2\right)^\circ$ and $m\angle B = \left(2x + 120\right)^\circ$. Find x.



18 Let $m\angle A = (2x + 24)^{\circ}$. Find x.



19 Q is the center of the circle, and $m\angle A = 25^{\circ}$. Find the measure of $\overset{\frown}{AB}$.

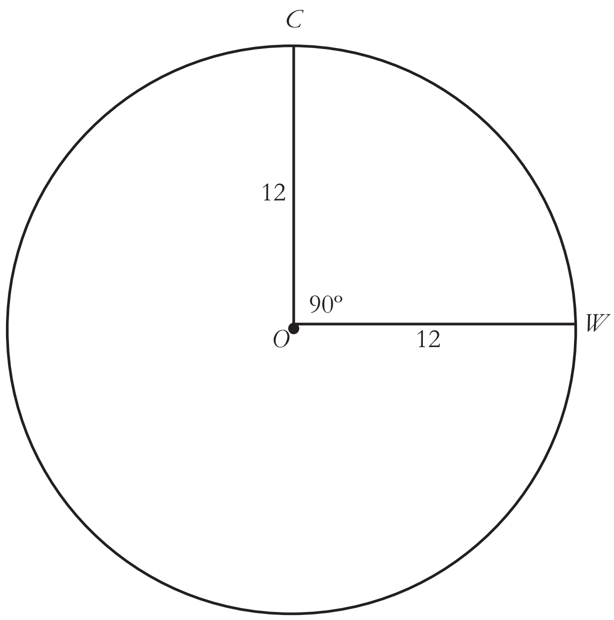


20 Circle O has radius 12. Find

a. The measure of $\overset{\frown}{CW}$.

b. The actual length of $\overset{\frown}{CW}$.

c. The *area* of the **sector** of the circle COW. (What do you suppose a **sector** is?)

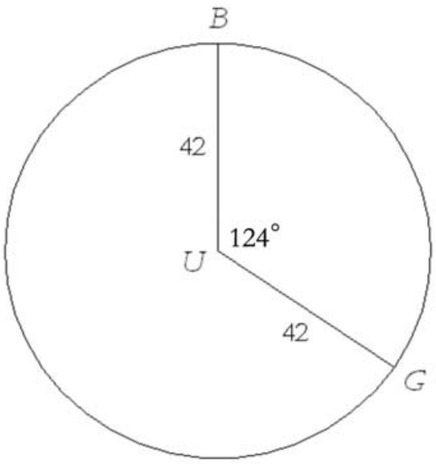


Problems

21 The circle below has radius 42. Find

a. The length of $\overset{\frown}{BG}$

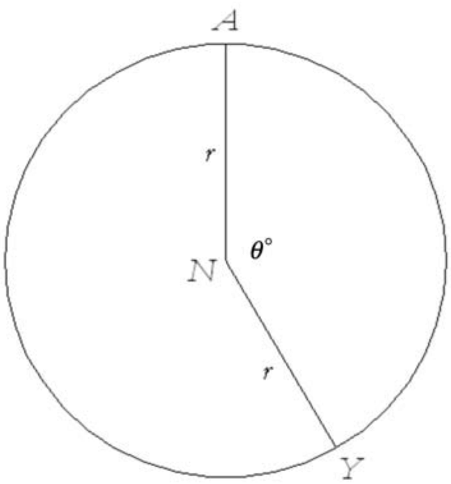
b. The area of sector *BUG*



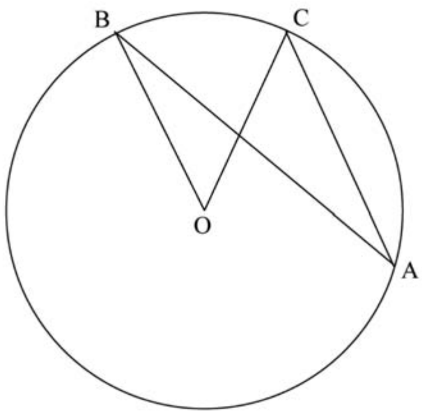
22 The circle below has radius r, and the central angle is θ. Find

a. The length of $\overset{\frown}{AY}$.

b. The area of sector ANY.



23 Here's another diagram showing a central angle and an inscribed angle that intercept the same arc.



a. Why don't the methods of proof you used in problems 10 and 11 work for this inscribed angle/central angle pair?

b. Prove that $m\angle BOC$ is twice $m\angle BAC$. As is often the case, you'll need to act on the diagram first.

c. Now have you proved this result for all inscribed angles, or are there other positions for inscribed angles for which none of your proofs would work?

d. When you need to do separate proofs depending on the situation, that is called "proof by cases." For example, if you wanted to prove a result in number theory, you might do different proofs depending on whether your original number was even or odd, but you would have to do both proofs in order to prove that the result always worked. Describe in words the three cases you considered when you did the inscribed angle proof.

24 Draw line segment AB, then accurately draw its perpendicular bisector.

a. Pick a few points on the perpendicular bisector and label them C, D, E, etc. Compare CA vs. CB. Does this result seem to hold up consistently?

b. Investigate whether the converse of your conjecture in part a is true. If you are having trouble figuring out what the converse should be, try rewriting your original conjecture in "if/then" form.

25 The radius of a circle is 16.5 inches.

a. What's the area of a 90-degree slice of the circle?

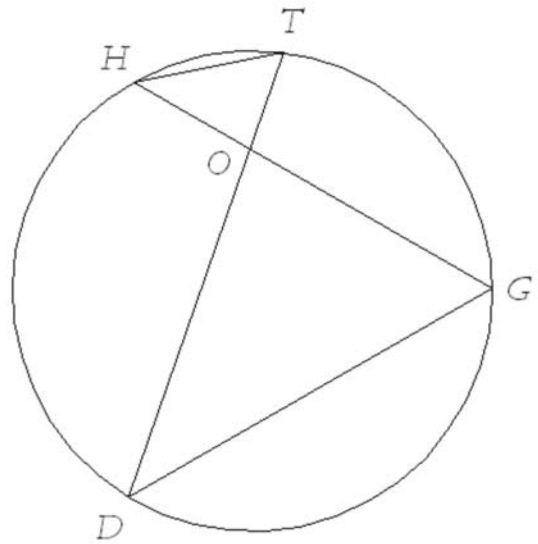
b. What's the area of a 1-degree slice of the circle?

c. What's the area of a 53.5-degree slice of the circle?

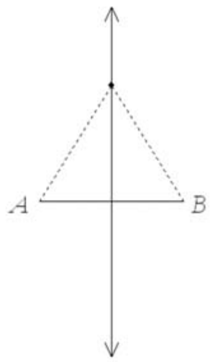
d. What's the area of an *n*-degree slice of the circle?

26 In your notebook, draw $\angle XYZ$. Suppose you begin to walk a path "inside the angle," starting at Y. You walk in such a way that you are always the same distance away from $\overrightarrow{YX}$ and $\overrightarrow{YZ}$. Using precise mathematical language, make a conjecture about your path.

27 Prove that $\triangle HOT \cong \triangle DOG$.



28 Here is a figure designed to help you with a proof. Two proofs, actually.



a. Use the figure to prove that points on the perpendicular bisector of a segment are equidistant from the endpoints of the segment. Be careful in selecting the given.

b. Now use the figure to prove the converse. Again, be careful in selecting the given and what is to be proved.

29 If you had only a compass and straightedge, but no ruler, suggest a method for drawing in the perpendicular bisector of any line segment *AB*. No fair folding the paper.

30 You cut a pizza into 10 congruent slices.

a. If the area of one slice is 6.5 square inches, what's the radius of the pizza?

b. If instead the perimeter of one slice is 17 inches, what's the radius of the pizza? (If you're stuck, you could write an equation first: 17=…)

31 Can the area of a circle ever be an integer? Why or why not?

32 Here is Problem 1, reprinted:

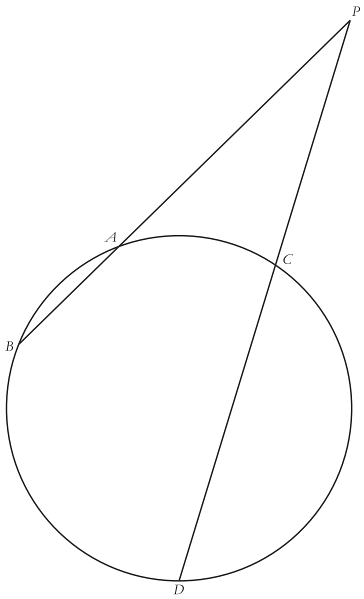
In your notebook, use a compass to draw a circle. Then draw two chords that cross somewhere inside the circle; they needn't cross at the center of the circle. Label the endpoints of one chord A and B, and the endpoints of another chord C and D. The chords cross at point E. Now measure segment AE, EB, CE, and ED. Try calculating $AE \cdot EB$ and $CE \cdot ED$. What do you find?

Prove the conjecture you made in Problem 1. Planning this proof is a great opportunity to **work backward**. What geometry tools do you have to show that two products equal one another?

33 Is it possible for the lengths of the four segments formed by two intersecting chords in a circle to be 4 consecutive integers?

34 Suppose that you have a triangle inscribed in a circle in such a way that one of the triangle's sides is the circle's diameter. What kind of triangle must you have, and why?

35 There is a theorem very similar to the one dealt with in Problem 32 about line segments drawn from a point outside a circle to a circle. (You could also think of this as a case where lines cross outside of a circle, instead of inside it.) As in the case of crossed chords, there is a product of lengths that remains constant. Which lengths are they?



36 State and prove a theorem based on your findings in Problem 26.

37 In the previous problem, did you prove a result corresponding to Problem 26, or did you prove the converse? Either way, state and prove the converse of the theorem you proved in the previous problem.

38 Suggest a way to construct an angle bisector using only a compass and straightedge.

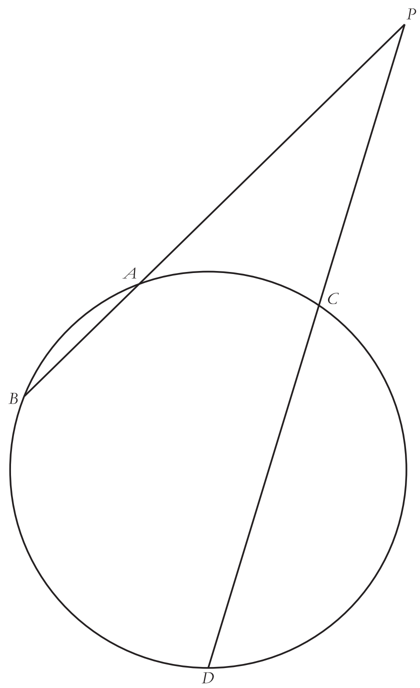
39 Given triangle ABC, let F be the point where segment BC meets the bisector of angle BAC. Draw the line through B that is parallel to segment AF, and let E be the pointwhere this parallel meets the extension of segment CA.

a. Find the four congruent angles in your diagram.

b. How are the lengths $EA, AC, BF,$ and FC related?

c. How are the lengths $AB, AC, BF,$ and FC related?

40 In the figure below, prove that $PB \cdot PA = PD \cdot PC$. Here are a few hints.

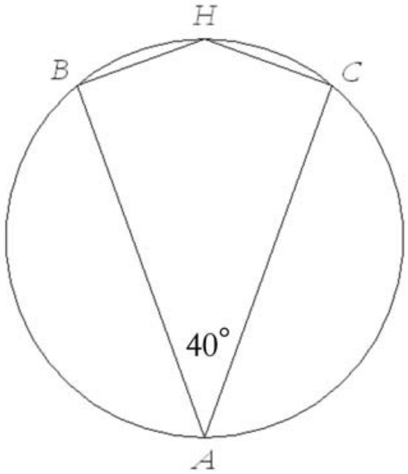


a. Show that the result you want is equivalent to $\dfrac{PB}{PC} = \dfrac{PD}{PA}$.

b. Part a is a sign that you should be looking for similar triangles where $\overline{PB}$ and $\overline{PD}$ are corresponding sides, and so are $\overline{PC}$ and $\overline{PA}$. Add some lines to the diagram so that triangles exist with these sides.

c. Now prove the result.

41 *BACH* is a kite inscribed in a circle. Find all the angles of the kite. (Hint: find some arc measures along the way.)



42 Don't use a calculator for this problem.

a. Solve for x if $\dfrac{2x-4}{x^2+3x+1}=0$

b. Simplify $\sqrt{72}$

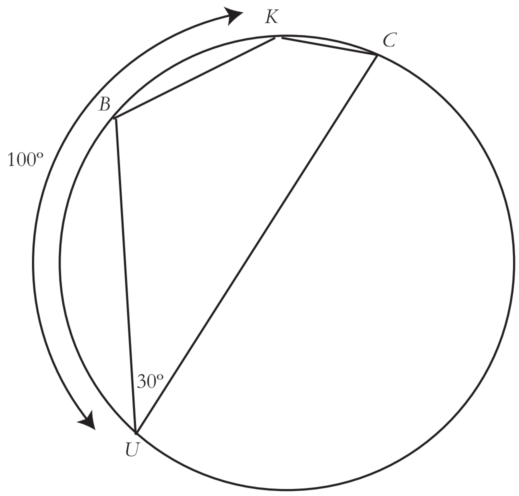
c. At what point do the lines $y=2x-4$ and $y=\dfrac{-4}{7}x+5$ cross?

d. Solve for x if $x^2+6x-2=0$.

e. If $x+y=1729$ and $\dfrac{1}{x}+\dfrac{1}{y}=1729$, find xy.

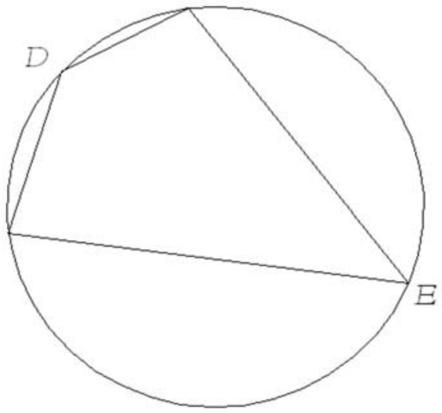
43 Draw several triangles of varying shapes and sizes, then construct the perpendicular bisectors of each side of each triangle. Coincidence? You be the judge.

44 *BUCK* is not a kite. Find all of its angles.



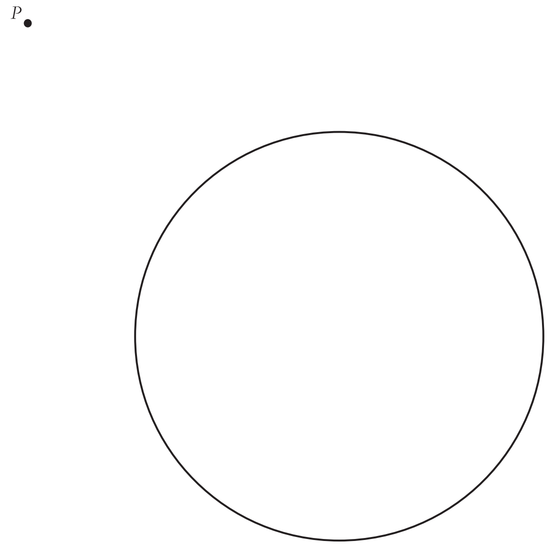
45 Make a generalization about the opposite angles of a quadrilateral inscribed in a circle.

46 Let $m\angle D = \left(x^2 - 10x + 60\right)^\circ$ and $m\angle E = \left(8x + 85\right)^\circ$. Find all possible values of x.



47 You know results about central angles and their intercepted arcs, as well as inscribed angles and their intercepted arcs. How about angles with their vertex not on the center or on the edge of a circle, but inside the circle? Make a conjecture. Hint: One way to get angles with their vertex inside a circle is to draw a pair of crossed chords.

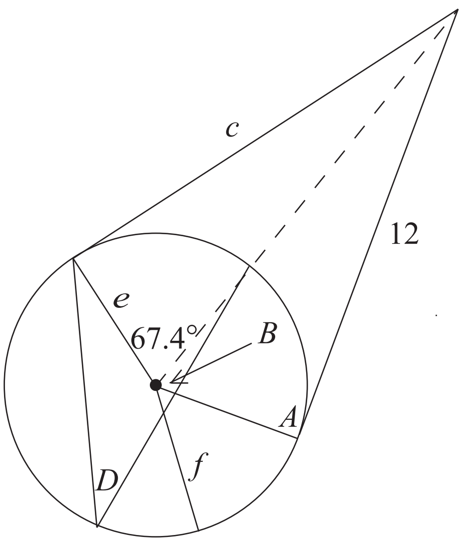
48 How many lines can you draw through point P that are tangent to the circle? Try drawing lines from a point other than P. What about a point inside or on the circle?



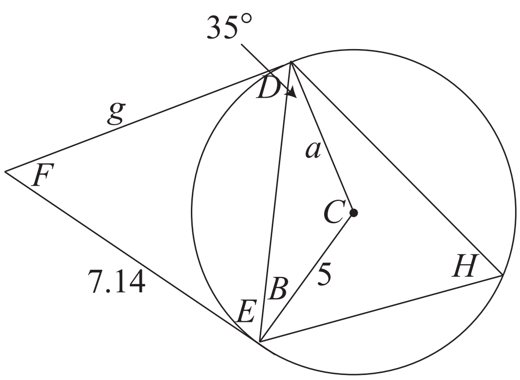
49 Rotate the diagram above so that P is on the bottom and the whole figure looks like an ice-cream cone. Make a conjecture about the lengths of the tangents from P.

50 Prove the conjecture you made in the previous problem. Do you need to have additional information? How can you create some?

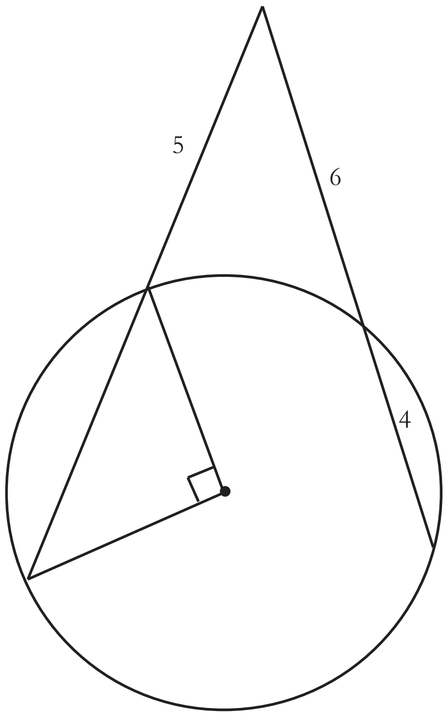
51 The dotted line segment has a length of 13. Find the lengths and angles marked with letters. Give a reason each time you deduce something. The center is marked with a dot; lines that appear to be tangents are tangents.



52 Find the lengths and angles marked with letters. (Uppercase for angles; lowercase for lengths) Give a reason each time you deduce something. The center is marked with a dot; lines that appear to be tangents are tangents.



53 Find the exact value of the radius of the circle below.

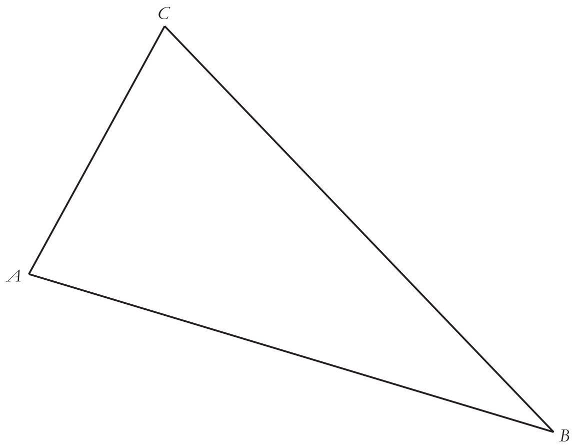


54 Prove that the measure of an angle formed by crossed chords in a circle is the average of the measures of the two arcs intercepted by the two angles of that size.

55 Find the probability that 3 points chosen at random from n points evenly spaced on a circle will form a right triangle. (Hint: simplify the problem.)

Exploring in Depth

56 Draw in the perpendicular bisectors for sides AB and BC of the triangle below. Then label the point where they cross D.

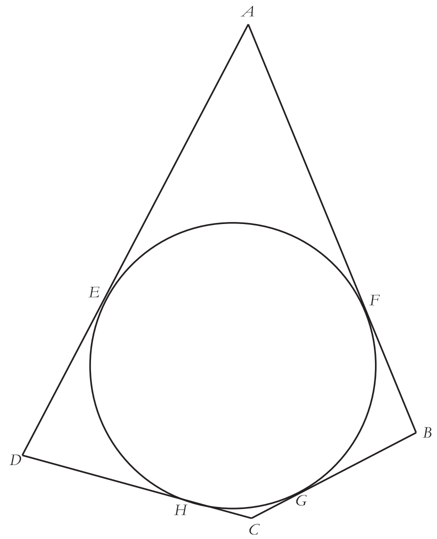


a. Is the point D closer to A or B?

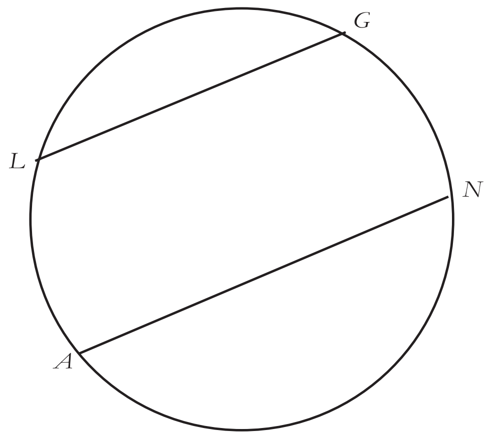
b. Is the point D closer to B or C?

c. Suppose, now, that you were going to draw the perpendicular bisector of AC as well. Can you make an argument that it should pass through the point D, even before you draw it?

57 In this figure, $AB = 20$, $BC = 11$, and $DC = 14$. Find AD. (Hint: pick a side length to call x, then try to write the other lengths in terms of x.)



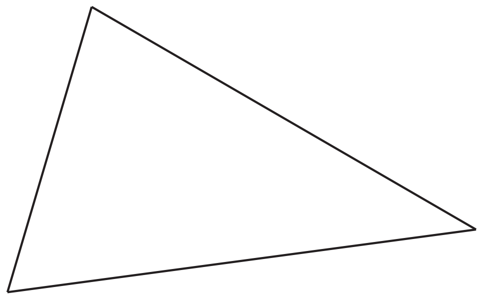
58 Given: $\overparen{NG} \cong \overparen{LA}$
Prove: $\overline{AN} \parallel \overline{GL}$



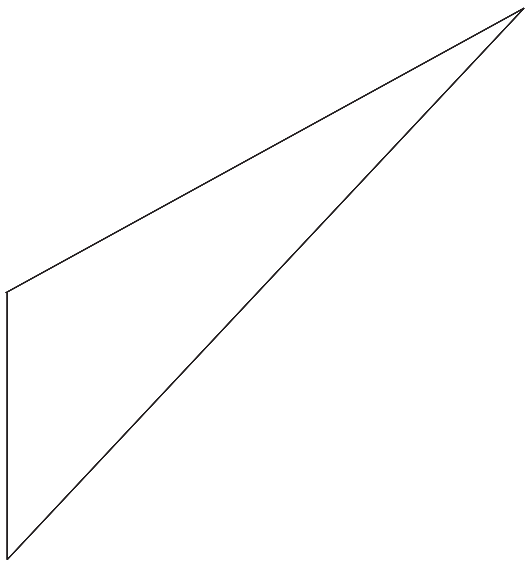
You know that every triangle has three different heights (or "altitudes").

59 Use a ruler and protractor to draw in the three different altitudes for each triangle:

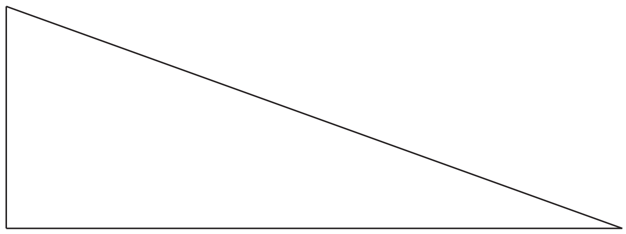
a.



b.



c.



60 Come up with an argument for why, in a right triangle, the three altitudes must all meet in one point.

61 In each of the triangles in problem 59, the three altitudes should have met in one point. Do you think this is true for all triangles? See if you can find a counterexample.

62 Draw a large triangle, then draw its midpoint triangle. Draw the perpendicular bisectors of the sides of the large triangle. These lines have a significance for the midpoint triangle as well. What's the significance, and why is this always the case?

63 Can you now prove that the three altitudes of any triangle meet at one point? Do so or explain why there still might be counterexamples.

64 Remind yourself of the result about angle bisectors you proved earlier in this lesson. Use it to prove that the three angle bisectors in a triangle meet at one point.

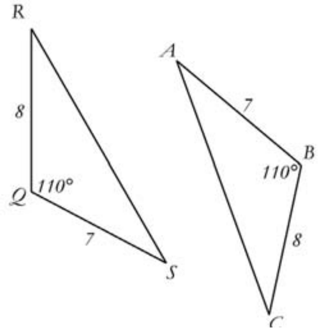
65 Prove that an angle formed by a chord and a tangent to a circle is half the measure of its intercepted arc.

66 Draw a point P outside of a circle. Then draw two line segments from P: a tangent segment that hits the circle at point A, and a segment that intersects the circle twice, at point B (farther from P) and C (closer to P). Prove that $PA^2 = PB \cdot PC$.

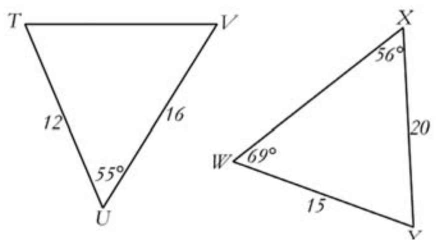
SUMMARY AND REVIEW

1 In each of the following problems 2 triangles are given with some information about them. Determine if the triangles can be proven similar, congruent, or neither, **and** give the **reason** (e.g. SSS congruence, AA Similarity, etc.) and the **exact similarity or congruence** (e.g. $\triangle ABC \sim \triangle DEF$).

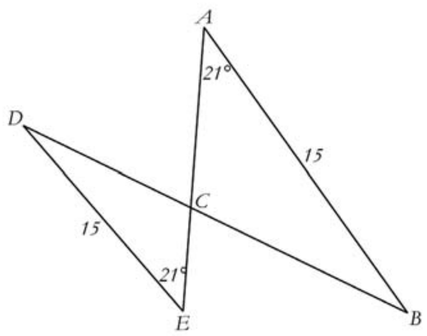
a.



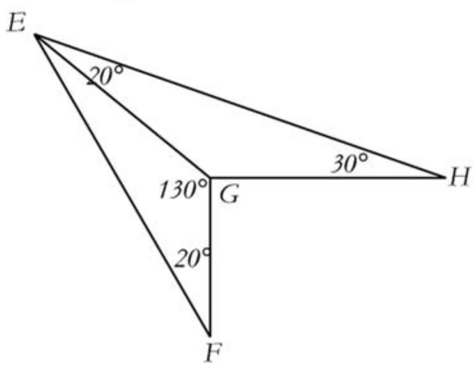
b.



c.



d.



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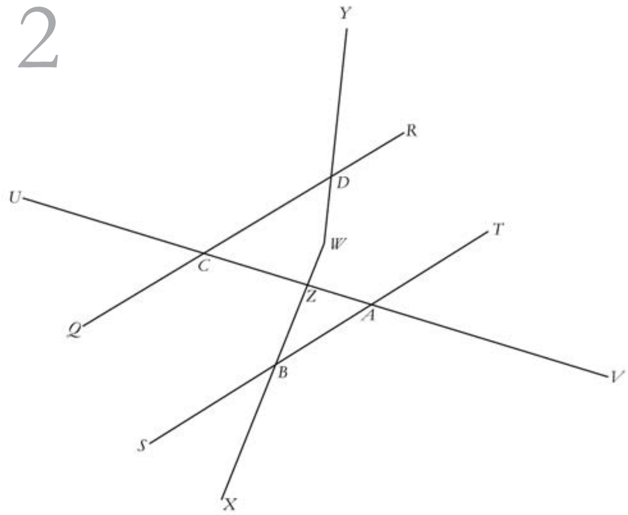


In the above figure, RQ and TS are parallel segments.

a. Name 2 pairs of congruent alternate interior angles.

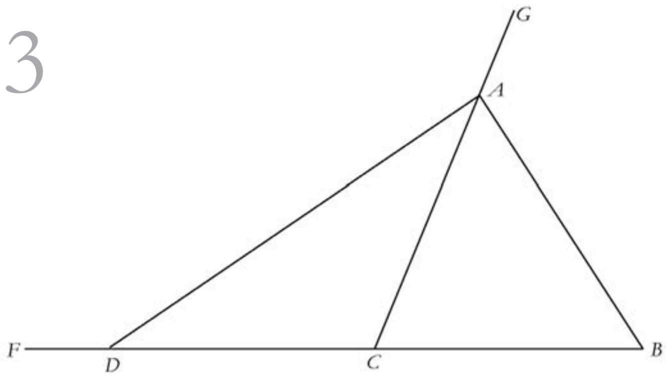
b. Are $\angle RDW$ and $\angle SBZ$ congruent? Explain.

c. Name 1) a pair of congruent corresponding angles, 2) a pair of congruent alternate exterior angles, and 3) a pair of congruent angles that are not the same type of angles as either of the previous two (or the two from part a).

d. Name 2 angles that add up to 180 degrees that do NOT together form a straight line.

e. 3 angles that add up to 180 degrees, and then name a completely different 3 angles that add up to 180 degrees.

3



Given: $\angle ACB \cong \angle ABC$

$DC = AB$

Prove: $\angle FDA \cong \angle GAD$

4 Zardon says: "If I don't eat sardines for lunch my wife will find me attractive!"

a. What is the converse of that statement?

b. If Zardon's wife finds him attractive, what can one conclude, if anything?

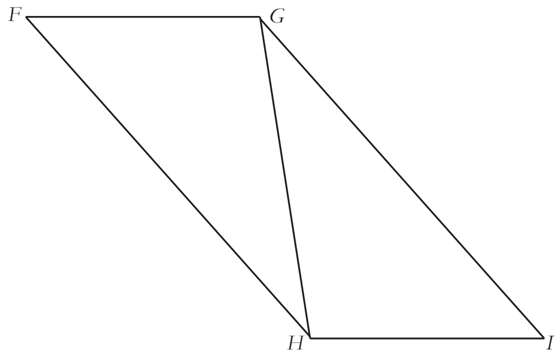
c. Zardon then proceeds to eat sardines anyway. Will Zardon's wife necessarily find him unattractive? Explain your answer.

d. If Zardon's wife finds him unattractive, what can one conclude, if anything?

5 For a given quadrilateral, its diagonals bisect each other.

Can one say that this quadrilateral is necessarily a parallelogram? If so, prove it. If not, explain why and draw a plausible counter-example.

6

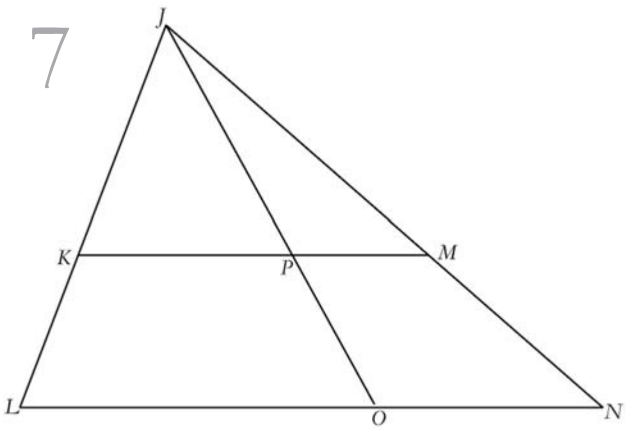


Given: $\angle FHG = \angle HGI$

FG is parallel to HI

Prove: $\overline{FG} = \overline{HI}$

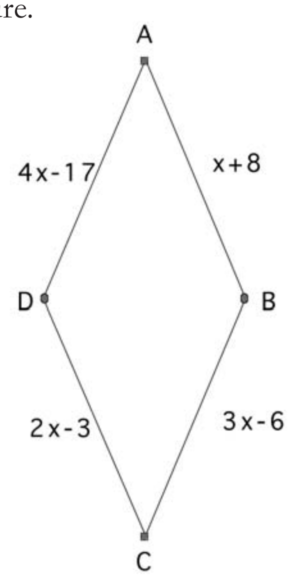
7



Given: $\angle JMK = \angle JNL$

Prove: $\dfrac{KP}{KJ} = \dfrac{LO}{LJ}$

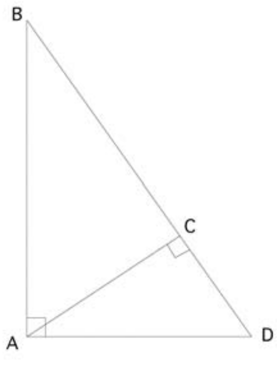
8 The figure below is a kite (a kite has two pairs of consecutive congruent sides). Determine the lengths of the 4 sides in the figure.



9

a. In the figure below, list as many pairs of similar triangles as you can. Why are they similar?

b. The two smaller triangles have three congruent angles and share a congruent side. Why, then, aren't they congruent?

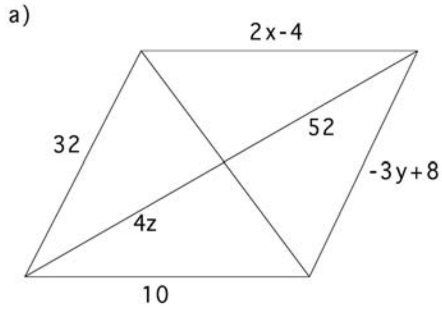


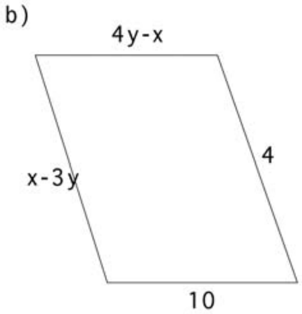
10 Explain, as if to someone who has only had middle school Geometry:

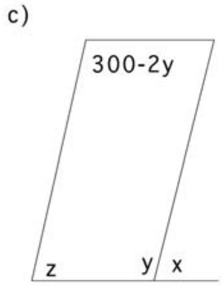
a. Why the exterior angle of a triangle is equal to the sum of the remote interior angles.

b. Why the sum of the angles of an "n" sided quadrilateral is equal to $180(n\text{-}2)$.

11 Find the values for each of the unknowns in each figure to ensure that each of them is a parallelogram:







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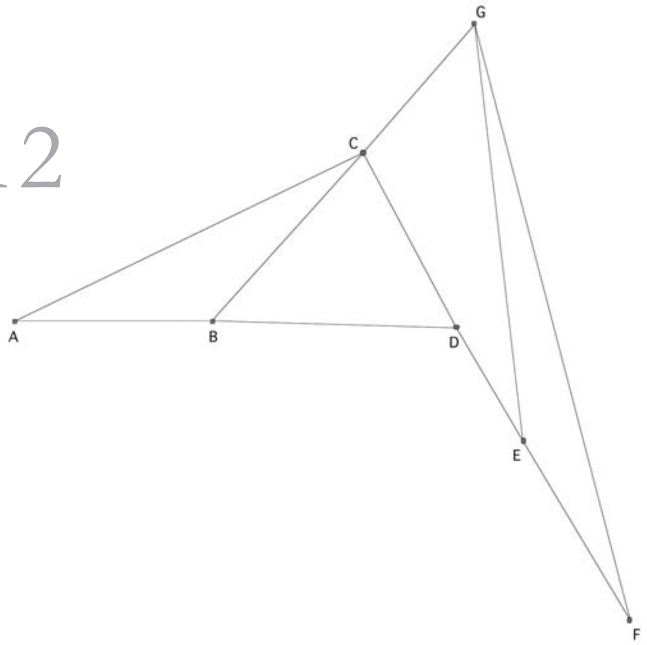
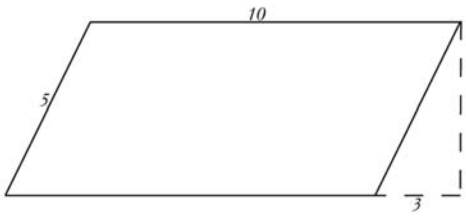


a. Explain why $\angle FEG$ must be greater than $\angle ACB$. (Hint: Think about what you know about exterior angles of a triangle)

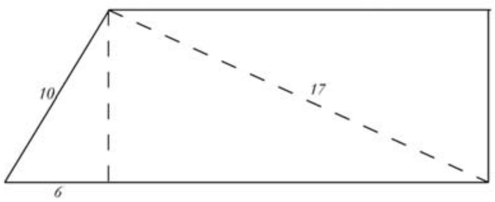
b. Explain why $\angle ABC$ must be greater than $\angle EFG$.

13 Find the perimeter and area of the following figures (Angles that look like right angles are indeed, and in c) the curve is a circular arc):

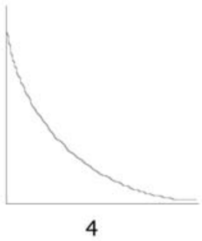
a.



b.



c.



14 For each of the following pairs of statements, determine if any "theorem" can be proven as a consequence. If not, write "No Theorem Proven".

a. Turkeys are fun to be around if it is Thanksgiving Day.
If a Turkey is fun to be around, then it must be late November.

b. Those who eat marshmallows are wise.
Drinkers of Diet Coke with Lime are foolish (that is, not wise).

15 Consider the statement "all sophomores are awesome."

a. Write the statement in if/then form.

b. Write the converse of the statement.

16 Consider the statement "You're a real Lacrosse fan if you wear spandex."

a. Write the statement in if/then form.

b. Suppose someone is a Lacrosse fan. According to the statement, can you predict they'll be wearing spandex?

17 Consider the statement "No one who is tall is handsome."

a. Write the statement in if/then form.

b. Write the converse of the statement.

c. Assuming that the original statement is true (isn't it?), does the converse have to be true as well?

18 For each of the following groups of statements, determine if any "theorem" can be proven if the given statements are assumed to be true. If not, write "No Theorem".

a. If I study over 3 hours for a math test, I'm sure to do well.
If my parents gave me a brand new shovel, I did well on my math test.

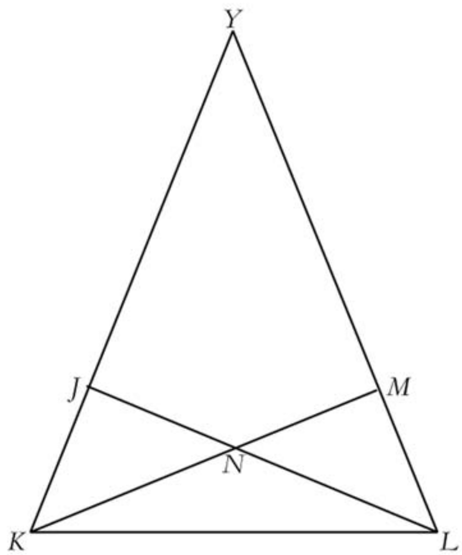
b. Potato chips taste salty if you are an astronaut.
If you don't like taffy, then potato chips don't taste salty.

c. If I have lost my marbles, then I can't play with my friends.
After dinnertime, I always watch my favorite TV show.
If it is not after dinnertime, I'm allowed to play with my friends.

d. If I don't love hats, then I am not a haberdasher.
If I not a cookieholic, then I don't eat fig newtons.
If I love hats, I like to eat fig newtons.

19 Given: $KM \perp YL$ and $JL \perp KY$
KM $\cong$ JL

Prove: $\triangle KYL$ is isosceles.



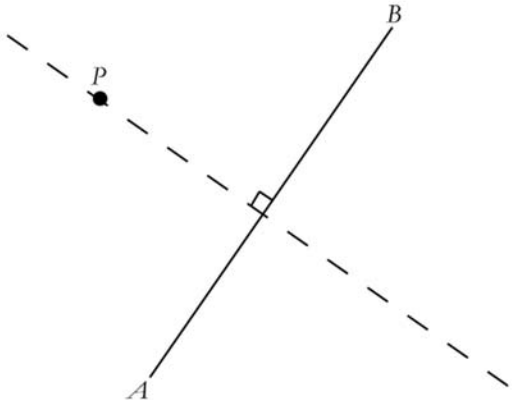
20 a. Prove that the diagonals of a rhombus are perpendicular.

b. If a quadrilateral has diagonals that bisect each of the 4 angles of the quadrilateral, does that mean this is a special kind of quadrilateral? If so, prove which special kind of quadrilateral you think it is. If not, explain why not.

c. Must a trapezoid have two pairs of consecutive angles that add up to 180 degrees?

d. Prove that the diagonals of a rectangle are congruent.

21 a. Use congruent triangles to prove that, if a point lies on the perpendicular bisector of a line segment, then the point is equidistant from the endpoints. (See figure below)

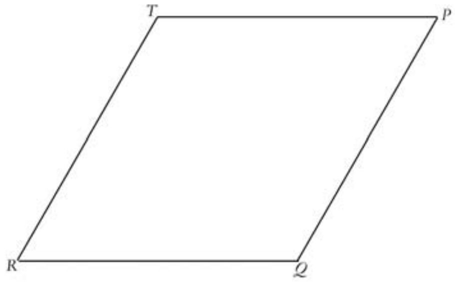


b. Draw another diagram and write a proof of the fact that, if a point is equidistant from two other points, then the point lies on the perpendicular bisector of the line segment joining those points.

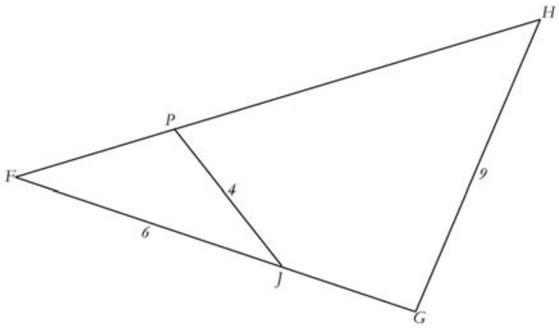
22 Say that a quadrilateral has diagonals that are congruent AND perpendicular. Must this be a special kind of quadrilateral? If so, prove which special kind of quadrilateral you think it is. If not, explain why not.

23 State two properties that each are necessary, but neither are sufficient, for a quadrilateral to be a parallelogram. Then state two properties that each are sufficient, but neither are necessary, for a quadrilateral to be a parallelogram. Finally, state two properties that, taken together, are necessary and sufficient for a quadrilateral to be a parallelogram.

24 PQRT is a rhombus. Diagonal PR = 16 and Diagonal QT = 12. Find the Area of the Rhombus.



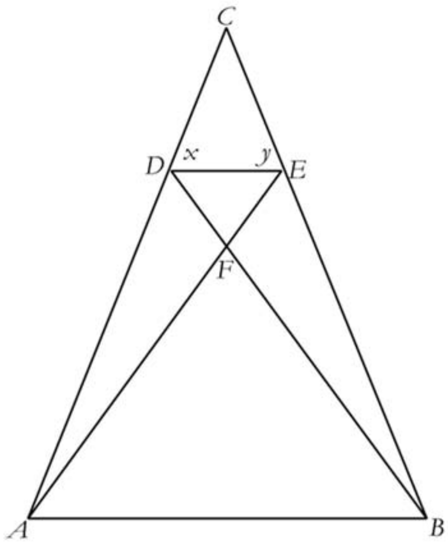
25 For the figure below, $\angle FPJ = \angle FGH$. Find the length of FH, making clear your reasoning in doing so.



26 Given: $DF = EF$ and $\angle x = \angle y$.

Prove: $\triangle AFB$ is isosceles.

Now also prove that $\angle EDF = \angle BAF$.

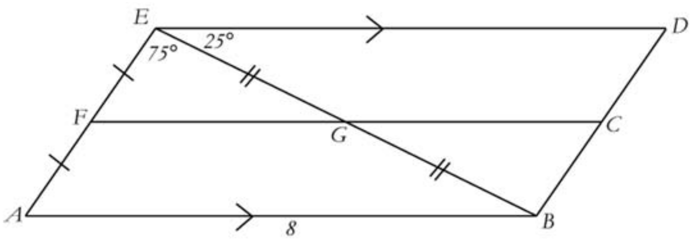


27 In the figure below, determine the following sides and angles, or say "not possible."

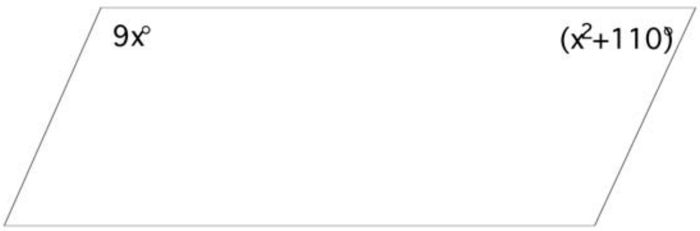
a. FG b. ED

c. $\angle EBA$ d. $\angle EBD$

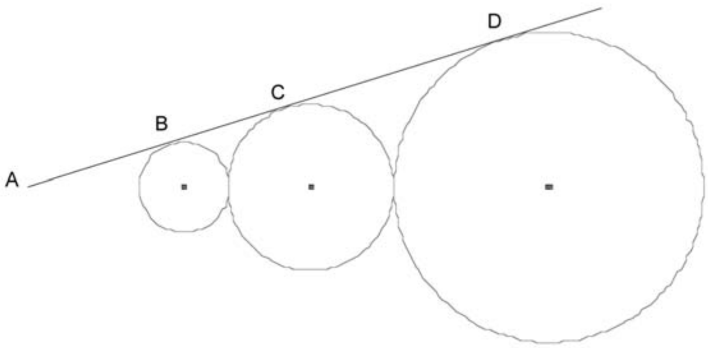
e. $\angle EGC$ f. $\angle EFG$



28 What are the possible values for the four angles in this parallelogram?

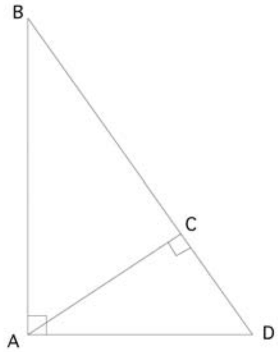


29 In the picture below, the circles have radii 1.01, 1.85, and 5.47. Their centers are collinear (lie on one straight line). Also, AB has length 3.34. Find AD.

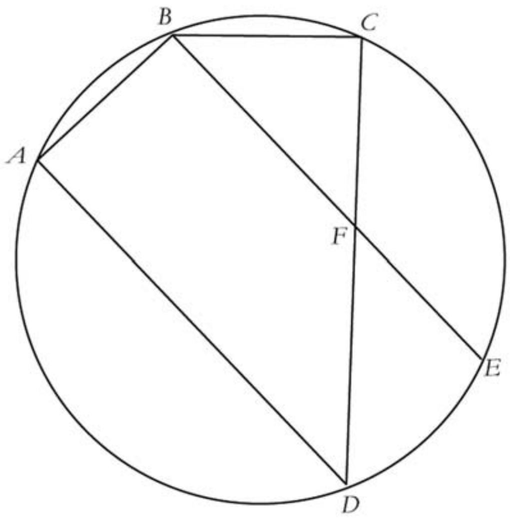


30 a. In the figure below, list as many pairs of similar triangles as you can. Why are they similar?

b. The two smaller triangles have three congruent angles and share a congruent side. Why, then, aren't they congruent?



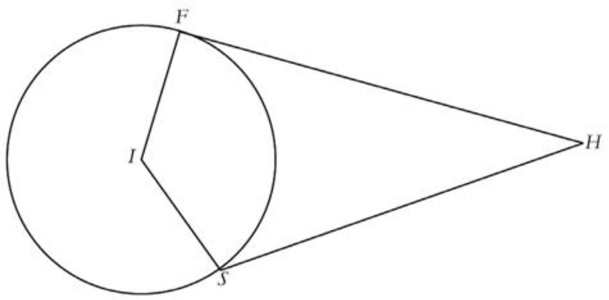
31 ABCD is a kite (with $AB = BC$). $\overset{\frown}{AB} = 40°$, $\angle ABF = 90°$, $BF = 4.9$, $FE = 3.3$, and $CF = 3.1$. Find all the remaining lengths, arc measures, and angles in the figure below. (Do not use trigonometry)



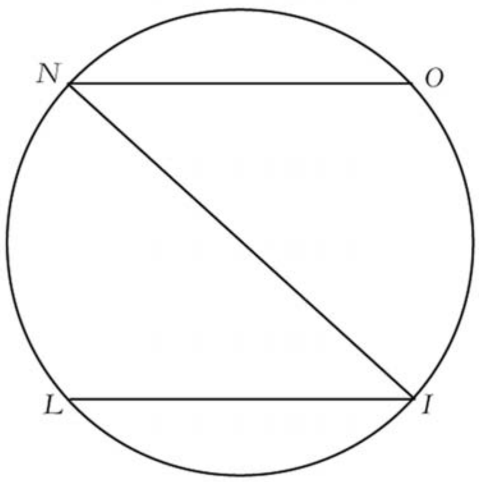
32 For each question, say whether the answer is "definitely yes," "definitely no," or "possibly." Give reasons.

a. Can you "circumscribe" (draw a circle that hits each vertex of) a quadrilateral whose angles are, in order, 70°, 80°, 130°, and 80°?

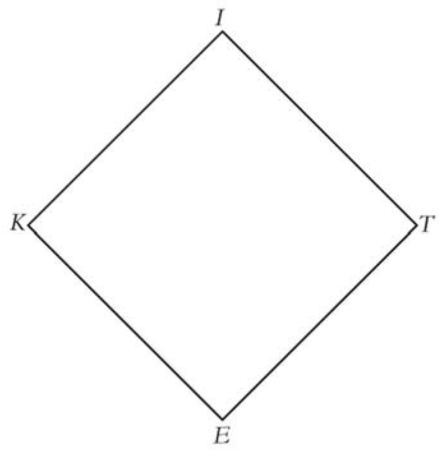
b. $\overline{FH}$ and $\overline{SH}$ are tangents and I is the center of the circle. Is FISH a kite?



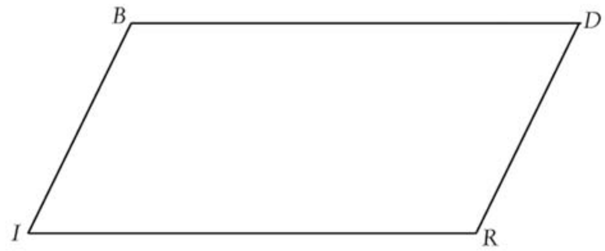
c. $\overset{\frown}{NL} \cong \overset{\frown}{OI}$. Is $\overline{LI} \parallel \overline{ON}$?



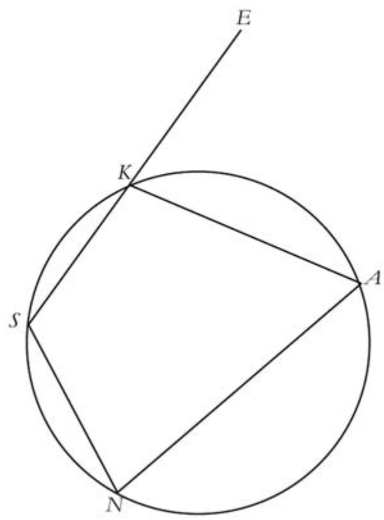
d. KITE is a kite. Is $\angle K \cong \angle T$?



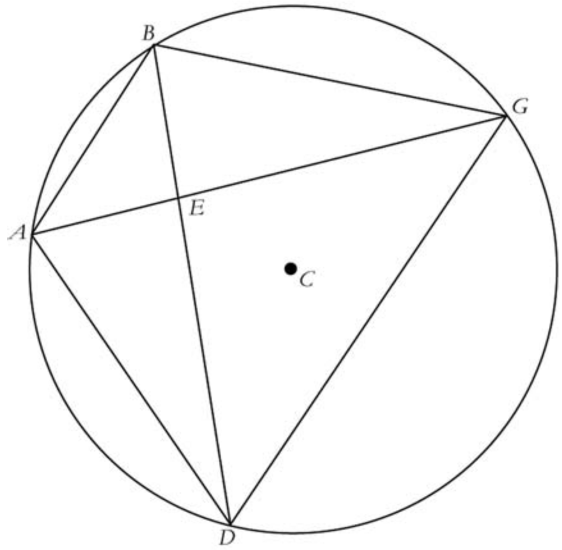
e. BIRD is a parallelogram with $BD \parallel IR$. Does BR bisect $\angle IBD$?



f. Does $\angle EKA \cong \angle SNA$?



33 In the circle (with Center C) below, $m\overset{\frown}{BA} = 66°$, $\angle BEG = 77°$, and $m\overset{\frown}{BG} = 72°$. Also, AG = 10, AE = 4, and DE = 8.



Find:

a. $\angle BDA$

b. $\angle ADG$

c. $m\overset{\frown}{GD}$

d. $\angle BGA$

e. $m\overset{\frown}{AD}$

f. BE

SELECTED ANSWERS

LESSON 1: PARALLEL LINES AND FIRST PRINCIPLES

Problem	Answer
1	All acute angles are x. All obtuse angles are y.
2	$x+y=180^\circ$
3 a	4 and 2, 5 and 7, 6 and 8
3 b	6 and 2
3 c	6 and 7
4	Answers may vary.
5	Answers may vary.
6	They add up to 180°
7	Answers may vary.
8	Answers may vary.
10	Answers may vary.
11	Answers may vary.
13	1 and 4, 2 and 5, 6 and 3
14	15, 12, 7, 9, and 4
15	For example, equal: 2 and 5, 5 and 10, 6 and 9. Supplementary: 1 and 2, 5 and 6, 15 and 16.
21	For a polygon with n sides the sum is $180^\circ(n-2)$.
22	21
23	Julia
24	360°
25	360°
27	In alphabetical order: 50°, 70°, 60°, 50°, 60°, 50°, 130°, 130°.
28	False
34	Yes
35	26
36	48
37	10
38	120
39	80
40	52.5°
41	6 and 8
42 a	$\sqrt{3}/2$
42 b	$3y^2/4$
42 c	$(x+1)(x+2)$
42 d	$\frac{x-4}{x-7}$

42 e	7
45	They have equal areas.
46	It is 4 times the initial area.
47	Twice those of the original figure.
49	Sometimes
51	Yes
52 c	No

LESSON 2: CONGRUENT TRIANGLES

Problem	Answer
1 a	One $3\sqrt{2}$, $3\sqrt{2}$, 6 triangle. Two 3, 3, $3\sqrt{2}$ triangles. Two 6, 6, $6\sqrt{2}$ triangles. One 3, $3\sqrt{2}$ parallelogram.
1 b	Identical in shape
2 a	No
2 b	No
4	She might in Case SSS
6	AAA, AAS, ASA, SSA, SAS, SSS
9	Jackson
11 a	True
11 b	False
12 a	No, No, Yes
12 b	No, Yes, Yes
14 a	8
14 b	4
14 c	8
15 a	S

15 b	C
15 c	C
15 d	S
15 e	S
15 f	C
15 g	I
15 h	S
16 a	Inconsistent
16 b	Inconsistent
16 c	Not necessarily true
17 a	They are congruent by SSS.
17 b	They are congruent.
19 a	False
19 b	False
19 c	True
19 d	False
19 e	False
19 f	False
19 g	True
20	Only case a is possible.

21 a	The first ant
21 b	The first path
22 a	Going directly from D to E.
22 b	Going directly from D to S.
22 c	Going directly from D to S.
23 a	The one with the greatest magnitude
23 b	The one with the shortest magnitude
24 a	$y > x > z$
24 b	$s > t > r$
25 a	Yes
25 b	No
25 c	Yes
25 d	No
26	6 cm and $60°$
27	25
28	11.83 approx.
29	$l > m > n$
31	Side lengths: 2.65 and 5.3. Angles: $60°$, $79.1°$, and $40.9°$ in both triangles.
33 a	They are congruent.
33 b	Yes

33 c	$c^2 + 2b\sqrt{c^2 - b^2} \Big/ 2$
34 a	$\triangle OPE \cong \triangle RPN$ $\triangle OSN \cong \triangle RSE$
34 b	$\triangle ABE \cong \triangle CBD$ $\triangle ADC \cong \triangle CEA$ $\triangle APD \cong \triangle CPE$
34 d	Yes
35 a	$\overline{BG}$
35 c	No
36 a	5.5
36 b	Yes
37 a	39° and 13
37 b	They are the same.
37 c	Total: $39.26 approx.
39 a	108°
39 b	180°
40 a	$\sqrt{3}/4$
40 b	For side length n: # of tiles $2n^2$ and area $n^2\sqrt{3}/2$
40 c	128
41	9.66 approx.
42 a	2
42 b	a^2c^5/bd^2
42 c	$3\sqrt{3}/4$

42 d	$y = \frac{1}{3}x + 7$
42 e	$(26/15, 2/9)$
43	1055.06 approx.
44 a	$3/4$
44 c	3
44 d	150
44 e	$61/121$
45 a	$(5,2)$
45 b	$(-2,2)$
46 a	7.5
46 b	46.04 square units (the top is open)
46 c	22.5 cubic units
47	16 minutes, 40 seconds
48	80°
50 b	13.75 km from A
51 b	$\triangle ADC \sim \triangle ACB \sim \triangle CDB$
52 a	$\text{area}(c_0 2) = \text{area}(c_0 1) + \text{area}(c_0 3)$
52 b	Congruent
52 c	Area of original triangle plus area at Stage 0

52 d	In each of the 3 triangles at Stage 1 the altitude to the hypotenuse split the triangle into two triangles. At Stage 2, a pair of triangles congruent to these two triangles is added by each triangle at Stage 1.
52 e	Area at Stage 1
52 f	From Stage 2 on, the area of the previous stage
54 a	x and z
54 b	They are congruent.
54 c	$z + x + y$
56 a	60° and 120°
56 b	z, x, and y
58	Yes
59	Yes
60 a	Area: $n \tan(\pi/n)$ Perimeter: $2n \tan(\pi/n)$
60 b	Yes
60 c	Yes
60 d	It becomes circular.
60 e	πr^2 and $2\pi r$

LESSON 3: GEOMETRIC PROOF

Problem	Answer
9 a	$AD = DB$, $AE = EC$
16 a	False
16 b	True
16 c	True
16 d	False
17	Side lengths: 5.2 and 2 Angles: 42° and 15°
18	20°, 140°
19	29
21	$\triangle XUZ \cong \triangle XQY$
28 a	No
28 b	Yes
28 c	Yes
29 a	Multiple
29 b	Exactly one
29 c	Exactly one
29 d	It might not exist.
29 e	Exactly one
29 f	It might not exist.

31	Square, kite, rhombus
34	Side lengths: 5, 5, 4 Most-right triangle's angles: 89°, 50°, 41°
35	10.5
36	Yes
38 b	$\sqrt{51}$
39	9
40	1, 2, $4\pi/3$, $8\pi/3$
42 a	y^2/x^5
42 b	$25x^2/16$
42 c	$x/(2y-3x)$
42 d	$(x+1)/(x-1)$
42 e	-1729, and 1
47	Possible

LESSON 4: LOGIC AND GEOMETRY

Problem	Answer
1	The potatoes in this dish are old.
2 a	If a person is savvy, then s/he is a businessperson.
4	Statement: If a quadrilateral is a parallelogram, then both pairs of opposite sides are congruent. Converse: If a quadrilateral has both pairs of opposite sides congruent, then it is a parallelogram.
5	Given: ABCD is a quadrilateral; $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{DA}$. To prove: ABCD is a parallelogram.
7	Necessary
8	Necessary and sufficient
9	No. Yes.
10	Yes
11	No
12	Yes
13 a	Yes
13 b	No
14	No. Yes. Yes.
15 a	False

15 b	True
15 c	False
15 d	#1: blue, not red, not hollow, not cube. #2: hollow, not blue, not red, not cube. #3: hollow, cube, not red, not blue.
16 a	No
16 b	Yes
16 c	No
16 d	Yes
18	No babies can manage a crocodile.
19 a	If a shape is a circle, then it is not a parallelogram.
19 c	Converse: If a shape is not a parallelogram, then it is a circle. Inverse: If a shape is not a circle, then it is a parallelogram. Contrapositive: If a shape is a parellelogram, then it is not a circle.
20 a	Yes
21 a	If you draw a line from the center of a circle to the midpoint of a chord on that circle, then the line will be perpendicular to the chord.
22 a	They are converses of one another.
24 a	They argue that it is necessary
24 b	He argues that it is not sufficient.

27 c	Yes
28	$\frac{2}{11}$
29 c	Both
33 a	No: you don't need *a* to be odd.
33 b	John's condition is sufficient, but not necessary.
34 a	Yes.
34 b	Both necessary and sufficient.
38	If you are on Park's basketball team, then you can't go on the Pirate ride at Six Flags.
39	If you live in Baltimore, then you like baseball.
40	No conclusion.
41	No conclusion.
42 a	$2a$
42 b	$(x-4)(x+7)$
42 c	5
42 d	$x^3+6x^2+12x+8$
42 e	$\frac{w^6y^4}{x^{12}}$
43	If Jean bought groceries on her way home, then she didn't commit the murder.
44	If you study for hours, then you are not a brilliant mathematician.
45	No conclusion.

46	None of your poems are popular among people of real taste.

LESSON 5: BUILDING UP MORE THEOREMS

Problem	Answer
1	---
2	---
3	180 mm; 270 mm; 150 mm
4	$135°$
5	$157.5°$
6	$313°$
7	---
8	It doesn't.
9	---
10 a	$2x$
10 b	---
11	---
12	No
13 a	$120°$
13 b	$60°$
13 c	$120°$
13 d	It's half of it, so, $60°$
14 a	Angles in the figure are $148°$ and $32°$; Arcs in the figure are also $148°$ and $32°$
14 b	Angles in the figure are $130°$ and $50°$; Arcs in the figure are also $130°$ and $50°$
14 c	Angles in the figure are $5°$, $30°$ and $35°$; Arcs in the figure are $70°$ and $170°$
14 d	Angles in the figure are $57.5°$, $72.5°$, and $107.5°$; Arcs in the figure are $130°$ and $30°$
15	They are equal
16	---
17	$x = 12$ or $x = -10$
18	$x = 6$ or $x = -4$
19	$130°$
20 a	$90°$
20 b	6π
20 c	36π
21 a	$\frac{434}{15}\pi \approx 28.933\pi \approx 90.897$
21 b	$\frac{3038}{5}\pi = 607.6\pi \approx 1908.832$
22 a	$\frac{\theta}{360°} \cdot 2\pi r$

22 b	$\frac{\theta}{360°} \cdot \pi r^2$
23 abcd	---
24 a	Yes.
24 b	---
25 a	68.0625π
25 b	$.75625\pi$
25 c	40.459π
25 d	$\frac{n}{360°} \cdot \pi(16.5)^2$
26	---
27	---
28 ab	---
29	---
30 a	4.5486 inches
30 b	6.468 inches
31	Yes. For example, if $r = \frac{2}{\sqrt{\pi}}$.
33	No
34	A right triangle
35	PB · PA = PD · PC

36 and 37	A path that is equidistant from the sides of an angle is that angle's bisector. The points along an angle bisector are equidistant from the sides of the angle.
38	*Hint:* Find a point that is equidistant from the sides of the angle.
39a	Isosceles
41	$m\angle ABH = 90°$; $m\angle BHC = 140°$; $m\angle HCA = 90°$
42 a	$x = 2$
42 b	$6\sqrt{2}$
42 c	(3.5, 3)
42 d	$x = \sqrt{11} - 3$ or $x = -\sqrt{11} - 3$
42 e	xy = 1
44	$m\angle UBK = 130°$; $m\angle BKC = 150°$; $m\angle KCU = 50°$
46	x = 7 or -5
48	2; none; 1
51	$m\angle A = 90°$; $m\angle B = 67.4°$; $c = 12$; $m\angle D = 33.7°$; $e = 5$; $f = 5$
52	$a = 5$; $m\angle B = 35°$; $m\angle C = 110°$; $m\angle D = m\angle E = 55°$; $m\angle F = 70°$; $g = 7.14$; $m\angle H = 55°$

53	$\dfrac{7}{\sqrt{2}}$
55	If n is odd, 0. If n is even, $\dfrac{3}{n-1}$.
56 a	It is equidistant.
57	23
58	*Hint:* Use geometric tinkering and draw in some lines.
65	*Hint:* Use geometric tinkering and draw in some lines.
66	*Hint:* Can you create similar triangles?

SUMMARY & REVIEW

Problem	**Answer**
1 a	$\triangle RQS \cong \triangle CBA$ by S.A.S. Congruence
1 b	Because $\angle Y = 55°$, $\triangle TUV \cong \triangle WYX$ by S.A.S. Similarity
1 c	$\angle DCE \cong \angle BCA$ because they are vertical angles; so $\triangle DCE \cong \triangle BCA$ by A.A. S. Congruence.
1 d	$\triangle FGE \sim \triangle EGH$ by A.A. Similarity; note that they are not congruent by A.A.S. because side EG, which is the shared side, is not opposite corresponding angles in the triangles—i.e. it is opposite the $20°$ angle in one and opposite the $30°$ angle in the other.
2 a	$\angle DCZ \cong \angle BAZ, \angle TAZ \cong \angle QCZ$
	No… they look like alternate interior angles, but since there is a "crook"at W they are not really connected by a transversal.
2 c	1) $\angle UCQ$ and $\angle ZAB$, or $\angle QCZ$ and $\angle BAV$ 2) $\angle UCQ$ and $\angle TAV$, or $\angle UCR$ and 3) $\angle YDR$ and $\angle CDW$, or some other pair of vertical angles.

2 d	For example $\angle RCZ \cong \angle ZAT$
2 e	$\angle BZA, \angle ZAB, \angle ABZ$, and $\angle CZW, \angle SBX, \angle TAV$
3	$\angle ACB \cong \angle ABC$ (Given) $AC \cong AB$ (If 2 angles in a $\triangle$ are =, the opposite sides are =) $DC \cong AB$ (Given) $DC \cong AC$ (Substitution) $\angle ADC \cong \angle CAD$ (If 2 sides in a $\triangle$ are =, the opposite angles are =) $\angle FDA = 180 - \angle ADC$, $\angle GAD = 180 - \angle CAD$ (Two angles that make up a straight line add to 180) $\angle FDA \cong \angle GAD$ $(180 - \angle ADC = 180 - \angle CAD$ (as $\angle ADC \cong \angle CAD$))
4 a	If Zardon's wife finds him attractive, then he didn't eat sardines for lunch.
4 b	Nothing can be concluded—the converse is not necessarily true.
4 c	No, for the reason given in part b.
4 d	He ate sardines for lunch.
5	If the diagonals bisect each other, then $DE = EB$ and $AE = EC$. By vertical angles, $\angle AED = \angle CEB$, and $\angle AEB = \angle CED$. So, by S.A.S Congruence, $\triangle AED \cong \triangle CEB$ and also $\triangle AEB \cong \triangle CED$. Since Clones have identical body parts, $\angle EAB = \angle ECD$ and $\angle DAE = \angle BCE$. But this means that AB is parallel to DC, because if alternate interior angles are congruent, the lines are parallel. For the same reason, AD must be parallel to BC, and so ABCD has been proven to be a parallelogram.
6	Angle FGH = Angle IHG because FG and HI are parallel and where GH is the transversal. So triangle FGH and triangle IHG are congruent by ASA. So FH=IG by CPCTC.
7	$\angle JMK = \angle JNL$ (Given) KM is parallel to LN (If corresponding angles are =, the lines are parallel) $\angle JKP = \angle JLO$ $\angle JPK = \angle JOL$ (If lines are parallel, corresponding angles are =) $\triangle JKP \sim \triangle JLO$ (A.A. Similarity) $\frac{KP}{KJ} = \frac{LO}{LJ}$ (Similar Triangles are proportional)

8	All lengths are 15.
9 a	$\triangle ABC \sim \triangle DAC$, $\triangle ABD \sim \triangle CAD$, $\triangle ABD \sim \triangle CBA$.
9 b	The pairs of angles that are congruent aren't corresponding angles.
10 a,b	Review your class notes.
11 a	$x = 7,\ \ y = -8,\ \ z = 13$
11 b	$x = 46,\ \ y = 14$
11 c	$x = 80,\ \ y = 100,\ \ z = 80$
12	The basis of this problem is to realize that since the exterior angle of a triangle is equal to the sum of the two remote interior angles, it therefore MUST be GREATER than either one of the individual remote interior angles. So for parts a and b, an exterior angle is greater than a remote interior angle, but in turn that remote interior angle is in fact also an exterior angle to another triangle, so it is greater than that other triangle's remote interior angles, and so on…
12 a	$\angle FEG > \angle GCE > \angle CBD > \angle ACB$
12b	$\angle ABC > \angle DCB > \angle GEC > \angle EFG$
13 a	Perimeter = 30, Area = 40
13 b	Perimeter = 54, Area = 144

13 c	Perimeter = 14.28, Area = 3.43
14 a	If it is Thanksgiving Day, then it must be late November.
14 b	If you eat marshmallows, then you do not drink Diet Coke with Lime.
15 a	If you are a sophomore, then you are awesome.
15 b	If you are awesome, then you are a sophomore.
16 a	If you wear spandex, then you're a real Lacrosse fan.
16 b	No. The converse of a statement is not necessarily true.
17 a	If someone is tall, then they are not handsome.
17 b	If someone is not handsome, then they are not tall.
17 c	No!
18 a	No Theorem.
18 b	If you are an astronaut, then you like taffy. (Or: If you don't like taffy, then you are not an astronaut.)
18 c	If I have lost my marbles, then I always watch my favorite TV show. (Or: If I don't watch my favorite TV show, then I have not lost my marbles)

18 d	If I'm a haberdasher, then I'm a cookieholic. (Or: If I'm not a cookieholic, then I'm not a haberdasher)
19	$\angle KMY$ and $\angle LJY = 90°$ (Given) $\angle Y = \angle Y$ (Obvious , A.K.A. "Reflexive") $KM \cong JL$ (Given) $\triangle KMY \cong \triangle LJY$ (A.A.S) $KY \cong LY$ (CPCTC) $\triangle KYL$ is isosceles (Triangles that have two equal sides are isosceles)
20 a	-----
20 b	It's a rhombus.
20 c	Yes, because the each transversal will generate two angles that must add up to 180 degrees.
20 d	-----
21	-----
22	It doesn't have to be anything special. You can draw pictures of quadrilaterals that fit this condition that don't fit any special category.

23	There are many possible answers. Here are some possibilities. a)A pair of opposite sides is parallel. A pair of opposite sides is congruent. b)All angles equal 90 degrees. All sides are congruent. c)A pair of opposite sides is parallel and congruent.
24	96
25	13.5
26 a	Since DF=EF, angle EDF and angle FED are equal. This means that angle CDF and CEF are equal. And this means that angle FDA and angle FEB are equal, since when added to angle CDF and angle CEF they add up to 180 degrees. Angles DFA and EFB are equal by vertical angles. Thus, triangle AFD and triangle BFE are congruent by A.S.A. So AF=FB by CPCTC. This means that triangle AFB is isosceles.
26 b	-----
27	The key to this problem is to understand that quadrilateral AEDB may not be a parallelogram; ED may be longer or shorter than AB, and BD need not be parallel to AE.
27	4
27 b	Not Possible

27 c	25°
27 d	Not Possible
27 e	155°
27 f	80°
28	45 degrees and 135 degrees
29	Approximately 12.53
30 a, b	See answers to 9 a and b.
31	DF= 5.2, AD= 8.3, angle ADC = 40 degrees, angle BAD and BCD are 90 degrees, arc BC = 40 degrees, angle CBF = 50 degrees, angle CFB = 40 degrees, angle BFD is 140 degrees, chords AB and BC are 3.79 (part of right triangles), arc AD is 140 degrees, arc DE is 40 degrees, arc CE is 100 degrees.
32 a	No. Opposite angles must add up to 180 degrees in an inscribed quadrilateral.
32 b	Yes. Radii are equal, as are the external tangents.
32 c	Yes. Angle NIL must equal angle ONI, as they are inscribed in equal arcs. But they are alternate interior angles to transversal NI, so LI is parallel to ON.
32 d	Possibly. Draw a random kite; are **both** pairs of opposite angles congruent?

32 e	Possibly. Only if the parallelogram is a rhombus.
32 f	Yes. Opposite angles in an inscribed quadrilateral add up to 180 degrees (Can you prove this fact? You should be able to!), so angle SNA and angle SKA add up to 180. But angle SKA and angle EKA add up to 180 as well. So SNA and EKA must be equal.
33 a	33°
33 b	69°
33 c	140°
33 d	33°
33 e	82°
33 f	3