HURWITZ THEORY, ORBIFOLD GW THEORY AND APPLICATIONS

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CONTENTS

1. INTRODUCTION

Rough version of the notes!

These notes are intended to accompany the mini-course on Hurwitz theory, Orbifold-Witten theory and applications to at BCIMR in Beijing in July 2012. The goal of this course is to introduce a circle of ideas in and around Hurwitz theory to an audience of students that have an interest in geometry/ algebraic geometry but have not necessarily had yet a prolong or in depth exposure to these areas. Hurwitz theory studies holomorphic/algebraic maps between Riemann surfaces/algebraic curves. This is a classical subject of study that over the course of the last century and a half has made contact and had applications in several areas of mathematics, from algebraic/arithmetic/differential geometry to representation theory, from combinatorics to the theory of integrable systems. This mini-course intends to present an aspect of Hurwitz theory which applies to the study of the geometry of moduli spaces of curves, and of maps. Major goals of this course are:

(1) To establish in an effective way a modern dictionary between the classical theory of ramified covers of curves and the study of orbifold maps to classifying spaces (and more general of global quotients).
(2) To describe the interaction between Hurwitz numbers/loci and tautological classes in the moduli space of curves.
(3) To present the algebraic/combinatorial structure of certain families of Hurwitz numbers/loci.
(4) To illustrate Atyiah-Bott localization as an effective tool for the computation of Hurwitz-Hodge integrals.

The mini-course will assume a basic geometric/algebraic background. We will start by quickly reviewing the fundamental facts of Riemann surface/curve theory that will be needed throughout the course. We will introduce the concept of a moduli space and describe a few example of moduli spaces related to curves (moduli space of curves, of (relative) stable maps, of admissible covers). We will introduce Atyiah-Bott localization as a tool to approach the intersection theory on moduli spaces of maps endowed with a torus action. As an application we well compute some generating functions for Hodge integrals on the hyperelliptic locus, and we will prove the ELSV formula, establishing a connection between simple Hurwitz numbers and Hodge integrals on the moduli space of curves.

Next we will study the case of double Hurwitz loci/numbers, i.e. maps to the projective line with two special (non generic) branch points. Such numbers/loci exhibit interesting algebraic structure: families of such loci are piecewise polynomial in the special ramification profiles, with modular wall crossing formulas. The study of these properties can be approached via a translation to algebraic combinatorics involving graph counting. Again, the remarkable algebraic structure of these classes suggests interesting connections to the tautological intersection theory of the moduli space of curves, or of some related moduli spaces. At the end of the course we hope to have the time to discuss some open problems, conjectures and observations on the matter.

This mini-course is going to be very much related, and yet somewhat complementary, to the mini-course by Paul Johnson that will follow.

2. Classical Hurwitz Theory

2.1. **Curves/Riemann Surfaces 101.** In this section we recall some basic facts in the theory of algebraic curves and Riemann Surfaces. There are several excellent references that can be looked at, for example [ACGH85], [Mir95], or [HM98].

The object of our study can be viewed equivalently as algebraic of complex analytic objects. It is very useful to keep in mind this equivalence.

Definition 2.1 (for algebraic geometers)**.** A (projective) **curve** is equivalently:

- a projective algebraic variety (over the complex numbers) of dimension 1.
- a field extension of $\mathbb{C}$ of transcendence degree 1.

Note: For a passionately pure algebraic geometer there is no need to have $\mathbb{C}$ as the ground field. Most features of the theory will hold over k an algebraically closed field of characteristic 0. Many surprises make the day of arithmetic geometers electing to work over finite fields or fields of positive characteristics. Here we do not dare to venture into this mysterious yet fascinating territory.

Definition 2.2 (for complex analysts)**.** A (compact) **Riemann Surface** is a compact complex analytic manifold of dimension 1.

We abuse of notation and allow Riemann Surface to have nodal singularities. It is a remarkable feature of the theory that we do not need to consider any worse type of degenerations of the smooth objects to have compact moduli spaces.

Exercise 1. A Riemann Surface is orientable. Check that the Cauchy-Riemann equations imply that any holomorphic atlas is a positive atlas.

Topologically a smooth Riemann surface is just a connected sum of g tori. The number g, the genus, is an important discrete invariant. Simple things become extremely confusing when one starts to deal with nodal or disconnected curves, so we spell out once and for all the relevant definitions.

Definition 2.3.

(1) If X is a smooth curve, the **genus** of X is equivalently:
   - the number of holes/handles of the corresponding topological surface.
   - $h^0(X, K_X)$: the dimension of the space of global sections of the canonical line bundle.
   - $h^1(X, \mathcal{O}_X)$.

(2) If X is a nodal, connected curve, the **geometric genus** of X is the genus of the normalization of X (i.e. the genus of the smooth curve obtained by pulling the nodes apart).

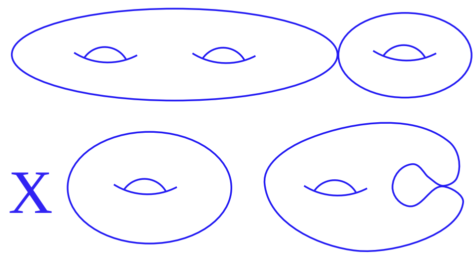


Figure 1. A (disconnected) curve of arithmetic genus 4 and geometric genus 5.

(3) If X is a nodal, connected curve, the **arithmetic genus** of X is $h^1(X, \mathcal{O}_X) = h^0(X, \omega_X)$ (i.e. the genus of the curve obtained by smoothing the node).
(4) If X is a disconnected curve, the **geometric genus** is the sum of the genera of the connected components.
(5) If X is a disconnected curve, the **arithmetic genus**

$$g := 1 - \chi(\mathcal{O}_X) = 1 - h^0(X, \mathcal{O}_X) + h^1(X, \mathcal{O}_X).$$

In other words, one subtracts one for every additional connected component beyond the first.

The arithmetic genus is constant in families, and therefore we like it best. Unless otherwise specified genus will always mean arithmetic genus. See figure 2.1 for an illustration.

Exercise 2. Check that all definitions are consistent when X is a smooth connected curve.

Fact. If L“ = ”$\mathcal{O}(D)$ is a line bundle (or invertible sheaf) on a smooth curve X, then:

Serre duality for curves:

$$H^i(X, \mathcal{O}_X(D)) \cong H^{1-i}(X, \mathcal{O}_X(K_X - D))^\vee$$

Riemann-Roch theorem for curves:

$$h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g$$

or equivalently:

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = c_1(L)$$

Exercise 3. For X a smooth connected curve, check that K_X has degree $2g - 2$.

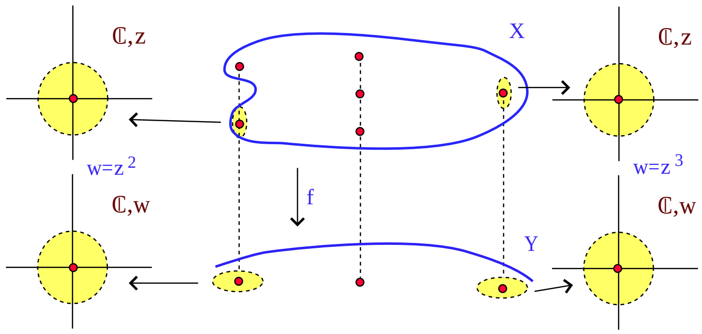


FIGURE 2. A ramified cover of degree 3.

2.2. **Maps of Curves.** Of the several excellent references for this section, my favorite is [Ful95]. It is a simple fact from complex analysis that any map of Riemann Surfaces can be given local expression $z \mapsto z^k$, with $k > 1$ only at a finite number of points.

Definition 2.4. A map $f : X \to Y$ of Riemann Surfaces is a **ramified cover**(see Figure 2):

- $B \subset Y$ is a finite set called **branch locus**;
- $f_{|X \setminus f^{-1}(B)} : f^{-1}(B) \to Y \setminus B$ is a degree d topological covering map;
- for $x \in f^{-1}(B)$ the local expression of f at x is $F(z) = z^k$; the number $k := r_f(x)$ is the **ramification order** of f at x.
- $R \subseteq f^{-1}(B) := \{x \in X \ s.t. \ r_f(x) > 1\}$ is called the **ramification locus**.

Viceversa, every branched cover identifies a unique map of Riemann Surfaces:

Fact (Riemann Existence Theorem)**.** If Y is a compact Riemann Surface and $f^\circ : X^\circ \to Y \setminus B$ a topological cover, then there exist a unique smooth compact Riemann Surface X, obtained from the topological surface X° by adding a finite number of points and a unique map f of Riemann Surfaces extending f°.

Finally, a theorem which is fundamental for us and relates the various discrete invariants of curves and maps introduced so far.

Theorem 2.5 (Riemann-Hurwitz)**.** *For a map of smooth Riemann Surfaces $f : X \to Y$:*

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X} (r_f(x) - 1). \tag{1}$$

Exercise 4. Prove the Riemann Hurwitz theorem in two ways:

(1) Topologically: compute the euler characteristic of X by lifting a triangulation on Y where the branch locus is contained in the set of vertices of the triangulation.
(2) Analytically/Algebro Geometrically: compute the degree of the divisor of the pullback via f of a meromorphic one-form on Y. Note that this gives the degree of K_X.

Definition 2.6. Let $f : X \to Y$ be a map of Riemann Surfaces, $y \in Y$, $\{x_1, \ldots, x_n\} = f^{-1}(x)$, then the (unordered) collection of integers $\{r_f(x_1), \ldots, r_f(x_n)\}$ is called the **ramification profile** of f at y. We think of this set as a partition of d and denote it $\eta(v)$ (or simply η). If $\eta(y) = (2, 1, \ldots, 1)$, then f has **simple ramification** over y.

We are ready for our first definition of Hurwitz numbers.

Definition 2.7 (Geometry)**.** Let $(Y, p_1, \ldots, p_r, q_1, \ldots, q_s)$ be an $(r+s)$-marked smooth Riemann Surface of genus h. Let $\underline{\eta} = (\eta_1, \ldots, \eta_s)$ be a vector of partitions of the integer d. We define the *Hurwitz number*:

$$H^r_{g\to h,d}(\underline{\eta}) \quad := \quad \text{weighted number of} \left\{ \begin{array}{c} \textit{degree } d \textit{ covers} \\ X \xrightarrow{f} Y \textit{ such that}: \\ \bullet \text{ X } \textit{is connected of genus } \text{g}; \\ \bullet\ f \textit{ is unramified over} \\ X \setminus \{p_1, \ldots, p_r, q_1, \ldots, q_s\}; \\ \bullet\ f \textit{ ramifies with profile } \eta_i \textit{ over } q_i; \\ \bullet\ f \textit{ has simple ramification over } p_i; \\ \circ \textit{ preimages of each } q_i \textit{ with same} \\ \textit{ramification are distinguished by} \\ \textit{appropriate markings.} \end{array} \right\}$$

Each cover is weighted by the number of its automorphisms.

Figure 3 might help visualize the features of this definition.

Remarks:

(1) For a Hurwitz number to be nonzero, r,g, h and $\underline{\eta}$ must satisfy the Riemann Hurwitz formula (1). The above notation is always redundant, and it is common practice to omit appropriate unnecessary invariants.
(2) The last condition $\circ$ was recently introduced in [GJV03], and it is well tuned to the applications we have in mind. These Hurwitz numbers differ by a factor of $\prod \mathrm{Aut}(\eta_i)$ from the classically defined ones where such condition is omitted.
(3) One might want to drop the condition of X being connected, and count covers with disconnected domain. Such Hurwitz

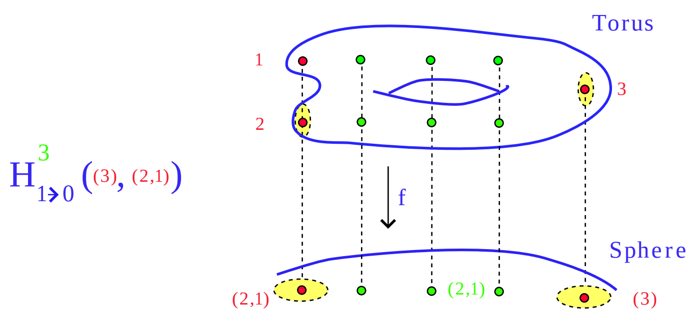


FIGURE 3. The covers contributing to a given Hurwitz Number.

numbers are denoted by $H^\bullet$ (To my knowledge Okounkov and Pandharipande started the now common convention of using $\bullet$ to denote a disconnected theory).

Example 2.8.

- $$H^0_{0\to 0,d}((d),(d)) = \frac{1}{d}$$
- $$H^4_{1\to 0,2} = \frac{1}{2}$$
- $$H^3_{1\to 0,2}((2),(1,1)) = 1$$

This is a very beautiful geometric definition, but it is extremely impractical. A reasonably simple Hurwitz number such as $H^4_{1\to 0,3}((3))$ is already out of our reach.

2.3. **Representation Theory.** The problem of computing Hurwitz numbers is in fact a discrete problem and it can be approached using the representation theory of the symmetric group. A standard reference here is [FH91].

Given a branched cover $f : X \to Y$, pick a point y_0 not in the branch locus, and label the preimages $1, \dots, d$. Then one can naturally define a group homomorphism:

$$\begin{array}{cccc} \varphi_f : & \pi_1(Y \setminus B, y_0) & \to & S_d \\ & \gamma & \mapsto & \sigma_\gamma : \{i \mapsto \tilde{\gamma}_i(1)\}, \end{array}$$

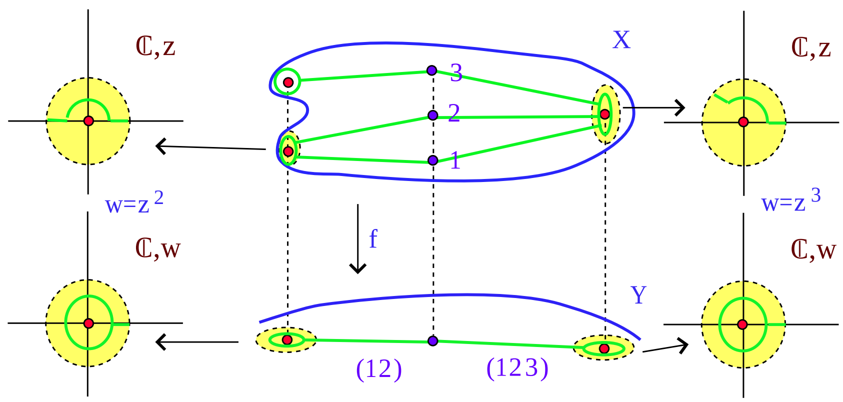


FIGURE 4. Sketch of the construction of the monodromy representation for the cover f.

where $\tilde{\gamma}_i$ is the lift of γ starting at i ($\tilde{\gamma}_i(0) = i$). This homomorphism is called the **monodromy representation**, and its construction is illustrated in Figure 4.

Remarks:

(1) A different choice of labelling of the preimages of y_0 corresponds to composing φ_f with an inner automorphism of S_d.

(2) If $\rho \in \pi_1(Y \setminus B, y_0)$ is a little loop winding once around a branch point with profile η, then σ_ρ is a permutation of cycle type η.

Viceversa, the monodromy representation contains enough information to recover the topological cover of $Y \setminus B$, and therefore, by the Riemann existence theorem, the map of Riemann surfaces. To count covers we can count instead (equivalence classes of) monodromy representations. This leads us to the second definition of Hurwitz numbers.

Definition 2.9 (Representation Theory)**.** Let $(Y, p_1, \ldots, p_r, q_1, \ldots, q_s)$ be an $(r+s)$-marked smooth Riemann Surface of genus g, and $\underline{\eta} = (\eta_1, \ldots, \eta_s)$ a vector of partitions of the integer d:

$$H^r_{g\to h,d}(\underline{\eta}) := \frac{|\{\underline{\eta}\text{-monodromy representations } \varphi^{\underline{\eta}}\}|}{|S_d|} \prod \mathrm{Aut}\eta_i, \tag{2}$$

where an $\underline{\eta}$-monodromy representation is a group homomorphism

$$\varphi^{\underline{\eta}} : \pi_1(Y \setminus B, y_0) \to S_d$$

such that:

• for ρ_{q_i} a little loop winding around q_i once, $\varphi^{\underline{\eta}}(\rho_{q_i})$ has cycle type η_i.

• for ρ_{p_i} a little loop winding around p_i once, $\varphi^{\underline{\eta}}(\rho_{p_i})$ is a transposition.
$\star$ $\mathrm{Im}(\varphi^{\underline{\eta}}(\rho_{q_i}))$ acts transitively on the set $\{1, \ldots, d\}$.

Remarks:

(1) To count disconnected Hurwitz numbers just remove the last condition $\star$.
(2) Dividing by $d!$ accounts simoultaneously for automorphisms of the covers and the possible relabellings of the preimages of y_0.
(3) $\prod \mathrm{Aut}\eta_i$ is non-classical and it corresponds to condition $\circ$ in Definition 2.7.

Exercise 5. Check with this definition the Hurwitz numbers in Example 2.8. Compute $H^4_{1\to 0,3}((3)) = 9$ and $H^4_{0\to 0,3} = 4$.

2.4. $h = 0$**, Disconnected, Unlabelled.** We restrict our attention to the target genus 0, disconnected theory, where the connection with representation theory can be carried even further. Also, it is more convenient to work with the classical definition of Hurwitz numbers, so in this section we drop condition $\circ$ of Definition 2.7. In this case Definition 2.9 can be reformulated:

$$H^\bullet_g(\underline{\eta}) = \frac{1}{d!}|\{(\sigma_1, \ldots, \sigma_s, \tau_1, \ldots, \tau_r) \ s.t. \ \sigma_1 \ldots \sigma_s \tau_1 \ldots \tau_r = Id\}|, \tag{3}$$

where:

- σ_i has cycle type η_i;
- τ_i is a transposition.

Equation (3) recasts the computation of Hurwitz numbers as a multiplication problem in the class algebra of the symmetric group. Recall that $\mathcal{Z}(\mathbb{C}[S_d])$ is a vector space of dimension equal the number of partitions of d, with a natural basis indexed by conjugacy classes of permutations.

$$\mathcal{Z}(\mathbb{C}[S_d]) = \bigoplus_{\eta \vdash d} \mathbb{C} \ C_\eta,$$

where

$$C_\eta = \sum_{\sigma \in S_d \text{ of cycle type } \eta} \sigma.$$

We use $|C_\eta|$ to denote the number of permutations of cycle type η. We also use the notation $C_{Id} = Id$ and $C_\tau = C_{(2,1^{d-2})}$. Then the Hurwitz number is the coefficient of the identity in the appropriate product of elements of the class algebra:

$$H^\bullet_g(\underline{\eta}) = \frac{1}{d!}[C_{Id}] C_{\eta_1} \cdot \ldots \cdot C_{\eta_s} \cdot C_\tau^r. \tag{4}$$

It is a classical fact that $\mathcal{Z}(\mathbb{C}[S_d])$ is a semisimple algebra with semisimple basis indexed by irreducible representations of S_d, and change of bases essentially given by the character table:

$$\mathrm{e}_\lambda = \frac{\dim\lambda}{d!}\sum_{\eta\vdash d}\mathcal{X}_\lambda(\eta)C_\eta \tag{5}$$

and

$$C_\eta = |C_\eta|\sum_{\lambda \text{ irrep. of } S_d}\frac{\mathcal{X}_\lambda(\eta)}{\dim\lambda}\mathrm{e}_\lambda. \tag{6}$$

Assuming without loss of generality that $r = 0$, we can finally rewrite equation (4):

$$\begin{aligned} H_g^\bullet(\underline{\eta}) &= \frac{1}{d!}[C_{Id}]\sum_\lambda\prod_{i=1}^{s}\left(|C_{\eta_i}|\frac{\mathcal{X}_\lambda(\eta_i)}{\dim\lambda}\right)\mathrm{e}_\lambda \\ &= \frac{1}{d!}\sum_\lambda\prod_{i=1}^{s}\left(|C_{\eta_i}|\frac{\mathcal{X}_\lambda(\eta_i)}{\dim\lambda}\right)\frac{\dim\lambda}{d!}\mathcal{X}_\lambda(Id) \\ &= \left(\frac{1}{d!}\right)^2\sum_\lambda(\dim\lambda)^{2-n}\prod_{i=1}^{s}|C_{\eta_i}|\mathcal{X}_\lambda(\eta_i). \end{aligned} \tag{7}$$

Example 2.10. Let us revisit the computation of $H_1((3))$. In this case the condition of a point with full ramification forces all covers to be connected, so $H = H^\bullet$. The symmetric group S_3 has three irreducible representations, the trivial and alternating one dimensional representations, and a two dimensional representation obtained by quotienting the permutation representation by the invariant small diagonal line. In Table 2.4 we recall the character table of S_3 and the transformations from the conjugacy class basis to the representation basis.

We have:

$$\begin{aligned} H_1((3)) &= \frac{1}{6}[C_{Id}]C_{(3)}C_\tau^4 \\ &= \frac{1}{6}[C_{Id}](2\cdot 3^4\mathrm{e}_1 + 2\cdot(-3)^4\mathrm{e}_{-1}) \\ &= \frac{1}{6}\left(\frac{2\cdot 3^4}{6}+\frac{2\cdot 3^4}{6}\right) = 9 \end{aligned}$$

2.5. **Disconnected to Connected: the Hurwitz Potential.** The character formula is an efficient way to describe disconnected Hurwitz numbers (provided one has a good handle on the characters of the appropriate symmetric group, which is in itself a complicated matter).

$\mathcal{X}_\lambda(C_\eta)$	C_{Id}	C_τ	$C_{(3)}$			
1	1	1	1	$e_1 = \frac{1}{6}(C_{Id} + C_\tau + C_{(3)})$		$C_{Id} = e_1 + e_{-1} + e_P$
-1	1	-1	1	$e_{-1} = \frac{1}{6}(C_{Id} - C_\tau + C_{(3)})$		$C_\tau = 3e_1 - 3e_{-1}$
P	2	0	1	$e_P = \frac{1}{3}(2C_{Id} - C_{(3)})$		$C_{(3)} = 2e_1 + 2e_{-1} - e_P$

TABLE 1. All you have always wanted to know about S_3 (and never dared to ask).

We now investigate how to relate the disconnected theory to the connected theory. Let us begin by observing a simple example:

Example 2.11. We have seen in Exercise 5 that $H_{0,3} = 4$. From the character formula:

$$H^\bullet_{0,3} = \frac{1}{36}(2 \cdot 3^4) = \frac{9}{2} = 4 + \frac{1}{2}$$

The last $\frac{1}{2}$ is the contribution of disconnected covers, consisting of an ellyptic curve mapping to the line as a double cover and of a line mapping isomorphically. The relationship between connected and disconnected Hurwitz numbers is systematized in the language of generating functions.

Definition 2.12. The **Hurwitz Potential** is a generating function for Hurwitz numbers. As usual we present it with as many variables as possible, keeping in mind that in almost all applications one makes a choice of the appropriate variables to mantain:

$$\mathcal{H}(p_{i,j}, u, z, q) := \sum H^r_{g\to 0,d}(\underline{\eta})\ p_{1,\eta_1} \cdot \ldots \cdot p_{s,\eta_s} \frac{u^r}{r!} z^{1-g} q^d,$$

where:

- $p_{i,j}$, for i and j varying among non-negative integers, index ramification profiles. The first index i keeps track of the branch point, the second of the profile. For a partition η the notation $p_{i,\eta}$ means $\prod_j p_{i,\eta_j}$.
- u is a variable for unmarked simple ramification. Division by $r!$ reflects the fact that these points are not marked.
- z indexes the genus of the cover (more precisely it indexes the euler characteristic, which is additive under disjoint unions).
- q keeps track of degree.

Similarly one can define a disconnected Hurwitz potential $\mathcal{H}^\bullet$ encoding all disconnected Hurwitz numbers.

Silly but Important Convention: we choose to set $p_{i,1} = 1$ for all i. This means that an unramified poing sitting above a branch point is not "recorded". With this convention, the monomial in p_i's (for a fixed i) has weighted degree at most (but not necessarily equal) the exponent of the variable q.

Fact. The connected and disconnected potentials are related by exponentiation:

$$1 + \mathcal{H}^\bullet = \mathrm{e}^{\mathcal{H}} \tag{8}$$

Exercise 6. Convince yourself of equation (8). To me, this is one of those things that are absolutely mysterious until you stare at it long enough that, all of a sudden, it becomes absolutely obvious...

Example 2.11 revisited: the information we observed before is encoded in the coefficient of $u^4 z q^3$ in equation (8):

$$H^\bullet_{0,3}\frac{u^4}{4!}zq^3 = H_{0,3}\frac{u^4}{4!}zq^3 + \frac{1}{2!}2\left(H_{1,2}\frac{u^4}{4!}q^2\right)(H_{0,1}zq)\,.$$

Exercise 7. Check equation (8) in the cases of $H^\bullet_{-1,4}$, $H^\bullet_{-1}((2,1,1),(2,1,1))$ and $H^\bullet_{-1}((2,1,1),(2,1,1),(2,1,1),(2,1,1)$. All these Hurwitz numbers equal $\frac{3}{4}$.

Remark 2.13. Unfortunately I don't know of any particulary efficient reference for this section. The book [Wil06] contains more information that one might want to start with on generating functions; early papers of various subsets of Goulden, Jackson and Vakil contain the definitions and basic properties of the Hurwitz potential.

2.6. **Higher Genus Target.** Hurwitz numbers for higher genus targets are determined by genus 0 Hurwitz numbers. In fact something much stronger holds true, i.e. target genus 0, 3-pointed Hurwitz numbers suffice to determine the whole theory. The key observation here are the degeneration formulas.

Theorem 2.14. *Let $\mathfrak{z}(\nu)$ denote the order of the centralizer of a permutation of cycle type ν. Then:*

(1)

$$H^{0,\bullet}_{g\to 0}(\eta_1,\ldots,\eta_s,\mu_1,\ldots,\mu_t) = \sum_{\nu\vdash d}\mathfrak{z}(\nu)H^{0,\bullet}_{g_1\to 0}(\eta_1,\ldots,\eta_s,\nu)H^{0,\bullet}_{g_2\to 0}(\nu,\mu_1,\ldots,\mu_t)$$

with $g_1 + g_2 + \ell(\nu) - 1 = g$.

(2)

$$H^{0,\bullet}_{g\to 1}(\eta_1,\ldots,\eta_s) = \sum_{\nu\vdash d}\mathfrak{z}(\nu)H^{0,\bullet}_{g-\ell(\nu)\to 0}(\eta_1,\ldots,\eta_s,\nu,\nu).$$

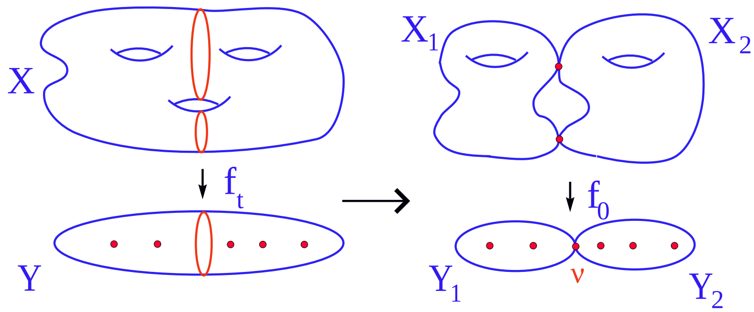


FIGURE 5. Degeneration of a cover to a nodal cover. Note that source and target degenerate simoultaneously and the ramification orders on both sides of the node match.

These formulas are called degeneration formulas because geometrically they correspond to simoultaneously degenerating the source and the target curve, as illustrated in Figure 5. Proving the degeneration formulas geometrically however gives rise to subtle issues of infinitesimal automorphisms (that explain the factor of $\mathfrak{z}\nu$). However a combinatorial proof is straightforward.
Proof of (1): recall that

$$d!H^{0,\bullet}_{g\to 0}(\eta_1,\ldots,\eta_s,\mu_1,\ldots,\mu_t) = |\{\sigma_1,\ldots,\sigma_s,\tilde{\sigma}_1,\ldots,\tilde{\sigma}_t\}|,$$

where the permutations have the appropriate cycle type, and the product of all permutations is the identity. Define $\pi = \sigma_1 \ldots \sigma_s$, then

$$|\{\sigma_1,\ldots,\sigma_s,\pi^{-1},\pi,\tilde{\sigma}_1,\ldots,\tilde{\sigma}_t\}| = \sum_{\nu\vdash d}\frac{1}{|C_\nu|}|\{\sigma_1,\ldots,\sigma_s,\pi_1\}||\{\pi_2,\tilde{\sigma}_1,\ldots,\tilde{\sigma}_t\}|$$

where in the RHS π_1 and π_2 have cycle type ν and we require the products of the permutations in the two sets to equal the identity. We must divide by $|C_\nu|$ because in the LHS we want the two newly introduced permutations to be inverses of each other, and not just in the same conjugacy class. But now we recognize that the term on the RHS is:

$$\sum_{\nu\vdash d}\frac{1}{|C_\nu|}d!H^{0,\bullet}_{g_1\to 0}(\eta_1,\ldots,\eta_s,\nu)d!H^{0,\bullet}_{g_2\to 0}(\nu,\mu_1,\ldots,\mu_t)$$

The proof is finally concluded by observing the identity $|C_\nu|\mathfrak{z}(\nu) = d!$.

Exercise 8. Prove part (2) of Theorem 5.

It is now immediate to observe that applying iteratively the two recursions above one can describe a formula for all Hurwitz numbers.

Combining this with formula (7) one obtains the general character formula for Hurwitz numbers, sometimes referred in the literature as Burnside's formula:

$$H^{0,\bullet}_{g\to h}(\underline{\eta}) = \sum_{\lambda}\left(\frac{d!}{\dim\lambda}\right)^{2h-2}\prod_{i=1}^{s}\frac{|C_{\eta_i}|\mathcal{X}_\lambda(\eta_i)}{\dim\lambda}$$

3. Moduli Spaces

3.1. **Quick and Dirty Introduction.** The concept of moduli space is central in algebraic geometry. In a sense, the point of view of modern algebraic geometry is that every space should be thought as a moduli space. While it is impossible to do justice to such a rich subject in a few pages, I wish to give some intuitive ideas that might help read the more rigorous literature on the field. Another very friendly introduction is given by the first chapter of [HM98].

Informally, a moduli space for (equivalence classes of) geometric objects of a given type consists of:

(1) a set $\mathcal{M}$ whose points are in bijective correspondence with the objects we wish to parameterize;
(2) the notion of functions to $\mathcal{M}$, described (functorially) in terms of families of objects. For any B:

$$Hom(B,\mathcal{M}) \quad \leftrightarrow \quad \left\{\begin{array}{ccc} X_b & \to & X \\ \downarrow & & \downarrow \\ b & \to & B \end{array}\right\}$$

Definition 3.1. A **family** over the base B is a morphism $\pi : X \to B$ in the category the objects live in such that the preimage of every point $b \in B$ belongs to the class of objects parameterized.

Remark 3.2. In modern language we are describing the (scheme) structure of $\mathcal{M}$ by describing its **functor of points**.

A family of objects naturally gives rise to a function to $\mathcal{M}$, but the other implication is much trickier. When this is the case we say that $\mathcal{M}$ is a **fine** moduli space.

Exercise 9. If $\mathcal{M}$ is a fine moduli space there is a family $\mathcal{U} \to \mathcal{M}$ (called **universal family**) such that the fiber over each $m \in \mathcal{M}$ is the object parameterized by m. Also, every family is obtained by pullback from the universal family.

It is often the case that it is not possible to obtain a fine moduli space for a given moduli problem (this typically happens when the objects one wishes to parameterize have automorphisms). In this case one must make a choice:

(1) be satisfied with a scheme $\mathcal{M}$ whose points are in bijective correspondence with the objects to parameterize, plus some universality condition (for any other space $\mathcal{N}$ whose points have such a property, there exists a unique map $\mathcal{M} \to \mathcal{N}$). In this case we say $\mathcal{M}$ is a **coarse** moduli space.
(2) forget the idea of $\mathcal{M}$ being a scheme, and allow it to be some categorical monstruosity (in modern language called a **stack**[1]), that has the property of recovering the equivalence between families and functions to $\mathcal{M}$. The mantra here is that to do (a good amount of) geometry on a stack one very seldom has to meddle with the categorical definitions but uses the above equivalence to translate geometric questions from the stack to families of objects.

Exercise 10. Familiarize yourself with these concepts by looking at the following more or less silly examples/exercises.

(1) Any scheme X is a fine moduli space...for itself, i.e. for the functor describing families of points of X.
(2) Note that the points of the cuspidal cubic $X = \{y^2 = x^3\}$ are in bijection with the points of $\mathbb{P}^1$, but X is not a coarse moduli space for "families of points of $\mathbb{P}^1$".
(3) Consider the moduli space for equivalence classes of unit length segments in the real plane up to rigid motions. What is the coarse moduli space? Show that this is not a fine moduli space by constructing two non-isomorphic families of segments. Understand that the lack of fine-ness comes from the fact that you can flip the segment.
(4) Let us introduce, given a group G, the fine moduli stack $\mathcal{B}G$ of principal G bundles. We do not define what it is, but remark that functions $B \to \mathcal{B}G$ correspond to principal G-bundles $P \to B$. Convince yourself that the moduli space considered in (3) is $\mathcal{B}\mathbb{Z}_2$.

3.2. **Various Moduli Spaces Related to Curves.** We are concerned with the interactions among different moduli spaces that have to do with curves. Here we introduce the characters. In general $\mathcal{M}$ denotes a moduli space parameterizing smooth objects and $\overline{\mathcal{M}}$ denotes

[1] I am of course exaggerating here, stacks are not all that monstrous - if for no other reason that they have become a necessary and hence standard notion in the study of moduli spaces. At this point there are both friendly introductory references such as [Fan01], or comprehensive ones such as [BCE+06] or [dJa] to be found at least in the ether.

some suitable compactification obtained by allowing mild degenerations of the objects. Curves can acquire only nodal singularities, and **stable** always means "with finitely many automorphisms". These moduli spaces are typically (Deligne-Mumford) stacks (hence denoted with script letters). We use non-script fonts to emphasize the cases that are actually schemes.

$\overline{\mathcal{M}}_{g,n}$**:** the moduli space of (isomorphism classes of) **stable curves** of genus g with n marked points. Here stability means that every rational component must have at least three special points (nodes or marks), and that a smooth ellptic curve needs to have at least one mark. This is a smooth stack of dimension $3g-3+n$, connected, irreducible. See [HM98] for more.

$\overline{\mathcal{M}}_{g,n}(\alpha_1,\ldots,\alpha_n)$**:** in **weighted stable curves** ([Has03]) one tweaks the stability of a pointed curve $(X=\cup_j X_j, p_1,\ldots,p_n)$ by assigning weights α_i to the marked points and requiring the restriction to each X_j of $\omega_X+\sum\alpha_i p_i$ to be ample (this amounts to the combinatorial condition that $\sum_{p_i\in X_j}\alpha_i+n_j>2-2g_j$, where n_j is the number of shadows of nodes on the j-th component of the normalization of X and g_j is the geometric genus of such component). In these spaces "light" points can collide with each other until a "critical mass" is reached that forces the sprouting of new components.

When $g=0$, two points are given weight 1 and all other points very small weight, the space $\overline{M}_{0,2+r}(1,1,\varepsilon,\ldots,\varepsilon)$ is classically known as the *Losev-Manin* space [LM00]: it parameterizes chains of $\mathbb{P}^1$'s with the heavy points on the two terminal components and light points (possibly overlapping amongst themselves) in the smooth locus of the chain.

$\overline{\mathcal{M}}_{g,n}(X,\beta)$**:** the space of **stable maps** to X of degree $\beta\in H_2(X)$. A map is stable if every contracted rational component has three special points. If $g=0$ and X is convex then these are smooth schemes, but in general these are nasty creatures even as stacks. They are singular and typically non-equidimensional. Luckily deformation theory experts can construct a Chow class of degree in the expected dimension, and many of the formal properties of the fundamental class, called a **virtual fundamental class**. Intersection theory on these spaces is then rescued by capping all classes with the virtual fundamental class. Good references for people interested in these spaces are [KV07] and [FP97].

$Hurw_{g\to h,d}(\underline{\eta})\subset Adm_{g\to h,d}(\underline{\eta})$**:** the **Hurwitz spaces** parameterize degree d covers of smooth curves of genus h by smooth curves of genus g. A vector of partitions of d specifies the ramification profiles over marked points on the base. All other ramification is required to be simple. Hurwitz spaces are typically smooth schemes (unless the ramification profiles are chosen in very particular ways so as to allow automorphisms), but they are obviously non compact. The **admissible cover** compactification, consisting of degenerating simoultaneosly target and cover curves, was introduced in [HM82]. In [ACV01], the normalization of such space is interpreted as a (component of a) space of stable maps to the stack $\mathcal{B}S_d$. Without going into the subtleties of stable maps to a stack, we understand that by admissible cover we always denote the corresponding smooth stack.

$\overline{\mathcal{M}}_{g,n}(X, \beta; \alpha D)$**:** spaces of **relative stable maps** relative to a divisor D with prescribed tangency conditions([LR01, Li02]). We are especially interested in the case when X is itself a curve. In this case giving relative conditions is equivalent to specifying ramification profiles over some marked points of the target: spaces of relative stable maps are a "hybrid" compactification that behaves like admissible covers over the relative points and as stable maps elsewhere. See [Vak08] for a more detailed description of the boundary degenerations.

Remark 3.3. When the target space is $\mathbb{P}^1$, an important variation of spaces of (relative) stable maps is the so called space of **rubber** maps, or maps to an unparameterized $\mathbb{P}^1$, where two maps are considered equivalent when they agree up to an automorphism of the base $\mathbb{P}^1$ preserving 0 and ∞ (in other words a $\mathbb{C}^*$ scaling of the base). It will be clear later why we care about these spaces.

Remark 3.4. If these many moduli spaces already feel a bit overwhelming, imagine that the possibilities increase when mixing the various features and variations introduced above (*weighted relative stable maps* to a rubber $\mathbb{P}^1$? No problem). However this should not be considered a confusing feature but rather part of the richness of this theory, as one can fine tune the choice of moduli space to the geometric question he is intending to study. The geometry of these moduli spaces is richly interlaced by a number of natural maps, such as evaluation maps at marked points, gluing maps, maps that forget points, functions, target, source etc... we will introduce these maps as the need arises.

Exercise 11. What is $\overline{M}_{0,3}$ (by this meaning what scheme represents the functor of stable families of three labelled points on a rational curve)? What is $\overline{M}_{0,4}$? And their universal families?

Exercise 12. Convince yourself that $\overline{\mathcal{M}}_{g,n}$ is isomorphic to a moduli space of weighted stable curves where all the points have weight 1. Further, understand that there are morphisms (called **reduction maps**) among spaces of weighted stable maps when the weights decrease, but not (necessarily) when the weights increase. How do these maps behave geometrically? (*Hint:* they contract some boundary strata...)

Exercise 13. A nice feature of all these spaces is that the boundary is modular, i.e. it is built up of similar types of moduli spaces, but with smaller type of invariants. Understand this statement for the spaces we introduced.

Exercise 14. Familiarize yourself with the statement that the universal family for $\overline{\mathcal{M}}_{g,n}$ is isomorphic to $\overline{\mathcal{M}}_{g,n+1}$. Describe the $(n+1)$-pointed curves parameterized by the image of a section $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \mathcal{U}$. We call this (boundary!) divisor D_i.

Exercise 15. Observe that there are natural branch morphisms from spaces of admissible covers and spaces of (relative) stable maps to a target curve $\mathbb{P}^1$, recording the branch divisor of the covering. Where do these branch maps take value? Or rather, for each of those spaces what is the most appropriate(i.e. the one that retains the most information) moduli space where these maps can take value? What is the degree of these branch maps?

Exercise 16. Describe the moduli space $M_g(\mathbb{P}^1, 1)$ and the stable maps compactification $\overline{M}_g(\mathbb{P}^1, 1)$.

Exercise 17. The **hyperelliptic locus** is the subspace of $\overline{M}_g$ parameterizing curves that admit a double cover to $\mathbb{P}^1$. Understand the hyperelliptic locus as the moduli space $Adm_{g\to 0,2}((2), \ldots, (2))$ and subsequently as a stack quotient of $\overline{\mathcal{M}}_{0,2g+2}$ by the trivial action of $\mathbb{Z}_2$.

3.3. **Tautological Bundles on Moduli Spaces.** We define bundles on our moduli spaces by describing them in terms of the geometry of families of objects. In other words, for any family $X \to B$, we give a bundle on B constructed in some canonical way from the family X. This insures that this assignment is compatible with pullbacks (morally means that we are thinking of B as a chart and that the bundle patches along various charts). This is the premium example of the philosophy of doing geometry with stacks. We focus on two particular bundles that will be important for our applications.

3.3.1. *The Cotangent Line Bundle and ψ classes.* An excellent reference for this section, albeit unfinished and unpublished, is [Koc01].

Definition 3.5. The **i-th cotangent line bundle** $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ is globally defined as the restriction to the i-th section of the relative dualizing sheaf from the universal family:

$$\mathbb{L}_i := \sigma_i^*(\omega_\pi).$$

The first Chern class of the cotangent line bundle is called ψ **class**:

$$\psi_i := c_1(\mathbb{L}_i).$$

This definition is slick but unenlightening, so let us chew on it a bit. Given a family of marked curves $f : X \to B (= \varphi_f : B \to \overline{\mathcal{M}}_{g,n})$, the cotangent spaces of the fibers X_b at the i-th mark naturally fit together to define a line bundle on the image of the i-th section, which is then isomorphic to the base B. This line bundle is the pullback $\varphi_f^*(\mathbb{L}_i)$. Therefore informally one says that the cotangent line bundle is the line bundle whose fiber over a moduli point is the cotangent line of the parameterized curve at the i-th mark.

The cotangent line bundle arises naturally when studying the geometry of the moduli spaces, as we quickly explore in the following exercises.

Exercise 18. Convince yourself that the normal bundle to the image of the i-th section in the universal family is naturally isomorphic to $\mathbb{L}_i^\vee$ (This is sometimes called the i-th tangent line bundle and denoted $\mathbb{T}_i$).

Exercise 19. Consider an irreducible boundary divisor $D \cong \overline{M}_{g_1,n_1+\bullet} \times \overline{M}_{g_2,n_2+\star}$. Then the normal bundle of D in the moduli space is naturally isomorphic to the tensor product of the tangent line bundles of the components at the shadows of the node:

$$N_{D/\overline{M}_{g,n}} \cong \mathbb{L}_\bullet^\vee \boxtimes \mathbb{L}_\star^\vee$$

Is this statement consistent with the previous exercise? Why?

When two moduli spaces admitting ψ classes are related by natural morphisms, a natural question to ask is how the corresponding ψ classes compare (more precisely, how a ψ class in one space compares with the pull-back via the natural morphism of the corresponding ψ class on the other space). The answer is provided by the following Lemma.

Lemma 3.6. *The following comparisons of ψ classes hold.*

(1) Let $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful morphism, and $i \neq n+1$. Then

$$\psi_i = \pi_{n+1}^*\psi_i + D_{i,n+1},$$

where $D_{i,n+1}$ is the boundary divisor parameterizing curves where the i-th and the $n+1$-th mark are the only two marks on a rational tail (or the image of the i-th section, if you think of $\overline{\mathcal{M}}_{g,n+1}$ as the universal family of $\overline{\mathcal{M}}_{g,n}$).

(2) *Let $\pi : \overline{\mathcal{M}}_{g,1}(X,\beta) \to \overline{\mathcal{M}}_{g,1}$ be the natural forgetful morphism. Then*

$$\psi_1 = \pi^*\psi_1 + D_1,$$

where D_1 is the divisor of maps where the mark lies on a contracting rational tail.

(3) *Let $r : \overline{\mathcal{M}}_{g,n}(\alpha_1, \dots, \alpha_n) \to \overline{\mathcal{M}}_{g,n}(\alpha'_1, \dots, \alpha'_n)$ be the natural reduction morphism. Then*

$$\psi_i = r^*\psi_i + D,$$

where D is the boundary divisor parameterizing curves where the i-th mark lies on a component that is contracted in $\overline{\mathcal{M}}_{g,n}(\alpha'_1, \dots, \alpha'_n)$.

In all cases the intuitive idea is that the "difference" in the cotangent line bundles is supported on the locus where the mark lives on a curve in the first space that gets contracted in the second space. To make a formal proof one has to observe how the universal family of the first space is obtained by appropriately blowing up the pull-back of the universal family on the second space, and what effect that has on the normal bundle to a section. Since we are lazy and it's a hot summer, we leave the details as an exercise.

Exercise 20. Prove Lemma 3.6

Exercise 21. Show that Lemma 3.6 gives sufficient information to determine ψ classes for every $\overline{M}_{0,n}$. In particular show it gives the following useful combinatorial boundary description of a ψ class. Let i, j, k be three distinct marks. The class ψ_i is the sum of all boundary divisors parameterizing curves where the i-th mark is on one component, the j-th and k-th marks are on the other. Note that such a boundary description is not unique, as it depends on the choice of j and k!

3.3.2. *The Hodge Bundle.*

Definition 3.7. The **Hodge bundle** $\mathbb{E}(= \mathbb{E}_{g,n})$ is a rank g bundle on $\overline{\mathcal{M}}_{g,n}$, defined as the pushforward of the relative dualizing sheaf from the universal family. Over a curve X, the fiber is canonically $H^0(X, \omega_X)$ (i.e. the vector space of holomorphic 1-forms if X is smooth). The Chern classes of $\mathbb{E}$ are called λ classes:

$$\lambda_i := c_i(\mathbb{E}).$$

We recall the following properties([Mum83]):

Vanishing: ch_i can be written as a homogeneous quadratic polynomial in λ classes. Thus:

$$\mathrm{ch}_i = 0 \quad \text{for} \quad i > 2g. \tag{9}$$

Mumford Relation: the total Chern class of the sum of the Hodge bundle with its dual is trivial:

$$c(\mathbb{E} \oplus \mathbb{E}^\vee) = 1. \tag{10}$$

Hence $\mathrm{ch}_{2i} = 0$ if $i > 0$.

Separating nodes:

$$\iota^*_{g_1,g_2,S}(\mathbb{E}) \cong \mathbb{E}_{g_1,n_1} \oplus \mathbb{E}_{g_2,n_2}, \tag{11}$$

where with abuse of notation we omit pulling back via the projection maps from $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}$ onto the factors.

Non-separating nodes:

$$\iota^*_{irr}(\mathbb{E}) \cong \mathbb{E}_{g-1,n} \oplus \mathcal{O}. \tag{12}$$

Remark 3.8. We define the Hodge bundle and λ classes on moduli spaces of stable maps and Hurwitz spaces by pulling back via the appropriate forgetful morphisms.

Exercise 22. Use the above properties to show vanishing properties of λ-classes:

(1) $\lambda_g^2 = 0$ if $g > 0$.
(2) $\lambda_g\lambda_{g-1}$ vanishes on the boundary of $\overline{\mathcal{M}}_g$. If now we allow marked points, then the vanishing holds on "almost all" the boundary, but one needs to be more careful. Describe the vanishing locus of $\lambda_g\lambda_{g-1}$ in this case.
(3) λ_g vanishes on the locus of curves not of compact type (i.e. where the geometric and arithmetic genera are different).

4. Orbifolds

Although we have been dealing with orbifolds (even though we called them "DM stacks" [2]) throughout the last section, we now take a quick look at the foundations in order to develop Gromov-Witten theory for orbifolds.

[2] The word *stack* is closer to the heart of algebraic geometers, while pretty much the rest of the world prefers *orbifold*. Usually when using the word orbifold one insists on finite isotropy in codimension at least one. Technically then gerbes are not orbifolds, while they are perfectly legit DM stacks. The lines are blurring as we speak though, perhaps thanks to the physicists.

Ever since the times of Klein and Hilbert mathematicians have been studying quotients of spaces by the action of groups by studying invariant functions. Orbifolds are a modern take on the subject that allow such quotients to be local charts. Just armed with the slogan *an orbifold is a space locally modelled as the quotient of a manifold by the (smooth) action of a finite group*, any mathematician familiar with manifolds and groups could easily recover the correct foundations of orbifold theory by generalizing the notion of an atlas to charts of the form $\mathbb{C}^n/G$ and making all functions that appear in the process equivariant with respect to the relevant groups. Such formulation turns out to be very intuitive but also unfortunately very clumsy to work with, as it is laden with interminable compatibility conditions that one has to carry around throughout the process. The categorical language of **Lie groupoids** turns out to be a much more efficient way of encoding all such compatibilities to study global geometry. We will leave the task of developing this language to Paul (and for those of you who are too impatient to wait you can look up the handy reference [ALR07]), and focus here just on some strictly necessary aspects of theory.

4.1. **Global Quotient Orbifolds.** Given a manifold X and a finite group G acting smoothly on it, we define the global quotient orbifold $[X/G]$ to simply be the datum of such pair. So far this is neither very exciting nor very geometric - but it is the only thing that we can do given that we wish to remember all information about our group action. The interesting task is to develop how to do geometry with such datum. The action of G induces an equivalence relation on points of X (equivalence classes are called **orbits**) and we denote by $|X/G|$ the **orbit space**(also called the **coarse moduli space** of the orbifold). If the action of G is free then the orbit space is a manifold. Of course, whatever geometric notions we develop for the orbifold $[X/G]$, we require that they specialize to the corresponding notions for the quotient manifold when the G action is free.

Definition 4.1. Let $x \in X$. We define the **isotropy group** (or the **local group**) of $[X/G]$ at the point x to be the stabylizer of x: $G_x = \{g \in G : gx = x\}$.

The isotropy groups of different points in the orbit of x are related by conjugation, and hence the isomorphism class of G_x is an invariant of the orbit. A very crude albeit often effective way to think of an orbifold is to think of its orbit space plus the additional information that each point of the orbit space carries an isomorphism class of groups. From the point of view of intersection theory, every such point $[x]$ should be given the fractional weight of $1/|G_x|$.

To describe maps between two global quotient orbifolds $f : [Y/H] \to [X/G]$ one is obviously tempted to ask for a group homomorphism $\phi : H \to G$ and an equivariant map $F : Y \to X$; certainly such data does induce a map of the underlying orbifolds - however this notion is too restrictive. Consider the simple example where $Y = S^1$, $H = \{e\}$, $X = \mathbb{R}$, $G = \mathbb{Z}$ and integers act by translations. The action is free and the orbit space is itself a circle, so we would expect the identity map of the circle (as $[Y/H]$) to itself (presented as $[X/G]$) to appear among our maps of orbifolds. However all maps F from S^1 to $\mathbb{R}$ are homotopically trivial: no such F can descend to the desired identity map. The issue is that a global quotient is a specific "presentation" of an orbifold: in order to witness all maps or orbifolds one has to be able to replace the source orbifold with a different equivalent way to present it. This leads to the notion of Morita equivalence and will be explored later. For now we content ourselves to study maps from a manifold to a global quotient orbifold.

Definition 4.2. Let Y be a manifold and $[X/G]$ a global quotient orbifold. Then a function $f : Y \to [X/G]$ consists of the following data:

(1) a principal G bundle $p : P \to Y$;
(2) a G-equivariant map $F : P \to X$.

Remark 4.3. This definition is a special case of what was fuzzily discussed earlier: P with the G action is a (Morita) equivalent way to present the orbifold Y (which happens to be a manifold, so be it!), and the map F is a "naive" map of global quotient orbifolds.

Example 4.4. Recall the moduli space $\mathcal{B}G$ we defined in Exercise 10. We can now observe that if G is a finite group then $\mathcal{B}G$ is the global quotient orbifold $[pt./G]$.

Example 4.5. The next class of examples are representations of finite groups. Gorenstein representations are particularly liked as the natural volume form on $\mathbb{C}^n$ descends to a volume form on the orbifold. Among these examples are all ADE surface singularities and $[\mathbb{C}^3/\mathbb{Z}_3]$ (three copies of the same non-trivial representation of $\mathbb{Z}_3$), which is obtained from the canonical line bundle of $\mathbb{P}^2$ by shrinking the 0-section.

4.2. **Orbicurves.** The other class of orbifolds we wish to introduce are "orbi-curves", which are the natural generalization of Riemann Surfaces in the orbifold world. Given a Riemann Surface C we obtain an orbicurve $\mathcal{C}$ by endowing a finite number of points of C with a cyclic quotient orbifold structure. We describe now how to give orbifold structure to one point in a Riemann Surface - of course repeating the process a finite number of times one obtains a general orbicurve. Consider the unit disk D and the natural multiplicative action of the cyclic group of n-th roots of unity (we should really call this μ_n but orbifold people have picked up the habit of using the notation $\mathbb{Z}_n$, I believe mostly to annoy number theorists). The orbit space $D_n := [D/\mathbb{Z}_n]$ is still diffeomorphic to the unit disk. The only point that carries non-trivial isotropy is the (image of the) origin, which has local group $\mathbb{Z}_n$. Given a Riemann Surface C with an atlas that contains a unique chart $\varphi_p : D \to C$ covering a given point p, one can safely replace such chart by $\tilde{\varphi}_p : D_n \to C$. The new atlas presents an orbifold where the point p has acquired $\mathbb{Z}_n$ orbifold structure, i.e. an orbi-curve with precisely one non-trivial orbi-point.

A map from an orbicurve $\mathcal{C}$ to $\mathcal{B}G$ corresponds to a principal G bundle over the orbi-curve. This is an honest principal bundle over the coarse space of the curve minus the set of non-trivial orbi-points. Over the orbi-points the cover acquires ramification. For example a hyperelliptic cover becomes an etale map (hence a principal $\mathbb{Z}_2$ bundle) over an orbi-$\mathbb{P}^1$ where the $2g+2$ branch points have been given $\mathbb{Z}_2$ orbifold structure (it is also common to say have been twisted by $\mathbb{Z}_2$). When the group G is the symmetric group, then a principal S_d bundle is a priori a $d!$ cover of the base curve. By letting S_d act naturally on a set of d points and taking the associated bundle, one recovers an "ordinary" d-fold cover (ramified over the orbi-points). This Hurwitz theory is equivalent to the theory of maps from orbi-curves to $\mathcal{B}S_d$.

Remark 4.6 (Nodal orbi-curves). When a Riemann surface has a node, we allow it to have orbifold structure at the node. This means that if we normalize the node, both shadows of the node have the same orbifold structure and that the representations induced by a generator of the cyclic group on the tangent spaces are dual to each other (in other words if the node has local equation $xy = 0$ one can give a cyclic action by $\eta \cdot x = \eta x$, $\eta \cdot y = \eta^{-1} y$ and then take the quotient via this action.)

4.3. **Inertia Orbifold.** On a very basic level orbifolds are spaces whose points "carry" the information of a group (of automorphisms is the orbifold is thought of as a moduli space or a category). However the orbifold itself cannot distinguish elements of these automorphism groups and when such refined information is needed there is a partner orbifold that is called to the task: the **inertia Orbifold**. On a very basic and imprecise level, points of the inertia orbifolds are pairs (x, g), where

x is a point of the orbifold and $g \in G_x$ is an element of the isotropy group of x. There is naturally a projection function from the inertia orbifold down to the original orbifold.

One can define the inertia orbifold more formally in a few different ways. We mention a couple here in passing, but then focus on the case of global quotient orbifolds, where we can be a lot more explicit. For a general orbifold $\mathcal{X}$, the inertia orbifold is:

(1) the space of constant loops, i.e. maps from S^1 to $\mathcal{X}$ whose image is one point ([ALR07]Definition 2.49).
(2) the fiber product of $\mathcal{X}$ with itself over the diagonal ([Abr08]).

Definition 4.7. Let $[X/G]$ be a global quotient orbifold, and for $g \in G$ denote X^g the set of points fixed by the element g. Then the inertia orbifold is defined to be the global quotient

$$\mathcal{I}\mathcal{X} = \left(\coprod_{g \in G} X^g\right)/G, \tag{13}$$

where the action is defined as follows: for $x \in X^g$ and $h \in G$, the point hx is the point hx in $X^{hgh^{-1}}$.

Exercise 23. The above presentation of the inertia orbifold of a global quotient can be refined by observing that you can identify fixed loci by elements that are conjugate to each other. Let A be a subset of G containing one element for each conjugacy class of G. Then prove that the inertia orbifold can be presented as:

$$\mathcal{I}\mathcal{X} = \coprod_{g \in A} \left(X^g/C(g)\right), \tag{14}$$

where $C(g)$ denotes the centralizer of g in G.

Example 4.8.

$$\mathcal{I}\mathcal{B}S_3 \cong \mathcal{B}S_3 \cup \mathcal{B}\mathbb{Z}_2 \cup \mathcal{B}\mathbb{Z}_3$$

Remark 4.9. The orbifold $[X/G]$ is always (isomorphic to) a connected component of the inertia orbifold, namely the **untwisted sector**, corresponding to the identity automorphism. The other components of the inertia orbifold, indexed by non-trivial conjugacy classes, are called the **twisted sectors**.

4.4. **Chen-Ruan Orbifold Cohomology.** Given an orbifold $\mathcal{X}$, the Chen-Ruan Cohomology of $\mathcal{X}$ is graded ring. As a vector space it is just the (ordinary) cohomology of the orbit space of $\mathcal{I}\mathcal{X}$. However the grading is appropriately shifted, and the product is constructed out of three point invariants ([ALR07]) or via a representation theoretic obstruction bundle ([JKK07]). We don't concern ourselves too much with the product here, but we briefly introduce the age grading.

Definition 4.10. Given $g \in G$ and $x \in X^g$, the representation induced by g on T_xX can be written as a diagonal matrix with entries η_j roots of unity. Assign to each root of unity $\eta_j = e^{2\pi i q_j}$ the rational number $0 \leq q_j < 1$. The sum of all the q_j's only depends on the conjugacy class of g and it is called the **age** of the twisted sector X^g. As a graded vector space, the Chen-Ruan cohomology of a global quotient is written as:

$$H^*_{CR}(\mathcal{X}) = \bigoplus_{g\in A} H^{*-2age(X^g)}(|X^g/C(g)|).$$

Example 4.11.

$$H^*_{CR}(\mathcal{B}G) = H^0_{CR}(\mathcal{B}G) = \mathbb{C}^{|A|}$$

$$H^*_{CR}([\mathbb{C}^3/\mathbb{Z}_3]) = H^0 \oplus H^2 \oplus H^4 = \mathbb{C}_{id} \oplus \mathbb{C}_\omega \oplus \mathbb{C}_{\bar{\omega}}$$

Remark 4.12. We note that the the age is an integer grading precisely when the orbifold has Gorenstein singularities.

4.5. **Twisted Stable Maps.** The study of Gromov-Witten invariants of orbifolds is developed by Chen and Ruan in [CR02] and [CR04]. The algebraic point of view is established in [AGV]. In order to obtain a good mathematical theory (i.e. a compact and reasonably well behaved moduli space, equipped with a virtual fundamental class) they study representable morphisms from orbi-curves, and require insertions to take value in the Chen-Ruan cohomology of the target.

With these two modifications in place, the moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ is a proper Deligne Mumford stack and just about any desirable (and undesirable) feature of ordinary Gromov-Witten theory carries over to the orbifold setting.

Exercise 24. Note that allowing curves to have orbifold structure is not just a philosphical niceity, or a desire to enrich the theory - but a necessity: even if one only cares about maps from smooth curves to an orbifold, in order to obtain a compact moduli space one has to allow nodal degenerations of smooth curves to acquire orbifold structure at the node. Convince yourself of this by thinking of a family of maps from elliptic curves to $\mathcal{B}\mathbb{Z}_2$ degenerating to a nodal elliptic curve. Construct one such family where the limit curve must have $\mathbb{Z}_2$ orbifold structure at the node.

Rather than spending too much time on the generalities of orbifold Gromov-Witten theory, we explore the situation that we are most intersted in, i.e. the case of maps to a classifying space $\mathcal{B}G$, parameterized by the space $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G, 0)$ (where the curve class $0 \in H_2(\mathcal{B}G)$ is typically omitted from the notation).

The space $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ is a (smooth) orbifold (DM stack) of dimension $3g-3+n$. It parameterizes G-covers of curves of genus g that are etale away from the n-marked points. The space has many connected components: on a first level, because one can specify different conjugacy classes - i.e. ramification profiles, for the marked points. But even after making one such choice one could have more than one connected component, as we illustrate in Example 4.13. There are two natural structure morphisms from this moduli space: one to $\overline{\mathcal{M}}_{g,n}$ which is a finite cover, and another to the moduli space $\cup_h \overline{\mathcal{M}}_h^{\bullet}$, where h is the genus of the cover curve and it varies depending on the twisting of the n marks on the base curve.

Example 4.13. Consider $\overline{\mathcal{M}}_g(\mathcal{B}\mathbb{Z}_2)$, parameterizing etale double covers of genus g curves. From the Riemann-Hurwitz formula the genus of the cover curves is $h = 2g-1$. The structure map to $\overline{\mathcal{M}}_{2g-1}^{\bullet}$ has degree 2^{2g-1}, corresponding to the number of homomorphisms from $H_1(C)$ to $\mathbb{Z}_2$ for C any curve of genus g, divided by 2 because each cover has a nontrivial involution. The space however consists of two connected components: the component of trivial covers, parameterizing disconnected double covers; from a group theoretic perspective this happens when all loops are sent to the identity element in $\mathbb{Z}_2$. And the component of connected double covers, occurring when the $\mathbb{Z}_2$ representation of the fundamental group is non-trivial. The trivial component is isomorphic to (a $\mathcal{B}\mathbb{Z}_2$ gerbe over) $\overline{\mathcal{M}}_g$.

Abramovich-Corti and Vistoli in [ACV01] establish the equivalence between spaces of twisted stable maps to $\mathcal{B}S_d$ and (the normalization of) moduli spaces of admissible covers as introduced by Harris and Mumford ([HM82]).

Exercise 25. Understand the space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1 \times \mathcal{B}G, 1)$ as the moduli space of admissible covers to a parameterized $\mathbb{P}^1$.

4.6. **Hurwitz-Hodge Integrals.** Moduli spaces of twisted stable maps to $\mathcal{B}G$ have two structure morphisms, and therefore can have via pullback two natural Hodge bundles, one corresponding to the base curves and one to the covers. While the “base” Hodge bundle may seem more natural, it is also less exciting, as any intersection theoretic question concerning these kind of Hodge classes can be made equivalent via projection formula, to a question about "classical" Hodge integrals. Therefore we focus our attention on the Hodge bundle pulled back from the space of covers: to make this explicit we call this the (total) **Hurwitz-Hodge bundle**.

The total Hurwitz-Hodge bundle inherits a natural G action and splits into subbundles corresponding to irreducible representations of G. Given an irrep ρ, we denote by $\mathbb{E}_\rho$ the subbundle whose fiber is a bunch of copies of ρ. In formula:

$$\mathbb{E}^h = \bigoplus_\rho \mathbb{E}_\rho$$

The $\mathbb{E}_\rho$'s are often referred as the Hurwitz-Hodge bundles and their Chern classes as Hurwitz-Hodge classes.

A formal and more orbifold theoretic definition of Hurwitz-Hodge bundles can be given by remembering that vector bundles on $\mathcal{B}G$ correspond to representations of G. Given an irrep ρ, let L_ρ denote the corresponding vector bundle on $\mathcal{B}G$. Let $\pi : \mathcal{U} \to \overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ be the universal family and $f : \mathcal{U} \to \mathcal{B}G$ be the universal map. Then we have:

$$R^1\pi_* f^*(L_\rho) = (\mathbb{E}_\rho)^\vee = \mathbb{E}^\vee_{\rho^\vee}.$$

We can use the Riemann-Roch formula for twisted curves from [AGV, 7.2.1] to compute the rank of the Hurwitz-Hodge bundles:

$$\chi(\mathcal{E}) = rk(\mathcal{E})\chi(\mathcal{O}_\mathcal{C}) + \deg(\mathcal{E}) - \sum_{j=1}^{k+\ell} \mathrm{age}_{p_j}(\mathcal{E}),$$

where $\mathcal{E}$ is a vector bundle on a twisted curve $\mathcal{C}$ and $p_1, \ldots, p_{k+\ell}$ are the twisted points.

Exercise 26. Compute the ranks of $\mathbb{E}_1$ and $\mathbb{E}_{-1}$ on $\overline{\mathcal{M}}_g(\mathcal{B}\mathbb{Z}_2)$. You can do this in two different ways, either by using the Riemann-Roch formula above or by geometric considerations. Pay attention: the ranks vary on different components of the moduli space!

Exercise 27. Compute the ranks of $\mathbb{E}_1$, $\mathbb{E}_\omega$ and $\mathbb{E}_{\bar{\omega}}$ on $\overline{\mathcal{M}}_{g,4}(\mathcal{B}\mathbb{Z}_3)$. Again note that there are many different components where the ranks are different.

In [BGP05] it is shown that a version of Mumford's relation holds among Hurwitz-Hodge bundles. This useful relation among Hurwitz-Hodge classes is often called **G-Mumford**:

$$c_t(\mathbb{E}_\rho \oplus \mathbb{E}^\vee_\rho) = 1.$$

Similarly we can define ψ classes on these moduli spaces. Again, there are a few variants that can be adopted. Here unless otherwise specified we adopt the convention that when we refer to a ψ class on $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ we intend the corresponding class pulled back from $\overline{\mathcal{M}}_{g,n}$.

Polynomials giving rise to zero dimensional classes in Hurwitz-Hodge classes and ψ classes are called Hurwitz-Hodge integrals. Besides being interesting objects on their own, they are relevant to orbifold Gromov-Witten theory: every time a target orbifold admits a torus action with a finite number of isolated fixed points, orbifold Gromov-Witten invariants can be expressed in terms of Hurwitz-Hodge integrals and combinatorics. This is through the technique of Atyiah-Bott localization, which we proceed to illustrate next.

5. Atyiah-Bott Localization

The localization theorem of [AB84] is a powerful tool for the intersection theory of moduli spaces that can be endowed with a torus action. In this section we present the basics of this techniques following [HKK$^+$03] and focus on one particular application, the evaluation of the hyperelliptic locus in the tautological ring of $\mathcal{M}_g$.

5.1. **Equivariant Cohomology.** Let G be group acting on a space X. According to your point of view G might be a compact Lie group or a reductive algebraic group. Then G-equivariant cohomology is a cohomology theory developed to generalize the notion of the cohomology of a quotient when the action of the group is not free. The idea is simple: since cohomology is homotopy invariant, replace X by a homotopy equivalent space $\tilde{X}$ on which G acts freely, and then take the cohomology of $\tilde{X}/G$. Rather than delving into the definitions that can be found in [HKK$^+$03], Chapter 4, we recall here some fundamental properties that we use:

(1) If G acts freely on X, then
$$H^*_G(X) = H^*(X/G).$$
(2) If X is a point, then let EG be any contractible space on which G acts freely, $BG := EG/G$, and define:
$$H^*_G(pt.) = H^*(BG).$$
(3) If G acts trivially on X, then
$$H^*_G(X) = H^*(X) \otimes H^*(BG).$$

Example 5.1. If $G = \mathbb{C}^*$, then $EG = S^\infty$, $BG := \mathbb{P}^\infty$ and
$$H^*_{\mathbb{C}^*}(pt.) = \mathbb{C}[\hbar],$$
with $\hbar = c_1(\mathcal{O}(1))$.

Remark 5.2. Dealing with infinite dimensional spaces in algebraic geometry is iffy. In [Ful98], Fulton finds an elegant way out by showing that for any particular degree of cohomology one is interested in, one can work with a finite dimensional approximation of BG. Another route is to instead work with the stack $\mathcal{B}G = [pt./G]$. Of course the price to pay is having to formalize cohomology on stacks...here let us just say that $\mathcal{O}(1) \to \mathcal{B}\mathbb{C}^*$, pulled back to the class of a point, is a copy of the identity representation $Id : \mathbb{C}^* \to \mathbb{C}^*$.

Let $\mathbb{C}^*$ act on X and let F_i be the irreducible components of the fixed locus. If we-push forward and then pull-back the fundamental class of F_i we obtain

$$i^* i_*(F_i) = e(N_{F_i/X}).$$

Since $N_{F_i/X}$ is the moving part of the tangent bundle to F_i, this euler class is a polynomial in $\hbar$ where the $\hbar^{codim(F_i)}$ term has non-zero coefficient. This means that if we allow ourselves to invert $\hbar$, this euler class becomes invertible. This observation is pretty much the key to the following theorem:

Theorem 5.3. *The maps:*

$$\bigoplus_i H^*(F_i)(\hbar) \overset{\sum \frac{i_*}{e(N_i)}}{\to} H^*_{\mathbb{C}^*}(X) \otimes \mathbb{C}(\hbar) \overset{i^*}{\to} \bigoplus_i H^*(F_i)(\hbar)$$

are inverses (as $\mathbb{C}(\hbar)$-algebra homomorphisms) of each other. In particular, since the constant map to a point factors (equivariantly!) through the fixed loci, for any equivariant cohomology class α:

$$\int_X \alpha = \sum_i \int_{F_i} \frac{i^*(\alpha)}{e(N_{F_i/X})}$$

In practice, one can reduce the problem of integrating classes on a space X, which might be geometrically complicated, to integrating over the fixed loci (which are hopefully simpler).

Example 5.4 (The case of $\mathbb{P}^1$)**.** Let $\mathbb{C}^*$ act on a two dimensional vector space V by:

$$t \cdot (v_0, v_1) := (v_0, tv_1)$$

This action defines an action on the projectivization $\mathbb{P}(V) = \mathbb{P}^1$. The fixed points for the torus action are $0 = (1:0)$ and $\infty = (0:1)$. The canonical action on $T_{\mathbb{P}}$ has weights $+1$ at 0 and -1 at ∞. Identifying $V \setminus 0$ with the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$ minus the zero section, we get a canonical lift of the torus action to $\mathcal{O}_{\mathbb{P}^1}(-1)$, with weights $0, 1$. Also, since $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(-1)^\vee$, we get a natural linearization for

$\mathcal{O}_{\mathbb{P}^1}(1)$ as well (with weights $0, -1$). Finally, by thinking of $\mathbb{P}^1$ as the projectivization of an equivariant bundle over a point, we obtain:

$$H^*_{\mathbb{C}^*}(\mathbb{P}^1) = \frac{\mathbb{C}[H, \hbar]}{H(H-\hbar)}.$$

The Atyiah-Bott isomorphism now reads:

$$\begin{array}{ccc}
\mathbb{C}(\hbar)_0 \oplus \mathbb{C}(\hbar)_\infty & \leftrightarrow & H^*_{\mathbb{C}^*}(\mathbb{P}) \otimes \mathbb{C}(\hbar) \\
(1,0) & \rightarrow & \frac{H}{\hbar} \\
(0,1) & \rightarrow & \frac{H-\hbar}{-\hbar} \\
(1,1) & \leftarrow & 1 \\
(\hbar, 0) & \leftarrow & H
\end{array}$$

5.2. **Applying the Localization Theorem to Spaces of Maps.** Kontsevich first applied the localization theorem to smooth moduli spaces of maps in [Kon95]. Graber and Pandharipande ([GP99]) generalized this technique to the general case of singular moduli spaces, showing that localization "plays well" with the virtual fundamental class. Several subsequent applications by Okounkov-Pandharipande, Graber-Vakil, Bertram and many other have crowned it as an extremely powerful technique for intersection theory on the moduli space of stable maps. In [Cav06b], [Cav05], the author began applying localization to moduli spaces of admissible covers, technique that was subsequently framed into the larger context of orbifold Gromov Witten theory via the foundational work of [AGV06].

Let X be a space with a $\mathbb{C}^*$ action, admitting a finite number of fixed points P_i, and of fixed lines l_i (NOT pointwise fixed). Typical examples are given by projective spaces, flag varieties, toric varieties... Then:

(1) A $\mathbb{C}^*$ action is naturally induced on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ by postcomposition.
(2) The fixed loci in $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parameterize maps from nodal curves to the target such that (see Figure **??**):
    - components of arbitrary genus are contracted to the fixed points P_i.
    - rational components are mapped to the fixed lines as d-fold covers fully ramified over the fixed points.

    In particular
$$F_i \cong \prod \overline{\mathcal{M}}_{g_j, n_j} \times \prod \mathcal{B}\mathbb{Z}_{d_k}.$$
(3) The **"virtual"** normal directions to the fixed loci correspond essentially to either smoothing the nodes of the source curve

(which by exercise 19 produces sums of ψ classes and equivariant weights), or to deforming the map out of the fixed points and lines. This can be computed using the deformation exact sequence ([HKK$^+$03], (24.2)), and produces a combination of equivariant weights and λ classes.

MAKE PICTURE!!

The punchline is, one has reduced the tautological intersection theory of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to combinatorics, and Hodge integrals (i.e. intersection theory of λ and ψ classes). From a combinatorial point of view this can be an extremely complicated and often unmanageable problem, but in principle application of the Grothendieck-Riemann-Roch Theorem and of Witten Conjecture/Kontsevich's Theorem completely determine all Hodge integrals. Carel Faber in [Fab99] explained this strategy and wrote a Maple code that can handle efficiently integrals up to a certain genus and number of marks.

6. Evaluation of The Hyperelliptic Locus

We apply localization to moduli spaces of admissible covers to give a proof of a theorem of Faber and Pandharipande that bypasses the use of Grothendieck-Riemann-Roch and only relies on the combinatorics of simple Hurwitz numbers. This proof is independent of the original proof and it is the $d=2$ case of Theorem 0.2 in [BCT06].

Theorem 6.1 ([FP00], Corollary of Proposition 3)**.** *Denote by H_g a $(2g+2)!$ cover of the hyperlliptic locus in $\mathcal{M}_g$ obtained by marking the Weierstrass points. Then:*

$$\sum_{g=1}^{\infty}\left(\int_{H_g}\lambda_g\lambda_{g-1}\right)\frac{x^{2g-1}}{(2g-1)!}=\frac{1}{2}\tan\left(\frac{x}{2}\right). \tag{15}$$

Observations.

(1) Since the class $\lambda_g\lambda_{g-1}$ vanishes on the boundary of $\overline{\mathcal{M}}_g$, the above integral can be performed on the closure of the hyperelliptic locus. We choose the open statement because of the original application that led to study the problem: the class of the hyperelliptic locus is a generator for the socle in the tautological ring of $\mathcal{M}_g$.

(2) Choosing the appropriate generating function packaging is key to solving these questions. While (15) is probably the most appealing form of the result, we prove the equivalent integrated

version:

$$\mathcal{D}_1(x) := \sum_{g=1}^{\infty} \left(\int_{H_g} \lambda_g \lambda_{g-1} \right) \frac{x^{2g}}{(2g)!} = -\ln\cos\left(\frac{x}{2}\right).$$

(3) We essentially use the identification described in Exercise 17:

$$\overline{H_g} \cong Adm_{g\to 0,2}((2),\ldots,(2)) \cong \text{“}\frac{1}{2}\text{”}\overline{\mathcal{M}}_{0,2g+2}.$$

to translate the geometric problem into a combinatorial one.

6.1. **Outline of proof.** We first introduce generating functions for other Hodge integrals on the Hyperelliptic locus:

$$\mathcal{D}_i(x) := \sum_{g\geq i} \left(\int_{\overline{H}_g} \lambda_g \lambda_{g-i} \psi^{i-1} \right) \frac{x^{2g}}{2g!}. \tag{16}$$

For reasons that will become evident in a few lines, we also define

$$\mathcal{D}_0(x) := \frac{1}{2}. \tag{17}$$

Now the proof of (15) follows from combining the following two ingredients. First a way to describe all $\mathcal{D}_i$'s in terms of $\mathcal{D}_1$.

Lemma 6.2 ([Cav06a],Theorem 1)**.**

$$\mathcal{D}_i(x) = \frac{2^{i-1}}{i!}\mathcal{D}_1^i(x).$$

or equivalently

$$\sum_{i=0}^{\infty} \mathcal{D}_i(x) = \frac{1}{2}\ e^{2\mathcal{D}_1(x)}$$

Second, an interesting way to write the identity $0 = 0$:

Lemma 6.3. *The integral*

$$\int_{Adm_{g\to\mathbb{P}^1}} e(R^1\pi_* f^*(\mathcal{O}\oplus\mathcal{O}(-1))) = 0 \tag{18}$$

implies the relation:

$$\frac{1}{2}(\cos(x)-1) = \frac{1}{2}\sin(x)\left(\sum_{i=0}^{\infty} \int \mathcal{D}_i(x)\right) \tag{19}$$

Exercise 28. Given the two lemmas, conclude the proof. This is in fact a Calc II exercise!

Remarks.

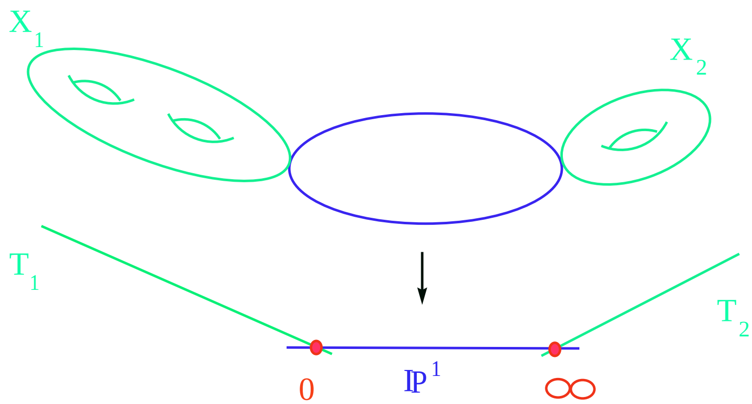


FIGURE 6. The fixed loci for integrals of admissible covers to a parameterized $\mathbb{P}^1$ consist of covers where all ramification happens over 0 and ∞ - or rather over rational tails sprouting from these points.

(1) Note that the auxiliary integral (18) is on a moduli space of admissible covers of a parameterized $\mathbb{P}^1$. This allows the moduli space to have a $\mathbb{C}^*$ action. The fixed loci however are boundary strata consisting of products of admissible cover spaces of an unparameterized rational curve, illustrated in Figure 6.

(2) Lemma 6.2 generalizes nicely to the case of multiple ψ insertions. For I a multi-index of size $i-1$

$$\sum_{g\geq i}\left(\int_{\overline{H}_g}\lambda_g\lambda_{g-i}\psi^I\right)\frac{x^{2g}}{2g!}=\binom{i-1}{I}\frac{2^{i-1}}{i!}\mathcal{D}_1^i(x). \tag{20}$$

Formula (20) was experimentally discovered independently by the author and Danny Gillam in 2007. A proof of (20) was given by the author and his PhD student Dusty Ross in [CR11].

6.1.1. *Proof of Lemma 6.2.* We compute via localization the following auxiliary integrals:

$$A_k := \int_{Adm_{g\to\mathbb{P}^1}}\lambda_g\lambda_{g-k}ev_1^*(0)ev_2^*(0)ev_3^*(\infty)=0$$

For $k>1$, the integral is 0 for trivial dimension reasons. Computing the integral via localization however yields a non-trivial relation among Hodge integrals. We observe the following vanishings:

(1) The class λ_g vanishes on curves not of compact type. Therefore all fixed loci having contracted components over both 0 and

∞ and where the double cover over the main component is unramified vanish.

(2) Since we are asking ramification both over 0 and ∞ the fixed loci having only one contracting component and unramified double cover over the main component also do not contribute.

(3) Since we are asking for (at least) two branch points to have gone to 0 we cannot have no contracted component over infinity.

The contributing fixed loci consist of a double cover of the main component, ramified over 0 and ∞, and either a lone genus g curve contracting over 0 (this gives us the principal part in the relation) or two contracting curves over 0 and ∞ of positive genera adding to g. For a fixed genus g the localization then relation reads:

(21)
$$\int_{\overline{H}_g} \lambda_g \lambda_{g-k} \psi^{k-1} = 2 \sum_{\substack{g_1+g_2=g \\ g_1, g_2 > 0}} \binom{2g-1}{2g_2} \sum_{i=1}^{k-1} (-1)^i \left(\int_{\overline{H}_{g_1}} \lambda_{g_1} \lambda_{g_1-k+i} \psi^{k-i-1} \right) \left(\int_{\overline{H}_{g_2}} \lambda_{g_2} \lambda_{g_2-i} \psi^{i-1} \right)$$

Relation (21) computes the integral of level k inductively in terms of lower values of k. To conclude the proof of Lemma 6.2 it suffices to package the relations coming from all genera as the relation of generating functions,

$$\sum_{i=0}^{k} (-1)^i \mathcal{D}'_{k-i}(x) \mathcal{D}_i(x) = 0 \tag{22}$$

and observe that the formula of Lemma 6.2 verifies (22) by Newton's binomial theorem.

6.1.2. *Proof of Lemma 6.3.* Deriving relation (19) from the auxiliary integral (18) requires an appropriate choice of lifting of the torus action to the bundles $\mathcal{O}$ and $\mathcal{O}(-1)$. We choose to linearize the trivial bundle with weight 1 everywhere, and the tautological bundle with weight -1 over 0 and 0 over ∞.

The zero weight for $\mathcal{O}(-1)$ over ∞ forces contributing localization graphs to be one-valent over infinity.

The opposite weights for the two bundles over 0 give, by Mumford relation (10) a sign and a Hurwitz number contribution by the moduli spaces of the covers sprouting over 0. Over infinity we obtain have the integrals that we wish to compute. The relation consists of a linear term, corresponding to a unique curve of genus g attaching over 0 to an etale double cover of the main $\mathbb{P}^1$, and a quadratic term, correspond to all fixed loci consisting of a ramified double cover of the main component with one or two curves of genera adding to g, weighted by some combinatorial coefficients encoding how many ways the branch points can arrange themselves over 0 and ∞ in order to give a given pair of genera.

Writing the relation for all genera in generating function form one obtains (19).

Exercise 29. Fill in the details of this proof. It's kind of fun...

7. Simple Hurwitz Numbers and the ELSV Formula

The name **simple Hurwitz number** (denoted $H_g(\eta)$) is reserved for Hurwitz numbers to a base curve of genus 0 and with only one special point where arbitrary ramification is assigned. In this case the number of simple ramification, determined by the Riemann-Hurwitz formula, is

$$r = 2g + d - 2 + \ell(\eta). \tag{23}$$

The combinatorial definition (3) of Hurwitz number simplifies further to count (up to an appropriate multiplicative factor) the number of ways to factor a (fixed) permutation $\sigma \in C_\eta$ into r transpositions that generate S_d:

(24)

$$H_g(\eta) = \frac{1}{\prod \eta_i} |\{(\tau_1, \ldots, \tau_r \ s.t. \ \tau_1 \cdot \ldots \cdot \tau_r = \sigma \in C_\eta, \langle \tau_1, \ldots, \tau_r \rangle = S_d)\}|$$

Exercise 30. Prove that (24) is indeed equivalent to (3).

The first formula for simple Hurwitz number was given and "sort of" proven by Hurwitz in 1891 ([Hur91]):

$$H_0(\eta) = r! d^{n-3} \prod \frac{\eta_i^{\eta^i}}{\eta_i!}.$$

Particular cases of this formula were proven throughout the last century, and finally the formula became a theorem in 1997 ([GJ97]). In studying the problem for higher genus, Goulden and Jackson made the following conjecture.

Conjecture. *For any fixed values of $g, n := \ell(\eta)$:*

$$H_g(\eta) = r! \prod \frac{\eta_i^{\eta^i}}{\eta_i!} P_{g,n}(\eta_1 \ldots, \eta_n), \tag{25}$$

where $P_{g,n}$ is a symmetric polynomial in the η_i's with:

- *deg $P_{g,n} = 3g - 3 + n$;*

- *$P_{g,n}$ doesn't have any term of degree less than $2g-3+n$;*
- *the sign of the coefficient of a monomial of degree d is $(-1)^{d-(3g+n-3)}$.*

In [ELSV01] Ekehdal, Lando, Shapiro and Vainshtein prove this formula by establishing a remarkable connection between simple Hurwitz numbers and tautological intersections on the moduli space of curves.

Theorem 7.1 (ELSV formula)**.** *For all values of $g, n = \ell(\eta)$ for which the moduli space $\overline{\mathcal{M}}_{g,n}$ exists:*

$$H_g(\eta) = r! \prod \frac{\eta_i^{\eta^i}}{\eta_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \ldots + (-1)^g \lambda_g}{\prod (1 - \eta_i \psi_i)}, \tag{26}$$

Remark 7.2. Goulden and Jackson's polynomiality conjecture is proven by showing the coefficients of $P_{g,n}$ as tautological intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Using our standard multi-index notation:

$$P_{g,n} = \sum_{k=0}^{g} \sum_{|I_k| = 3g-3+n-k} (-1)^k \left(\int \lambda_k \psi^{I_k} \right) \eta^{I_k}$$

Remark 7.3. The polynomial $P_{g,n}$ is a generating function for all linear (meaning where each monomial has only one λ class) Hodge integrals on $\overline{\mathcal{M}}_{g,n}$, and hence a good understanding of this polynomial can yield results about the intersection theory on the moduli space of curves. In fact the *ELSV* formula has given rise to several remarkable applications:

[OP09]**:** Okounkov and Pandharipande use the ELSV formula to give a proof of Witten's conjecture, that an appropriate generating function for the ψ intersections satisfies the KdV hierarchy. The ψ intersections are the coefficients of the leading terms of $P_{g,n}$, and hence can be reached by studying the asymptotics of Hurwitz numbers:

$$\lim_{N \to \infty} \frac{P_{g,n}(N\eta)}{N^{3g-3+n}}$$

[GJV06]**:** Goulden, Jackson and Vakil get a handle on the lowest order terms of $P_{g,n}$ to give a new proof of the λ_g conjecture:

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi^I = \binom{2g-3+n}{I} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2}$$

We sketch a proof of the *ELSV* formula following [GV03a]. The strategy is to evaluate an integral via localization, fine tuning the geomety in order to obtain the desired result.

We denote

$$\mathcal{M} := \overline{\mathcal{M}}_g(\mathbb{P}^1, \eta\infty)$$

the moduli space of relative stable maps of degree d to $\mathbb{P}^1$, with profile η over ∞. The degenerations included to compactify are twofold:

- away from the preimages of ∞ we have degenerations of "stable maps" type: we can have nodes and contracting components for the source curve, and nothing happens to the target $\mathbb{P}^1$;
- when things collide at ∞, then the degeneration is of "admissible cover" type: a new rational component sprouts from $\infty \in \mathbb{P}^1$, the special point carrying the profile requirement transfers to this component. Over the node we have nodes for the source curve, with maps satisfying the kissing condition.

The space $\mathcal{M}$ has virtual dimension $r = 2g + d + \ell(\eta) - 2$ and admits a globally defined branch morphism ([FP02]):

$$br : \mathcal{M} \to Sym^r(\mathbb{P}^1) \cong \mathbb{P}^r.$$

The simple Hurwitz number:

$$H_g(\eta) = \deg(br) = br^*(pt.) \cap [\mathcal{M}]^{vir}$$

can now interpreted as an intersection number on a moduli space with a torus action and evaluated via localization. The map br can be made $\mathbb{C}^*$ equivariant by inducing the appropriate action on $\mathbb{P}^r$. The key point is now to choose the appropriate equivariant lift of the class of a point in $\mathbb{P}^r$. Recalling that choosing a point in $\mathbb{P}^r$ is equivalent to fixing a branch divisor, we choose the $\mathbb{C}^*$ fixed point corresponding to stacking all ramification over 0. Then there is a unique fixed locus contributing to the localization formula, depicted in Figure 7, which is essentially isomorphic to $\overline{\mathcal{M}}_{g,n}$ (up to some automorphism factors coming from the automorphisms of the bubbles over $\mathbb{P}^1$).

The *ELSV* formula falls immediately out of the localization formula. The virtual normal bundle to the unique contributing fixed locus has a denominator part given from the smoothing of the nodes that produces the denominator with ψ classes in the ELSV formula. Then there is the equivariant euler class of the derived push-pull of $T\mathbb{P}^1(-\infty)$: when restricted to the fixed locus this gives a Hodge bundle linearized with weight 1, producing the polynomial in λ classes, and a bunch of trivial but not equivariantly trivial bundles corresponding to the restriction of the push-pull to the trivial covers of the main components. The equivariant euler class of such bundles is just the product of the corresponding weights, and gives rise to the combinatorial pre-factors before the Hodge integral.

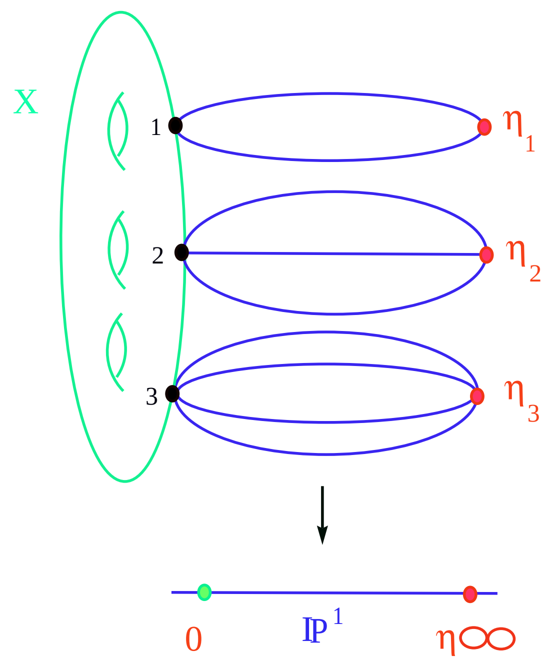


FIGURE 7. the unique contributing fixed locus in the localization computation proving the *ELSV* formula.

Remark 7.4. An abelian orbifold version of the ELSV formula has been developed by Johnson, Pandharipande and Tseng in [JPT11]. In this case the connection is made between Hurwitz-Hodge integrals and wreath Hurwitz numbers. I will leave it to Paul to explain this work.

8. Double Hurwitz Numbers

Double Hurwitz numbers count covers of $\mathbb{P}^1$ with special ramification profiles over two points, that for simplicity we assume to be 0 and ∞. Double Hurwitz numbers are classically denoted $H_g^r(\mu, \nu)$; in [CJM11] we start denoting double Hurwitz numbers $H_g^r(\mathbf{x})$, for $\mathbf{x} \in H \subset \mathbb{R}^n$ an integer lattice point on the hyperplane $\sum x_i = 0$. The subset of positive coordinates corresponds to the profile over 0 and the negative coordinates to the profile over ∞. We define $\mathbf{x_0} := \{x_i > 0\}$ and $\mathbf{x}_\infty := \{x_i < 0\}$.

The number r of simple ramification is given by the Riemann-Hurwitz formula,

$$r = 2g - 2 + n$$

and it is independent of the degree d. In [GJV03], Goulden, Jackson and Vakil start a systematic study of double Hurwitz numbers and in particular invite us to consider them as a function:

$$H_g^r(-) : \mathbb{Z}^n \cap H \to \mathbb{Q}. \tag{27}$$

They prove some remarkable combinatorial property of this function:

Theorem 8.1 (GJV). *The function $H_g(-)$ is a piecewise polynomial function of degree $4g-3+n$.*

And conjecture some more:

Conjecture (GJV). *The polynomials describing $H_g^r(-)$ have degree $4g-3+n$, lower degree bounded by $2g-3+n$ and are even or odd polynomials (depending on the parity of the leading coefficient).*

Later, Shapiro, Shadrin and Vainshtein explore the situation in genus 0. They describe the location of all walls, and give a geometrically suggestive formula for how the polynomials change when going across a wall.

Theorem 8.2 (SSV). *The chambers of polynomiality of $H_g^r(-)$ are bounded by* ***walls*** *corresponding to the* ***resonance*** *hyperplanes W_I, given by the equation*

$$W_I = \left\{ \sum_{i \in I} x_i = 0 \right\},$$

for any $I \subset \{1, \dots, n\}$.

Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two chambers adjacent along the wall W_I, with $\mathfrak{c}_1$ being the chamber with $x_I < 0$. The Hurwitz number $H_g^r(\mathbf{x})$ is given by polynomials, say $P_1(\mathbf{x})$ and $P_2(\mathbf{x})$, on these two regions. A wall crossing formula is a formula for the polynomial

$$WC_I^r(\mathbf{x}) = P_2(\mathbf{x}) - P_1(\mathbf{x}).$$

Genus 0 wall crossing formulas have the following inductive description:

$$WC_I^r(\mathbf{x}) = \delta \binom{r}{r_1, r_2} H^{r_1}(\mathbf{x_I}, \delta) H^{r_2}(\mathbf{x_{I^c}}, -\delta), \tag{28}$$

where $\delta = \sum_{i \in I} x_i$ is the distance from the wall at the point we evaluate the wall crossing.

Remarks.

(1) This formula appears not to depend on the particular choice of chambers $\mathfrak{c}_1$ and $\mathfrak{c}_2$ that border on the wall, but only upon the wall W_I; however the polynomials for the simpler Hurwitz numbers in the formula depend on chambers themselves.
(2) The walls W_I correspond to values of $\mathbf{x}$ where the cover could potentially be disconnected, or where $\mathbf{x}_i = 0$. In the first case the formula reminds of a boundary divisor degeneration formula, and somehow begs for a geometric understanding.
(3) Crossing this second type of wall corresponds to moving a ramification between 0 and ∞. In the traditional view of double Hurwitz numbers, these were viewed as separate problems: the lenght of the profiles over 0 and ∞ were fixed separately, rather than just the total length. However, here we see that it is natural to treat them as part of the same problem: in genus 0 the wall crossing formula for $\mathbf{x}_i = 0$ is trivial - and as such identical to all other wall crossing formulas. This motivates our $\mathbf{x}$ replacing μ, ν in our notation. Note that I am note just being cute here. Let me preview that in higher genus this second type of wall crossing are not trivial any more, while still obeying the same wall crossing formulas as wall crossing of the first type.

The way Goulden, Jackson and Vakil prove their result is similar to [OP09]: they compute double Hurwitz numbers by counting decorated ribbon graphs on the source curve. A ribbon graph is obtained by pulling back a set of segments from the base curve (connecting 0 to the simple ramification points) and then stabylizing. Each ribbon graph comes with combinatorial decorations that are parameterized by integral points in a polytope with linear boundaries in the x_i's. Standard algebraic combinatorial techniques then show that such counting yields polynomials so long as the topology of the various polytopes does not change. The downside of this approach is that these are pretty "large" polytopes and it is hard to control their topology.

Shapiro, Shadrin and Vainshtein go at the problem with a geometric angle, and are able to prove the wall crossing formulas using some specific properties of intersections of ψ classes in genus 0. Since descendants become quickly more mysterious in higher genus, their approach didn't generalize.

In what follows I'd like to present the approach of [CJM10] to this problem. Motivated by tropical geometry, we are able to compute double Hurwitz numbers in terms of some trivalent polynomially weighted graphs (that can be thought as tropical covers, even though this point of view is not necessary other than to give the initial motivation) that are, in a sense, "movies of the monodromy representation". These graphs give a straightforward and clean proof of the genus 0 situation. In [CJM11] we show that in higher genus each graph Γ comes together with a polytope P_Γ (with homogenous linear boundaries in the x_i) and we have to sum the polynomial weight of the graph over the integer lattice points of P_Γ. It is again standard (think of it as a discretization of integrating a polynomial over a polytope) to show that this contribution is polynomial when the topology of the polytope does not change. The advantage here is that we have shoved most of the complexity of the situation in the polynomial weights of the graph: our polytopes are only g dimensional and it is possible to control their topology. Thus in [CJM11], we are able to give a complete description of the situation for arbitrary genus.

8.1. **The Cut and Join Recursions and Tropical Hurwitz Numbers.** The *Cut and Join equations* are a collection of recursions among Hurwitz numbers. In the most elegant and powerful formulation they are expressed as one differential operator acting on the Hurwitz potential. Our use of cut and join here is unsophisticated, so we limit ourselves to a basic discussion, and refer the reader to [GJ99] for a more in-depth presentation.

Let $\sigma \in S_d$ be a fixed element of cycle type $\eta = (n_1, \ldots, n_l)$, written as a composition of disjoint cycles as $\sigma = c_l \ldots c_1$. Let $\tau = (ij) \in S_d$ vary among all transpositions. The cycle types of the composite elements $\tau\sigma$ are described below.

cut: if i, j belong to the same cycle (say c_l), then this cycle gets "cut in two": $\tau\sigma$ has cycle type $\eta' = (n_1, \ldots, n_{l-1}, m', m'')$, with $m' + m'' = n_l$. If $m' \neq m''$, there are n_l transpositions giving rise to an element of cycle type η'. If $m' = m'' = n_l/2$, then there are $n_l/2$.

join: if i, j belong to different cycles (say c_{l-1} and c_l), then these cycles are "joined": $\tau\sigma$ has cycle type $\eta' = (n_1, \ldots, n_{l-1} + n_l)$. There are $n_{l-1}n_l$ transpositions giving rise to cycle type η'.

Example 8.3. Let $d = 4$. There are 6 transpositions in S_4. If $\sigma = (12)(34)$ is of cycle type $(2, 2)$, then there are 2 transpositions ((12) and (34)) that "cut" σ to give rise to a transposition and $2 \cdot 2$ transpositions $((13), (14), (23), (24))$ that "join" σ into a four-cycle.

For readers allergic to notation, Figure **??** illustrates the above discussion. MAKE FIGURE in PDF

Let us now specialize our definition of Hurwitz number by counting monodromy representations to the case of double Hurwitz numbers.

$$H_g^r(\mathbf{x}) := \frac{|\mathrm{Aut}(\mathbf{x_0})||\mathrm{Aut}(\mathbf{x}_\infty)|}{d!}|\{\sigma_0, \tau_1, \ldots, \tau_r, \sigma_\infty \in S_d\}|$$

such that:

- σ_0 has cycle type $\mathbf{x_0}$;
- τ_i's are simple transpositions;
- σ_∞ has cycle type $\mathbf{x}_\infty$;
- $\sigma_0\tau_1 \ldots \tau_r\sigma_\infty = 1$
- the subgroup generated by such elements acts transitively on the set $\{1, \ldots, d\}$.

The key insight of [CJM10] is that we can organize this count in terms of the cycle types of the composite elements

$$C_{\mathbf{x_0}} \ni \sigma_0, \sigma_0\tau_1, \sigma_0\tau_1\tau_2, \ldots, \sigma_0\tau_1\tau_2 \ldots \tau_{r-1}, \sigma_0\tau_1\tau_2 \ldots \tau_{r-1}\tau_r \in C_{\mathbf{x}_\infty}$$

At each step the cycle type can change as prescribed by the cut and join recursions, and hence for each possibility we can construct a graph with edges weighted by the multiplicites of the cut and join equation. In [CJM11] we call such graphs monodromy graphs.

Definition 8.4. For fixed g and $\mathbf{x} = (x_1, \ldots, x_n)$, a graph Γ is a **monodromy graph** if:

1. Γ is a connected, genus g, directed graph.
2. Γ has n 1-valent vertices called *leaves*; the edges leading to them are *ends*. All ends are directed inward, and are labeled by the weights $x_1, \ldots, x_n$. If $x_i > 0$, we say it is an *in-end*, otherwise it is an *out-end*.
3. All other vertices of Γ are 3-valent, and are called *internal vertices*. Edges that are not ends are called *internal edges*.
4. After reversing the orientation of the out-ends, Γ does not have directed loops, sinks or sources.
5. The internal vertices are ordered compatibly with the partial ordering induced by the directions of the edges.
6. Every internal edge e of the graph is equipped with a *weight* $w(e) \in \mathbb{N}$. The weights satisfy the *balancing condition* at each internal vertex: the sum of all weights of incoming edges equals the sum of the weights of all outgoing edges.

So we can compute Hurwitz number as a weighted sum over monodromy graphs. Keeping in account the various combinatorial factors one obtains:

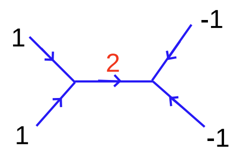
Lemma 8.5 ([CJM10], Lemma 4.1)**.** *The Hurwitz number is computed as:*

$$H_g(\mathbf{x}) = \sum_{\Gamma} \frac{1}{|Aut(\Gamma)|}\varphi_\Gamma, \tag{29}$$

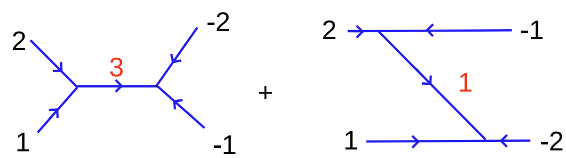
where the sum is over all monodromy graphs Γ *for* g *and* $\mathbf{x}$*, and* φ_Γ *denotes the product of weights of all internal edges.*

Example 8.6. Here are a couple silly examples computed using formula 29.

$$H^0((1,1),-(1,1)) = 2$$



$$H^0((2,1),-(2,1)) = 4$$



Now instead of individual numbers, we want to compute the Hurwitz functions. In figure **??** we illustrate the situation. We change a little the notation, using y's for the coordinates at ∞. We observe that different graphs contribute according to the sign of $x_1 + y_1$. This gives us two different polynomials, the difference of which is given by formula 28 (in a trivial way since three pointed genus zero double Hurwitz numbers are trivially seen to be 1).

8.2. **Genus** 0**.** In this section we show how this point of view leads to fairly elementary proofs of Theorems 8.1 and 8.2.

(1) Each edge weight is linear homogeneous in the x_i's and we have $3g-3+n$ internal edges. Therefore each contributing graph gives a polynomial of the appropriate degree. A graph contributes if and only if all edge weights are positive. It's clear then that the regions of polynomiality are precisely where the signs of all edge weights persist.
(2) For a given wall W_I, graphs that appear on both sides of the wall give no contribution to the wall crossing. A graph appears on both sides of the wall if and only if no edge changes sign across the wall, i.e. if and only if there is no edge with weight $\delta = \pm\sum_{i\in I} x_i$.
(3) So in order to compute the wall crossing formula, we only need to focus on graphs that contain an edge labelled $\pm\delta$. In particular if the edge is labelled δ, the graph appears on one side of

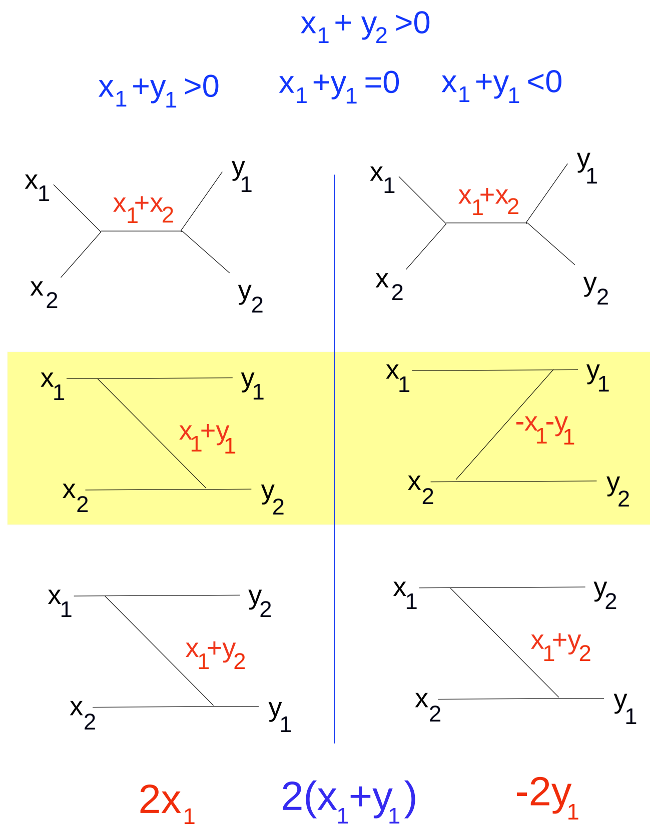


FIGURE 8. Computing double Hurwitz numbers using Lemma 8.5 and observing the wall crossing.

the wall, if it is labelled $-\delta$ the graph appears on the other side of the wall. Keeping in mind that the wall crossing formula is the subtraction of the Hurwitz polnomyals on either side of the wall, the polynomial contribution from any graph (no matter on which side of the wall it is on) Γ is δ times the product of the weights of all other internal edges.

(4) Now look at the RHS of (28). Each of the Hurwitz numbers appearing can be computed as a weighted sum over the appropriate monodromy graphs, and therefore the product of Hurwitz numbers can be computed as a weighted sum over pairs of graphs. For a fixed pair of graphs the polynomial contribution is δ times the product of internal edges of both graphs.

(5) We now prove formula (28) by exhibiting a bijection that preserves polynomial contributions between:
   (a) the set L of graphs contributing to the wall crossing.

(b) a set R of cardinality $\binom{r}{r_1, r_2}$ times the cardinality of the set of pairs of graphs contributing to the product of Hurwitz numbers on the right.

First we describe R as the set of triples $(\Gamma_1, \Gamma_2, \mathfrak{o})$, where Γ_1 and Γ_2 are graphs contributing to the respective Hurwitz numbers, and $\mathfrak{o}$ is a total ordering ther vertices of $\Gamma_1 \cup \Gamma_2$ compatible with the individual vertex orderings of both graphs.

Exercise 31. Convince yourself that R has indeed the desired cardinality.

Now consider the functions

$$Cut : L \to R$$

that cuts each graph along the edge labelled δ, and

$$Glue : R \to L$$

that glues the two graphs Γ_1, Γ_2 along the ends labelled δ orienting the new edge as prescribed by the total order $\mathfrak{o}$.

Exercise 32. Convince yourself that Cut and $Glue$ are inverses to each other.

9. Higher Genus

Lemma 8.5 is not special to genus 0, and gives us a combinatorial recipe for computing arbitrary double Hurwitz numbers. However two significant complications arise when trying to generalize the previous theorems to higher genus:

(1) In genus 0 the balancing condition and the weights at the ends determine uniquely the weights of all internal edges. It is easy to see that in higher genus this is not true, and that in fact each genus adds a degree of freedom in the choice of weights. In other words, given a directed, vertex ordered graph with labelled ends, there is a g-dimensional polytope parameterizing internal edge weights compatible with the end weights and the balancing condition. The bounds of this polytope are linear in the x_i's.

(2) In genus 0 a graph contributing to the wall crossing had a unique edge labelled δ, and consequently a unique way to be disconnected into two smaller graphs. In higher genus this is not the case any more: there are multiple edges that can be "cut" when crossing the wall, and multiple ways to disconnect the graph. A careful project of inclusion/exclusion is then required to obtain the result.

To recover Theorem **??** one has to deal with (1), and it is really not too bad. Our Hurwitz number is still expressed as a finite sum over graphs, except now for each graph the contribution is a homogenous polynomial of degree $3g-3+n$ in the x_i's plus g new variables, that we need to "integrate" over the integer lattice points of a g dimensional polytope. You can think of this as a generalization of continuous integration, or as a repeated application of the power sum formulas; in either case it is not hard to see that the result is locally a polynomial in the x_i's of degree $4g-3+n$.

Where does piecewise polynomiality kick in? Well, as you move the x_i's around the various graph polytopes change their topology (some face could get hidden or uncovered by other faces translating around). That is precisely where the walls are. Again a little bit of analysis shows that this happen precisely when multiple hyperplanes defining the polytope intersect non-transversally, and this happens when the graph can be disconnected.

Here we wish to focus on the much more subtle (and interesting) case of generalizing the wall crossing formula. Rather than a complete discussion of the proof (a hopefully reasonable outline of which can be found in section 4 of [CJM11]), here we would like to illustrate the salient ideas through two examples, each one specifically tuned to illustrate dealing with points (1) and (2) above. But before we embark into this endeavour, let us start with stating the result.

Theorem 9.1.
(30)
$$WC^r_I(\mathbf{x}) = \sum_{\substack{s+t+u=r\\ |\mathbf{y}|=|\mathbf{z}|=|\mathbf{x}_I|}} (-1)^t \binom{r}{stu} \frac{\prod \mathbf{y}_i}{\ell(\mathbf{y})!}\frac{\prod \mathbf{z}_j}{\ell(\mathbf{z})!} H^s(\mathbf{x}_I,\mathbf{y})H^{t\bullet}(-\mathbf{y},\mathbf{z})H^u(\mathbf{x}_{I^c},-\mathbf{z})$$

Here $\mathbf{y}$ *is an ordered tuple of* $\ell(\mathbf{y})$ *positive integers with sum* $|\mathbf{y}|$*, and similarly with* $\mathbf{z}$*.*

9.1. **The simple cut: Hyperplane Arrangements and The "Cut-Glue" Geometric Bijection.** In this section we focus on an example of how a geometric bijection can be defined between graphs contributing to the wall crossing, and graphs contributing to the degeneration formula on the RHS of (30). The idea distills to its simplest in the case of graphs that can be cut in only one way.

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $r = 6$ (i.e. $g = 2$). Let $I = \{1, 3\}$, and let $\mathfrak{c}_1$ be an H-chamber next to the wall W_I satisfying $x_1+x_3 \leq 0$. $\mathfrak{c}_2$ is the opposite H-chamber. In Figure 9 we consider a graph Γ contributing to the wall crossing for this particular wall. Really, we want to think of Γ as an un-directed graph. The direction of the edges we show in this picture are to be thought of as a choice of reference orientation, and we understand that putting a positive weight on an edge preserves the reference orientation, while a negative weight reverses it. Now we observe:

- For any choice of labeling of the ends, there is a 2-dimensional plane of possible weights for the edges satisfying the balancing condition.
- This plane is subdivided by hyperplanes (lines) whose equations are given precisely by the internal edge weights.
- Chambers for this hyperplane arrangement correspond to orientations of the edges of the graph. Unbounded chambers correspond to orientations with directed loops, bounded chambers correspond to orientation with NO directed loop.
- We give each chamber a multiplicity consisting of the number of vertex orderings of the directed graph compatible with the directions of the edges. Note that unbounded chambers have multiplicity 0.

Figure 9 illustrate this situation over two points living on either side of wall W_I.

At the wall the the three red hyperplanes corresponding to the three edges that can be cut to disconnect the graph, meet in codimension 2.

On one side of the wall, these three hyperplanes form a simplex which vanishes when we hit the wall (*vanishing F-chamber*). A new simplex reappears on the other side of the wall (an *appearing F-chamber*).

The directed graph corresponding to the appearing chamber has a flow from top to bottom, that can only be realized when $x_1+x_3 \geq 0$, i.e. on side "2" of the wall. Thus we can see from the graphs whether an F-chamber is vanishing/appearing or not. The 6 neighboring chambers appear on both sides of the wall.

When computing the Hurwitz number, for each directed graph we need to sum the product of internal edges over the integer points of the corresponding chamber. Note that for all directed graphs with the same underlying graph, the product of internal edges ($= \varphi_\Gamma$) differs at most by a sign, and that this signs alternates in adjacent chambers. To understand the contribution to P_2, we sum the polynomial $\varphi_{\mathcal{A}}$ (weighted with sign and multiplicity) over the lattice points in each of the chambers A, B, ..., G. For the polynomial P_1, we have to play the same game with the chambers B', ..., H' on top, however, we evaluate this polynomial now at the point $\mathbf{x}_2$ which is not in $\mathfrak{c}_1$ but in $\mathfrak{c}_2$. Thus, we have to "carry" the chambers $B', \ldots, H'$ over the wall, i.e. we

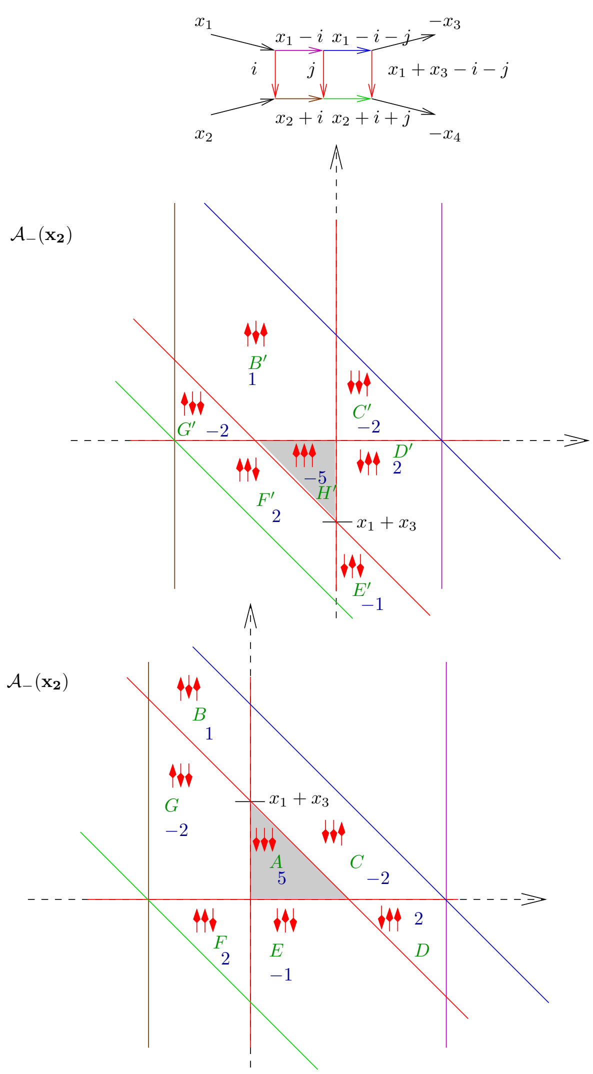


FIGURE 9. Hyperplane arrangements corresponding to orientations of edges of a graph.

need to interpret the region bounded e.g. by the defining hyperplanes of B' on the other side of the wall in terms of the chambers $A, \dots, G$. **Note:** there are several things to be careful about signs in these discussions, but I would like to make the expositional choice of completely ignoring any sign issue, and ask for your trust that everything pans out as desired at the end of the day. Or rather, if you are really hardcore...

Exercise 33. Check my signs! They should be defined in a natural and consisten way till the end of the paper...hopefully, that is.

We now express each of the chambers $B', \dots, H'$ as a formal signed sum of the chambers $A, \dots, G$.

$$\begin{array}{llll} H' = A & B' = B - A & C' = C + A & D' = D - A \\ E' = E + A & F' = F - A & G' = G + A. & \end{array}$$

The only chamber on side 1 which contains B in its support when interpreted on side 2 is B', and the B coefficient for B' is $+1$. In the difference $P_2(\mathbf{x}_2) - P_1(\mathbf{x}_2)$ the two summands $\sum_B 1 \cdot \varphi_{\Gamma_B} - \sum_B 1 \cdot \varphi_{\Gamma_B}$ cancel. This is a general fact.

Fact. Only appearing chambers contribute to the wall crossing. We must be careful though. Chambers on side 1 of the wall (which are certainly not appearing!), when transported to side 2 DO contribute to an appearing chamber.

Since the polynomial we are integrating over A is always (up to sign) the same, what we really must be concerned with is the multiplicities of the contributions to A by A itself and by the chambers on side 1 that map to A.

$$\begin{aligned} &\sum_A \big(5-(-5)+1-(-2)+2-(-1)+2-(-2)\big)\varphi_{\mathcal{A}} \\ &= \sum_A 20 \cdot \varphi_{\mathcal{A}} = \sum_A \binom{6}{3} \cdot \varphi_{\mathcal{A}}. \end{aligned} \tag{31}$$

If we cut the graph Γ at the three edges, then the upper part Γ_u contributes to the Hurwitz number $H^3(x_1, x_3, -i, -j, -x_1 - x_3 + i + j)$ and the lower part Γ_l contributes to the Hurwitz number $H^3(x_2, x_4, i, j, -x_2 - x_4 - i - j)$. In fact the pair (Γ_u, Γ_l) appears 6 times in the product of Hurwitz numbers, corresponding to all ways of labelling the three cut edges. Then note that to compute the pair of Hurwitz numbers we must sum over all $i \geq 0$, $j \geq 0$ and $x_1 + x_3 - i - j \geq 0$ (the simplex A) the product of internal edges of the two connected components times the connecting edges, hence just the polynomial $\varphi_{\mathcal{A}}$. Then the contribution to the right hand side of (30) by pair of graphs that glue to Γ is $6\sum_A \binom{6}{3} \cdot \frac{\varphi_{\mathcal{A}}}{6}$, i.e. (31).

We want to take this a little further, and interpret this equality geometrically. The factor $\binom{6}{3}$ counts the ways to merge two orderings of the vertices of Γ_1 and Γ_2 to a total ordering of all vertices. Then re-gluing the cut graphs with the extra data of this merging gives a bijection with the directed, vertex-ordered graphs, contributing to (31). So in this case we have a direct generalization of the $Cut - Glue$ correspondence of section 8.2

Remark 9.2. Or course to turn this idea into a proof one needs to formalise things. In [CJM11], we interpret the bounded chambers above as a basis of the g-th relative homology of the hyperplane arrangement, and the process of "carrying the chambers over the wall" as a Gauss Manin connection on the corresponding homological bundle. Then the core of our idea is a combinatorial formula for this Gauss Manin connection in terms of cutting and regluing of graphs.

9.2. **The egg: Inclusion/Exclusion.** In section 9.1 we focused on a graph with a wealth of possible edge orientations and hence a very interesting associated hyperplane arrangement, but only one way to disconnect along the wall. Here we go to the other extreme and observe a graph with trivial orientation choices but many possible ways to disconnect. Consider "the egg" in Figure 10, and again the wall W_I given by the equation $x_1 + x_3 = 0$.

There is only one possible orientation for the egg on each side of the wall and one possible vertex ordering for each of these orientations. The chamber on the left hand side is an appearing chamber, and the chamber on the left, when transported to the right, covers it with multeplicity -1.

The coefficient in front of $\sum_C \varphi_G amma$ for the egg contribution to the wall crossing is therefore:

$$1 - (-1) = 2$$

Good news first: let us check that our formula indeed gives us a 2. If we disconnect the egg in all possible ways that give us at most three connected component (see Figure 11) and look at the appopriate coefficients, we get:

$$\binom{4}{1} + \binom{4}{3} + \binom{4}{2} - \binom{4}{1,1,2} - \binom{4}{2,1,1} + \binom{4}{1,2,1} = 2$$

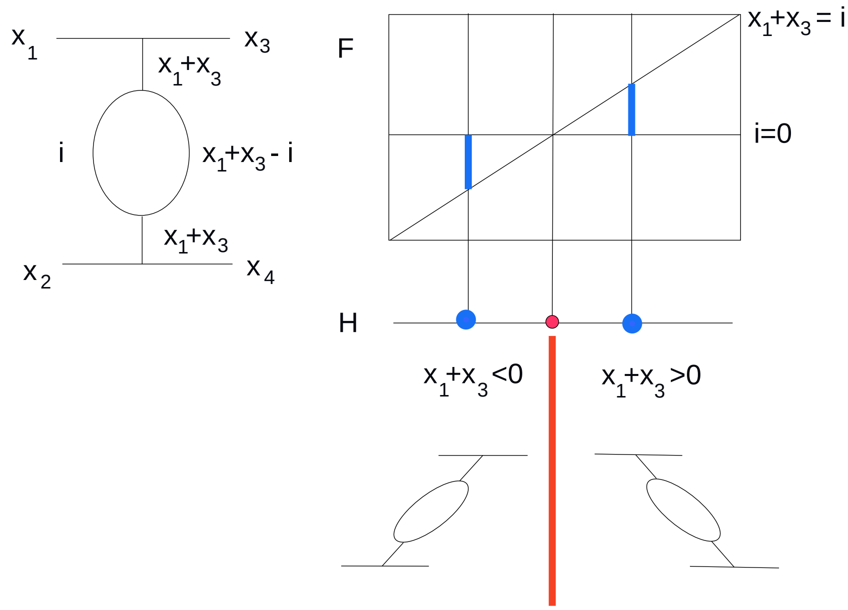


FIGURE 10. The egg graph and its orientations on either side of the wall.

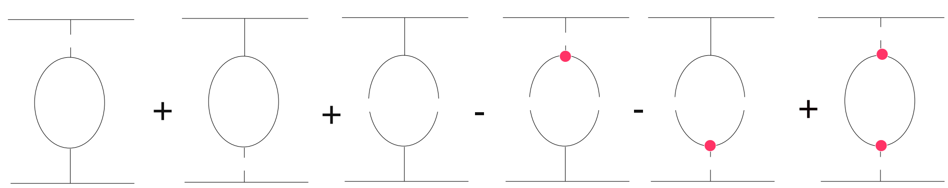


FIGURE 11. The light cuts of the egg graph.

But now for the bad news. We would like to prove this by giving a geometric bijection between the graphs contributing to the wall crossings and the regluings of the cut graphs where we allow to reorient the cut edges. To go from the left side egg to the right side egg we need to reorient all internal edges, and there is no cut in Figure **??** that allows us to do that!

After much crying and gnashing of teeth, this lead us to think that maybe we should allow ourselves to do more general cuts, in fact to cut the graphs in all possible ways, and organize our inclusion/exclusion in terms of the number of connected components of the cut graph.

Luckily, in this example we obtain the desired 2, as shown in Figure 12.

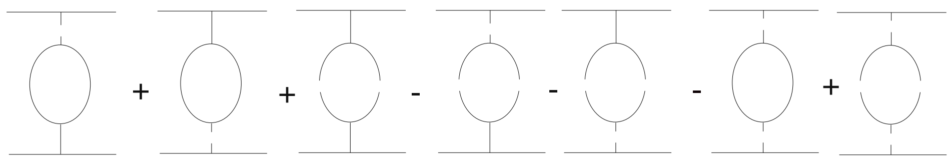
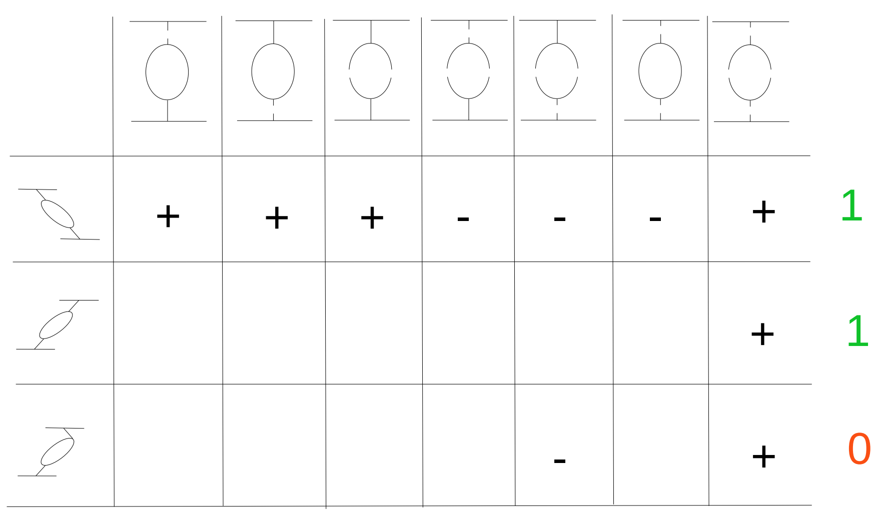


FIGURE 12. The general cuts of the egg graph.

$$\binom{4}{1}+\binom{4}{3}+\binom{4}{2}-\binom{4}{1,1,2}-\binom{4}{2,1,1}-\binom{4}{1,2,1}+\binom{4}{1,1,1,1}=2$$

In this cut/glue inclusion exclusion process one introduces a huge number of other graphs (with sinks/sources etc) whose contribution should clearly vanish. In Figure 9.2, we check that the contribution to the inclusion/exclusion process by each of the two contributing egg graphs is indeed 1 (corresponding to the number of vertex orderings of the "good eggs"), and the contribution for a "bad egg" (i.e. an egg with a sink) indeed vanishes.



Therefore, to prove Theorem 9.1, we first prove that our combinatorial recipe for the Gauss Manin connection proves a "heavy formula"', and then show via yet another inclusion/exclusion argument that the two formulas are equivalent.

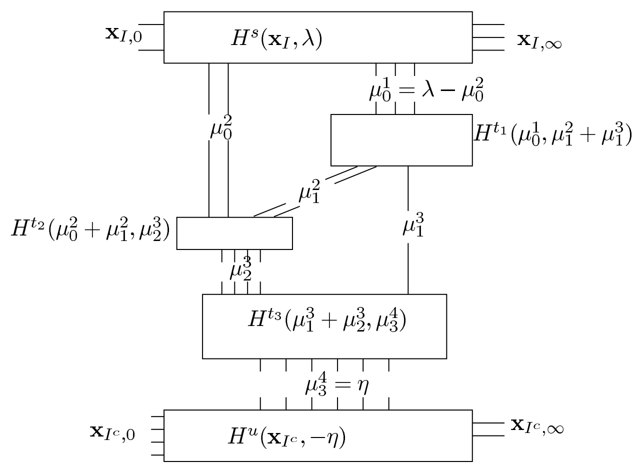


Figure 13. The data denoted by $\star$ in the heavy formula, Theorem 9.3

Theorem 9.3 (Heavy Formula)**.**

$$WC_I^r(\mathbf{x}) = \sum_{N=0}^{\infty} \sum_{\substack{s+(\sum_{j=1}^N t_j)+u=r \\ |\lambda|=|\eta|=d \\ data\ in\ \star}} (-1)^N \binom{r}{s,t_1,\ldots,t_N,u} \frac{\prod \mu_i^j}{\prod \ell(\mu_i^j)!} \tag{32}$$

$$H^s(\mathbf{x}_I,\lambda)\left(\prod_{j=1}^N H^{t_j}(\star)\right) H^u(\mathbf{x}_{I^c},-\eta)$$

The data denoted by $\star$ is illlustrated in Figure 13: it consists in disconnecting a graph with the right numerical invariants in all possible legal ways, where legal means that the graph obtained by shrinking all connected components to vertices and mantaining the cut edges as edges has no directed loops. The μ_i^j denote the partitions of weights of the edges connecting the i-th to the j-th connected component.

10. Current Questions

So far we have focused our attention mostly on Hurwitz numbers, and investigated their combinatorial properties as well as their relationship with the geometry of moduli spaces of curves. A natural point of view on Hurwitz numbers is that they really are the degree of a zero-dimensional cycle, defined as the pull-back of (the class of) a point

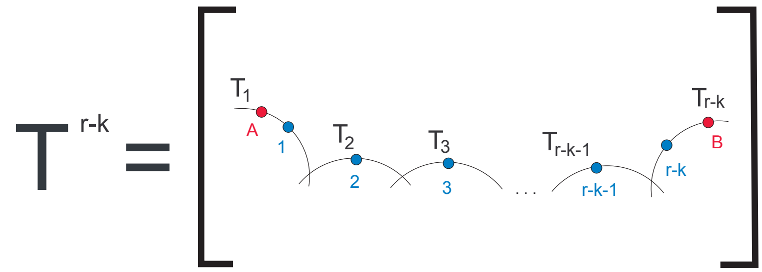


FIGURE 14. The union of strata we denote by $T^{r-k} \subset \overline{M}_{0,2+r}(1,1,\varepsilon,\dots,\varepsilon)$ parametererizes chains of $r-k$ rational curves, where the i-th component hosts the i-th (light) marked point, and the remaining k points are distributed in all possible ways among the various twigs.

via the natural branch morphism. Similarly one can define higher dimensional Hurwitz loci: in genus zero they are simple to describe and enjoy similar polynomiality and wall crossing properties than the zero-dimensional objects. The simplicity is given by the fact that the genus zero space of relative stable maps is birational to $\overline{M}_{0,n}$. In higher genus the determination of the class of the (pushforward of the) moduli space of maps (or rather of a suitable compactification thereof) is already an interesting and challenging question known as Eliashberg's problem. The determination of such a class would lead us to extend polynomiality results in arbitrary degree and develop an analogue of the ELSV formula for double Hurwitz number. In the following sections we collect what is known of this story and highlight a good number of open questions and speculations.

10.1. **Rational Double Hurwitz Classes.** In this section we define Hurwitz classes and study their (piecewise) polynomial properties. We say that a family of Chow cycles $\alpha(\mathbf{x})$ in the moduli space of rational pointed stable curves is polynomial of degree d and dimension k if $\alpha(\mathbf{x}) \in Z_k(\overline{M}_{0,n})[x_1,\dots,x_n]_d$. This is equivalent to α having an expression as a combination of dimension k boundary strata with coefficients polynomials in the x_i's of degree d.

Definition 10.1. Let $\mathbf{x} \in \mathcal{H}$. Consider the union of boundary strata $T^{r-k} \subset \overline{M}_{0,2+r}(1,1,\varepsilon,\dots,\varepsilon)$ parameterizing chains of $r-k$ projective lines, where the i-th component hosts the i-th (light) marked point, as illustrated in Figure 14. Referring to Figure 15 for the names of the natural morphisms, we define the k-dimensional Hurwitz cycle:

$$\mathbb{H}_k(\mathbf{x}) := st_*(br^*(T^{r-k})) \in Z_k(\overline{M}_{0,n}) \tag{33}$$

$$\begin{tikzcd}
\tilde{\mathbb{H}}_k(\mathbf{x}) \arrow[r] \arrow[d] & \overline{M}_0^{\sim}(\mathbf{x}, t_1, \ldots, t_j) \arrow[r, "st"] \arrow[d, "br"] & \overline{M}_{0,n} \\
{[T^{r-k}]} \arrow[r] & {[\overline{M}_{0,2+r}(1,1,\varepsilon,\ldots,\varepsilon)/S_k]} &
\end{tikzcd}$$

FIGURE 15. The k-dimensional Hurwitz locus is the inverse image via the branch map of the stratum T^{r-k}.

Sometimes we want to look at the k-dimensional locus in the appropriate space of maps. We make the definition:

$$\tilde{\mathbb{H}}_k(\mathbf{x}) := br^{-1}(T^{r-k}) \subseteq \overline{M}_0^{\sim}(\mathbf{x}, t_1, \ldots, t_{r-k}) \tag{34}$$

Remark 10.2. Hurwitz loci were introduced in [GV03b, Section 4.4]. The only difference in our definition is that we "mark" the branch points that we are fixing. The use of the more refined branch morphism to the Losev-Manin space gives us a more convenient expression of the locus in terms of the pull-back of a boundary stratum in $\overline{M}_{0,2+r}(1,1,\varepsilon,\ldots,\varepsilon)$.

10.1.1. *Multiplicities of boundary strata.* Boundary strata in moduli spaces of relative stable maps corresponding to breaking the target are naturally described in terms of products of other moduli spaces of relative stable maps. It is important to keep careful track of various multiplicities coming both from combinatorics of the gluing and infinitesimal automorphisms (see [GV03b, Theorem 4.5]). Let S be a boundary stratum in $\overline{M}_0^{\sim}(\mathbf{x})$, parameterizing maps to a chain T^N of N projective lines. S can be seen as the image of:

$$gl : \prod_{i=1}^{N} \mathcal{M}_i^{\bullet} \to S \subset \overline{M}_0^{\sim}(\mathbf{x}),$$

where the $\mathcal{M}_i^{\bullet}$ are moduli spaces of possibly disconnected relative stable maps, where the relative condition imposed at the point ∞ in the i-th line matches the condition at 0 in the $(i+1)$-th line. We denote by $\mathbf{z_i} = (z_i^1, \ldots, z_i^{r_i})$ such relative condition and by abuse of notation we say it is the relative condition at the i-th node of T^N. Then,

$$[S] = \prod_{i=1}^{N-1} \frac{\prod_{j=1}^{r_i} z_i^j}{|Aut(\mathbf{z_i})|} \left[gl_* \left(\prod_{i=1}^{N} \mathcal{M}_i^{\bullet} \right) \right]. \tag{35}$$

Equation (35) seems horrendous, but it amounts to the following recipe: the general element in S is represented by a map from a nodal curve X to T^N, with matching ramification on each side of each node of

X. The multiplicity $m(S)$ is the product of ramification orders for each node of X divided by the (product of the order of the group of) automorphisms of each partition of the degree prescribing the ramification profile over each node of T^N.

The description of Hurwitz loci in terms of boundary strata in spaces of maps, together with an analysis of the multiplicities of the pushforwards to $\overline{M}_{0,n}$ naturally leads to discover the (piecewise) polynomiality of the Hurwitz cycles.

Theorem 10.3. *For $\mathbf{x} \in \mathfrak{c}$, $\mathbb{H}_k(\mathbf{x})$ is a homogeneous polynomial cycle of degree $n-3-k$.*

This theorem is established by three facts: first a simple vanishing statement, noting that boundary strata contribute to the Hurwitz locus only if they have exactly one non-trivial component over the chain they cover; second, the observation that given any boundary stratum in $\overline{M}_{0,n}$ of the right dimension, it appears in the image of the pushforward of some component of $\tilde{\mathbb{H}}_k(\mathbf{x})$. Finally there is again an analysis of the boundary multiplicities. We omit the first fact which is a simple dimension count, and discuss the other two.

Lemma 10.4. *Let $\mathbf{x} \in \mathfrak{c}$, $\Gamma \in \mathcal{T}_{r-k}(\mathbf{x})$. Then there exists an irreducible component $\tilde{\Delta}$ in $\tilde{\mathbb{H}}_k(\mathbf{x})$ such that the stabilization of the source curve of the general element of $\tilde{\Delta}$ has dual graph Γ.*

Proof. This statement becomes transparent after noting that boundary strata in $\overline{M}_0^{\sim}(\mathbf{x}, t_1, \ldots, t_{r-k})$ are in bijective correspondence with their "tropical dual graphs". Given a boundary stratum $[S]$ whose general element is given by a map $f : X \to T^{r-k}$:

- the chain T^{r-k} is replaced by the interval $[0, r-k+1]$ with the i-th projective line corresponds to the point i and i-th node corresponding to the segment $(i, i+1)$; the points 0 and $r-k+1$ correspond to the relative points.
- over point i draw one vertex for each connected component of $f^{-1}(T_i)$;
- over segment $(i, i+1)$ draw one edge for each node of X above the i-th node of T^{r-k}; the edge connects the appropriate vertices and is weighted with the ramification order at the node;
- over $[0, 1)$ and $(r-k, r-k+1]$ draw edges corresponding to the relative condition $\mathbf{x}$.

We call the graph thus obtained the *pre-stable* tropical dual of $[S]$. We stabilize the graph by forgetting all two talent vertices to obtain what we call the tropical dual of $[S]$. This procedure is illustrated in

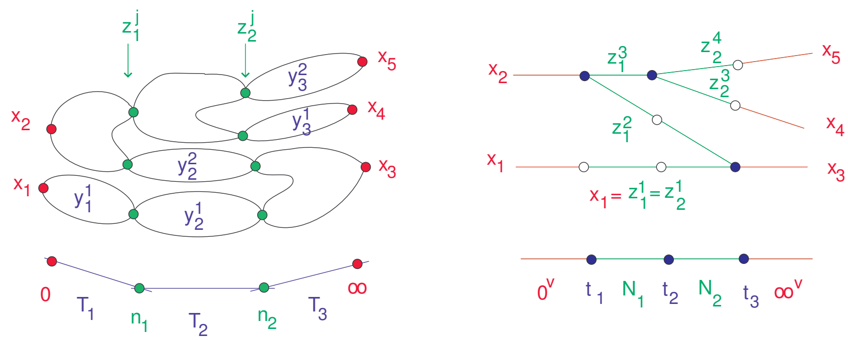


FIGURE 16. The construction of the tropical dual graph associated to a boundary stratum of relative maps. The y_i^j are the degrees of the trivial covers of the i-th twig, and the z_i^j the ramification orders over the i-th node. Note that the z's on either side of a trivial cover are equal to the corresponding y (e.g $x_1 = y_1^1 = z_1^1 = y_2^1 = z_2^1$).This makes it possible to erase the two valent vertices and have the z_i^j become the weights of the edges of the tropical dual graph.

Figure 16 and clearly it can be inverted to identify a boundary stratum of maps given a tropical cover. With this translation Lemma 10.4 is equivalent to the purely combinatorial statement of Lemma **??**. □

Lemma 10.5. *Let* $\mathbf{x} \in \mathfrak{c}$, $\Gamma \in \mathcal{T}_{r-k}(\mathbf{x})$ *and* Δ_Γ *the corresponding boundary stratum in* $\overline{M}_{0,n}$. *Let* $m_{\mathfrak{c}}(\Gamma)$ *and* $\varphi(\Gamma)$ *as in Definitions* **??**, **??** *and* $f : \mathcal{T}^{\mathfrak{c}}_{r-k}(\mathbf{x}) \to \mathcal{T}_{r-k}(\mathbf{x})$ *be the natural forgetful morphism. Then*

$$\mathbb{H}_k(\mathbf{x}) = \sum_{\tilde{\Gamma} \in \mathcal{T}^{\mathfrak{c}}_{r-k}(\mathbf{x})} \varphi(\tilde{\Gamma}) \prod_v (val(v) - 2)\ \Delta_{f(\tilde{\Gamma})}$$

$$(36) \qquad = \sum_{\Gamma \in \mathcal{T}_{r-k}(\mathbf{x})} m_{\mathfrak{c}}(\Gamma) \varphi(\Gamma) \prod_v (val(v) - 2) \Delta_\Gamma.$$

Proof. We observed that for each boundary stratum Δ_Γ in $\overline{M}_{0,n}$ there are precisely $m_{\mathfrak{c}}(\Gamma)$ boundary strata in $\tilde{\mathbb{H}}_k(\mathbf{x})$ pushing forward to it. It remains to show that each such stratum $[S]$ pushes forward with multiplicity $\varphi(\Gamma) \prod_v (val(v) - 2)$. We obtain this multiplicity by adapting formula (35) to the simple shape of the boundary strata we are observing. Over each T_i in the chain there is only one connected component of X that maps non-trivially. Therefore we can replace the moduli space of possibly disconnected covers $\mathcal{M}_i^\bullet$ of (35) by the unique moduli space of non-trivial connected covers to T_i, call it $\mathcal{M}_i$, times an automorphism factor of $1/y_i^j$ for every connected component of degree y_i^j mapping trivially to T_i; we denote by $triv_i$ the number of trivial covers of the i-th component. Also the factors of $1/|Aut(\mathbf{z_i})|$ need to be replaced by the automorphisms of each sub-partition of $\mathbf{z_i}$ corresponding to nodes on the same pair of curves on the two sides of the i-th node of T^{r-k}. But X is a rational curve and hence all such sub-partitions have length one and trivially no automorphisms. Hence (35) becomes:

$$[S] = \frac{\prod_{i=1}^{r-k-1}\prod_{j=1}^{r_i} z_i^j}{\prod_{i=1}^{r-k}\prod_{j=1}^{triv_i} y_i^j}\left[gl_*\left(\prod_{i=1}^{r-k}\mathcal{M}_i\right)\right]. \tag{37}$$

Equation (36) is deduced from (37) via the following observations:

(1) There is a factor of y_i^j for each 2-valent vertex of the pre-stable tropical dual of $[S]$, and y_i^j is equal to the weight of (either) edge on each side of the vertex. Therefore all y_i^j's cancel with some of the z's. The remaining multiplicity from the first part of (37) is therefore the product of weights of all internal edges of the tropical dual graph: by definition this is $\varphi(\Gamma)$.

(2) Each $\mathcal{M}_i$ is a moduli space of connected, rubber relative stable maps, with one simple transposition marked. We know st maps $\mathcal{M}_i$ onto $\overline{M}_{0,n}$ with degree $r = \binom{r}{1}$. If we call v_i the vertex of the tropical dual graph corresponding to $\mathcal{M}_i$, then in this case $n = val(v_i)$ and thus $r = val(v_i) - 2$.

□

To obtain Theorem 10.3 it is now sufficient to remark that the weights of the edges of the tropical dual graph are linear homogeneous polynomials in the x_i's and there are precisely $n-3-k$ internal edges in the tropical dual graph of any stratum in $\widetilde{\mathbb{H}}_k(\mathbf{x})$ that pushes forward non-trivially to $\overline{M}_{0,n}$.

We conclude this section by noting that our arguments do not apply exclusively to Hurwitz cycles, but to any cycle obtained by pull-pushing a boundary stratum from Losev-Manin to $\overline{M}_{0,n}$. We thus obtain the following:

Corollary 10.6. *Let $\mathbf{x} \in \mathfrak{c}$ and $\overline{\Delta}$ be a union of boundary strata of dimension k in $\overline{M}_{0,2+r}(1,1,\varepsilon,\ldots,\varepsilon)$. Then $st_*br^*(\overline{\Delta})$ is a homogeneous polynomial class of degree $n-3-k$ in $\mathfrak{c}$.*

Hurwitz cycles are polynomials in each chamber, and, like for Hurwitz numbers, the way the polynomials change is described by an inductive formula.

Theorem 10.7 (Wall Crossing)**.** *Let $I \subseteq \{1, \ldots, n\}$ and consider the wall $W_I = \{\epsilon := \sum_{i \in I} x_i = 0\}$. Then:*

(38)
$$WC_{I,k}(\mathbf{x}) = \epsilon \sum_{j=\max\{0,1+k-r_2\}}^{\min\{k,r_1-1\}} \binom{r-k}{|I|-1-j} \mathbb{H}_j(\mathbf{x}_I, -\epsilon) \boxtimes \mathbb{H}_{k-j}(\mathbf{x}_{I^c}, \epsilon)$$

The proof of Theorem 10.7 is parallel to [CJM10, Theorem 6.10]: it essentially boils down to observing that three-valency of the vertices of the tropical dual graphs does not play a role in the wall crossing phenomenon.

10.2. **Eliashberg's Problem.** In 2001 Yasha Eliashberg, in laying the foundations of Symplectic Field Theory, asked the following question to Richard Hain.

Question (Eliashberg's problem)**.** *Given an n-tuple of integers $\mathbf{x}$ adding to 0 determine the class in $\overline{\mathcal{M}}_{g,n}$ of pointed curves $(C, p_1, \ldots, p_n)$ such that the divisor $D = \sum x_i p_i$ is principal.*

A few observations are in order:

(1) Perhaps the first thing to clarify is why does this question belong in this mini-course. If C is a smooth curve, then asking for a divisor to be principal is equivalent to having a meromorphic function with zeroes and poles with prescribed orders, i.e. a map to $\mathbb{P}^1$ with specified ramification profile over 0 and ∞. On the open moduli space hence Eliashberg is asking for a description of a double Hurwitz locus.

(2) There is a certain degree of ambiguity with which the question is (intentionally) written, as the concept of principal divisor becomes iffy for a singular curve (more specifically for a curve not of compact type). There is various ways in which Eliashberg's question can be rigorously formulated - each of which I believe is of interest.

   (a) Describe the above locus as a class on a partial compactification of $\mathcal{M}_{g,n}$. A natural partial compactification where this class is define is $\mathcal{M}_{g,n}^{ct}$, the locus of curves of compact type. The domain of definition of the class is $\overline{\mathcal{M}}_{g,n} - \Delta_0^{sing}$, the complement of the singular locus in Δ_0.

(b) Describe some meaningful class on the closed moduli space that restricts to the appropriate class: this could be simply the closure of the class discussed in the previous item, or the class of some meaningful compactification of the moduli problem implicitly posed in Eliashberg's question. We already have two natural candidates at hand: moduli spaces of admissible covers and moduli spaces of relative stable maps.

We therefore can reformulate Eliashberg's question as follows.

Question. *Describe the push-forward (via the natural stabilization morphism to $\overline{\mathcal{M}}_{g,n}$) of $[\overline{\mathcal{M}}_g^{\sim}(\mathbb{P}^1, \mathbf{x})]^{vir}$ and of $[Adm_{g\to 0}(\mathbf{x})]$. Compare the two classes.*

Recently Richard Hain ([Hai11]) has described a class, that we call Hain's class, answering Eliashberg's question on $\mathcal{M}^{ct}_{g,n}$. The key ingredient is that the class of the zero section of the universal Jacobian over curves of compact type is $1/g!$ times the g-th power of the Θ divisor. The question is then reduced to pulling back Θ via the natural section $\sigma(\mathbf{x}) : \mathcal{M}^{ct}_{g,n} \to \mathcal{J}_{g,n}$. Hain accomplishes the task using the theory of normal functions, whereas in [GZ12], Grushevski and Zacharov use test curves to determine the coefficients of the pullback of the divisor. We present a very pretty version of the class of the pull-back of Θ that, to my knowledge, has been formulated by Dimitri Zvonkine:

$$\sigma(\mathbf{x})^*(\Theta) = \frac{1}{4} \sum_{I\subset\{1,\ldots,n\}} \sum_{h=0}^{g} x_I x_{I^c} \overline{D}_{h,g-h}(I|I^c), \tag{39}$$

where $\overline{D}_{h,g-h}(I|I^c)$ denotes the divisor (of compact type) parameterizing curves with one node and two components of genus h and $g-h$, with marks in subset I on the component of genus h. Then $x_I := \sum_{i\in I} x_i$. The following abuses of notation are also in place:

$$\overline{D}_{0,g}(\{i\}|I-\{i\}) := -\psi_i. \tag{40}$$

Formula (40) is homogeneous quadratic in the x_i's, thus making Hain's class a polynomial class which is homogeoneous of degree $2g$. In [CMW12] we have exploited this polynomiality to compute Hain's class in genus 1 via localization on moduli spaces of relative stable maps. We obtain the result in the following form. We first describe Hain's class in $\mathcal{M}^{ct}_{1,3}$: we assume x_1 and x_2 are positive and use the relation $x_3 = -x_1 - x_2$ to eliminate x_3 from our formula. Then Hain's class is the restriction of $\mathfrak{H}_1(x_1, x_2, x_3) = A_1 x_1^2 + A_2 x_2^2 + B x_1 x_2$, with

(41)
$$A_1 = \psi_3 - \overline{D}_{1,0}(1|2,3) \qquad A_2 = \psi_3 - \overline{D}_{1,0}(2|1,3) \qquad B = \psi_3 - \overline{D}_{1,0}(3|1,2).$$

Exercise 34. Check that equations (40) and (41) agree on $\mathcal{M}_{1,3}^{ct}$. Further they agree with Hain's original formula which in this specific case is:

$$\mathfrak{H}_1(x_1, x_2, x_3) = -x_2 x_3 \overline{D}_{1,0}(1|2,3) - x_1 x_3 \overline{D}_{1,0}(2|1,3) - x_1 x_2 \overline{D}_{1,0}(3|1,2,3) \\ -(x_2 x_3 + x_1 x_3 + x_1 x_2) \overline{D}_{1,0}(\phi|1,2,3).$$

For more than three marks the class is then simply obtained by pulling back the coefficients via the appropriate forgetful morphisms. We note that the class $\mathfrak{H}_1(x_1, x_2, x_3)$ is well defined over all of $\overline{M}_{1,3}$, however it is not the pushforward of the virtual fundamental class from the moduli space of relative stable maps. The difference, somewhat suggestively, is precisely $-\lambda_1$ (the closure of the self intersection of the zero section in the universal Jacobian).

Exercise 35. In genus one λ_1 generates the space of divisors not of compact type, hence the two classes above must differ by a multiple of λ_1. Hence you can check that

$$[\overline{\mathcal{M}}_1^{\sim}(\mathbb{P}^1, \mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})]^{vir} = \mathfrak{H}_1(x_1, x_2, x_3) - \lambda_1$$

by an intersection theoretic computation using the following facts:

- The Double Hurwitz number formula ([GJV03]):
$$H_1(x_1, x_2, x_3) = \frac{3!}{24}(x_1 + x_2)^2 (x_1^2 + x_2^2 - 1).$$
- The fact that we will show in section 10.3:
$$[\overline{\mathcal{M}}_1^{\sim}(\mathbb{P}^1, \mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})]^{vir} \cdot 3!(x_1 + x_2)^2 \psi_3^2 = H_1(x_1, x_2, x_3)$$

10.3. **ELSV Formula for Double Hurwitz Numbers.** The combinatorial structure of double Hurwitz numbers seems to suggest the existence of an *ELSV* type formula, i.e. an intersection theoretic expression that explains the polynomiality properties of the This proposal was initially made in [GJV03] for the specific case of *one-part* double Hurwitz numbers, where there are no wall-crossing issues. After [CJM11], we propose an intriguing, albeit maybe excessively bold strengthening of Goulden-Jackson-Vakil's original conjecture.

Conjecture (Bayer-Cavalieri-Johnson-Markwig). *For* $\mathbf{x} \in \mathbb{Z}^n$ *with* $\sum x_i = 0$,

$$H_g(\mathbf{x}) = \int_{\overline{P}(\mathbf{x})} \frac{1 - \Lambda_2 + \ldots + (-1)^g \Lambda_2 g}{\prod (1 - x_i \psi_i)}, \tag{42}$$

$$\begin{tikzcd}
\mathcal{M}(x) = \overline{\mathcal{M}}^{\sim}_{g}(x) \arrow[r, "stab"] \arrow[d, "br"] & \overline{\mathcal{M}}_{g,n} \\
\mathcal{M}_{br} = \overline{\mathcal{M}}_{0,2+r}(1,1,\varepsilon,\dots,\varepsilon)/S_r &
\end{tikzcd}$$

FIGURE 17. The tautological morphisms from the space of rubber stable maps.

where,

1. *$\overline{P}(\mathbf{x})$ is a moduli space (depending on $\mathbf{x}$) of dimension $4g-3+n$.*
2. *$\overline{P}(\mathbf{x})$ is constant on each chamber of polynomiality.*
3. *The parameter space for double Hurwitz numbers can be identified with a space of stability conditions for a moduli functor and the $\overline{P}(\mathbf{x})$ with the corresponding compactifications.*
4. *Λ_{2i} are tautological Chow classes of degree $2i$.*
5. *ψ_i's are cotangent line classes.*

Goulden, Jackson and Vakil, in the one part double Hurwitz number case, propose that the mystery moduli space may be some compactification of the universal Picard stack over $\overline{\mathcal{M}}_{g,n}$. They verify that such a conjecture holds for genus 0 and for genus 1 by identifying $\overline{Pic}_{1,n}$ with $\overline{\mathcal{M}}_{1,n+1}$.

An alternative approach to a more geometric view of the structure of double Hurwitz numbers can be obtained by reasoning on the polynomiality of Hain's class. The following discussion stems from work with Arend Bayer. Consider the diagram of spaces in Figure 10.3:

The double Hurwitz number $H_g(x)$ is the degree of br:

$$H_g(x)[pt.] = br^*([pt])$$

We rewrite this expression in terms of ψ classes. We have three different kinds of ψ classes, and we have to be very careful to not mix them up:

1. $\hat{\psi}_0$: this is the psi class on the target space at the relative divisor 0, i.e. the first chern class of the cotangent line bundle at the relative point 0 on the universal target.
2. $\tilde{\psi}_i$: these are the psi classes on the space of rubber stable maps at the i-th mark. Remember that we are marking the preimages of the relative divisors.
3. ψ_i is the ordinary psi class on the moduli space of curves.

Lemma 10.8.

$$\hat{\psi}^{2g-3+n} = \frac{1}{r!}[pt.]$$

This follows immediately by combining the following facts:

(1) Any ψ class on an (ordinary) $\overline{M}_{0,n}$ has top self intersection $1[pt.]$.

(2) The fact that the point 0 has weight 1 means that no twigs containing the point 0 get contracted because of the small weigths at the other points. Therefore if we consider the contraction map $c : \overline{M}_{0,r+2} \to \overline{\mathcal{M}}_0(1,1,\varepsilon,\ldots,\varepsilon)$, we have that $c^*(\psi_1) = \hat{\psi}_0$.

(3) The $r!$ factor comes from the fact that the branch space is a S_r quotient of $\overline{\mathcal{M}}_0(1,1,\varepsilon,\ldots,\varepsilon)$.

Lemma 10.9.

$$br^*(\hat{\psi}_0) = x_i\tilde{\psi}_i$$

where it is understood that the i-th mark is a preimage of 0.

Consider the diagram:

$$\begin{tikzcd}
\mathcal{U}(\mathbf{x}) \arrow[d, "f"] \arrow[dr] & \\
\mathcal{U}_{br} \arrow[dr] & \mathcal{M}(\mathbf{x}) \arrow[ul, bend right, "s_i"'] \arrow[d, "br"] \\
& \mathcal{M}_{br} \arrow[ul, bend left, "0"]
\end{tikzcd}$$

Then:

$$br^*(\hat{\psi}_0) = -br^*0^*(0) = -s_i^* f^*(0) = -s_i^*(x_i s_i) = x_i\tilde{\psi}_i$$

Now the key fact is that

$$H_g(\mathbf{x}) = r!br^*(\hat{\psi}^{2g-3+n}) = r!x_i^{2g-3+n}\tilde{\psi}_i^{2g-3+n}$$

This is a small step in the right direction: we are expressing the Hurwitz number as a globally polynomial function over a family of moduli spaces. The wall crossings for double Hurwitz numbers are then explained as wall crossings for the psi classes on the various moduli spaces. But we still have a dependence of the moduli spaces which is way too intrinsically related to the Hurwitz problem, and also the degree of the polynomial is $2g$ off by what it should be...just saying that the top intersection of psi classes on rubber stable maps is a rational function that corrects the x_i^{top} and produces the double Hurwitz number.

Everything is much more interesting if we notice that the tilda-psi classes are pull-backs of ordinary psi classes plus some corrections, namely by divisors in the spaces of relative stable maps parameterizing curves where the mark lies on an unstable component of the curve. Then one can use projection formula to obtain a class which, assuming $st_*[\overline{\mathcal{M}}_g^{\sim}(\mathbb{P}^1, \mathbf{x})]^{vir}$ to be an even polynomial of degree g gives the desired polynomiality properties for the Hurwitz numbers in each chamber. We make this analysis for the one part double Hurwitz numbers (where we recover the appropriate formulas and explain the high divisibility by d observed in [GJV03]).

10.3.1. *One part double Hurwitz numbers.* The key observation here is that the $\tilde{\psi}$ class is a pullback of an ordinary psi class:

Lemma 10.10. *Let x_1 be the unique positive part. Then:*

$$\tilde{\psi}_1 = stab^* \psi_1$$

This follows from a natural isomorphisms of the cotangent line bundles, due to the fact that in the space of rubber maps the first mark point never lies on a bubble that gets contracted by the stabilization morphism. In other words, observing the diagram,

$$\begin{tikzcd}
\mathcal{U}(\mathbf{x}) \arrow[d, "U"] \arrow[dr] & \\
\mathcal{U} \arrow[dr] & \mathcal{M}(\mathbf{x}) \arrow[ul, bend right, "\tilde{s}_1"'] \arrow[d, "stab"] \\
& \overline{\mathcal{M}}_{g,n} \arrow[ul, bend left, "s_1"]
\end{tikzcd}$$

we have that

$$\tilde{s}_1 = U^*(s_1)$$

and hence:

$$stab^*(\psi_1) = -stab^* s_1^*(s_1) = -\tilde{s}_1^* U^*(s_1) = -\tilde{s}_1^*(\tilde{s}_1) = \tilde{\psi}_1$$

Now by projection formula:

(43)
$$H_g(\mathbf{x}) = r! stab_*(x_1^{2g-3+n} \tilde{\psi}_1^{2g-3+n}) = r! stab_*([\mathcal{M}(\mathbf{x})]^{vir}) x_1^{2g-3+n} \psi_1^{2g-3+n}$$

Remark: Formula (43) explains the comments made in GJV - paragraph after Corollary 3.2. (the divisibility of the polynomial by x_1^{2g-3+n}, and the appearence of constants related to the moduli spaces of curves 1/24, 1/5760 etc...).

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