Hydraulics I, Spring 2006

A Properties of fluids

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Tutorial

Recommended Readings:

Hamill, *Understanding Hydraulics*, 2nd Edition, Palgrave.

Massey, *Mechanics of Fluids*, 7th Edition, Spon Press.

White, *Fluid Mechanics*, 5th Edition, McGraw-Hill.

Chadwick, Morfett and Borthwick, *Hydraulics in Civil and Environmental Engineering*, 4th Edition, Spon Press.

A Properties of fluid

A.1 Concept of continuum

On a macroscopic scale, large compared with the distance between molecules, the molecular structure of a fluid does not explicitly influence the variations of fluid properties. Thus, the macroscopic behaviour of fluid is the same as if it were perfectly continuous in structure, i.e. physical quantities, such as density, temperature and velocity, will be regarded as varying continuously in space. Such a fluid is called *a continuum*. This hypothesis implies that fluid properties can be regarded as point functions, and differential calculus can be applied to describe their variations in time and in space. (So are equations established.)

A.2 Density

Density ρ = mass/volume, e.g. kg/m^3. Density in a fluid flow, generally speaking, can vary in position and in time, i.e. $\rho(\boldsymbol{x}, t)$, where $\boldsymbol{x} = (x, y, z)$. However, density in a liquid is nearly constant under most circumstances. For example, $\rho_{water} = 1000$ kg/m^3, and only increases 1% if the pressure is increased by a factor of 220. So, liquids are usually 'incompressible'.

$$\textbf{Specific weight} = \frac{\text{weight}}{\text{volume}} \quad \text{i.e.} \quad \gamma = \rho g$$

$$\textbf{Relative density} = \frac{\rho_{fluid}}{\rho_{reference}}$$

Relative density is dimensionless, and is sometimes called the 'specific gravity'. For gas, it is rarely used.

A.3 Pressure

The *normal stress* on any plane through a fluid element is called the fluid pressure, p in N/m^2, taken as positive for compression by convention. A fluid at rest cannot withstand any tangential stresses (shear stresses), but it still has the pressure – the *hydrostatic pressure* which is the subject of topic B.

At any one point in a fluid, the pressure is the same in all directions, i.e. pressure is a scalar.

To see this, let us consider a triangular slab of fluid, see figure 1. The fluid may be stationary, or accelerating in time but without relative motion (i.e. free of shear stress) at a_x and a_z in the x and z directions.

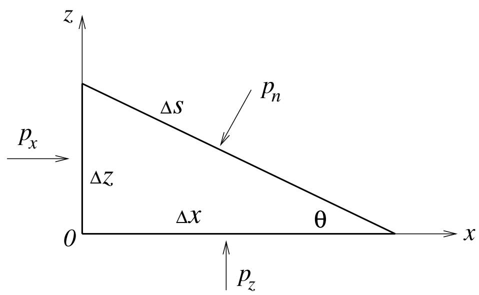


Figure 1: The end view of a fluid slab.

Applying Newton's second law in the x and y directions, one can show that

$$p_x = p_z = p_n = p$$

So, *at a given time, there can only be one value of fluid pressure at a particular point.*

Vapour pressure

It is the pressure at which a liquid boils, and is in equilibrium with its own vapour. If the liquid pressure falls below the vapour pressure, bubbles begin to appear in the liquid. This can occur when a liquid undergoes rapid accelerations, a phenomenon called 'cavitation' (fluid-induced boiling). Cavitation can severely damage fluid machines, such as turbines and pumps.

A.4 Viscosity

Viscosity is a quantitative measure of a fluid's resistance to flow when a shear stress is applied. It relates the local stress in a moving fluid to the strain rate of the fluid element. Consider a fluid element (a small square) under a horizontal shear stress τ, which results in a horizontal velocity $u = \delta u$, see figure 2(a).

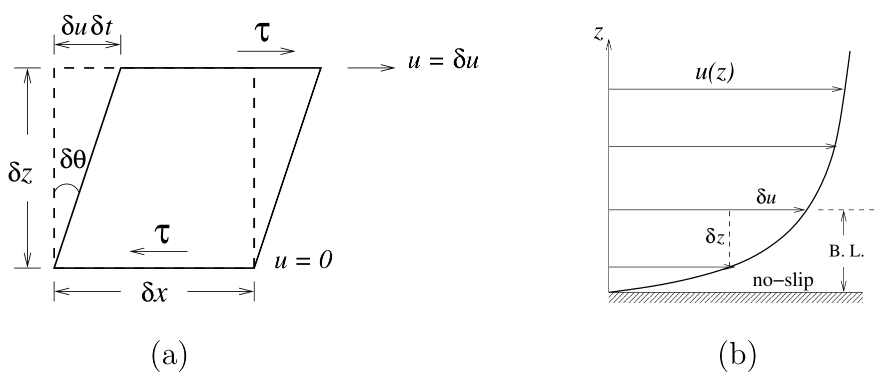


Figure 2: (a) A fluid element under a horizontal shear. (b) A velocity profile.

Common fluids, such as water, air and oil, show a linear relation between applied shear and the resulting strain rate, i.e.

$$\tau \propto \frac{\delta\theta}{\delta t}.$$

For small angle $\delta\theta$,

$$\delta\theta \simeq \tan\delta\theta = \frac{\delta u \delta t}{\delta z} \quad \Longrightarrow \quad \frac{\delta\theta}{\delta t} = \frac{\delta u}{\delta z} \quad \left(\text{i.e.} \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}z}\right)$$

Thus, $\tau \propto \mathrm{d}u/\mathrm{d}z$.

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}z}, \tag{A.1}$$

where μ is **coefficient of viscosity**. For *newtonian* fluids, shear stress is linearly proportional to the velocity gradient.

From the velocity profile above a solid wall, see figure 2(b), we observe:

(i) Because of viscosity, $u = 0$ at a solid boundary. This is the so-called **'no-slip' boundary condition**, and true for any viscous fluid.

(ii) Since u must increase to a non-zero value just above the wall, the velocity gradient, $\mathrm{d}u/\mathrm{d}z$, is usually the greatest at the wall. So is the shear stress τ. This is the manifestation of 'friction' at a solid boundary.

(iii) A layer, crossing which the velocity changes sharply and thus the viscous force is important, is termed as the 'boundary layer'. It can occur in the interior of a flow (i.e. without the presence of a solid boundary).

μ has a unit of *stress–time*. For water, $\mu = 10^{-3}$ kg/(m · s); for air, $\mu = 1.8 \times 10^{-5}$ kg/(m · s); but for glycerin, $\mu = 1.5$ kg/(m · s). Temperature can have a strong effect on the viscosity of a fluid.

Kinematic viscosity

$$\nu = \frac{\mu}{\rho}, \tag{A.2}$$

and has a unit of *length*^{2}/*time*. For water, $\nu \simeq 10^{-6}$ m^2/s.

Reynolds number

$$Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu} = \frac{\text{inertial acceleration force}}{\text{viscous force}}, \tag{A.3}$$

where U and L are, respectively, the characteristic velocity and length of a flow. It measures the importance of viscous effects relative to inertial accelerations (non-frictional). It is dimensionless.

Example 1 A simple shear flow (plane Couette flow)

Consider a layer of viscous fluid between two parallel plates. The top plate is pulled by a steady force such that it moves at a constant speed V. The bottom plate is at rest. The fluid velocity varies linearly in z, see figure 3. What is the shear applied at the top plate? Evaluate the shear for $\mu = 0.3$ kg/(m $\cdot$ s), $V = 3.0$ m/s and $h = 2$ cm. (*Ans:* $\tau = 45$ Pa)

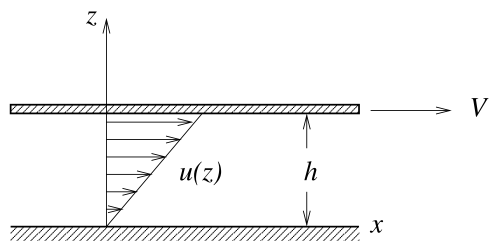


Figure 3: Parallel flow between two plates.

A.5 Compressibility

All fluids, like most everything else, decrease their volumes when placed under pressure. The compressibility of a fluid is measured by the

bulk modulus of elasticity

$$K = -\frac{\delta p}{\delta V/V} = \frac{\delta p}{\delta \rho/\rho} \tag{A.4}$$

where δp is a change of pressure, V the volume of the fluid and δV the change of the volume due to δp. Clearly, K has a unit of pressure. *Note*: $\delta\rho/\rho = -\delta V/V$, since mass, $m = \rho V$, is conserved.

The bulk modulus of a liquid is very high, indicating that most liquids are incompressible at moderate pressure. However, in circumstances where change of pressure is very large or very sudden, as in 'water hammers' in pipes, the compressibility of a liquid must be taken into account.

A.6 Surface tension

A small drop of liquid in air, or an air bubble in water, always forms a sphere; on a solid (clean) surface, a drop of mercury tends to be a sphere, however a drop of water forms a lens (partial sphere). All these facts are due to *surface tension*, which arises from the intermolecular cohesive forces. Its origin indicates that surface tension is most important at small scales. The recent expansion in areas, such as bio-engineering, micro- and nano-technology, is increasingly making the phenomena of surface tension of engineering interest.

It is necessary to know that the free surface of a liquid, or the interface between two liquids which do not mix, acts as if it were in a state of uniform tension – surface tension σ. It is a force per unit length, i.e.

> **Across any line drawn on the interface there is exerted a force of magnitude σ per unit length in a direction normal to the line and tangential to the interface.**

Surface tension may also be regarded as a surface energy per unit area. For a clean surface at 20°C,

$$\begin{aligned} \sigma &= 0.073\text{N/m} && \text{pure water – air;} \\ \sigma &= 0.480\text{N/m} && \text{mercury – air;} \\ \sigma &= 0.023\text{N/m} && \text{alcohol – air} \end{aligned}$$

The existence of surface tension means that there is a pressure jump across a curved surface of fluids, the pressure being higher on the concave side. For an arbitrarily curved interface whose principal radii of curvature are R_1 and R_2, a force balance normal to the surface will show that

the pressure increase on the concave side is

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{A.5}$$

Example 2

What is the pressure jump across the surface of a spherical droplet, of radius R, due to surface tension? Where is the higher pressure? Neglect the weight of the fluid. (*Ans:* $\Delta p = 2\sigma/R$)

Contact angle

In the case of a liquid interface intersecting with a solid surface, another important surface effect, in addition to surface tension σ, is the *contact angle* θ, see figure 4.

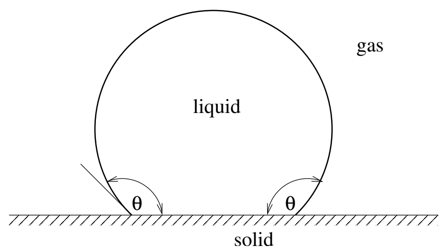


Figure 4: Contact angle θ at a liquid-solid-air interface.

If $\theta < 90°$, the liquid 'wets' the solid.

If $\theta > 90°$, the liquid is non–wetting.

The force balance would involve both σ and θ, which are both very sensitive to the physical and chemical conditions of the solid-liquid interface.

Example 3 Capillary tube and capillary rise

Derive an expression for the capillary rise h in a circular tube of liquid with surface tension σ and contact angle θ. Evaluate h for water-air-glass interface, $D = 2$ mm and $\theta \simeq 0$. (Ans: $h = 4\sigma\cos\theta/(\rho g D)$)

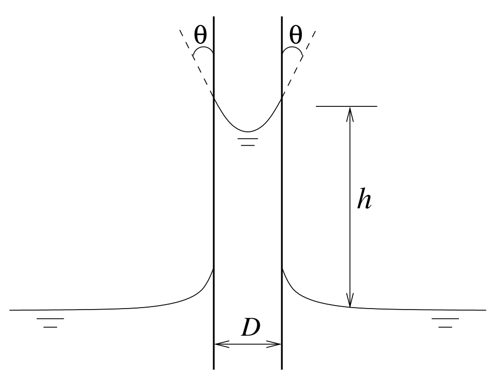


Figure 5: Capillary rise in a capillary tube.

A.7 Equation of state for perfect gas

The molecules of a perfect gas do not exert forces on each other except at collisions, and their volumes are negligible. For a perfect gas in equilibrium, there is an

equation of state:

$$p = R\rho T, \tag{A.6}$$

where T is absolute temperature (degree Kelvin, $T°\mathrm{K} \ = T°\mathrm{C} + 273.16$) and R is a constant (independent of temperature). Each gas has its own constant R. If M is the relative molecular mass of a gas, i.e. the ratio of the mass of the molecule to the mass of a normal hydrogen atom, then

$$MR = 8314 \text{ J/(kg K)} \qquad \text{– the } \textit{universal gas constant}$$

Example 4

A mass of air, at a pressure of 200 kPa and a temperature 27°C, occupies a volume of 3 m^3. For air, the relative molecular mass $M = 28.97$. Determine:

(a) the gas constant R for air;

(b) the density of the air;

(c) its mass.

B Hydrostatics

Hydrostatics concerns primarily the problems when a fluid is at rest, specifically the balance of forces. Under the hydrostatic conditions,

(i) there is no shear stress, but only the normal stress (pressure) on any plane through a fluid element;
(ii) in the *horizontal* direction, there is no pressure variation;
(iii) in the *vertical* direction, there is a change in pressure due to gravity.

B.1 The hydrostatic equation

It is not the pressure but the *spatial variation of the pressure that causes a net force* on a fluid. Consider a small fluid tube below.

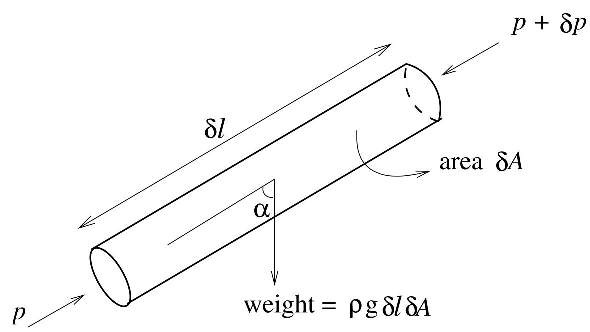


Figure 6: Force balance of a fluid tube in equilibrium.

For equilibrium, the *net force in any direction must vanish.* Thus, in the direction along the tube, the force balance is written as

$$[(p+\delta p)\,\delta A - p\delta A] + \rho g \delta l \delta A \cos\alpha = 0,$$

$$\therefore \qquad \delta p = -\rho g\,(\delta l \cos\alpha) = -\rho g \delta z. \tag{B.1}$$

This leads to the

hydrostatic equation:

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g. \tag{B.2}$$

The negative sign is due to the z–axis being pointing upward.

It is this vertical pressure gradient that produces a net force, balancing the gravity force (the weight). That is to say,

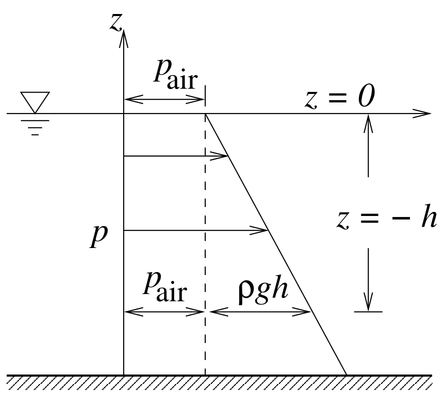
As a result of the weight of a fluid, there is a change in pressure with depth.

B.2 Pressure in fluid of uniform density

Equation (B.2) is valid for any fluids at rest. For incompressible fluid, ρ is constant, and (B.2) gives

$$p + \rho g z = constant, \tag{B.3}$$

in a fluid of uniform density. The *constant* is the **piezometric pressure**.



For incompressible fluid, *the hydrostatic pressure increases linearly with depth*, regardless of the shape of the container. The pressure is the same at all points on a given horizontal plane (same z).

Figure 7: Pressure increases with depth.

B.3 Absolute, gauge and vacuum pressure

When the pressure measurement is specified with respect to a zero pressure reference (i.e. that of a vacuum), it is the *absolute pressure*. For example, in figure 7, $p = p_{air} + \rho g h$ is the absolute pressure at $z = -h$.

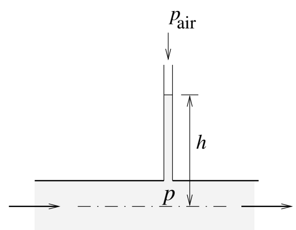
When the measured pressure is higher than the local atmospheric pressure, the *difference* is called the *gauge pressure*, e.g. $\rho g h$ in figure 7 is the gauge pressure at $z = -h$; if lower, the difference is called the *vacuum pressure* (or negative gauge pressure). The atmospheric pressure is approximately 1 bar or 1000 kN/m^2.

The term $p/\rho g$ is called the *pressure head* or simply 'head'.

B.4 Applications to pressure measurements

Piezometer

The pressure in a liquid, relative to p_{air}, can be measured using a *piezometer tube*, see figure 8.



piezometric head h:

$$p - p_{air} = \rho g h.$$

Figure 8: A piezometer tapped into a pipe.

Manometers

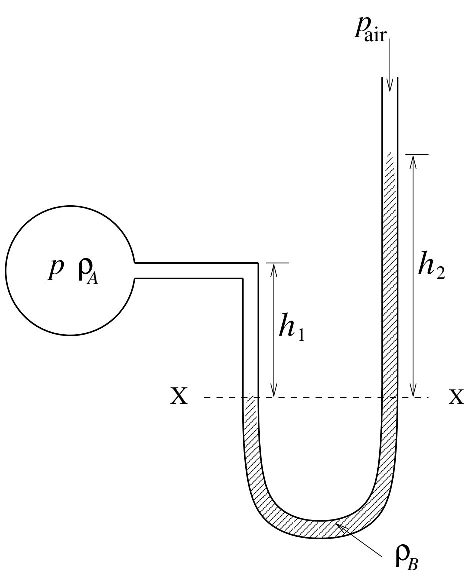
From the hydrostatic equation (B.2),

$$p_2 - p_1 = -\rho g \left(z_2 - z_1 \right) \tag{B.4}$$

i.e. the pressure difference between two points is proportional to the vertical distance between the points. Thus, a static column of one or more fluids can be used to measure the pressure difference between two points (or relative to the surrounding atmosphere). Such a device is called a *manometer*. Some examples are seen below.

Example 5

(a) A simple U-tube manometer



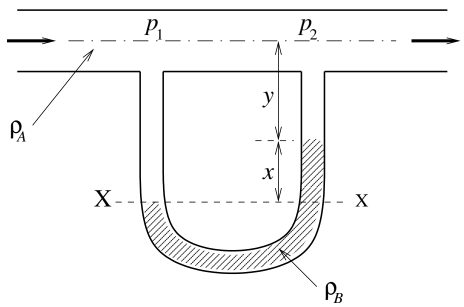
Pressures at X–X are the same.

$$p + \rho_A g h_1 = p_{air} + \rho_B g h_2.$$

Setting $p_{air} = 0$ (i.e. we are interested in the pressure relative to p_{air}, not the absolute pressure), the pressure in the container is

$$p = \rho_B g h_2 - \rho_A g h_1.$$

(b) Differential manometer



By equaling the pressures at X–X,

$$p_1 + \rho_A g(x+y) = p_2 + \rho_A g y + \rho_B g x,$$

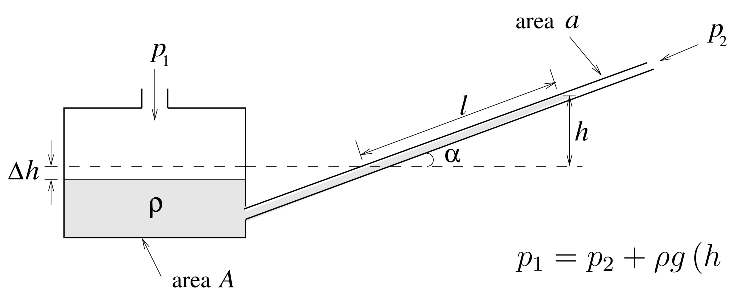
$$\therefore \qquad p_1 - p_2 = (\rho_B - \rho_A)\, g x.$$

Converting to pressure head,

$$h_1 - h_2 = \frac{p_1 - p_2}{\rho_A g} = \left(\frac{\rho_B}{\rho_A} - 1\right) x.$$

This is the head loss from point 1 to point 2, due to wall friction. If A is water and B is mercury, $\rho_B/\rho_A = 13.6$ and $h_1 - h_2 = 12.6x$.

(c) An inclined manometer for measuring small pressure



$p_1 = p_2 + \rho g\,(h + \Delta h)$

$h = l \sin\alpha$

$A\Delta h = al \qquad$ (mass conservation)

$p_1 - p_2 = \rho g l \left(\sin\alpha + \frac{a}{A}\right)$

If $a/A \to 0, \quad p_1 - p_2 = \rho g l \sin\alpha$

B.5 Hydrostatic force on plane surfaces

The pressure force on a submerged surface is always normal to the surface, regardless of the shape and orientation of the surface.

Vertical rectangle

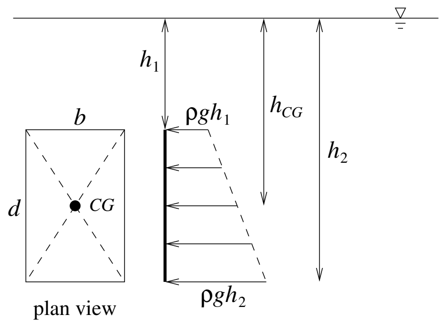


Figure 9: Pressure diagram for a vertical rectangle.

$$\begin{aligned} \text{F} &= \text{area under pressure diagram} \\ &\qquad \times \text{ width} \\ &= \underbrace{\frac{\rho g h_1 + \rho g h_2}{2}}_{p_{ave} = \rho g h_{CG}} d \times b \end{aligned}$$

$$F = \textbf{average pressure} \times \textit{immersed} \text{ area} = \rho g h_{CG} A \qquad \text{(B.5)}$$

Note: The centroid CG of the immersed rectangle is located at depth h_{CG} below the water surface.

Example 6 Pressure force on a dam

Calculate the force on a dam in a reservoir of depth 10 m.

(a) The wet face is vertical.

(b) The wet face is sloped at $\alpha = 45°$.

(a) $h_{CG} = \frac{1}{2}h = 5$ m. Area/per width is $A = h \times 1 = 10$ m/per width.

$$F = \rho g h_{CG} A = 1000 \times 9.81 \times 5 \times 10 = 490500\text{N/m width}$$

(b) $h_{CG} = 5$ m, however, the area/per width is

$$A = \frac{h}{\sin \alpha} \times 1 = \frac{10}{\sin 45°} = 14.14 \text{ m/per width}$$

$$F = \rho g h_{CG} A = 1000 \times 9.81 \times 5 \times 14.14 = 693567\text{N/m width}$$

General plane surface

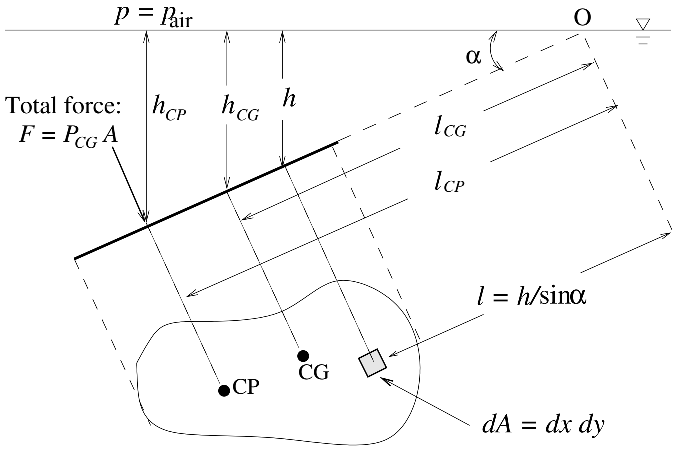


Figure 10: Hydrostatic force and centre of pressure on an arbitrary plane surface of area A inclined at an angle α.

- *Total force:*

Force on an element area: $\mathrm{d}F = p \times \mathrm{d}A = \rho g h \mathrm{d}A$

Force of the surface: $F = \int \mathrm{d}F = \int p\,\mathrm{d}A$ (i.e. sum of p over A) $= p_{ave} \times A$

Since $p_{ave} =$ *the pressure at the centroid* $= \rho g h_{CG}$,

$$\boxed{\begin{aligned} F &= \text{pressure at the plate centroid } \times \text{ the area} \\ &= \rho g h_{CG} A = \rho g \left(l_{CG} \sin\alpha \right) A \end{aligned}} \quad \text{(B.6)}$$

Equation (B.6) is true for any plane submerged in fluid of uniform density, regardless of the shape of the plane and its orientation.

- *Center of pressure h_{CP}:*

The total force F, however, does *not* act through the centroid. The centre of pressure CP is generally *below* the centroid toward the high pressures, in order to balance the bending moment portion of the stress. Extending the immersed surface, it intersects the water surface at a line, denoted as OX.

Moment of the element force about OX axis: $p\mathrm{d}A \times l = \rho g \left(l \sin\alpha \right) \mathrm{d}A l$

Moment of the resultant force: $F l_{CP} = \int p\mathrm{d}A \times l = \int \rho g \sin\alpha\, l^2 \mathrm{d}A$. So,

$$l_{CP} = \frac{\int l^2 \mathrm{d}A}{A\, l_{CG}} = \frac{\text{second moment of area about } OX}{A\, l_{CG}}.$$

The parallel axes theorem states

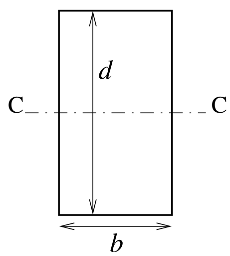
$$\int l^2 \mathrm{d}A = I_{CG} + A\, l_{CG}^2$$

I_{CG} = second moment of an area about its centroid. Thus,

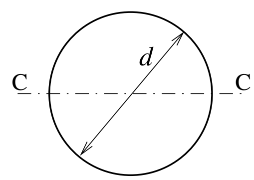
centre of pressure:

$$l_{CP} = \frac{I_{CG}}{A\, l_{CG}} + l_{CG} \qquad h_{CP} = l_{CP} \sin\alpha \tag{B.7}$$

Common examples are



$I_{CG} = \frac{1}{12} b d^3$



$I_{CG} = \frac{1}{64} \pi d^4$

Example 7 Center of pressure

Calculate the centre of pressure for the two cases in example 6.

(a) $d = 10$ m, $b = 1$ (unit width). $l_{CG} = h_{CG} = 5$ m.

$$I_{CG} = \tfrac{1}{12} b d^3 = \frac{1000}{12} \text{ m}^3 \text{ per width}$$

$$h_{CP} = \frac{I_{CG}}{A h_{CG}} + h_{CG} = \frac{1000}{12\,(10 \times 1)\,5} + 5 = 1.67 + 5 = 6.67 \text{ m}$$

(b) $d = 10/\sin 45^\circ = 14.14$ m, $b = 1$ (unit width).

$h_{CG} = 5$ m, $l_{CG} = h_{CG}/\sin 45^\circ = 7.07$ m.

$$I_{CG} = \tfrac{1}{12} b d^3 = \frac{(14.14)^3}{12} \text{ m}^3 \text{ per width}$$

$$l_{CP} = \frac{I_{CG}}{A l_{CG}} + l_{CG} = \frac{(14.14)^3}{12\,(14.14 \times 1)\,7.07} + 7.07 = 9.43 \text{ m}$$

$$h_{CP} = l_{CP} \sin 45^\circ = 6.67\text{m}$$

B.6 Hydrostatic force on curved surfaces

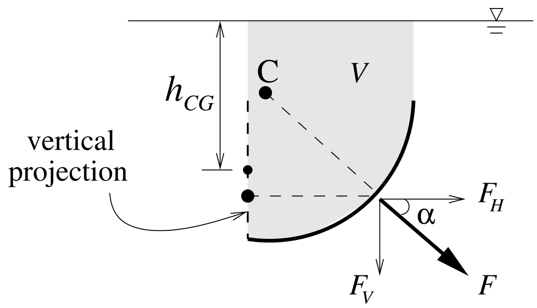


Figure 11: Pressure force on a curved surface.

– The total force F is normal to the curved surface, so it always passes through the centre of curvature.

– It is simplest to resolve horizontal and vertical components of F.

- *Horizontal component F_H:*

  Project the curved surface onto a vertical plane. A_{proj} = area of the vertical projection, and its centroid is located at h_{CG} below the water surface.

$$\begin{aligned} F_H &= \textbf{force on the vertical projection} \\ &= \rho g h_{CG} A_{proj}. \end{aligned} \tag{B.8}$$

  F_H **acts through the centre of pressure of** A_{proj}**.**

- *Vertical component F_H:*

$$V = \text{ the volume directly above curved surface}$$

$$\begin{aligned} F_V &= \textbf{weight of water in } V \\ &= \rho g V. \end{aligned} \tag{B.9}$$

  F_V **acts through the centre of gravity of the volume** V**.**

- *The direction of F*

$$\tan \alpha = F_V / F_H \tag{B.10}$$

Buoyancy

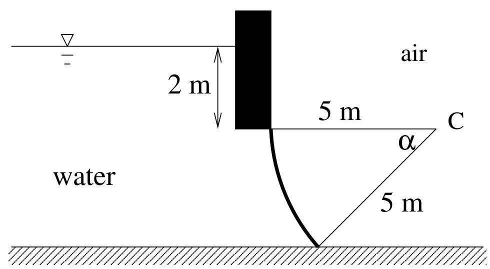
One can think that a body is a volume enclosed by an upper surface and a lower one. Applying (B.9) to these two surfaces of a submerged body, the results are the two laws of buoyancy, following Archimedes.

– A body immersed in a fluid experiences a vertical *buoyant force equal to the weight of the fluid it displays.*

– A floating body displays its own weight in the fluid in which it floats.

Example 8 Pressure force on a curved gate

A sector gate, of radius 5 m and length 3.5 m, controls the flow of water in a channel. $\alpha = 30°$. Determine the total force on the gate.



Tutorial

Hydrostatic forces on gates, and calculation of moments.