Hyperbolic Geometry in Parameterization of Surfaces

Directed Reading Program Project

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1 Introduction

Parameterization is the process to find explicit equations of a curve ($\mathbb{R}^2$) or a surface ($\mathbb{R}^3$), or a manifold ($\mathbb{R}^n$) in general. Intuitively, a parameterization of a geoemtry shape is to represent the shape in terms of "parameters". In this paper, I will refresh the concept of parameterization from calculus, introduce a specific problem exists in the parameterization in $\mathbb{R}^3$, and provide a solution.

2 Parameterization, and Reparameterization in $\mathbb{R}^2$

2.1 Parameterization

I believe readers have seen the parameterization of a circle before in multi-variable calculus. Therefore, let's review the common definition of parameterization.

Definition: A *parameterized curve* in $\mathbb{R}^n$ is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.

The symbol (α, β) denotes the open interval

$$(\alpha, \beta) = \{t \in \mathbb{R} \mid \alpha < t < \beta\}$$

Therefore, a *parameterized curve* in $\mathbb{R}^2$ is

$$\gamma(t) = \sigma(u(t), v(t))$$

2.2 Reparameterization

Now, as we can imagine, there exists a process to reparameterize the curve, which is to express the curve with a different set of "parameters".

Definition: A *parameterized curve* curve $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ is a *reparamatrization* of a parameterized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth bijective map $\theta : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ (the *reparameterization map*) such that the inverse map $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is also smooth and

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) \text{ for all } \tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$$

Note that, since ϕ has a smooth inverse, γ is a reparameterization of $\tilde{\gamma}$:

$$\tilde{\gamma}(\phi^{-1}(t)) = \gamma(\phi(\phi^{-1}(t))) = \gamma(t) \text{ for all } t \in (\alpha, \beta)$$

Two curves that are reparameterizations of each other have the same image, so they should have the same geometric properties. Informally, we can perceive this as a changing of the "speed" of t.

2.3 Example: Parameterization of a Circle

I believe the form of parameterization function must be familiar to the readers, we recall the implicit formula of a circle is in the form of

$$x^2 + y^2 = 1$$

The unit circle has a parameterization

$$\tilde{\gamma}(t) = (\sin t, \cos t)$$

Then we could find a reparameterization of σ, we have to find a reparameterization map ϕ such that

$$(\cos \phi(t), \sin \phi(t)) = (\sin t, \cos t)$$

One solution could be $\theta(t) = \pi/2 - t$. The images before reparameterization and after the transition maintain the same shapes and geometric properties.

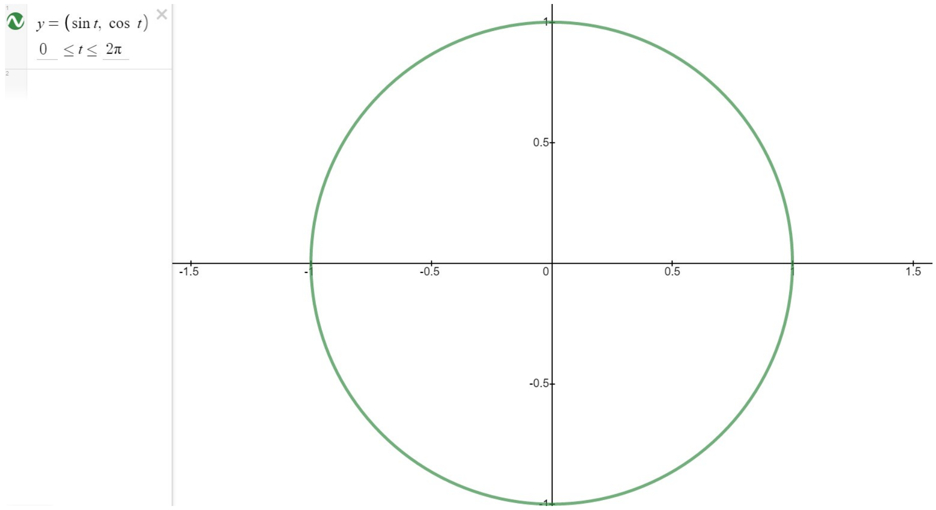

Figure 1: Parameterization of a Circle

3 Parameterization in $\mathbb{R}^3$

3.1 Surface and Surface Patch (Parameterization)

Definition: A subset $\mathcal{S}$ of $\mathbb{R}^3$ is a *surface* if, for every point $\mathbf{p} \in \mathcal{S}$, there is an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing $\mathbf{p}$ such that $\mathcal{S} \cap W$ is homemorphic to U.

Similarly, we could find a parameterization, or surface patch for $\mathbb{R}^3$ as we did for $\mathbb{R}^2$.

Definition: A homemorphism $\sigma : U \to \mathcal{S} \cap W$ as in this definiton is called a *surface patch* or *parameterization* of the open subset $\mathcal{S} \cap W$ of $\mathcal{S}$.

Definition: A collection of such surface patches whose images cover the whole of $\mathcal{S}$ is called an *atlas* of S.

As we can see, there is analogy between parameterization in $\mathbb{R}^2$ and $\mathbb{R}^3$. In fact, the most important application in $\mathbb{R}^3$ is UV mapping, commonly used in Computer Graphics and Visual Effects. As you can see, this is also a bijective mapping.

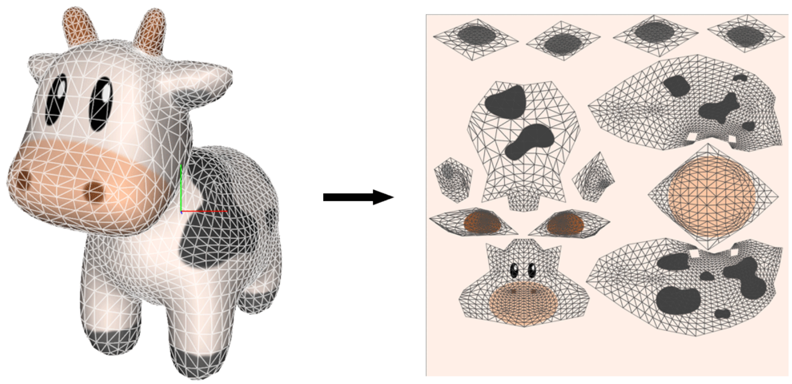

Figure 2: Parameterization of a 3D Model, credit to Keenan Crane

3.2 Genus 1 Surface and the Parameterization of a Torus

The most common form of a Genus 1 Surface is a torus. Given the parameterization equations:

$$x(\theta, \varphi) = (a + b\cos\theta)\cos\varphi$$
$$y(\theta, \varphi) = (a + b\cos\theta)\sin\varphi$$
$$z(\theta, \varphi) = b\sin\theta$$

Therefore,

$$\sigma(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta)$$

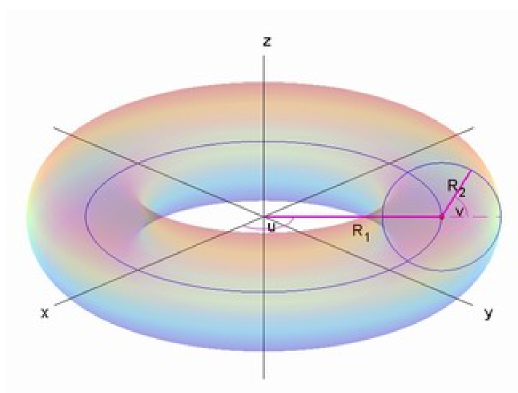

Figure 3: Parameterization of a Torus, credit to Rob Womersley

To get this parameterization, we could imagine the first two terms define the circle, then the third term construct another circle whose radius is perpendicular to the first circle at any given point on the first circle.

4 The Problem in the Parameterization of A Genus 2 Surface

4.1 Genus 2 Surface and its Common Construction

Since Genus 1 Surface is a Torus, Genus 2 surface, occasionally called Double Torus, is a a surface formed by the connected sum of 2 tori. A famous exemple of a non-orientable surface genus two is the Klein bottle.

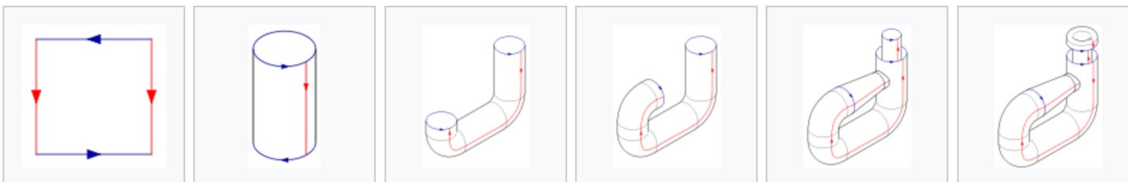

Figure 4: Klein Bottle and Its Fundamental Polygon, credit to Wikipedia user Inductiveload

The common construction of a double torus is a octagon with opposite identified.

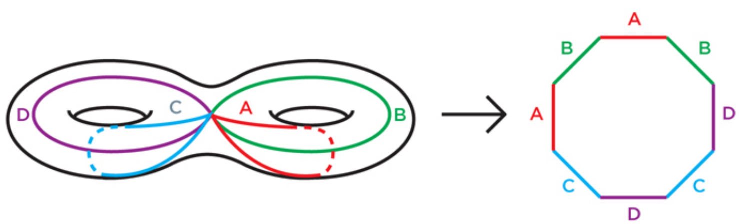

Figure 5: A Octagon to Double-Torus

By our definition of surface patch, this process is a parameterization of the genus surface in $\mathbb{R}^3$.

4.2 the Importance of Euclidean Intuition

As we all know, we live in a world where Euclidean space is the "norm". However, it is impossible to always obtain an understandable Euclidean graph when study geometry. In the case mentioned in the last sub-section, we could obtain 4 lines intersect one point, which implies the sum of 4 angles produced is equal to 360° since genus 2 surface is locally Euclidean. Apparently, the octagon is in $\mathbb{R}^2$ and the sum of its interior angles is 1080°. Let us assume the octagon is an equilateral octagon, then each angle has the degree of 135°. Apparently, the angles deformed during the surface patch since the sum of angles changed.

However, as we have mentioned before, we want to preserve this Euclidean intuition to help us solving more questions. Then it is clear we want a better way to that preserve the angle and length of the $\mathbb{R}^2$ open set in surface patch.

5 The Solution to the Problem

5.1 Conformal Mapping and Its Limitation

Firstly, let us think within the box: start with $\mathbb{R}^2$, there exists a mapping that can preserve angles.

Definition: If $\mathcal{S}_1$ and $\mathcal{S}_2$ are surfaces, a *conformal map* $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a local diffeomorphism s.t. if γ_1 and $\hat{\gamma}_1$ are any two curves on $\mathcal{S}_1$ that intersect, say at a point $p \in \mathcal{S}_1$, and if γ_2 and $\hat{\gamma}_2$ are their images under f, the angle of intersection of γ_1 and $\hat{\gamma}_1$ at $\mathbf{p}$ are equal to the angle of intersection of γ_2 and $\hat{\gamma}_2$ at $f(\mathbf{p})$.

However, conformal map does not preserve the length of our equilateral octagon, here is an example of in Computer Graphics: as you can see, the disproportional stretch of polygons is not ideal to our mapping.

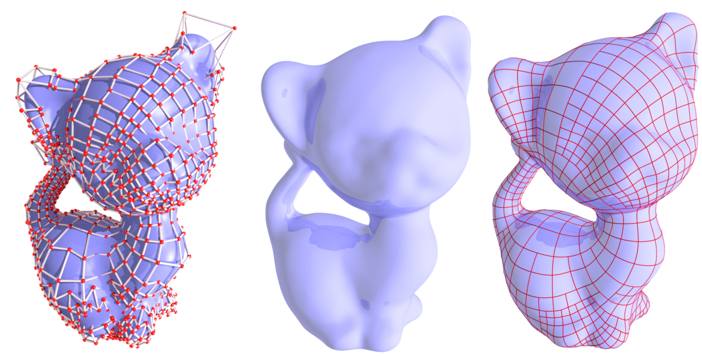

Figure 6: An Example of Conformal Mapping in Computer Graphics, credit to CCGL

5.2 Poincaré Disk Model and Hyperbolic Geometry

The only difference between Hyperbolic Geometry and Euclidean Geometry is the one postulate: the parallel postulate (Playfair's axiom). In the following of this paper, I will use the Poincaré Disk Model to demonstrate some properties of hyperbolic geometry.
Definition: Here is the five **axioms** of *Hyperbolic Geometry*:

(1) To draw a straight line from any point to any point.

(2) To produce (extend) a finite straight line continuously in a straight line.

(3) To describe a circle with any centre and distance (radius).

(4) That all right angles are equal to one another.

(5) **For any given line R and point P not on R, in the plane containing both line R and point P there are at least two distinct lines through P that do not intersect R.** An alternative way to say it is there exists two or more parallel line to a line through a given point that is not on the line.

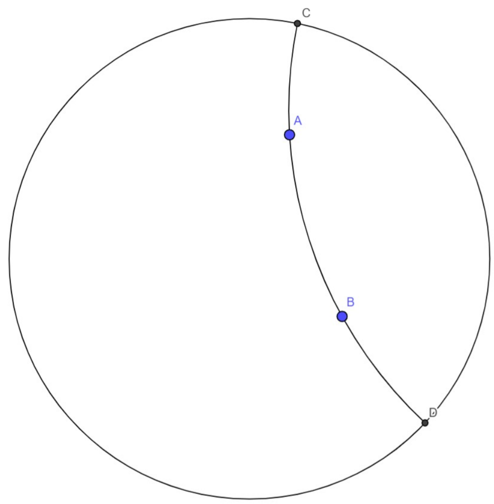

Figure 7: A Straight Line in Hyperbolic Plane $\mathbb{H}^2$

Now, let us denote hyperbolic space as $\mathbb{H}$. Based on Poincaré disk model, $\mathbb{H}^2$ is an open disk and $\mathbb{H}^2$ is a subspace of $\mathbb{R}^2$. This implies we could use Euclidean intuition to construct hyperbolic lines/shape ($\mathbb{H}^2$) in $\mathbb{R}^2$.

5.3 Properties of Hyperbolic Geometry in $\mathbb{H}^2$

Back to the equilateral octagon we used in double torus parameterization, we could reconstruct the equilateral octagon in hyperbolic plane $\mathbb{H}^2$ that preserve the same length yet have a different sum of interior angles.

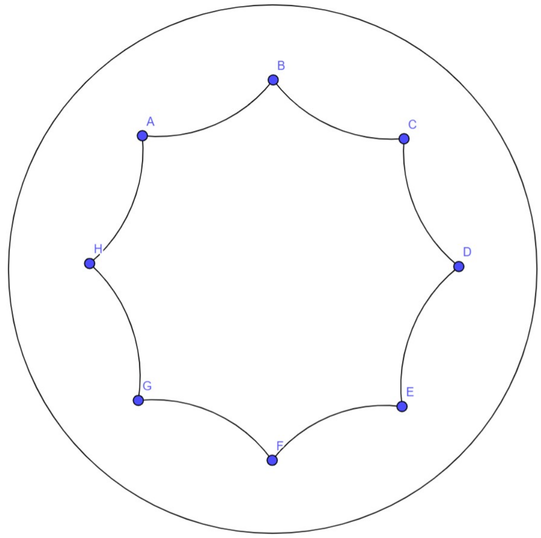

Figure 8: An equilateral octagon with each interior angles equal to 90° in $\mathbb{H}^2$

Now, let me use triangles to demonstrate some "magical" properties on hyperbolic plane. It

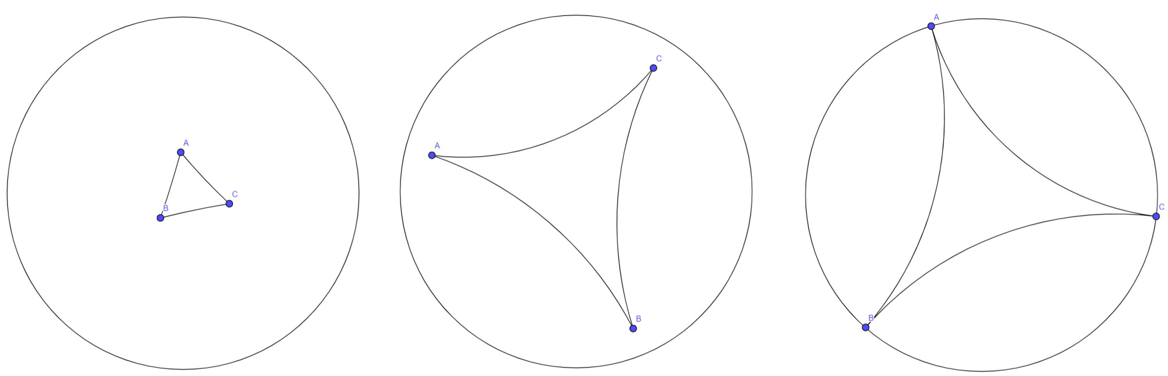

Figure 9: Three equilateral triangles in $\mathbb{H}^2$ with different degree of interior angle

is easy to notice that all triangles are equilateral, but the closer the triangle to the center of hyperbolic plane, the more geometrically similar it is to its counterpart in Euclidean plane. Vice versa: if the triangle is far from the center, its shape will "deformed" (in an Euclidean intuition) and the sum of interior angles will approach to 0. To put it simply: while all three triangles are equilateral triangles, they have different interior angle sum.

5.4 The Solution: An Octagon on A Hyperbolic Plane

In summary, instead of traditional $\mathbb{R}^2$ based shapes, we choose an equilateral octagon on $\mathbb{H}^2$. Using the similar idea we used in last sub-section, we could construct arbitrary equilateral octagon with ideal interior angle degree. Therefore, even after the surface patch, the sum of interior angles of our hyperbolic shape can be preserved. Furthermore, with careful construction, we could potentially preserve the length as well.

6 Conclusion

Paramatrization of a surface in $\mathbb{R}^3$ can be challenging. In this paper, I provided an interesting solution to the problem of angle deformations in $\mathbb{R}^2 \to \mathbb{R}^3$ surface patch with introducing hyperbolic geometry. Lastly and most importantly, thanks to my DRP mentor Neža's patience and effort in helping me compiling this note.

Appendices

A Basic Topology Definition

Definition	Quantifier Statement	Notation
p is an interior point of G	$(\exists r > 0)(N_r(p) \in G)$	$p \in G^\circ$
p is in the closure of E	$(\forall r > 0)(N_r(p) \cap E \neq)$	$p \in \overline{E}$
G is open	N/A	$G^\circ = G$
F is closed	N/A	$F \not\subset F$ or $\overline{F} = F$

Figure 10: Basic Topology Definition Chart

References

[1] Andrew Pressley, *Elementary Differential Geometry* (Springer, 2001. ISBN 1-85233-152-6)