STUDENT NUMBER:

HAND IN

Answers recorded on exam paper.

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUEEN'S UNIVERSITY AT KINGSTON

MATH 121 - DEC 2018

Sections 002, 003 - MAIN CAMPUS Students ONLY

Instructors: A. Ableson, G. Gottesman

INSTRUCTIONS:

- Write your student number at the top of this page where indicated ("STUDENT NUMBER:") before you begin.
- This examination is 3 HOURS in length.
- Only **CASIO fx-991** calculators are permitted.
- Answer **all questions**, writing clearly in the space provided. If you need more room, continue your answer on one of the blank pages at the back, **providing clear directions to the marker**.
- The last page of the exam has a bubble sheet. **Multiple choice answers for Section I must be recorded on the last page.**
- For full marks, you must show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise.
- Wherever appropriate, **include units in your answers**.
- When drawing graphs, **add labels and scales on all axes**.

PLEASE NOTE: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

I	II	III	IV	V	VI	VII	VIII	IX	X	Total
20	8	8	7	7	8	10	10	12	10	100

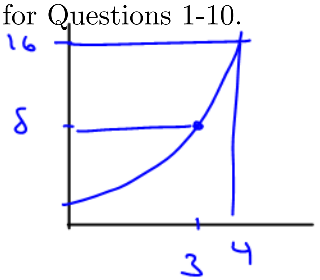
Section I. Multiple Choice (10 questions, 2 marks each)

Each question has four possible answers, labeled (A), (B), (C), and (D). Choose the most appropriate answer.

On the last page of the exam, fill in the bubble corresponding to your answer for Questions 1-10.

(1) Find the average value of the function $f(x) = 2^x$ over the interval $x \in [0, 4]$.

- (A) The average value is greater than 16.
- (B) The average value is above 12 and less than or equal to 16.
- (C) The average value is above 8 and less than or equal to 12.
- [✓ (D) The average value is below 8.]



f(x) below 8 for most of the interval

Check w/ definition $A = \frac{1}{4}\int_0^4 2^t\,dt$

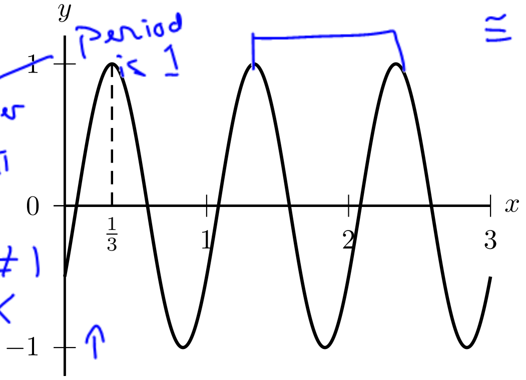
$= \frac{1}{4}\left(\frac{2^t}{\ln 2}\right)\Big|_0^4 = \frac{1}{4}\frac{1}{\ln(2)}(16-1)$

$\cong 5.41$

(2) Determine the formula for the trigonometric function shown to the right.

- (A) $y = \cos\left(\pi x - \frac{\pi}{3}\right)$ ✗
- (B) $y = \cos\left(\pi x - \frac{\pi}{6}\right)$ ✗
- (C) $y = \cos\left(2\pi x - \frac{\pi}{3}\right)$ @ $x = 1/3$, $y = \cos(\pi/3) \neq 1$ ✗
- [✓ (D) $y = \cos\left(2\pi x - \frac{2\pi}{3}\right)$] @ $x = 1/3$, $y = \cos(0) = 1$ ✓

x multiplier is $\frac{2\pi}{1} = 2\pi$



Period is 1

$x = 1/3$ should give $y = 1$

(3) The integral $\int_{-2}^{2} 4(f(x) - 1)\,dx = 4$. What is the value of $\int_{-2}^{2} f(x)\,dx$?

- (A) $\int_{-2}^{2} f(x)\,dx$ is less than or equal to 2.
- (B) $\int_{-2}^{2} f(x)\,dx$ is greater than 2, and less than or equal to 4.
- [✓ (C) $\int_{-2}^{2} f(x)\,dx$ is greater than 4, and less than or equal to 8.]
- (D) $\int_{-2}^{2} f(x)\,dx$ is greater than 8.

Carefully separate integral:

$$\int_{-2}^{2}(4f(x) - 4)\,dx = 4$$

$$\underbrace{4\int_{-2}^{2} f(x)\,dx}_{\text{want this}} - \underbrace{\int_{-2}^{2} 4\,dx}_{= 4x|_{-2}^{2} = 16} = 4$$

$$4\int_{-2}^{2} f(x)\,dx = 4 + 16 = 20$$

$$\int_{-2}^{2} f(x)\,dx = \boxed{5}$$

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(4) The graph of a function $g(t)$ is given in the figure to the right.
Which of the following is correct?

(A) $\int_0^2 g(t)\,dt < 30.9$

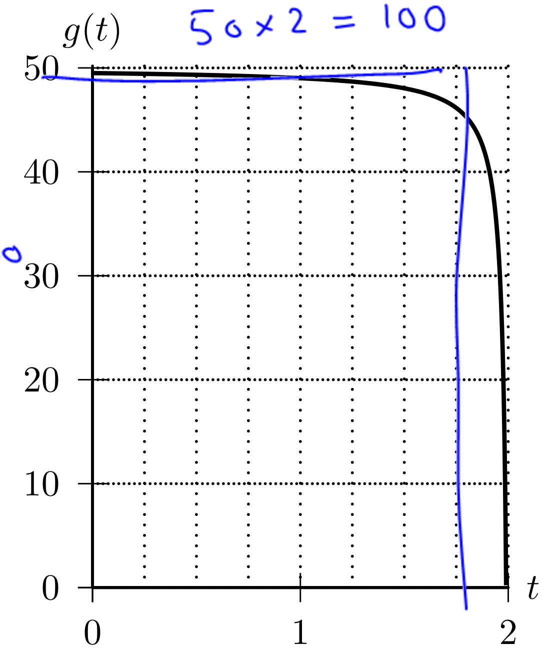
(B) $30.9 < \int_0^2 g(t)\,dt < 69.2$

[✓ (C) $69.2 < \int_0^2 g(t)\,dt < 105.6$]

(D) $105.6 < \int_0^2 g(t)\,dt$

area a little less than 100

$50 \times 2 = 100$



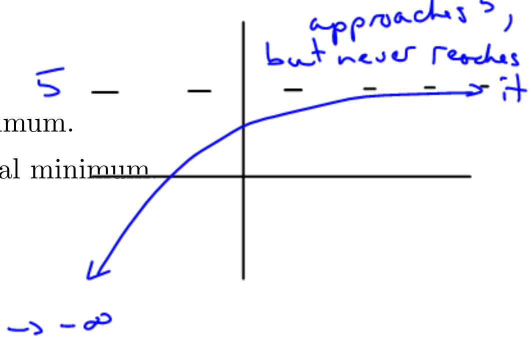
(5) Identify whether the function $h(x) = 5 - e^{-2x}$ has a global minimum and/or global maximum on its domain. Hint: a sketch might help.

[✓ (A) $h(x)$ has **neither** a global maximum, nor a global minimum.]

(B) $h(x)$ **has** a global maximum, but does **not** have a global minimum.

(C) $h(x)$ does **not** have a global maximum, but **does** have a global minimum.

(D) $h(x)$ has **both** a global maximum, and a global minimum.



(6) The function $f(x) = e^{\frac{-(x-a)^2}{b}}$ has a global maximum at the point $(3, 1)$. What can be inferred about the values of a and b?

↳ critical point

$f'(x) = \frac{2(x-a)}{b} e^{\frac{(x-a)^2}{b}}$

(A) Without more information, neither a nor b can be uniquely determined.

[✓ (B) $a = 3$, but the specific value of b cannot be determined from the information provided.]

(C) $b = \frac{1}{2}$, but the specific value of a cannot be determined from the information provided.

(D) $a = 3$, and $b = \frac{1}{2}$.

$f' = 0$ @ $x = 3$

$0 = \frac{\cancel{2}(3-a)}{\cancel{b}} \cancel{e^{\frac{(3-a)^2}{b}}}$

$3 - a = 0$ $\boxed{a = 3}$

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(7) Consider the table of data shown on the right (with entries rounded to 2 decimal places). Which of the functions is growing exponentially with time?

t	$f(t)$	$g(t)$	$h(t)$
0	10.00	1.00	10.00
1	15.00	1.25	15.25
2	22.50	5.00	31.00
3	33.75	21.25	57.25
4	50.63	65.00	94.00

$\times 1.5$

f(t) multiplied by 1.5 each step.

[✓ (A) $f(t)$ is growing exponentially.]

(B) $g(t)$ is growing exponentially.

(C) $h(t)$ is growing exponentially.

(D) None of the three functions is growing exponentially.

(8) A graph is defined by the relation $\underset{\checkmark}{\overset{d/dx}{\sin(4y)}} = \overset{d/dx}{(1 - e^x)}$. What is the slope of a tangent line to this graph at the point $(0, 0)$?

needs implicit derivative

(A) The slope is in the range $(-\infty, -2]$.

[✓ (B) The slope is in the range $(-2, 0]$.]

(C) The slope is in the range $(0, 2]$.

(D) The slope is in the range $(2, \infty)$.

$$\cos(4y) \cdot 4 \frac{dy}{dx} = -e^x$$

$$\frac{dy}{dx} = \frac{-e^x}{4\cos(4y)}$$

$$@ (0,0),\ \boxed{dy/dx = \frac{-1}{4}}$$

(9) Compute the integral of the function $x(t) = \cos(5t) + t^4$

(A) The integral equals $\frac{\sin(5t)}{5} + 4t^5 + C$.

(B) The integral equals $\frac{-\sin(5t)}{5} + 4t^5 + C$.

[✓ (C) The integral equals $\frac{\sin(5t)}{5} + \frac{t^5}{5} + C$.]

(D) The integral equals $\frac{-\sin(5t)}{5} + \frac{t^5}{5} + C$.

$$\int \cos(5t) + t^4 \, dt$$

$$= \frac{\sin(5t)}{5} + \frac{t^5}{5} + C$$

d/dt ✓

(10) Consider the function $y = 3 - 6e^{-3x}$. For what value of x is $y = 0$?

(A) The x value will be below $x = -0.5$.

(B) The x value will be equal to or above $x = -0.5$, but below $x = 0$.

[✓ (C) The x value will be equal to or above $x = 0$, but below $x = 0.5$.]

(D) The x value will be equal to or above $x = 0.5$.

$$0 = 3 - 6e^{-3x}$$

$$6e^{-3x} = 3$$

$$e^{-3x} = 3/6 = 1/2$$

$$-3x = \ln(1/2)$$

$$x = \frac{\ln(1/2)}{-3} \simeq 0.23$$

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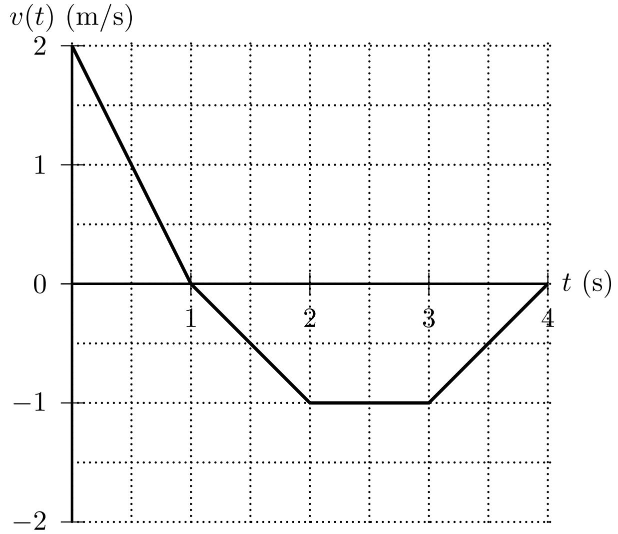
Section II. Graphing

[/8]

A particle's velocity is measured for 4 seconds, and the graph of that velocity is shown on the graph to the right. The velocity is measured in m/s, and the time is in seconds.



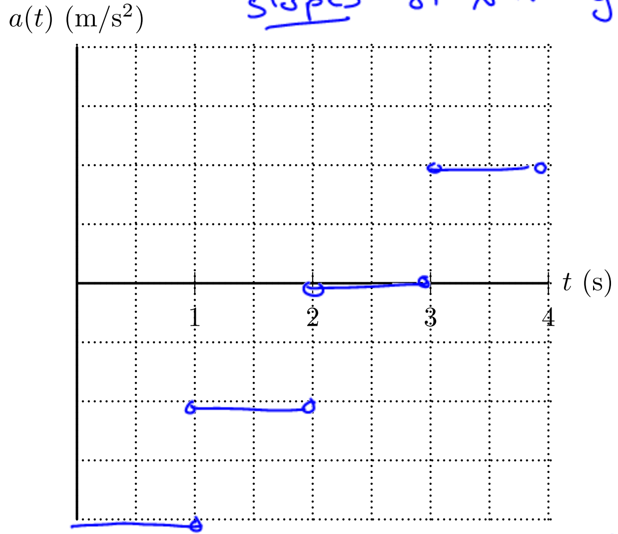
(a) On the axes to the right, sketch the **acceleration** of the particle, $a(t)$.

Recall that $\frac{d}{dt}v(t) = a(t)$.

Clearly indicate the vertical scale on your axes.

Use the fact that $a(1.5) = -1$ in your graph.

slopes of v(t) graph



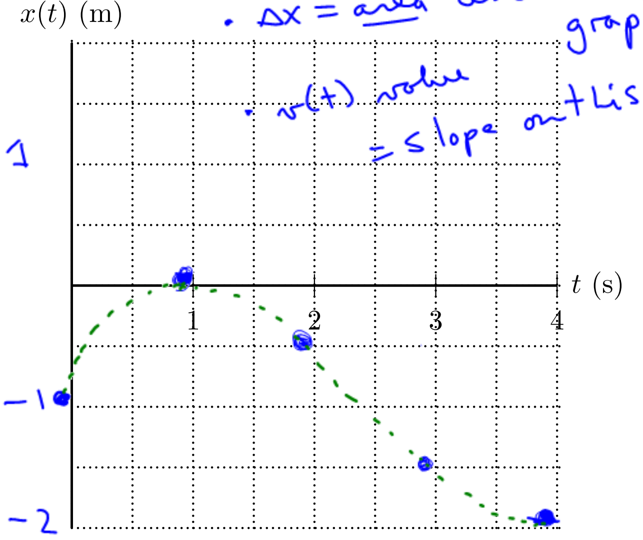
(b) On the axes to the right, plot the **position** of the particle, $x(t)$ at times $t = 0$, $t = 1$, $t = 2$, and $t = 3$.

Recall that $\frac{d}{dt}x(t) = v(t)$.

Clearly indicate the vertical scale on your axes.

Use the fact that $x(1.0) = 0$ in your graph.

- Δx = area under v(t) graph
- v(t) value = slope on this graph



Section III. Flight Efficiency *[/8]*

A major design goal of modern aircraft construction is to improve their fuel efficiency. However, there are several ways to compute and report this efficiency.

Let $f(v)$ be the fuel consumption reported in **kg of fuel/km flown** of a plane flying at velocity v (in km/h). In other words, $f(v)$ tells you how many kg of fuel the plane uses to go one kilometer, if it is traveling at velocity v km/h.

You are told that for a new Boeing 737MAX airplane,

$$f(750) = 3.04 \text{ kg/km, and } f'(750) = 0.0005 \text{ (kg/km)/(km/h)}$$

We now define $h(v)$ as the plane's gas consumption in **kg of fuel/h**. In other words, $h(v)$ tells you how many kilograms of fuel the plane uses in one hour of flight, if it is flying at velocity v.

(a) Write an equation that relates $h(v)$, $f(v)$ and v.

$$\boxed{h(v) = f(v) \cdot v}$$

$$(\text{kg}/\text{h}) \qquad \text{kg}/\text{km} \qquad \frac{\text{km}}{\text{h}}$$

(b) Use your equation and the earlier information about $f(v)$ to compute $h(750)$. Include units in your answer.

$$\begin{aligned} h(750) &= f(750) \cdot 750 \\ &= (3.04) \cdot 750 \\ &= 2280 \quad (\text{kg}/\text{h}) \end{aligned}$$

(c) Compute the value of $h'(750)$ (or $\frac{dh}{dv}(750)$). Include units in your answer.

$$\frac{dh}{dv}(v) = f'(v) \cdot v + f(v) \cdot 1$$

product rule, and $\frac{dv}{dv} = 1$

$$\begin{aligned} h'(750) &= (0.0005)(750) + 3.04 \\ &= 3.415 \quad (\text{kg}/\text{h}) / \text{km}/\text{h} \end{aligned}$$

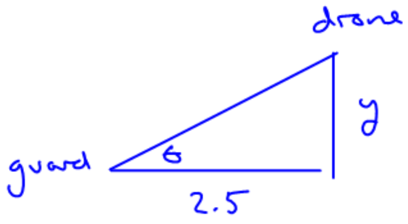
(d) Express the meaning of your answer to part (c) in a sentence.

For each 1 km/h faster the plane flies, it will use 3.415 kg/h more fuel. [relative to current speed of 750 km/h)

Section IV. Just beyond an airport, a security guard observed a drone (a remote controlled flying vehicle) rising straight up from the ground; i.e. the drone moved only vertically, not horizontally. The drone started on the ground 2.5 km from the guard. A few seconds after lifting off,

- the angle between the observer's view of the drone and horizontal (ground) was $\frac{\pi}{12}$, and
- the angle was increasing at a rate of 0.112 rad/min.

How fast is the drone rising at that moment? Include units in your answer. *[/7]*



From the diagram,

$$\tan\theta = \frac{y}{2.5}$$

so $$\frac{d}{dt}\left(\tan\theta\right) = \frac{d}{dt}\left(\frac{1}{2.5}\,y\right)$$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{2.5}\,\frac{dy}{dt}$$

sec = 1/cos

so $$\frac{dy}{dt} = \frac{1}{\cos^2\theta}\cdot\frac{d\theta}{dt}\,(2.5)$$

At the particular moment of interest, $\frac{d\theta}{dt} = 0.112$, $\theta = \pi/12$

$$\frac{dy}{dt} = \frac{1}{\cos^2(\pi/12)}\cdot 0.112\,(2.5)$$

$$\cong 0.300\ \frac{\text{km}}{\text{min}} \quad \left(\text{or } 300\ \frac{\text{m}}{\text{min}} \text{ or } 5\ \frac{\text{m}}{\text{s}}\right)$$

Section V. Critical Point Analysis [/7]

Consider the function $f(x) = xe^{-kx}$, where k can be any real number. This function has either one or zero critical points, depending on the value of k.

(a) For what k values does the function have one critical point?

Look for critical points: $f'(x) = e^{-kx} + x\left[-k e^{-kx}\right]$

Set $f' = 0 = e^{-kx}(1 - kx)$

$1 = kx$

$x = 1/k$

This will have a solution for all k values $\neq 0$

(or $(-\infty, 0) \cup (0, \infty)$,

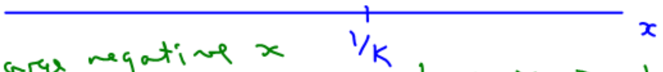
or $x < 0 \cup x > 0$)

(b) For what k values will there be one critical point which is a local **maximum**?

Assume $k \neq 0$. Can use either first or second derivative test.

1st: $f'(x) = \underset{\substack{\uparrow \\ (+)}}{e^{-kx}}(1 - kx)$

k>0 leads to the critical point being a local max



large negative x ⇒ $1 - kx$ is pos if $k > 0$, neg if $k < 0$

⌒ $k > 0$ ◡ $k < 0$

large x ⇒ $1 - kx$ is neg if $k > 0$, pos if $k < 0$

2nd: $f'' = -ke^{-kx}(1 - kx) + e^{-kx}(-k)$

$= e^{-kx}(-k + k^2x - k)$

@ $x = \frac{1}{k}$, $f'' = e^{-kx}\left(-2k + \frac{k^2}{k}\right)$

$= -ke^{-kx}$

$k > 0 \Rightarrow f''$ neg → local max

$k < 0$, f'' pos → local min

(c) When k is chosen so that $f(x)$ has no critical points, describe the graph of $f(x)$.

If $k = 0$, $f(x) = xe^0 = x$

f(x) is a straight line w/ slope 1.

Section VI. Taylor Polynomials and Limits *[/8]*

(a) Find the second degree (that is, quadratic) Taylor polynomial approximation to $f(x) = \cos(3x)$ at the point $x = \pi$.

$f(x) = \cos(3x)$

$f'(x) = -3\sin(3x)$

$f''(x) = -9\cos(3x)$

@ $x = \pi$

$f(\pi) = \cos(3\pi) = -1$

$f'(\pi) = -3\sin(3\pi) = 0$

$f''(\pi) = -9\cos(3\pi) = +9$

$$P_2(x) = -1 + 0\cdot(x-\pi) + \frac{9}{2}(x-\pi)^2$$

$$= -1 + \frac{9}{2}(x-\pi)^2$$

(b) Evaluate the limit $\lim\limits_{x\to\infty} \dfrac{3x - 4x^3}{5x^3 - 10}$. Reminder: show all your work.

dividig by x^3/x^3

$$= \lim_{x\to\infty} \frac{3/x^2 - 4}{5 - 10/x^3}$$

$$= \frac{-4}{5}$$

or l'Hôpital,

l'Hop

$$= \lim_{x\to\infty} \frac{3 - 12x^2}{15x^2} \left(\to \frac{-\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{-24\cancel{x}}{30\cancel{x}} = \frac{-24}{30} = \frac{-4}{5}$$

(c) Evaluate the limit $\lim\limits_{t\to\infty} \dfrac{t}{\ln(t)}$.

$\to \frac{\infty}{\infty}$

l'Hop

$$= \lim_{t\to\infty} \frac{1}{1/t} = \lim_{t\to\infty} t \to \infty$$ or DNE.

Section VII. Area Optimization *[/10]*

A rectangle is drawn in the first quadrant of the xy plane, with:

- its base on the x axis, and
- its left side on the y axis, and
- its upper right corner on the graph of $y = \dfrac{1}{(x+3)^2}$.

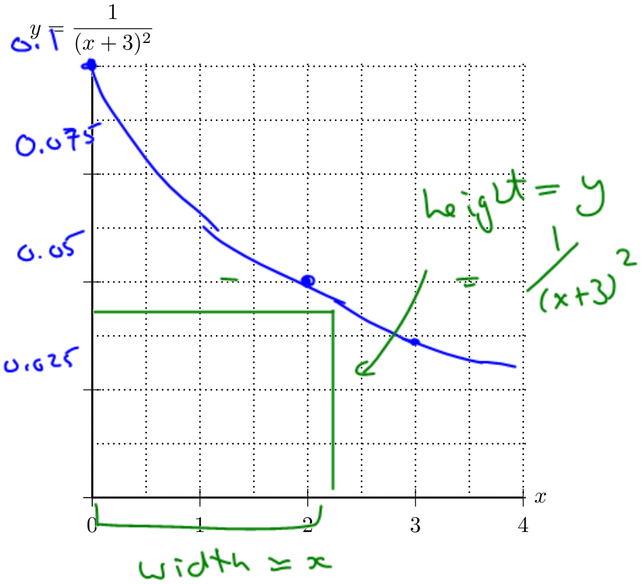
(a) What are the height and width of such a rectangle with the greatest possible area? Hint: a sketch might help to visualize the scenario.

Rectangle area = $(h)(w)$

$$A(x) = \left(\frac{1}{(x+3)^2}\right)(x) = \frac{x}{(x+3)^2}$$

To find optimal A, look for critical points

$$\frac{dA}{dx} = \frac{1\cdot(x+3)^2 - x\left(2(x+3)\right)}{(x+3)^4}$$



Set $=0$: $$0 = \frac{x^2+\cancel{6x}+9-(2x^2+\cancel{6x})}{\cancel{(x+3)^4}}$$

$$0 = -x^2+9$$

$$x^2 = 9$$

$x = \pm 3$, but only $x=3$ is in the first quadrant.

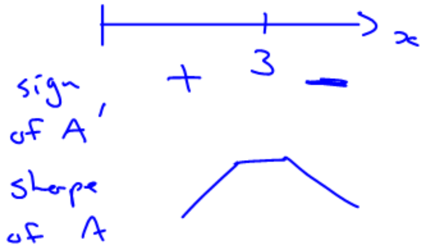
@ $x=3$, $y = \dfrac{1}{(3+3)^2} = \dfrac{1}{36}$

optimal:
width = 3
height = $\dfrac{1}{36}$

(b) Provide evidence that the dimensions you found correspond to a local maximum for the area.

1st or 2nd derivative test will work

1st derivative test $A'(x) = \dfrac{9-x^2}{(x+3)^4}$



$\Rightarrow$ A(3) is a local maximum for the area.

Section VIII. Greenhouse Gas Emissions *[/10]*

The following table tracks the rate of emission of greenhouse gasses by Canadians, $E(t)$, in megatonnes (Mt) of carbon dioxide per year. All dates represent the effective rate on Jan 1st of that year.

$\Delta t = 1$

Date (Jan 1st of)	2009	2010	2011	2012	2013	2014	2015	2016
E (Mt/year)	682	694	700	707	716	717	714	704

Data source: Environment Canada.

(a) Use the trapezoidal rule with 7 intervals to estimate the total carbon dioxide emissions by Canadians over the period Jan 1 2009- Jan 1 2016 (7 year period). Include units in your answer.

$$\text{TRAP}(7) = \overbrace{\left[\frac{682+694}{2}\right](1) + \left[\frac{694+700}{2}\right](1) + \dots + \left[\frac{714+704}{2}\right](1)}^{7 \text{ terms}}$$

$$= \sum \left(\underset{\text{rate}}{\text{avg emission}} \right)(\Delta t) \qquad \boxed{= 4941 \quad \text{Mt}}$$

(b) If we define $t = 0$ as January 1st 2009, then a good approximation to the data shown is $E(t) = -1.51\ t^2 + 14.24\ t + 680.63$ Mt/year where t is measured in years. Use this function and an integral to generate a second estimate of the total carbon dioxide emissions over the period Jan 1 2009-Jan 1 2016 (7 year period). Include units in your answer.

$$\text{Total emissions} = \int_0^7 (-1.51\ t^2 + 14.24\ t + 680.63)\ dt$$

$$= -\frac{1.51}{3} t^3 + \frac{14.24}{2} t^2 + 680.63 \cdot t \Big|_0^7$$

$$\cong 4940.65 \text{ Mt}$$

(c) Over the last few years, the rate of carbon dioxide emission is decreasing. **Assuming the trend from 2015/16 continues in a linear way**, estimate the emission rate we would expect on Jan 1 2020.

Several answers are possible, and any reasonable estimate will be accepted. Clearly indicate how you arrived at your estimate, and include units in your answer.

B/w 2015 and 2016, rate dropped by 10 (Mt/y)

$\Rightarrow$ by 2020, 4 years later, rate will drop by 4 (10) = 40 Mt/y

$\Rightarrow$ E (2020) will be forecast as 704 − 40

= 664 Mt/y

Section IX. Integration *[/12]*

Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

By substitution, let $\boxed{w = \ln(x)}$

so $\dfrac{dw}{dx} = \dfrac{1}{x}$ or $\boxed{x\,dw = dx}$

$$= \int \frac{\cos(w)}{\cancel{x}} \left[\cancel{x}\,dw\right]$$

$$= \int \cos(w)\,dw$$

$$= \sin(w) + C$$

back to x's

$$= \sin(\ln(x)) + C$$

(b) Determine by differentiating whether or not your answer to question (a) is correct. State your conclusion clearly for the grader.

$$\frac{d}{dx}\left(\sin(\ln(x)) + C\right) = \cos(\ln(x)) \cdot \frac{1}{x}$$

This matches the original integrand, so our integral is correct.

(Integration continued)

(c) $\int t^2 \ln(t)\ dt$

By parts.

$$\begin{tikzcd}
\text{let } u = \ln(t), \arrow[d, "d/dt"] \arrow[dr] & \int dv = \int t^2\ dt \arrow[d] \\
du = \frac{1}{t}\ dt & v = t^3/3 \arrow[l]
\end{tikzcd}$$

$$= \ln(t)\left[\frac{t^3}{3}\right] - \int \frac{t^3}{3}\cdot\frac{1}{t}\ dt$$

tidy

$$= \frac{1}{3}t^3\ln(t) - \frac{1}{3}\int t^2 dt$$

$$= \frac{1}{3}t^3\ln(t) - \frac{1}{3}\ t^3/3 + C$$

$$= \frac{1}{3}t^3\ln(t) - \frac{1}{9}t^3 + C$$

(d) Determine by differentiating whether or not your answer to question (c) is correct. Again state your conclusions clearly for the grader.

$$\frac{d}{dt}\left(\frac{1}{3}t^3\ln(t) - \frac{1}{9}t^3 + C\right)$$

$$= \left[t^2\ln(t) + \frac{1}{3}t^3\cdot\left(\frac{1}{t}\right)\right] - \frac{1}{9}\left(3t^2\right)$$

$$= t^2\ln(t) + \cancel{\frac{1}{3}t^2} - \cancel{\frac{1}{3}t^2}$$

$$= t^2\ln(t)$$

Equals original integrand

$\Rightarrow$ integral <u>is</u> correct.

Section X. Population Model *[/10]*

A population reproduces according to a discrete-time updating function, defined by

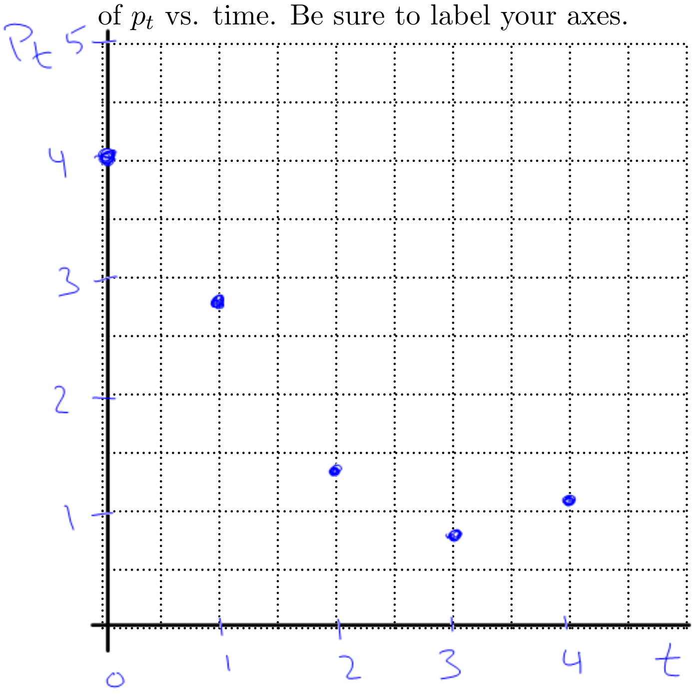
$$p_{t+1} = \frac{2}{5}(p_t)^2 - \frac{7}{5}p_t + 2$$

where p is the population in thousands, t is in generations. For this population, $0 \leq p \leq 5$.

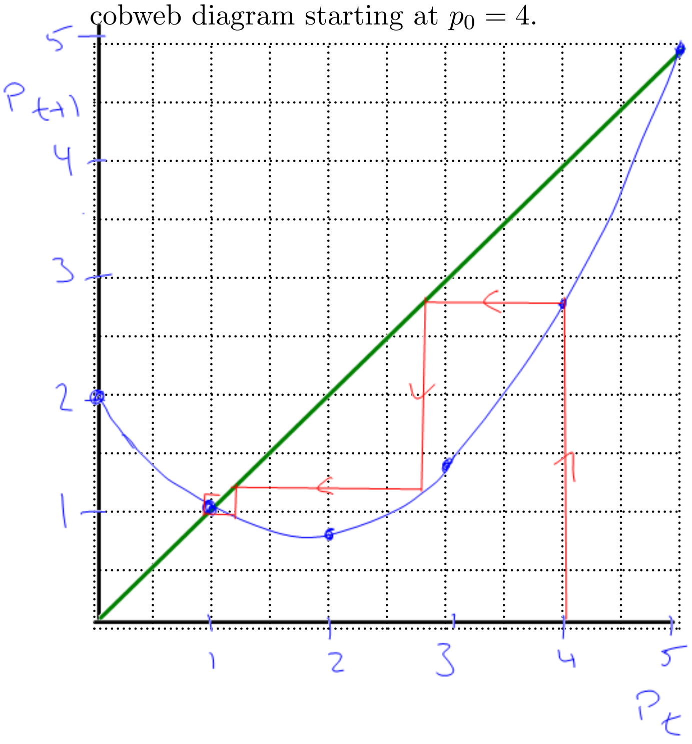
(a) Assuming that the initial population level is $p_0 = 4$, predict the population level for the next 4 generations (up to and including p_4). Keep 3 digits after the decimal for your calculations.

t	P_t
0	4
1	2.8
2	1.216
3	0.889
4	1.075

(b) Using your answer to part (a), sketch a graph of p_t vs. time. Be sure to label your axes.



(c) Sketch the graph of p_{t+1} vs. p_t, and draw a cobweb diagram starting at $p_0 = 4$.



STUDENT NUMBER:

Section X continued. $$p_{t+1} = \frac{2}{5}(p_t)^2 - \frac{7}{5}p_t + 2$$

(d) Use the updating formula to find all equilibrium value(s) of p_t.

Set $P_t = P_{t+1} = \boxed{P}$

$$P = \frac{2}{5}p^2 - \frac{7}{5}p + 2$$

$$0 = \frac{2}{5}p^2 - \frac{12}{5}p + 2 \qquad (\times 5)$$

$$0 = 2p^2 - 12p + 10$$

$$0 = (2p-2)(p-5)$$

$$\boxed{p = 1, 5}$$

or

$$p = \frac{12 \pm \sqrt{12^2 - 4(2)(10)}}{2(2)}$$

$$= 3 \pm \frac{\sqrt{64}}{4}$$

$$= 3+2,\ 3-2$$

$$\boxed{p = 5, 1}$$

(e) Classify each of the equilibrium level(s) you found in part (d) as either stable or unstable, using the slope test for equilibria.

$$f(p) = \frac{2}{5}p^2 - \frac{7}{5}p + 2$$

so $f'(p) = \frac{4}{5}p - \frac{7}{5}$

@ $p=1$, $f'(1) = \frac{4}{5}(1) - \frac{7}{5} = \frac{-3}{5}$

This is between -1 and +1 slope

$p=1$ is a stable equilibrium.

@ $p=5$, $f'(5) = \frac{4}{5}(5) - \frac{7}{5} = \frac{13}{5}$

This outside the slope range of -1 to +1, so

$p=5$ is an unstable equilibrium.

Space for additional work. **Indicate clearly which Section you are continuing if you use this space.**

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