IB HL Mathematical Exploration

Modelling the surface area of a ceramic pot

Personal code: glt101

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Introduction

Rationale

Personally, creating art has always been something I have been drawn to as it allows me to figuratively pour my feelings out onto paper. Taking visual arts as an IB subject was only natural for me; what was not natural for me was having to create art across a range of different art forms as an IB requirement, as I was only comfortable working with paint and pencil. It was only until I began to experiment ceramics at the start of IB that I became interested in other forms of art. The glazed ceramic pot shown below in **Figure 1** was made and painted early into Grade 11. Though simplistic in design, it is of great significance to me and hence, I wanted to find a way in which I could tie it together with my Maths Exploration. Hence, I decided to centre my investigation on finding the surface area of my ceramic pot. This topic has wider implications; the surface area of a ceramic body determines the amount of glaze that needs to be applied to it prior to firing, to seal and protect the fired clay piece ("The Basics of Glaze - Kiln Arts", 2017). Ceramic glazes can also be pricey, especially if they have precious metal components. Thus, this exploration has practical applications as well if I want to roughly approximate the cost of coating future ceramics I make with a certain glaze and to determine if it would be an economically viable option to use that glaze.

Figure 1: Front view of ceramic pot

Aim

During Maths HL class, we were taught how to utilise integral calculus in order to find the volume of a solid of revolution in the interval $[a, b]$. The formula given to us was:

$$V = \int_a^b \pi y^2 dx$$

Hence, I wondered if there was a similar way in which the surface area of a solid of revolution could be found through calculus. After doing some research, I found a formula that would allow me to find the surface area, A, of the ceramic pot in the interval $[a, b]$:

$$A = \int_a^b 2\pi y \, ds$$

Where $y = f(x) > 0$, $a \leq x \leq b$ and $ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$

This formula only applies to solids of revolution, which is just as well since the ceramic pot I have has a circular cross section as shown below in **Figure 2**. Thus, the aim of my exploration will be to learn the how to mathematically derive the formula for the surface area of a solid of revolution and then, apply it to approximating the surface area of the ceramic pot.

Figure 2: Aerial view of ceramic pot with circular cross section

Exploring the formula for surface area

A solid of revolution is made by rotating a continuous a continuous function $y = f(x)$ about the x-axis in the interval $[a, b]$. The solid of revolution can be divided into an infinite number of frustums, created by taking a line segment and rotating it around the x-axis, with equal width Δdx. The number of frustums is taken as infinite as this will allow the solid to be modelled as closely as possible. An example of a solid revolution being divided into frustums at equal intervals of width Δdx to reflect its shape is shown below with an image of my ceramic pot, although only to 4 frustums.

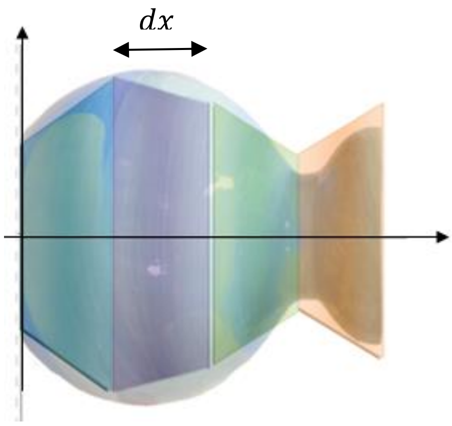

Figure 3: Ceramic pot divided into 4 frustums

As shown, using only 4 frustums does not reflect the shape of the ceramic pot accurately. Thus, an infinite number of frustums will have to be considered to calculate the surface area.

To derive the formula for the surface area of a solid of revolution, I have to start with the formula for the surface area of a frustum which is given by:

$$A = 2\pi rl$$

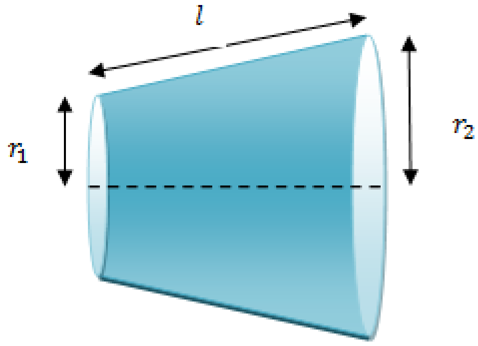

Figure 4: Diagram of a frustum

Where $r = \frac{r_1+r_2}{2}$ as the frustum has sides of different radii and l is the slant height of the frustum as shown in **Figure 4** above. l can be equated to ds which is the curve length of the function. This gives me:

$$A = 2\pi rl$$

$$= 2\pi(\frac{r_1 + r_2}{2})ds$$

$r_1 = f(x_{i-1})$and $r_2 = f(x_i)$. Since Δx is small and $f(x)$ is a continuous function, $f(x_{i-1})$ and $f(x_i)$ can both be approximated to be $f({x_i}^*)$. These values are then substituted back into the original formula for the surface area of a frustum. Thus, the surface area of a frustum in the interval $[x_{i-1}, x_i]$ is:

$$A = 2\pi\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right)ds$$

$$= 2\pi\left(\frac{f({x_i}^*) + f({x_i}^*)}{2}\right)ds$$

$$= 2\pi f({x_i}^*)ds$$

Since the definite integral of $f(x)$ in the interval $[a, b]$ is defined as:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,dx$$

The surface area of the solid of revolution with an infinite number of frustums can thus be approximated to be:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i{}^*)ds = \int_a^b 2\pi f(x)ds$$

I then need to find ds. Since dx and dy are small, the curve length,ds, can be taken as a straight line; the curved solid of revolution can then be said to be comprised of an infinite number of straight lines. ds can then be equated to l which is the slant height of the frustum. To find l and by extension ds, the Pythagorean Theorem can be used. dx and dy in relation to ds can be shown through a right-angled triangle.

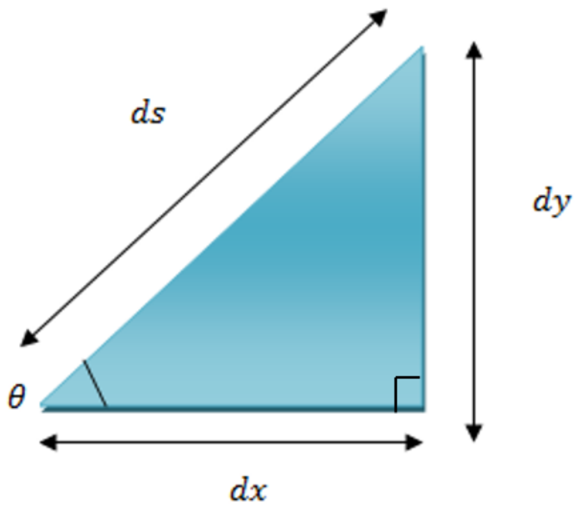

Figure 5: Diagram of a right-angled triangle

Hence, utilising Pythagorean Theorem, I can then find ds:

$$ds^2 = dx^2 + dy^2$$

I then square root both sides to find ds:

$$ds = \sqrt{dx^2 + dy^2}$$

I then factorise dx^2 from the terms inside the square root and simplify to find the equation of ds:

$$ds = \sqrt{dx^2\left(1 + \frac{dy^2}{dx^2}\right)}$$

$$= \sqrt{1 + (\frac{dy}{dx})^2 dx}$$

Thus, the formula for the surface area of a solid of revolution in the interval $[a, b]$ is shown to be:

$$A = \int_a^b 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1 + (\frac{dy}{dx})^2} dx$$

Graphing the ceramic pot

To calculate the surface area of the ceramic pot, the formula indicates that the equation of the curve must be found. Prior to finding the functions of the graph of the ceramic pot, I had to first record the coordinates of several points on the ceramic pot in order to plot out its graph. This was initially done by wrapping a tape measure made of ribbon around the several points on the ceramic pot to find their circumferences. However, I had difficulty doing so as the tape measure kept slipping as the ceramic pot was too smooth. Hence, I decided instead to only measure the circumferences of the bottom and top tips of the ceramic pot as this was easier. After finding the circumferences, the formula for the area of a circle was used to find the radius of each circular cross section:

$$Circumference\ of\ a\ circle = 2\pi r$$

The radius of the circle, which then corresponds to its positive$x$-coordinate on the graph, is then given by:

$$r = \frac{Circumference\ of\ a\ circle}{2\pi}$$

For example, the circumference of the bottom cross section of the ceramic pot gives a radius of:

$$r = \frac{13}{2\pi} \approx 2.1\ cm\ (1\ d.p.)$$

The radii of the bottom and top cross sections of the ceramic pot are recorded below in **Table 1**.

	Circumference (cm)	Radius (cm)
Bottom	13	2.1
Top	15	2.4

Table 1: Circumference and radius of bottom and top tips

The values of the radii, 13 cm and 15 cm, give the upper and lower bounds of the domain of the graph respectively. I used Adobe Photoshop to superimpose the photograph of the ceramic pot onto the background layer of a math grid template found online. The opacity of the layer with the ceramic pot photograph was reduced to 70% so that the grid lines would show through clearly and the coordinates of the curves could be recorded accurately. Finding the radii values of the bottom and top cross sections of the ceramic pot allowed me to scale the image of the pot accurately onto the math grid. The rest of the points were then recorded using the graph rather than by hand using the tape measure which reduced the margin for human error.

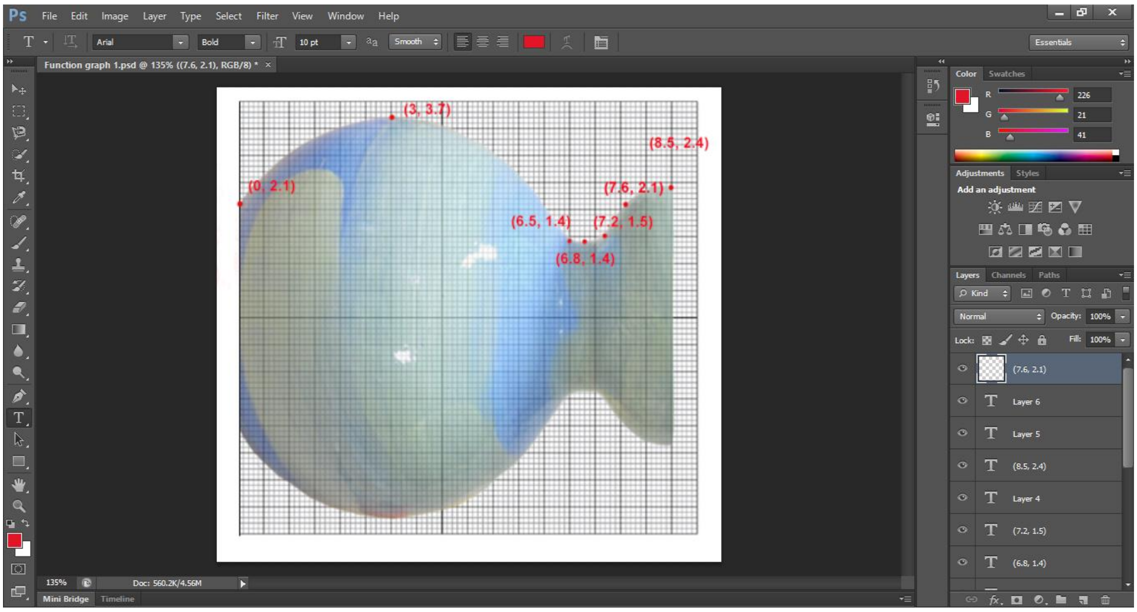

Figure 6: Using Adobe Photoshop to plot the graph of the ceramic pot

In total, 7 points were chosen to derive a piecewise function that will allow me to calculate the total surface area of the ceramic pot. I chose points located at the bottom/top tips of the ceramic pot or where shape of the ceramic pot curves noticeably to form points of inflexion. The x-coordinates and y-coordinates of the points were then plotted onto the graph of the ceramic pot as shown below in **Figure 7**. The x-coordinates and y-coordinates of the points chosen are shown in below in **Table 2**.

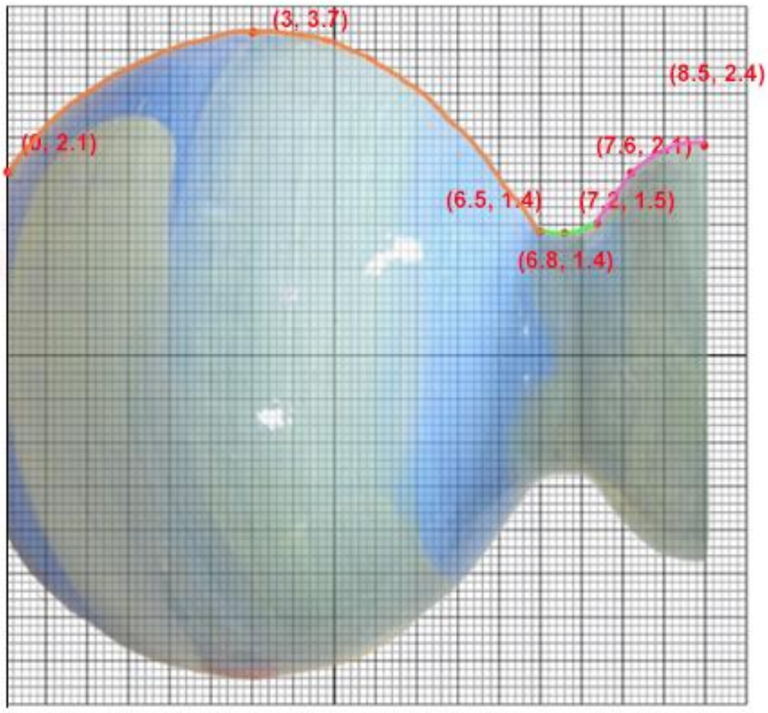

Figure 7: Graph of the ceramic pot

Coordinate	x	y
1	0	2.1
2	3	3.7
3	6.5	1.4
4	6.8	1.4
5	7.2	1.5
6	7.6	2.1
7	8.5	2.4

Table 2: Coordinates of the graph

There will be three different function equations to be found, divided into the bottom, middle and top section of ceramic pot. The bottom section is represented by the orange coloured curve line in **Figure 7**, the middle section represented by the green coloured curve line and the top section by the pink coloured curve line. Each section is comprised of three coordinates that will be used to find their respective curve equations.

Finding the functions

The three function equations were found analytically using the Lagrange interpolation formula. I assumed that the functions are all polynomial functions as the Lagrange interpolation formula only applies to polynomial functions. I felt that this method was appropriate as I recorded down values of x at unequal intervals which the formula takes into account. The formula is as follows that for a unique polynomial $P(x)$:

$$P(x) = \sum_{i=0}^{n} y_i \frac{(x - x_0) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

Where $0 \leq i \leq n$ and $P(x_i) = y_i$

Bottom section:

The polynomial function of the bottom section of the ceramic pot is first found. From the shape of the curve, it appears to be a quadratic function; the graph is concave downwards with one maximum point at $(3.0, 3.7)$. I could have chosen a function of a higher-degree polynomial, such as a cubic polynomial, since the characteristics of the curve also apply to part of a cubic graph. But although the accuracy of the interpolated data points would have been greater with a cubic polynomial, this does not necessarily mean that the graph will be a smooth fit for the curve of the ceramic pot. Thus, I chose simplicity over accuracy and decided to use a quadratic function instead. The Lagrange interpolation formula finds the equation of a unique polynomial of order n passing through $(n + 1)$ data points. Hence, three data points were chosen since I am finding a polynomial of order 2. Their x and y coordinates are shown below.

Coordinates:

i	x	y
0	0	2.1
1	3.0	3.7
2	6.5	1.4

Table 3: Coordinates of bottom section

When $i = 2$ and for a sequence of x coordinates of values $\{0, 3, 6.5\}$, the function $P_1(x)$ is given by:

$$P_1(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$= \frac{(x - 3)(x - 6.5)}{(0 - 3)(0 - 6.5)}(2.1) + \frac{(x - 0)(x - 6.5)}{(3 - 0)(3 - 6.5)}(3.7) + \frac{(x - 0)(x - 3)}{(6.5 - 0)(6.5 - 3)}(1.4)$$

$$\approx -0.183150x^2 + 1.082784x + 2.100000 \ (6\ d.p.)$$

Thus, the function of the bottom section of the ceramic pot is given by the equation $-0.183150x^2 + 1.082784x + 2.100000$.

Note: Expanding the algebraic expressions individually utilising a calculator would have been tedious. Thus, the expressions were expanded to give the function using C programming. I wrote a simple code that allowed me to find the coefficients of the quadratic polynomial. According to the Lagrange interpolation formula, for a polynomial of order 2 in the form $ax^2 + bx + c$:

$$a = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$$b = \frac{-x_2 y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{-x_1 y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{-x_2 y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{-x_0 y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{-x_1 y_2}{(x_2 - x_0)(x_2 - x_1)} + \frac{-x_0 y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$$c = \frac{x_1 x_2 y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{x_0 x_2 y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{x_0 x_1 y_2}{(x_2 - x_0)(x_2 - x_1)}$$

I then merely needed to give the values of each x and y coordinate as my input data. The C programme for the code was then compiled and run at https://www.codechef.com/ide. The output for the code gave me the values of a, b and c. This is illustrated below in **Figure 8**. The output values are precise up to 6 decimal places. I was more satisfied with the degree of precision for the programme compared to several online algebraic expression expansion calculators online that were only precise up to 5 decimal places.

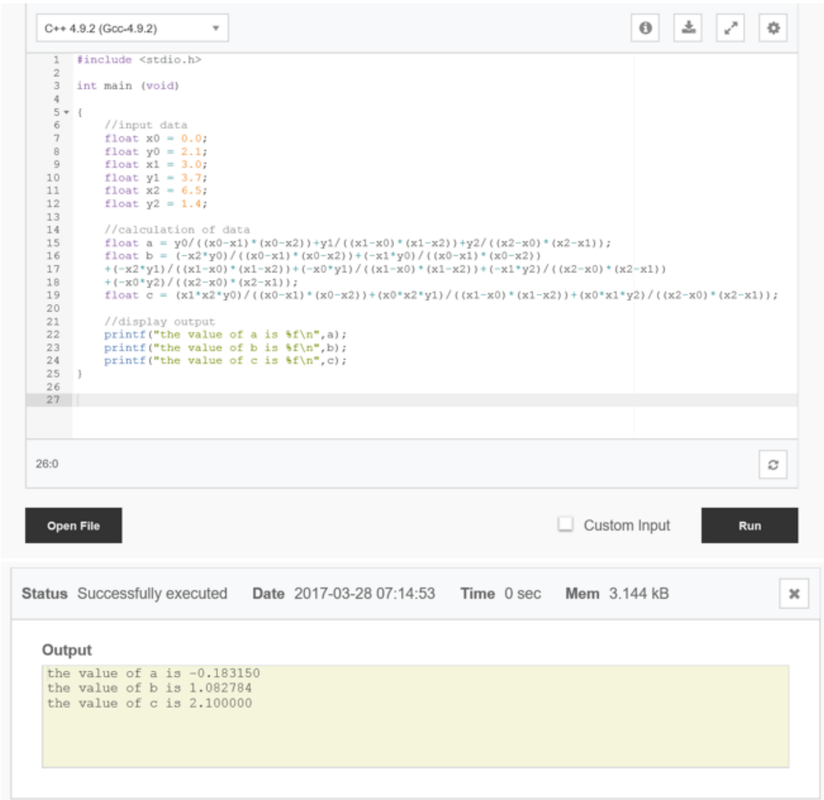

Figure 8: Illustration of the code and output it yields

Middle section:

The same method I illustrated to find the function of the bottom section of the ceramic pot was applied to find the function of the middle section. Similarly, the graph appears to be a quadratic function from the shape of the curve as it is concave upwards with one minimum point at $(6.8, 1.4)$. Hence, three points were chosen. Their x and y coordinates are shown below.

Coordinates:

i	x	y
0	6.5	1.4
1	6.8	1.4
2	7.2	1.5

Table 4: Coordinates of middle section

I then utilised the Lagrange interpolation formula and expanded the algebraic expressions with the C programme. Thus, the function of the middle section of the ceramic pot, $P_2(x)$, is given by the equation $0.357143x^2 - 4.750008x + 17.185715$, precise up to 6 decimal places.

Top section:

The same method I illustrated to find the function of the bottom section of the ceramic pot was applied to find the function of the middle section. Similarly, the graph appears to be a quadratic function from the shape of the curve as it is concave downwards with one maximum point at $(8.5, 2.4)$. Hence, three points were chosen. Their x and y coordinates are shown below.

Coordinates:

i	x	y
0	7.2	1.5
1	7.6	2.1
2	8.5	2.4

Table 5: Coordinates of top section

I then utilised the Lagrange interpolation formula and expanded the algebraic expressions with the C programme. Thus, the function of the middle section of the ceramic pot, $P_3(x)$, is given by the equation $-0.8974385x^2 + 14.782037x - 58.407654$, precise up to 6 decimal places.

The graph of the ceramic pot can then be represented by the piecewise function:

$$f(x) = \begin{cases} -0.183150x^2 + 1.082784x + 2.100000 & 0 \le x \le 6.5 \\ 0.357143x^2 - 4.750008x + 17.185715 & 6.5 \le x \le 7.2 \\ -0.8974385x^2 + 14.782037x - 58.407654 & 7.2 \le x \le 8.5 \end{cases}$$

I then graphed the three functions using an online graphing calculator from:
https://www.desmos.com/calculator

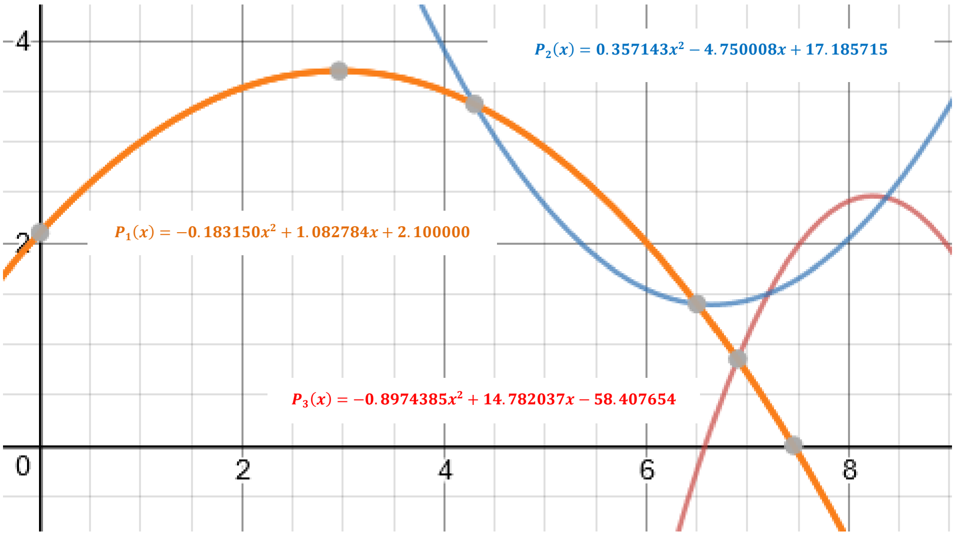

Figure 9: Graphs of the bottom, middle and top sections of the ceramic

The three function graphs were then superimposed onto the image of the ceramic pot using Adobe Photoshop. This is shown below in **Figure 10**. The green curve line drawn represents the piecewise function. As shown, the green curve models the shape of the ceramic pot very closely. This proves that my decision to use a lower degree polynomial of order 2 instead of higher degree polynomials for the three functions was safe.

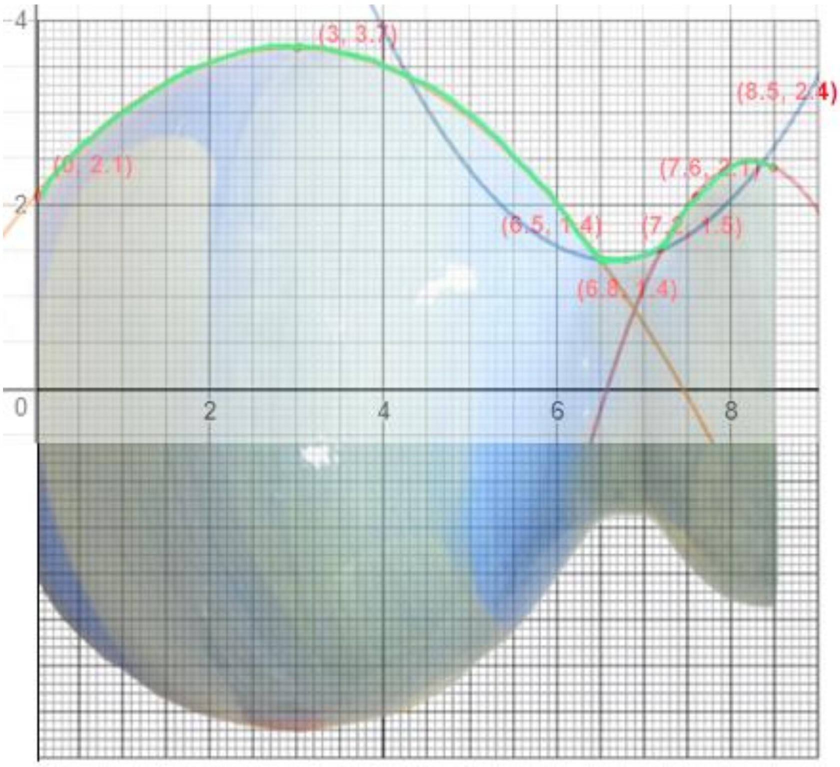

Figure 10: Superimposed image of the graph functions onto the ceramic pot

Calculating the surface area of the ceramic pot

The surface area of the ceramic pot was then calculated with the formula for the surface area of a solid of revolution I derived:

$$A = \int 2\pi y ds = \int 2\pi y \sqrt{1 + (\frac{dy}{dx})^2} dx \text{ since } ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$$

Bottom section:

To find ds, the derivative of the equation of the function $P_1(x) = -0.183150x^2 + 1.082784x + 2.100000$ needs to be found. This gives me:

$$\frac{dy}{dx} = -0.3663x + 1.082784$$

As $A = \int 2\pi y ds = \int 2\pi y \sqrt{1 + (\frac{dy}{dx})^2} dx$ since $ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$, $\frac{dy}{dx} = -0.3663x + 1.082784$ can be substituted into the formula for surface area. Thus, the surface area of the bottom section of the ceramic pot is given by:

$$A_B = \int_0^{6.5} 2\pi(- 0.183150x^2 + 1.082784x + 2.100000)\sqrt{1 + (-0.3663x + 1.082784)^2} dx$$

I then used a graphic display calculator (GDC) to evaluate A_B. This gives me:

$$A_B = 145.209\ cm^2$$

$$\approx 145\ cm^2 (3\ s.f.)$$

Middle section:

I used the same method illustrated above to find the surface area of the middle section of the ceramic pot. Thus, the surface area of the middle section of the ceramic pot represented by the function $P_2(x) = 0.357143x^2 - 4.750008x + 17.185715$ is given by:

$$A_M = \int_{6.5}^{7.2} 2\pi(0.357143x^2 - 4.750008x + 17.185715)\sqrt{1 + (0.714286x - 4.750008\ )^2} dx$$

I then used a GDC to evaluate $A_{M.}$This gives me:

$$A_M = 6.37808\ cm^2$$

$$\approx 6.38\ cm^2 (3\ s.f.)$$

Top section:

I used the same method illustrated above to find the surface area of the top section of the ceramic pot. Thus, the surface area of the top section of the ceramic pot represented by the function $P_3(x) = -0.8974385x^2 + 14.782037x - 58.407654$ is given by:

$$A_T = \int_{7.2}^{8.5} 2\pi(-0.8974385x^2 + 14.782037x - 58.407654)\sqrt{1 + (-1.794877x + 14.782037)^2}dx$$

I then used a GDC to evaluate A_T .This gives me:

$$A_T = 23.4394\ cm^2$$

$$\approx 23.4\ cm^2 (3\ s.f.)$$

Total surface area:

The value for the total surface area of the ceramic pot is given by adding up the values of the surface area of the top, middle and bottom sections together. This gives me:

$$A_{total} = A_B + A_M + A_T$$

$$= 145 + 6.38 + 23.4$$

$$\approx 175 cm^2 (3\ s.f.)$$

Thus, the total surface area of the ceramic pot can be approximated as 175 cm^2.

Limitations and Improvements

However, there were some limitations present as I carried out my exploration, the most significant being that the value for the surface area of the ceramic pot I calculated is a mere estimate, thus limiting its accuracy to three significant figures. Also, the C programme I utilised to expand and simplify the function equations of the ceramic pot was only precise to 6 decimal places despite the equations going on for many more decimals. Admittedly, a calculator or mathematical tool that rounded off to more than five decimal places could have been found to improve the accuracy of my exploration. But with the limited tools I had at my disposal and after checking through several online calculators, the coding programme I used seemed the most precise. Furthermore, this exploration could have been subject to human error. I had to first measure the circumference of the bottom and top edges of the ceramic pot with a tape measure and hence, could have read off the wrong values. Also, the measurements for the circumference of the bottom and top edges had to be rounded off to 0.1 cm as this was the smallest division for the measuring tape I used. I could have used an instrument like a vernier caliper to measure the radius instead as it is precise up to 0.01cm.

Conclusion

In conclusion, I was gratified to be able to apply the calculus and the formula for the surface area of a solid of revolution to real life. Moreover, it was satisfying to be able to uncover the mathematical reasoning behind the formula as well as understand it intuitively. Though my exploration was based off too many assumptions to allow for the surface area calculated to be a truly accurate value, I felt that it was enough for me to be able to tie together mathematics and my personal interest in art. Personally, I am also interested in taking this further and perhaps investigating the formula for finding the volume of a solid of revolution or using the surface area calculated to calculate the volume of the ceramic pot. This gives rise to other sorts of possibilities such as the volume of clay needed to mould such a ceramic pot which would be applicable if I wanted to extend my art project and make a series of such ceramic pots.

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