Identification of the parameters of the equation of Richards by the genetic algorithms

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Abstract

In this article it is question of identification of the parameters in the equation of Richards modelling the flow in unsaturated porous medium. The equation considered is strongly nonlinear and is in dimension one of space. The method of inversion developed is based on the genetic algorithms. The direct problem was solved by the finite difference method. The function cost used is built by using the infiltration. A test on real data has been carried. The software used is Matlab.

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1 Introduction

The fluid movement in unsaturated porous medium is characterized by parameters which take into account the type of soil, the initial condition and the boundary conditions of the considered soil. However these parameters are, in most cases, badly known, hence the importance of the inverse modeling.
There exist many methods to solve the inverse problems [1,2]. But most these methods require a good knowledge of the solution. Indeed, these algorithms can not detect a global optimum and can stop with a local optimum. Moreover, these algorithms require a certain regularity of the functions to be optimized. However, this regularity is not always checked.
The genetic algorithms developed by John Holland in 1975 [3] and popularized by

Goldberg in 1989 [4], are adapted better to the problems of optimization in which the size of the space of research is important, where the parameters interact in a complex way and where very little information on the function to be optimize disavailable. They do not require a particular assumption on the regularity of the function objective. The genetic algorithms do not use in particular the successive derivative of the functions to be optimized; no assumption on continuity is not necessary. The function to be optimized can thus be the result of a simulation. These algorithms are often much more robust in their capacity to identify the total optimum with less sensitivity to the initial condition.

In this work, we adapt the genetic algorithm Non-dominated Sorting Genetic Algorithm-II (NSGA-II) proposed by Deb and al(2002)[5] for the optimization of a multi-objective problem to the resolution of a mono-objective problem without constraints.

Our paper is organized as follows: in the first part, we present the algorithm used; the second part is devoted to the equation of Richards in one dimension, his resolution by finite differences method and the calculation of the cumulated infiltration; A third part is devoted to the identification of the parameters of the equation of Richards by our algorithm.

2 Presentation of the algorithmused

This algorithm makes it possible to maximize a positive function f called *fitness* or evaluation function of the individual. The individuals represent the variables.

2.1 Coding and creation of the initial population

The real-type coding used consists in directly representing the actual values of the variable. We subdivided the eligible domain in several sub-domains. And the initial population was created in a random way by using the uniform law in each under field. That make sit possible to have a diversified population from the beginning and convergency-accelerating. The size of the population is n. We create a table of n variables.

2.2 Operation of selection

We used the selection by caster of Goldberg [6]. The parents are selected according to their performance. In this method the probability p with which an individual i represented by a variable x_i of fitness f_i (evaluation of the function in x_i) reintroduced in a new population of size n is:

$$p = \frac{f_i}{\sum_{j=1}^{n} f_j}$$

2.3 Operation of crossing

The barycentric crossing is used but we did not use a probability of crossing. In this kind of crossing, two genes $P1(i)$ and $P2(i)$ are selected from each parent to the same

position *i*. They define two new genes *C*1(*i*) and *C*2(*i*) by linear combination:
*C*1(*i*)=*aP*1(*i*)+(1−*a*)*P*2(*i*) ; *C*2(*i*)=(1−*a*)*P*1(*i*)+*aP*2(*i*) ; *a*∈]0,1[.
In this document, we crossed the whole mother population to get a child population of size *n*.

2.4 Operation of mutation

Mutation of a Gaussian type is applied to the population. One selects an individual *x* under a probability *p*. If *p* is lower than the probability of mutation p_m, one adds a Gaussian noise to *x* i.e. one replaces *x* by *x*+ε, where ε are a random value obtained according to the law from Gauss of standard deviation σ. The newly-createdindividual replaces the former one if it is better and if it is in the acceptable field.

2.5 New population

After the operations of selection, crossing and change, an intermediate population of size 2*n* is created by gathering the parent and child populations. The new parent population is obtained by keeping the *N* better individuals.
Finally the algorithm used is:

Algorithm 1 Algorithmused

For each iteration t do

To calculate the score of each individual of P_t

To generate a new population of child Q_t by applying the operators of selection, crossing and mutation

$R_t = P_t \cup Q_t$ (add Q_t to P_t)

To classify the individuals of R_t from decreasing order according to the score of each individual

To keep *n* best individuals of R_t to form a new population of parent P_{t+1}

$t = t + 1$ (To increment the counter of the generation)

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3 Direct problem
3.1 Mathematical model

There exists everal formulations of the equation of Richards who modelling the flow in unsaturated porous medium but in this work, we use the mixed formulation pressure head-moisture content because the numerical solutions obtained with this mixed formulation are more precise [7,8,9].
In one dimension, the mixed formulation is given by :

$$\begin{cases} \dfrac{\partial \theta}{\partial t} - \dfrac{\partial}{\partial z}\left[K(h)\left(\dfrac{\partial h}{\partial z}-1\right)\right]=f \quad \text{in} \qquad [0,Z]\times[0,T] \\[2mm] h(z,0)=h_0(z) \ \text{in} \qquad [0,Z] \\[2mm] h(0,t)=h_{\text{sup}}(t) \qquad \text{in} \qquad [0,T] \quad \text{(bottom limit)} \\[2mm] h(Z,t)=h_{\text{inf}}(t) \qquad \text{in} \qquad [0,T] \quad \text{(top limit)} \end{cases} \qquad (1)$$

with :
- z denotes the vertical dimension;
- $h[L]$ the pressure head;
- $\theta[L^3/L^3]$ the moisture content which given by

$$\theta(h)=\dfrac{\theta_s-\theta_r}{1+\ \alpha\,|h|^{\ n}{}^{\ m}}+\theta_r \qquad (2)$$

Where θ_s the moisture content to saturation (L^3/L^3), θ_r the residual moisture content (L^3/L^3), α a parameter of form related to the mean size of the pores (L^{-1}), n a parameter related to the distribution of the sizes of pores $[-]$. According to Mualem, we have $m=1-1/n$ [10].
- $K(h)$ is the insaturated hydraulic conductivity $[L/T]$. We use the relation of Van Genuchten [11] given by

$$K(S_e)=K_s S_e^{1/2}\ \left[1-\ \left(1-S_e^{1/m}\right)^{\ m}\right]^2 \qquad (3)$$

With K_S the saturated hydraulic conductivity $[L/T]$

S_e the effective saturation given by:

$$S_e=\begin{cases} \dfrac{\theta-\theta_r}{\theta_s-\theta_r} & \text{if} \qquad h<0 \\[3mm] 1 & \text{if} \qquad h\geq 0 \end{cases} \qquad (4)$$

h and θ are related by the moisture capacity function $C(h)$ $[1/L]$ defined by

$$C(h)=\dfrac{\partial \theta}{\partial h} \qquad (5)$$

Whats gives

$$C(h)=-\alpha n(\theta_r-\theta_s)sign(h)(\dfrac{1}{n}-1)(\alpha\,|h|)^{n-1}(1+(\alpha\,|h|)^n)^{1/n-2} \qquad (6)$$

Remark 1: *We suppose thatthereis no contribution i.e. f=0.*
To solve the problem (1), the five parameters should be known: α, θ_s, θ_r, n et K_S.

3.2 Numerical resolution by the finite difference method
In this part, we propose to solve numerically the equation (1). We will use the finite differences method for the space discretization. A implicit scheme of Euler will be used for the temporal discretization.

The equation being strongly non-linear, we will use the method of Picard or method of the fixed point.

3.2.1 Discretization of the equation (1)

To discretize the domain $[0,Z]\times[0,T]$, we introduce a step of space $\Delta z = \dfrac{Z}{N_z+1}$ (N_z ainteger strictly positive) and a step of time $\Delta t = \dfrac{T}{M}$ (*M* a integer strictly positive), and we define the nodes of a regular meshing:

$(z_i, t_j) = (i\Delta z, j\Delta t)$ for $(i, j) \in \{0,1,\ldots,N_z+1\}\times\{0,1,\ldots,M\}$

We denote $h_i^j \approx h(x_i, t_j)$, $\theta_i^j \approx \theta(x_i, t_j)$. The discretization of (1) by a implicit scheme gives

$$\frac{\theta_i^{j+1}-\theta_i^j}{\Delta t} = \frac{1}{\Delta z}\left[K_{i+\frac{1}{2}}^{j+1}\left(\frac{h_{i+1}^{j+1}-h_i^{j+1}}{\Delta z}-1\right) - K_{i-\frac{1}{2}}^{j+1}\left(\frac{h_i^{j+1}-h_{i-1}^{j+1}}{\Delta z}-1\right) \right] \tag{7}$$

$$i = 1,\ldots,N_z, j = 0,\ldots,M-1$$

$$K_{i-\frac{1}{2}} = \sqrt{K(h_i^{j+1})\times K(h_{i-1}^{j+1})} \text{ et } K_{i+\frac{1}{2}} = \sqrt{K(h_i^{j+1})\times K(h_{i+1}^{j+1})}$$

By applying the method of Picard the equation (7) iswritten:

$$\frac{\theta_i^{j+1,k+1}-\theta_i^j}{\Delta t} = \frac{1}{\Delta z}K_{i+\frac{1}{2}}^{j+1,k}\left(\frac{h_{i+1}^{j+1,k+1}-h_i^{j+1,k+1}}{\Delta z}-1\right)$$

$$-\frac{1}{\Delta z}K_{i-\frac{1}{2}}^{j+1,k}\left(\frac{h_i^{j+1,k+1}-h_{i-1}^{j+1,k+1}}{\Delta z}-1\right) \tag{8}$$

$$i = 1,\ldots,N_z, j = 0,\ldots,M-1$$

k is the index of iteration of the method of Picard.

We pose $h^j = (h_1^j,\ldots,h_{N_z}^j)$ *and* $\theta^j = (\theta_1^j,\ldots,\theta_{N_z}^j)$, the development in Taylor series of θ respect to *h*at point $h^{j+1,k}$ gives [7]

$$\theta^{j+1,k+1} = \theta^{j+1,k} + \frac{d\theta}{dh}\Big|^{j+1,k}(h^{j+1,k+1}-h^{j+1,k})+0(\delta^2)$$

$h^{j+1,k+1}$ *and* $\theta^{j+1,k+1}$ represent respectively the vector of pressure and the vector of the moisture content to the step of time *j*+1 and the iteration *k*+1.

By truncating and by using the relation (5), we have:

$$\theta^{j+1,k+1} \approx \theta^{j+1,k} + C^{j+1,k}(h^{j+1,k+1}-h^{j+1,k})$$

The equation (7) becomes:

$$C_i^{j+1,k}\left(\frac{h_i^{j+1,k+1} - h_i^{j+1,k}}{\Delta t}\right) - \frac{1}{\Delta z^2} K_{i+\frac{1}{2}}^{j+1,k}.h_{i+1}^{j+1,k+1}$$

$$-\frac{1}{\Delta z^2}\left[\left(K_{i+\frac{1}{2}}^{j+1,k} + K_{i-\frac{1}{2}}^{j+1,k}\right)h_i^{j+1,k+1} + K_{i-\frac{1}{2}}^{j+1,k}.h_{i-1}^{j+1,k+1}\right] \tag{9}$$

$$= -\frac{(\theta_i^{j+1,k} - \theta_i^{j})}{\Delta t} - \frac{1}{\Delta z}\left(K_{i+\frac{1}{2}}^{j+1,k} + K_{i-\frac{1}{2}}^{j+1,k}\right)$$

Posing $\delta_i^k = h_i^{j+1,k+1} - h_i^{j+1,k}$, the system (9) becomes:

$$C_i^{j+1,k}\frac{\delta_i^k}{\Delta t} - \frac{1}{(\Delta z)^2}\left[K_{i+\frac{1}{2}}^{j+1,k}\ \delta_{i+1}^k - \delta_i^k\ - K_{i-\frac{1}{2}}^{j+1,k}\ \delta_i^k - \delta_{i-1}^k\right]$$

$$= \frac{1}{(\Delta z)^2}\left[K_{i+\frac{1}{2}}^{j+1,k}\ h_{i+1}^k - h_i^k\ - K_{i-\frac{1}{2}}^{j+1,k}\ h_i^k - h_{i-1}^k\right] \tag{10}$$

$$-\frac{(\theta_i^{j+1,k} - \theta_i^{j})}{\Delta t} + \frac{1}{\Delta z}\left(K_{i+\frac{1}{2}}^{j+1,k} + K_{i-\frac{1}{2}}^{j+1,k}\right)$$

By regrouping we have:

$$\frac{K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}\delta_{i-1}^k + \left[\frac{C_i^{j+1,k}}{\Delta t} - \left(\frac{K_{i+\frac{1}{2}}^{j+1,k} + K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}\right)\right]\delta_i^k - \frac{K_{i+\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}\delta_{i+1}^k$$

$$= \frac{1}{(\Delta z)^2}\left[K_{i+\frac{1}{2}}^{j+1,k}\ h_{i+1}^k - h_i^k\ - K_{i-\frac{1}{2}}^{j+1,k}\ h_i^k - h_{i-1}^k\right] \tag{11}$$

$$-\frac{(\theta_i^{j+1,k} - \theta_i^{j})}{\Delta t} + \frac{1}{\Delta z}\left(K_{i+\frac{1}{2}}^{j+1,k} + K_{i-\frac{1}{2}}^{j+1,k}\right)$$

What is written in matrix form, with each iteration k of Picard
$$A^k \delta^k = R^k \tag{12}$$
Where A^k is a tridiagonal matrix of size $N_z \times N_z$, and R^k is a vector of size N_z, concretely

$$A_{i,i}^k = \frac{C_i^{j+1,k}}{\Delta t} - \left(\frac{K_{i+\frac{1}{2}}^{j+1,k} + K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}\right), i = 1,\ldots,N_z$$

$$A_{i-1,i}^k = \frac{K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}, i = 2,\ldots,N_z$$

$$A_{i,i+1}^k = \frac{K_{i-\frac{1}{2}}^{j+1,k}}{(\Delta z)^2}, i = 1,\ldots,N_z - 1$$

and

$$R_i^k = \frac{1}{(\Delta z)^2}\left[K_{i+\frac{1}{2}}^{j+1,k}\left(h_{i+1}^k - h_i^k \right) - K_{i-\frac{1}{2}}^{j+1,k}\left(h_i^k - h_{i-1}^k \right) \right]$$

$$-\frac{(\theta_i^{j+1,k} - \theta_i^j)}{\Delta t} + \frac{1}{\Delta z}\left(K_{i+\frac{1}{2}}^{j+1,k} + K_{i-\frac{1}{2}}^{j+1,k} \right), i = 1,\ldots,N_z$$

In summary the resolution by the iterative method of Picard is given by the algorithm 2.

Algorithm 2 Algorithm of Picard

At the iteration j+1 of time, to do

1-Initialize $h^{j+1,0} = h^j$ (pressure the iteration j)

2-k=0

3-To do

i-To build the system (12)

ii-To solve the system(12)

iii-To built the new solution: $h^{j+1,k+1} = \delta^{j+1,k} + h^{j+1,k}$

iv-k=k+1

v-While $|\delta^{j+1,k}| < \varepsilon$

vi-if not convergence and that k≤I$_m$, to change the step of time and to return to 1

ε is the tolerance.

We choose the absolute error like criterion. I_m is the maximal number of the iteration of Picard.

3.2.2 Adaptation of the step of time

The calculation of the step of time is done according to the algorithm 3

Algorithm 3 calculation of step of time

One sets a maximum number of iterations of Picard I_m

For each step of time:

- If the method converged, the step of time is accepted.
- If the method converged in less than $0.3 \times I_m$ iterations, the step of time is multiplied by 1.3.
- If the method converged with more $0.7 \times I_m$ iterations, the step of time is multiplied by 0.7.
- If the method did not converge, the step of time is rejected. A new test is carried out with a step of time 3 times smaller.

3.3 Calculation of the infiltration

One of the objectives of the modeling of the flow in unsaturated porous medium is the estimate of the quantity of water which infiltrates to reach the saturated zone. The infiltration describes the process of water penetrating in the ground starting from its surface. In a general way, for a variable initial condition $\theta(0,z)$, the cumulative infiltration I_{cum} is defined by:

$$I_{cum}(t) = \int_0^Z q(t,z)dz$$

$q(z,t)$ is the rate of infiltration and Z is the depth of the ground considered. If the initial condition is constant $\theta_{ini}=cste$, one will have:

$$I_{cum}(t) = \int_0^Z (\theta(t,z) - \theta_{ini})dz \tag{13}$$

$\theta(t,z)$ is the moisture content. In discrete form $I_{cum}(t_j)$ is obtained by making an approximation of (13) by the formula of the trapezoids:

$$I_{cum}(t_j) = \Delta z \left[\frac{1}{2} \; \theta_{sup} - 2\theta_{ini} + \theta_{inf} \; + \sum_{i=1}^{N_z} \theta_i^j - \theta_{ini} \right] \tag{14}$$

θ_{inf} is the moisture content at the bottom and θ_{sup} is the moisture content at the top.

4 Inverse problem
4.1 Method of inversion

We will use the function of infiltration because its values are easy to measure on the ground.

Let thus M observations of values of infiltration $I_{obs}(t_j)$ at the moments t_j; $j=1,\cdots,M$.

Let thus J the functional defined by

$$\begin{aligned} J(U) &= \frac{\Delta t}{2} \sum_1^M \left(I_{cum}(t_j) - I_o bs(t_j) \right)^2 \\ &= \frac{\Delta t}{2} \sum_1^M \left(\Delta z \left[\frac{1}{2} \; \theta_{sup} - 2\theta_{ini} + \theta_{inf} \; + \sum_{i=1}^{N_z} \theta_i^j - \theta_{ini} \; \right] - I_{obs}(t_j) \right)^2 \end{aligned} \tag{15}$$

U is the vector of parameters $(\alpha, n, \theta_r, \theta_s, K_s)$.

The problem of identification consists in finding the vector of parameters $U^* = (\alpha^*, n^*, \theta_r^*, \theta_s^*, K_s^*)$ so that the infiltration observed on the ground is as close as possible to the computed value. What amounts minimizing the functional J defined in (15).

The functional J being positive, to minimize J is equivalent to maximize $F(U) = \dfrac{1}{J(U)}$.

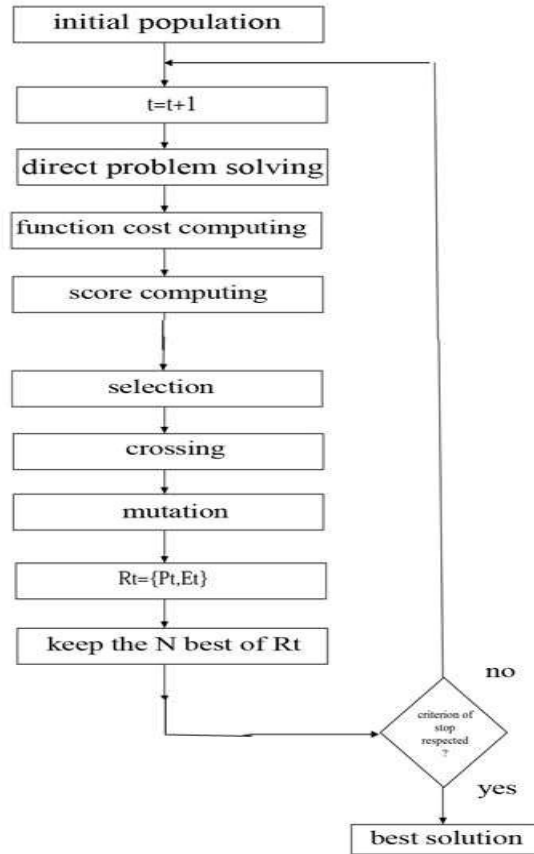The diagram of figure 1summarizes the steps of inversion.

Figure 1: Steps of inversion

4.2 Identification of the parameters of the equation of Richards

The data used were measured on a clay soil on a column of $1m$ long. The values of the infiltration were recorded all the 5 *mn* during 2 hours.

We applied our algorithm with the following parameters:

Table 1: Configuration of algorithm

P_m	size	generations	σ
0.0001	200	30	0.5

The identified values are given in table 2.

Table 2: Results of the identification

Parameters	Interval	Identified values
θ_s	[0;5]	0.0238
θ_r	[0;5]	0.379
$\alpha[cm^{-1}]$	[0;1]	0.0879
n	[0;10]	1.1359
$K_s(cm\times s^{-1})$	$[0;15]\times 10^{-5}$	1.75×10^{-5}

Value of the function cost J:0.0027

According to the figure 2, the infiltrations observed coincide with the simulated infiltrations.

The values of the infiltrations observed at the moments t=30mn, t=1h and t=1h30mn were not used in the process of identifications. They were used like values test:

Table 3: Comparisonat the points test

Times	Calculated	Observed
30mn	1.5732	1.5943
1h	2.2901	2.3220
1h30mn	2.7939	2.8143

That proves that the identified values as well as possible adjust the cumulated infiltrations observed.

The figures 3 and 4 are respectively the variation respect to depth z of the pressure head and of the moisture content.
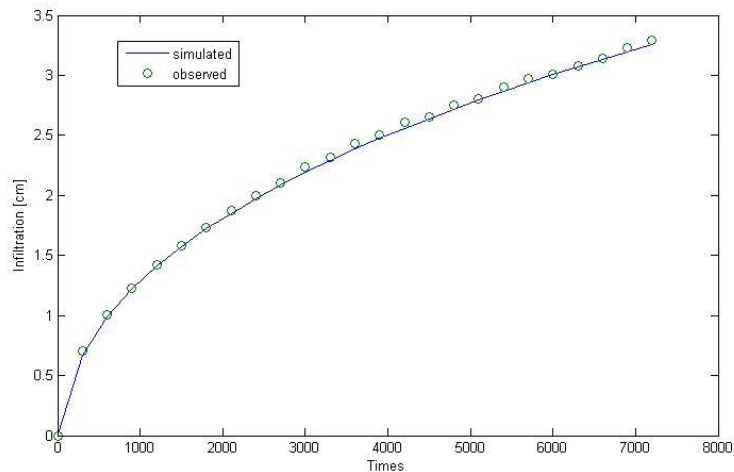
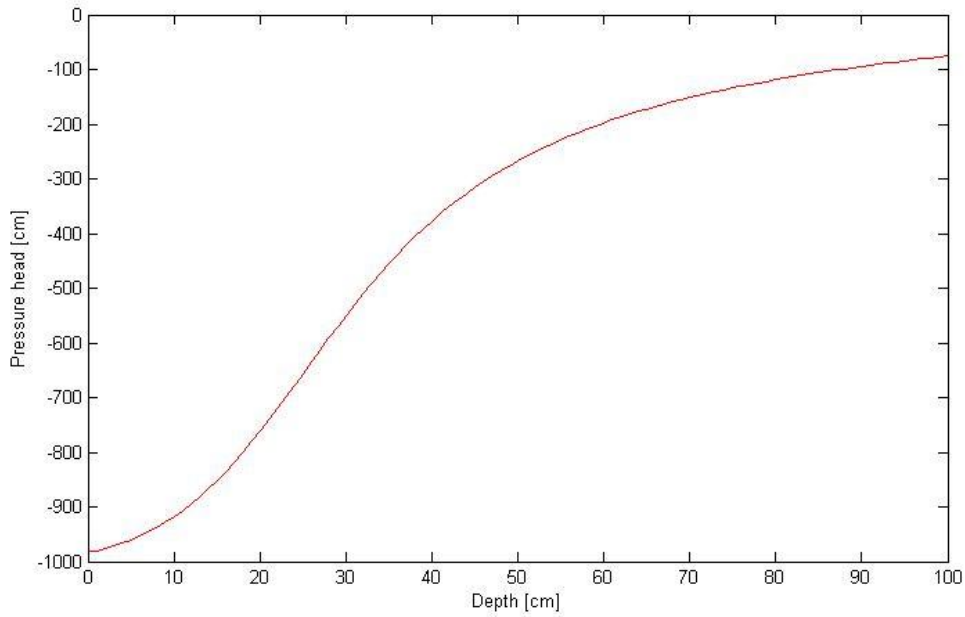**Figure 2: Curves of infiltration observed and simulated**
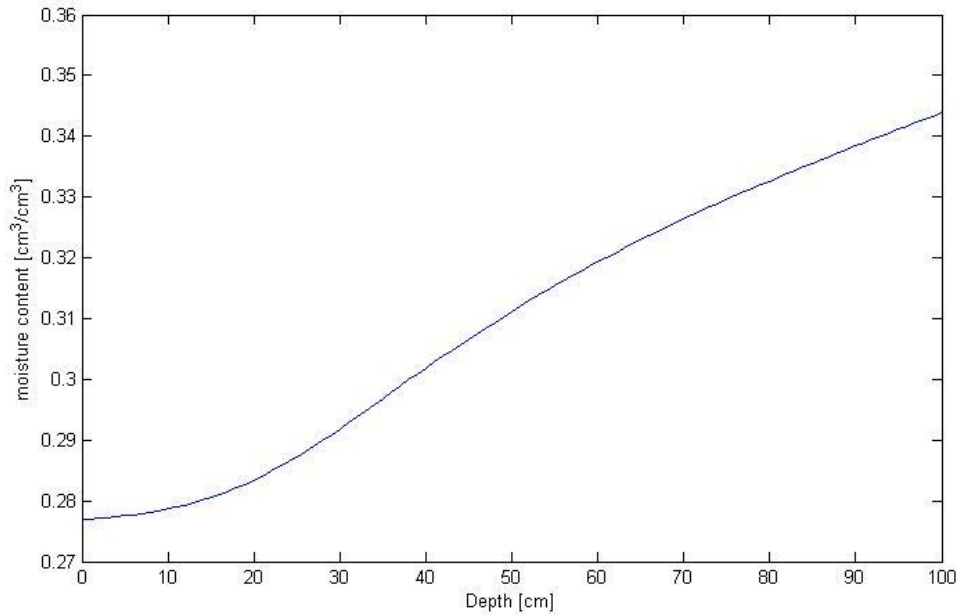
Figure 3: Curve of the pressure

Figure 4: Curve of the moisture content

5 Conclusion

In this article, we presented an efficient and robust method for the identification of the parameters in the equation of Richard who is a strongly non linear equation. The advantage of this method which is based on the genetic algorithms is that it is not blocked on the local optima. In more it is not too sensitive to initialization.

An application on real data of infiltration made it possible to check the efficiency of our method.

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