CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Identifying Similar Triangles

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Alpha Version

For more details, visit: http://map.mathshell.org

Identifying Similar Triangles

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how students reason about geometry, and in particular, how well they are able to:

- Use facts about the angle sum and exterior angles of triangles to calculate missing angles.
- Apply angle theorems to parallel lines cut by a transversal.
- Interpret geometrical diagrams using mathematical properties to identify similarity of triangles.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8G: Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and ability to reason using geometrical properties. You then review their responses and create questions for students to consider when improving their work.
- After a whole-class introduction, students work in small groups on a collaborative discussion task, categorizing diagrams of pairs of triangles based on their similarity. Throughout their work students justify and explain their thinking and reasoning.
- Students review their work by comparing their categorizations with their peers.
- In a whole-class discussion, students review their work and discuss what they have learned.
- In a follow-up lesson, students review their initial work on the assessment task and work alone on a task similar to the introductory task.

MATERIALS REQUIRED

- Each student will need a mini-whiteboard, pen, and wipe, and copies of the assessment tasks *Puzzling Triangles* and *Puzzling Triangles (revisited)*.
- Each small group of students will need a copy of *Sorting Triangles*, a pencil, a marker, a large sheet of poster paper, a pair of scissors and a glue stick.

There are also some *Blank Cards* for students, to create some new cards for others to work on, that finish the collaborative lesson activity.

TIME NEEDED

15 minutes before the lesson, a 75-minute lesson, and 15 minutes in a follow-up lesson (or for homework). These timings are not exact. Exact timings will depend on the needs of your students.

BEFORE THE LESSON

Assessment task: *Puzzling Triangles* (15 minutes)

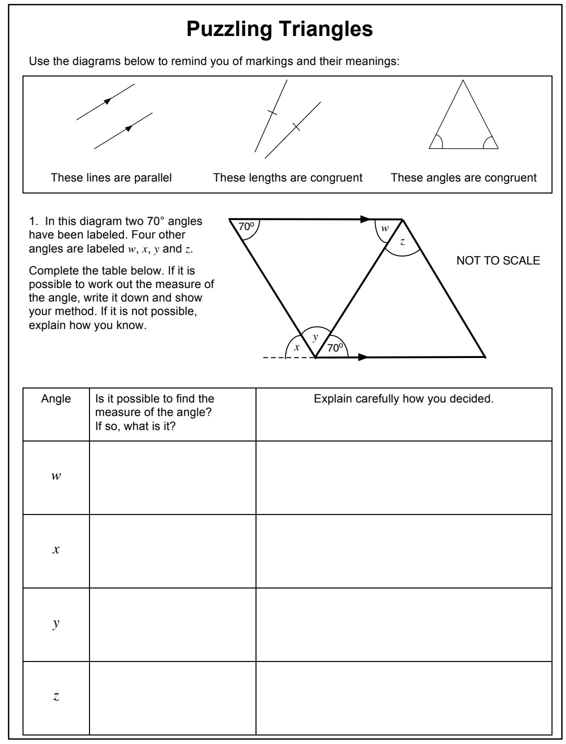
Ask students to complete this task in class or for homework a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of the assessment task *Puzzling Triangles* and briefly introduce the task:

> *In this task you are asked to find the measure of some missing angles and to decide whether or not pairs of triangles are mathematically similar.*
>
> *There is an information box at the start of the task. Use this to help you to interpret the diagrams.*
>
> *Make sure you explain your answers clearly.*

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Students should not worry too much if they cannot understand or do everything because in the next lesson they will work on a similar task that should help them. Explain to students that by the end of the next lesson they should be able to answer questions such as these confidently. This is their goal.



Puzzling Triangles

Use the diagrams below to remind you of markings and their meanings:

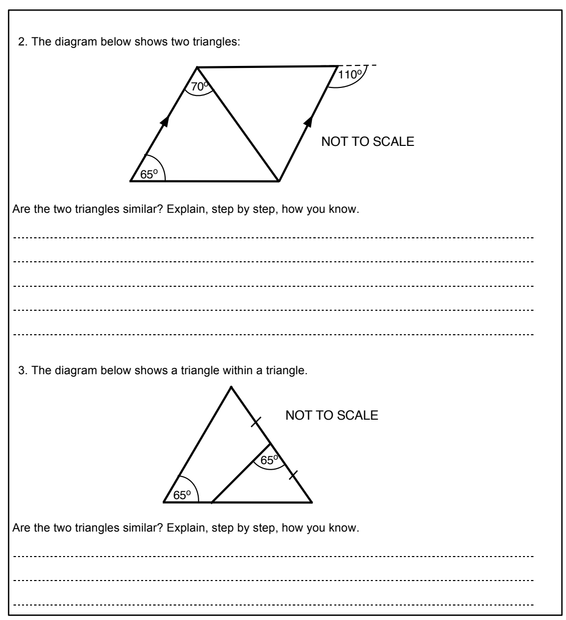
These lines are parallel — These lengths are congruent — These angles are congruent

1. In this diagram two 70° angles have been labeled. Four other angles are labeled w, x, y and z.

Complete the table below. If it is possible to work out the measure of the angle, write it down and show your method. If it is not possible, explain how you know.

NOT TO SCALE

Angle	Is it possible to find the measure of the angle? If so, what is it?	Explain carefully how you decided.
w		
x		
y		
z		



2. The diagram below shows two triangles:

NOT TO SCALE

Are the two triangles similar? Explain, step by step, how you know.

3. The diagram below shows a triangle within a triangle.

NOT TO SCALE

Are the two triangles similar? Explain, step by step, how you know.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their different ways of reasoning. The purpose of doing this is to forewarn you of issues that may arise during the lesson itself, so that you can prepare carefully.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics. Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table that follows. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- Write one or two questions on each student's work, or
- Give each student a printed version of your list of questions, and highlight appropriate questions for each student.

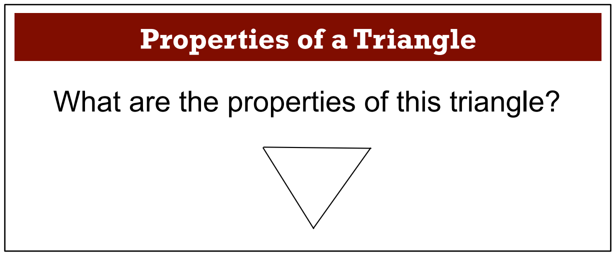
If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these on the board when you return the work to the students

Common issues:	Suggested questions and prompts:
Student has difficulty getting started	• What properties of parallel lines do you know? • What triangle properties do you know? • Can you use these properties on the diagrams given?
Student relies on visual appearance For example: The student measures the missing angles. Or: The student says angles are/are not equal because they look/do not look the same. Or: The student assumes that lines that are not marked as parallel are parallel because they look a constant distant apart.	• The diagram has not been drawn to scale. Can you use mathematics to calculate the missing angle/show that the angles are the same? • These lines look like they might be parallel but are not marked as parallel. How can we check that they are or are not parallel? Which angles would be the same if they were parallel?
Student does not provide reasons for assertions For example: The student provides numerical answers with no justification. Or: The student makes a claim of equality without justification.	• Suppose someone did not believe your answer. How could you convince them you are correct? • Why do you think that (these angles are equal)? Explain how you know your answer is correct. • What properties are written on the diagram? Can you explain how you have used these properties?
Student provides incorrect reasons for assertions For example: The student refers to a theorem that does not apply to the given situation.	• Look at the diagram carefully. What features would the diagram need to have for this theorem to apply? • What other properties/theorems do you know that might apply to this diagram? How could you check? • Describe this situation using math properties. What else do you know about these properties?
Student produces only short chains of reasoning For example: The student can derive one claim, but does not combine derivations to produce a full solution.	• Write what you have found so far in detail. Do you see any connections? • What do you already know? What do you want to find out?
Student relies on just one form of reasoning For example: The student attempts to answer all questions using angle theorems.	• Are there any other properties that can be used in these diagrams? • What do you know about the angles in a triangle/on a straight line?

SUGGESTED LESSON OUTLINE

Whole-class introduction (15 minutes)

Give each student a mini-whiteboard, pen, and wipe. Display Slide P-1 of the projector resource, showing a triangle:



What can you tell me about this triangle? What properties does it have?
[3 sides, 3 angles, sum of interior angles is 180°.]

Some students may assume that the triangle is equilateral. If they do, hold a discussion about being unable to rely on visual appearances:

It looks like the sides of this triangle are congruent. What do we call a triangle with all three sides congruent? [Equilateral.]

What other properties does an equilateral triangle have? [Congruent angles.]

BUT, we cannot rely on the visual appearance of a diagram.

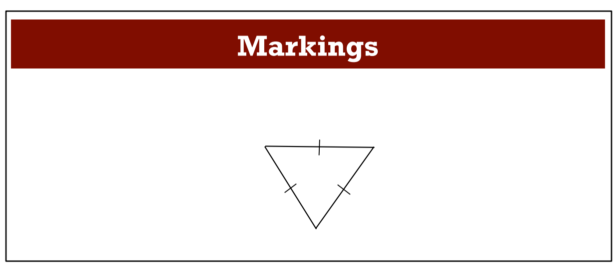
Display Slide P-2, showing a regular hexagon with the same triangle inscribed.

Here is the same triangle drawn inside a regular hexagon. What properties does a regular hexagon have? [Congruent sides and angles.]

The triangle no longer looks like it is equilateral.

We can only assume that a sketch of a triangle is meant to be equilateral if two angles of the triangle are shown to be 60°, or sides are marked as equal in length like this:

Display Slide P-3:

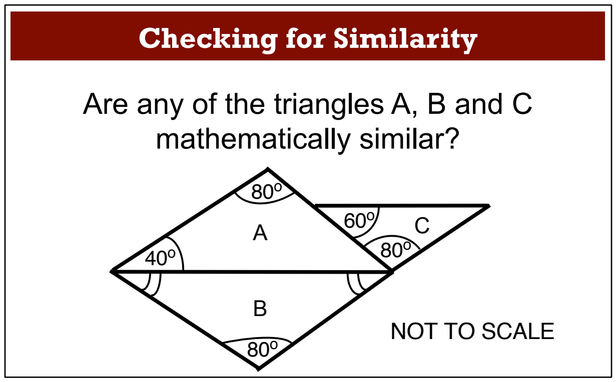


Explain the focus of today's lesson.

None of the diagrams that you will be looking at today are drawn to scale so we cannot rely on what we can see or measure.

In today's lesson you are going to be using mathematical reasoning to identify mathematically similar triangles.

Check that students understand what is meant by ‘mathematically similar’. Display Slide P-4 of the projector resource:



Here are three triangles.
We want to decide whether or not any of them are mathematically similar.

What does it mean to say that two triangles are mathematically similar?
[One may be transformed into the other by a sequence of rotations, reflections, translations, and dilations.]

We have been given information about their angles but not about their lengths.
How do we show that two triangles are similar based on their angles alone?
[If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.]

Work with the class to fill in any missing angles. Encourage them to use their mini-whiteboards as they work. Check that they understand the markings showing congruent angles in the bottom triangle. Students should be able to conclude that triangles A and C are similar.

It may be appropriate to extend the discussion further by asking students to consider what extra information can be found out about the diagram by filling in the missing angles.

Which lines are parallel? [The lines that appear horizontal are parallel.]
How do you know? [The alternate interior angles are both 60°.]

Do triangles B and C lie on a straight line? [No.]
How can you tell? [The three internal angles add to 190°: 80°+60°+50°.]

What markings can be added to triangle B?
[The two congruent angles imply an isosceles triangle, so two congruent sides may be marked.]

Collaborative small group work (30 minutes)

Ask students to work in groups of two or three. Give each group a copy of *Sorting Triangles*, a pencil, a pair of scissors, a glue stick, a marker and a large sheet of poster paper.

Explain to students that in their groups they will be looking at some diagrams of pairs of triangles. They will be deciding whether the triangles in the diagram are ‘similar’, ‘not similar’, or it ‘cannot be determined’.

Ask students to divide their large sheet of poster paper into three columns and head separate columns with the words: ‘*Similar*’, ‘*Not Similar*’, ‘*Cannot be Determined*’.

For two triangles to be ‘similar’, what do we need to show?
[That two angles of one triangle are congruent to two triangles of the other.]

For two triangles to be 'not similar' what do we need to show?
[That the triangles contain at most one congruent angle.]

To show that it 'cannot be determined' whether the two triangles are similar or not you will need to show that we do not have enough information to show that the triangles have two congruent angles.

Most of the triangles have angles labeled but some do not so students will need to use angle theorems to label angles as congruent rather then calculating missing angles for these diagrams. If you think students will struggle with this, encourage them to work with diagrams A and B first.

Now explain how students are to work together:

Take it in turns to:

1. *Select a diagram, cut it out and place it in one of the columns, explaining your reasoning.*
2. *If you think two triangles are similar (or not), fill in missing angles to show that this is the case. If you think that similarity cannot be determined, fill in as many angles as you can and explain why no other angles can be found.*
3. *Your partner must challenge your explanation if they disagree, or describe it in their own words if they agree.*
4. *Once agreed, glue the diagram onto the poster and write your explanation in pencil next to it.*

Continue to take turns until all the diagrams are placed.

Slide P-5 of the projector resource summarizes these instructions.

The purpose of this structured work is to encourage each student to engage with their partner's explanations and to take responsibility for their partner's understanding. Students should use their mini-whiteboards for calculations and to explain their thinking to each other.

It does not matter if students do not manage to place all the diagrams. It is more important that everyone in each group understands the categorization of each diagram.

While students are working, you have two tasks: to notice their approaches to the task, and to support student problem solving.

Notice different student approaches to the task

Listen to and watch students carefully. In particular, notice whether students are addressing the difficulties they experienced in the assessment task. Are students engaging with mathematical properties or are they relying on perceptual reasoning about surface features? Do students fully understand the notations they are working with? Are they noting the properties they read from diagrams? Do they notice any implications or connections of those properties?

Do students reason by deriving new information from known facts? Do they use theorems about the sum of angles on a straight line, the sum of angles in a triangle, angles formed when a transversal crosses a pair of parallel lines?

How do students refer to diagrams and properties? Do they use 'this' and 'that' frequently, or name the objects and properties? Do partners understand what the student is pointing to?

Support student problem solving

Help students to work constructively together. Remind them to look at the slide for instructions on how to take turns. Check that one student listens to another by asking the listener to paraphrase the speaker's statements. Check that students are recording their discussions as rough notes on their mini-whiteboards.

Try not to solve students' problems for them or to add structure to longer problems and do the reasoning for them. Instead, you might ask strategic questions to suggest ways of moving forward.

What do you know? What else can you find out from the diagram?

What do you need to find out? What math do you know that connects to that?

What's the same about these two diagrams? What is different?

Support students in developing written explanations. Suggest that they write rough notes on their mini-whiteboards as they make decisions. Suggest that they refer to these notes as they write a fuller, clearer version of the explanation on their poster. Suggest that they are careful to name objects so that the reader understands what they are writing about.

If any students finish early, they should be encouraged to use the *Blank Cards* provided to create some new cards for others to work on.

Sharing work (10 minutes)

As students finish their posters, get them to critique each other's work by asking one student from each group to visit another group. It may be helpful for the students visiting another group to first jot down a list of the categorized diagrams (i.e. Similar: A, D, E etc.) on their mini-whiteboard.

*Now, **one** person from each group jot down your categorizations on your mini-whiteboard and then go to another group's desk and compare your categorizations with their's.*

If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

If you have categorized diagrams in the same columns, compare your methods and check that you understand each other's explanations.

If you are staying at your desk, be ready to explain the reasons for your group's decisions.

Slide P-6 of the projector resource, *Sharing Work*, summarizes these instructions.

Poster Review (5 minutes)

Once students have had chance to share their work, and discuss their categorizations and reasoning with their peers, give them a few minutes to review their posters.

Now that you have discussed your work with someone else, as a group you need to consider whether or not to make any changes to your work.

If you think a diagram is in the wrong column, draw an arrow on your poster to show which column it should move to. If you are confident with your decisions, go over your work in pen (or make amends in pen if you have changed your mind.)

Whole-class discussion (15 minutes)

Organize a whole-class discussion about what has been learned and explore the different methods of justification used when categorizing diagrams.

You may want to first select a diagram that most groups categorized correctly, as this may encourage good explanations.

Which column did you put this diagram in? Can you explain your decision?

Can anyone improve this explanation?

Does anyone have a different explanation?

Which explanation do you prefer? Why?

Try to include a discussion of at least one diagram from each of the three columns.

Give me a diagram that shows triangles that are similar/not similar/where their similarity cannot be determined.

Why did you put this diagram in this column?

Did anyone put this diagram in a different column?

Once one group has justified their choice for a particular diagram, ask other students to contribute ideas of alternative approaches and their views on which reasoning was easier to follow. To help students explain their work, there are slides in the projector resource showing each of the diagrams A to H from the lesson task (Slides P-7 to P-14).

Ask students what they learned by looking at other students' work and whether or not this helped them with diagrams they had found difficult to categorize or were unsure about:

Which diagram did you find the most difficult to categorize? Why do you think this was?

Did seeing where another group had placed this diagram help? If so, in what way did it help?

In what ways did having another group critique your poster help?

Did looking at another group's poster help you with your own work? Can you give an example?

During the discussion, draw out any issues you noticed as students worked on the activity, making specific reference to the misconceptions you noticed. You may want to use the questions in the *Common issues* table to support your discussion.

Follow-up lesson: reviewing the assessment task (15 minutes)

Give each student a copy of the review task, *Puzzling Triangles (revisited)* and their scripts from the original assessment task, *Puzzling Triangles*. If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your original responses and the questions [on the board/written on your script.] Think about what you have learned.

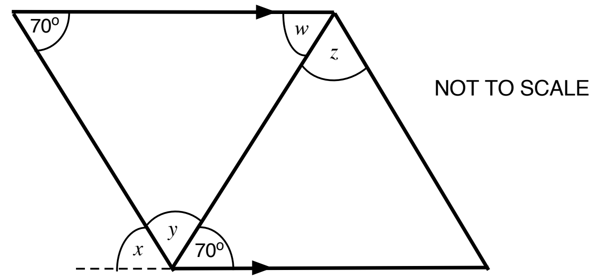
Now look at the new task sheet, Puzzling Triangles (revisited). Can you use what you have learned to answer these questions?

Some teachers give this as homework.

SOLUTIONS

Assessment Task: Puzzling Triangles

1. The transversals are not parallel. Students may assume that they are, leading to incorrect reasoning. It is important that students identify a mathematical structure from which to derive their numerical answers.



The given 70° angle (at the bottom of the diagram) and w are alternate interior angles formed by a transversal crossing a pair of parallel lines and so are equal. Therefore $w = 70°$.

Similarly, with the other given 70° angle (at the top of the diagram) and angle x so $x = 70°$ also.

Angles x, y and the given 70° angle are on a straight line so they sum to 180°. So $y = 180° - x - 70° = 180° - 70° - 70° = 40°$. (The sum of the angles in a triangle may also be used to show that $y = 40°$, once angle w has been calculated.)

The measure of angle z cannot be deduced from the diagram, as the transversals are not parallel.

2. All of the angles in the two triangles can be calculated to show that the two triangles are not similar. The missing angle in the first triangle is 45° and the three angles in the second triangle are 40°, 70° and 70°.

3. The two triangles have a shared angle and a 65° angle in common. If two of the angles are the same the third angle must also be the same so the two triangles are similar.

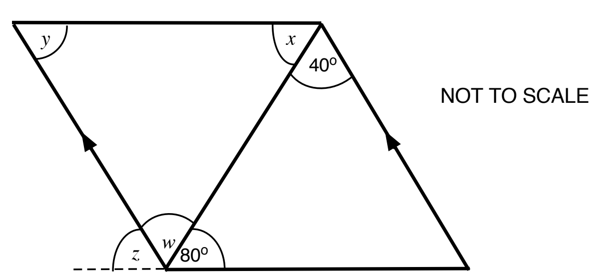
Sorting Triangles

The quality of the written reasoning with references to angle theorems and triangle properties is the focus of this task. However, we provide a list of categorizations for your convenience.

Similar	Not similar	Cannot be determined
A	B	C
D		F
E		G
H		

Assessment Task: Puzzling Triangles (revisited)

1. The given 40° angle and w are alternate interior angles formed by a transversal crossing a pair of parallel lines and so are equal. Therefore $w = 40°$.



Angles z, w and the given 80° angle are on a straight line so they sum to 180°. So $z = 180° - w - 80° = 180° - 40° - 80° = 60°$.

The measures of angle x and y cannot be deduced from the diagram.

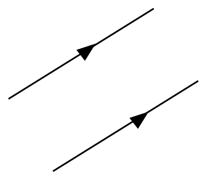
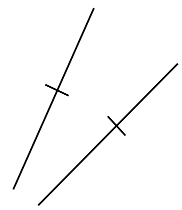
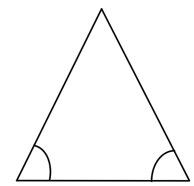
2. Using alternate angles gives us the angle in the larger triangle that has been divided into two as 70° in total. The line therefore bisects this angle and so both of the smaller triangles contain a 35° angle.

We can use the fact that the larger triangle is isosceles to find the other two angles in the larger triangle as each being 55°. The smaller triangles therefore each contain a 55° angle and a 35°. If two of the angles are the same the third angle must also be the same so the two triangles are similar.

3. All of the angles in the two triangles can be calculated to show that the two triangles are not similar. The missing angles in the first triangle are 40° and 70°. The three angles in the second triangle are 30°, 70° and 80°.

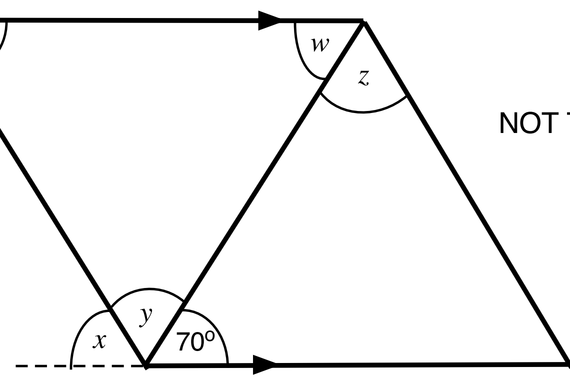
Puzzling Triangles

Use the diagrams below to remind you of markings and their meanings:

		
These lines are parallel	These lengths are congruent	These angles are congruent

1. In this diagram two 70° angles have been labeled. Four other angles are labeled w, x, y and z.

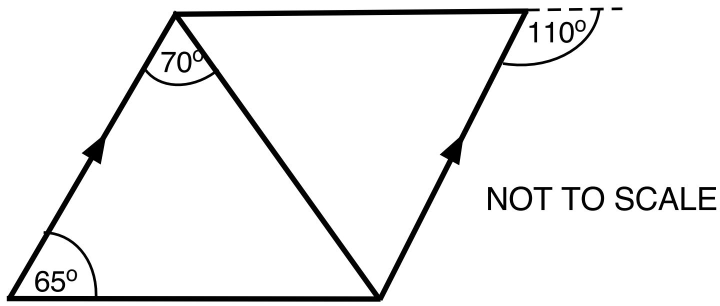
Complete the table below. If it is possible to work out the measure of the angle, write it down and show your method. If it is not possible, explain how you know.



NOT TO SCALE

Angle	Is it possible to find the measure of the angle? If so, what is it?	Explain carefully how you decided.
w		
x		
y		
z		

2. The diagram below shows two triangles:



Are the two triangles similar? Explain, step by step, how you know.

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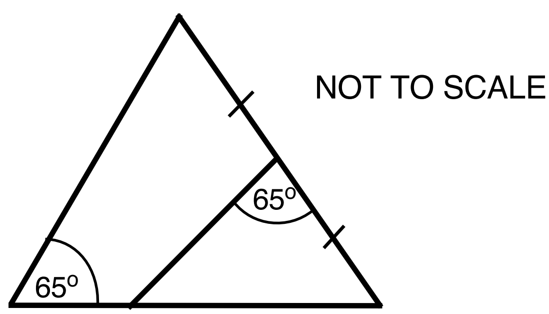
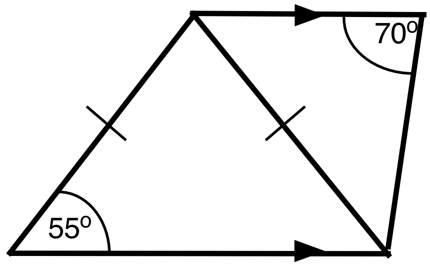
3. The diagram below shows a triangle within a triangle.



Are the two triangles similar? Explain, step by step, how you know.

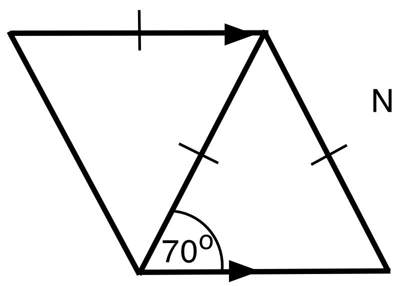
Sorting Triangles

A



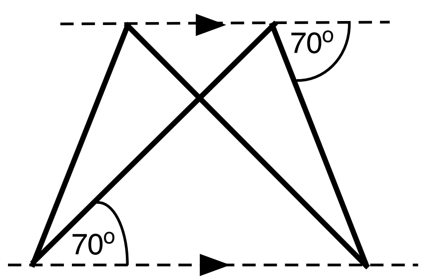
NOT TO SCALE

B



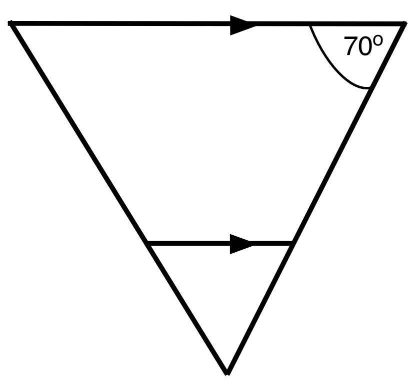
NOT TO SCALE

C



NOT TO SCALE

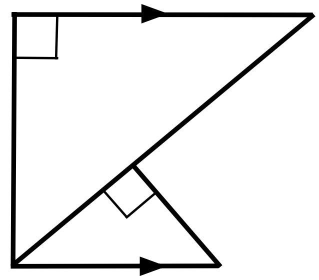
D



NOT TO SCALE

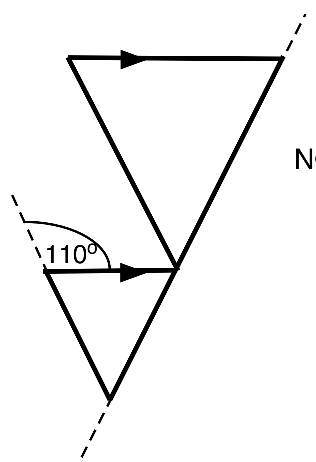
Sorting Triangles (continued)

E



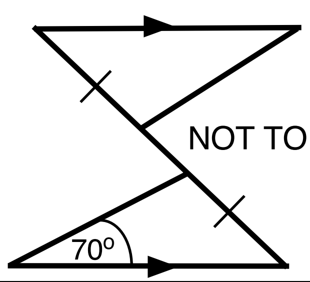
NOT TO SCALE

F



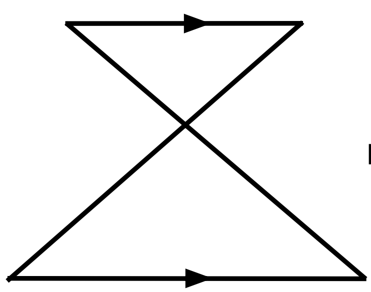
NOT TO SCALE

G



NOT TO SCALE

H



NOT TO SCALE

Blank Cards

Blank
Blank
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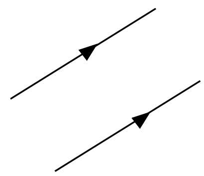
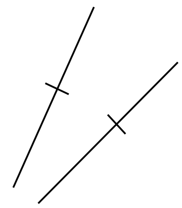
Puzzling Triangles (revisited)

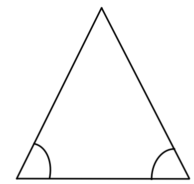
Use the diagrams below to remind you of markings and their meanings:



These lines are parallel



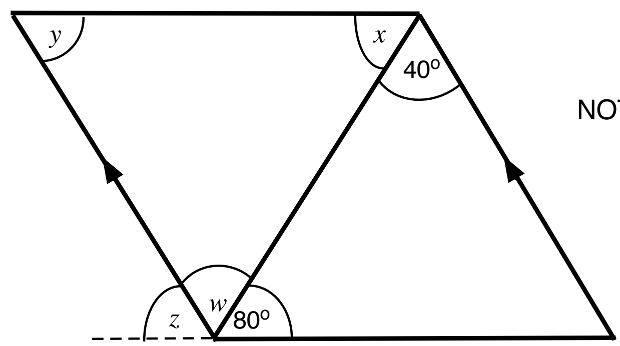
These lengths are congruent



These angles are congruent

1. In this diagram an 80° angle and a 40° angle have been labeled. Four other angles are labeled w, x, y and z.

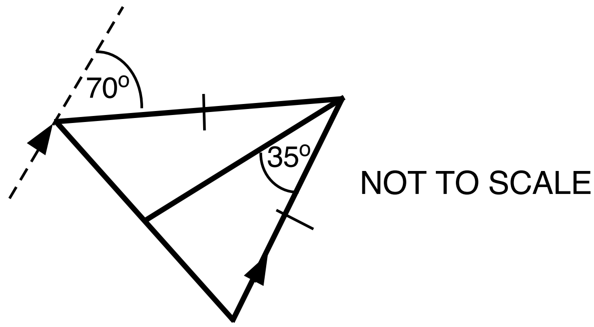
Complete the table below. If it is possible to work out the measure of the angle, write it down and show your method. If it is not possible, explain how you know.



NOT TO SCALE

Angle	Is it possible to find the measure of the angle? If so, what is it?	Explain carefully how you decided.
w		
x		
y		
z		

2. The diagram below shows two triangles:



Are the two triangles similar? Explain, step by step, how you know.

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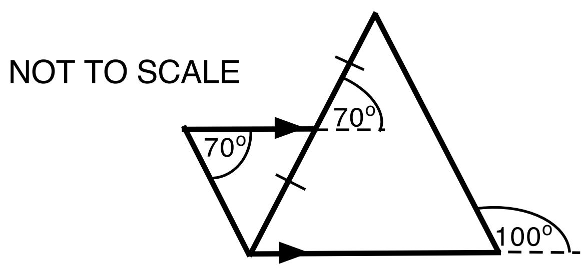
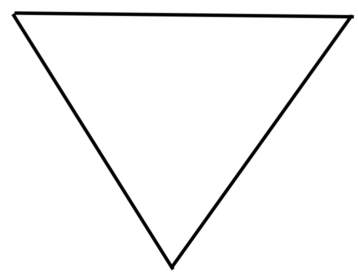
3. The diagram below shows two triangles:



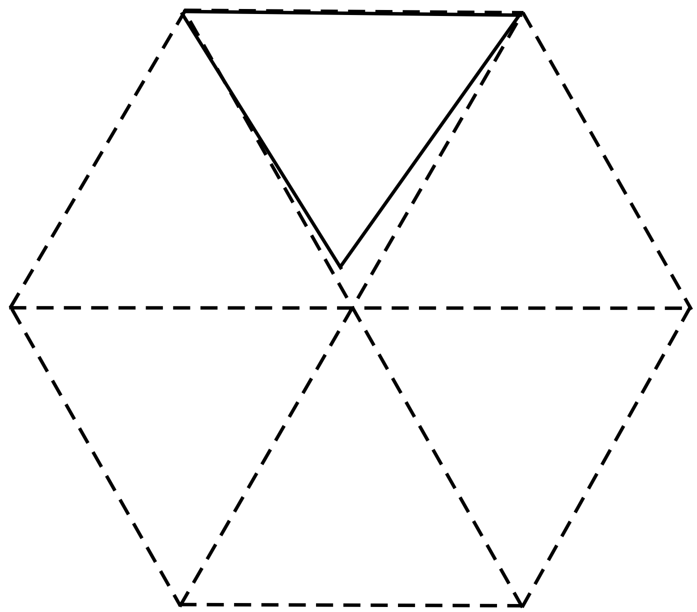
Are the two triangles similar? Explain, step by step, how you know.

Properties of a Triangle

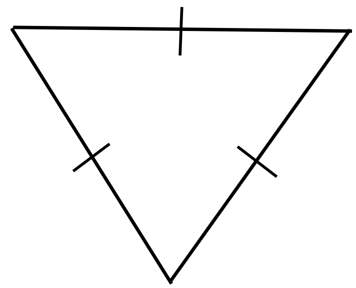
What are the properties of this triangle?



Regular Hexagon

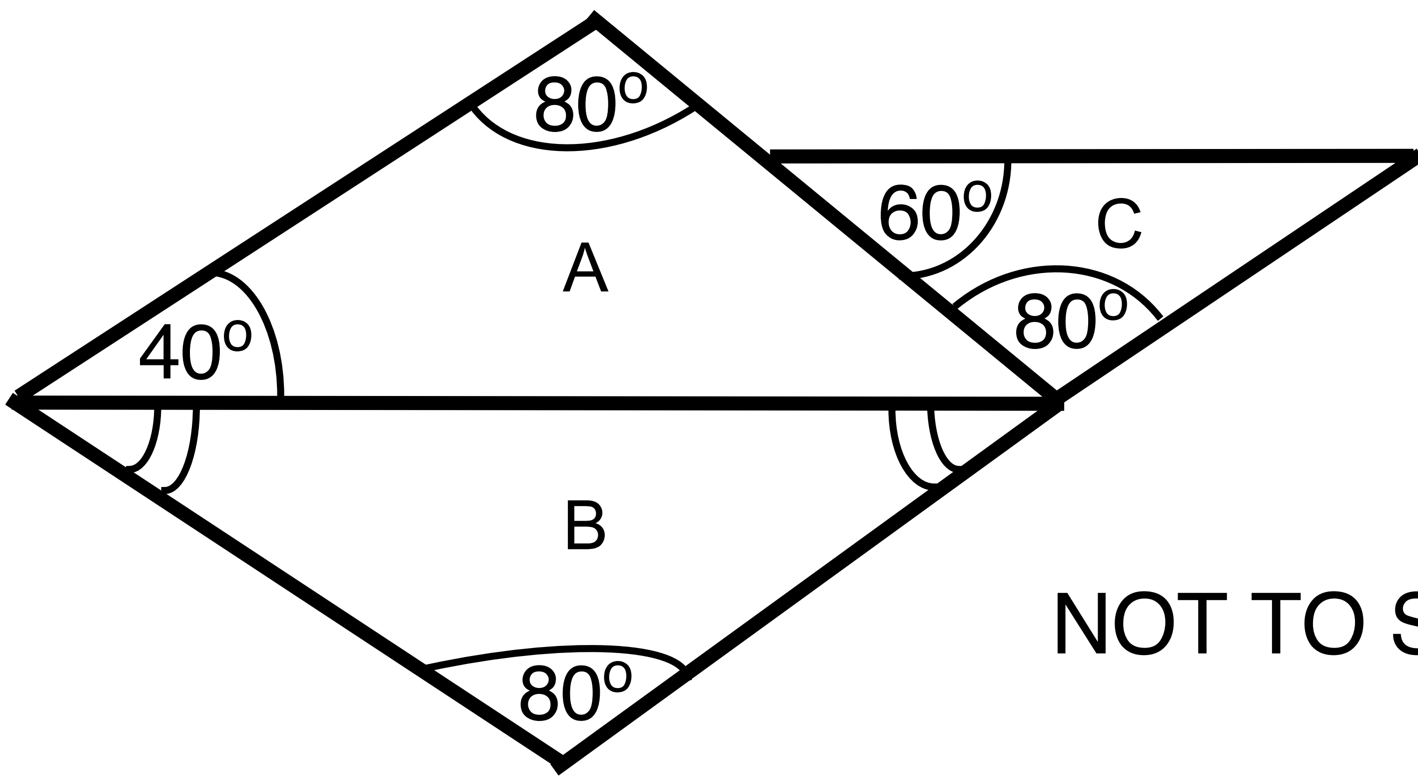


Markings



Checking for Similarity

Are any of the triangles A, B and C mathematically similar?



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Working Together

Take it in turns to:

1. Select a diagram, cut it out and place it in one of the columns, explaining your reasoning.
2. If you think two triangles are similar (or not), fill in missing angles to show this is the case. If you think that similarity cannot be determined, fill in as many angles as you can and explain why no other angles can be found.
3. Your partner must challenge your explanation if they disagree or describe it in their own words if they agree.
4. Once agreed, glue the diagram onto the poster and write your explanation **in pencil** next to it.

Continue to take turns until all the diagrams are placed.

Sharing Work

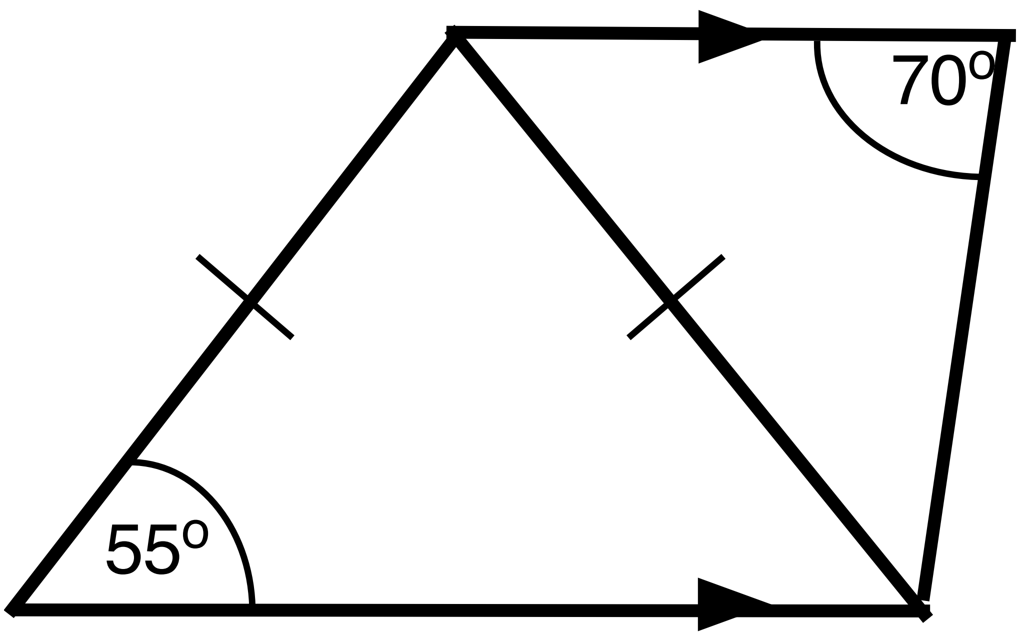
1. **One** person in your group jot down your diagram categorizations on your mini-whiteboard and then go to another group's desk and check your work against their categorizations.

2. If there are differences, ask for an explanation. If you still don't agree, justify your own thinking and explain which column you think the diagram needs moving to.

3. If you have categorized diagrams in the same columns, compare your methods and check that you understand each other's explanations.

4. If you are staying at your desk, be ready to explain the reasons for your group's decisions.

Card A



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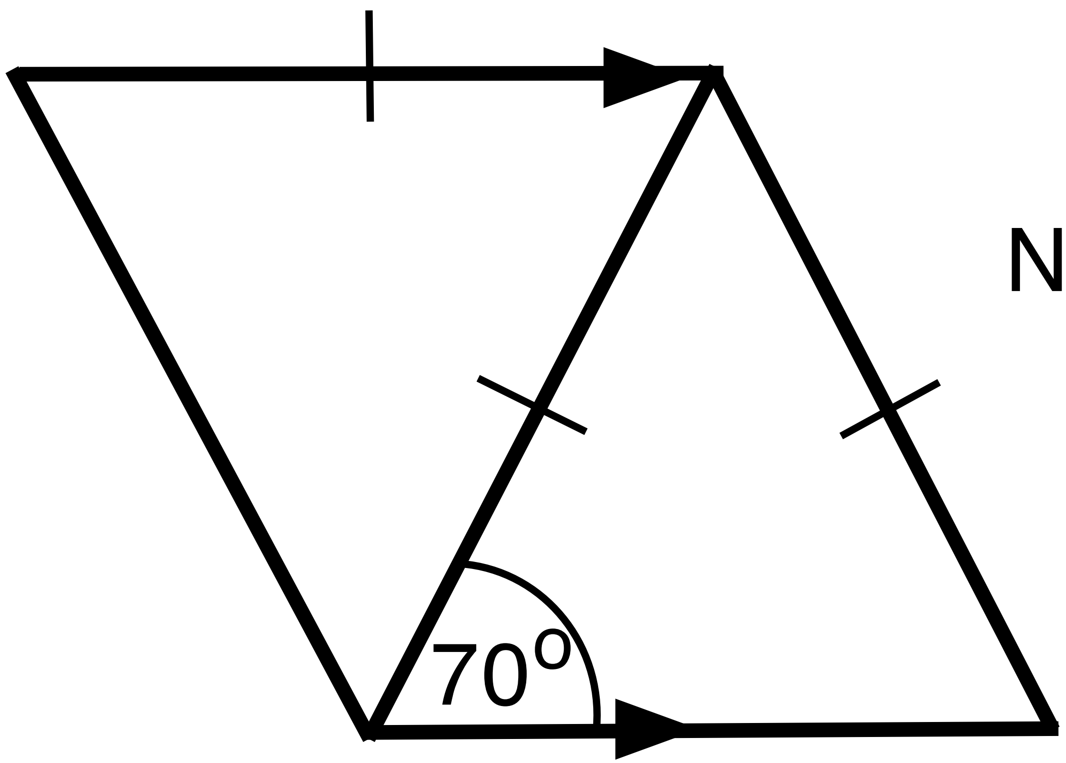
Card B



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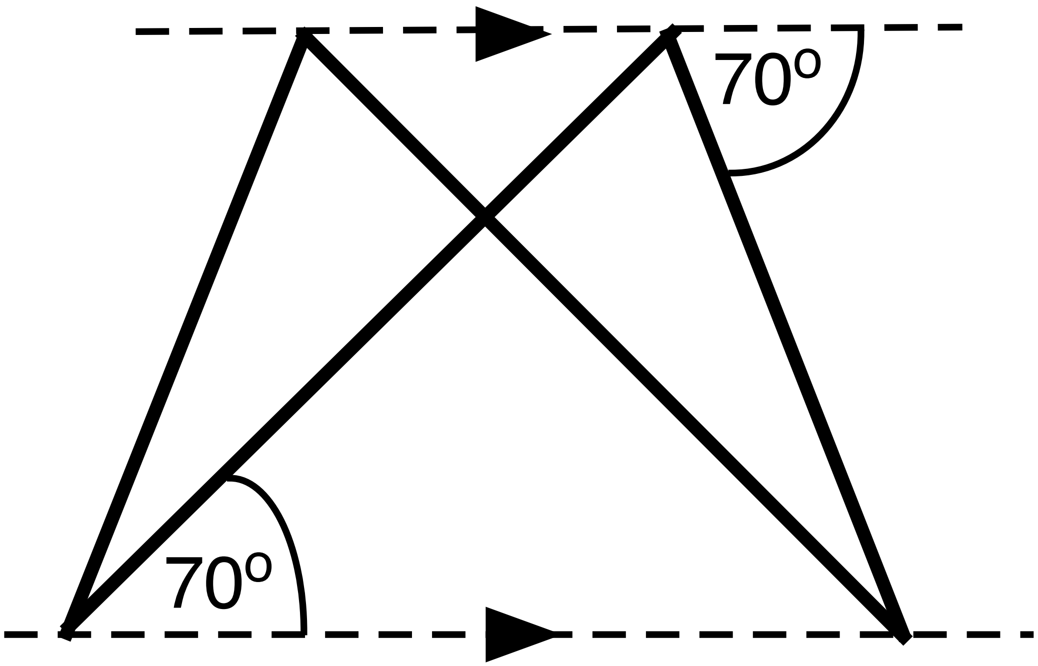
Card C



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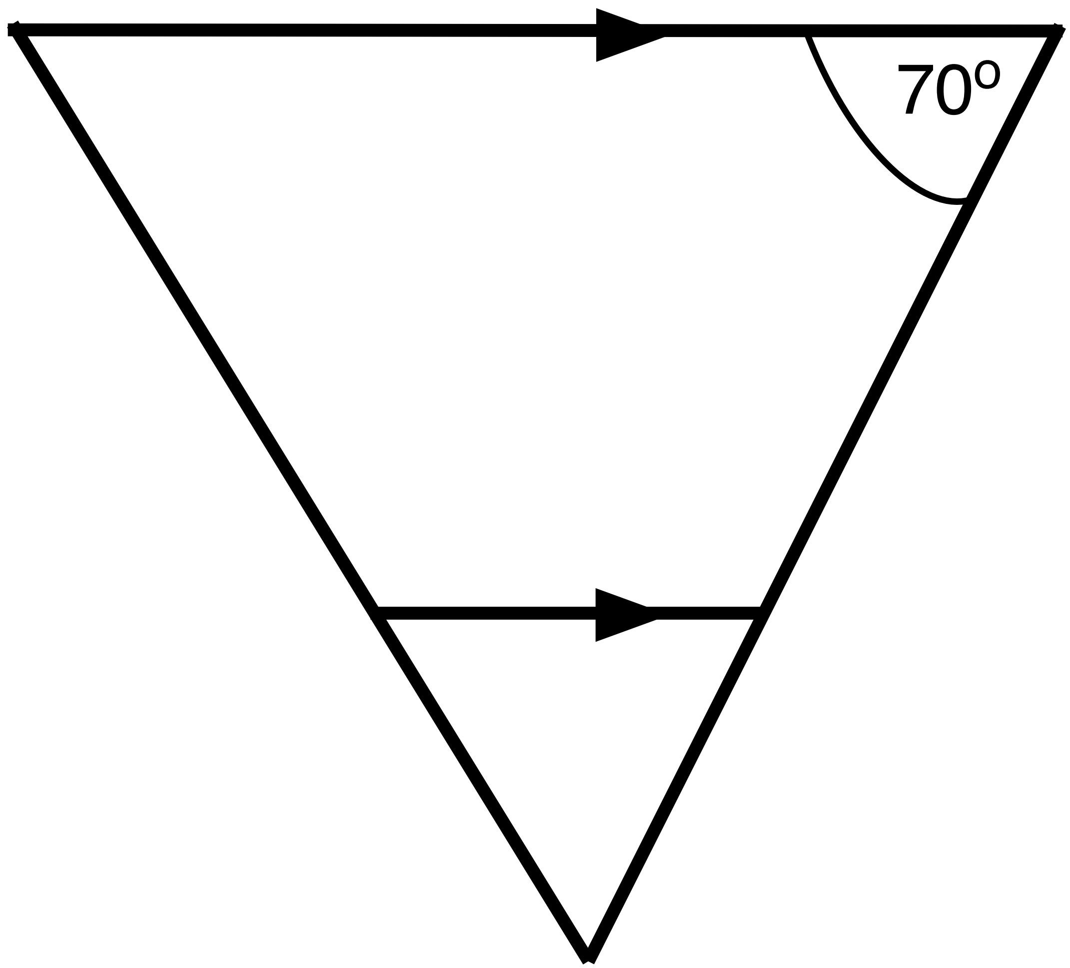
Card D



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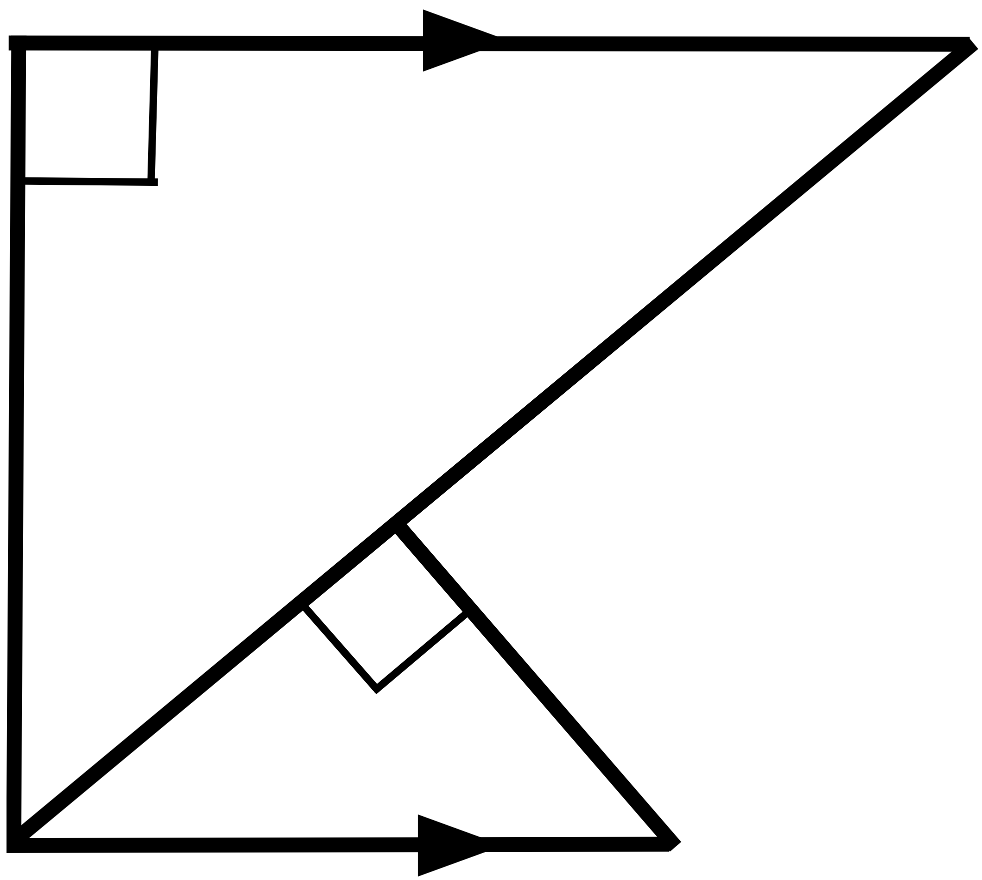
Card E



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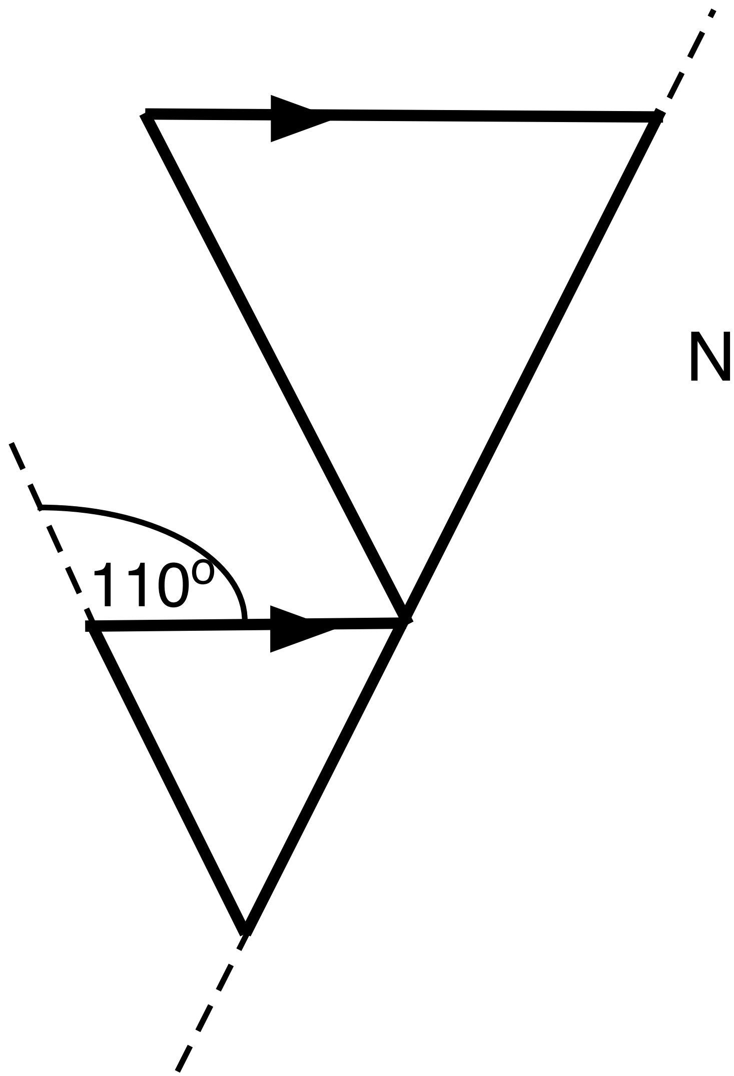
Card F



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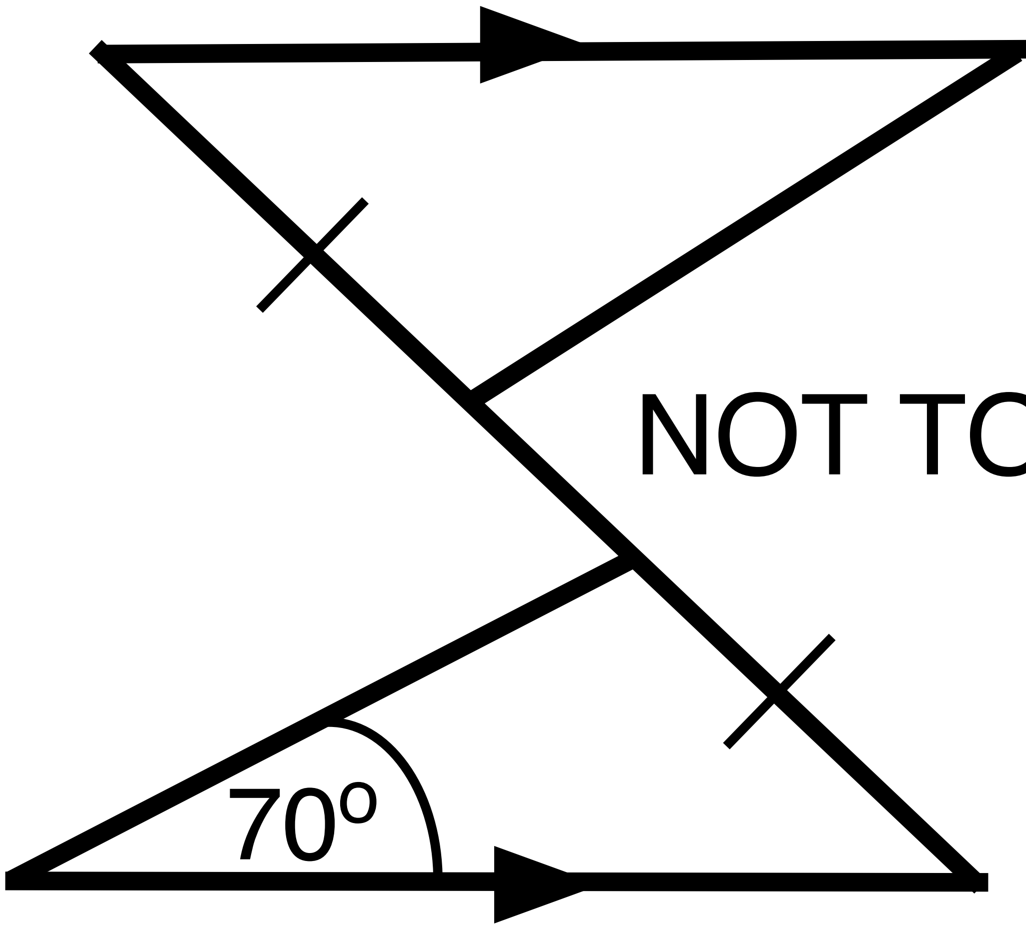
Card G



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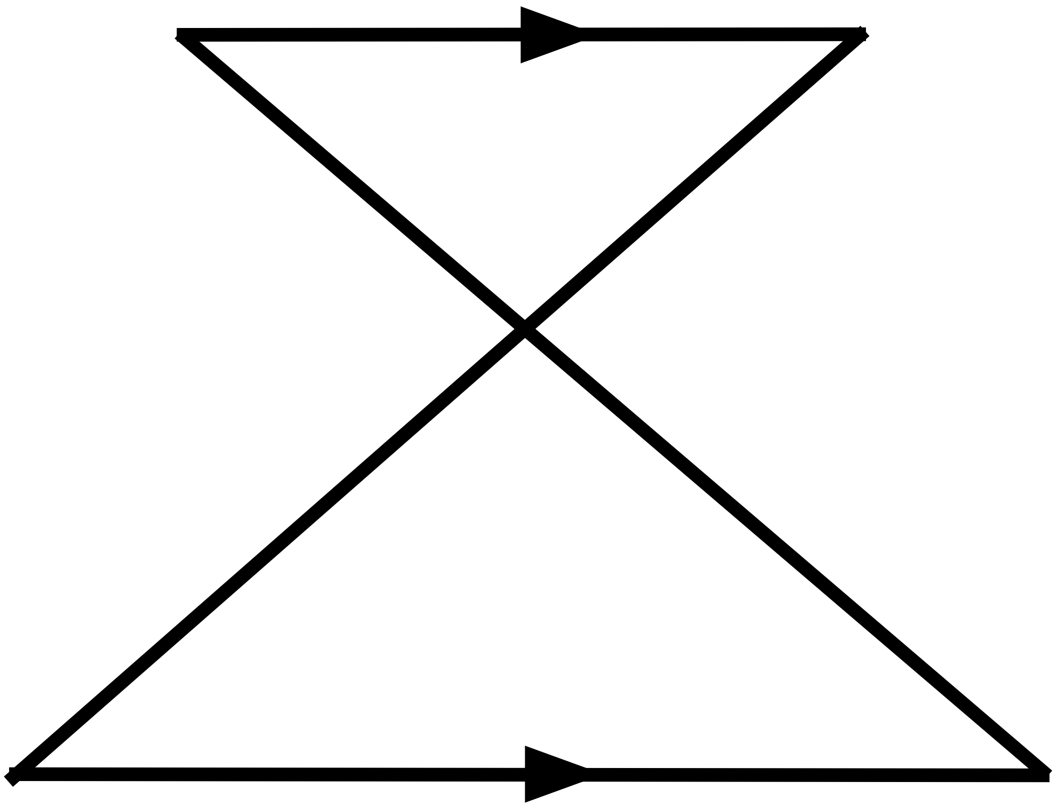
Card H



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