Stability of Dynamical Systems: A Constructive Approach

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***Abstract*—A set A of $n\times n$ complex matrices is *stable* if for every neighborhood of the origin $U\subset C^n$, there exists another neighborhood of the origin V, such that for each $M\in A'$ (the set of finite products of matrices in A), $MV\subseteq U$. Matrix and Liapunov stability are related.**

***Theorem:* A set of matrices A is stable if and only if there exists a norm, $|\ \ |$, such that for all $M\in A$, and all $z\in C^n$, $|Mz|\leqslant|z|$.**

The norm is a *Liapunov function* for the set A. It need not be smooth; using smooth norms to prove stability can be inadequate. A novel central result is a *constructive* algorithm which can determine the stability of A based on the following.

***Theorem:* $A=\{M_0,M_1,\cdots,M_{m-1}\}$ is a set of m distinct complex matrices. Let W_0 be a bounded neighborhood of the origin. Define for $k>0$,**

$$W_k=\text{convex hull}\left[\bigcup_{i=0}^{\infty}M_{k'}^{i}W_{k-1}\right]$$

where $k'=(k-1)\bmod m$. Then A is stable if and only if $W^*\equiv\cup_{i=0}^{\infty}W_i$ is bounded.

W^* is the norm of the first theorem. The constructive algorithm represents a convex set by its extreme points and uses linear programming to construct the successive W_k. Sufficient conditions for the finiteness of constructing W_k from W_{k-1}, and for stopping the algorithm when either the set is proved stable or unstable are presented. A is generalized to be any convex set of matrices. A dynamical system of differential equations is stable if a corresponding set of matrices—associated with the logarithmic norms of the matrices of the linearized equations—is stable. The concept of the stability of a set of matrices is related to the existence of a matrix norm. Such a norm is an induced matrix norm if and only if the set of matrices is maximally stable (i.e., it cannot be enlarged and remain stable).

I. Introduction

DETERMINATION of stability is a fundamental concern which arises in virtually any real situation capable of being modeled by a dynamical system. Although the methods of this paper are developed for time-varying difference equations, they apply to questions of the Liapunov stability of nonlinear differential equations. The procedure would be to view the differential equation

$$dx/dt=f(x)\tag{1}$$

as a "linear" system:

$$dx/dt=M(x)x$$

where $M(x)$ is a matrix. Applying Euler's method we obtain

$$x_{n+1}=x_n+h_nM(x_n)x_n$$

where h_n is the current step-size (i.e., $h_n\equiv t_{n+1}-t_n$). If we let S_M denote the matrices $M(x_n)$ obtained when x_n varies over all possible values, then the equation becomes a linear time-varying difference equation

$$x_{n+1}=(I+h_nM_n)x_n,\qquad M_n\in S_M.\tag{2}$$

In Section X we prove that if (2) is stable for all sequences $\{h_n\}$ where $0\leqslant h_n\leqslant h'$, then (1) is stable. Although the set of matrices $\{I+h_nM_n|0\leqslant h_n\leqslant h',\ M_n\in S_M\}$ is infinite, the methods of Sections IV, V, and XI permit a constructive approach to the question of the stability of (2), and hence the stability of (1).

In [1], the stability of linear multistep methods was investigated where the integration step-size varied with each step. When applied to the stable test differential equation $dx/dt=\lambda x$, Re $\lambda\leqslant 0$, the general multistep method

$$\sum_{i=0}^{k}\alpha_ix_{n+1-i}+h_n\sum_{i=0}^{k}\beta_idx_{n+1-i}/dt=0\tag{3}$$

gives rise to a linear time-varying difference equation

$$\sum_{i=0}^{k}(\alpha_i+\lambda h_n\beta_i)x_{n+1-i}=0\tag{4}$$

where h_n is the step-size. The coefficients α_i, β_i depend on the last k step-size ratios $h_n/h_{n-1},\cdots,h_{n-k+1}/h_{n-k}$. They are computed so that (3) is accurate to some order $p\geqslant 1$.

Equation (4) written in vector form becomes (with $q\equiv\lambda h_n$)

$$z_{n+1}=\begin{bmatrix}\dfrac{\alpha_1+q\beta_1}{-\alpha_0-q\beta_0} & \dfrac{\alpha_2+q\beta_2}{-\alpha_0-q\beta_0} & \cdots & \dfrac{\alpha_k+q\beta_k}{-\alpha_0-q\beta_0}\\ 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & \cdots 1 & 0\end{bmatrix}z_n\tag{5}$$

where $z_n^T\equiv(x_n,x_{n-1},\cdots,x_{n-k})$. Since q, α, and β generally depend on the h_i, the question of the stability of (5) is that of the stability of a linear time-varying difference equation.

Willson in [2] was concerned with the design of a digital frequency shift keying oscillator which produced the difference equation

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$$z_{n+1}=\begin{pmatrix}0 & 1\\ & \beta\alpha_n\end{pmatrix}z_n. \tag{6}$$

This equation represents the linear region of operation; the time variation of α_n causes a shift in the frequency of oscillation. For higher amplitudes, a saturation effect stabilizes the oscillation, so that the requirement on (6) is that it be unstable. To this effect, Willson notes that $z_{n+1}=M_nz_n$ is unstable provided that $w_{n+1}=(M_n^T)^{-1}w_n$ is asymptotically stable.

Thus (2), (5), and (6) each represent problems concerning the stability of a system of linear time-varying difference equations

$$z_{n+1}=M_nz_n \tag{7}$$

where each M_n is a stable matrix (i.e., eigenvalues $\lambda(M_n)$ satisfy $|\lambda(M_n)|\leqslant 1$, with simple eigenvalues on the unit circle).

A standard method for settling stability questions for such difference equations would be to follow nonlinear theory and seek a quadratic Liapunov function in order to prove stability. This approach would require a positive definite matrix L such that

$$z_{n+1}^TLz_{n+1}\leqslant z_n^TLz_n$$

for all z_n and $z_{n+1}=M_nz_n$ or

$$z_n^TM_n^TLM_nz_n\leqslant z_n^TLz_n.$$

Thus the matrix

$$L-M_n^TLM_n$$

is required to be nonnegative definite for *all* positive integers n. Willson was successful with this approach for sets $\{A_i\}$, where

$$A_i=\begin{pmatrix}a_i & 1\\ b & 0\end{pmatrix},\qquad i=1,2.$$

In fact, Willson was able to produce stability regions (Fig. 1 solid lines) based on the *best possible quadratic* Liapunov function. We have since determined the actual stability regions; hence it is possible in this case to view the exact amount of deficiency in the quadratic function approach.

In Sections II and III, we examine the following problem: given a difference equation (7) where $M_n\in A$, and A is some set, finite or infinite, of stable matrices, when can we find a *Liapunov function* for A, that is a function $w(z)$ such that

$$w(M_nz)\leqslant w(z) \tag{8}$$

for all $M_n\in A$? Some such results were given in [1] and we initially follow the line of investigation set up in that paper. In Section IV, we examine an approach which under certain conditions either *constructs* a Liapunov function for A, or determines that none such exists. This approach is then used in Section VI on Willson's example to determine the exact stability regions. In Section V some finiteness and stopping criteria are established for the algorithm and a complete outline of the algorithm is given in Section XI. Sections VII and VIII demonstrate that the algorithm applies to complex matrices and to infinite sets of matrices. In Section IX the concepts of stability and matrix norms are related. In Section X, the transition from discrete systems to continuous systems is made. Two examples, given in Sections XII and XIII, illustrate how these techniques can be used on a variety of problems with interesting results.

II. Stability of a Set of Matrices: The Concept

With a set of matrices A we associate the larger set A' (the semigroup generated by matrices in A) consisting of all finite products of matrices in A. In general, A' is an infinite set.

Definition

Matrix M is *stable* if there exists K such that $\|M^i\|\leqslant K$ for all i. Thus M is stable if and only if $|\lambda(M)|\leqslant 1$, and the eigenvalues of M on the unit circle are simple.

Definition

A *set* of $n\times n$ complex matrices A is *stable* if for every neighborhood of the origin $U\subset C^n$, there exists another neighborhood of the origin V such that, for each $M\in A'$, $MV\subseteq U$.

Lemma 1

A set of matrices A is stable if and only if there is a bounded neighborhood of the origin W such that for each $M\in A'$, $MW\subseteq W$. Furthermore, W can be chosen to be convex and balanced.[1]

For proof, see [1]. ■

Lemma 2:

If A is stable, then there exists a norm, $|\ \ |$, such that

$$|Mz|\leqslant|z|$$

for all $M\in A$, $z\in C^n$.

Proof: Since A is stable, by Lemma 1 there exists a convex balanced W such that $MW\subseteq W$. Let ∂W denote the boundary of W. For $z\in C^n$, there exists a unique $z'\in\partial W$ and scalar $\alpha>0$ such that $z=\alpha z'$ (since W is a convex neighborhood of the origin). Define $w(z)\equiv\|z\|/\|z'\|=\alpha$ for $z\neq 0$, and $w(0)\equiv 0$. Because Mz is linear in z, we have

$$Mz=\alpha Mz'=w(z)Mz'.$$

Further, since W is balanced, w has the property that $w(\gamma z)=|\gamma|w(z)$ for any complex number γ. Thus

$$w(Mz)=w(z)w(Mz').$$

Since $Mz'\in W$ for $M\in A$, then $w(Mz')\leqslant 1$. Hence,

$$w(Mz)\leqslant w(z).$$

[1] W is *balanced* means that if $z\in W$, then $z\exp(i\theta)\in W$. The term *equilibrated* is also used.

The continuity of w follows from the convexity of W. Finally, because w is a convex function,

$$w(\lambda u+(1-\lambda)v) \leqslant \lambda w(u)+(1-\lambda)w(v).$$

Substituting $u=x/\lambda$, $v=y/(1-\lambda)$ gives

$$w(x+y) \leqslant w(x)+w(y).$$

Hence w is a norm. ■

Lemma 3

A is unstable if and only if for each $K>0$, there exists $M \in A'$ and $z \in C^n$ such that

$$\frac{\|Mz\|}{\|z\|}>K.$$

Thus A is stable if and only if A' is bounded.

Proof: If A is unstable, then there exists a neighborhood of the origin U such that for every neighborhood of the origin $V \subseteq U$, there exists $M \in A'$ for which $MV \not\subseteq U$. Let

$$\alpha=\inf \|z\|, \qquad \text{for } z \notin U$$

and let $V=\{z \mid \|z\|<\alpha K^{-1}\}$. Since $MV \not\subseteq U$, there exists $z' \in V$ such that $Mz' \notin U$. Then

$$\frac{\|Mz'\|}{\|z'\|} \geqslant \frac{\alpha}{\|z'\|} > \frac{\alpha}{\alpha K^{-1}}=K.$$

Conversely, if A is stable, then there exists W such that for each matrix $M \in A'$, $MW \subseteq W$. Let $\alpha=\sup \|z\|$, $z \in \partial W$ and $\beta=\inf \|z\|$, $z \in \partial W$. Then $\|Mz\|/\|z\|<\alpha/\beta$ for all $M \in Z'$, $z \in C^n$. ■

Interestingly, Butcher [3] has given the following example of an unstable set A where each $M \in A'$ is stable. Let

$$A=\begin{pmatrix} e^{i\theta} & a_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where θ/π is irrational. We will show that the set $A \equiv \{A,B\}$ is unstable but that any matrix $M \in A'$ is stable. We first note that

$$A^2=\begin{pmatrix} e^{2i\theta} & (1+e^{i\theta})a_1 \\ 0 & 1 \end{pmatrix}.$$

Consider any $M \in A'$. Then

$$M=\begin{pmatrix} e^{i\psi} & \alpha \\ 0 & 1 \end{pmatrix}$$

for some α and ψ, and ψ/π is irrational. Thus M is stable. Next consider the sequence of matrices $A_k \in A'$ where $A_1=A$, and

$$A_k \equiv \begin{pmatrix} e^{i\theta_k} & a_k \\ 0 & 1 \end{pmatrix}$$

$$A_{k+1}=\begin{cases} A_k^2, & \text{if } \operatorname{Re} e^{2i\theta_k} \geqslant 0 \\ A_k^2 B, & \text{otherwise} \end{cases}.$$

Note that

$$A_k^2=\begin{pmatrix} e^{2i\theta_k} & (1+e^{i\theta_k})a_k \\ \cdot & \cdot \end{pmatrix}.$$

The definition of A_{k+1} ensures that $\operatorname{Re} e^{i\theta_{k+1}} \geqslant 0$. Thus

$$|a_{k+1}|=|1+e^{i\theta_k}|\,|a_k| = \sqrt{2+2\operatorname{Re} e^{i\theta_k}}\,|a_k| \geqslant \sqrt{2}\,|a_k|.$$

Since $\lim_{k\to\infty}|a_k|=\infty$, then $\{A,B\}$ is unstable.

Lemma 3 demonstrates that the instability of a set of matrices coincides with the occurrence of unbounded growth. However, this unbounded growth need not be due to the instability of a particular matrix of A', but may occur only through a nonrepeating application of matrices in A.

We now relate the concepts of Liapunov and matrix stability.

Theorem 1

A is stable if and only if there exists a norm, $|\ \ |$, such that for all $M \in A$, $z \in C^n$,

$$|Mz| \leqslant |z|.$$

The proof follows directly from the three lemmas and is omitted. ■

III. The Form of the Norm

In many cases, one might try to find, as did Willson, a quadratic Liapunov function to prove stability. Indeed, if A consists of a single stable matrix M, we know that there exists a positive definite Hermitian matrix M_H such that

$$w(z) \equiv z^* M_H z$$

is a Liapunov function for A. We previously noted that Willson [2] found quadratic Liapunov functions in his case $A=\{M_1,M_2\}$, where M_1 and M_2 are 2×2 matrices. If, however, matrix $M \in A$ has an eigenvalue on the unit circle, it is possible that A is stable but no quadratic or even differentiable Liapunov function exists. For deriving this result we use the following [4].

Definition

A *support plane* of a closed set S is a plane $\pi \equiv \{z \mid z^*c=b\}$ such that 1) there exists $z' \in S \cap \pi$ and 2) $z \in S$ implies $z^*c \leqslant b$. π is called a *support plane of S at z'*.

If S is a closed convex neighborhood of the origin, then there exists a support plane of S at z for all $z \in \partial S$. Further, if $w(z)$ is the function associated with S and w is differentiable, then the support plane at z' is unique and is given by

$$\pi=\{z \mid (\partial w/\partial z)^*(z-z')=0\}.$$

Lemma 4

Suppose A is a stable set of matrices. Assume there exists, for some $M \in A$, a single eigenvalue λ on the unit circle. Let the corresponding eigenvector be ξ, let W be the set of Lemma 1, and let ξ be normalized so that $\xi \in \partial W$. Then the plane π passing through ξ and parallel to the eigenspace of

Proof: Suppose π is not a support plane at ξ. Then there exists $z' \in W_I \cap \pi$ (where W_I denotes the interior of W). Since A is stable and λ is the only eigenvalue on the unit circle, then $|\lambda_i| < 1$ for the remaining eigenvalues of M, $i = 2, \cdots, n$. With these eigenvalues is associated an invariant space π_0 (i.e., $\pi - \xi$), and z' can be expressed uniquely as

$$z' = \xi + y$$

where $y \in \pi_0$. Consider the point $z = z' + \epsilon\xi$. Since $z' \in W_I$, then for some $\epsilon > 0$, $z \in W_I$. Now

$$M^k z = M^k y + (1+\epsilon)\lambda^k \xi$$

or

$$\lambda^{-k} M^k z = \lambda^{-k} M^k y + (1+\epsilon)\xi.$$

Since $\lambda^{-k} M^k y \in \pi_0$, then $\|\lambda^{-k} M^k y\| \to 0$ as $k \to \infty$. Hence,

$$\lambda^{-k} M^k z \to (1+\epsilon)\xi, \qquad \text{as } k \to \infty.$$

However, $(1+\epsilon)\xi \notin W$, because $\xi \in \partial W$ and W is convex. Since, by Lemma 1 and the fact that W is balanced, $\lambda^{-k} M^k z \in W$, hence it is impossible that $\lambda^{-k} M^k z \to (1+\epsilon)\xi \notin W$. Hence π must be a support plane of W at ξ. ■

Theorem 2

Suppose A is a stable set of matrices and there exists $M_i \in A$, $i = 1,2$, such that $M_i \xi = \lambda_i \xi$, $|\lambda_i| = 1$. Assume that λ_i is the single eigenvalue on the unit circle for M_i. Let η_i be such that $\eta_i^* M_i = \lambda_i \eta_i^*$; assume that η_1 and η_2 are independent, and each η_i is normalized so that $\eta_i^* \xi = 1$. Then any Liapunov function $w(z)$ that exists for A must be nondifferentiable.

Proof: The eigenspace π_i of M_i complementary to ξ is given by

$$\pi_i = \{ z \mid \exists z' \text{ s.t. } z = (I - \xi\eta_i^*) z' \}.$$

The two planes at ξ, $\pi_i + \xi$, $i = 1,2$, are support planes at ξ by Lemma 4, and are not equal because η_1 and η_2 are independent. Hence the support plane at ξ is not unique, and therefore $(\partial w)/(\partial z)(\xi)$ cannot exist (see comment preceding Lemma 4). ■

The situation described in Theorem 2 requires that a Liapunov function, which we know exists by Theorem 1, must possess a "corner" of the associated invariant set W at ξ. One candidate for the form of $w(z)$ might be

$$w(z) = \|Mz\|_\infty = \max_i \left\{ \left| \sum_j M_{ij} z_j \right| \right\}$$

where M is some matrix to be chosen. If M has the property that $M\xi = (1, 1, \cdots, 1)^T$, then the set

$$S = \{ z \mid w(z) \leq 1 \}$$

has a corner at ξ, thus allowing for several support planes at ξ. Another possibility for $w(z)$ is

$$w(z) = \|Mz\|_1 = \sum_i \left| \sum_j M_{ij} z_j \right|$$

however, in the next part of this paper, we *construct* w (or equivalently a norm), so there is no need to *guess* its form. The construction allows for corners; in fact, generally the constructed w will have many corners.

IV. A Constructive Theorem for Stability

There is no widely applicable theoretical procedure for establishing the form of the Liapunov function as described in Theorem 1. However, an iterative algorithm will now be presented which can determine the stability of a set of matrices in virtually all cases. We do not believe that any other extant approach is as simple, or as powerful as the one which follows.

Theorem 3 (Constructive Theorem)

Given a finite set $A = \{M_0, M_1, \cdots, M_{m-1}\}$ of m distinct $n \times n$ complex matrices. Let $W_0 \subset C^n$ be a bounded neighborhood of the origin. Define, for $k > 0$,

$$W_k \equiv \mathcal{H}\left[\bigcup_{i=0}^{\infty} M_{k'}^i W_{k-1} \right], \qquad \text{where } k' \equiv (k-1) \bmod m.$$

Then A is stable if and only if $W^* \equiv \cup_{i=0}^{\infty} W_i$ is bounded.[2]

Proof: Let W^* be bounded. Observe that $W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W^*$. We will show, by induction on p, that for any $z \in W^*$ and for any M of the form

$$M = M_{j_1}^{k_1} M_{j_2}^{k_2} \cdots M_{j_p}^{k_p} \in A'$$

(with $k_a \neq 0$, and $j_a \neq j_b$ if $|a-b| = 1$), $Mz \in W^*$.

Say $z \in W^*$; then $z \in W_{i_0}$ for some i_0 and hence $z \in W_{i_0+1}, W_{i_0+2}, \cdots, W_{i_0+m}$. Hence $M_{k'}^i z \in W^*$ for $k' = 0, 1, \cdots, m-1$. Thus $Mz \in W^*$ for $p = 1$. Assume that for $p = r$, $Mz \in W^*$. Let

$$M' = M_{j_1}^{k_1} M_{j_2}^{k_2} \cdots M_{j_r}^{k_r} M_{j_{r+1}}^{k_{r+1}}$$

with conditions as above. Then

$$M'z = \left(M_{j_1}^{k_1} M_{j_2}^{k_2} \cdots M_{j_r}^{k_r} \right)\left(M_{j_{r+1}}^{k_{r+1}} z \right)$$
$$= \left(M_{j_1}^{k_1} M_{j_2}^{k_2} \cdots M_{j_r}^{k_r} \right) z' \in W^*$$

by the induction hypothesis, since $z' = M_{j_{r+1}}^{k_{r+1}} \in W^*$. So $MW^* \subseteq W^*$ for all $M \in A'$. Since W^* is bounded, then by Lemma 1, A is stable.

Conversely, suppose A is stable. By Lemma 1, there exists a bounded neighborhood of the origin W such that, for each $M \in A'$, $MW \subseteq W$. Choose $\rho > 0$ so that $\rho W_0 \subseteq W$. If $\rho W_k \subseteq W$, then

$$\rho W_{k+1} = \rho \mathcal{H}\left[\bigcup_{i=0}^{\infty} M_{j_{k+1'}}^i W_k \right] = \mathcal{H}\left[\bigcup_{i=0}^{\infty} M_{j_{k+1'}}^i (\rho W_k) \right] \subseteq W.$$

Thus by induction

$$\rho W^* = \rho \bigcup_{i=0}^{\infty} W_i = \bigcup_{i=0}^{\infty} (\rho W_i) \subseteq W$$

and W^* must be bounded. ■

There is, alternatively, a simpler proof following from the definition of stability and the observation that $W^* = \mathcal{H}[\cup MW_0, M \in A']$ (see (9)). However, the above proof

[2] $\mathcal{H}$ denotes the *convex hull.*

elucidates the actual constructive process, suggesting a computational algorithm for determining stability. (For a detailed description, see Section XI). We further note that the theorem would have remained valid had W_k been defined simply by

$$W_k = M_{k'} W_{k-1}, \qquad k' = (k-1) \bmod m.$$

Practice, however, has suggested that the formulation formerly given involves fewer calculations.

The constructive theorem may be related directly to the norm of Theorem 1. If A is stable, Theorem 1 assures us of the existence of a norm $w(z)$ for A. Since W^* has the property of the neighborhood W of Lemma 1, we can therefore construct $w(z)$ by simply defining $w(z) \equiv \alpha$, where z is uniquely expressed as $z = \alpha z'$, $z' \in \partial W^*$ (see proof of Lemma 2).

Theorem 3 by itself is not totally constructive. It requires the computation of $M_{k'}^i W_{k-1}$, where W_{k-1} is a convex neighborhood of the origin. However if W_{k-1} is a polyhedral region, then it has a finite number of extreme points and the next two lemmas show that, in this case, the computation of $M_{k'}^i W_{k-1}$ is constructive.

Lemma 5

If z is an extreme point of W_k, then $z = M_{k'}^j u$, where u is an extreme point of W_{k-1}.

Proof: $W_k = \cup_{i=0}^{\infty} \mathcal{H}[M_{k'}^i W_{k-1}]$. Hence $E(W_k) \subseteq E(\cup_{i=0}^{\infty} M_{k'}^j W_{k-1})$. Suppose $z \in E(W_k)$. Then $z \in E(M_{k'}^j W_{k-1})$, for some j. But then $z = M_{k'}^j u$, for some $u \in E(W_{k-1})$. ■

Lemma 5 allows us to deal with W_{k-1} given as the convex hull of a finite number of points. Then each extreme point of W_k will be obtained only from an extreme point of W_{k-1}. Thus it is sufficient to compute with *only* the extreme points; each W_k is completely represented by its extreme points. If $E(W_k)$ is finite, then this part of the process is constructive.

In practice, we will generate extreme points for W_k and add these to the set of extreme points of W_{k-1}. At each step we have a candidate set of extreme points for W_k. The following lemma gives us a stopping criterion for this procedure.

Lemma 6

Let $z_i = M_{k'}^j u_i$ for some j and some $u_i \in E(W_{k-1})$. Then $\mathcal{H}\{z_1, \cdots, z_r\} = W_k$ (defined by Theorem 3) if and only if $M_{k'} z_i \in \mathcal{H}\{z_1, \cdots, z_r\}$, for $i = 1, \cdots, r$.

Proof: Let $U = \mathcal{H}\{z_1, \cdots, z_r\}$. If $M_{k'} z_i \in U$, then $M_{k'} U \subseteq U$. Thus $M_{k'}^j U \subseteq U$ and hence $W_k = \mathcal{H}\{\cup_{i=0}^{\infty} M_{k'}^i W_{k-1}\} \subseteq U$. Since $M_{k'} W_k \subseteq W_k$, if there exists a point $z \in U$ but $z \notin W_k$, there must be an extreme point of U not in W_k. However this contradicts the fact that the extreme points of U, being of the form $M_{k'}^j u_i$ where $u_i \in E(W_{k-1})$, must be in W_k. Thus $U = W_k$.

Conversely suppose $W_k = \mathcal{H}\{z_1, \cdots, z_r\}$. Since $M_{k'} W_k \subseteq W_k$, then $M_{k'} z_i \in W_k$. ■

Another possibly nonconstructive aspect to Theorem 3 is that the union of an infinite number of regions would have to be formed. The next section deals with some common situations in which the construction is shown to be finite.

V. Finiteness Criteria for the Stability Algorithm

Lemma 7

Let M be a matrix whose eigenvalues $\lambda_i(M)$ have magnitude $|\lambda_i(M)| < 1$; then there exists k such that $\cup_{i=0}^{\infty} M^i R = \cup_{i=0}^{k} M^i R$ for any bounded neighborhood of the origin R.

Proof: Let P transform M to Jordan form $J \equiv P^{-1}MP$. Then $M^k R = P(P^{-1}M^k P)(P^{-1}R) = PJ^k R'$, where $R' = P^{-1}R$. Let $v' \in R'$. Since $|\lambda_i(M)| < 1$, $\|J^k\| \to 0$ as $k \to \infty$. Therefore, there exists k such that $\|J^k v'\| < d \equiv \min \|w\|$, $w \notin R'$, for *all* $v' \in R'$. For such a k, $J^k R' \subseteq R'$ and hence $M^r R = PJ^r R' \subseteq R = PR'$ for $r \geq k$. ■

Thus if W_0 is a polyhedron, and each matrix in A satisfies the condition of Lemma 7, then each subsequent region W_k will be the convex hull of a finite number of points (a polyhedron) and can therefore be determined in a finite number of steps.

The next two lemmas provide sufficient conditions for the boundedness and unboundedness, respectively, of the region W^*.

Lemma 8

If there exists a k such that

$$\{\partial W_0 \cap \partial W_k\} = \phi$$

then W^* is unbounded.

Proof: Suppose W^* is bounded. Then A is stable and, by Lemma 1, there exists a convex bounded neighborhood of the origin W such that $MW \subseteq W$ for all $M \in A'$. Choose $\rho > 0$ so that $\{\partial W \cap \partial \rho W_0\} \neq \phi$ and $\rho W_0 \subseteq W$. Now for any matrix M and any set of points P,

$$M \mathcal{H}[P] = \mathcal{H}[MP]. \tag{9}$$

Hence,

$$W_k = \mathcal{H}\left[\bigcup_{i_{k-1}=0}^{\infty} \cdots \bigcup_{i_0=0}^{\infty} \left(M_k^{i_{k-1}} \cdots M_0^{i_0}\right)(\partial W_0)\right].$$

Then since $\rho(\partial W_0) \subset W$, $\rho W_k \subseteq W$ by the nature of W. But then $\rho W_0 \subseteq \rho W_k \subseteq W$ and $\{\partial W \cap \partial \rho W_0\} \neq \phi$ imply $\{\partial \rho W_k \cap \partial \rho W_0\} \neq \phi$, a contradiction. ■

Lemma 9

If for some $k \geq m$, $W_k = W_{k-m}$, then W^* is bounded.

The proof is obvious. ■

VI. Example: 2×2 Real Matrices

We apply this new method to find the true regions of stability for Willson's set $A = \{M_1, M_2\}$, where

$$M_1 = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} ra & 1 \\ b & 0 \end{pmatrix}, \qquad |r| \leq 1, \ a, b \in \mathbb{R}.$$

The algorithm, based on the criteria mentioned in the previous section, was computer-implemented in APL.

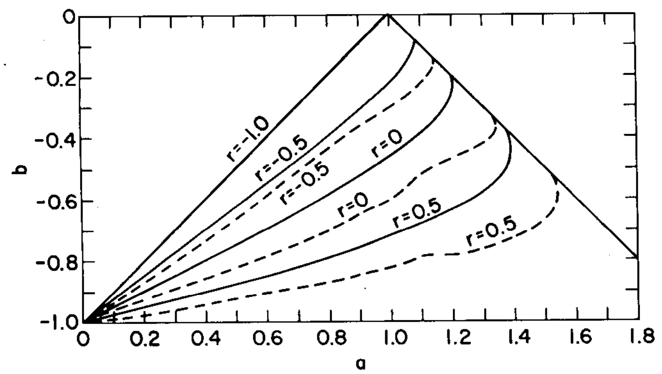

Fig. 1. Curves showing boundary of stability region in (a,b)-plane for four values of r. Solid curves were obtained using optimal quadratic Liapunov functions [2]. Dashed curves show true boundary. Stability region contains (0,0), is symmetric about $a=0$, and lies in triangle whose vertices are $\{(0,1),(2,-1),(-2,-1)\}$.

The triangle with vertices at (0,1), $(2,-1)$, $(-2,-1)$ is the region inside which the eigenvalues of both M_1 and M_2 have magnitudes less than 1. On the border, at least one eigenvalue of M_1 has magnitude 1. Thus all points (a,b) outside the triangle are unstable.[3] As one might expect from the proof of Lemma 7, the number of computations needed to determine each region W_k increases as the point (a,b) approaches the border of the triangle. Interestingly, although we only showed the sufficiency of Lemmas 8 and 9, we did not come across a case inside the triangle in which the algorithm was not terminated by one of these two conditions. However, the amount of computer time increased significantly as we approached a point (a,b) on the boundary of the stability region for S_r (the set of all points (a,b) for fixed r that give rise to a stable set A).

In Fig. 1, several stability regions are shown for various values of r. The stability regions obtained by Willson using an optimal quadratic Liapunov function are shown by solid lines. Any point (a,b) inside the stability region for a particular value of r gives rise to a stable set of matrices $\{A,B\}$. Only the lower right half of the triangle is shown since the stability region is symmetric about the y-axis, and the region of the triangle with $b \geqslant 0$ is stable for any value of r, $-1 \leqslant r \leqslant 1$.

To obtain the dotted curves in Fig. 1, the constructive algorithm was used to determine, for a particular point (a,b), if the pair of matrices was stable or unstable. The triangle was subdivided into rectangles of size $\Delta a = 0.0125$, $\Delta b = 0.006$, and a two-dimensional binary search algorithm was used to determine the rectangles in which boundary points of the stability region were located. By comparing the solid curve with the dotted curve (for the same value of r), we can thus see how much the quadratic Liapunov function approach falls short of obtaining the true stability region. Another interesting point about Fig. 1 is that the true stability region is not convex.

[3]We shall use (a,b) and *the set of matrices defined by* (a,b) interchangeably.

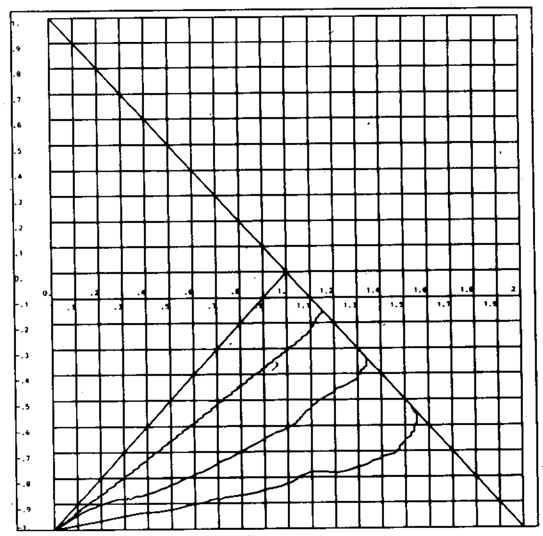

Fig. 2. Computer drawn curves of the boundary of the true stability region for different values of r.

Fig. 2 shows the computer drawn curves of the boundary of the true stability region for different values of r.

VII. Realizing the Complex Case

Willson's matrices happen to be real. In general, we would like to have an easy way of dealing with complex matrices; we find from experience that a method which does not involve complex space is to be preferred to one which does. With this in mind we have the following theorem.

Theorem 4

Let M be a complex matrix $M = M_r + iM_i$, where M_r, M_i are real $n \times n$ matrices. Define

$$\sigma M = \begin{pmatrix} M_r & -M_i \\ M_i & M_r \end{pmatrix}.$$

Then if A is a finite set of complex matrices, A is stable if and only if σA is stable.

Proof: Suppose A is stable. Then by Lemma 1, there exists a neighborhood of the origin W such that $MW \subseteq W$ for all $M \in A'$. Define the mapping $\sigma' : C^n \to \mathbb{R}^{2n}$ by

$$\sigma'(\nu) = \begin{pmatrix} \nu_r \\ \nu_i \end{pmatrix}$$

where $\nu_r = \mathrm{Re}\ (\nu)$, $\nu_i = \mathrm{Im}\ (\nu)$. Let $W' = \sigma' W$, and let $\nu' \in W'$. Then there exists $\nu \in W$ such that $\sigma' \nu = \nu'$. Now,

$$M\nu = (M_r + M_i i)(\nu_r + \nu_i i) = (M_r \nu_r - M_i \nu_i) + (M_i \nu_r + M_r \nu_i) i.$$

Thus

$$(\sigma M)\nu' = (\sigma M)(\sigma' \nu) = \begin{pmatrix} M_r & -M_i \\ M_i & M_r \end{pmatrix} \begin{pmatrix} \nu_r \\ \nu_i \end{pmatrix}$$

$$= \begin{pmatrix} M_r \nu_r - M_i \nu_i \\ M_i \nu_r + M_r \nu_i \end{pmatrix} = \sigma'(M\nu) \subseteq W'$$

since $Mv \in W$. Hence W' is a neighborhood of the origin with the property that $(\sigma M)W' \subseteq W'$ for all $M \in A'$; therefore, by Lemma 1, σA is stable.

The argument is completely reversible. ■

VIII. Stability of an Infinite Set of Matrices

The constructive method was seemingly restricted to a finite set of matrices A. The following theorem shows that it applies to infinite sets as well.

Theorem 5

Let A be a set of matrices, and let $E(A)$ be the set of extreme matrices of A. Then $\mathcal{H}A$ is stable if and only if $E(A)$ is stable.

Proof: Since $E(A) \subset A$, the assertion—if A is stable then $E(A)$ is stable—is obvious. Assume $\mathcal{H}A$ is unstable, and suppose $E(A)$ is stable. Then there exists $K > 0$ such that, for all $M \in E(A)'$, we have $\|Mx\| \leqslant K\|x\|$ for all $x \in \mathbb{R}^n$. Since $\mathcal{H}A$ is unstable, then by Lemma 3, there exists $L \in (\mathcal{H}A)'$ and $x \in \mathbb{R}^n$ such that $\|Lx\| > K\|x\|$. Now L is a finite product of matrices in $\mathcal{H}A$, say

$$L = M_1 M_2 \cdots M_r, \qquad M_i \in \mathcal{H}A.$$

Since $\mathcal{H}A$ is convex and $E(A)$ its extreme points, then each M_i can be represented as

$$M_i = \Sigma \lambda_{ij} S_j$$

where $S_j \in E(A)$, $\lambda_{ij} \geqslant 0$, $\Sigma_{j=1}^{m} \lambda_{ij} = 1$. Multiplying this out, we obtain

$$L = \sum_{i=1}^{p} \mu_i S_i'$$

where $\mu_i \geqslant 0$, $\Sigma \mu_i = 1$, and $S_i' \in E(A)'$. Now

$$\|Lx\| = \|\Sigma \mu_i S_i' x\| \leqslant \Sigma \| \mu_i S_i' x\| = \Sigma \| \mu_i K x\| = \|Kx\|$$

since each $S_i' \in E(A)'$. Hence, $\|Lx\| \leqslant K\|x\|$, a contradiction. Thus $E(A)$ is unstable. ■

Since $A \subseteq \mathcal{H}A$, a set of matrices is stable if and only if its extreme points are stable. The convex hull of a set of matrices can be approximated to any arbitrary degree by a polytope containing it; since the set of extreme points of a polytope is finite, the stability of a set can be determined through a polyhedral approximation of that set, conjoined with use of the above theorem. (Unless, of course, the set happens to be *maximally stable.*)

IX. Maximal Stability Regions and Matrix Norms

Knowing the stability of a set of matrices evokes a natural question—can the set be enlarged, and remain stable? We will see that the concept of the stability of a set of matrices is equivalent to the existence of a matrix norm, and maximally stable sets of matrices are equivalent to the existence of matrix norms induced by vector norms.

Definition

A *matrix norm* is a norm defined on matrices with the additional property that $\|AB\| \leqslant \|A\| \, \|B\|$.

Definition

An *induced matrix norm* is one defined by $\|M\| \equiv \sup (|Mx|/|x|)$, where $| \ |$ is any vector norm. This is also called the *bound* for M with respect to $| \ |$ (see Householder [5]).

Theorem 6

Let $\| \ \|$ be the matrix norm induced by some vector norm $| \ |$. Then the set of matrices

$$L_k \equiv \{M \mid \|M\| \leqslant k\}$$

is stable if and only if $k \leqslant 1$.

Proof: Suppose $k=1$. Note that $L_1 = L_1'$; hence L_1' is bounded and by Lemma 3, therefore, L_1 is stable. Since $kI \in L_k$ and kI is unstable if $k > 1$, then L_k is unstable for $k > 1$. ■

Theorem 7

Let S be a convex, balanced set of matrices containing a neighborhood of 0 and such that $S = S'$. The S is stable if and only if there exists a matrix norm $\| \ \|$ such that $S = \{M \mid \|M\| \leqslant 1\}$.

Proof: Assume S is stable. Then S' is bounded and $\|A\| \equiv \inf \{\alpha \mid A \in \alpha S'\}$ defines a norm since S' is balanced, convex, and contains a neighborhood of 0. Clearly $\|A\| \leqslant 1$ if and only if $A \in S'$. Let $A, B \in S'$. Then $A/\|A\| \in S'$ and $B/\|B\| \in S'$; hence $(AB/\|A\|\,\|B\|) \in S'$ which implies that $\|AB\| \leqslant \|A\| \, \|B\|$. Thus $\| \ \|$ is a matrix norm.

Suppose that $\| \ \|$ is a matrix norm such that $S' = \{M \mid \|M\| \leqslant 1\}$. Then S' is bounded and by Lemma 3 therefore S is stable. ■

Definition

A set of matrices S is *maximally stable* if there exists no stable set strictly containing S.

Theorem 8

$\| \ \|$ is an induced matrix norm if and only if the set

$$S = \{M \mid \|M\| \leqslant 1\}$$

is maximally stable.

The proof requires the following theorem of Schneider and Strang [6].

If $\| \ \|_a$ and $\| \ \|_b$ are two induced matrix norms such that $\|M\|_a \leqslant \|M\|_b$ for all matrices M, then in fact $\| \ \|_a = \| \ \|_b$.

Proof of Theorem 8: Suppose S is maximally stable. Define $| \ |_y$ by

$$|x|_y \equiv \|xy^T\|$$

where $y \neq 0$. Let $\| \ \|_y$ be the matrix norm induced by

Further if $M \in S$, then

$$\|M\|_y \equiv \max_x \{|Mx|_y/|x|_y\}$$

$$= \max_x \{\|Mxy^T\|/\|xy^T\|\} \leqslant \|M\| \leqslant 1$$

and hence $M \in S_1$ proving $S \subseteq S_1$. Since S is maximally stable, then $S = S_1$ and $\| \ \|_y = \| \ \|$, i.e., $\| \ \|$ is an induced norm.

Suppose $\| \ \|$ is a matrix norm induced by $| \ |$. Then by Theorem 6, S is stable. Assume S is not maximally stable and let S_1 be a maximally stable set strictly containing S: $\{\partial S \cap \partial S_1\} = \phi$. Using the first half of this theorem there is a norm, say, $| \ |_a$, which induces a matrix norm $\| \ \|_a$ such that $S_1 = \{M \mid \|M\|_a \leqslant 1\}$. Since $\{\partial S \cap \partial S_1\} = \phi$, then $\| \ \| \leqslant \| \ \|_a$, and because these are both induced norms, we conclude that $\| \ \| = \| \ \|_a$, i.e., $S = S_1$. Since this is a contradiction, it must be that S is maximally stable. ■

Note that in the above proof, $\| \ \| = \| \ \|_y$ for any $y \neq 0$. This agrees with a theorem of Lyubich [7].

Suppose a vector norm $| \ |$ is defined by a set of r points $R = \{x_i\}$ in the following manner:

$$\{x \mid |x| \leqslant 1\} \equiv \mathcal{H}[R]. \tag{10}$$

This is always possible provided $0 \in (\mathcal{H}[R])_I$. Let $\| \ \|$ be the matrix norm induced by $| \ |$ and suppose we have a set of matrices A such that

$$A \subset L_1 \equiv \{M \mid \|M\| \leqslant 1\}. \tag{11}$$

A can be extended to include more of L_1 using the following construction.

Choose $B \in \partial \mathcal{H}(A)$ and let C be any matrix strictly contained in $\mathcal{H}(A)$. Then the r linear constraints (one for each x_i in R),

maximize μ_i such that

$$(C + \mu_i(B - C))x_i = \sum_{j=1}^{r} \lambda_j x_j, \qquad \lambda_j \geqslant 0, \quad \Sigma \lambda_j = 1$$

will locate the matrix $B_C \equiv C + \mu_{\min}(B - C)$, where $\mu_{\min} = \min\{\mu_i\}$, on the "line" joining B and C which satisfies $\|B_C\| = 1$, that is, $B_C \in \partial L_1$.

Thus, for a particular vector norm $| \ |$ defined by (or arbitrarily closely by) a finite set of points R (10), we can determine L_1 arbitrarily closely by choosing an initial set A (of dimension n^2) satisfying (11), and extending it as many times as desired by, say, fixing C and (strategically) varying B over $\partial \mathcal{H}(A)$ without repetition.

Note that Theorem 8 implies that the set of matrices L_1 of Theorem 6 cannot be enlarged even by a single matrix and remain stable, i.e., for any $M \notin L_1$, $L_1 \cup \{M\}$ is unstable. Alternately, given a matrix norm, the linear program above can be used to determine if the norm is an induced matrix norm; if the unit norm body of matrices can be enlarged then the matrix norm is not an induced matrix norm.

X. Application to Nonlinear Differential Equations

Let $dx/dt = f(x)$ and assume (without loss of generality) that the origin is an equilibrium point, i.e., $f(0) = 0$.

Theorem 9

Suppose $f(x)$ is uniformly Lipschitz in E_N and

$$x_{n+1} = x_n + h_n f(x_n) \tag{12}$$

is stable uniformly for all sequences $\{h_n\}$ such that $0 \leqslant h_n \leqslant h'$. Then

$$dx/dt = f(x) \tag{13}$$

is stable.

Proof: Choose a neighborhood U of the origin and choose U' such that $U \subset U'$ and

$$\min \|x - y\| = \delta > 0, \qquad x \in U, \quad y \notin U'.$$

Assume (13) is not stable. Then for every neighborhood V of the origin there exists $x' \in V$ and t' such that $x(t') \notin U'$. Let V be the neighborhood such that $x_0 \in V$ implies $x_n \in U$. This exists independent of the sequence $\{h_i\}$, $0 \leqslant h_i \leqslant h'$ of (12), since (12) is assumed stable uniformly for all such sequences. Let $x' \in V$ and t' be chosen so that $x(0) = x'$ and $x(t') \notin U'$. t' and x' exist because we assumed (13) was not stable. Theorem 3.5 of Henrici [8] states that for fixed step $h_i = h$, the solution of (12), $\{x_{n,h}\}$ with $x_{0,h} = x'$ converges to the solution of (13), $\{x(t_n)\}$, with $x(0) = x'$, i.e.,

$$\sup \|x_{n,h} - x(t_n)\| \to 0 \text{ as } h \to 0, \text{ and } n \leqslant t'/h.$$

However, $\|x(t') - x_{t'/h,h}\| \geqslant \delta$ for all h; hence we have a contradiction and conclude that $dx/dt = f(x)$ must be stable. ■

Theorem 9 can be used with the constructive method to prove the stability of nonlinear systems of differential equations. We first express (12) in an equivalent matrix form and prove the stability of the corresponding infinite set of matrices,

$$A \equiv \{I + hM, 0 \leqslant h \leqslant h', M \in S\}$$

using the methods of this paper. S is a set of matrices with the property that, for each x, there exists $M \in S$ such that $Mx = f(x)$. Set S is not unique, but should preferably be chosen to be minimal. (For example, we may let S be the set of Jacobians, $J(x)$.) If A is a stable set of matrices then (12) is stable uniformly for all sequences $\{h_n\}$ and hence (13) is a stable system of differential equations.

We note that

$$\lim_{h \to 0} \frac{(\|I + hM\| - 1)}{h}$$

is the logarithmic norm of M (see [9]).

Definition

A function $w(x)$ is a *Liapunov function* for the *system* $dx/dt = f(x)$ if for any solution $x(t)$,

$$w(x(t_2)) \leqslant w(x(t_1)), \qquad \text{for all } t_2 \geqslant t_1.$$

Theorem 10

Let $S=\{M|\forall x\in\mathbb{R}^n$, there exists M s.t. $f(x)=Mx\}$. Suppose W is a convex, bounded neighborhood of the origin such that $AW\subseteq W$ for all $A\in A\equiv\{I+h'M|M\in S\}$. Then $w(x)\equiv|x|_w$ defined by Lemma 2 is a Liapunov function for the system $dx/dt=f(x)$.

Proof: Define $A_0\equiv\{I+hM|0\leqslant h\leqslant h', M\in S\}$. Since $(I+h'M)w\in W$ for $w\in W$, therefore, $(I+hM)w\in W$, $0\leqslant h\leqslant h'$, by the convexity of W. Hence, $BW\subseteq W$, for all $B\in A_0$. Suppose $x(t_1)\in\partial W$. Then for $t_2>t_1$, there exists a sequence of matrices $B_i\in A_0'$ such that

$$\lim_{i\to\infty}(B_ix(t_1))=x(t_2).$$

Since $B_ix(t_1)\in W$, then $x(t_2)\in\overline{W}$. Hence, $w(x(t_2))\leqslant w(x(t_1))$ for $t_2>t_1$. ■

XI. The Algorithm

The final form of the computational algorithm is as follows.

Given a SET of m $n\times n$ real or complex matrices $\{M_i\}$ with eigenvalues $\lambda_j(M_i)$ of magnitude $|\lambda_j(M_i)|<1$.

I. Initialization

- a) If the SET is complex, let SET $=\sigma$SET and let $n=2n$ (Theorem 4).
- b) Form the vertex set $E(W_0)=\{x||x_i|=1$, where $x=(x_1,x_2,\cdots,x_n)\}$. (*W_0 is an n-dimensional hypercube*).
- c) Set $k=0$, $i=1$.

II. Formation of the new vertex set $E(W_{k+1})$ (Lemma 5).

- a) Set $j=1$; set TEMP $=E(W_k)$; set $V=E(W_k)\cap E(W_{\max(k-m,0)})$; set FLAG $=0$.
- b) Exit *stable* if $V=$ TEMP and $k\geqslant m$ (Lemma 9).
- c) Set POINT $=$ point j of $E(W_k)$.
- d) Go to g) if POINT $\in E(W_0)$.
- e) Use a linear program to decide whether POINT $\in\mathcal{K}$[TEMP $-$ POINT]. If it isn't, go to h).
- f) Drop POINT from TEMP; go to n).
- g) Use a linear program to decide whether POINT is strictly contained in $\mathcal{K}$[TEMP]; if so, go to f).
- h) Set FLAG $=0$. (FLAG $=1$ iff a NEWPT has been added to TEMP.)
- i) If POINT $\in V$, go to n). (See Remark 4 below.)
- j) Set NEWPT $=$ POINT.
- k) Set NEWPT $=M_i$(NEWPT); set FLAG $=0$.
- l) Use a linear program to decide whether NEWPT $\in\mathcal{K}$[TEMP]. If so, go to n).
- m) Otherwise, add NEWPT to TEMP; set FLAG $=1$.
- n) Exit *unstable* if $\{E(W_0)\cap$ TEMP$\}=\phi$ (Lemma 8).
- o) If FLAG $=1$, go to k).
- p) Set $j=j+1$. Go to c) if $j\leqslant$ number of points in $E(W_k)$.
- q) Set $i=(i+1)$ mod m; set $k=k+1$. Go to a).

Remarks:

1) Each pass through (II) takes a finite number of steps (Lemma 7).

2) Although it is difficult to prove the *correctness* of an algorithm, we point out that, on the one hand, the idea behind the algorithm is taken in a straightforward manner from the ideas presented previously; on the other, the realization of the program is centered about steps IIe), g), and l)—that is, the vast majority of actual computations takes place in a linear program. The validity of the latter should thus strongly support the results of the former.

3) We do not yet have a general guarantee that the algorithm will terminate in a finite number of steps. We only know that if $|\lambda(M)|<1$ then the step of constructing W_k from W_{k-1} will be finite. The algorithm can be applied to a set of matrices that contain matrices M with $|\lambda(M)|=1$. This has been done successfully, as in the example of Section XIII.

4) In step i of the algorithm, we stop applying powers of M_i to POINT whenever $(M_i)^k$(POINT) $\in V$ for some k. After W_{k+1} is completely formed, this guarantees that $M_ix\in W_{k+1}$ for any $x\in W_{k+1}$, since this holds for the extreme points of W_{k+1} (Lemma 4).

5) The linear program to determine if $x\in\mathcal{K}\{x_i\}$ is

$$\text{minimize } 0 \text{ subject to}$$
$$x=\Sigma\lambda_ix_i \qquad \lambda_i\geqslant 0 \qquad \Sigma\lambda_i=1.$$

It only uses the feasible solution part of the linear program.

XII. Example: Systems of Differential Equations

Theorem 11

Consider the system

$$\dot{x}=Ax+By$$
$$\dot{y}=Cx+f(y) \tag{14}$$

where the eigenvalues of A, $\lambda(A)$, satisfy Re $\lambda(A)<0$. *Assume that $C(A+i\omega I)^{-1}B$ is symmetric for all ω real, and $\partial f/\partial y$ is symmetric. Let S be the set of all matrices $S(y)$ where*

$$S(y)=\begin{pmatrix} A & B \\ C & D(y)\end{pmatrix}$$

and

$$D(y)=\frac{\partial f(y)}{\partial y} \quad \text{or} \quad D(y)y=f(y).$$

If the set

$$\{(I+hS), 0\leqslant h\leqslant h', S\in S\} \tag{15}$$

is stable, then every solution of (14) *decays to an equilibrium point as $t\to\infty$.*

Proof: In [10] it was proved that under the conditions that 1) Re $\lambda(A)<0$, 2) $C(A+i\omega I)^{-1}B$ and $\partial f/\partial y$ symmetric, and 3) the matrices $S(y)$, using $D(y)=\partial f/\partial y$, satisfy Re $\lambda(S(y))<0$, then each *bounded* solution of (14) decays to an equilibrium point. Therefore, we need only show that all solutions of (14) are bounded, and that condition stable, we conclude that all solutions of (14) are bounded. We note that $\{I+hS,\ S\in \boldsymbol{S}\}$ is stable. The eigenvalues of $I+hS$ are $1+h\lambda(S)$ and hence $|1+h\lambda(S)|>1$ if $\operatorname{Re}\lambda(S)\geqslant 0$, $\lambda(S)\neq 0$. This contradicts the fact that $\{I+hS\}$ is stable. Hence $\operatorname{Re}\lambda(S)<1$ if $\lambda\neq 0$. ■

Example: Consider the equation

$$\begin{aligned}\dot{x}&=-x-y\\ \dot{y}&=x-f(y)\end{aligned}\tag{16}$$

where x, y, and f are scalar, and $f(0)=0$. Then the conditions of Theorem 11 hold. Let $f(y)$ be such that

$$\alpha_1\leqslant\frac{\partial f}{\partial y}\leqslant\alpha_2.$$

Then it follows since $f(0)=0$, that

$$\alpha_1\leqslant\frac{f(y)}{y}\leqslant\alpha_2.$$

We have a convex set of matrices $\boldsymbol{S}$ to consider

$$\boldsymbol{S}=\left\{I+h\begin{pmatrix}-1 & -1\\ 1 & -\alpha\end{pmatrix}\middle|\,\alpha_1\leqslant\alpha\leqslant\alpha_2,\ 0\leqslant h\leqslant h'\right\}.$$

The extreme matrices of $\boldsymbol{S}$ are

$$B(\alpha_1,\alpha_2,h')\equiv\left\{I+h'\begin{pmatrix}-1 & -1\\ 1 & -\alpha_1\end{pmatrix},\ I+h'\begin{pmatrix}-1 & -1\\ 1 & -\alpha_2\end{pmatrix},\ I\right\}.$$

Using our constructive procedure, we compute that $B(-0.78,0.8,0.1)$ is stable but $B(-0.79,0.8,0.1)$ is unstable. Thus the system (16) is asymptotically stable for all $f(y)$ such that

$$-0.78\leqslant\frac{\partial f}{\partial y}\leqslant 0.8.$$

Note that boundedness of (16) is obtained under the weaker condition

$$-0.78\leqslant\frac{f(y)}{y}\leqslant 0.8.$$

XIII. Example: Numerical Methods for Differential Equations

The following set of matrices arises in examining the stability of the variable-step second-order backward differentation method applied to the test equation $\dot{x}=\lambda x$,

$$A(\mu,z)=\begin{pmatrix}z(1+\mu) & -z\mu\\ 1 & 0\end{pmatrix}$$

where $z=(1-\nu\lambda h)^{-1}$, $\nu=(r_1+1)/(2r_1+1)$, and $\mu=r_1^2\nu/(r_1+1)$. r_1 is the ratio of the current step size, h_n, to the previous step size, h_{n-1} (see [1]). The test equation is stable if $\operatorname{Re}\lambda\leqslant 0$. We note that

$$A=\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1+\mu & -\mu\\ 1 & 0\end{pmatrix}$$

and use the following lemma.

Lemma 10

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are convex sets of matrices, then $\boldsymbol{AB}\equiv\{AB\,|\,A\in\boldsymbol{A},\ B\in\boldsymbol{B}\}$ is a convex set, and $E(\boldsymbol{AB})\subseteq\{A_iB_j\,|\,A_i\in E(\boldsymbol{A}),\ B_j\in E(\boldsymbol{B})\}$.

Consequently $\boldsymbol{AB}$ is stable if $\{A_iB_j\}$ is stable. Also note that $(\boldsymbol{AB})'\subseteq(\boldsymbol{A}\cup\boldsymbol{B})'$ so that $\boldsymbol{AB}$ could be stable even if $\boldsymbol{A}\cup\boldsymbol{B}$ is unstable.

Proof: Choose $C=AB$. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are convex, then $A=\sum\lambda_iA_i$ where $A_i\in E(\boldsymbol{A})$ and $B=\sum\mu_jB_j$, $B_j\in E(\boldsymbol{B})$. Thus

$$C=\sum_{ij}\lambda_i\mu_jA_iB_j.\tag{17}$$

Since $\sum_{ij}\lambda_i\mu_j=1$ and $\lambda_i\mu_j\geqslant 0$, then C is a convex combination of matrices in $\boldsymbol{AB}$. Since (17) holds for any matrix $C\in\boldsymbol{AB}$, then $\boldsymbol{AB}$ is convex and $E(\boldsymbol{AB})\subseteq\{A_iB_j\,|\,A_i\in E(\boldsymbol{A}),\ B_j\in E(\boldsymbol{B})\}$. ■

Suppose we investigate the stability of the set $\boldsymbol{A}$ for $\lambda=-\rho e^{i\theta}$, where $0\leqslant\rho\leqslant\infty$, $-\alpha\leqslant\theta\leqslant\alpha$ and $0\leqslant r_1\leqslant\delta$. Then the implied restrictions on z and μ are

$$0\leqslant\mu\leqslant\frac{\delta^2}{2\delta+1}\equiv\delta_0$$

and

$$z=(1+\sigma e^{i\theta})^{-1}$$

where $0\leqslant\sigma<\infty$. The set

$$\{z\,|\,z=(1+\sigma e^{i\theta})^{-1},\ 0\leqslant\sigma<\infty,\ -\alpha\leqslant\theta\leqslant\alpha\}$$

is contained in the set $\mathcal{H}\{0,0.5(1\pm i\tan\alpha),1\}$. Let $z_0(\alpha)\equiv 0.5(1+i\tan\alpha)$,

$$A(\alpha)=\left\{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}z_0 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}\bar{z}_0 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$$

and

$$B(\delta_0)=\left\{\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}1+\delta_0 & -\delta_0\\ 1 & 0\end{pmatrix}\right\}.$$

Let

$$C_1=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$$C_2=\begin{pmatrix}z_0 & 0\\ 1 & 0\end{pmatrix}$$

$$C_3=\begin{pmatrix}z_0(1+\delta_0) & -z_0\delta_0\\ 1 & 0\end{pmatrix}$$

$$C_4=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$$

$$C_5=\begin{pmatrix}1+\delta_0 & -\delta_0\\ 1 & 0\end{pmatrix}.$$

Then by Lemma 10, $A(\alpha)B(\delta_0)$ is stable if $\{C_1,C_2,C_3,C_4,C_5,\bar{C}_2,\bar{C}_3\}$ is stable. We compute using the constructive procedure with $\alpha=\pi/4$, $\delta_0=0.42353$ (corresponds to $r_1\leqslant 1.2$) that this set is stable. (This was done by using the idea of Section VII to convert the complex case into the real case of dimension 4. The computed W^* had 76 vertex points.) Thus the second-order variable-step backward differentation method is stable for the test equation

$$\dot{x}=\lambda x\qquad\lambda=-\rho e^{i\theta}\qquad -\frac{\pi}{4}\leqslant\theta\leqslant\frac{\pi}{4}\qquad 0\leqslant\rho<\infty$$

for all sequences of time steps $\{h_i\}$ where $0 \leqslant h_i < \infty$ and $0 < h_i/h_{i-1} \leqslant 1.2$.

In this example, it was necessary to use Lemma 10, since we also computed that $A \cup B$ was unstable. The example also illustrates the use of a slightly larger set of matrices, but with a finite number of extreme points, in order to apply the constructive procedure.

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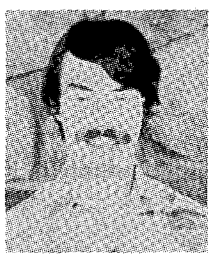

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