Chapter Five

Impedance Matching and Tuning

This chapter marks a turning point, in that we now begin to apply the theory and techniques of previous chapters to practical problems in microwave engineering. We start with the topic of *impedance matching*, which is often an important part of a larger design process for a microwave component or system. The basic idea of impedance matching is illustrated in Figure 5.1, which shows an impedance matching network placed between a load impedance and a transmission line. The matching network is ideally lossless, to avoid unnecessary loss of power, and is usually designed so that the impedance seen looking into the matching network is Z_0. Then reflections will be eliminated on the transmission line to the left of the matching network, although there will usually be multiple reflections between the matching network and the load. This procedure is sometimes referred to as *tuning*. Impedance matching or tuning is important for the following reasons:

- Maximum power is delivered when the load is matched to the line (assuming the generator is matched), and power loss in the feed line is minimized.
- Impedance matching sensitive receiver components (antenna, low-noise amplifier, etc.) may improve the signal-to-noise ratio of the system.
- Impedance matching in a power distribution network (such as an antenna array feed network) may reduce amplitude and phase errors.

As long as the load impedance, Z_L, has a positive real part, a matching network can always be found. Many choices are available, however, and we will discuss the design and performance of several types of practical matching networks. Factors that may be important in the selection of a particular matching network include the following:

- *Complexity*—As with most engineering solutions, the simplest design that satisfies the required specifications is generally preferable. A simpler matching network is usually cheaper, smaller, more reliable, and less lossy than a more complex design.
- *Bandwidth*—Any type of matching network can ideally give a perfect match (zero reflection) at a single frequency. In many applications, however, it is desirable to match a load over a band of frequencies. There are several ways of doing this, with, of course, a corresponding increase in complexity.

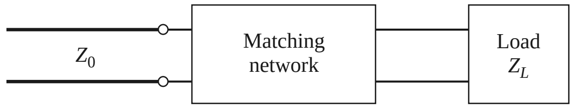


FIGURE 5.1 A lossless network matching an arbitrary load impedance to a transmission line.

- *Implementation*—Depending on the type of transmission line or waveguide being used, one type of matching network may be preferable to another. For example, tuning stubs are much easier to implement in waveguide than are multisection quarter-wave transformers.
- *Adjustability*—In some applications the matching network may require adjustment to match a variable load impedance. Some types of matching networks are more amenable than others in this regard.

5.1 MATCHING WITH LUMPED ELEMENTS (*L* NETWORKS)

Probably the simplest type of matching network is the *L-section*, which uses two reactive elements to match an arbitrary load impedance to a transmission line. There are two possible configurations for this network, as shown in Figure 5.2. If the normalized load impedance, $z_L = Z_L/Z_0$, is inside the $1 + jx$ circle on the Smith chart, then the circuit of Figure 5.2a should be used. If the normalized load impedance is outside the $1 + jx$ circle on the Smith chart, the circuit of Figure 5.2b should be used. The $1 + jx$ circle is the resistance circle on the impedance Smith chart for which $r = 1$.

In either of the configurations of Figure 5.2, the reactive elements may be either inductors or capacitors, depending on the load impedance. Thus, there are eight distinct possibilities for the matching circuit for various load impedances. If the frequency is low enough and/or the circuit size is small enough, actual lumped-element capacitors and inductors can be used. This may be feasible for frequencies up to about 1 GHz or so, although modern microwave integrated circuits may be small enough such that lumped elements can be used at higher frequencies as well. There is, however, a large range of frequencies and circuit sizes where lumped elements may not be realizable. This is a limitation of the L-section

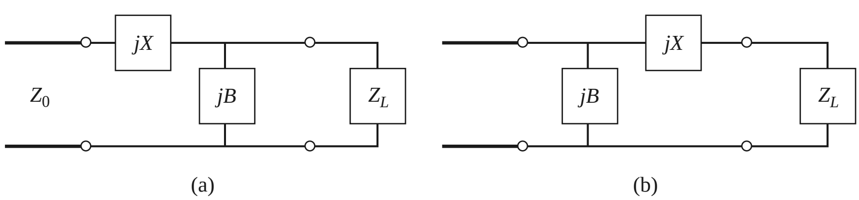


FIGURE 5.2 L-section matching networks. (a) Network for z_L inside the $1 + jx$ circle. (b) Network for z_L outside the $1 + jx$ circle.

matching technique. We will first derive analytic expressions for the matching network elements of the two cases in Figure 5.2, and then illustrate an alternative design procedure using the Smith chart.

Analytic Solutions

Although we will discuss a simple graphical solution using the Smith chart, it is also useful to have simple expressions for the L-section matching network components. These expressions can be used in a computer-aided design program for L-section matching, or when it is necessary to have more accuracy than the Smith chart can provide.

Consider first the circuit of Figure 5.2a, and let $Z_L = R_L + jX_L$. We stated that this circuit would be used when $z_L = Z_L/Z_0$ is inside the $1 + jx$ circle on the Smith chart, which implies that $R_L > Z_0$ for this case. The impedance seen looking into the matching network, followed by the load impedance, must be equal to Z_0 for an impedance-matched condition:

$$Z_0 = jX + \frac{1}{jB + 1/(R_L + jX_L)}. \tag{5.1}$$

Rearranging and separating into real and imaginary parts gives two equations for the two unknowns, X and B:

$$B(XR_L - X_L Z_0) = R_L - Z_0, \tag{5.2a}$$

$$X(1 - BX_L) = BZ_0 R_L - X_L. \tag{5.2b}$$

Solving (5.2a) for X and substituting into (5.2b) gives a quadratic equation for B. The solution is

$$B = \frac{X_L \pm \sqrt{R_L/Z_0}\sqrt{R_L^2 + X_L^2 - Z_0 R_L}}{R_L^2 + X_L^2}. \tag{5.3a}$$

Note that since $R_L > Z_0$, the argument of the second square root is always positive. Then the series reactance can be found as

$$X = \frac{1}{B} + \frac{X_L Z_0}{R_L} - \frac{Z_0}{BR_L}. \tag{5.3b}$$

Equation (5.3a) indicates that two solutions are possible for B and X. Both of these solutions are physically realizable since both positive and negative values of B and X are possible (positive X implies an inductor and negative X implies a capacitor, while positive B implies a capacitor and negative B implies an inductor). One solution, however, may result in significantly smaller values for the reactive components, or may be the preferred solution if the bandwidth of the match is better, or if the SWR on the line between the matching network and the load is smaller.

Next consider the circuit of Figure 5.2b. This circuit is used when z_L is outside the $1 + jx$ circle on the Smith chart, which implies that $R_L < Z_0$. The admittance seen looking into the matching network, followed by the load impedance, must be equal to $1/Z_0$ for an impedance-matched condition:

$$\frac{1}{Z_0} = jB + \frac{1}{R_L + j(X + X_L)}. \tag{5.4}$$

Rearranging and separating into real and imaginary parts gives two equations for the two unknowns, X and B:

$$BZ_0(X + X_L) = Z_0 - R_L, \tag{5.5a}$$

$$(X + X_L) = BZ_0R_L. \tag{5.5b}$$

Solving for X and B gives

$$X = \pm\sqrt{R_L(Z_0 - R_L)} - X_L, \tag{5.6a}$$

$$B = \pm\frac{\sqrt{(Z_0 - R_L)/R_L}}{Z_0}. \tag{5.6b}$$

Because $R_L < Z_0$, the arguments of the square roots are always positive. Again, note that two solutions are possible.

In order to match an arbitrary complex load to a line of characteristic impedance Z_0, the real part of the input impedance to the matching network must be Z_0, while the imaginary part must be zero. This implies that a general matching network must have at least two degrees of freedom; in the L-section matching circuit these two degrees of freedom are provided by the values of the two reactive components.

Smith Chart Solutions

Instead of the above formulas, the Smith chart can be used to quickly and accurately design L-section matching networks. The procedure is best illustrated by an example.

EXAMPLE 5.1 *L*-SECTION IMPEDANCE MATCHING

Design an L-section matching network to match a series RC load with an impedance $Z_L = 200 - j100\ \Omega$ to a 100 Ω line at a frequency of 500 MHz.

Solution

The normalized load impedance is $z_L = 2 - j1$, which is plotted on the Smith chart of Figure 5.3a. This point is inside the $1 + jx$ circle, so we use the matching circuit of Figure 5.2a. Because the first element from the load is a shunt susceptance, it makes sense to convert to admittance by drawing the SWR circle through the load, and a straight line from the load through the center of the chart, as shown in Figure 5.3a. After we add the shunt susceptance and convert back to impedance, we want to be on the $1 + jx$ circle so that we can add a series reactance to cancel jx and match the load. This means that the shunt susceptance must move us from y_L to the $1 + jx$ circle on the *admittance* Smith chart. Thus, we construct the rotated $1 + jx$ circle as shown in Figure 5.3a (center at $r = 0.333$). (A combined ZY chart may be convenient to use here, if it is not too confusing.) Then we see that adding a susceptance of $jb = j0.3$ will move us along a constant-conductance circle to $y = 0.4 + j0.5$ (this choice is the shortest distance from y_L to the shifted $1 + jx$ circle). Converting back to impedance leaves us at $z = 1 - j1.2$, indicating that a series reactance of $x = j1.2$ will bring us to the center of the chart. For comparison, the formulas (5.3a) and (5.3b) give the solution as $b = 0.29,\ x = 1.22$.

This matching circuit consists of a shunt capacitor and a series inductor, as shown in Figure 5.3b. For a matching frequency of 500 MHz, the capacitor has a value of

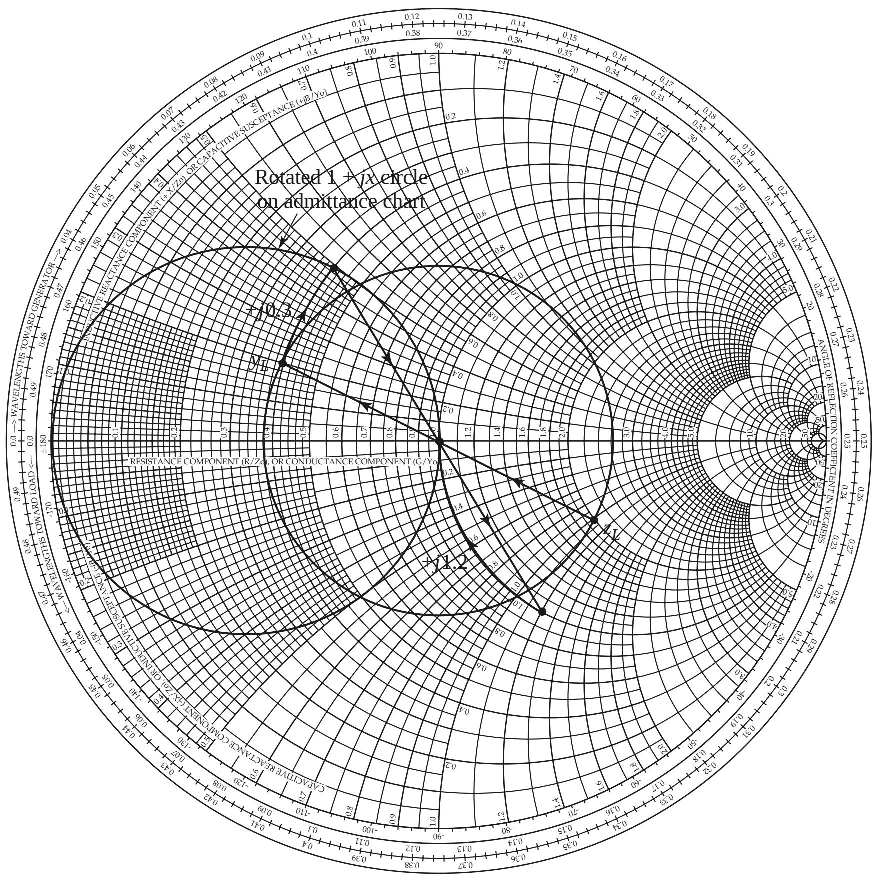
$$C = \frac{b}{2\pi f Z_0} = 0.92 \text{ pF},$$

and the inductor has a value of

$$L = \frac{x Z_0}{2\pi f} = 38.8 \text{ nH}.$$

It is also interesting to look at the second solution to this matching problem. If instead of adding a shunt susceptance of $b = 0.3$, we use a shunt susceptance of $b = -0.7$, we will move to a point on the lower half of the shifted $1 + jx$ circle, to $y = 0.4 - j0.5$. Then converting to impedance and adding a series reactance of $x = -1.2$ leads to a match as well. Formulas (5.3a) and (5.3b) give this solution as $b = -0.69$, $x = -1.22$. This matching circuit is also shown in Figure 5.3b, and is seen to have the positions of the inductor and capacitor reversed from the first matching network. At a frequency of $f = 500$ MHz, the capacitor has a value of

$$C = \frac{-1}{2\pi f x Z_0} = 2.61 \text{ pF},$$



(a)

FIGURE 5.3 Solution to Example 5.1. (a) Smith chart for the L-section matching networks.

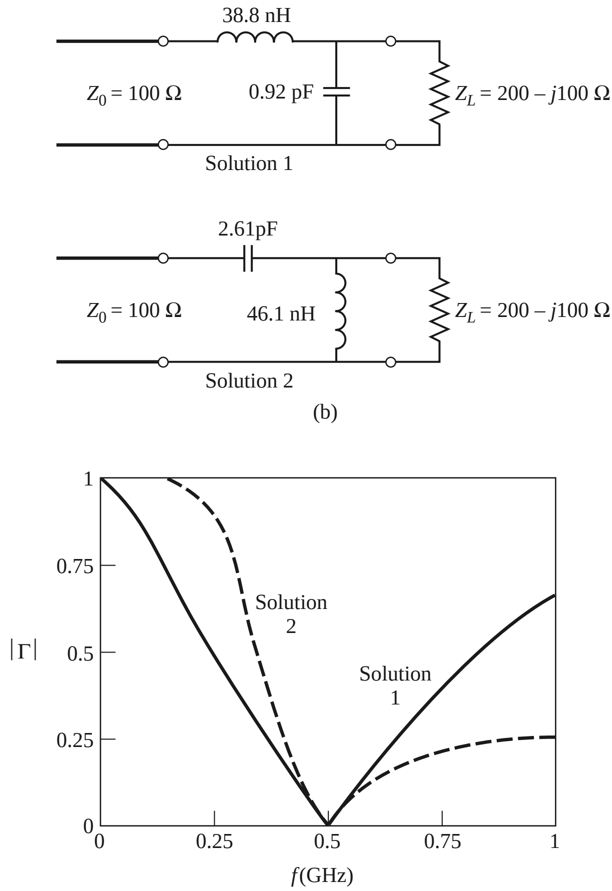


FIGURE 5.3 Continued. (b) The two possible L-section matching circuits. (c) Reflection coefficient magnitudes versus frequency for the matching circuits of (b).

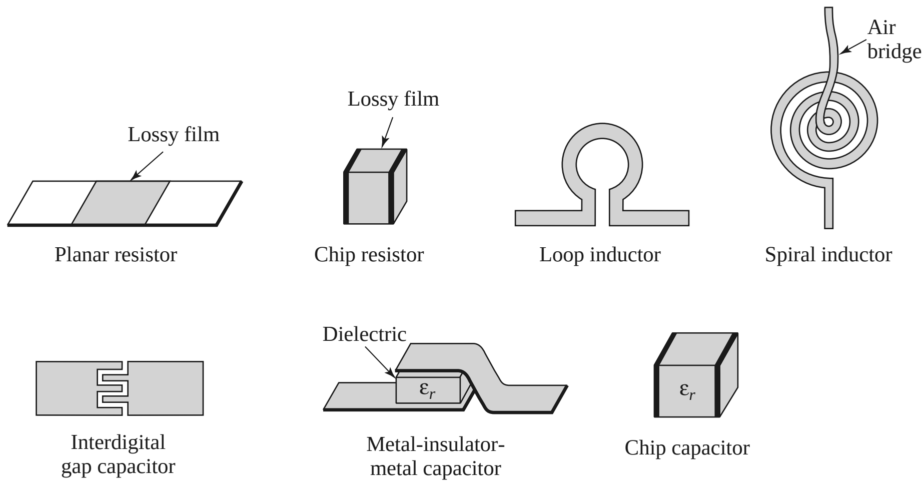
while the inductor has a value of

$$L = \frac{-Z_0}{2\pi f b} = 46.1 \text{ nH}.$$

Figure 5.3c shows the reflection coefficient magnitude versus frequency for these two matching networks, assuming that the load impedance of $Z_L = 200 - j100\ \Omega$ at 500 MHz consists of a 200 Ω resistor and a 3.18 pF capacitor in series. There is not a substantial difference in bandwidth for these two solutions. ■

POINT OF INTEREST: Lumped Elements for Microwave Integrated Circuits

Lumped R, L, and C elements can be practically realized at microwave frequencies if the length, ℓ, of the component is very small relative to the operating wavelength. Over a limited range of values, such components can be used in hybrid and monolithic microwave integrated circuits at frequencies up to 60 GHz, or higher, if the condition that $\ell < \lambda/10$ is satisfied. Usually, however, the characteristics of such an element are far from ideal, requiring that undesirable effects such as parasitic capacitance and/or inductance, spurious resonances, fringing fields, loss, and perturbations caused by a ground plane be incorporated in the design via a CAD model (see the Point of Interest concerning CAD).



Resistors are fabricated with thin films of lossy material such as nichrome, tantalum nitride, or doped semiconductor material. In monolithic circuits such films can be deposited or grown, whereas chip resistors made from a lossy film deposited on a ceramic chip can be bonded or soldered in a hybrid circuit. Low resistances are hard to obtain.

Small values of inductance can be realized with a short length or loop of transmission line, and larger values (up to about 10 nH) can be obtained with a spiral inductor, as shown in the following figures. Larger inductance values generally incur more loss and more shunt capacitance; this leads to a resonance that limits the maximum operating frequency.

Capacitors can be fabricated in several ways. A short transmission line stub can provide a shunt capacitance in the range of 0–0.1 pF. A single gap, or an interdigital set of gaps, in a transmission line can provide a series capacitance up to about 0.5 pF. Greater values (up to about 25 pF) can be obtained using a metal-insulator-metal sandwich in either monolithic or chip (hybrid) form.

5.2 SINGLE-STUB TUNING

Another popular matching technique uses a single open-circuited or short-circuited length of transmission line (a *stub*) connected either in parallel or in series with the transmission feed line at a certain distance from the load, as shown in Figure 5.4. Such a *single-stub tuning* circuit is often very convenient because the stub can be fabricated as part of the transmission line media of the circuit, and lumped elements are avoided. Shunt stubs are preferred for microstrip line or stripline, while series stubs are preferred for slotline or coplanar waveguide.

In single-stub tuning the two adjustable parameters are the distance, d, from the load to the stub position, and the value of susceptance or reactance provided by the stub. For the shunt-stub case, the basic idea is to select d so that the admittance, Y, seen looking into the line at distance d from the load is of the form $Y_0 + jB$. Then the stub susceptance is chosen as $-jB$, resulting in a matched condition. For the series-stub case, the distance d is selected so that the impedance, Z, seen looking into the line at a distance d from the load is of the form $Z_0 + jX$. Then the stub reactance is chosen as $-jX$, resulting in a matched condition.

As discussed in Chapter 2, the proper length of an open or shorted transmission line section can provide any desired value of reactance or susceptance. For a given susceptance or reactance, the difference in lengths of an open- or short-circuited stub is $\lambda/4$.

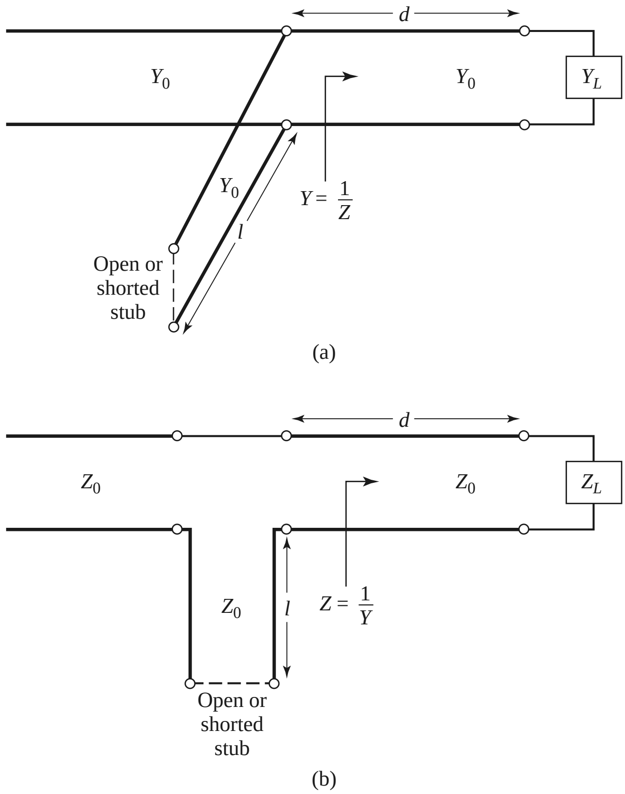


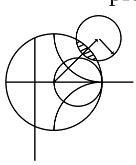
FIGURE 5.4 Single-stub tuning circuits. (a) Shunt stub. (b) Series stub.

For transmission line media such as microstrip or stripline, open-circuited stubs are easier to fabricate since a via hole through the substrate to the ground plane is not needed. For lines like coax or waveguide, however, short-circuited stubs are usually preferred because the cross-sectional area of such an open-circuited line may be large enough (electrically) to radiate, in which case the stub is no longer purely reactive.

We will discuss both Smith chart and analytic solutions for shunt- and series-stub tuning. The Smith chart solutions are fast, intuitive, and usually accurate enough in practice. The analytic expressions are more precise, and are useful for computer analysis.

Shunt Stubs

The single-stub shunt tuning circuit is shown in Figure 5.4a. We will first discuss an example illustrating the Smith chart solution and then derive formulas for d and ℓ.



EXAMPLE 5.2 SINGLE-STUB SHUNT TUNING

For a load impedance $Z_L = 60 - j80\ \Omega$, design two single-stub (short circuit) shunt tuning networks to match this load to a 50 Ω line. Assuming that the load is matched at 2 GHz and that the load consists of a resistor and capacitor in series, plot the reflection coefficient magnitude from 1 to 3 GHz for each solution.

Solution

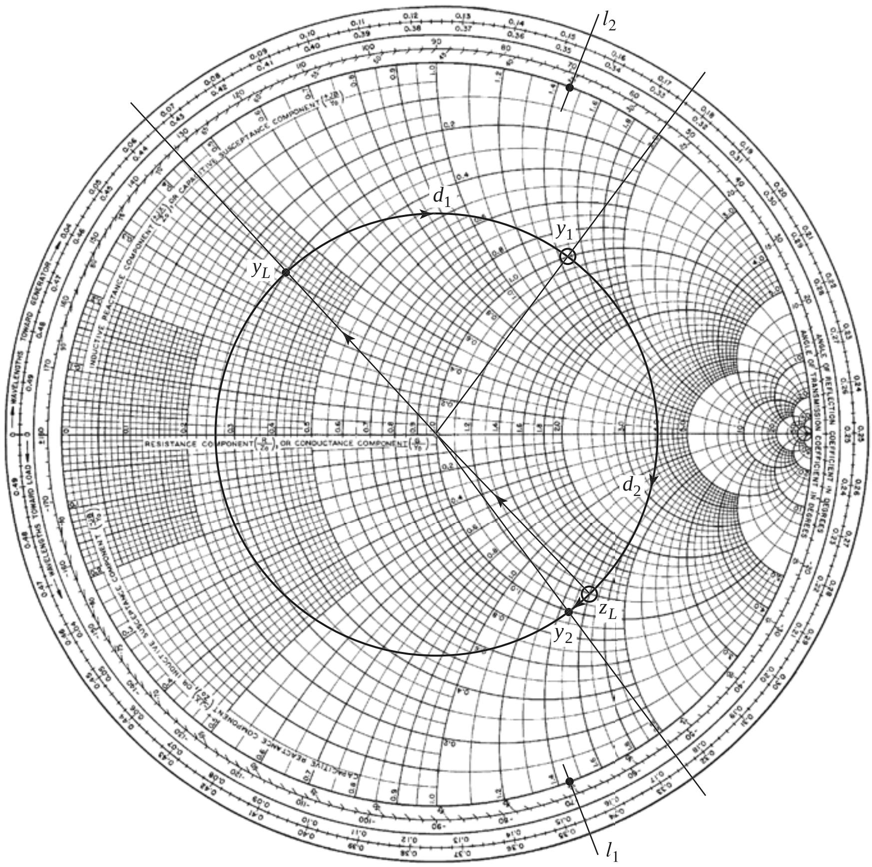
The first step is to plot the normalized load impedance $z_L = 1.2 - j1.6$, construct the appropriate SWR circle, and convert to the load admittance, y_L, as shown on the Smith chart in Figure 5.5a. For the remaining steps we consider the Smith chart as an admittance chart. Notice that the SWR circle intersects the $1 + jb$ circle at two points, denoted as y_1 and y_2 in Figure 5.5a. Thus the distance d from the load to the stub is given by either of these two intersections. Reading the WTG scale, we obtain

$$d_1 = 0.176 - 0.065 = 0.110\lambda,$$
$$d_2 = 0.325 - 0.065 = 0.260\lambda.$$

Actually, there is an infinite number of distances d around the SWR circle that intersect the $1 + jb$ circle. Usually it is desired to keep the matching stub as close as possible to the load to improve the bandwidth of the match and to reduce losses caused by a possibly large standing wave ratio on the line between the stub and the load.

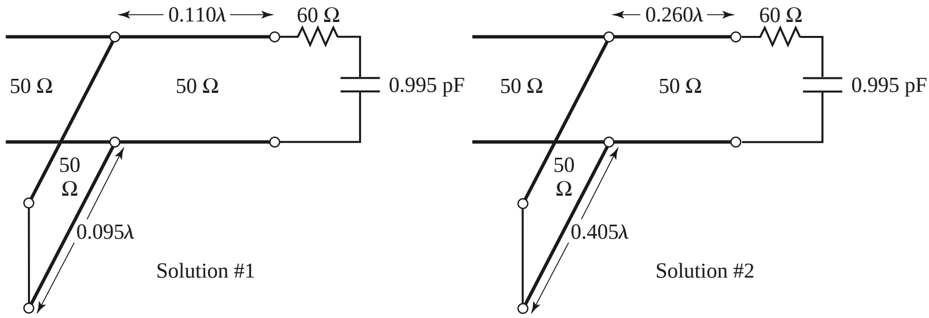
At the two intersection points, the normalized admittances are

$$y_1 = 1.00 + j1.47,$$
$$y_2 = 1.00 - j1.47.$$



(a)

FIGURE 5.5 Solution to Example 5.2. (a) Smith chart for the shunt-stub tuners.



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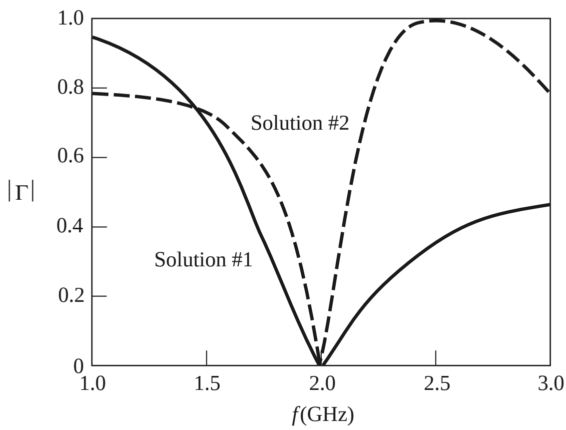


(c)

FIGURE 5.5 Continued. (b) The two shunt-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

Thus, the first tuning solution requires a stub with a susceptance of $-j1.47$. The length of a short-circuited stub that gives this susceptance can be found on the Smith chart by starting at $y = \infty$ (the short circuit) and moving along the outer edge of the chart ($g = 0$) toward the generator to the $-j1.47$ point. The stub length is then

$$\ell_1 = 0.095\lambda.$$

Similarly, the required short-circuit stub length for the second solution is

$$\ell_2 = 0.405\lambda.$$

This completes the two tuner designs.

To analyze the frequency dependence of these two designs, we need to know the load impedance as a function of frequency. The series-RC load impedance is $Z_L = 60 - j80\ \Omega$ at 2 GHz, so $R = 60\ \Omega$ and $C = 0.995$ pF. The two tuning circuits are shown in Figure 5.5b. Figure 5.5c shows the calculated reflection coefficient magnitudes for these two solutions. Observe that solution 1 has a significantly better bandwidth than solution 2; this is because both d and ℓ are shorter for solution 1, which reduces the frequency variation of the match. ■

To derive formulas for d and ℓ, let the load impedance be written as $Z_L = 1/Y_L = R_L + jX_L$. Then the impedance Z down a length d of line from the load is

$$Z = Z_0 \frac{(R_L + jX_L) + jZ_0 t}{Z_0 + j(R_L + jX_L)t}, \tag{5.7}$$

where $t = \tan \beta d$. The admittance at this point is

$$Y = G + jB = \frac{1}{Z},$$

where

$$G = \frac{R_L(1 + t^2)}{R_L^2 + (X_L + Z_0 t)^2}, \tag{5.8a}$$

$$B = \frac{R_L^2 t - (Z_0 - X_L t)(X_L + Z_0 t)}{Z_0 \left[R_L^2 + (X_L + Z_0 t)^2 \right]}. \tag{5.8b}$$

Now d (which implies t) is chosen so that $G = Y_0 = 1/Z_0$. From (5.8a), this results in a quadratic equation for t:

$$Z_0(R_L - Z_0)t^2 - 2X_L Z_0 t + \left(R_L Z_0 - R_L^2 - X_L^2\right) = 0.$$

Solving for t gives

$$t = \frac{X_L \pm \sqrt{R_L \left[(Z_0 - R_L)^2 + X_L^2 \right] / Z_0}}{R_L - Z_0} \quad \text{for } R_L \neq Z_0. \tag{5.9}$$

If $R_L = Z_0$, then $t = -X_L/2Z_0$. Thus, the two principal solutions for d are

$$\frac{d}{\lambda} = \begin{cases} \dfrac{1}{2\pi} \tan^{-1} t & \text{for } t \geq 0 \\ \dfrac{1}{2\pi}(\pi + \tan^{-1} t) & \text{for } t < 0. \end{cases} \tag{5.10}$$

To find the required stub lengths, first use t in (5.8b) to find the stub susceptance, $B_s = -B$. Then, for an open-circuited stub,

$$\frac{\ell_o}{\lambda} = \frac{1}{2\pi} \tan^{-1}\left(\frac{B_s}{Y_0}\right) = \frac{-1}{2\pi} \tan^{-1}\left(\frac{B}{Y_0}\right), \tag{5.11a}$$

and for a short-circuited stub,

$$\frac{\ell_s}{\lambda} = \frac{-1}{2\pi} \tan^{-1}\left(\frac{Y_0}{B_s}\right) = \frac{1}{2\pi} \tan^{-1}\left(\frac{Y_0}{B}\right). \tag{5.11b}$$

If the length given by (5.11a) or (5.11b) is negative, $\lambda/2$ can be added to give a positive result.

Series Stubs

The series-stub tuning circuit is shown in Figure 5.4b. We will illustrate the Smith chart solution by an example, and then derive expressions for d and ℓ.

EXAMPLE 5.3 SINGLE-STUB SERIES TUNING

Match a load impedance of $Z_L = 100 + j80$ to a 50 Ω line using a single series open-circuit stub. Assuming that the load is matched at 2 GHz and that the load consists of a resistor and inductor in series, plot the reflection coefficient magnitude from 1 to 3 GHz.

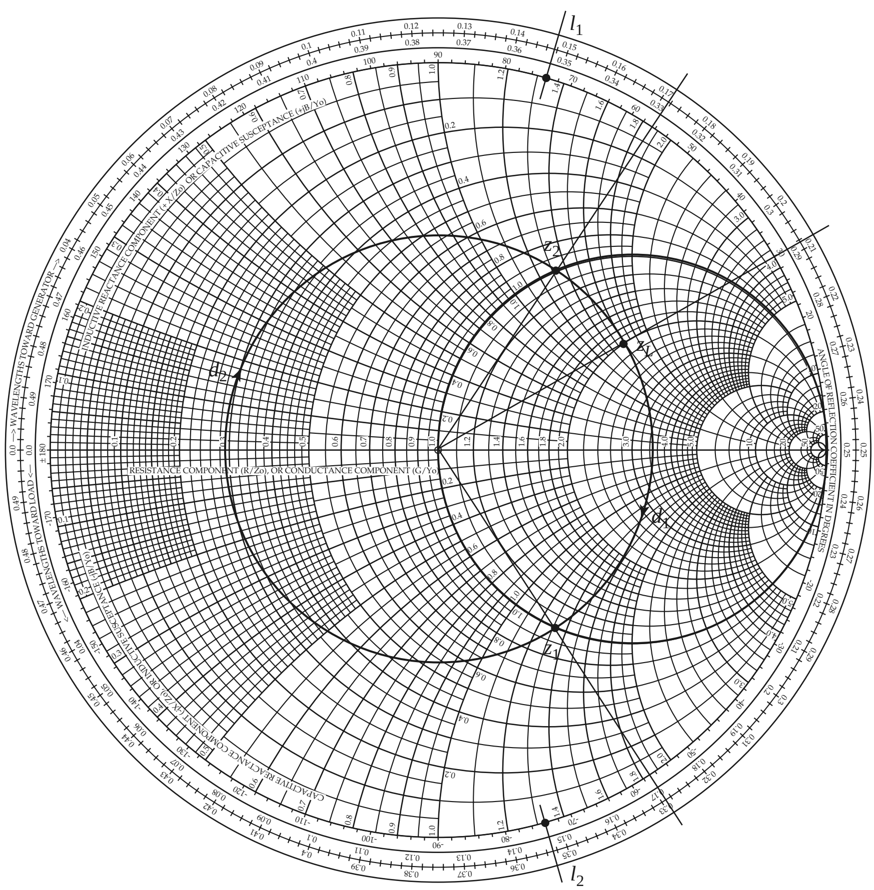
Solution
First plot the normalized load impedance, $z_L = 2 + j1.6$, and draw the SWR circle. For the series-stub design the chart is an impedance chart. Note that the SWR circle intersects the $1 + jx$ circle at two points, denoted as z_1 and z_2 in Figure 5.6a. The shortest distance, d_1, from the load to the stub is, from the WTG scale,

$$d_1 = 0.328 - 0.208 = 0.120\lambda,$$

and the second distance is

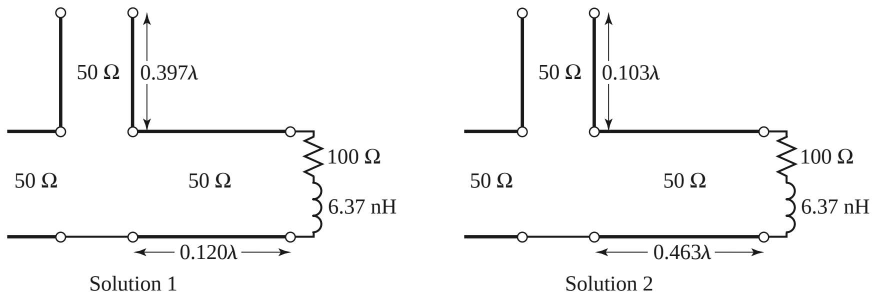
$$d_2 = (0.5 - 0.208) + 0.172 = 0.463\lambda.$$

As in the shunt-stub case, additional rotations around the SWR circle lead to additional solutions, but these are usually not of practical interest.



(a)

FIGURE 5.6 Solution to Example 5.3. (a) Smith chart for the series-stub tuners.



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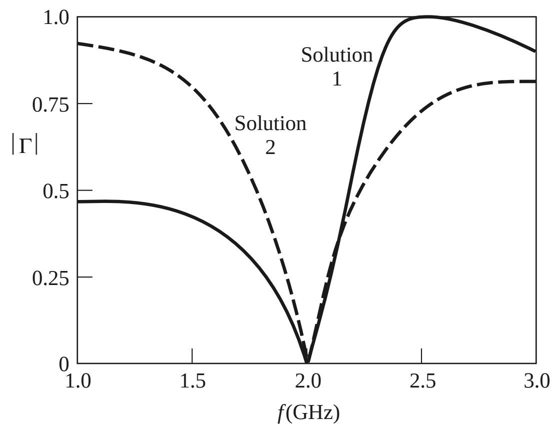


(c)

FIGURE 5.6 Continued. (b) The two series-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

The normalized impedances at the two intersection points are

$$z_1 = 1 - j1.33,$$
$$z_2 = 1 + j1.33.$$

Thus, the first solution requires a stub with a reactance of $j1.33$. The length of an open-circuited stub that gives this reactance can be found on the Smith chart by starting at $z = \infty$ (open circuit), and moving along the outer edge of the chart ($r = 0$) toward the generator to the $j1.33$ point. This gives a stub length of

$$\ell_1 = 0.397\lambda.$$

Similarly, the required open-circuited stub length for the second solution is

$$\ell_2 = 0.103\lambda.$$

This completes the tuner designs.

If the load is a series resistor and inductor with $Z_L = 100 + j80\ \Omega$ at 2 GHz, then $R = 100\ \Omega$ and $L = 6.37$ nH. The two matching circuits are shown in Figure 5.6b. Figure 5.6c shows the calculated reflection coefficient magnitudes versus frequency for the two solutions. ■

To derive formulas for d and ℓ for the series-stub tuner, let the load admittance be written as $Y_L = 1/Z_L = G_L + jB_L$. Then the admittance Y down a length d of line from the load is

$$Y = Y_0 \frac{(G_L + jB_L) + jtY_0}{Y_0 + jt(G_L + jB_L)}, \tag{5.12}$$

where $t = \tan \beta d$ and $Y_0 = 1/Z_0$. The impedance at this point is

$$Z = R + jX = \frac{1}{Y},$$

where

$$R = \frac{G_L(1 + t^2)}{G_L^2 + (B_L + Y_0 t)^2}, \tag{5.13a}$$

$$X = \frac{G_L^2 t - (Y_0 - tB_L)(B_L + tY_0)}{Y_0 \left[G_L^2 + (B_L + Y_0 t)^2\right]}. \tag{5.13b}$$

Now d (which implies t) is chosen so that $R = Z_0 = 1/Y_0$. From (5.13a), this results in a quadratic equation for t:

$$Y_0(G_L - Y_0)t^2 - 2B_L Y_0 t + \left(G_L Y_0 - G_L^2 - B_L^2\right) = 0.$$

Solving for t gives

$$t = \frac{B_L \pm \sqrt{G_L\left[(Y_0 - G_L)^2 + B_L^2\right]/Y_0}}{G_L - Y_0} \quad \text{for } G_L \neq Y_0. \tag{5.14}$$

If $G_L = Y_0$, then $t = -B_L/2Y_0$. Then the two principal solutions for d are

$$d/\lambda = \begin{cases} \dfrac{1}{2\pi}\tan^{-1} t & \text{for } t \geq 0 \\ \dfrac{1}{2\pi}(\pi + \tan^{-1} t) & \text{for } t < 0. \end{cases} \tag{5.15}$$

The required stub lengths are determined by first using t in (5.13b) to find the reactance X. This reactance is the negative of the necessary stub reactance, X_s. Thus, for a short-circuited stub,

$$\frac{\ell_s}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{X_s}{Z_0}\right) = \frac{-1}{2\pi}\tan^{-1}\left(\frac{X}{Z_0}\right), \tag{5.16a}$$

and for an open-circuited stub,

$$\frac{\ell_o}{\lambda} = \frac{-1}{2\pi}\tan^{-1}\left(\frac{Z_0}{X_s}\right) = \frac{1}{2\pi}\tan^{-1}\left(\frac{Z_0}{X}\right). \tag{5.16b}$$

If the length given by (5.16a) or (5.16b) is negative, $\lambda/2$ can be added to give a positive result.

5.3 DOUBLE-STUB TUNING

The single-stub tuner of the previous section is able to match any load impedance (having a positive real part) to a transmission line, but suffers from the disadvantage of requiring a variable length of line between the load and the stub. This may not be a problem for a fixed matching circuit, but would probably pose some difficulty if an adjustable tuner was

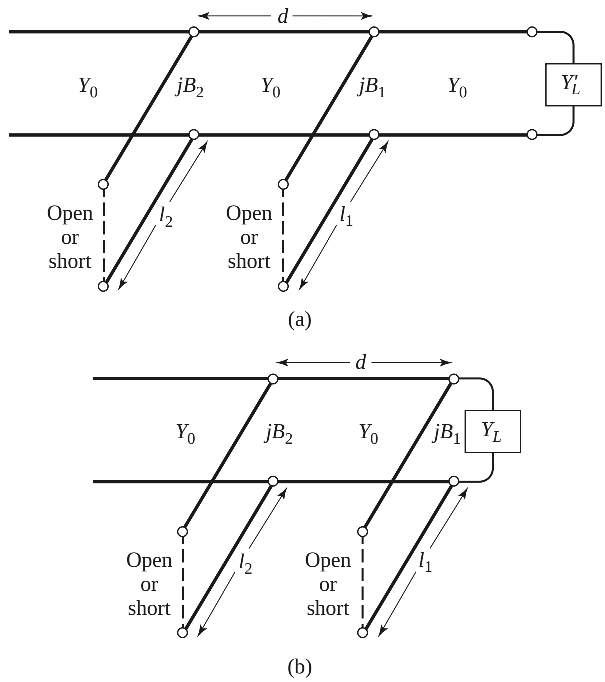


FIGURE 5.7 Double-stub tuning. (a) Original circuit with the load an arbitrary distance from the first stub. (b) Equivalent circuit with the load transformed to the first stub.

desired. In this case, the *double-stub tuner*, which uses two tuning stubs in fixed positions, can be used. Such tuners are often fabricated in coaxial line with adjustable stubs connected in shunt to the main coaxial line. We will see, however, that a double-stub tuner cannot match all load impedances.

The double-stub tuner circuit is shown in Figure 5.7a, where the load may be an arbitrary distance from the first stub. Although this is more representative of a practical situation, the circuit of Figure 5.7b, where the load Y_L' has been transformed back to the position of the first stub, is easier to deal with and does not lose any generality. The shunt stubs shown in Figure 5.7 can be conveniently implemented for some types of transmission lines, while series stubs are more appropriate for other types of lines. In either case, the stubs can be open-circuited or short-circuited.

Smith Chart Solution

The Smith chart of Figure 5.8 illustrates the basic operation of the double-stub tuner. As in the case of the single-stub tuner, two solutions are possible. The susceptance of the first stub, b_1 (or b_1', for the second solution), moves the load admittance to y_1 (or y_1'). These points lie on the rotated $1 + jb$ circle; the amount of rotation is d wavelengths toward the load, where d is the electrical distance between the two stubs. Then transforming y_1 (or y_1') toward the generator through a length d of line leaves us at the point y_2 (or y_2'), which must be on the $1 + jb$ circle. The second stub then adds a susceptance b_2 (or b_2'), which brings us to the center of the chart and completes the match.

Notice from Figure 5.8 that if the load admittance, y_L, were inside the shaded region of the $g_0 + jb$ circle, no value of stub susceptance b_1 could ever bring the load point to intersect the rotated $1 + jb$ circle. This shaded region thus forms a forbidden range of load admittances that cannot be matched with this particular double-stub tuner. A simple way

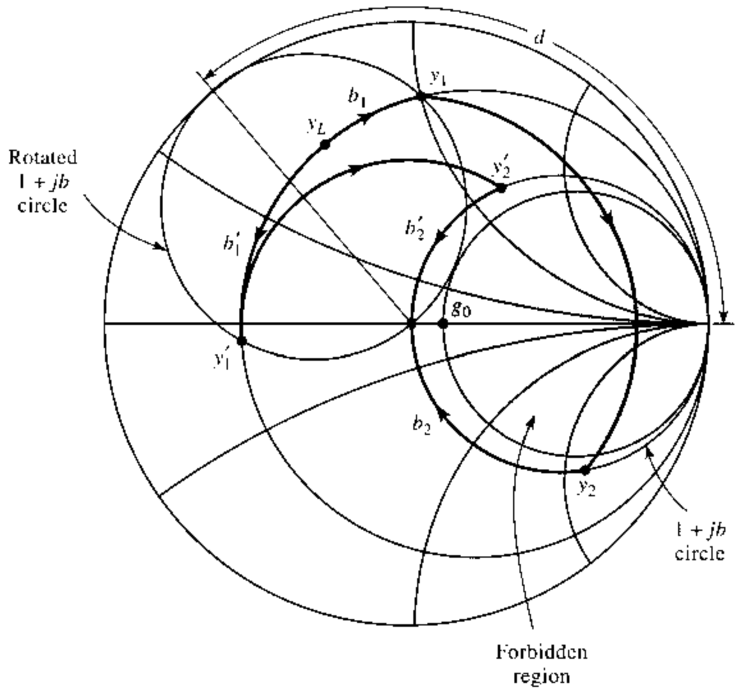


FIGURE 5.8 Smith chart diagram for the operation of a double-stub tuner.

of reducing the forbidden range is to reduce the distance d between the stubs. This has the effect of swinging the rotated $1 + jb$ circle back toward the $y = \infty$ point, but d must be kept large enough for the practical purpose of fabricating the two separate stubs. In addition, stub spacings near 0 or $\lambda/2$ lead to matching networks that are very frequency sensitive. In practice, stub spacings are usually chosen as $\lambda/8$ or $3\lambda/8$. If the length of line between the load and the first stub can be adjusted, then the load admittance y_L can always be moved out of the forbidden region.

EXAMPLE 5.4 DOUBLE-STUB TUNING

Design a double-stub shunt tuner to match a load impedance $Z_L = 60 - j80\ \Omega$ to a 50 Ω line. The stubs are to be open-circuited stubs and are spaced $\lambda/8$ apart. Assuming that this load consists of a series resistor and capacitor and that the match frequency is 2 GHz, plot the reflection coefficient magnitude versus frequency from 1 to 3 GHz.

Solution

The normalized load admittance is $y_L = 0.3 + j0.4$, which is plotted on the Smith chart of Figure 5.9a. Next we construct the rotated $1 + jb$ conductance circle by moving every point on the $g = 1$ circle $\lambda/8$ toward the load. We then find the susceptance of the first stub, which can be one of two possible values:

$$b_1 = 1.314 \quad \text{or} \quad b_1' = -0.114.$$

We now transform through the $\lambda/8$ section of line by rotating along a constant-radius (SWR) circle $\lambda/8$ toward the generator. This brings the two solutions to the following points:

$$y_2 = 1 - j3.38 \quad \text{or} \quad y_2' = 1 + j1.38.$$

Then the susceptance of the second stub should be

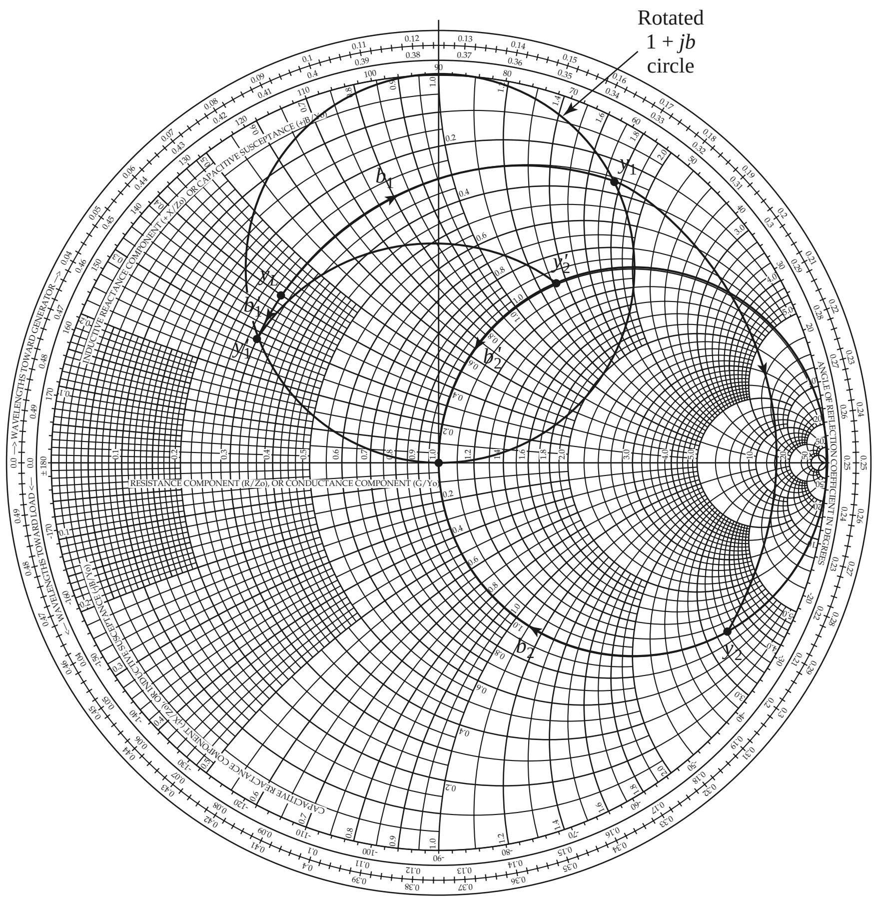
$$b_2 = 3.38 \quad \text{or} \quad b_2' = -1.38.$$

The lengths of the open-circuited stubs are then found as

$$\ell_1 = 0.146\lambda, \ell_2 = 0.204\lambda \quad \text{or} \quad \ell_1' = 0.482\lambda, \ell_2' = 0.350\lambda.$$

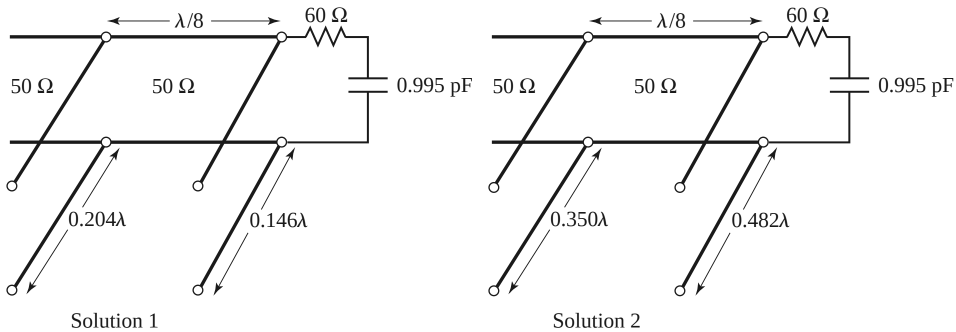
This completes both solutions for the double-stub tuner design.

At $f = 2$ GHz the resistor-capacitor load of $Z_L = 60 - j80\ \Omega$ implies that $R = 60\ \Omega$ and $C = 0.995$ pF. The two tuning circuits are then as shown in Figure 5.9b, and the reflection coefficient magnitudes are plotted versus frequency in Figure 5.9c. Note that the first solution has a much narrower bandwidth than the second (primed) solution due to the fact that both stubs for the first solution are somewhat longer (and closer to $\lambda/2$) than the stubs of the second solution. ■

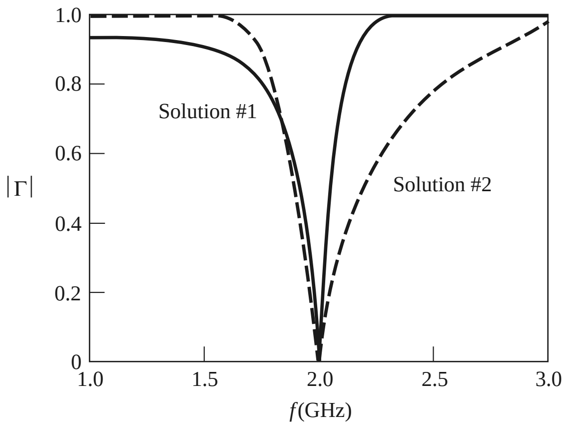


(a)

FIGURE 5.9 Solution to Example 5.4. (a) Smith chart for the double-stub tuners.



(b)



(c)

FIGURE 5.9 Continued. (b) The two double-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

Analytic Solution

The admittance just to the left of the first stub in Figure 5.7b is

$$Y_1 = G_L + j(B_L + B_1), \tag{5.17}$$

where $Y_L = G_L + jB_L$ is the load admittance, and B_1 is the susceptance of the first stub. After transforming through a length d of transmission line, we find that the admittance just to the right of the second stub is

$$Y_2 = Y_0 \frac{G_L + j(B_L + B_1 + Y_0 t)}{Y_0 + jt(G_L + jB_L + jB_1)}, \tag{5.18}$$

where $t = \tan \beta d$ and $Y_0 = 1/Z_0$. At this point the real part of Y_2 must equal Y_0, which leads to the equation

$$G_L^2 - G_L Y_0 \frac{1+t^2}{t^2} + \frac{(Y_0 - B_L t - B_1 t)^2}{t^2} = 0. \tag{5.19}$$

Solving for G_L gives

$$G_L = Y_0 \frac{1+t^2}{2t^2}\left[1 \pm \sqrt{1 - \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y_0^2(1+t^2)^2}}\right]. \tag{5.20}$$

Because G_L is real, the quantity within the square root must be nonnegative, and so

$$0 \leq \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y_0^2(1+t^2)^2} \leq 1.$$

This implies that

$$0 \leq G_L \leq Y_0 \frac{1+t^2}{t^2} = \frac{Y_0}{\sin^2 \beta d}, \tag{5.21}$$

which gives the range on G_L that can be matched for a given stub spacing d. After d has been set, the first stub susceptance can be determined from (5.19) as

$$B_1 = -B_L + \frac{Y_0 \pm \sqrt{(1+t^2)G_L Y_0 - G_L^2 t^2}}{t}. \tag{5.22}$$

Then the second stub susceptance can be found from the negative of the imaginary part of (5.18) to be

$$B_2 = \frac{\pm Y_0 \sqrt{Y_0 G_L(1+t^2) - G_L^2 t^2} + G_L Y_0}{G_L t}. \tag{5.23}$$

The upper and lower signs in (5.22) and (5.23) correspond to the same solutions. The open-circuited stub length is found as

$$\frac{\ell_o}{\lambda} = \frac{1}{2\pi} \tan^{-1}\left(\frac{B}{Y_0}\right), \tag{5.24a}$$

and the short-circuited stub length is found as

$$\frac{\ell_s}{\lambda} = \frac{-1}{2\pi} \tan^{-1}\left(\frac{Y_0}{B}\right), \tag{5.24b}$$

where $B = B_1$ or B_2.

5.4 THE QUARTER-WAVE TRANSFORMER

As introduced in Section 2.5, the quarter-wave transformer is a simple and useful circuit for matching a real load impedance to a transmission line. An additional feature of the quarter-wave transformer is that it can be extended to multisection designs in a methodical manner to provide broader bandwidth. If only a narrow band impedance match is required, a single-section transformer may suffice. However, as we will see in the next few sections, multisection quarter-wave transformer designs can be synthesized to yield optimum matching characteristics over a desired frequency band. We will see in Chapter 8 that such networks are closely related to bandpass filters.

One drawback of the quarter-wave transformer is that it can only match a real load impedance. A complex load impedance can always be transformed into a real impedance, however, by using an appropriate length of transmission line between the load and the transformer, or an appropriate series or shunt reactive element. These techniques will usually alter the frequency dependence of the load, and this often has the effect of reducing the bandwidth of the match.

In Section 2.5 we analyzed the operation of a quarter-wave transformer from both an impedance viewpoint and a multiple reflection viewpoint. Here we will concentrate on the bandwidth performance of the transformer as a function of the load mismatch; this

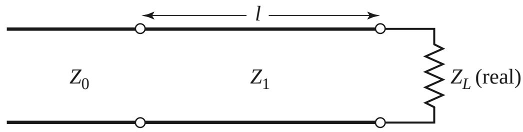


FIGURE 5.10 A single-section quarter-wave matching transformer. $\ell = \lambda_0/4$ at the design frequency f_0.

discussion will also serve as a prelude to the more general case of multisection transformers in the sections to follow.

The single-section quarter-wave matching transformer circuit is shown in Figure 5.10, with the characteristic impedance of the matching section given as

$$Z_1 = \sqrt{Z_0 Z_L}. \tag{5.25}$$

At the design frequency, f_0, the electrical length of the matching section is $\lambda_0/4$, but at other frequencies the length is different, so a perfect match is no longer obtained. We will derive an approximate expression for the resulting impedance mismatch versus frequency.

The input impedance seen looking into the matching section is

$$Z_{\text{in}} = Z_1 \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t}, \tag{5.26}$$

where $t = \tan \beta\ell = \tan\theta$, and $\beta\ell = \theta = \pi/2$ at the design frequency f_0. The resulting reflection coefficient is

$$\Gamma = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \frac{Z_1(Z_L - Z_0) + jt\left(Z_1^2 - Z_0 Z_L\right)}{Z_1(Z_L + Z_0) + jt\left(Z_1^2 + Z_0 Z_L\right)}. \tag{5.27}$$

Because $Z_1^2 = Z_0 Z_L$, this reduces to

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0 + j2t\sqrt{Z_0 Z_L}}. \tag{5.28}$$

The reflection coefficient magnitude is

$$\begin{aligned} |\Gamma| &= \frac{|Z_L - Z_0|}{\left[(Z_L + Z_0)^2 + 4t^2 Z_0 Z_L\right]^{1/2}} \\ &= \frac{1}{\left\{(Z_L + Z_0)^2/(Z_L - Z_0)^2 + [4t^2 Z_0 Z_L/(Z_L - Z_0)^2]\right\}^{1/2}} \\ &= \frac{1}{\left\{1 + [4Z_0 Z_L/(Z_L - Z_0)^2] + [4Z_0 Z_L t^2/(Z_L - Z_0)^2]\right\}^{1/2}} \\ &= \frac{1}{\left\{1 + [4Z_0 Z_L/(Z_L - Z_0)^2]\ \sec^2\theta\right\}^{1/2}}, \end{aligned} \tag{5.29}$$

since $1 + t^2 = 1 + \tan^2\theta = \sec^2\theta$.

If we assume that the operating frequency is near the design frequency f_0, then $\ell \simeq \lambda_0/4$ and $\theta \simeq \pi/2$. Then $\sec^2\theta \gg 1$, and (5.29) simplifies to

$$|\Gamma| \simeq \frac{|Z_L - Z_0|}{2\sqrt{Z_0 Z_L}} |\cos\theta| \quad \text{for } \theta \text{ near } \pi/2. \tag{5.30}$$

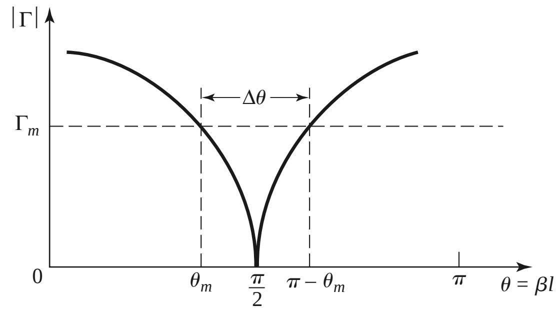


FIGURE 5.11 Approximate behavior of the reflection coefficient magnitude for a single-section quarter-wave transformer operating near its design frequency.

This result gives the approximate mismatch of the quarter-wave transformer near the design frequency, as sketched in Figure 5.11.

If we set a maximum value, Γ_m, for an acceptable reflection coefficient magnitude, then the bandwidth of the matching transformer can be defined as

$$\Delta\theta = 2\left(\frac{\pi}{2} - \theta_m\right), \tag{5.31}$$

since the response of (5.29) is symmetric about $\theta = \pi/2$, and $\Gamma = \Gamma_m$ at $\theta = \theta_m$ and at $\theta = \pi - \theta_m$. Equating Γ_m to the exact expression for the reflection coefficient magnitude in (5.29) allows us to solve for θ_m:

$$\frac{1}{\Gamma_m^2} = 1 + \left(\frac{2\sqrt{Z_0 Z_L}}{Z_L - Z_0}\sec\theta_m\right)^2,$$

or

$$\cos\theta_m = \frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}. \tag{5.32}$$

If we assume TEM lines, then

$$\theta = \beta\ell = \frac{2\pi f}{v_p}\frac{v_p}{4 f_0} = \frac{\pi f}{2 f_0},$$

and so the frequency of the lower band edge at $\theta = \theta_m$ is

$$f_m = \frac{2\theta_m f_0}{\pi},$$

and the fractional bandwidth is, using (5.32),

$$\begin{aligned}\frac{\Delta f}{f_0} &= \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{2 f_m}{f_0} = 2 - \frac{4\theta_m}{\pi}\\ &= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right].\end{aligned} \tag{5.33}$$

Fractional bandwidth is usually expressed as a percentage, $100\Delta f/f_0\%$. Note that the bandwidth of the transformer increases as Z_L becomes closer to Z_0 (a less mismatched load).

The above results are strictly valid only for TEM lines. When non-TEM lines (such as waveguides) are used, the propagation constant is no longer a linear function of frequency, and the wave impedance will be frequency dependent. These factors serve to complicate

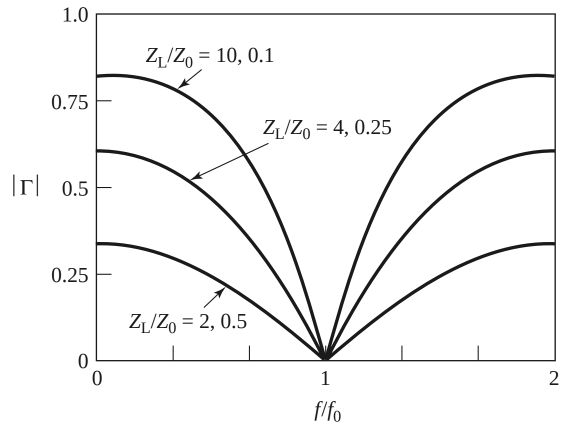
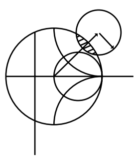


FIGURE 5.12 Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.

the general behavior of quarter-wave transformers for non-TEM lines, but in practice the bandwidth of the transformer is often small enough that these complications do not substantially affect the result. Another factor ignored in the above analysis is the effect of reactances associated with discontinuities when there is a step change in the dimensions of a transmission line. This can often be compensated by making a small adjustment in the length of the matching section.

Figure 5.12 shows a plot of the reflection coefficient magnitude versus normalized frequency for various mismatched loads. Note the trend of increased bandwidth for smaller load mismatches.



EXAMPLE 5.5 QUARTER-WAVE TRANSFORMER BANDWIDTH

Design a single-section quarter-wave matching transformer to match a 10 Ω load to a 50 Ω transmission line at $f_0 = 3$ GHz. Determine the percent bandwidth for which the SWR ≤ 1.5.

Solution

From (5.25), the characteristic impedance of the matching section is

$$Z_1 = \sqrt{Z_0 Z_L} = \sqrt{(50)(10)} = 22.36\ \Omega,$$

and the length of the matching section is $\lambda/4$ at 3 GHz (the physical length depends on the dielectric constant of the line). An SWR of 1.5 corresponds to a reflection coefficient magnitude of

$$\Gamma_m = \frac{\text{SWR} - 1}{\text{SWR} + 1} = \frac{1.5 - 1}{1.5 + 1} = 0.2.$$

The fractional bandwidth is computed from (5.33) as

$$\begin{aligned}\frac{\Delta f}{f_0} &= 2 - \frac{4}{\pi} \cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right] \\ &= 2 - \frac{4}{\pi} \cos^{-1}\left[\frac{0.2}{\sqrt{1-(0.2)^2}} \frac{2\sqrt{(50)(10)}}{|10-50|}\right] \\ &= 0.29, \text{ or } 29\%.\end{aligned}$$

■

5.5 THE THEORY OF SMALL REFLECTIONS

The quarter-wave transformer provides a simple means of matching any real load impedance to any transmission line impedance. For applications requiring more bandwidth than a single quarter-wave section can provide, *multisection transformers* can be used. The design of such transformers is the subject of the next two sections, but prior to that material we need to derive some approximate results for the total reflection coefficient caused by the partial reflections from several small discontinuities. This topic is generally referred to as the *theory of small reflections* [1].

Single-Section Transformer

We will derive an approximate expression for the overall reflection coefficient, Γ, for the single-section matching transformer shown in Figure 5.13. The partial reflection and transmission coefficients are

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \tag{5.34}$$

$$\Gamma_2 = -\Gamma_1, \tag{5.35}$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}, \tag{5.36}$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2}, \tag{5.37}$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2}. \tag{5.38}$$

We can compute the total reflection, Γ, seen by the feed line using either the impedance method, or the multiple reflection method, as discussed in Section 2.5. For our present

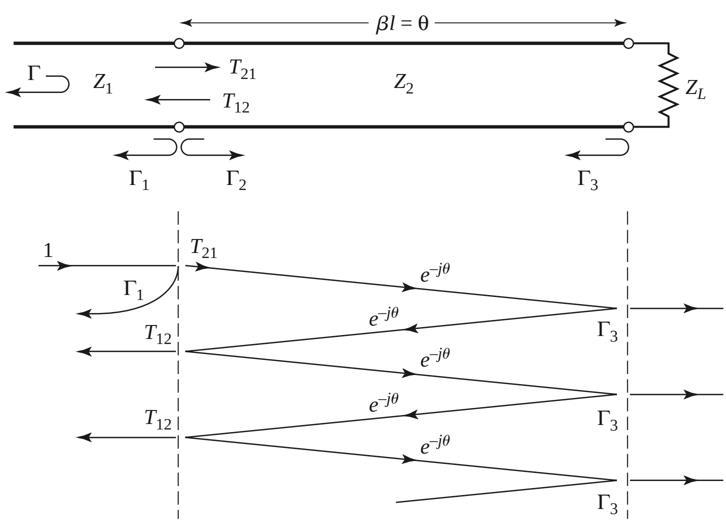


FIGURE 5.13 Partial reflections and transmissions on a single-section matching transformer.

purpose the latter technique is preferred, so we express the total reflection as an infinite sum of partial reflections and transmissions as follows:

$$\begin{aligned}\Gamma &= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-4j\theta} + \cdots \\ &= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta}\sum_{n=0}^{\infty}\Gamma_2^n\Gamma_3^n e^{-2jn\theta}.\end{aligned} \tag{5.39}$$

The summation of the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1$$

allows us to express (5.39) in closed form as

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-2j\theta}}{1 - \Gamma_2\Gamma_3 e^{-2j\theta}}. \tag{5.40}$$

From (5.35), (5.37), and (5.38), we use $\Gamma_2 = -\Gamma_1$, $T_{21} = 1 + \Gamma_1$, and $T_{12} = 1 - \Gamma_1$ in (5.40) to give

$$\Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-2j\theta}}{1 + \Gamma_1\Gamma_3 e^{-2j\theta}}. \tag{5.41}$$

If the discontinuities between the impedances Z_1, Z_2 and Z_2, Z_L are small, then $|\Gamma_1\Gamma_3| = 1$, so we can approximate (5.41) as

$$\Gamma \simeq \Gamma_1 + \Gamma_3 e^{-2j\theta}. \tag{5.42}$$

This result expresses the intuitive idea that the total reflection is dominated by the reflection from the initial discontinuity between Z_1 and Z_2 (Γ_1), and the first reflection from the discontinuity between Z_2 and Z_L ($\Gamma_3 e^{-2j\theta}$). The $e^{-2j\theta}$ term accounts for the phase delay when the incident wave travels up and down the line. The accuracy of this approximation is illustrated in Problem 5.14.

Multisection Transformer

Now consider the multisection transformer shown in Figure 5.14, which consists of N equal-length (*commensurate*) sections of transmission lines. We will derive an approximate expression for the total reflection coefficient Γ.

Partial reflection coefficients can be defined at each junction, as follows:

$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \tag{5.43a}$$

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}, \tag{5.43b}$$

$$\Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}. \tag{5.43c}$$

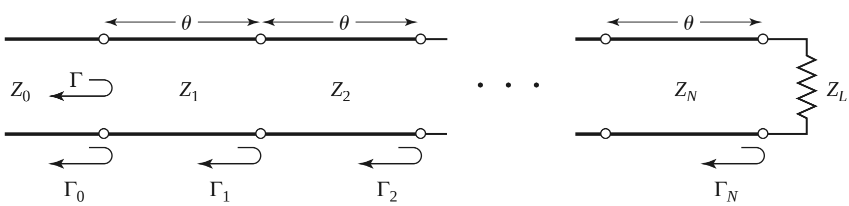


FIGURE 5.14 Partial reflection coefficients for a multisection matching transformer.

We also assume that all Z_n increase or decrease monotonically across the transformer and that Z_L is real. This implies that all Γ_n will be real and of the same sign ($\Gamma_n > 0$ if $Z_L > Z_0$; $\Gamma_n < 0$ if $Z_L < Z_0$). Using the results of the previous section allows us to approximate the overall reflection coefficient as

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_N e^{-2jN\theta}. \tag{5.44}$$

Further assume that the transformer can be made symmetrical, so that $\Gamma_0 = \Gamma_N,\ \Gamma_1 = \Gamma_{N-1},\ \Gamma_2 = \Gamma_{N-2}$, and so on. (Note that this does *not* imply that the Z_n are symmetrical.) Then (5.44) can be written as

$$\Gamma(\theta) = e^{-jN\theta}\left\{\Gamma_0[e^{jN\theta} + e^{-jN\theta}] + \Gamma_1[e^{j(N-2)\theta} + e^{-j(N-2)\theta}] + \cdots\right\}. \tag{5.45}$$

If N is odd, the last term is $\Gamma_{(N-1)/2}(e^{j\theta} + e^{-j\theta})$, while if N is even, the last term is $\Gamma_{N/2}$. Equation (5.45) is seen to be of the form of a finite Fourier cosine series in θ, which can be written as

$$\begin{aligned}\Gamma(\theta) &= 2e^{-jN\theta}\bigg[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta \\ &\quad + \cdots + \frac{1}{2}\Gamma_{N/2}\bigg] \quad \text{for } N \text{ even,}\end{aligned} \tag{5.46a }$$

$$\begin{aligned}\Gamma(\theta) &= 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta \\ &\quad + \cdots + \Gamma_{(N-1)/2} \cos\theta] \quad \text{for } N \text{ odd.}\end{aligned} \tag{5.46b}$$

The importance of these results lies in the fact that we can synthesize any desired reflection coefficient response as a function of frequency (θ) by properly choosing the Γ_n and using enough sections (N). This should be clear from the realization that a Fourier series can approximate an arbitrary smooth function if enough terms are used. In the next two sections we will show how to use this theory to design multisection transformers for two of the most commonly used passband responses: the *binomial* (maximally flat) response, and the *Chebyshev* (equal-ripple) response.

5.6 BINOMIAL MULTISECTION MATCHING TRANSFORMERS

The passband response (the frequency band where a good impedance match is achieved) of a binomial matching transformer is optimum in the sense that, for a given number of sections, the response is as flat as possible near the design frequency. This type of response, which is also known as *maximally flat*, is determined for an N-section transformer by setting the first $N-1$ derivatives of $|\Gamma(\theta)|$ to zero at the center frequency, f_0. Such a response can be obtained with a reflection coefficient of the following form:

$$\Gamma(\theta) = A(1 + e^{-2j\theta})^N. \tag{5.47}$$

Then the reflection coefficient magnitude is

$$\begin{aligned}|\Gamma(\theta)| &= |A||e^{-j\theta}|^N |e^{j\theta} + e^{-j\theta}|^N \\ &= 2^N |A||\cos\theta|^N\end{aligned} \tag{5.48}$$

Note that $|\Gamma(\theta)| = 0$ for $\theta = \pi/2$, and that $d^n|\Gamma(\theta)|/d\theta^n = 0$ at $\theta = \pi/2$ for $n = 1, 2, \ldots, N-1$. ($\theta = \pi/2$ corresponds to the center frequency, f_0, for which $\ell = \lambda/4$ and $\theta = \beta\ell = \pi/2$.)

We can determine the constant A by letting $f \to 0$. Then $\theta = \beta\ell = 0$, and (5.47) reduces to

$$\Gamma(0) = 2^N A = \frac{Z_L - Z_0}{Z_L + Z_0},$$

since for $f = 0$ all sections are of zero electrical length. The constant A can then be written as

$$A = 2^{-N} \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{5.49}$$

Next we expand $\Gamma(\theta)$ in (5.47) according to the binomial expansion:

$$\Gamma(\theta) = A(1 + e^{-2j\theta})^N = A \sum_{n=0}^{N} C_n^N e^{-2jn\theta}, \tag{5.50}$$

where

$$C_n^N = \frac{N!}{(N-n)!n!} \tag{5.51}$$

are the binomial coefficients. Note that $C_n^N = C_{N-n}^N$, $C_0^N = 1$, and $C_1^N = N = C_{N-1}^N$. The key step is now to equate the desired passband response, given by (5.50), to the actual response as given (approximately) by (5.44):

$$\Gamma(\theta) = A \sum_{n=0}^{N} C_n^N e^{-2jn\theta} = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_N e^{-2jN\theta}.$$

This shows that the Γ_n must be chosen as

$$\Gamma_n = AC_n^N. \tag{5.52}$$

where A is given by (5.49) and C_n^N is a binomial coefficient.

At this point, the characteristic impedances, Z_n, can be found via (5.43), but a simpler solution can be obtained using the following approximation [1]. Because we assumed that the Γ_n are small, we can write

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \simeq \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n},$$

since $\ln x \simeq 2(x-1)/(x+1)$ for x close to unity. Then, using (5.52) and (5.49) gives

$$\ln \frac{Z_{n+1}}{Z_n} \simeq 2\Gamma_n = 2AC_n^N = 2(2^{-N}) \frac{Z_L - Z_0}{Z_L + Z_0} C_n^N \simeq 2^{-N} C_n^N \ln \frac{Z_L}{Z_0}, \tag{5.53}$$

which can be used to find Z_{n+1}, starting with $n = 0$. This technique has the advantage of ensuring self-consistency, in that Z_{N+1} computed from (5.53) will be equal to Z_L, as it should.

Exact design results, including the effect of multiple reflections in each section, can be found by using the transmission line equations for each section and numerically solving for the characteristic impedances [2]. The results of such calculations are listed in Table 5.1, which gives the exact line impedances for $N =$ 2-, 3-, 4-, 5-, and 6-section

TABLE 5.1 Binomial Transformer Design

	$N = 2$		$N = 3$			$N = 4$			
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.1067	1.3554	1.0520	1.2247	1.4259	1.0257	1.1351	1.3215	1.4624
2.0	1.1892	1.6818	1.0907	1.4142	1.8337	1.0444	1.2421	1.6102	1.9150
3.0	1.3161	2.2795	1.1479	1.7321	2.6135	1.0718	1.4105	2.1269	2.7990
4.0	1.4142	2.8285	1.1907	2.0000	3.3594	1.0919	1.5442	2.5903	3.6633
6.0	1.5651	3.8336	1.2544	2.4495	4.7832	1.1215	1.7553	3.4182	5.3500
8.0	1.6818	4.7568	1.3022	2.8284	6.1434	1.1436	1.9232	4.1597	6.9955
10.0	1.7783	5.6233	1.3409	3.1623	7.4577	1.1613	2.0651	4.8424	8.6110

	$N = 5$					$N = 6$					
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_5/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_5/Z_0	Z_6/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.0128	1.0790	1.2247	1.3902	1.4810	1.0064	1.0454	1.1496	1.3048	1.4349	1.4905
2.0	1.0220	1.1391	1.4142	1.7558	1.9569	1.0110	1.0790	1.2693	1.5757	1.8536	1.9782
3.0	1.0354	1.2300	1.7321	2.4390	2.8974	1.0176	1.1288	1.4599	2.0549	2.6577	2.9481
4.0	1.0452	1.2995	2.0000	3.0781	3.8270	1.0225	1.1661	1.6129	2.4800	3.4302	3.9120
6.0	1.0596	1.4055	2.4495	4.2689	5.6625	1.0296	1.2219	1.8573	3.2305	4.9104	5.8275
8.0	1.0703	1.4870	2.8284	5.3800	7.4745	1.0349	1.2640	2.0539	3.8950	6.3291	7.7302
10.0	1.0789	1.5541	3.1623	6.4346	9.2687	1.0392	1.2982	2.2215	4.5015	7.7030	9.6228

binomial matching transformers for various ratios of load impedance, Z_L, to feed line impedance, Z_0. The table gives results only for $Z_L/Z_0 > 1$; if $Z_L/Z_0 < 1$, the results for Z_0/Z_L should be used but with Z_1 starting at the load end. This is because the solution is symmetric about $Z_L/Z_0 = 1$; the same transformer that matches Z_L to Z_0 can be reversed and used to match Z_0 to Z_L. More extensive tables can be found in reference [2].

The bandwidth of the binomial transformer can be evaluated as follows. As in Section 5.4, let Γ_m be the maximum value of reflection coefficient that can be tolerated over the passband. Then from (5.48),

$$\Gamma_m = 2^N |A| \cos^N \theta_m,$$

where $\theta_m < \pi/2$ is the lower edge of the passband, as shown in Figure 5.11. Thus,

$$\theta_m = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right], \tag{5.54}$$

and using (5.33) gives the fractional bandwidth as

$$\begin{aligned}\frac{\Delta f}{f_0} &= \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi}\\ &= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right].\end{aligned} \tag{5.55}$$

EXAMPLE 5.6 BINOMIAL TRANSFORMER DESIGN

Design a three-section binomial transformer to match a 50 Ω load to a 100 Ω line and calculate the bandwidth for $\Gamma_m = 0.05$. Plot the reflection coefficient magnitude versus normalized frequency for the exact designs using 1, 2, 3, 4, and 5 sections.

Solution

For $N = 3$, $Z_L = 50\ \Omega$, and $Z_0 = 100\ \Omega$ we have, from (5.49) and (5.53),

$$A = 2^{-N}\frac{Z_L - Z_0}{Z_L + Z_0} \simeq \frac{1}{2^{N+1}}\ln\frac{Z_L}{Z_0} = -0.0433.$$

From (5.55) the bandwidth is

$$\begin{aligned}\frac{\Delta f}{f_0} &= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]\\ &= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{0.05}{0.0433}\right)^{1/3}\right] = 0.70, \text{ or } 70\%.\end{aligned}$$

The necessary binomial coefficients are

$$C_0^3 = \frac{3!}{3!0!} = 1,$$

$$C_1^3 = \frac{3!}{2!1!} = 3,$$

$$C_2^3 = \frac{3!}{1!2!} = 3.$$

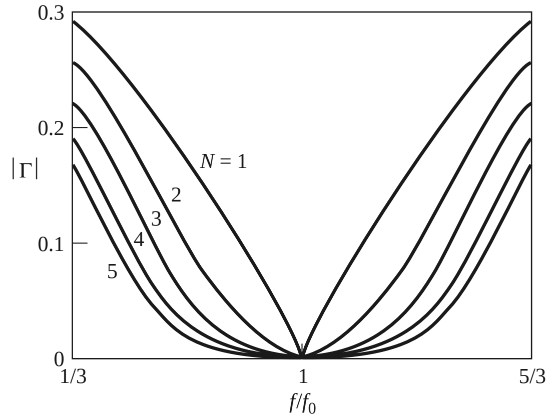


FIGURE 5.15 Reflection coefficient magnitude versus frequency for multisection binomial matching transformers of Example 5.6. $Z_L = 50\ \Omega$ and $Z_0 = 100\ \Omega$.

Using (5.53) gives the required characteristic impedances as

$$n = 0: \quad \ln Z_1 = \ln Z_0 + 2^{-N} C_0^3 \ln \frac{Z_L}{Z_0}$$
$$= \ln 100 + 2^{-3}(1) \ln \frac{50}{100} = 4.518,$$
$$Z_1 = 91.7\ \Omega;$$
$$n = 1: \quad \ln Z_2 = \ln Z_1 + 2^{-N} C_1^3 \ln \frac{Z_L}{Z_0}$$
$$= \ln 91.7 + 2^{-3}(3) \ln \frac{50}{100} = 4.26,$$
$$Z_2 = 70.7\ \Omega;$$
$$n = 2: \quad \ln Z_3 = \ln Z_2 + 2^{-N} C_2^3 \ln \frac{Z_L}{Z_0}$$
$$= \ln 70.7 + 2^{-3}(3) \ln \frac{50}{100} = 4.00,$$
$$Z_3 = 54.5\ \Omega.$$

To use the data in Table 5.1 we reverse the source and load impedances and consider the problem of matching a 100 Ω load to a 50 Ω line. Then $Z_L/Z_0 = 2.0$, and we obtain the exact characteristic impedances as $Z_1 = 91.7\ \Omega$, $Z_2 = 70.7\ \Omega$, and $Z_3 = 54.5\ \Omega$, which agree with the approximate results to three significant digits. Figure 5.15 shows the reflection coefficient magnitude versus frequency for exact designs using $N = 1, 2, 3, 4$, and 5 sections. Observe that greater bandwidth is obtained for transformers using more sections. ■

5.7 CHEBYSHEV MULTISECTION MATCHING TRANSFORMERS

In contrast with the binomial transformer, the multisection *Chebyshev matching transformer* optimizes bandwidth at the expense of passband ripple. Compromising on the flatness of the passband response leads to a bandwidth that is substantially better than that of the binomial transformer for a given number of sections. The Chebyshev transformer is designed by equating $\Gamma(\theta)$ to a Chebyshev polynomial, which has the optimum characteristics needed for this type of transformer. We will first discuss the properties of Chebyshev polynomials and then derive a design procedure for Chebyshev matching transformers using the small-reflection theory of Section 5.5.

Chebyshev Polynomials

The nth-order Chebyshev polynomial is a polynomial of degree n, denoted by $T_n(x)$. The first four Chebyshev polynomials are

$$T_1(x) = x, \tag{5.56a}$$

$$T_2(x) = 2x^2 - 1, \tag{5.56b}$$

$$T_3(x) = 4x^3 - 3x, \tag{5.56c}$$

$$T_4(x) = 8x^4 - 8x^2 + 1. \tag{5.56d}$$

Higher order polynomials can be found using the following recurrence formula:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x). \tag{5.57}$$

The first four Chebyshev polynomials are plotted in Figure 5.16, from which the following very useful properties of Chebyshev polynomials can be noted:

- For $-1 \leq x \leq 1$, $|T_n(x)| \leq 1$. In this range the Chebyshev polynomials oscillate between ± 1. This is the *equal-ripple* property, and this region will be mapped to the passband of the matching transformer.
- For $|x| > 1$, $|T_n(x)| > 1$. This region will map to the frequency range outside the passband.
- For $|x| > 1$, the $|T_n(x)|$ increases faster with x as n increases.

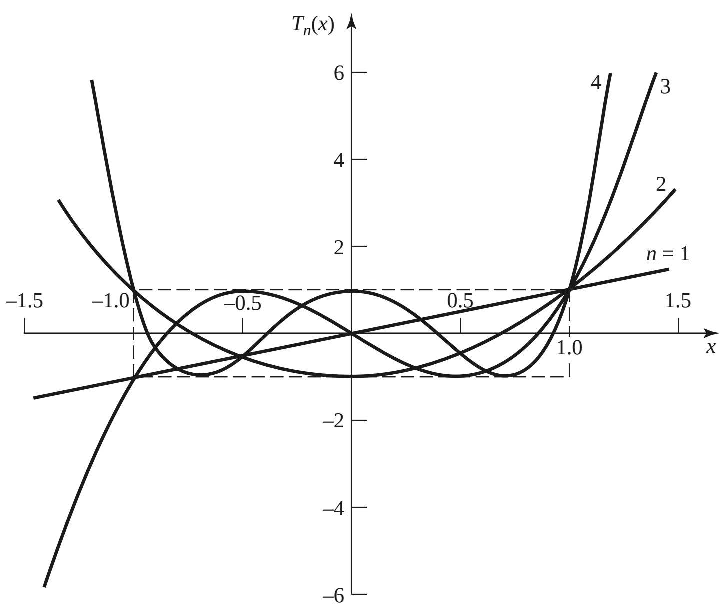


FIGURE 5.16 The first four Chebyshev polynomials, $T_n(x)$.

Now let $x = \cos\theta$ for $|x| < 1$. Then it can be shown that the Chebyshev polynomials can be expressed as

$$T_n(\cos\theta) = \cos n\theta,$$

or more generally as

$$T_n(x) = \cos(n \cos^{-1} x) \qquad \text{for } |x| < 1, \tag{5.58a}$$

$$T_n(x) = \cosh(n \cosh^{-1} x) \qquad \text{for } x > 1. \tag{5.58b}$$

We desire equal ripple for the passband response of the transformer, so it is necessary to map θ_m to $x = 1$ and $\pi - \theta_m$ to $x = -1$, where θ_m and $\pi - \theta_m$ are the lower and upper edges of the passband, respectively, as shown in Figure 5.11. This can be accomplished by replacing $\cos\theta$ in (5.58a) with $\cos\theta/\cos\theta_m$:

$$T_n\left(\frac{\cos\theta}{\cos\theta_m}\right) = T_n(\sec\theta_m \cos\theta) = \cos n\left[\cos^{-1}\left(\frac{\cos\theta}{\cos\theta_m}\right)\right]. \tag{5.59}$$

Then $|\sec\theta_m \cos\theta| \leq 1$ for $\theta_m < \theta < \pi - \theta_m$, so $|T_n(\sec\theta_m \cos\theta)| \leq 1$ over this same range.

Because $\cos^n\theta$ can be expanded into a sum of terms of the form $\cos(n - 2m)\theta$, the Chebyshev polynomials of (5.56) can be rewritten in the following useful form:

$$T_1(\sec\theta_m \cos\theta) = \sec\theta_m \cos\theta, \tag{5.60a}$$

$$T_2(\sec\theta_m \cos\theta) = \sec^2\theta_m(1 + \cos 2\theta) - 1, \tag{5.60b}$$

$$T_3(\sec\theta_m \cos\theta) = \sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3\sec\theta_m \cos\theta, \tag{5.60c}$$

$$\begin{aligned} T_4(\sec\theta_m \cos\theta) = {} & \sec^4\theta_m(\cos 4\theta + 4\cos 2\theta + 3) \\ & -4\sec^2\theta_m(\cos 2\theta + 1) + 1. \end{aligned} \tag{5.60d}$$

These results can be used to design matching transformers with up to four sections, and will also be used in later chapters for the design of directional couplers and filters.

Design of Chebyshev Transformers

We can now synthesize a Chebyshev equal-ripple passband by making $\Gamma(\theta)$ proportional to $T_N(\sec\theta_m \cos\theta)$, where N is the number of sections in the transformer. Thus, using (5.46), we have

$$\begin{aligned} \Gamma(\theta) &= 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n)\theta + \cdots] \\ &= Ae^{-jN\theta} T_N(\sec\theta_m \cos\theta), \end{aligned} \tag{5.61}$$

where the last term in the series of (5.61) is $(1/2)\Gamma_{N/2}$ for N even and $\Gamma_{(N-1)/2}\cos\theta$ for N odd. As in the binomial transformer case, we can find the constant A by letting $\theta = 0$, corresponding to zero frequency. Thus,

$$\Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = AT_N(\sec\theta_m),$$

so we have

$$A = \frac{Z_L - Z_0}{Z_L + Z_0}\frac{1}{T_N(\sec\theta_m)}. \tag{5.62}$$

If the maximum allowable reflection coefficient magnitude in the passband is Γ_m, then from (5.61) $\Gamma_m = |A|$ since the maximum value of $T_n(\sec\theta_m \cos\theta)$ in the passband is unity.

Then (5.62) gives

$$T_N(\sec\theta_m) = \frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|,$$

which, after using (5.58b) and the approximations introduced in Section 5.6, allows us to determine θ_m as

$$\begin{aligned}\sec\theta_m &= \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right] \\ &\simeq \cosh\left[\frac{1}{N}\cosh^{-1}\left(\left|\frac{\ln Z_L/Z_0}{2\Gamma_m}\right|\right)\right].\end{aligned} \tag{5.63}$$

Once θ_m is known, the fractional bandwidth can be calculated from (5.33) as

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi}. \tag{5.64}$$

From (5.61), the Γ_n can be determined using the results of (5.60) to expand $T_N(\sec\theta_m\cos\theta)$ and equating similar terms of the form $\cos(N-2n)\theta$. The characteristic impedances Z_n can be found from (5.43), although, as in the case of the binomial transformer, accuracy can be improved and self-consistency can be achieved by using the approximation that

$$\Gamma_n \simeq \frac{1}{2}\ \ln\frac{Z_{n+1}}{Z_n}.$$

This procedure will be illustrated in Example 5.7.

The above results are approximate because of the reliance on small-reflection theory but are general enough to design transformers with an arbitrary ripple level, Γ_m. Table 5.2 gives exact results [2] for a few specific values of Γ_m for $N = 2$, 3, and 4 sections; more extensive tables can be found in reference [2].

EXAMPLE 5.7 CHEBYSHEV TRANSFORMER DESIGN

Design a three-section Chebyshev transformer to match a 100 Ω load to a 50 Ω line with $\Gamma_m = 0.05$, using the above theory. Plot the reflection coefficient magnitude versus normalized frequency for exact designs using 1, 2, 3, and 4 sections.

Solution

From (5.61) with $N = 3$,

$$\Gamma(\theta) = 2e^{-j3\theta}\left(\Gamma_0\cos 3\theta + \Gamma_1\cos\theta\right) = Ae^{-j3\theta}T_3(\sec\theta_m\cos\theta).$$

Then $A = \Gamma_m = 0.05$, and from (5.63),

$$\begin{aligned}\sec\theta_m &= \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{\ln Z_L/Z_0}{2\Gamma_m}\right)\right] \\ &= \cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{\ln(100/50)}{2(0.05)}\right)\right] \\ &= 1.408,\end{aligned}$$

so $\theta_m = 44.7°$.

Using (5.60c) for T_3 gives

$$2\left(\Gamma_0\cos 3\theta + \Gamma_1\cos\theta\right) = A\sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3A\sec\theta_m\cos\theta.$$

TABLE 5.2 Chebyshev Transformer Design

	$N = 2$				$N = 3$					
	$\Gamma_m = 0.05$		$\Gamma_m = 0.20$		$\Gamma_m = 0.05$			$\Gamma_m = 0.20$		
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.1347	1.3219	1.2247	1.2247	1.1029	1.2247	1.3601	1.2247	1.2247	1.2247
2.0	1.2193	1.6402	1.3161	1.5197	1.1475	1.4142	1.7429	1.2855	1.4142	1.5558
3.0	1.3494	2.2232	1.4565	2.0598	1.2171	1.7321	2.4649	1.3743	1.7321	2.1829
4.0	1.4500	2.7585	1.5651	2.5558	1.2662	2.0000	3.1591	1.4333	2.0000	2.7908
6.0	1.6047	3.7389	1.7321	3.4641	1.3383	2.4495	4.4833	1.5193	2.4495	3.9492
8.0	1.7244	4.6393	1.8612	4.2983	1.3944	2.8284	5.7372	1.5766	2.8284	5.0742
10.0	1.8233	5.4845	1.9680	5.0813	1.4385	3.1623	6.9517	1.6415	3.1623	6.0920

	$N = 4$							
	$\Gamma_m = 0.05$				$\Gamma_m = 0.20$			
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.0892	1.1742	1.2775	1.3772	1.2247	1.2247	1.2247	1.2247
2.0	1.1201	1.2979	1.5409	1.7855	1.2727	1.3634	1.4669	1.5715
3.0	1.1586	1.4876	2.0167	2.5893	1.4879	1.5819	1.8965	2.0163
4.0	1.1906	1.6414	2.4369	3.3597	1.3692	1.7490	2.2870	2.9214
6.0	1.2290	1.8773	3.1961	4.8820	1.4415	2.0231	2.9657	4.1623
8.0	1.2583	2.0657	3.8728	6.3578	1.4914	2.2428	3.5670	5.3641
10.0	1.2832	2.2268	4.4907	7.7930	1.5163	2.4210	4.1305	6.5950

Equating similar terms in $\cos n\theta$ gives the following results:

$$\begin{aligned} \cos 3\theta: \quad 2\Gamma_0 &= A \sec^3 \theta_m, \\ \Gamma_0 &= 0.0698; \\ \cos \theta: \quad 2\Gamma_1 &= 3A(\sec^3 \theta_m - \sec \theta_m), \\ \Gamma_1 &= 0.1037. \end{aligned}$$

From symmetry we also have that

$$\begin{aligned} \Gamma_3 &= \Gamma_0 = 0.0698, \\ \Gamma_2 &= \Gamma_1 = 0.1037. \end{aligned}$$

Then the characteristic impedances are:

$$\begin{aligned} n = 0: \quad \ln Z_1 &= \ln Z_0 + 2\Gamma_0 \\ &= \ln 50 + 2(0.0698) = 4.051 \\ Z_1 &= 57.5\ \Omega; \\ n = 1: \quad \ln Z_2 &= \ln Z_1 + 2\Gamma_1 \\ &= \ln 57.5 + 2(0.1037) = 4.259 \\ Z_2 &= 70.7\ \Omega; \\ n = 2: \quad \ln Z_3 &= \ln Z_2 + 2\Gamma_2 \\ &= \ln 70.7 + 2(0.1037) = 4.466 \\ Z_3 &= 87.0\ \Omega. \end{aligned}$$

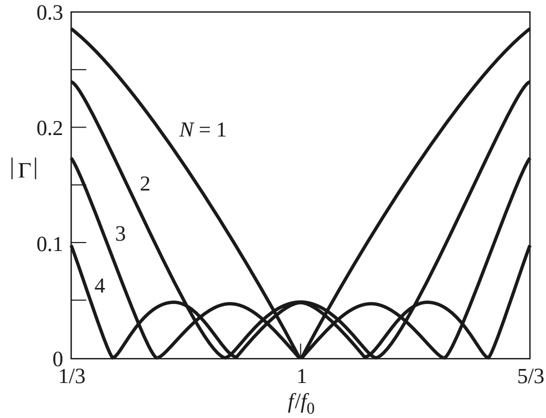


FIGURE 5.17 Reflection coefficient magnitude versus frequency for the multisection matching transformers of Example 5.7.

These values can be compared to the exact values from Table 5.2 of $Z_1 = 57.37\ \Omega$, $Z_2 = 70.71\ \Omega$, and $Z_3 = 87.15\ \Omega$. The bandwidth, from (5.64), is

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi} = 2 - 4\left(\frac{44.7^\circ}{180^\circ}\right) = 1.01,$$

or 101%. This is significantly greater than the bandwidth of the binomial transformer of Example 5.6 (70%), which involved the same impedance mismatch. The trade-off, of course, is a nonzero ripple in the passband of the Chebyshev transformer.

Figure 5.17 shows reflection coefficient magnitudes versus frequency for the exact designs from Table 5.2 for $N = 1$, 2, 3, and 4 sections. ■

5.8 TAPERED LINES

In the preceding sections we discussed how an arbitrary real load impedance could be matched to a line over a desired bandwidth by using multisection matching transformers. As the number N of discrete transformer sections increases, the step changes in characteristic impedance between the sections become smaller, and the transformer geometry approaches a continuously tapered line. In practice, of course, a matching transformer must be of finite length—often no more than a few sections long. This suggests that, instead of discrete sections, the transformer can be continuously tapered, as shown in Figure 5.18a. Different passband characteristics can be obtained by using different types of taper.

In this section we will derive an approximate theory, again based on the theory of small reflections, to predict the reflection coefficient response as a function of the impedance taper versus position, $Z(z)$. We will apply these results to a few common types of impedance tapers.

Consider the continuously tapered line of Figure 5.18a as being made up of a number of incremental sections of length Δz, with an impedance change $\Delta Z(z)$ from one

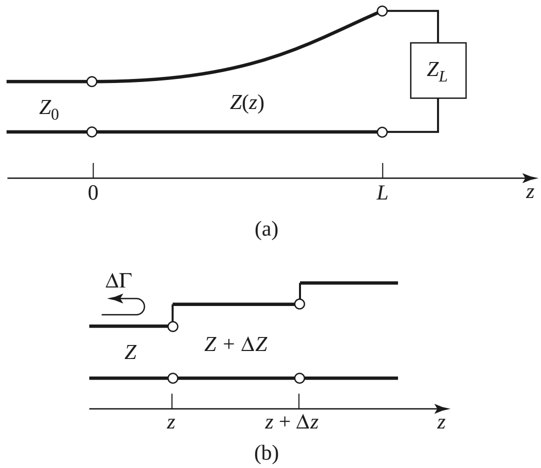


FIGURE 5.18 A tapered transmission line matching section and the model for an incremental length of tapered line. (a) The tapered transmission line matching section. (b) Model for an incremental step change in impedance of the tapered line.

section to the next, as shown in Figure 5.18b. The incremental reflection coefficient from the impedance step at z is given by

$$\Delta\Gamma = \frac{(Z + \Delta Z) - Z}{(Z + \Delta Z) + Z} \simeq \frac{\Delta Z}{2Z}. \tag{5.65}$$

In the limit as $\Delta z \to 0$ we have an exact differential:

$$d\Gamma = \frac{dZ}{2Z} = \frac{1}{2}\frac{d(\ln Z/Z_0)}{dz}dz, \tag{5.66}$$

since

$$\frac{d(\ln f(z))}{dz} = \frac{1}{f}\frac{df(z)}{dz}.$$

By using the theory of small reflections, we can find the total reflection coefficient at $z = 0$ by summing all the partial reflections with their appropriate phase shifts:

$$\Gamma(\theta) = \frac{1}{2}\int_{z=0}^{L} e^{-2j\beta z}\frac{d}{dz}\ln\left(\frac{Z}{Z_0}\right)dz, \tag{5.67}$$

where $\theta = 2\beta\ell$. If $Z(z)$ is known, $\Gamma(\theta)$ can be found as a function of frequency. Alternatively, if $\Gamma(\theta)$ is specified, then in principle $Z(z)$ can be found by inversion. This latter procedure is difficult, and is generally avoided in practice; the reader is referred to references [1] and [4] for further discussion of this topic. Here we will consider three special cases of $Z(z)$ impedance tapers, and evaluate the resulting responses.

Exponential Taper

Consider first an *exponential taper*, where

$$Z(z) = Z_0 e^{az} \quad \text{for } 0 < z < L, \tag{5.68}$$

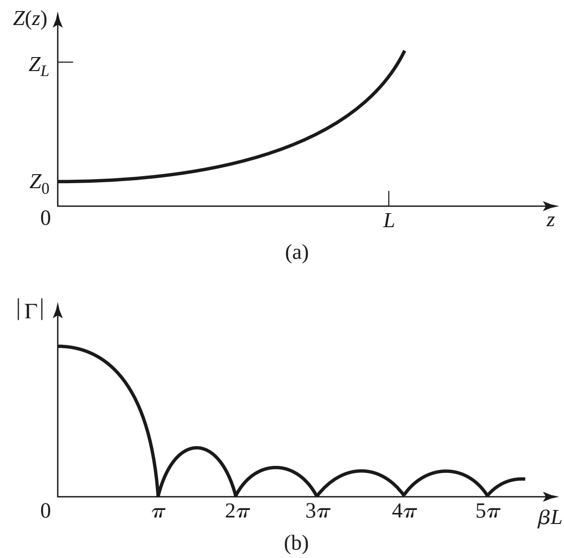


FIGURE 5.19 A matching section with an exponential impedance taper. (a) Variation of impedance. (b) Resulting reflection coefficient magnitude response.

as indicated in Figure 5.19a. At $z = 0$, $Z(0) = Z_0$, as desired. At $z = L$ we wish to have $Z(L) = Z_L = Z_0 e^{aL}$, which determines the constant a as

$$a = \frac{1}{L} \ln\left(\frac{Z_L}{Z_0}\right). \tag{5.69}$$

We find $\Gamma(\theta)$ by using (5.68) and (5.69) in (5.67):

$$\begin{aligned}\Gamma &= \frac{1}{2}\int_0^L e^{-2j\beta z}\frac{d}{dz}(\ln e^{az})dz \\ &= \frac{\ln Z_L/Z_0}{2L}\int_0^L e^{-2j\beta z}dz \\ &= \frac{\ln Z_L/Z_0}{2} e^{-j\beta L}\frac{\sin\beta L}{\beta L}.\end{aligned} \tag{5.70}$$

Observe that this derivation assumes that β, the propagation constant of the tapered line, is not a function of z—an assumption generally valid only for TEM lines.

The magnitude of the reflection coefficient in (5.70) is sketched in Figure 5.19b; note that the peaks in $|\Gamma|$ decrease with increasing length, as one might expect, and that the length should be greater than $\lambda/2$ $(\beta L > \pi)$ to minimize the mismatch at low frequencies.

Triangular Taper

Next consider a *triangular taper* for $d\ln(Z/Z_0)/dz$, that is,

$$Z(z) = \begin{cases} Z_0 e^{2(z/L)^2 \ln Z_L/Z_0} & \text{for } 0 \le z \le L/2 \\ Z_0 e^{(4z/L - 2z^2/L^2 - 1)\ln Z_L/Z_0} & \text{for } L/2 \le z \le L, \end{cases} \tag{5.71}$$

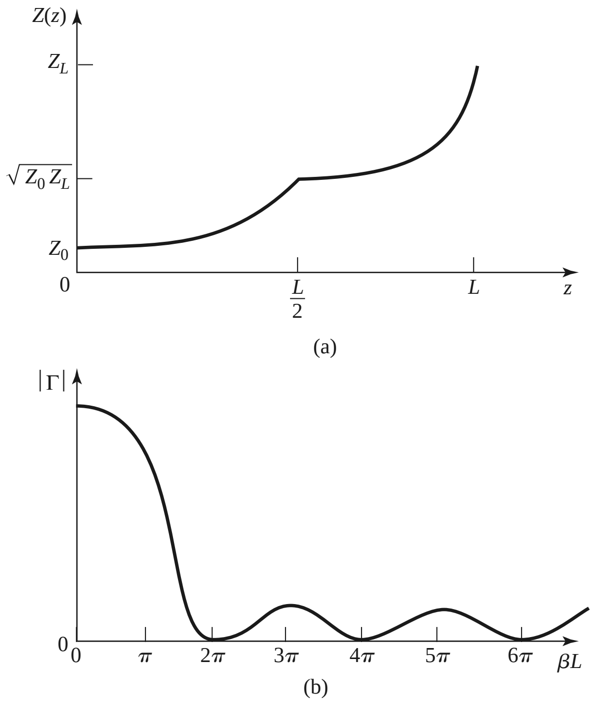


FIGURE 5.20 A matching section with a triangular taper for $d(\ln Z/Z_0)/dz$. (a) Variation of impedance. (b) Resulting reflection coefficient magnitude response.

so that the derivative is triangular in form:

$$\frac{d(\ln Z/Z_0)}{dz} = \begin{cases} 4z/L^2 \ln Z_L/Z_0 & \text{for } 0 \leq z \leq L/2 \\ (4/L - 4z/L^2) \ln Z_L/Z_0 & \text{for } L/2 \leq z \leq L. \end{cases} \tag{5.72}$$

$Z(z)$ is plotted in Figure 5.20a. Evaluating Γ from (5.67) gives

$$\Gamma(\theta) = \frac{1}{2} e^{-j\beta L} \ln\left(\frac{Z_L}{Z_0}\right) \left[\frac{\sin(\beta L/2)}{\beta L/2}\right]^2. \tag{5.73}$$

The magnitude of this result is sketched in Figure 5.20b. Note that, for $\beta L > 2\pi$, the peaks of the triangular taper are lower than the corresponding peaks of the exponential case. However, the first null for the triangular taper occurs at $\beta L = 2\pi$, whereas for the exponential taper it occurs at $\beta L = \pi$.

Klopfenstein Taper

Considering the fact that there is an infinite number of possibilities for choosing an impedance matching taper, it is logical to ask if there is a design that is "best." For a given taper length (greater than some critical value), the *Klopfenstein impedance taper* [4, 5] has been shown to be optimum in the sense that the reflection coefficient is minimum over the passband. Alternatively, for a maximum reflection coefficient specification in the passband, the Klopfenstein taper yields the shortest matching section.

The Klopfenstein taper is derived from a stepped Chebyshev transformer as the number of sections increases to infinity, and is analogous to the Taylor distribution of antenna array theory. We will not present the details of this derivation, which can be found in references [1] and [4]; only the necessary results for the design of Klopfenstein tapers are given in what follows.

The logarithm of the characteristic impedance variation for the Klopfenstein taper is given by

$$\ln Z(z) = \frac{1}{2}\ln Z_0 Z_L + \frac{\Gamma_0}{\cosh A} A^2 \phi(2z/L - 1,\ A) \quad \text{for } 0 \le z \le L, \tag{5.74}$$

where the function $\phi(x, A)$ is defined as

$$\phi(x, A) = -\phi(-x,\ A) = \int_0^x \frac{I_1(A\sqrt{1-y^2})}{A\sqrt{1-y^2}} dy \quad \text{for } |x| \le 1, \tag{5.75}$$

where $I_1(x)$ is the modified Bessel function. The function of (5.75) has the following special values:

$$\phi(0,\ A) = 0$$
$$\phi(x,\ 0) = \frac{x}{2}$$
$$\phi(1,\ A) = \frac{\cosh A - 1}{A^2},$$

but otherwise (5.75) must be calculated numerically. A simple and efficient method for doing this is available [6].

The resulting reflection coefficient is given by

$$\Gamma(\theta) = \Gamma_0 e^{-j\beta L} \frac{\cos\sqrt{(\beta L)^2 - A^2}}{\cosh A} \quad \text{for } \beta L > A. \tag{5.76}$$

If $\beta L < A$, the $\cos\sqrt{(\beta L)^2 - A^2}$ term becomes $\cosh\sqrt{A^2 - (\beta L)^2}$.

In (5.74) and (5.76), Γ_0 is the reflection coefficient at zero frequency, given as

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} \simeq \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right). \tag{5.77}$$

The passband is defined as $\beta L \ge A$, and so the maximum ripple in the passband is

$$\Gamma_m = \frac{\Gamma_0}{\cosh A} \tag{5.78}$$

because $\Gamma(\theta)$ oscillates between $\pm\Gamma_0/\cosh A$ for $\beta L > A$.

It is interesting to note that the impedance taper of (5.74) has steps at $z = 0$ and L (the ends of the tapered section) and so does not smoothly join the source and load impedances. A typical Klopfenstein impedance taper and its response are given in the following example.

EXAMPLE 5.8 DESIGN OF TAPERED MATCHING SECTIONS

Design a triangular taper, an exponential taper, and a Klopfenstein taper (with $\Gamma_m = 0.02$) to match a 50 Ω load to a 100 Ω line. Plot the impedance variations and resulting reflection coefficient magnitudes versus βL.

Solution

Triangular taper: From (5.71) the impedance variation is

$$Z(z) = Z_0 \begin{cases} e^{2(z/L)^2 \ln Z_L/Z_0} & \text{for } 0 \le z \le L/2 \\ e^{(4z/L - 2z^2/L^2 - 1)\ln Z_L/Z_0} & \text{for } L/2 \le z \le L, \end{cases}$$

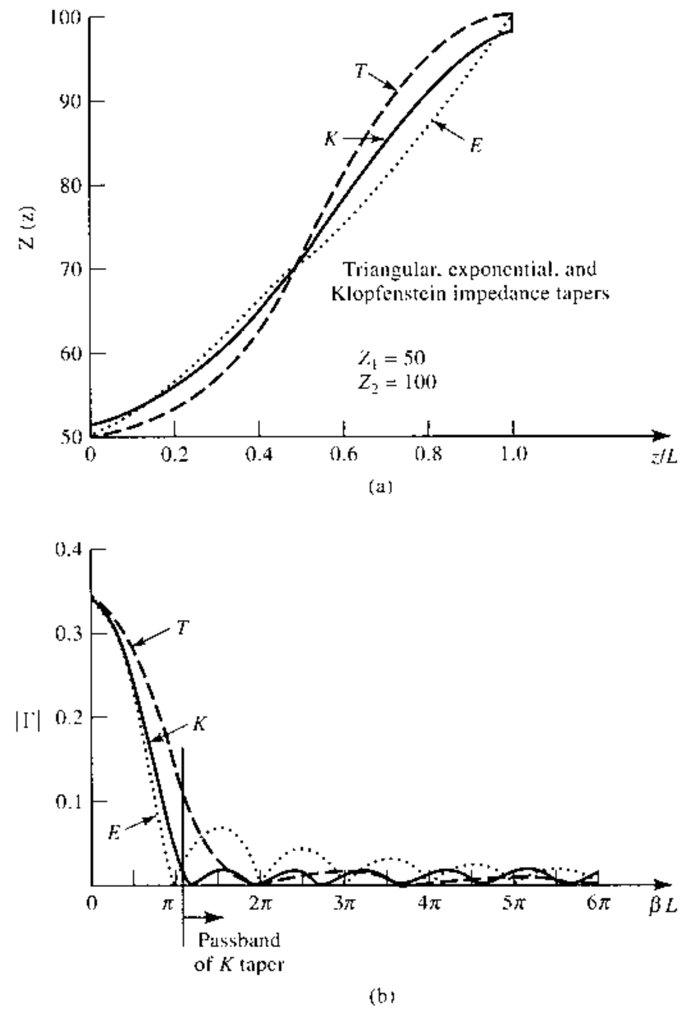


FIGURE 5.21 Solution to Example 5.8. (a) Impedance variations for the triangular, exponential, and Klopfenstein tapers. (b) Resulting reflection coefficient magnitude versus frequency for the tapers of (a).

with $Z_0 = 100\ \Omega$ and $Z_L = 50\ \Omega$. The resulting reflection coefficient response is given by (5.73):

$$|\Gamma(\theta)| = \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right)\left[\frac{\sin(\beta L/2)}{\beta L/2}\right]^2.$$

Exponential taper: From (5.68) the impedance variation is

$$Z(z) = Z_0 e^{az} \quad \text{for } 0 < z < L,$$

with $a = (1/L)\ln Z_L/Z_0 = 0.693/L$. The reflection coefficient response is, from (5.70),

$$|\Gamma(\theta)| = \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right)\frac{\sin\beta L}{\beta L}.$$

Klopfenstein taper: Using (5.77) gives Γ_0 as

$$\Gamma_0 = \frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right) = 0.346,$$

and (5.78) gives A as

$$A = \cosh^{-1}\left(\frac{\Gamma_0}{\Gamma_m}\right) = \cosh^{-1}\left(\frac{0.346}{0.02}\right) = 3.543.$$

The impedance taper must be numerically evaluated from (5.74). The reflection coefficient magnitude is given by (5.76):

$$|\Gamma(\theta)| = \Gamma_0 \frac{\cos\sqrt{(\beta L)^2 - A^2}}{\cosh A}.$$

The passband for the Klopfenstein taper is defined as $\beta L > A = 3.543 = 1.13\pi$.

Figure 5.21 shows the impedance variations (vs. z/L), and the resulting reflection coefficient magnitude (vs. βL) for the three types of tapers. The Klopfenstein taper gives the desired response of $|\Gamma| \leq \Gamma_m = 0.02$ for $\beta L \geq 1.13\pi$, which is smaller than the corresponding lengths of either the triangular or the exponential taper transformer. Also note that, like the stepped-Chebyshev matching transformer, the response of the Klopfenstein taper has equal-ripple lobes versus frequency in its passband. ■

5.9 THE BODE–FANO CRITERION

In this chapter we discussed several techniques for matching an arbitrary load at a single frequency, using lumped elements, tuning stubs, and single-section quarter-wave transformers. We presented multisection matching transformers and tapered lines as a means of obtaining broader bandwidths with various passband characteristics. We close our study of impedance matching with a somewhat qualitative discussion of the theoretical limits that constrain the performance of an impedance matching network.

We limit our discussion to the circuit of Figure 5.1, where a lossless network is used to match an arbitrary complex load, generally over a nonzero bandwidth. From a very general perspective, we might raise the following questions in regard to this problem:

- Can we achieve a perfect match (zero reflection) over a specified bandwidth?
- If not, how well can we do? What is the trade-off between Γ_m, the maximum allowable reflection in the passband, and the bandwidth?
- How complex must the matching network be for a given specification?

These questions can be answered by the Bode–Fano criterion [7, 8] which gives, for certain canonical types of load impedances, a theoretical limit on the minimum reflection coefficient magnitude that can be obtained with an arbitrary matching network. The Bode–Fano criterion thus represents an optimum result that can be ideally achieved, even though such a result may only be approximated in practice. Such optimal results are always important, however, because they specify an upper limit of performance, and so provide a benchmark against which a practical design can be compared.

Figure 5.22a shows a lossless network used to match a parallel RC load impedance. The Bode–Fano criterion states that

$$\int_0^\infty \ln \frac{1}{|\Gamma(\omega)|} d\omega \leq \frac{\pi}{RC}, \quad (5.79)$$

where $\Gamma(\omega)$ is the reflection coefficient seen looking into the arbitrary lossless matching network. The derivation of this result is beyond the scope of this text (the interested reader is referred to references [7] and [8]); our goal here is to discuss the implications of this result.

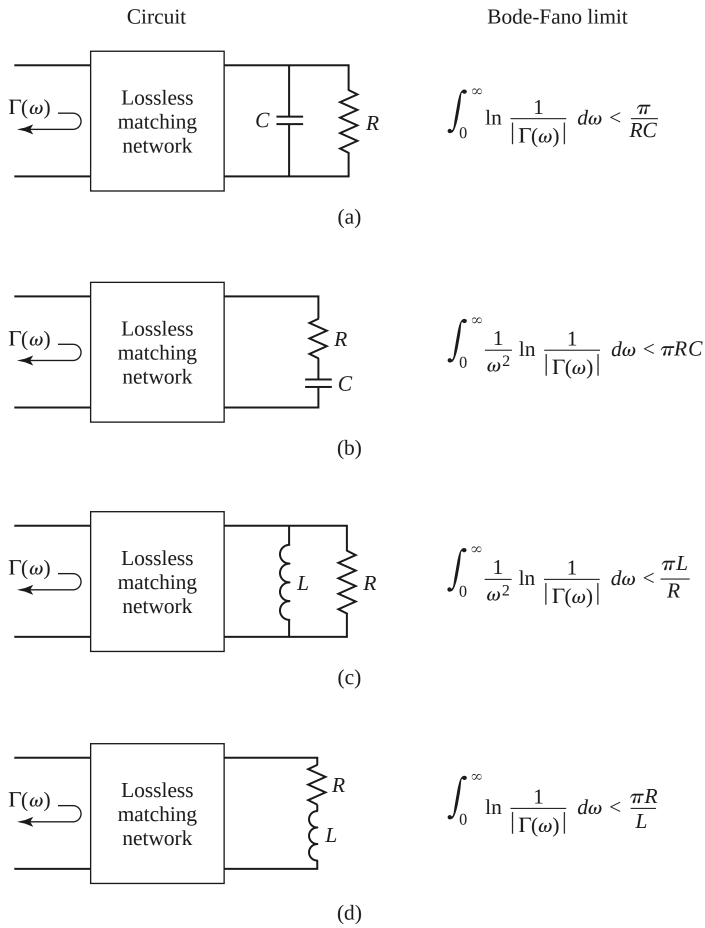


FIGURE 5.22 The Bode–Fano limits for *RC* and *RL* loads matched with passive and lossless networks (ω_0 is the center frequency of the matching bandwidth). (a) Parallel *RC*. (b) Series *RC*. (c) Parallel *RL*. (d) Series *RL*.

Assume that we desire to synthesize a matching network with a reflection coefficient response like that shown in Figure 5.23a. Applying (5.79) to this function gives

$$\int_0^\infty \ln \frac{1}{|\Gamma|} d\omega = \int_{\Delta\omega} \ln \frac{1}{\Gamma_m} d\omega = \Delta\omega \ln \frac{1}{\Gamma_m} \leq \frac{\pi}{RC}, \tag{5.80}$$

which leads to the following conclusions:

- For a given load (a fixed *RC* product), a broader bandwidth ($\Delta\omega$) can be achieved only at the expense of a higher reflection coefficient in the passband (Γ_m).
- The passband reflection coefficient, Γ_m, cannot be zero unless $\Delta\omega = 0$. Thus a perfect match can be achieved only at a finite number of discrete frequencies, as illustrated in Figure 5.23b.
- As R and/or C increases, the quality of the match ($\Delta\omega$ and/or $1/\Gamma_m$) must decrease. Thus, higher-Q circuits are intrinsically harder to match than are lower-Q circuits (we will discuss Q in Chapter 6).

Because $\ln(1/|\Gamma|)$ is proportional to the return loss (in dB) at the input of the matching network, (5.79) can be interpreted as requiring that the area between the return loss curve

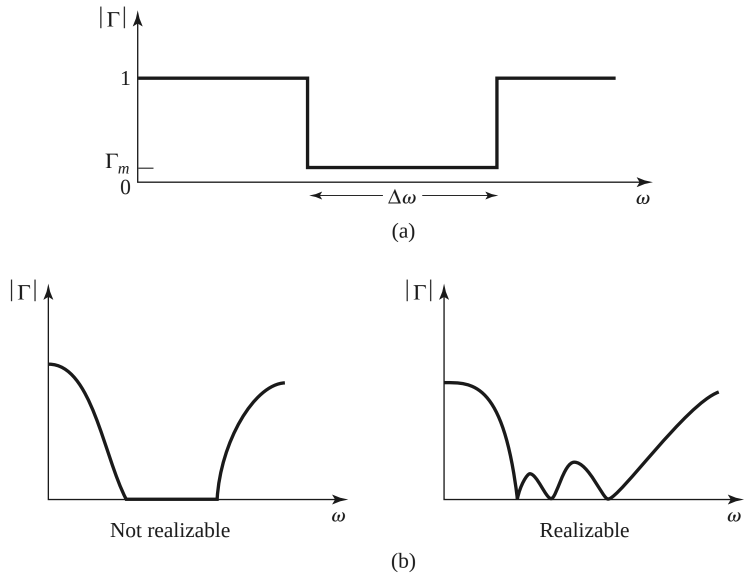


FIGURE 5.23 Illustrating the Bode–Fano criterion. (a) A possible reflection coefficient response. (b) Nonrealizable and realizable reflection coefficient responses.

and the $|\Gamma| = 1$ ($RL = 0$ dB) axis must be less than or equal to a particular constant. Optimization then implies that the return loss curve be adjusted so that $|\Gamma| = \Gamma_m$ over the passband and $|\Gamma| = 1$ elsewhere, as in Figure 5.23a. In this way, no area under the return loss curve is wasted outside the passband, or lost in regions within the passband for which $|\Gamma| < \Gamma_m$. The square-shaped response of Figure 5.23a is therefore the optimum response, but cannot be realized in practice because it would require an infinite number of elements in the matching network. It can be approximated, however, with a reasonably small number of elements, as described in reference [8]. Finally, note that the Chebyshev matching transformer can be considered as a close approximation to the ideal passband of Figure 5.23a when the ripple of the Chebyshev response is made equal to Γ_m. Figure 5.22 lists the Bode–Fano limits for other types of *RC* and *RL* loads.

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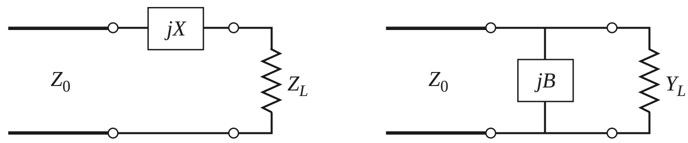
PROBLEMS

5.1 Design two lossless L-section matching circuits to match each of the following loads to a 100 Ω generator at 3 GHz. (a) $Z_L = 150 - j200\ \Omega$ and (b) $Z_L = 20 - j90\ \Omega$.

5.2 We have seen that the matching of an arbitrary load impedance requires a network with at least two degrees of freedom. Determine the types of load impedances/admittances that can be matched with the two single-element networks shown below.



(a) (b)

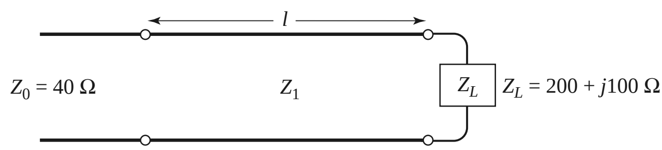
5.3 A load impedance $Z_L = 100 + j80\ \Omega$ is to be matched to a 75 Ω line using a single shunt-stub tuner. Find two designs using open-circuited stubs.

5.4 Repeat Problem 5.3 using short-circuited stubs.

5.5 A load impedance $Z_L = 90 + j60\ \Omega$ is to be matched to a 75 Ω line using a single series-stub tuner. Find two designs using open-circuited stubs.

5.6 Repeat Problem 5.5 using short-circuited stubs.

5.7 In the circuit shown below a load $Z_L = 200 + j100\ \Omega$ is to be matched to a 40 Ω line, using a length ℓ of lossless transmission line of characteristic impedance Z_1. Find ℓ and Z_1. Determine, in general, what type of load impedances can be matched using such a circuit.



5.8 An open-circuit tuning stub is to be made from a lossy transmission line with an attenuation constant α. What is the maximum value of normalized reactance that can be obtained with this stub? What is the maximum value of normalized reactance that can be obtained with a shorted stub of the same type of transmission line? Assume $\alpha\ell$ is small.

5.9 Design a double-stub tuner using open-circuited stubs with a $\lambda/8$ spacing to match a load admittance $Y_L = (0.4 + j1.2)Y_0$.

5.10 Repeat Problem 5.9 using a double-stub tuner with short-circuited stubs and a $3\lambda/8$ spacing.

5.11 Derive the design equations for a double-stub tuner using two series stubs spaced a distance d apart. Assume the load impedance is $Z_L = R_L + jX_L$.

5.12 Consider matching a load $Z_L = 200\ \Omega$ to a 100 Ω line, using single shunt-stub, single series stub, and double shunt-stub tuners, with short-circuited stubs. Which tuner will give the best bandwidth? Justify your answer by calculating the reflection coefficient for all six solutions at $1.1 f_0$, where f_0 is the match frequency, or use CAD to plot the reflection coefficient versus frequency.

5.13 Design a single-section quarter-wave matching transformer to match a 350 Ω load to a 100 Ω line. What is the percent bandwidth of this transformer, for SWR ≤ 2? If the design frequency is 4 GHz, sketch the layout of a microstrip circuit, including dimensions, to implement this matching transformer. Assume the substrate is 0.159 cm thick, with a relative permittivity of 2.2.

5.14 Consider the quarter-wave transformer of Figure 5.13 with $Z_1 = 100\ \Omega$, $Z_2 = 150\ \Omega$, and $Z_L = 225\ \Omega$. Evaluate the worst-case percent error in computing $|\Gamma|$ from the approximate expression (5.42), compared to the exact result.

5.15 A waveguide load with an equivalent TE_{10} wave impedance of 377 Ω must be matched to an air-filled X-band rectangular guide at 10 GHz. A quarter-wave matching transformer is to be used, and is to consist of a section of guide filled with dielectric. Find the required dielectric constant and physical length of the matching section. What restrictions on the load impedance apply to this technique?

5.16 A four-section binomial matching transformer is to be used to match a 12.5 Ω load to a 50 Ω line at a center frequency of 1 GHz. (a) Design the matching transformer, and compute the bandwidth for $\Gamma_m = 0.05$. Use CAD to plot the input reflection coefficient versus frequency. (b) Lay out the microstrip implementation of this circuit on an FR4 substrate having $\epsilon_r = 4.2$, $d = 0.158$ cm, and $\tan\delta = 0.02$, with copper conductors 0.5 mil thick. Use CAD to plot the insertion loss versus frequency.

5.17 Derive the exact characteristic impedance for a two-section binomial matching transformer for a normalized load impedance $Z_L/Z_0 = 1.5$. Check your results with Table 5.1.

5.18 Calculate and plot the percent bandwidth for an $N = 1$-, 2-, and 4-section binomial matching transformer versus $Z_L/Z_0 = 1.5$ to 6 for $\Gamma_m = 0.2$.

5.19 Design a four-section Chebyshev matching transformer to match a 50 Ω line to a 30 Ω load. The maximum permissible SWR over the passband is 1.25. What is the resulting bandwidth? Use the approximate theory developed in the text, as opposed to the tables. Use CAD to plot the input SWR versus frequency.

5.20 Derive the exact characteristic impedances for a two-section Chebyshev matching transformer for a normalized load impedance $Z_L/Z_0 = 1.5$. Check your results with Table 5.2 for $\Gamma_m = 0.05$.

5.21 A load of $Z_L/Z_0 = 1.5$ is to be matched to a feed line using a multisection transformer, and it is desired to have a passband response with $|\Gamma(\theta)| = A(0.1 + \cos^2\theta)$ for $0 \leq \theta \leq \pi$. Use the approximate theory for multisection transformers to design a two-section transformer.

5.22 A tapered matching section has $d\ln(Z/Z_0)/dz = A\sin\pi z/L$. Find the constant A so that $Z(0) = Z_0$ and $Z(L) = Z_L$. Compute Γ, and plot $|\Gamma|$ versus βL.

5.23 Design an exponentially tapered matching transformer to match a 100 Ω load to a 50 Ω line. Plot $|\Gamma|$ versus βL, and find the length of the matching section (at the center frequency) required to obtain $|\Gamma| \leq 0.05$ over a 100% bandwidth. How many sections would be required if a Chebyshev matching transformer were used to achieve the same specifications?

5.24 An ultra wideband (UWB) radio transmitter, operating from 3.1 to 10.6 GHz, drives a parallel RC load with $R = 75$ Ω and $C = 0.6$ pF. What is the best return loss that can be obtained with an optimum matching network?

5.25 Consider a series RL load with $R = 80$ Ω and $L = 5$ nH. Design a lumped-element L-section matching network to match this load to a 50 Ω line at 2 GHz. Plot $|\Gamma|$ versus frequency for this network to determine the bandwidth for which $|\Gamma| \leq \Gamma_m = 0.1$. Compare this with the maximum possible bandwidth for this load, as given by the Bode–Fano criterion. (Assume a square reflection coefficient response like that of Figure 5.23a.)