Chapter 11

Improper Integrals

Improper integrals are important to many applications where the concept of ∞ is used to implement some physical idealization. Or in probability, if you have a continuous random variable then the natural condition for the probability density function ϕ is $\int_{-\infty}^{\infty} \phi(x)dx = 1$. In physics the divergence of functions often represents a physically interesting quantity. For example, if $E(x)$ is the electric field in a one-dimensional system and $E(x) \to \infty$ as $x \to x_o^+$ then there is a positive electric charge at $x = x_o$. The mathematics of improper integrals are all made by combining the concept of the integral with the concept of a limit at a point or infinity or both. Once you understand the definitions in this Chapter they are entirely natural, with perhaps the exception of $\int_{-\infty}^{\infty} f(x)dx$.

Beyond infinity, or perhaps before it, we can use the concept of improper integration to carefully define integration of piecewise-continuous functions. The reason is subtle. Riemann integration is only defined for closed intervals. If $f(x)$ is piecewise defined to be $f_1(x)$ for $x \in [a, c)$ and on the other hand $f_2(x)$ for $x \in (c, b]$ then many calculus I instructors state that

$$\int_a^b f(x)dx = \int_a^c f_1(x)dx + \int_c^b f_2(x)dx.$$

However, it is cannot just mean the Riemann integral[1] of f_1 and f_2. At least without some qualification this equation is nonsense[2] because I cannot set up a partitition which allows us to evaluate $f_1(c)$ or $f_2(c)$. These points need not be in the domain of f and they certainly cannot be defined for both f_1 and f_2 and yet be distinct (if they were not equal then f would not be function). So, how should we define such an integral? It's not hard, we just have to introduce variable bounds and allow those bounds to tend to the point c rather than actually reach it. In practice, I don't force students to write these limits in calculus I because in the case of finite-jump discontinuities there is no danger of getting the wrong answer[3]. On the other hand, once the integrand has a point of discontinuity where there is a vertical asymptote the limit is an essential detail as application of L'Hospital's rule and other subtle limit calculations are totally possible.

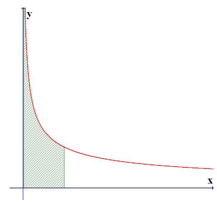
[1] not as we have defined it at least

[2] it will make perfect sense soon, we just need a simple definition to fix this hole. Actually, depending on the finer points of your definition of integration this may already be meaningful. The point I raise here may be too picky for the integration theory given in calculus I and II.

[3] I show we were correct later in this chapter, I show the limits are not really necessary for the bounded piecewise continuous case.

11.1 divergent integrands

Let me begin with a question: what is the meaning of $\int_0^1 \frac{dx}{\sqrt{x}}$? Notice it cannot just be the limit of a Riemann sum since the function is not even defined at the first partitition point $x_o = a = 0$ and that point gives division by zero for the integrand $1/\sqrt{x}$. On the other hand, if we consider the graph then we'll note there might be a chance that the area bounded by $x = 0$, $x = 1$, $y = 0$ and $y = 1/\sqrt{x}$ is finite since the shape gets very very narrow where it gets very very tall.



Furthermore, if $t > 0$ then clearly the integral $\int_t^1 \frac{dx}{\sqrt{x}}$ is meaningful for each such value t. I propose the natural definition for the integral in question is simply as follows:

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}}$$

The expresssion above defines the *improper* integral on the LHS in terms of concepts on the RHS which we have already given explicit and rigorous meaning.

Example 11.1.1.

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \left(2\sqrt{x}\Big|_t^1 \right) = \lim_{t\to 0^+} \left(2\sqrt{1} - 2\sqrt{t}\right) = \boxed{2.}$$

I'll give a careful definition that covers the preceding example. Again, this is the natural definition because it extends our concept in a way such that the integral still has the same essential meaning.

Definition 11.1.2. *Improper integral with divergence at edge of integration interval.*

Assume $a < b$ and $(a, b) \subseteq dom(f)$. The integral $\int_a^b f(x)dx$ is generalized to mean the following in the cases that either $a \notin dom(f)$ or $b \notin dom(f)$.

1. If $a \notin dom(f)$ however $b \in dom(f)$ then $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$.
2. If $b \notin dom(f)$ however $a \in dom(f)$ then $\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$.
3. If both $a, b \notin dom(f)$ then we choose $c \in (a, b)$ and define $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^c f(x)dx + \lim_{s\to b^-} \int_c^s f(x)dx$.

If any of the limits exist then we say the integral converges to the value of the limit. Moreover, if the integral converges to a value we say it is **convergent**. However, if any of the limits above do not exist then we say the integral is **divergent**. In the case the the limits diverge to $\pm\infty$ we will denote the divergence by $\int_a^b f(x)dx = \pm\infty$ in cases (1.) and (2.). However, in case (3.) we say that $\int_a^b f(x)dx = d.n.e.$ if one of the limits tends to ∞ while the other tends to $-\infty$. Finally, if both limits tend ∞ $(-\infty)$ then we say $\int_a^b f(x)dx$ diverges to ∞ $(-\infty)$.

You might expect that ∞ and $-\infty$ can cancel. While that is possible, it is not possible in the context of case (3.) above. The divegergence between the upper and lower bound are not connected in this definition.

Example 11.1.3.

$$\int_0^1 \frac{dx}{x} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} \left(\ln|x| \Big|_t^1 \right) = \lim_{t\to 0^+} \Big(2\ln 1 - 2\ln t \Big) = \boxed{\infty.}$$

This integral diverged to ∞.

Example 11.1.4.

$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \left(\ln|x-1| \Big|_t^1 \right) = \lim_{t\to 0^+} \Big(2\ln|t-1| - 2\ln|0-1| \Big) = \boxed{-\infty.}$$

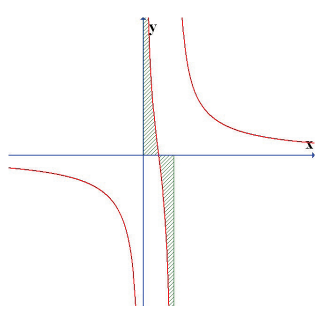
This integral diverged to ∞.

Example 11.1.5.

$$\begin{aligned}
\int_0^1 \left(\frac{1}{x} + \frac{1}{x-1}\right) dx &= \lim_{t\to 0^+} \int_t^{0.5} \left(\frac{1}{x} + \frac{1}{x-1}\right) dx + \lim_{s\to 1^-} \int_{0.5}^{s} \left(\frac{1}{x} + \frac{1}{x-1}\right) dx \\
&= \lim_{t\to 0^+} \left(\ln|x| + \ln|x-1| \Big|_t^{0.5} \right) + \lim_{s\to 1^-} \left(\ln|x| + \ln|x-1| \Big|_{0.5}^{s} \right) \\
&= \lim_{t\to 0^+} \Big(2\ln|0.5| - \ln|t| + \ln|t-1| \Big) + \lim_{s\to 1^-} \Big(\ln|s| + \ln|s-1| - 2\ln|0.5| \Big) \\
&= \infty - \infty \\
&= \boxed{d.n.e.}
\end{aligned}$$

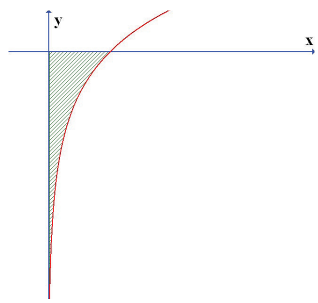
Notice that if we used some other middle point besides 0.5 the answer would not change. Moreover, if you notice the pattern of the terms following from the 0.5-evaluations it is clear this cancellation will also occur in the case of integrals which are not divergent. If this were not the case, if the integral depended on the choice of the middle point, then the definition we gave would be meaningless. I leave the proof that the ambiguity in the choice of c does not lead to ambiguity in the integral to the reader.

The area of the shaded region was shown to diverge in the last example.



Example 11.1.6. *Recall that IBP yields that* $\int \ln(x)dx = x\ln(x) - x + c$ *and consider:*

$$\begin{aligned}
\int_0^1 \ln(x)dx &= \lim_{t\to 0^+}\int_t^1 \ln(x)dx \\
&= \lim_{t\to 0^+}\Big(1\ln(1) - 1 - t\ln(t) - t\Big) \\
&= -1 + \lim_{t\to 0^+}\Big(-t\ln(t)\Big) \qquad \textit{note this is type } 0\infty \\
&= -1 + \lim_{t\to 0^+}\left(\frac{\ln(t)}{\frac{-1}{t}}\right) \qquad \textit{now is type } \infty/\infty \textit{ thus apply L'Hospital's Rule.} \\
&= -1 + \lim_{t\to 0^+}\left(\frac{\frac{1}{t}}{\frac{1}{t^2}}\right) \\
&= -1 + \lim_{t\to 0^+}(t) \\
&= \boxed{-1}.
\end{aligned}$$



Next we consider the case that the function has an isolated discontinuity in the domain of integration.

Definition 11.1.7. *Improper integral with discontinuities in the interior of integration interval.*

Assume $a < c < b$ and $\lim_{x\to c} f(x) \neq f(c)$. If both $\int_c^b f(x)dx \in \mathbb{R}$ and $\int_a^c f(x)dx \in \mathbb{R}$ then the integral $\int_a^b f(x)dx$ is generalized to mean $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. If either $\int_a^c f(x)dx$ or $\int_c^b f(x)dx$ diverge then we say $\int_a^b f(x)dx$ diverges. We say that $\int_a^b f(x)dx = d.n.e.$ if one of the improper integrals diverges to ∞ while the other diverges to $-\infty$. Finally, if both diverge to ∞ $(-\infty)$ then we say $\int_a^b f(x)dx$ diverges to ∞ $(-\infty)$.

Suppose $\{a_1, a_2, \dots, a_n\}$ is a finite set of discontinuities for an otherwise continuous function f and $a < a_1 < a_2 < \cdots < a_n < b$. are points of discontinuities for an otherwise continuous function f on $[a, b]$ then we define $\int_a^b f(x) = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \cdots + \int_{a_n}^b f(x)dx$.

Example 11.1.8.

$$\begin{aligned}
\int_{-2}^{3} \frac{1}{x} dx &= \int_{-2}^{0} \frac{dx}{x} + \int_{0}^{3} \frac{dx}{x} \\
&= \lim_{t\to 0^-} \int_{-2}^{t} \frac{dx}{x} + \lim_{s\to 0^+} \int_{s}^{3} \frac{dx}{x} \\
&= \lim_{t\to 0^-} (\ln|t| - \ln|-2|) + \lim_{s\to 0^+} (\ln|3| - \ln|s|) \\
&= -\infty + \infty \\
&= \boxed{d.n.e.}
\end{aligned}$$

Some students incorrectly think $\int_{-2}^{3} \frac{1}{x} dx$ is calculated by $\lim_{t\to 0^+}\left(\int_{-2}^{-t} \frac{dx}{x} + \int_{t}^{3} \frac{dx}{x}\right)$. Note:

$$\lim_{t\to 0^+}\left(\int_{-2}^{-t} \frac{dx}{x} + \int_{t}^{3} \frac{dx}{x}\right) = \lim_{t\to 0^+}\Big(\ln|-t| - \ln|-2| + \ln|3| - \ln|t|\Big) = \ln 3 - \ln 2$$

However, while this calculation is interesting, it is not the improper integral at zero. It is incorrect to assume that we approach zero in the same way from the left and right. These are independent limits.

Example 11.1.9.

$$\begin{aligned}
\int_{0}^{3} \frac{dx}{(x-1)^2} &= \int_{0}^{1} \frac{dx}{(x-1)^2} + \int_{1}^{3} \frac{dx}{(x-1)^2} \\
&= \lim_{t\to 1^-} \int_{0}^{t} \frac{dx}{(x-1)^2} + \lim_{s\to 1^+} \int_{s}^{3} \frac{dx}{(x-1)^2} \\
&= \lim_{t\to 1^-} \left(\frac{-1}{t-1} + \frac{1}{0-1}\right) + \lim_{s\to 1^+} \left(\frac{-1}{3-1} + \frac{1}{s-1}\right) \\
&= -(-\infty) + \infty \\
&= \boxed{\infty.}
\end{aligned}$$

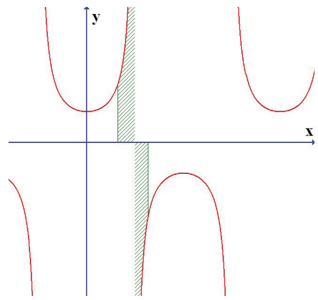
Example 11.1.10.

$$\int_{0}^{3} \frac{x^2+3x-2}{x(x^2-4)} dx = \lim_{t_1\to 0^+} \int_{t_1}^{1} \frac{x^2+3x-2}{x(x^2-4)} dx + \lim_{t_2\to 2^-} \int_{1}^{t_2} \frac{x^2+3x-2}{x(x^2-4)} dx + \lim_{t_3\to 2^+} \int_{t_3}^{3} \frac{x^2+3x-2}{x(x^2-4)} dx$$

Note that $x = -2, 2, 0$ are all points where the integrand is not defined due to division by zero hence we can only approach these points. However, only 0 and 2 are within the integration region.

Example 11.1.11. *Recall $\pi/2 \approx 1.57$ and $\sec(x) = 1/\cos(x)$ to understand what follows:*

$$\int_1^2 \sec(x)dx = \int_1^{\pi/2} \sec(x)dx + \int_{\pi/2}^2 \sec(x)dx.$$



11.1.1 on the integration of finite jump discontinuous functions

Example 11.1.12. *Notice $f(x) = x/|x|$ is discontinuous at $x = 0$ since $f(x) = x/-x = -1$ for $x < 0$ and $f(x) = x/x = 1$ for $x > 0$.*

$$\int_{-2}^3 f(x)dx = \int_{-2}^0 \frac{xdx}{|x|} + \int_0^3 \frac{xdx}{|x|} = \lim_{t\to 0^+} \int_{-2}^t (-dx) + \lim_{s \to 0^-} \int_s^3 dx = \lim_{t\to 0^+}(-2-t) + \lim_{s\to 0^-}(3-s) = 1.$$

Compare the example above to the following:

$$\int_{-2}^3 f(x)dx = \int_{-2}^0 \frac{xdx}{|x|} + \int_0^3 \frac{xdx}{|x|} = \int_{-2}^0 (-dx) + \int_0^3 dx = -x\Big|_{-2}^0 + x\Big|_0^3 = 0 - 2 + 3 - 0 = 1.$$

The calculation following the example is what is more common to find in calculus texts and courses, we ought to wonder if both of these procedures always yield the same result. The answer, in the case the function is bounded piecewise continuous, is yes. I explain why below: Suppose f has a finite jump-discontinuity at $x = c$ and is otherwise continuous on $[a, b]$ where $c \in (a, b)$. This means that $\lim_{x\to c^-} f(x) = L_1$ and $\lim_{x\to c^+} f(x) = L_2$ where $L_1, L_2 \in \mathbb{R}$ and $L_1 \neq L_2$. Define g by $g(x) = f(x)$ for $x \in [a, c)$ and $g(c) = L_1$. Observe that the function g is continuous on $[a, c]$. Apply FTC part I to obtain the existence of an antiderivative G for g then apply FTC part II to find $\int_a^t g(x)dx = G(t) - G(a)$ for any $t \in (a, c]$.

Next, define h by $h(x) = f(x)$ for $x \in (c, b]$ and define $h(c) = L_2$. Observe that the function h is continuous on $[c, b]$. Apply FTC part I to obtain the existence of an antiderivative H for h then apply FTC part II to find $\int_s^b h(x)dx = H(b) - H(s)$ for any $s \in [c, b)$.

Connsider,

$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\
&= \lim_{t\to c^+} \int_a^t f(x)dx + \lim_{s\to c^-} \int_s^b f(x)dx \\
&= \lim_{t\to c^+} \int_a^t g(x)dx + \lim_{s\to c^-} \int_s^b h(x)dx \\
&= \lim_{t\to c^+} (G(t) - G(a)) + \lim_{s\to c^-} (H(b) - H(s)) \\
&= G(c) - G(a) + H(b) - H(c) \\
&= \int_a^c g(x)dx + \int_c^b h(x)dx.
\end{aligned}$$

Therefore, to integrate a piecewise-defined function with simple jump-discontinuities we can break the integral into a sum of integrals where each integral is not of the function itself, but rather the continuous extension of the function onto the closure of the subinterval. Pragmatically, this means we need not write limits in the case of improper integrals of the finite-jump-discontinuity type. We can calculate the antiderivative of each piece, and simply sum the sub-integrals.

The especially picky reader will question how I made the first equality in the argument above. In truth, to really understand such questions we need a more careful theory of integration. The question I raised on this page probably is better left to a real analysis course.

The overall lesson I'd like you to take here is that we can ignore the contribution of a few points if the points are merely finite-jump-discontinuities. On the other hand, if the points are genuine divegences for the integrand then we have seen that more care is required.

Dr. Skoumbourdis sometimes emphasized "you can only integrate over a closed interval" and that is what got me started on the last page. I thought, when I break the piecewise integral into pieces those pieces are not closed intervals so what I am calling an integral must not quite be what I usually call an integral. Or at least, something subtle is being done with the points of discontinuity.

Those points of discontinuity are on the edge of the subintervals so it also occured to me that if we did not use the edges for our integral then we might be able to get around the problem. For example, the midpoint rule would never sample the endpoints so it would be well-defined even if the endpoints were not given for the integrand. It turns out that this is precisely the reason that the midpoint rule is interesting for numerical methods. If you want to calculate an improper integral then the midpoint rule is a nice choice because it does not sample endpoints where the function is often not even defined. In constrast, left, right, trapezoid and Simpsons all use boundary points of the integration region to approximate the integral.

11.2 divergent bounds

The other type of improper integral is where the bounds are infinite. The question we begin with is what should we mean by $\int_a^\infty f(x)dx$? The natural interpretation is to use this to denote the signed area bounded bounded by $y = f(x)$ for $x \geq a$. We can calculate the area out to $x = t$ by $\int_a^t f(x)dx$. We want to let $t \to \infty$ to catch all the signed-area bounded by $y = f(x)$. This brings us to the definition below:

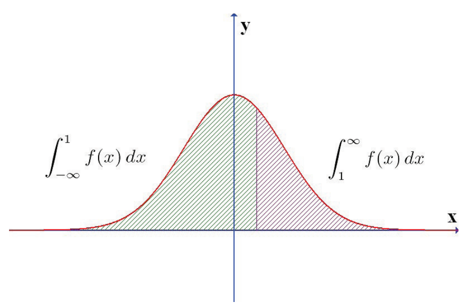
Definition 11.2.1. *Integrals with infinite bounds.*

1. If f is continuous on $[a, \infty)$ then we define $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$.
2. If f is continuous on $(-\infty, a]$ then we define $\int_{-\infty}^a f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx$.
3. If f is continuous on $\mathbb{R}$ then we define $\int_{-\infty}^\infty f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx + \lim_{s\to\infty} \int_a^s f(x)dx$.

In each case if the limit exists then we say the integral is convergent, whereas if the limit does not exist then we say it diverges. If the limit tends to $\pm\infty$ in cases (1.) or (2.) then we say the integral diverges to $\pm\infty$. In case (3.) if both limits tend to ∞ $(-\infty)$ then we say the integral diverges to ∞ $(-\infty)$. In case (3.) if one of the limits id divergent and the other is convergent then the integral diverges. Finally, if the function in question is not continuous at some finite number of points we combine this definition with the previous definition for improper integrals at points of discontinuity.

Sorry for the abundance of words, I'm simply trying to cover all the bases. One of the concepts I'm trying to impress on you is that the words "convergence" and "divergence" are now also applicable to improper integrals. Moreover, the actual calculation of of these improper integrals amounts to performing an integral with a variable bound(s) followed by taking the limit as that bound tends to $\pm\infty$. Graphically the idea is simple enough: note below that the entire area, if it is finite, is found from adding the two areas together

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$



Note the choice of $x = 1$ as the divinding line was simply a choice. Fortunately, the result will not change for different choices since an alternate choice amounts to cutting the same area into two different sized pieces[4].

Example 11.2.2.

$$\int_0^\infty xdx = \lim_{t\to\infty} \int_0^t xdx = \lim_{t\to\infty} \left[\frac{1}{2}t^2 - 0\right] = \boxed{\infty.}$$

[4]a more careful proof of this claim is left to the interested reader.

Example 11.2.3.

$$\int_{-\infty}^{0} e^x\,dx = \lim_{t\to-\infty}\int_t^0 e^x\,dx = \lim_{t\to\infty}\Big[e^0 - e^t\Big] = \boxed{1.}$$

Example 11.2.4.

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty}\Big[\tan^{-1}(t) - \tan^{-1}(0)\Big] = \boxed{\frac{\pi}{2}.}$$

See Example 2.7.11 for the background to make the last example meaningful. Notice that the integrand $\frac{1}{1+x^2} \to 0$ as $x\to\infty$. One is tempted to suppose this is enough for such an improper integral to exist. However, the next example shows that life is not so easy.

Example 11.2.5.

$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty}\Big[\ln(t) - \ln(1)\Big] = \boxed{\infty.}$$

Example 11.2.6. *Let $p>1$ and consider:*

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\left[\frac{1}{(-p+1)t^{p-1}} + \frac{1}{p-1}\right] = \boxed{\frac{1}{p-1}.}$$

The preceding pair of examples prove the nontrivial portion of the proposition below:

Theorem 11.2.7. *p-test for improper integrals.*

1. If $p \geq 1$ then $\int_1^\infty \frac{dx}{x^p}$ diverges.
2. If $p < 1$ then $\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$, hence it converges.

The case $p=1$ is important and we will see it again.

Example 11.2.8. *Refer to Example 9.3.7 for the details of how to calculate $\int \frac{dx}{4(x+1)^2+5}$.*

$$\begin{aligned}
\int_0^\infty \frac{dx}{4x^2+8x+9} &= \lim_{t\to\infty}\int_0^t \frac{dx}{4x^2+8x+9} \\
&= \lim_{t\to\infty}\left[\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2(t+1)}{\sqrt{5}}\right) - \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right] \\
&= \boxed{\frac{1}{2\sqrt{5}}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right].}
\end{aligned}$$

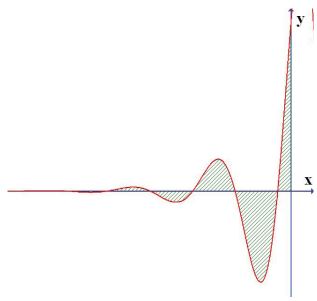
Example 11.2.9. *Refer to Example 9.1.9 for details on why*

$$\int e^x \sin(3x+1)dx = \frac{1}{10}e^x\big[\sin(3x+1) - \cos(3x+1)\big] + c.$$

I use the indefinite integral above to calculate the improper integral below:

$$\begin{aligned}
\int_{-\infty}^{0} e^x \sin(3x+1)dx &= \lim_{t\to-\infty}\int_t^0 e^x \sin(3x+1)dx \\
&= \lim_{t\to-\infty}\left[\frac{1}{10}e^0\left[\sin(1)-\cos(1)\right]-\frac{1}{10}e^t\left[\sin(3t+1)-\cos(3t+1)\right]\right] \\
&= \boxed{\frac{\sin(1)-\cos(1)}{10}.}
\end{aligned}$$

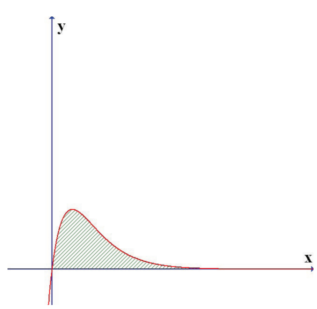
The area illustrated below is calculated by the integral above.



Example 11.2.10. *Note that* $\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -e^{-x}(x+1)+c$ *by IBP. Thus,*

$$\begin{aligned}
\int_0^{\infty} xe^{-x}dx &= \lim_{t\to\infty}\int_0^t xe^{-x}dx \\
&= \lim_{t\to\infty}\left[-e^{-t}(t+1)+e^{-0}(0+1)\right] \\
&= \lim_{t\to\infty}\left[-e^{-t}(t+1)\right]+1 \\
&= \lim_{t\to\infty}\left[-\frac{t+1}{e^t}\right]+1 \\
&= \lim_{t\to\infty}\left[-\frac{1}{e^t}\right]+1 \\
&= \boxed{1.}
\end{aligned}$$

We used L'Hospital's rule to simplify the limit of type ∞/∞ *going from the* 4^{th} *to* 5^{th} *line. The area illustrated below is calculated by the integral above.*



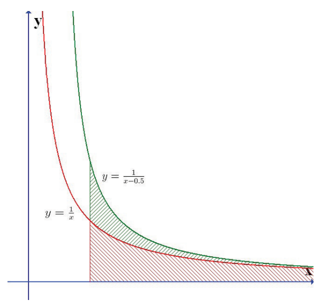
Theorem 11.2.11. *Comparison test for improper integrals.*

Suppose that f and g are continuous functions on $[a, \infty)$ such that $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

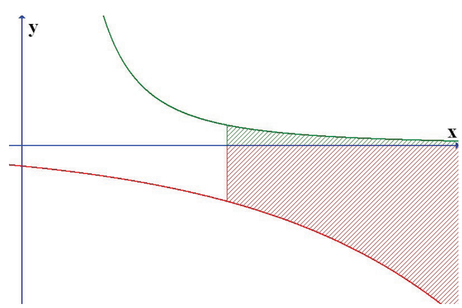
1. If $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ converges. Moreover, $\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$.
2. If $\int_a^\infty f(x)dx$ diverges then $\int_a^\infty g(x)dx$ diverges.

Proof: To prove (1.) we remind the reader that a property of definite integrals is that they preserve inequalities. In particular, $0 \leq f(x) \leq g(x)$ implies $0 \leq \int_a^t f(x)dx \leq \int_a^t g(x)dx$. Then, the limiting process likewise preserves this inequality hence $0 \leq \lim_{t\to\infty} \int_a^t f(x)dx \leq \lim_{t\to\infty} \int_a^t g(x)dx$. Note further that $\int_a^t f(x)dx$ is an increasing function of t since $\frac{d}{dt}\int_a^t f(x)dx = f(t) \geq 0$ for $t \geq a$. We have that $\int_a^t f(x)dx$ is an increasing function of t which is bounded between 0 and $\int_a^\infty g(x)dx \in \mathbb{R}$. It follows that $\lim_{t\to\infty}\int_a^t f(x)dx$ exists. Therefore, the convergence of $\int_a^\infty g(x)dx$ implies the convergence of $\int_a^\infty f(x)dx$. I leave the proof of (2.) to the reader. In fact, the reader may also need to supply the lemma which proves the limit of a bounded increasing function exists. I don't believe we have proved that result in these notes. $\square$

Example 11.2.12. *Does $\int_1^\infty \frac{dx}{x-0.5}$ converge or diverge? Notice that $0 \leq \frac{1}{x} \leq \frac{1}{x-0.5}$ for all $x \geq 1$. Furthermore $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x-0.5}$ are clearly continuous on $[1, \infty)$. Observe that $\int_1^\infty \frac{dx}{x} = \infty$ by the $p = 1$ test hence by the comparison theorem we find $\int_1^\infty \frac{dx}{x-0.5}$ diverges. The area illustrates the comparison theorem in action, the integral of the lower function diverges by the p-test for improper integrals thus the integral of the larger function must likewise diverge.*



Example 11.2.13. *Notice that $-e^x < \frac{1}{x^2}$ for $x \geq 1$. Furthermore, note that $\int_1^\infty \frac{dx}{x^2}$ converges by the $p = 2$ improper integral test. Therefore, the integral of a function larger than $-e^x$ converges hence by the comparison test $\int_1^\infty -e^x dx$ converges. Yet this is clearly false: note the area is obviously unbounded for the graph $y = -e^x$ for $x \geq 1$. What is the flaw in my logic? Fix this argument. Here is a picture of the problem:*



Theorem 11.2.14. *Tail of integral is key for improper integrals.*

Suppose that f is a continuous function on $[a, \infty)$ and $b > a$ then

1. $\int_a^\infty f(x)dx$ converges iff $\int_b^\infty f(x)dx$ converges.
2. $\int_a^\infty f(x)dx$ diverges iff $\int_b^\infty f(x)dx$ diverges.

Notice that the assumption of continuity is important since for a discontinuous functions we can find divergence at finite points. For example, $\int_1^\infty \frac{dx}{(x-4)^2}$ diverges whereas $\int_5^\infty \frac{dx}{(x-4)^2}$ converges. We cannot just discard the low values of x in the convergence analysis for $\int_1^\infty \frac{dx}{(x-4)^2}$. Ok, enough about what we can't do let's see what we can do[5]:

Example 11.2.15. *Does $\int_{10}^\infty \frac{dx}{x^3}$ converge or diverge? Notice*

$$\underbrace{\int_1^\infty \frac{dx}{x^3}}_{1/2} = \underbrace{\int_1^{10} \frac{dx}{x^3}}_{99/200} + \int_{10}^\infty \frac{dx}{x^3}$$

Furthermore, it is clear that $1/x^3$ is continuous on $[1, \infty)$ hence we conclude that this integral is just the tail of the $p = 3$ integral which converges hence $\int_{10}^\infty \frac{dx}{x^3}$ converges. Moreover, in a case such as this we can even calculate its value indirectly by $\int_{10}^\infty \frac{dx}{x^3} = 1/2 - 99/200 = 1/200$.

Remark 11.2.16. *alternate solutions possible.*

When analyzing convergence and divergence it is often possible to argue a point many different ways. In this section, I do not always seek an optimal solution. Instead, I am trying to illustrate the theorems with examples that are not overly difficult. One side-effect of this approach is that currently we have no need to use the Tail theorem or the comparison theorem for the examples above. We could have just straight calculated the improper integrals and came to the same conclusion with less thinking. However, we will soon consider examples where the integral is simply not calculatable in closed form. Instead our approach will be to see how it is like something we can calculate then use the tail, comparison and/or the linearity theorem (given later in this section) to circumvent direct calculation of the improper integral.

[5] proof of the tail theorem is left to the reader, I don't think this is a hard one

Example 11.2.17. *Does $\int_a^\infty \frac{dx}{x^4+8x^2+3}$ converge for a sufficiently large choice of a? Note, $x^4+8x^2+3>0$ thus we can choose $a=0$ in this case. The integrand $\frac{1}{x^4+8x^2+3}$ is clearly continuous on $[0,\infty)$. Moreover, notice that for $x \geq 0$*

$$\frac{1}{x^4+8x^2+3} \leq \frac{1}{x^4} \qquad \textit{[think about it, I made the denominator smaller].}$$

Observe $\int_0^\infty \frac{dx}{x^4}$ converges because it it the tail of the $p=4$ integral which we know converges. Furthermore, $0 \leq \frac{1}{x^4-8x^2-3} \leq \frac{1}{x^4}$ hence the comparison theorem applies and we find $\int_0^\infty \frac{dx}{x^4-8x^2-3}$ converges.

The inequality in the last example was pretty easy to see. The next example, which concerns $\int_a^\infty \frac{dx}{x^4-8x^2-3}$ is much harder due to the change of sign. We would like to compare this integral to the $p=4$ integral. However, the approach used in the last example doesn't quite work because we have the wrong inequality. Notice that for $x \geq a$

$$\frac{1}{x^4} \leq \frac{1}{x^4-8x^2-3} \qquad \text{[think about it, I made the denominator smaller on the RHS].}$$

This inequality is correct, it's just not what we need for our comparison theorem argument. The fact that the integral of the LHS converges doesn't say anything about integral of the RHS. We need to find a different argument for the example below.

Example 11.2.18. *Does $\int_a^\infty \frac{dx}{x^4-8x^2-3}$ converge for a sufficiently large choice of a? Consider that $x^4-8x^2-3=0$ is a quadratic equation in x^2 and the solutions are*

$$x^2 = \frac{8+\sqrt{64+12}}{2} \qquad \textit{or} \qquad x^2 = \frac{8-\sqrt{64+12}}{2}$$

Note that $\frac{8+\sqrt{76}}{2}>0$ whereas $\frac{8-\sqrt{76}}{2}<0$ thus we find only two real solutions to the equation $x^4-8x^2-3=0$. I'll give them labels for our convenience:

$$\lambda_1 = \sqrt{\frac{8+\sqrt{76}}{2}} \qquad \lambda_2 = -\sqrt{\frac{8+\sqrt{76}}{2}}.$$

We can calculate $\lambda_1<3$. Thus, we know there is no zero for the denominator of $\frac{1}{x^4-8x^2-3}$ on $[3,\infty)$ hence the function is continuous and positive. Moreover, if $x>3$ then clearly $x^2-\lambda_1>1$ thus

$$\frac{1}{x^4-8x^2-3} = \frac{1}{(x^2-\lambda_1)(x^2-\lambda_2)} < \frac{1}{x^2-\lambda_2} < \frac{1}{x^2}.$$

We apply the comparison theorem to argue that $\int_3^\infty \frac{dx}{x^4-8x^2-3}$ converges since $\int_3^\infty \frac{dx}{x^2}$ is the tail of the $p=2$ integral which is known to converge. We can apply the comparison theorem because we have shown the integrands are continuous and positive on $[3,\infty)$ and satisfy the needed inequality of $\frac{1}{x^4-8x^2-3} < \frac{1}{x^2}$ for all $x \in [3,\infty)$.

There are many techniques to use here. Certainly factoring polynomials is a great tool and sign charts help to unravel the non-linear inequalities. The idea that, for a positive fraction, increasing the denominator makes a fraction smaller whereas decreasing the denominator makes it larger is very important to keep in mind. Finally, the goal to reduce to a p-type integral for the purpose of comparison saves us the trouble of integration. However, there are other examples where an integration is likely the only way to go. For example:

Example 11.2.19. *Does the integral below converge or diverge?*

$$\int_1^{\infty} \frac{e^x dx}{\sin(x) + e^{2x}}$$

The sine function is bounded by $-1 \leq \sin(x) \leq 1$ *and* $e^{2x} \geq e^2$ *for* $x \geq 1$ *since the exponential function is increasing. We find that* $\sin(x) + e^{2x} \leq 1 + e^{2x}$ *on the other hand the function* $\frac{e^x}{\sin(x)+e^{2x}}$ *clearly positive since the* $\sin(x) + e^{2x} \geq -1 + e^{2x} > -1 + e^2 > 0$. *We argue that for all* $x \geq 1$

$$0 \leq \frac{e^x}{1 + e^{2x}} \leq \frac{e^x}{\sin(x) + e^{2x}}.$$

Notice that we can integrate the function above by making a $u = e^x$ *subsitution:*

$$\int \frac{e^x dx}{1 + e^{2x}} = \int \frac{du}{1 + u^2} = \tan^{-1}(e^x) + c.$$

Hence we calculate

$$\begin{aligned}
\int_1^{\infty} \frac{e^x dx}{1 + e^{2x}} &= \lim_{t \to \infty} \int_1^t \frac{e^x dx}{1 + e^{2x}} \\
&= \lim_{t \to \infty} \left[\tan^{-1}(e^t) - \tan^{-1}(e) \right] \\
&= \frac{\pi}{2} - \tan^{-1}(e).
\end{aligned}$$

Therefore, $\int_1^{\infty} \frac{e^x dx}{1+e^{2x}}$ *converges. It follows by the comparison theorem that* $\int_1^{\infty} \frac{e^x dx}{\sin(x)+e^{2x}}$ *converges. Moreover, due to our calculations here we can even reason that* $0 \leq \int_1^{\infty} \frac{e^x dx}{\sin(x)+e^{2x}} \leq \frac{\pi}{2} - \tan^{-1}(e)$.

Theorem 11.2.20. *Linearity for improper integrals.*

If $\int_a^{\infty} f(x)dx$ and $\int_a^{\infty} g(x)dx$ converge then for each $c \in \mathbb{R}$ we find $\int_a^{\infty} \left[f(x) + cg(x)\right]dx$ converges and

$$\int_a^{\infty} \left[f(x) + cg(x) \right] dx = \int_a^{\infty} f(x)dx + c \int_a^{\infty} g(x)dx.$$

On the other hand, if $\int_a^{\infty} f(x)dx$ diverges then $\int_a^{\infty} cf(x)dx$ also diverges for any $c \in \mathbb{R}$ such that $c \neq 0$. If $c > 0$ and $\int_a^{\infty} f(x)dx = \pm\infty$ then we likewise find $\int_a^{\infty} f(x)dx = \pm\infty$. If $c < 0$ and $\int_a^{\infty} f(x)dx = \pm\infty$ then we find $\int_a^{\infty} cf(x)dx = \mp\infty$.

I just made use of the mathching $\pm$ with $\mp$ convention. As an alternate example of its use note you can express $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ with just a single equation in the matching $\pm$ notation: $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$. Notations aside, I hope the theorem above is entirely reasonable in your opinion. The proof stems primarily from the corresponding linearity rules for limits at ∞ and the details are left to the reader. This is a very useful theorem. We could take any pair of examples and splice them together with this theorem. I conclude with just one such example:

Example 11.2.21. *Does the integral below converge or diverge?*

$$\int_1^{\infty} \left[\frac{1}{3x^2} + \frac{2e^x}{\sin(x) + e^{2x}} \right] dx.$$

We know this converges because this is just a linear combination of the example above the theorem and a simple $p = 2$ integral multiplied by $1/3$. In particular, using our work in the previous example,

$$\int_1^\infty \left[\frac{1}{3x^2} + \frac{2e^x}{\sin(x) + e^{2x}}\right] dx. = \frac{1}{3}\int_1^\infty \frac{dx}{x^2} + 2\int_1^\infty \frac{e^x}{\sin(x) + e^{2x}} dx$$
$$= \boxed{\frac{1}{9} + \pi - 2\tan^{-1}(e).}$$

The misuse the linearity theorem is often a source of logical weakness for students. For a bad example:

$$\int_1^\infty \left(\frac{1}{x} - x\right) dx = \int_1^\infty \frac{dx}{x} - \int_1^\infty x dx = -\infty + \infty$$

At which point some students will choose 0 and others will choose $-\infty$, yes others $-\infty/2$. There are many wrong choices, but they all stem from the assumption that you can apply the linearity theorem in the divergent case and it is simply false. We can only split an improper integral into a sum of new improper integrals when the new integrals actually converge. Otherwise, we beg all sorts of questions about how we should go about defining arithemetic for ∞. We will not engage in such a discussion. Well, I think this is a good start. I hope you learn more as you work the homework.

Example 11.2.22. *By the way, here's a problem I don't know how to solve with our current methods:*

$$\int_1^\infty \frac{dx}{x\sin(x)}.$$

I want to say it is convergent, but, the nonlinearity of the $\frac{1}{x}$ threatens the symmetry of the $\sin(x)$*-induced cancellation. On the other hand, what happens with*

$$\int_1^\infty \frac{\sin(x)dx}{x}?$$

Both of these are in some sense on the bubble of the $p = 1$ case which is where many of the most fascintating and challenging examples emerge.