Independent Events

Two definitions of independence

- Def.1
  - Two events, A and B are said to be independent if $P(A \cap B) = P(A)P(B)$
- Def. 2
  - Two events, A and B are said to be independent if $P(A/B) = P(A)$
- Note that they are algebraically equivalent

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)}$$

Intuitive meaning of independence

- $\mathrm{P}(A/B) = \mathrm{P}(A)$
  - Knowledge of B is irrelevant to A
    - P(Thunder/lightning) $\neq$P(Thunder)
    - P(Face coin1/Face coin2)= P(Face coin1)
  - Sample space of A **does not change** if B has happened.
    - For instance a sample space generated by the cartesian product of two sets.

$$\Omega_1 = \{A_1, A_2, \cdots A_n\}$$

$$\Omega_2 = \{B_1, B_2, \cdots B_m\}$$

$$\Omega = \Omega_1 \times \Omega_2$$

$$\Omega = \left\{A_1B_1, A_1B_2, \cdots A_1B_m, \cdots A_nB_m\right\}$$

Intuitive meaning of independence

- Sample space of A **does not change** if B has happened.
  - Sample space generated by the ***cartesian*** product of two sets.

$$P(A/B) = P(A)$$

$$P(A \cap B) = P(A)P(B)$$

$$\Omega_1 \qquad \Omega_2 \qquad \Omega = \Omega_1 \times \Omega_2$$

$$\begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{matrix} \times \begin{matrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{matrix} \qquad \begin{matrix} A_1B_1 \\ A_1B_2 \\ \vdots \\ A_1B_m \\ A_2B_1 \\ \vdots \\ A_nB_m \end{matrix}$$

$$p(A_1) = \frac{1}{n}$$

$$p(A_1B_1) = \frac{1}{nm}$$

$$p(B_1) = \frac{1}{m}$$

$$p(A_1 / B_1) = \frac{1}{n}$$

Explaination of dependent events by means of the sample space

- Sample space of A **does change** if B has happened. Eliminate possibilities

$$P(A/B) \neq P(A)$$

$$P(A \cap B) \neq P(A)P(B)$$

Ω_1		Ω_2	Ω_{New}
A_1		B_1	A_1B_1
A_2	$\times$	B_2	A_1B_2
$\vdots$		$\vdots$	$\vdots$
A_n		B_m	A_1B_m
			~~A_2B_1~~
			$\vdots$
			A_nB_m

$$p(A_1) = \frac{1}{n}$$

$$p(A_1B_1) = \frac{1}{nm-1}$$

$$p(B_1) = \frac{1}{m}$$

$$p(A_1/B_1) = \frac{m}{nm-1}$$

Explaination of dependent events by means of the sample space

- Sample space of A **does change** if B has happened.
  - Eliminated possibilities
  - Preferencial Attatchment

Model 1 of the problem

A1=Rain,A2=Sun shine

B1=Thunder

Model 2 of the problem

A1=Rain,A2=Sun shine

B1=Dressed with a rain coat

Ω_1		Ω_2	Ω_{New}
A_1		B_1	A_1B_1
A_2	$\times$	B_2	A_1B_2
$\vdots$		$\vdots$	$\vdots$
A_n		B_m	A_1B_m
			A_2B_1
			$\vdots$
			A_nB_m

$$p(A_1) = \frac{1}{n}$$

$$p(A_1B_1) = \frac{1}{nm-1}$$

$$p(B_1) = \frac{1}{m}$$

$$p(A_1/B_1) = \frac{m}{nm-1}$$

$$P(A/B) \neq P(A)$$

$$P(A\cap B) \neq P(A)P(B)$$

Intuitive meaning of independence

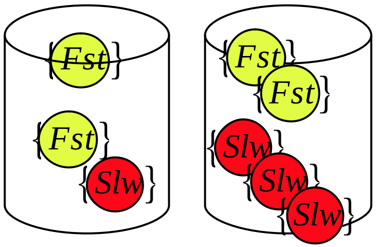
- Another case: <u>**Proportion**</u> of the sample space of A does not change if B has happened
  - Note: the condition is algebraic, not **physical**

$$P(S_1) = 1/2$$

$$P(S_2) = 5/12$$

$$P(Slw/S_1) = P(Slw)$$

$$P(Slw/S_1) = \frac{1}{3} = \frac{P(Slw \cap S_1)}{P(S_1)}$$



$$P(Slw/S_2) = \frac{2}{5} = \frac{P(Slw \cap S_2)}{P(S_2)}$$

$$P(Slw) = P(S_2)P(Slw/S_2) + P(S_1)P(Slw/S_1)$$

$$P(Slw) = \frac{1}{2}\frac{1}{3} + \frac{5}{12}\frac{2}{5} = \frac{1}{3}$$

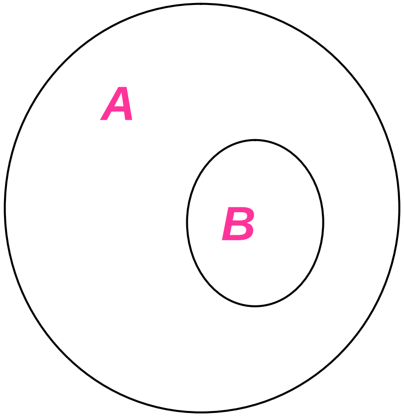
Independence by inclusion

Operations

Implication	$\rightarrow$	Inclusion	$\rightarrow$	Condition

- Preposition
  - If it rains, I'll bring the umbrella
- Sets
  - A={It rains}, B={Bring umbrella}

    $B \subset A$



- Probabilities
  - P(B/A)=P(B)
  - P(B/Not A)= $\varnothing$

Propositions $\rightarrow$ Relations between objects $\rightarrow$ Numbers

Intuitive meaning of independence

- **Proportion** of the sample space of A does not change if B has happened
  - Note: the condition is algebraic, not **physical**

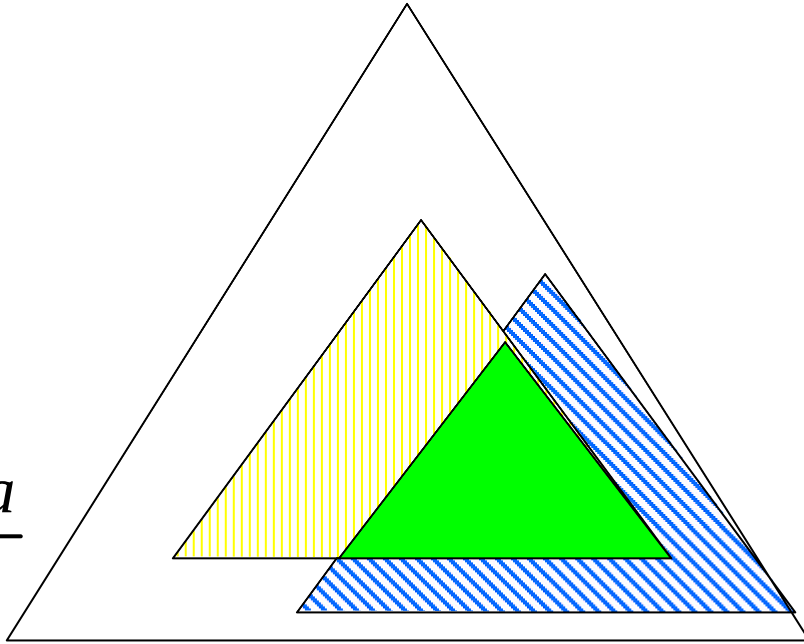
$$P(\Omega) = Total\ Area = 1$$

$$P(A) = \frac{Yellow\ Area}{Total\ Area}$$

$$P(B) = \frac{Blue\ Area}{Total\ Area}$$

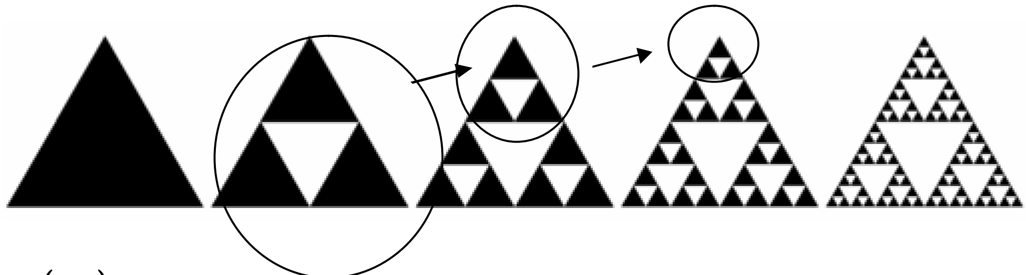
$$P(A/B) = \frac{Green\ \text{Area}}{Blue\ Area} = \frac{Yellow\ Area}{Total\ Area}$$

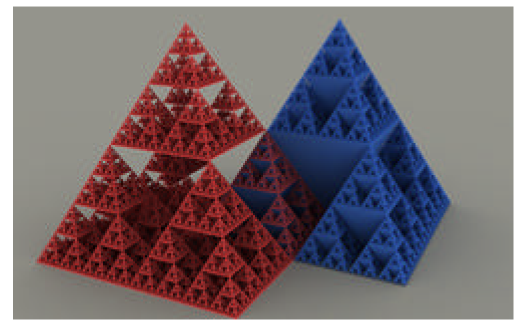
$$P(A/B) = P(A)$$



Application to Scale Free objects

- Application to fractal images and objects.
  - Sierpinski triangle





http://en.wikipedia.org/wiki/Sierpinski_triangle

$$P(\Omega) = Total\ \ Area = 1$$

$$P(A) = \frac{Yellow\ Area}{Total\ Area}$$

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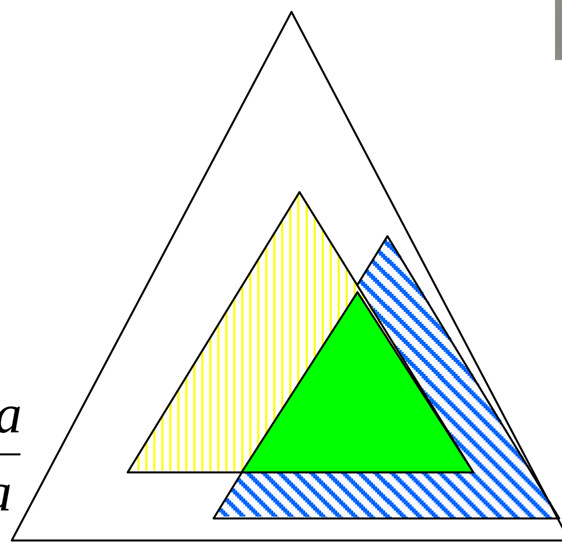
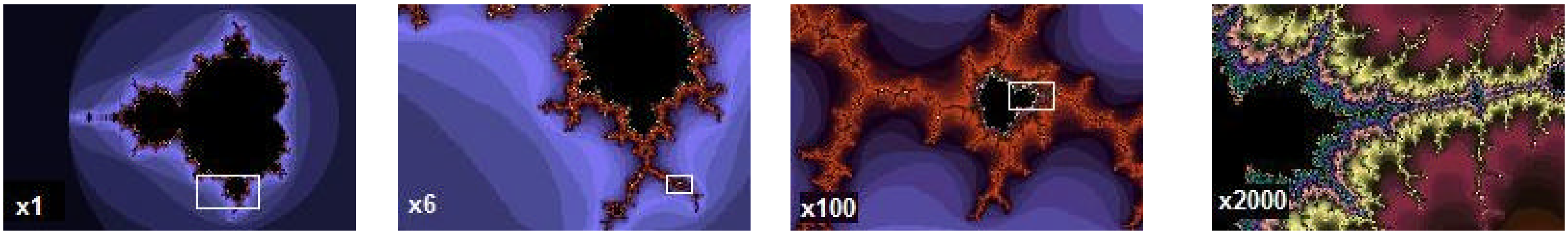
$$P(A/B) = \frac{Green\ Area}{Blue\ Area} = \frac{Yellow\ Area}{Total\ \ Area}$$

$$P(A/B) = P(A)$$



Application to Scale Free objects

- Application to fractal images and objects.



http://en.wikipedia.org/wiki/Fractals

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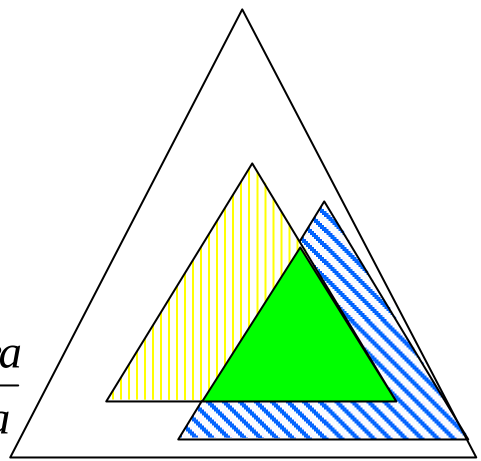
$$P(A/B) = P(A)$$



Application to Scale Free objects

- Application to internet traffic.

$A = \{q\,\%\ \text{change in the traffic}\}$

$B_0 = \{\text{time scale: month}\}$

$B_1 = \{\text{time scale: day}\}$

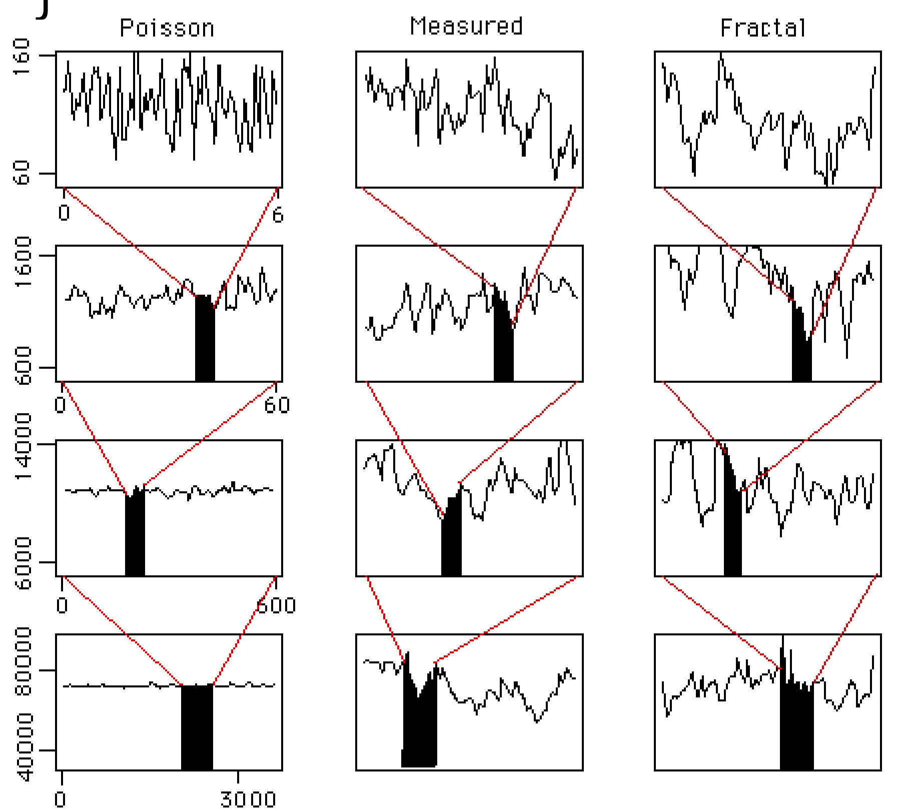
$B_2 = \{\text{time scale: hour}\}$

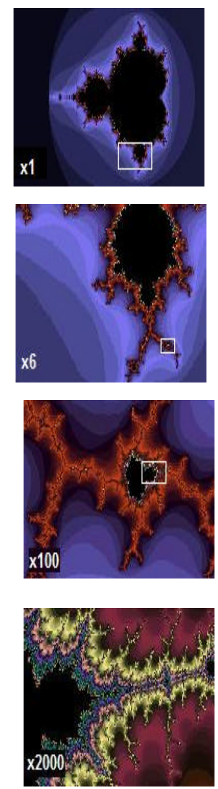
$B_3 = \{\text{time scale: seconds}\}$

$\forall \quad i,j$

$\mathrm{P}(A) = \mathrm{P}(A/B_i)$

$\mathrm{P}(A/B_j) = \mathrm{P}(A/B_i)$





http://classes.yale.edu/fractals/Panorama/ManuFractals/Internet/Internet4.html

Application to Scale Free objects

- Flips of coins. 10.000 vs. 1.000.000

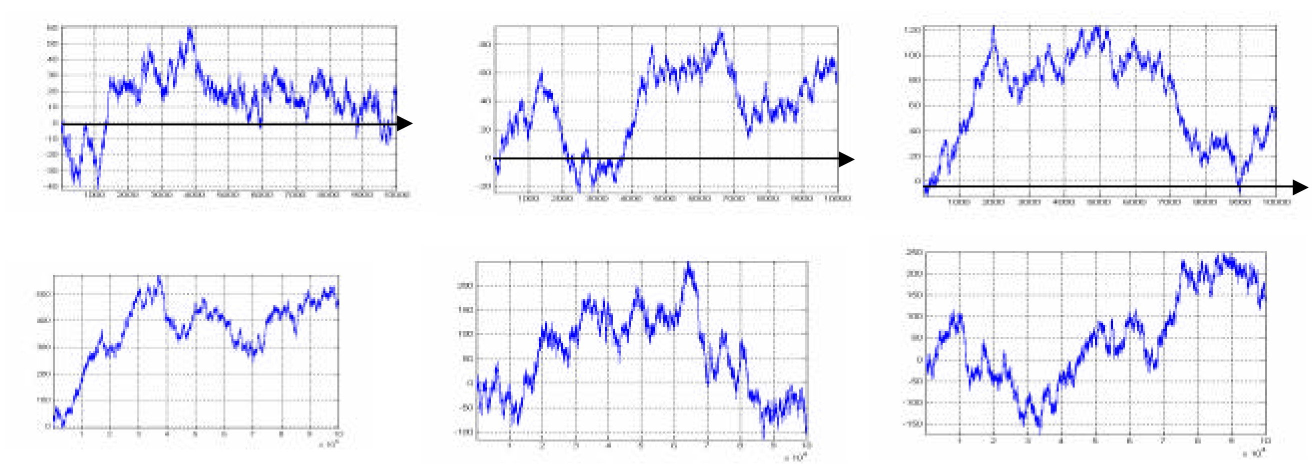
$$\Omega_1 = \{\text{set of all possible results in } 10.000 \text{ flips of a coin}\}$$

$$\Omega_2 = \{\text{set of all possible results in } 1.000.000 \text{ flips of a coin}\}$$

$$A = \{\text{Fraction of Time one player is winning}\}$$

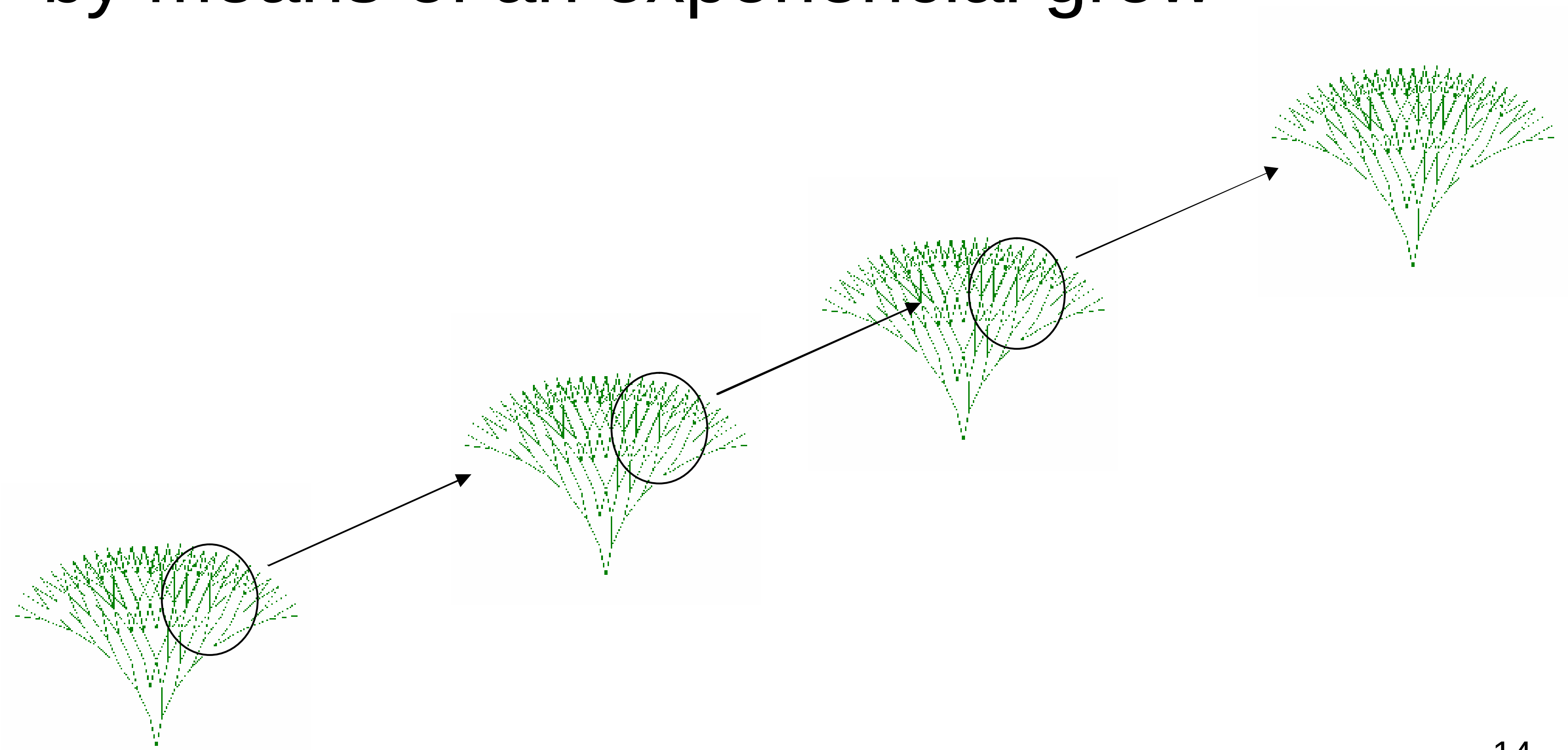
$$\mathrm{B} = \{\text{Scale of the experiment}\}$$

$$\mathrm{p}(A) = \mathrm{p}(A / B)$$



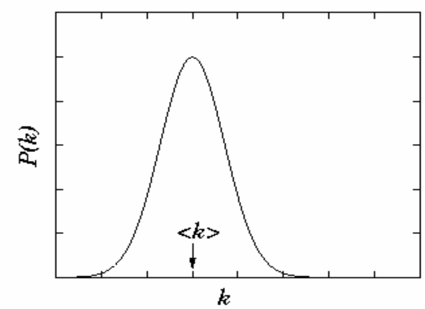
Application to Scale Free objects

- One way of creating Scale free objects, is by means of an exponencial grow

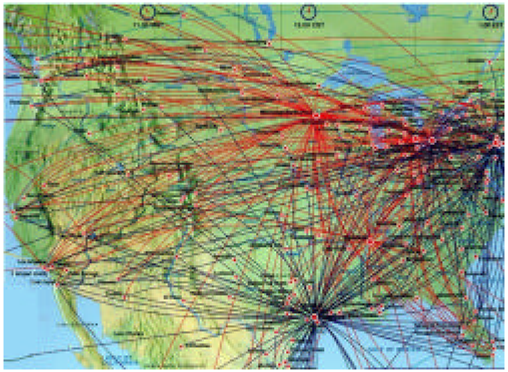


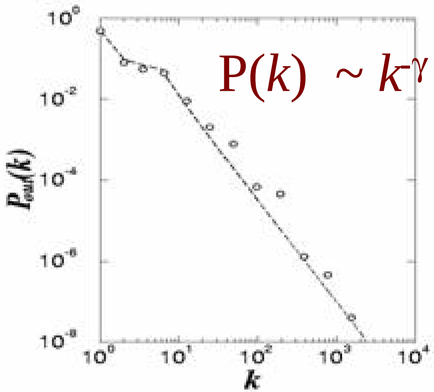
Application to Scale Free objects

Prefencial connexions (road to the nearest neighbour) ***vs***. indifferent conexions (can fly anywhere)



$$\Omega_1 = \{A_1, A_2, \cdots A_n\}$$
$$\Omega_2 = \{B_1, B_2, \cdots B_m\}$$
$$\Omega = \Omega_1 \times \Omega_2$$
$$\Omega = \{A_1B_1, A_1B_2, \cdots \xcancel{A_1B_m}, \cdots A_nB_m$$





$$\Omega_1 = \{A_1, A_2, \cdots A_n\}$$
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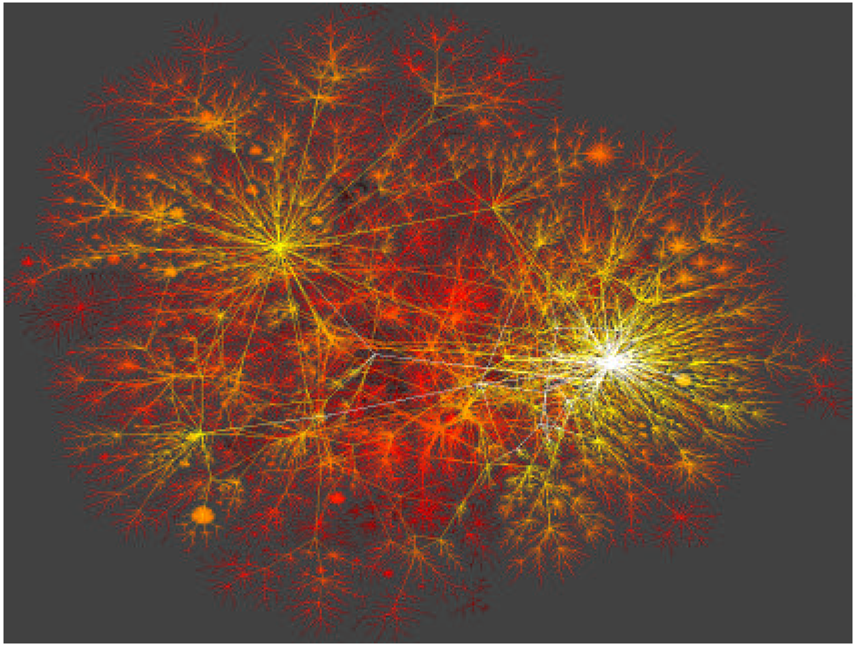
$$\Omega = \{A_1B_1, A_1B_2, \cdots A_1B_m, \cdots A_nB_m$$

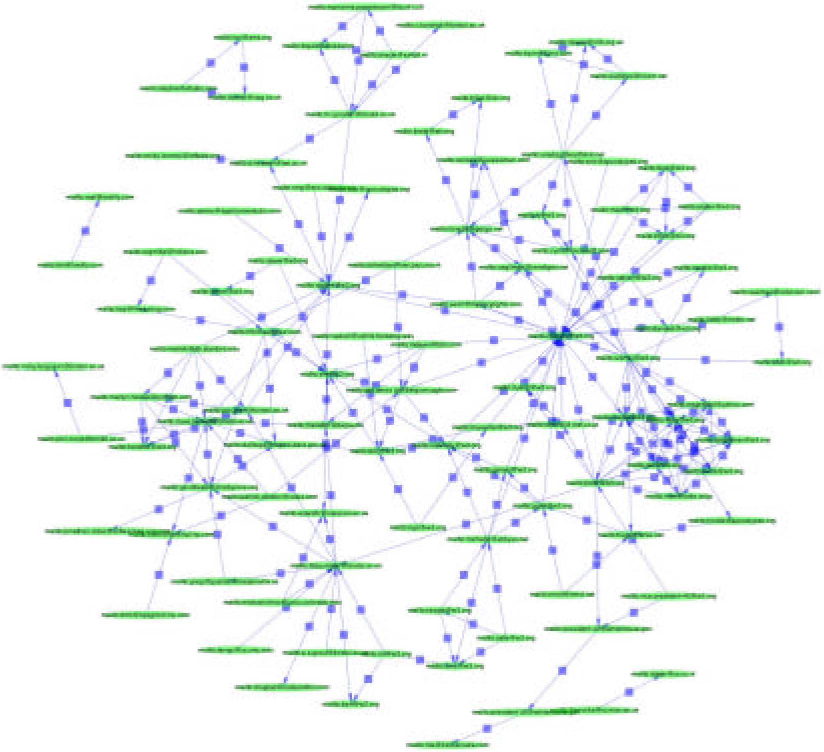
Taken from:

The architecture of complexity: From the topology of the www to the cell's genetic network

Similarities between natural graphs

- Semantic map vs. Physical connections in internet



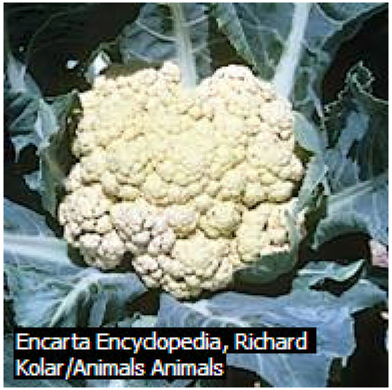


The architecture of complexity:
From the topology of the www to the cell's genetic network

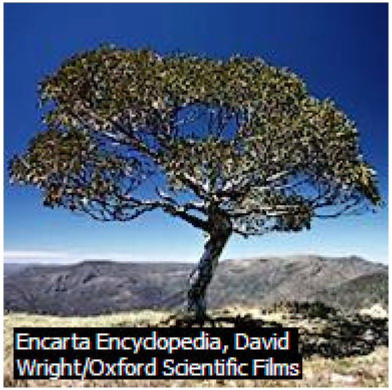
Albert-László **Barabási**

http://rdfweb.org/2002/02/foafpath/

Examples of Scale Free in biology



Broccoli



Eucalyptus Tree

Relation between independence and disjoint condition

- Independence does not imply disjointness
  - Condition of indepence $P(A \cap B) = P(A)P(B)$
  - Condition of disjointness $A \cap B = 0$
    - In probabilities means:

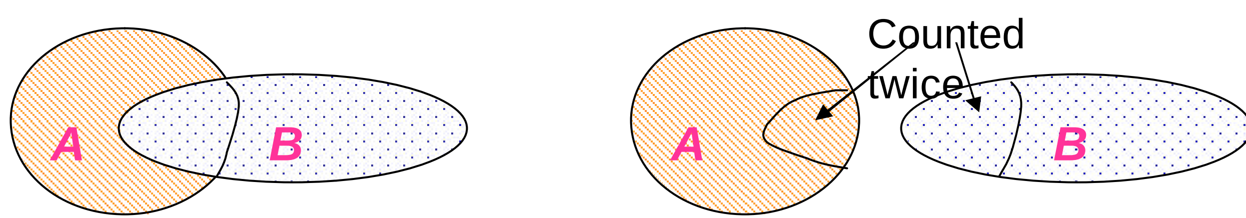
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B)$$

- What does $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B) = 0$ mean?

Relation between independence and disjoint condition

- What does $P(A\cap B)=P(A)P(B)=0$ mean?



$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Eliminated possibilities
Preferencial Attatchment

odel 1 of the problem
A1=Rain,A2=Sun shine
B1=Thunder

odel 2 of the problem
A1=Rain,A2=Sun shine
B1=Dressed with a rain coat

Ω_1		Ω_2	$\Omega=\Omega_1\times\Omega_2$
A_1		B_1	A_1B_1 ←
A_2	$\times$	B_2	A_1B_2
$\vdots$		$\vdots$	$\vdots$
A_n		B_m	A_1B_m
			A_2B_1 ←
			$\vdots$
			A_nB_m

$$p(A_1)=\frac{1}{n}$$

$$p(A_1B_1)=\frac{1}{nm}$$

$$p(B_1)=\frac{1}{m}$$

$$p(A_1/B_1)=\frac{1}{n}$$

Probability of the intersection of a set of independent events.

- Probability of the union of independent events $\Omega = \{A_1, A_2, \cdots A_n\}$
- Formally the union of all the elements, consists on the event:
  - E={Simultaneously of the elements of the set appear}

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$$

- Note:

$$\text{Propositions} \rightarrow \text{Relations between objects} \rightarrow \text{Numbers}$$

When intersection of sets corresonds to multiplication of probabilities?

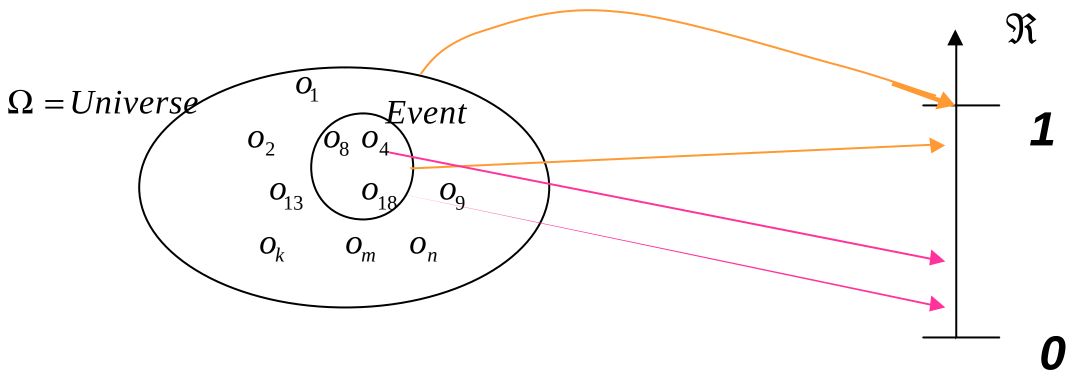
$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$$

Propositions → Relations between objects → Numbers

$$Logic = \{OR, AND, NOT, IMPLICATION\}$$

$$Sets = \{UNION, INTERSECTION, COMPLEMENT, INCLUSION\}$$

$$Sets = \{SUM, MULTIPLICATION, CONDITIONING\ (p(./.))\}$$



Probability of getting at least one event of a set of independent events

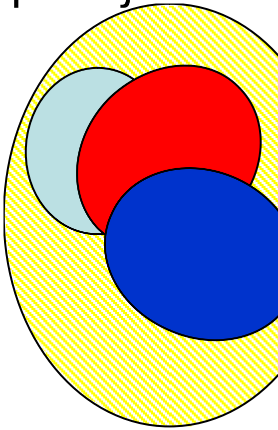
- Probability of the union of independent events $\Omega = \{A_1, A_2, \cdots A_n\}$
- Formally the union of all the elements, consists on the event:
  - E={At least one of the elements of the set appear}
  - $\overline{E}$={Not a single element of the set appears }
- Which is equivalent to $E = \{\Omega - \overline{E}\}$

Probability of getting at least one event of a set of independent events

- Probability of the union of independent events $\Omega = \{A_1, A_2, \cdots A_n\}$

$$E = \bigcup_{i=1}^{n} A_i$$

E={At least one of the elements of the set appear.}



$$\overline{E} = \bigcap_{i=1}^{n} (\Omega - A_i)$$

$\overline{E}$={Not a single element of the set appears }

$$E = \Omega - \bigcap_{i=1}^{n} (\Omega - A_i)$$

$$P(E) = P\left(\Omega - \bigcap_{i=1}^{n} (\Omega - A_i) \right) = 1 - P\left(\bigcap_{i=1}^{n} (\Omega - A_i) \right) =$$

$$P(E) = 1 - \prod_{i=1}^{n} \left[1 - P(A_i) \right]$$

Example 1

- A web page has two kind links. {A,B}
- ***M*** different users select randomly and independently of each other one of the links.
- What is the probability that at a link of kind A is visited least once?

  – For instance: Web based bookshop that also has CD, DVD, second hand books.

Example 1

- A web page has two kind links. {A,B}
- Sample space of the links

$$\Omega_1 = \{A_1, A_2, \cdots A_n\} \qquad P(A) = \frac{n}{n+m}$$

$$\Omega_2 = \{B_1, B_2, \cdots B_m\} \qquad P(B) = \frac{m}{n+m}$$

- Possible choises of the M users

$$\text{Possible of choices} = (\{A_{i_1}\ OR\ B_{j_1}\}, \{A_{i_2}\ OR\ B_{j_2}\}, \cdots \{A_{i_M}\ OR\ B_{j_M}\})$$

$$\text{Number of choices} = \quad 2 \quad \times \quad 2 \quad \times \cdots \quad 2 \quad = 2$$

Example 1

- Probability of a given selection:

$$P(\{A_{i_1}\ AND\ A_{i_2}\ AND\ B_{j_1} \cdots A_{i_L}\ AND\ B_{j_{M-L}}\}) = \left(\frac{n}{n+m}\right)^{L} \left(\frac{m}{n+m}\right)^{M-L}$$

$$P(A) = \frac{n}{n+m}$$

$$P(B) = \frac{m}{n+m}$$

- What is the probability that at a link of kind A is visited least once?

$$P(\{\text{At least an A}\}) = 1 - P(\{All\ \text{B}\}) = 1 - \left(\frac{m}{n+m}\right)^{M}$$

Example 1

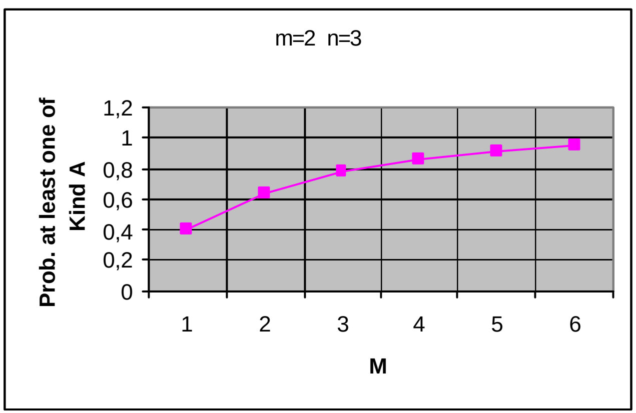
- What is the probability that at a link of kind A is visited least once? $P(\{\text{At least an A}\}) = 1 - P(\{All\ \text{B}\}) = 1 - \left(\frac{m}{n+m}\right)$

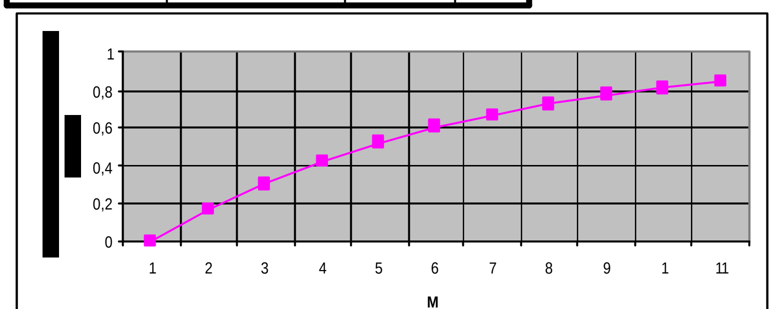
$$P(A) = \frac{n}{n+m}$$

$$P(B) = \frac{m}{n+m}$$

M	*m=2 n=3*	P(A)=	0,6
1	0,4	P(B)=	0,4
2	0,64		
3	0,784		
4	0,8704		
5	0,92224		
6	0,953344		

M	*m=10 n=3*	P(A)=	0,8
1	0,16666667	P(B)=	0,2
2	0,30555556		
3	0,4212963		
4	0,51774691		
5	0,59812243		
6	0,66510202		
7	0,72091835		
8	0,76743196		
9	0,8061933		
10	0,83849442		





Example 2

- Another way of deriving the formula:

$$P(\{\text{At least an A}\}) = 1 - P(\{All\ \text{B}\}) = 1 - \left(\frac{m}{n+m}\right)^M$$

- Throw a coin N times, what is the probability that heads occur on at least one trial?

$$P(\{Heads\ \text{at least in one trial}\}) = p + q^2 p + q^3 p \cdots + q^{M-1} p = p\frac{1-q^M}{1-q} = 1 - q^M$$

How?

Example 2

- Throw a coin N times, what is the probability that heads occur on at least one trial?

$$A_i = \{First \text{ Head occurs in the trial number i}\}$$

$$A_i = \{(i-1\,Tails \text{ followed by a Head}) \cup (\text{then anything else})\}$$

?

$$P(A_i) = q^{i-1}p + P(\{\text{then anything else}\}) = q^{i-1}p$$

$$P(\{Heads \text{ at least in one trial}\}) = P(A_1 \cup A_2 \cdots \cup A_M) = \sum_{i=1}^{M} A_i$$

$$P(\{Heads \text{ at least in one trial}\}) = p + q^2 p + q^3 p \cdots + q^{M-1} p = p\frac{1-q^M}{1-q} = 1 - q^M$$

Example 2

- P({then anything else})?

Case of 3

$$P(\{\text{then anything else}\}) = ppp + ppq + pqp + qpp + qqp + qpq + pqq + qqq$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + b^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

$$\{aab, aba, baa\}$$

$$(p+q)^n = 1$$

Example 2

- Applications of the formula:

$$P(\{\text{At least an A}\}) = 1 - P(\{All\ \text{B}\}) = 1 - \left(\frac{m}{n+m}\right)^M$$

- Carl Sagan an the probability of inteligent life in our galaxy
- Saddam's 'Plebicito' with a 99.9% of approval
- Other 'plebicito's' and elections