Inductors and Capacitors

- **Inductor** is a Coil of wire wrapped around a supporting (mag or non mag) core
- Inductor behavior related to magnetic field
- Current (movement of charge) is source of the magnetic field
- Time varying current sets up a time varying magnetic field
- Time varying magnetic field induces a voltage in any conductor linked by the field
- Inductance relates the induced voltage to the current

- **Capacitor** is two conductors separated by a dielectric insulator
- Capacitor behavior related to electric field
- Separation of charge (or voltage) is the source of the electric field
- Time varying voltage sets up a time varying electric field
- Time varying electric field generates a displacement current in the space of field
- Capacitance relates the displacement current to the voltage
- Displacement current is equal to the conduction current at the terminals of capacitor

Inductors and Capacitors (contd)

- Both inductors and capacitors can store energy (since both magnetic fields and electric fields can store energy)
- Ex, energy stored in an inductor is released to fire a spark plug
- Ex, Energy stored in a capacitor is released to fire a flash bulb
- L and C are passive elements since they do not generate energy

Inductor

- Inductance symbol L and measured in Henrys (H)
- Coil is a reminder that inductance is due to conductor linking a magnetic field

$$v = L\frac{di}{dt}$$

- First, if current is constant, v = 0
- Thus **inductor behaves as a short with dc current**
- Next, **current cannot change instantaneously in L** i.e. current cannot change by a finite amount in 0 time since an infinite (i.e. impossible) voltage is required
- In practice, when a switch on an inductive circuit is opened, current will continue to flow in air across the switch (arcing)

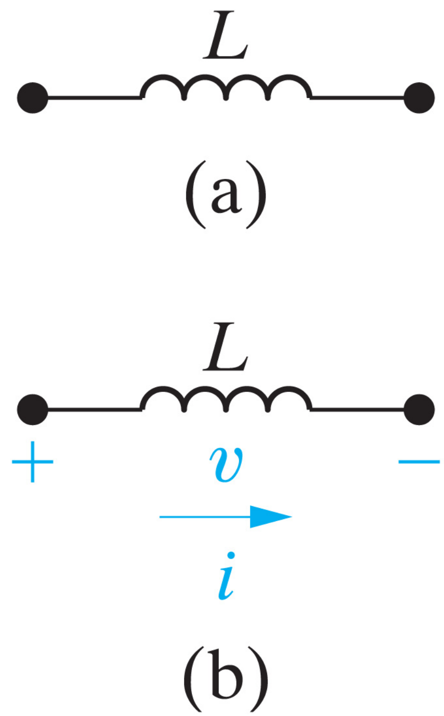
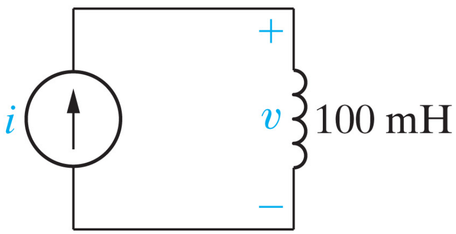


Figure: 06-01a,b

Inductor: Voltage behavior



$$i = 0, \qquad t < 0$$

$$i = 10te^{-5t}\,\text{A}, \qquad t > 0$$

Figure: 06-02Ex6.1

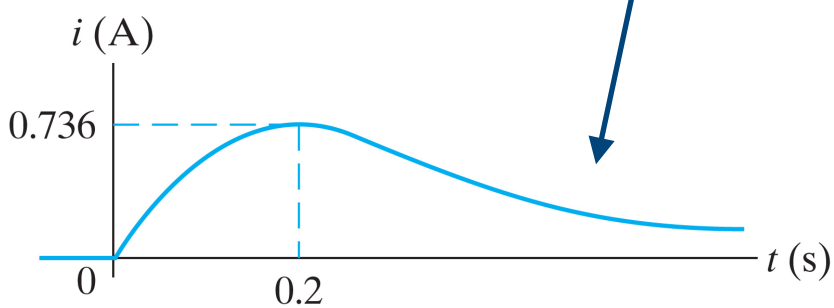
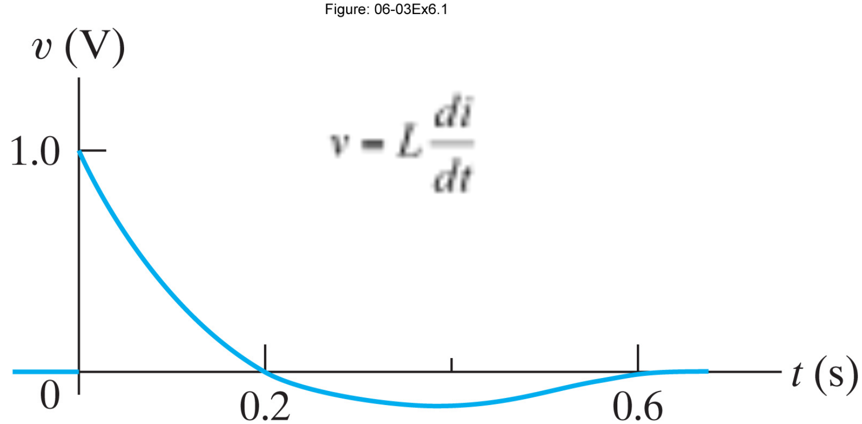


Figure: 06-03Ex6.1



$$v = L\frac{di}{dt}$$

Figure: 06-04Ex6.1

- Why does the inductor voltage change sign even though the current is positive? (slope)
- Can the voltage across an inductor change instantaneously? (yes)

Inductor: Current, power and energy

$$v = L\frac{di}{dt}$$

$$vdt = L\left(\frac{di}{dt}\right)dt$$

$$vdt = Ldi$$

$$Ldi = vdt$$

$$L\int_{i(t_0)}^{i(t)} dx = \int_{t_0}^{t} vd\tau$$

$$\therefore i(t) = \frac{1}{L}\int_{t_0}^{t} vd\tau + i(t_0)$$

$$t_0 = 0; i(t) = \frac{1}{L}\int_{0}^{t} vd\tau + i(0)$$

$$p = vi$$

$$p = \left(L\frac{di}{dt}\right)i$$

$$p = v\left[\frac{1}{L}\int_{t_0}^{t} vd\tau + i(t_0)\right]$$

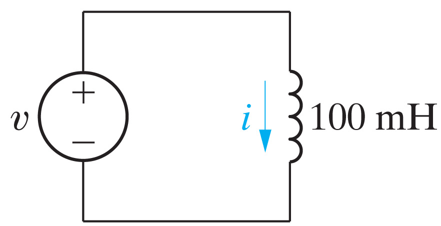
$$p = \frac{dw}{dt} = Li\frac{di}{dt}$$

$$\therefore dw = (Li)di$$

$$\int_{0}^{w} dx = L\int_{0}^{i} y\,dy$$

$$\therefore w = \frac{1}{2}Li^2$$

Inductor: Current behavior



$$v = 0, \qquad t < 0$$

$$v = 20te^{-10t}\,\text{V}, \quad t > 0$$

Figure: 06-05Ex6.2

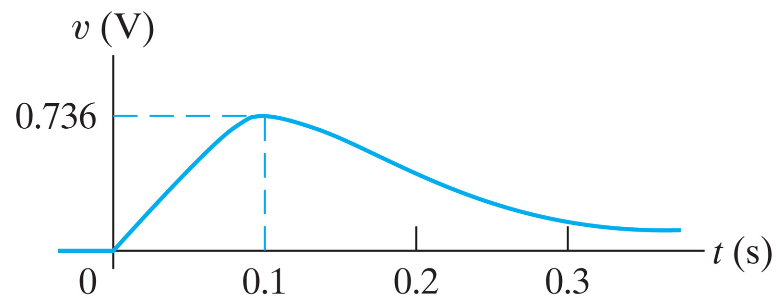


Figure: 06-06Ex6.2

$$i(t) = \frac{1}{L}\int_0^t v\,d\tau + i(0)$$

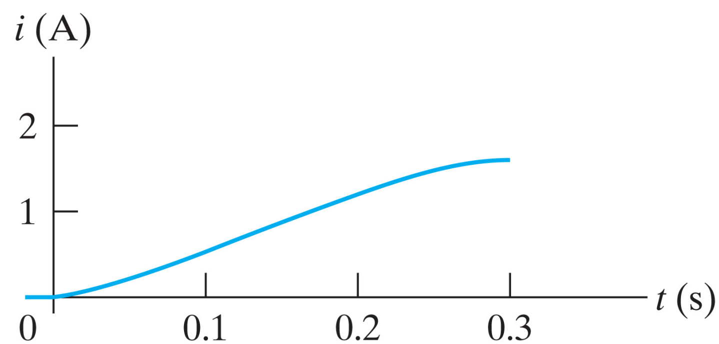
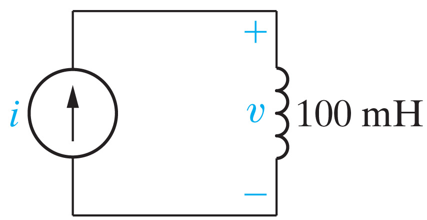


Figure: 06-07Ex6.2

- Why does the current approach a constant value (2A here) even though the voltage across the L is being reduced? (lossless element)

Inductor: Example 6.3, I source



$$i = 0, \qquad t < 0$$

$$i = 10te^{-5t}\text{A}, \qquad t > 0$$

Figure: 06-02Ex6.1

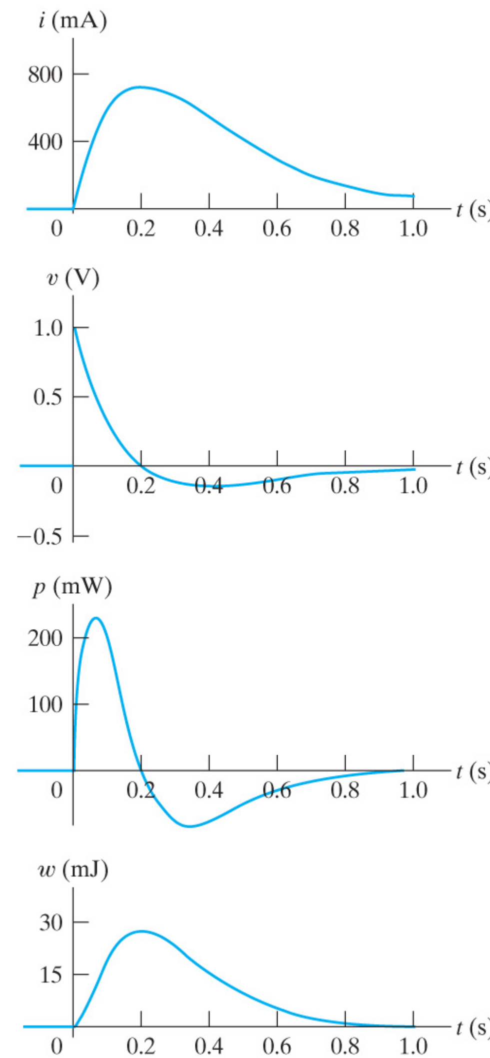
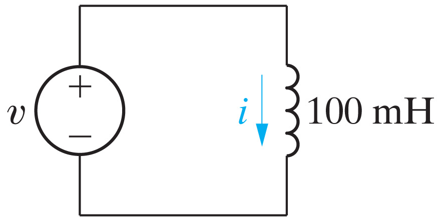


Figure: 06-081-4Ex6.3 -C

- In this example, the excitation comes from a current source
- Initially increasing current up to 0.2s is storing energy in the inductor, decreasing current after 0.2 s is extracting energy from the inductor
- Note the positive and negative areas under the power curve are equal. When power is positive, energy is stored in L. When power is negative, energy is extracted from L

Inductor: Example 6.3, V source



$$v = 0, \quad t < 0$$

$$v = 20te^{-10t}\text{ V}, \quad t > 0$$

Figure: 06-05Ex6.2

- In this example, the excitation comes from a voltage source
- Application of positive voltage pulse stores energy in inductor
- Ideal inductor cannot dissipate energy – thus a sustained current is left in the circuit even after the voltage goes to zero (lossless inductor)
- In this case energy is never extracted

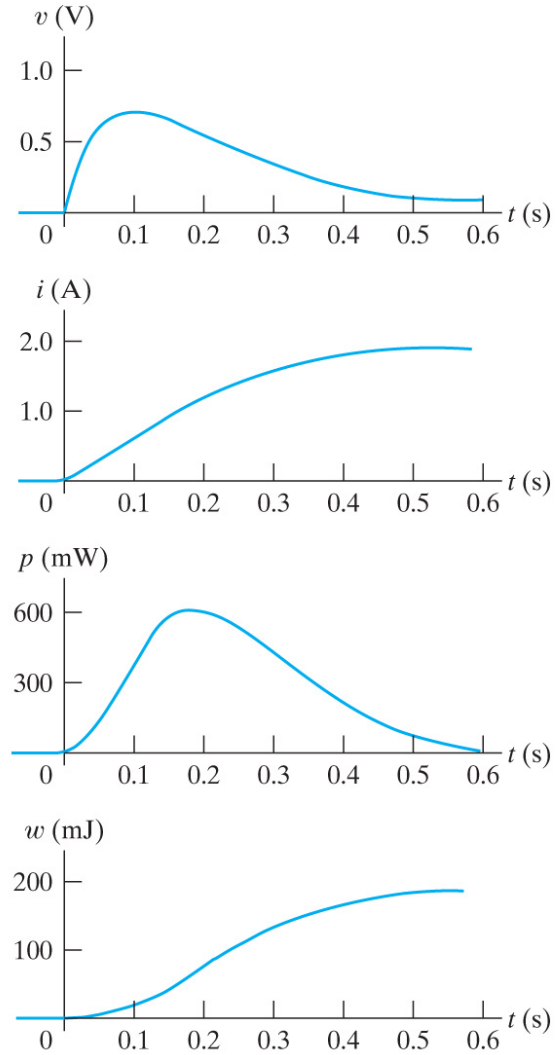


Figure: 06-091-4

Capacitor

- Capacitance symbol C and measured in Farads (F)
- Air gap in symbol is a reminder that capacitance occurs whenever conductors are separated by a dielectric
- Although putting a V across a capacitor cannot move electric charge through the dielectric, it can displace a charge within the dielectric → displacement current proportional to v(t)
- At the terminals, displacement current is similar to conduction current

$$i = C\frac{dv}{dt}$$

- As per above eqn, **voltage cannot change instantaneously across the terminals of a capacitor** i.e. voltage cannot change by a finite amount in 0 time since an infinite (i.e. impossible) current would be produced
- Next, for DC voltage, capacitor current is 0 since conduction cannot happen through a dielectric (need a time varying voltage v(t) to create a displacement current). Thus, **a capacitor is open circuit for DC voltages.**

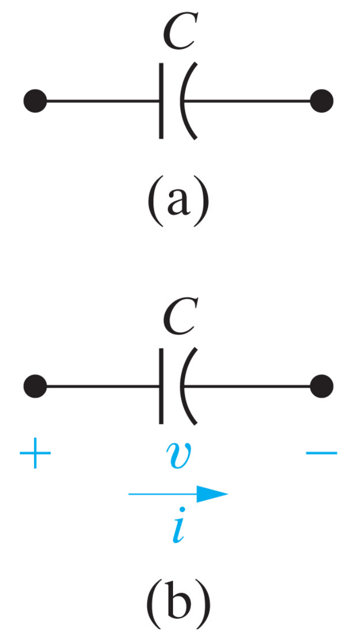


Figure: 06-10a,b

Capacitor: voltage, power and energy

$$i = C\frac{dv}{dt}$$

$$idt = C\left(\frac{dv}{dt}\right)dt$$

$$idt = Cdv$$

$$dv = \frac{1}{C}idt$$

$$\int_{v(t_0)}^{v(t)} dx = \frac{1}{C}\int_{t_0}^{t} id\tau$$

$$\therefore v(t) = \frac{1}{C}\int_{t_0}^{t} id\tau + v(t_0)$$

$$t_0 = 0; v(t) = \frac{1}{C}\int_{0}^{t} id\tau + v(0)$$

$$p = vi$$

$$p = v\left(C\frac{dv}{dt}\right)$$

$$p = i\left[\frac{1}{C}\int_{t_0}^{t} id\tau + v(t_0)\right]$$

$$p = \frac{dw}{dt}$$

$$\therefore dw = (Cv)dv$$

$$\int_{0}^{w} dx = C\int_{0}^{v} ydy$$

$$\therefore w = \frac{1}{2}Cv^2$$

Capacitor: Example 6.4, V source

$$v(t) = 0, t \le 0s;$$
$$v(t) = 4t, 0 \le t \le 1s;$$
$$v(t) = 4e^{-(t-1)}; t \ge 1s$$

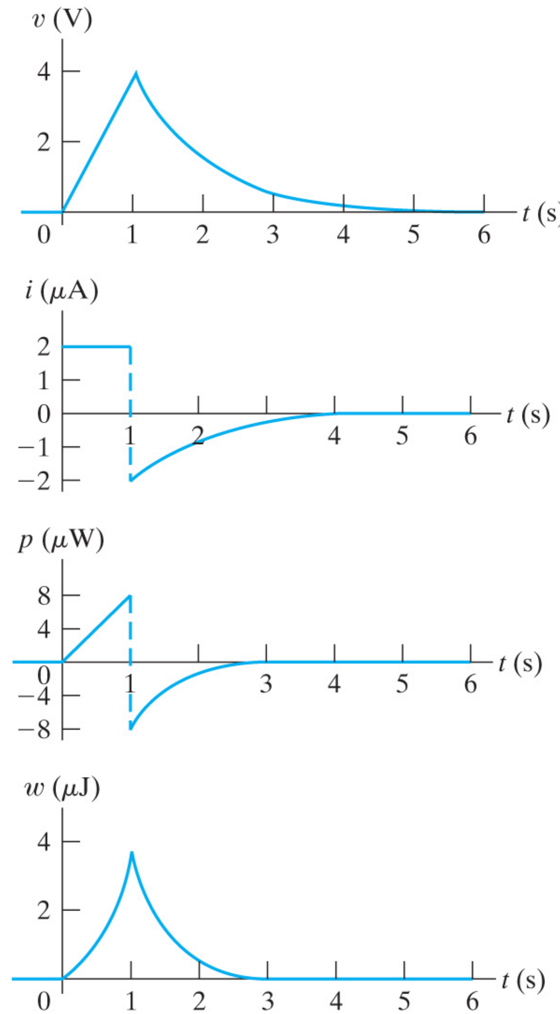


Figure: 06-111-4Ex6.4-C

- In this example, the excitation comes from a voltage source
- Energy is being stored in the capacitor whenever the power is positive and delivered when the power is negative
- Voltage applied to capacitor returns to zero with increasing time. Thus, energy stored initially (up to 1 s) is returned over time as well

Capacitor: Example 6.5, I source

$$i(t) = 0, t \le 0s;$$
$$i(t) = 5000t, 0 \le t \le 20\mu s;$$
$$i(t) = 0.2 - 5000t, 20\mu s \le t \le 40\mu s;$$
$$i(t) = 0; t \ge 40\mu s$$

- In this example, the excitation comes from a current source
- Energy is being stored in the capacitor whenever the power is positive
- Here since power is always positive, energy is continually stored in capacitor. When current returns to zero, the stored energy is trapped since ideal capacitor. Thus a voltage remains on the capacitor permanently (ideal lossless capacitor)
- Concept used extensively in memory and imaging circuits

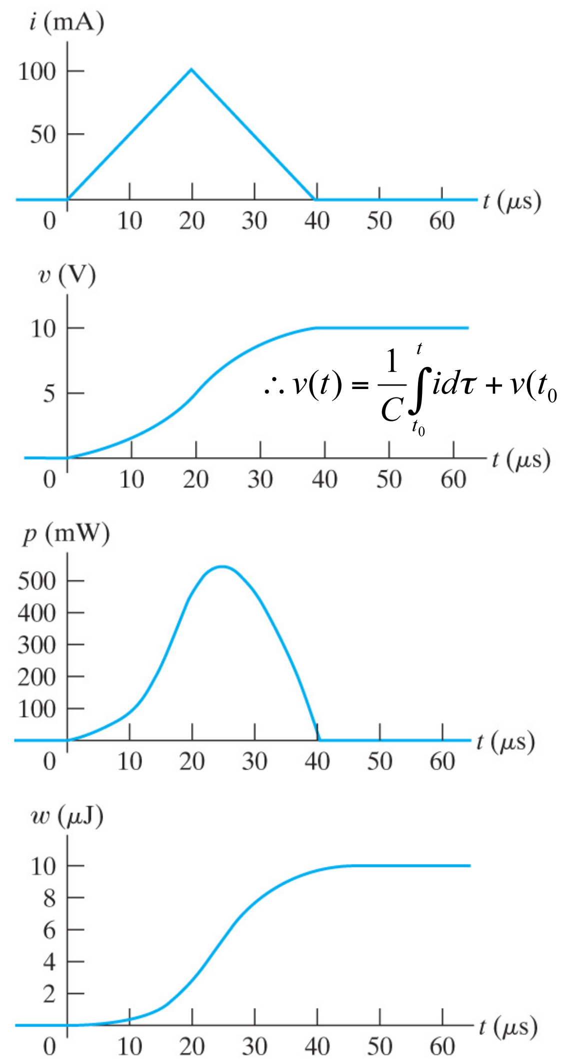
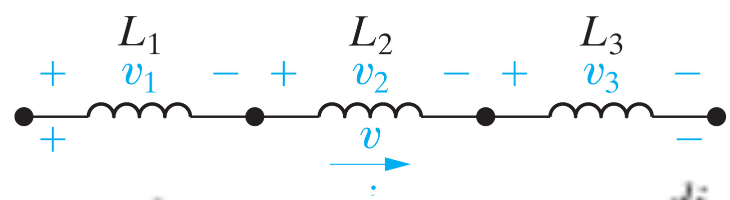


Figure: 06-121-4Ex6.5-C

Series-Parallel Combination (L)



$$v_1 = L_1 \frac{di}{dt} \quad v_2 = L_2 \frac{di}{dt} \quad v_3 = L_3 \frac{di}{dt}$$

$$v = v_1 + v_2 + v_3 = (L_1 + L_2 + L_3)\frac{di}{dt}$$

$$i_1 = \frac{1}{L_1}\int_{t_0}^{t} v\,d\tau + i_1(t_0)$$

$$i_2 = \frac{1}{L_2}\int_{t_0}^{t} v\,d\tau + i_2(t_0)$$

$$i_3 = \frac{1}{L_3}\int_{t_0}^{t} v\,d\tau + i_3(t_0)$$

$$i = i_1 + i_2 + i_3$$

$$i = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_{t_0}^{t} v\,d\tau + i_1(t_0) + i_2(t_0) + i_3(t_0)$$

$$i = \left(\frac{1}{L_{eq}}\right)\int_{t_0}^{t} v\,d\tau + i(t_0)$$

$$\therefore \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}; i(t_0) = i_1(t_0) + i_2(t_0) + i_3(t_0)$$

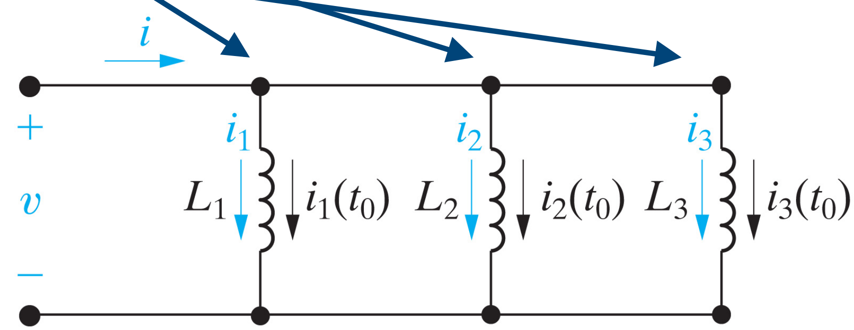
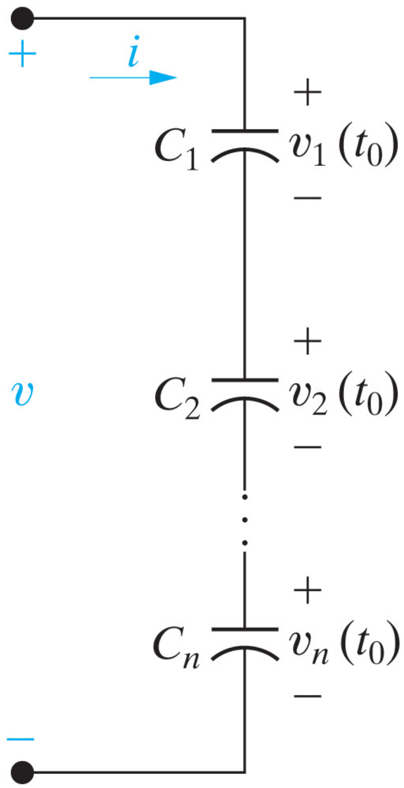


Figure: 06-15

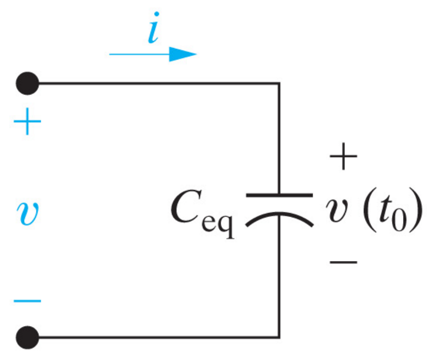
Series Combination (C)

$$v = \frac{1}{C_1}\int_0^t i + v_1(0) + \frac{1}{C_2}\int_0^t i\,dx + v_2(0) + \cdots$$

$$v = \left[\frac{1}{C_1} + \frac{1}{C_2} + \cdots\right]\int_0^t i\,dx + v_1(0) + v_2(0) + \cdots$$



(a)



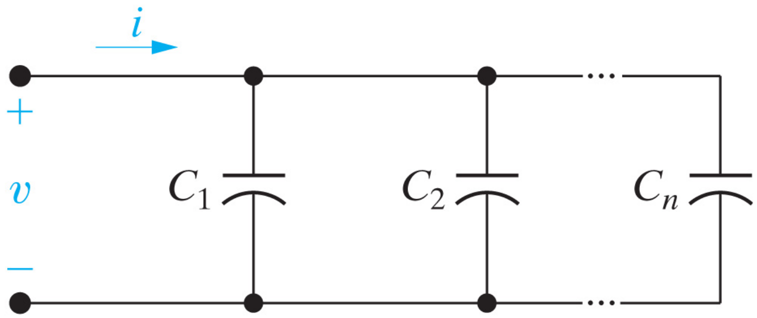
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_n(t_0)$$

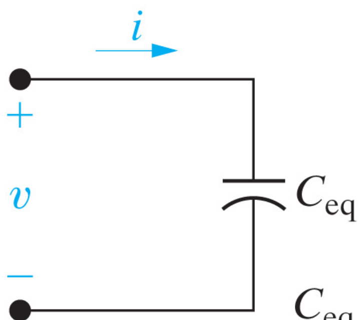
(b)

Figure: 06-17a,b

Parallel Combination (C)



(a)



$$C_{eq} = C_1 + C_2 + \cdots + C_n$$

(b)

Figure: 06-18a,b

$$i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + \cdots = [C_1 + C_2 + \cdots]\frac{dv}{dt}$$

Therefore $C_{eq} = C_1 + C_2 + \cdots$. Because the capacitors are in parallel, the initial voltage on every capacitor must be the same. This initial voltage would appear on C_{eq}.

First Order RL and RC circuits

- Class of circuits that are analyzed using first order ordinary differential equations
- To determine circuit behavior when energy is released or acquired by L and C due to an abrupt change in dc voltage or current.
- **Natural response**: i(t) and v(t) when energy is released into a resistive network (i.e. when L or C is disconnected from its DC source)
- **Step response**: i(t) and v(t) when energy is acquired by L or C (due to the sudden application of a DC i or v)

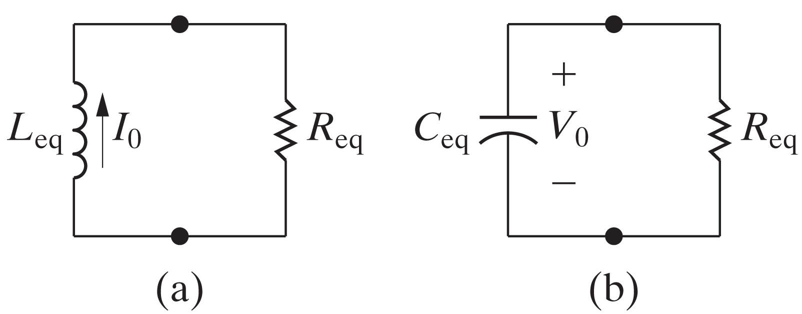


Figure: 07-01

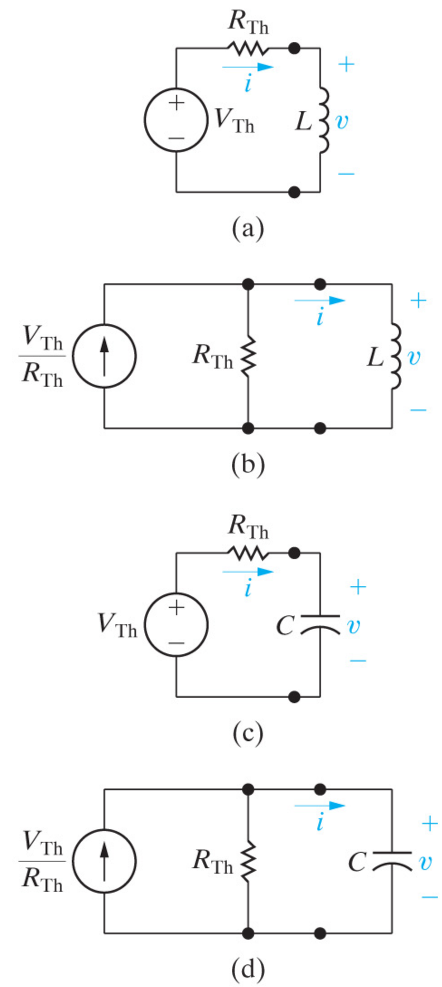


Figure: 07-02

Natural response: RL circuit

- Assume all currents and voltages in circuit have reached steady state (constant, dc) values

Prior to switch opening,

- L is acting as short circuit (i.e. since at DC)
- So all I_s is in L and none in R
- We want to find v(t) and i(t) for t>0

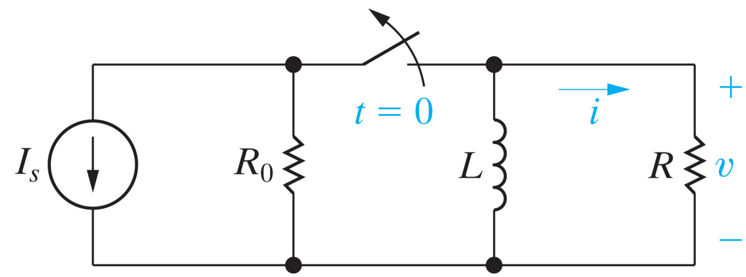


Figure: 07-03

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt}dt = -\frac{R}{L}idt$$

$$\frac{di}{i} = -\frac{R}{L}dt$$

$$\int_{i(t_0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_{t_0}^{t} dy; t_0 = 0$$

$$\ln\frac{i(t)}{i(0)} = -\frac{R}{L}t$$

$$\therefore i(t) = i(0)e^{-\left(\frac{R}{L}\right)t}$$

- Since current cannot change instantly in L, $i(0^-) = i(0^+) = I_0$

$$\therefore i(t) = I_0 e^{-\left(\frac{R}{L}\right)t}; t \geq 0$$

$$v = i(t)R = RI_0 e^{-\left(\frac{R}{L}\right)t}; t \geq 0^+$$

- $v(0^-) = 0$ but $v(0^+) = I_0R$

$$p = vi = i^2R = \frac{v^2}{R} = RI_0^2 e^{-2\left(\frac{R}{L}\right)t}; t \geq 0^+$$

$$w = \int_0^t p\,dx = \frac{1}{2}LI_0^2\left(1 - e^{-2\left(\frac{R}{L}\right)t}\right); t \geq 0$$

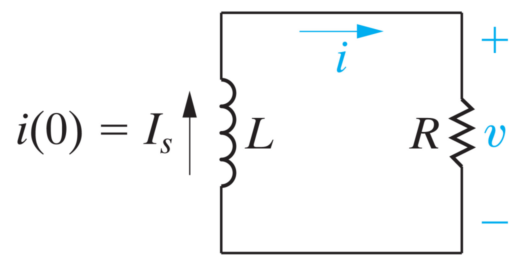


Figure: 07-04

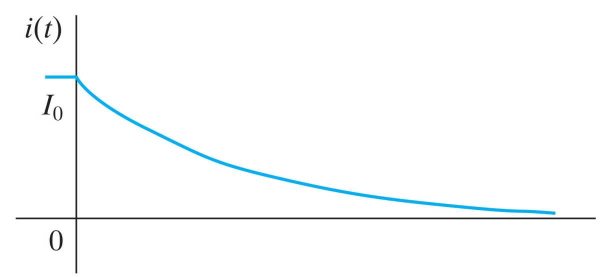


Figure: 07-05

Natural response time constant

- Both i(t) and v(t) have a term $e^{-\left(\frac{R}{L}\right)t}$
- Time constant τ is defined as $\tau = \left(\frac{L}{R}\right)$

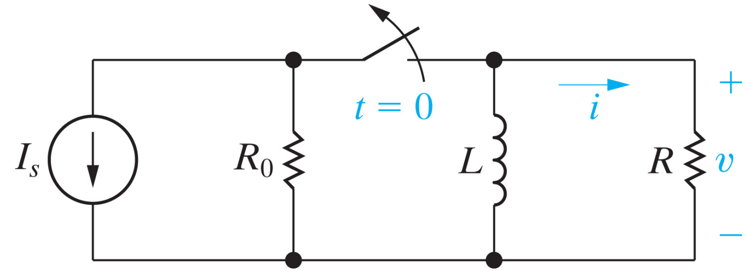


Figure: 07-03

$$\therefore i(t) = I_0 e^{-\frac{t}{\tau}}; t \geq 0$$

$$v(t) = RI_0 e^{-\frac{t}{\tau}}; t \geq 0^+$$

$$p = RI_0^2 e^{-\frac{2t}{\tau}}; t \geq 0^+$$

$$w = \frac{1}{2} LI_0^2 \left(1 - e^{-\frac{2t}{\tau}}\right); t \geq 0$$

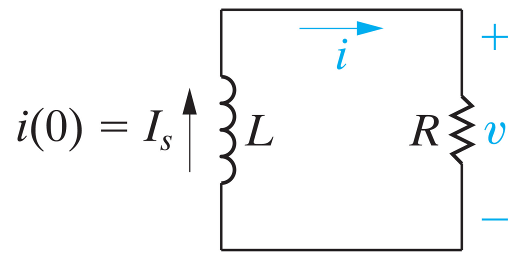


Figure: 07-04

- Think of τ as an integral parameter
- i.e. after 1 τ, the inductor current has been reduced to e^{-1} (or 0.37) of its initial value. After 5 τ, the current is less than 1% of its original value (i.e. steady state is achieved)
- The existence of current in the RL circuit is momentary – transient response. After 5τ, cct has steady state response

Extracting τ

- If R and L are unknown
- τ can be determined from a plot of the natural response of the circuit
- For example,

$$\frac{di}{dt}(0^+) = -\frac{1}{\tau} I_0 e^{-\frac{0^+}{\tau}} = -\frac{R}{L} I_0 = -\frac{I_0}{\tau}$$

- If i starts at I_0 and decreases at I_0/τ, i becomes

$$\therefore i = I_0 - \frac{I_0}{\tau} t$$

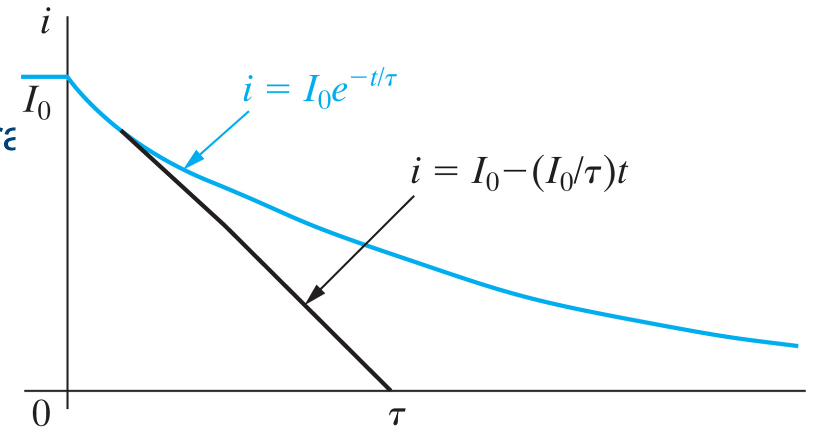


Figure: 07-06

- Then, drawing a tangent at t = 0 would yield τ at the x-axis intercept
- And if I_0 is known, natural response can be written as,

$$i(t) = I_0 e^{-\frac{t}{\tau}}$$

Example 7.1

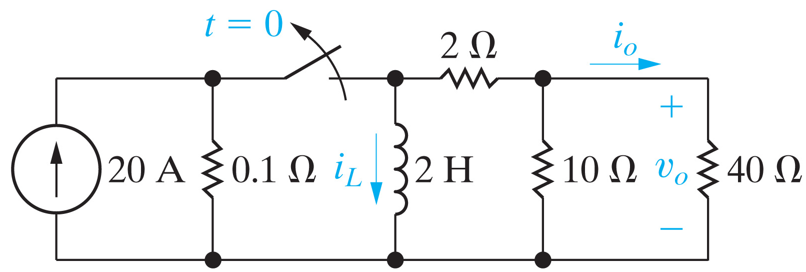


Figure: 07-07Ex7.1

- To find $i_L(t)$ for $t \geq 0$, note that since cct is in steady state before switch is opened, L is a short and all current is in it, i.e. $I_L(0^+) = I_L(0^-) = 20A$
- Simplify resistors with $R_{eq} = 2+40||10 = 10\Omega$
- Then $\tau = L/R = 0.2s$,

$$\therefore i_L(t) = 20e^{-\frac{t}{0.2}} A, t \geq 0$$

- With switch open,

$$i_0 = -i_L\left(\frac{10}{10+40}\right) = -0.2i_L$$

$$\therefore i_0(t) = -4e^{-\frac{t}{0.2}} A, t \geq 0^+$$

- voltage across 40 Ω and 10 Ω,

$$v_0(t) = 40i_0(t) = -160e^{-\frac{t}{0.2}} V, t \geq 0^+$$

- power dissipated in 10 Ω

$$p_{10\Omega}(t) = \frac{v_0(t)^2}{10} = 2560e^{-\frac{t}{0.1}} W, t \geq 0^+$$

- Energy dissipated in 10 Ω

$$w_{10\Omega}(t) = \int_0^\infty 2560e^{-\frac{t}{0.1}} dt = 256J$$

$$w(0) = \frac{1}{2}Li^2(0) = 400J$$

$$\% \frac{w_{10\Omega}(t)}{w(0)} = 64\%$$

Example 7.2

- Initial I in L_1 and L_2 already established by "hidden sources"
- To get i_1, i_2 and i_3, find v(t) (since parallel cct) with simplified circuit

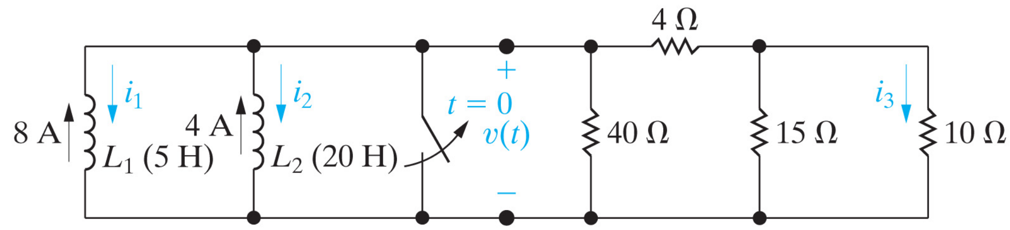


Figure: 07-08Ex7.2

$$L = 4H, R = 8\Omega,$$

$$\therefore i(t) = 12e^{-2t} A, t \geq 0$$

$$v_0(t) = 8i_0(t) = 96e^{-2t} V, t \geq 0^+$$

$$v_0(t) = 0, t < 0$$

$$\because i(t) = \frac{1}{L}\int_0^t v d\tau + i(t_0)$$

$$i_1(t) = \frac{1}{L}\int_0^t 96e^{-2x} dx - 8$$

$$i_1(t) = 1.6 - 9.6e^{-2t} A, t \geq 0$$

$$i_2(t) = -1.6 - 2.4e^{-2t} A, t \geq 0$$

$$i_3(t) = \frac{v(t)}{10}\frac{15}{25} = 5.76e^{-2t} A, t \geq 0^+$$

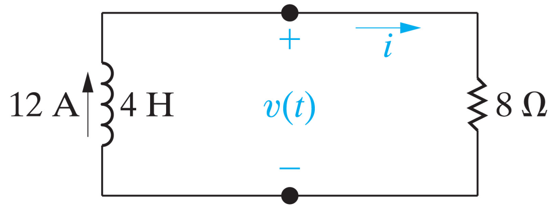


Figure: 07-09Ex7.2

- Note inductor current i_1 and i_2 are valid from t ≥ 0 since current in inductor cannot change instantaneously
- However, resistor current i_3 is valid only from t ≥ 0^+ since there is 0 current in resistor at t = 0 (all I is shorted through inductors in steady state)

Example 7.2 (contd)

- Initial energy stored in inductors

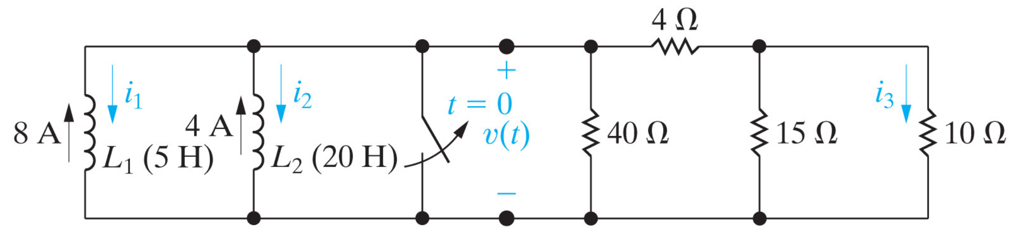


Figure: 07-08Ex7.2

$$w = \frac{1}{2}Li^2;\ w_{init} = w_{5H} + w_{20H}$$

$$w_{init} = \frac{1}{2}(5)(64) + \frac{1}{2}(20)(16) = 320J$$

$$If\ (t \to \infty, i_1 \to 1.6A, i_2 \to -1.6A)$$

$$w_{final} = \frac{1}{2}(5)(1.6)^2 + \frac{1}{2}(20)(-1.6)^2 = 32J$$

$$w_R = \int_0^\infty p\,dt = \int_0^\infty \left(\frac{v(t)^2}{R_{eq}}\right)dt = \int_0^\infty \left(\frac{(96e^{-2t})^2}{8}\right)dt =$$

$$w_R = \int_0^\infty 1152e^{-4t}dt = 1152\left.\frac{e^{-4t}}{-4}\right|_0^\infty = 288J$$

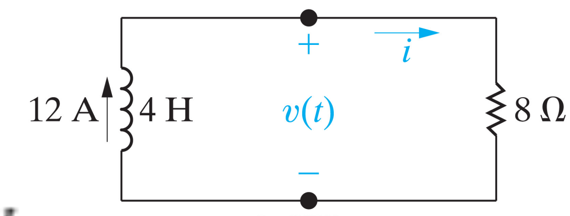


Figure: 07-09Ex7.2

- Note $w_R + w_{final} = w_{init}$
- w_R indicates energy dissipated in resistors after switch opens
- w_{final} is energy retained by inductors due to the current circulating between the two inductors (+1.6A and -1.6A) when they become short circuits at steady state again

Natural Response of RC circuit

- Similar to that of an RL circuit
- Assume all currents and voltages in circuit have reached steady state (constant, dc) values

Prior to switch moving from a to b,

- C is acting as open circuit (i.e. since at DC)
- So all of V_g appears across C since I = 0
- We want to find v(t) for t>0
- Note that since voltage across capacitor cannot change instantaneously, $V_g = V_0$, the initial voltage on capacitor

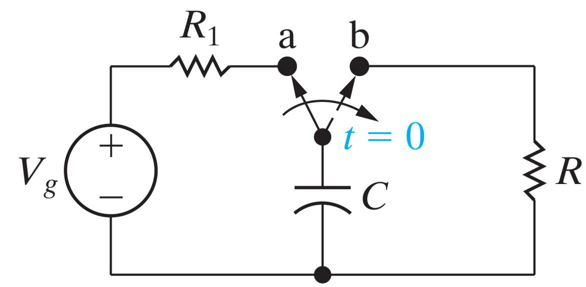


Figure: 07-10

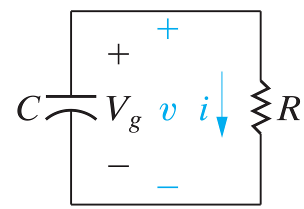


Figure: 07-11

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

Solving,

$$v(t) = v(0)e^{-\frac{t}{RC}}; t \geq 0$$

$$v(0^-) = v(0) = v(0^+) = V_g = V_0$$

$$\tau = RC$$

$$v(t) = V_0 e^{\frac{-t}{\tau}}; t \geq 0$$

$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R}e^{-\frac{t}{\tau}}; t \geq 0^+$$

$$p = vi = \frac{V_0^2}{R}e^{-\frac{2t}{\tau}}; t \geq 0^+$$

$$w = \int_0^t p\,dx = \int_0^t \frac{V_0^2}{R}e^{-\frac{2x}{\tau}}dx$$

$$w = \frac{1}{2}CV_0^2\left(1 - e^{-\frac{2t}{\tau}}\right); t \geq 0$$

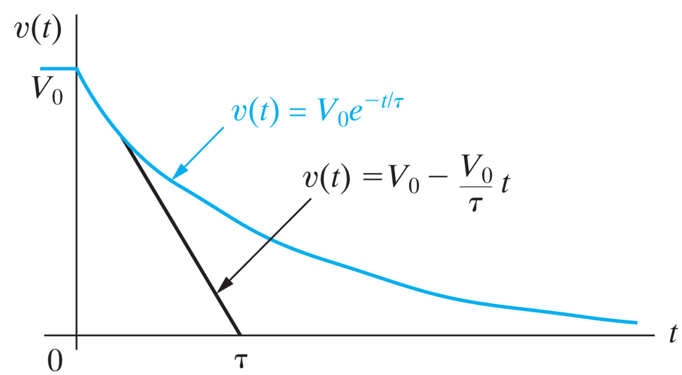


Figure: 07-12

Example 7.3

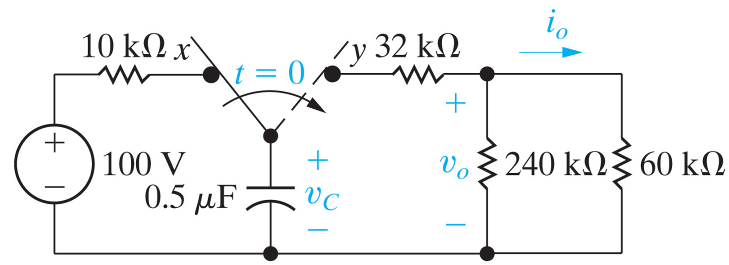


Figure: 07-13Ex7.3

- To find $v_C(t)$ for $t \geq 0$, note that since cct is in steady state before switch moves from x to y, C is charged to 100V. The resistor network can be simplified with a equivalent 80k resistor.
- Simplify resistors with R_{eq} = 32+240||60 = 80kΩ
- Then $\tau = RC = (0.5\mu F)(80k\Omega) = 40$ ms,

$$\therefore v_C(t) = 100e^{-25t}V, t \geq 0$$

- voltage across 240 kΩ and 60 kΩ,

$$v_0(t) = -v_C(t)\left(\frac{48}{32+48}\right) = 60e^{-25t}V, t \geq 0^+$$

- current in 60 kΩ resistor

$$i_0(t) = \frac{v_0(t)}{60k\Omega} = e^{-25t}mA, t \geq 0^+$$

- power dissipated in 60 kΩ

$$p_{60k\Omega}(t) = i_0(t)^2(60k\Omega) = 60e^{-50t}mW, t \geq 0^+$$

- Energy dissipated in 60 kΩ

$$w_{60k\Omega}(t) = \int_0^{\infty} i_0(t)^2(60k\Omega)dt = 1.2mJ$$

Example 7.4: Series capacitors

$$\therefore v(t) = 20e^{-t}V, t \geq 0$$

$$i(t) = \frac{v(t)}{250k\Omega} = 80e^{-t}\mu A, t \geq 0^{+}$$

$$\therefore v_1(t) = -\frac{1}{5\mu F}\int_0^t (80e^{-x}\mu A)dx - 4, t \geq 0$$

$$v_1(t) = (16e^{-t} - 20)V, t \geq 0$$

$$v_2(t) = -\frac{1}{20\mu F}\int_0^t (80e^{-x}\mu A)dx + 24, t \geq 0$$

$$v_2(t) = (4e^{-t} + 20)V, t \geq 0$$

$$w_{C_1} = 0.5CV^2 = 0.5(5\mu F)(4V)^2 = 40\mu J$$

$$w_{C_2} = 0.5(20\mu F)(24V)^2 = 5760\mu J$$

$$w_0 = 5800\mu J$$

$$t \to \infty, v_1 \to -20V, v_2 \to 20V$$

$$w_\infty = 0.5(25\mu F)(20)^2 = 5000\mu J$$

$$w_{250k\Omega} = \int_0^\infty p\,dt = \int_0^\infty \frac{(20e^{-t})^2}{250k\Omega}dt = 800\mu J$$

- Initial voltages established by "hidden" sources

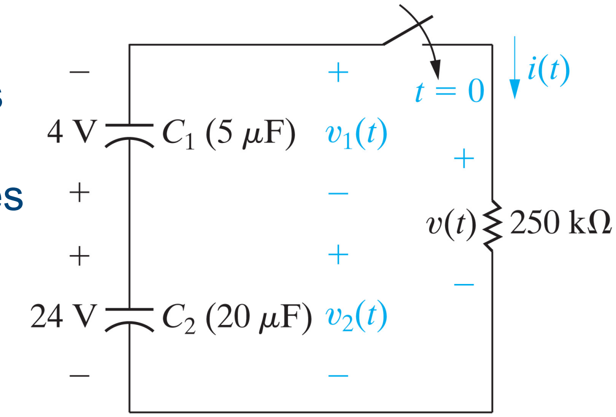


Figure: 07-14Ex7.4

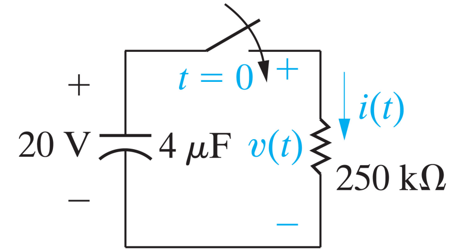


Figure: 07-15Ex7.4

Step response of RL circuits

$$KVL: V_s = iR + L\frac{di}{dt}$$

$$\frac{di}{dt} = \frac{-Ri + V_s}{L} = \frac{-R}{L}\left(i - \frac{V_s}{R}\right)$$

$$di = \frac{-R}{L}\left(i - \frac{V_s}{R}\right)dt$$

$$\frac{di}{i - \frac{V_s}{R}} = \frac{-R}{L}dt$$

$$\int_{I_0}^{i(t)} \frac{dx}{x - \frac{V_s}{R}} = \frac{-R}{L}\int_0^t dy$$

$$\ln\frac{i(t) - \frac{V_s}{R}}{I_0 - \frac{V_s}{R}} = \frac{-R}{L}t$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-\frac{R}{L}t}$$

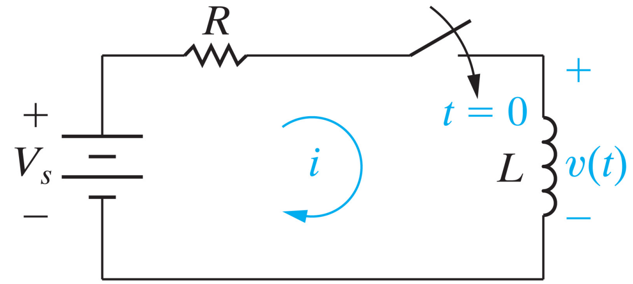


Figure: 07-16

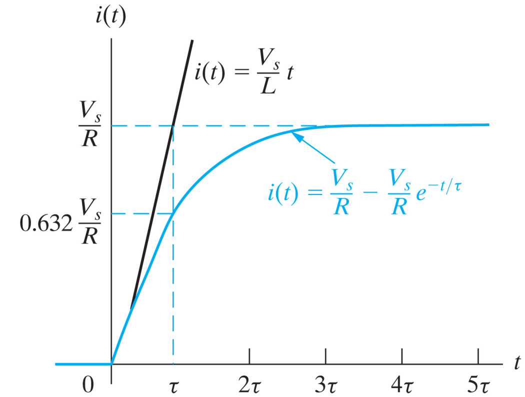


Figure: 07-17

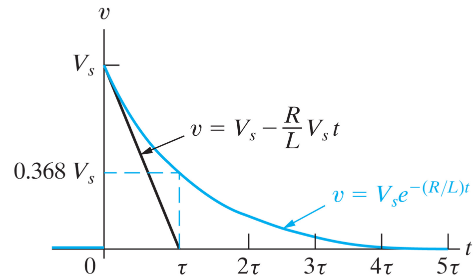


Figure: 07-18

Example 7.5: RL step response

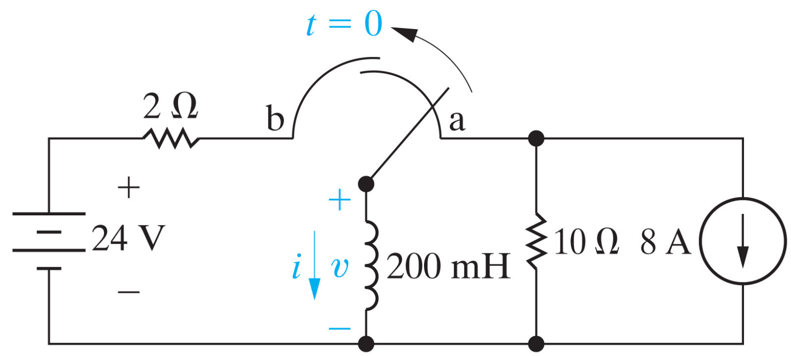


Figure: 07-19Ex7.5

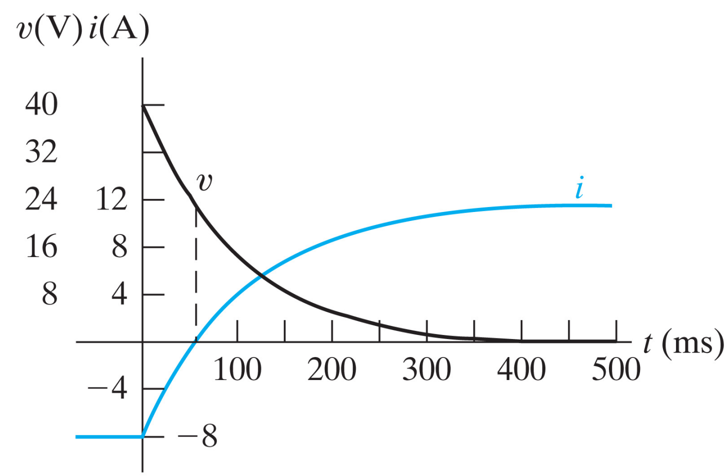


Figure: 07-20Ex7.5

Step response of RC circuits

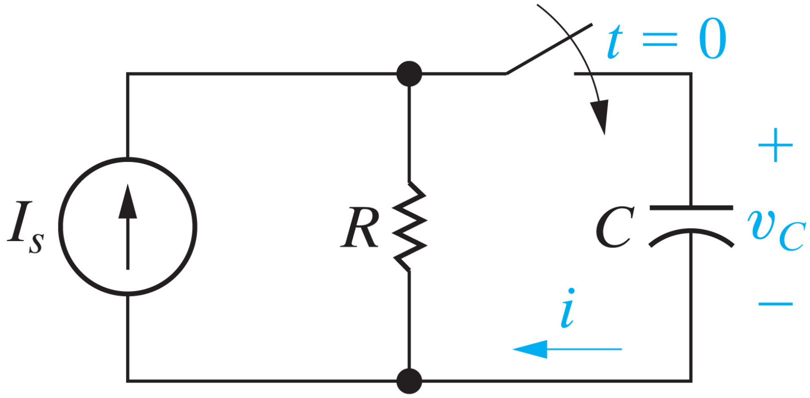


Figure: 07-21

Example 7.6: RC Step Response

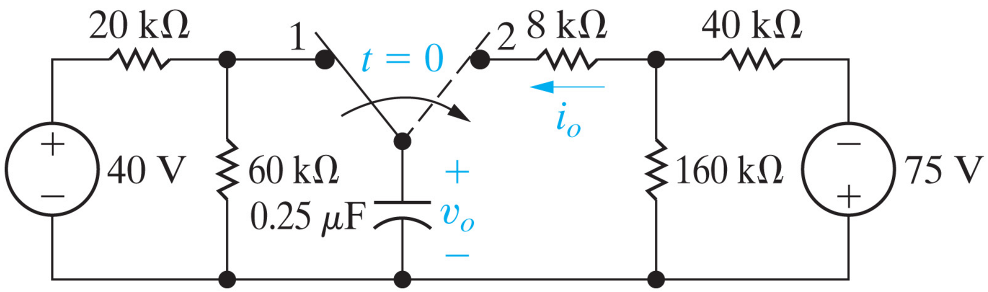


Figure: 07-22Ex7.6

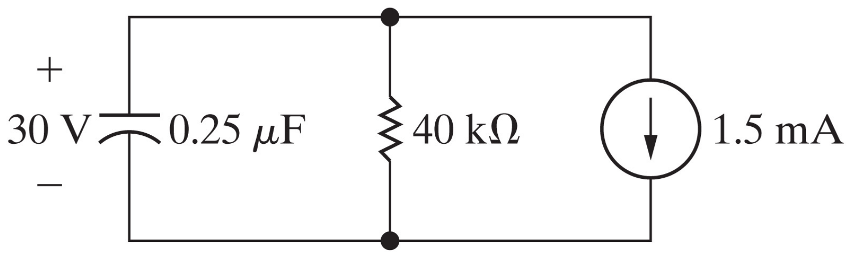


Figure: 07-23Ex7.6