
Electromagnetics

Inductors and the Wave Equation

ECE 111 Introduction to ECE

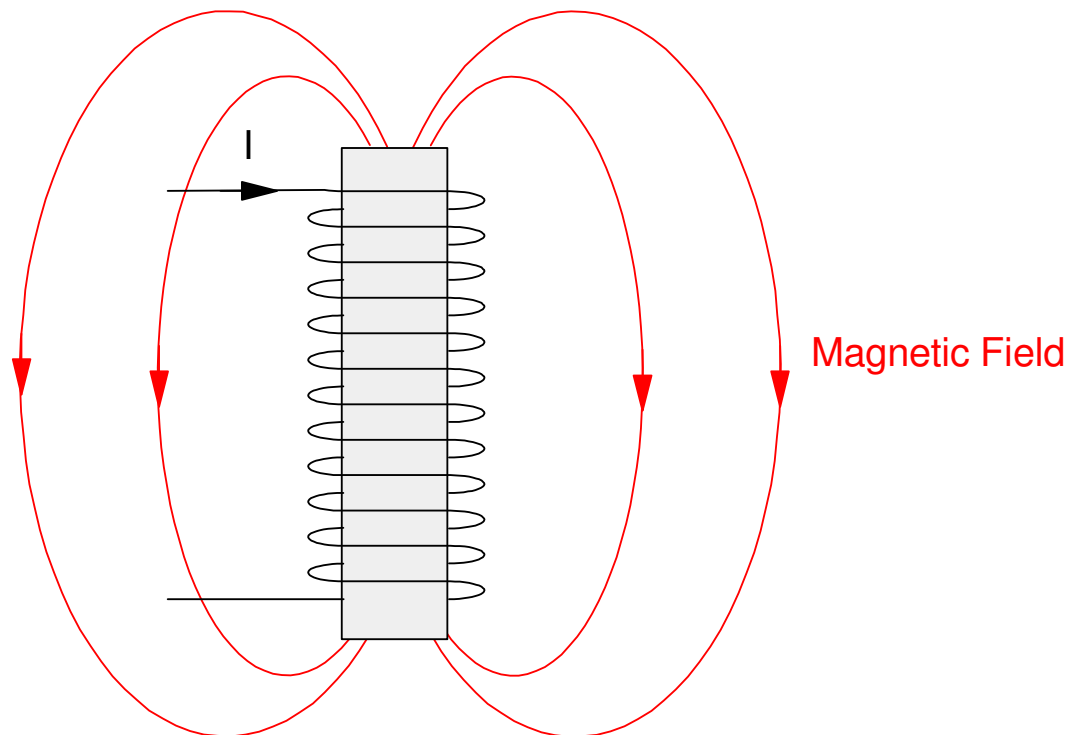
Jake Glower - Week #8

Please visit [Bison Academy](#) for corresponding
lecture notes, homework sets, and solutions

Inductors

Inductors store energy in a magnetic field.

- Units are Henries
- A coil of wire makes an inductor
- A long wire carrying current makes an inductor



Example 1: Find L and R

- 100 windings of 36 gage wire copper wire (1.38 Ohms / meter)
- Cross sectional area = 25mm
- Length = 10mm
- Air core (relative permeability of 1.000)

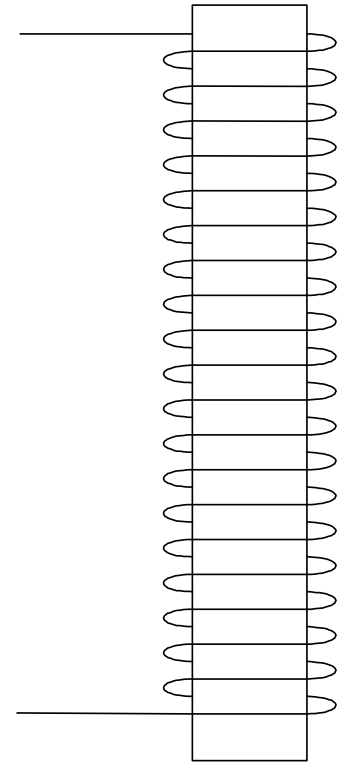
$$L = \mu N^2 A / l$$

$$L = \left(4\pi \cdot 10^{-7} \frac{H}{m} \right) (100)^2 (25 \cdot 10^{-6} m^2) / (0.01 m)$$

$$L = 31.42 \mu H$$

$$R = (100 \text{ windings}) \left(0.02 \frac{m}{\text{winding}} \right) (1.38 \frac{\Omega}{m})$$

$$R = 2.77 \Omega$$



Example 2: Determine the inductance of a copper transmission line:

- Length = 1km, radius = 1cm, f = 60Hz

From Wikipedia¹

$$L = \frac{\mu_0}{2\pi} l (A - B + C)$$

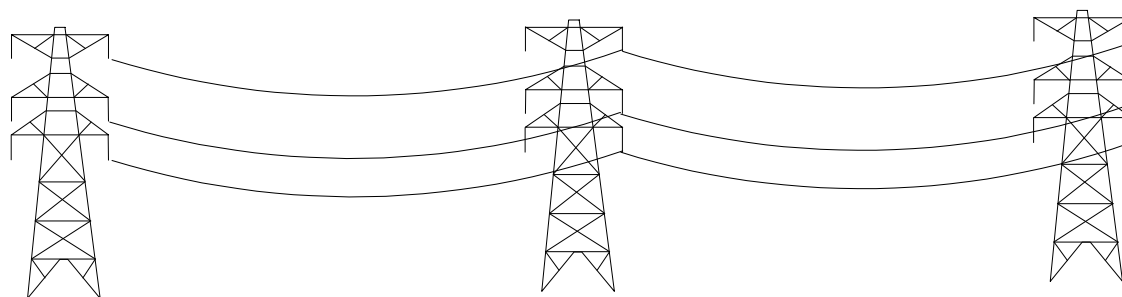
$$A = \ln \left(\frac{l}{r} + \sqrt{\left(\frac{l}{r}\right)^2 + 1} \right) = 12.20$$

$$B = \frac{1}{\frac{r}{l} + \sqrt{1 + \left(\frac{r}{l}\right)^2}} \approx 1$$

$$C = \frac{1}{4 + r \sqrt{\frac{2}{\rho} \omega \mu}} = 0.1569$$

giving

$$L = 2.30 \frac{mH}{km}$$



¹ https://en.wikipedia.org/wiki/Inductor#Inductance_formulas

VI Characteristics of Inductors

Inductors store energy in the magnetic field as

$$E = \frac{1}{2}LI^2 \quad \text{Joules}$$

The power absorbed is the derivative of energy

$$P = \frac{dE}{dt} \quad \text{Watts}$$

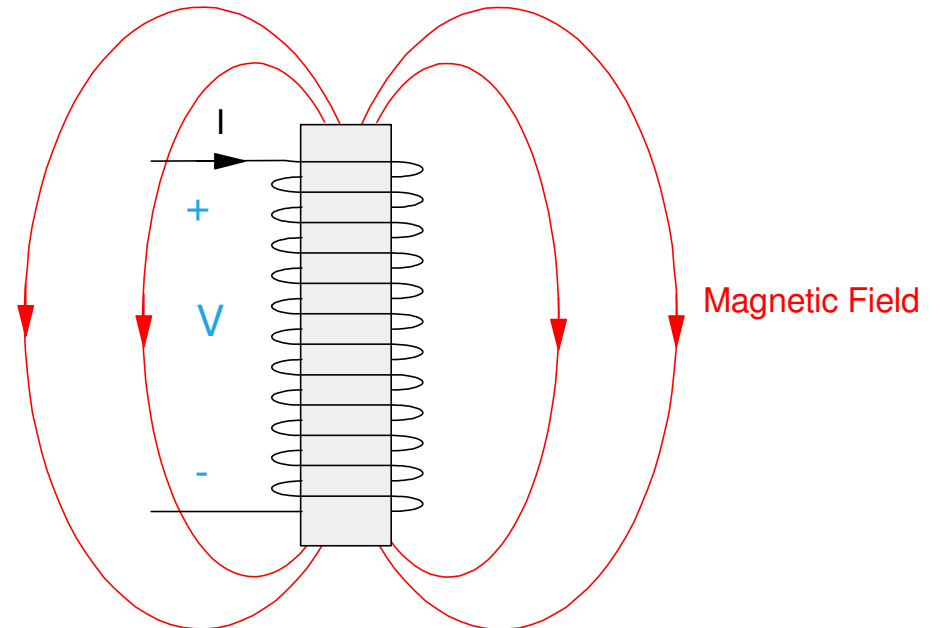
or

$$P = VI = LI \frac{dI}{dt}$$

Canceling terms gives the VI relationship for inductors:

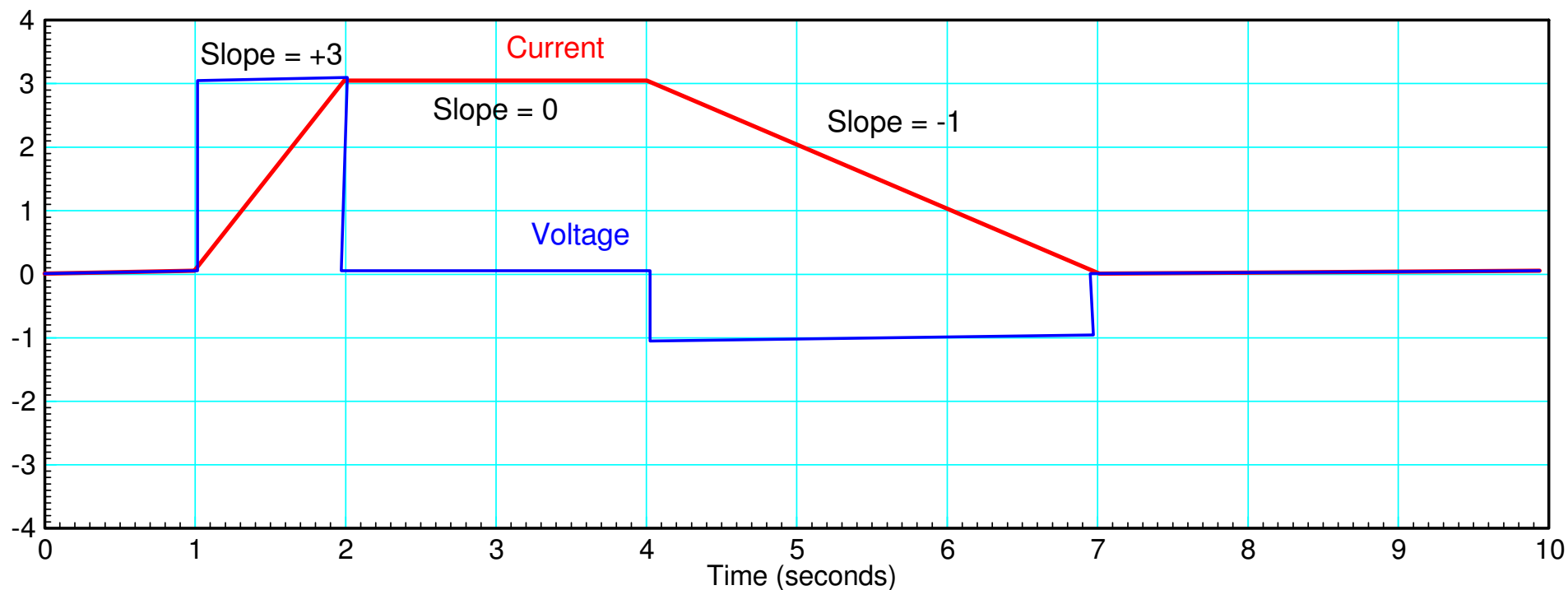
$$V = L \frac{dI}{dt}$$

Inductors act as differentiators.



Example: The current through a 1H inductor is shown below in red. Sketch the voltage.

Solution: Shown in blue. The derivative (times one) is the voltage.



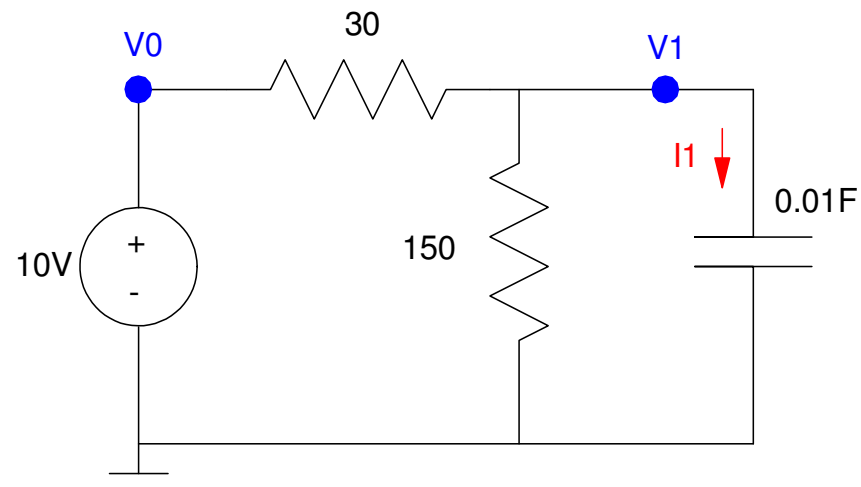
Heat Equation (recap)

One Element:

- One capacitor
- One pot on a stove

$$I_1 = C\dot{V}_1 = \left(\frac{V_0 - V_1}{30}\right) + \left(\frac{0 - V_1}{150}\right)$$

$$\dot{V}_1 = -4V_1 + 3.33V_0$$



N Elements (heat equation)

$$\dot{V}_i = aV_{i-1} - bV_i + aV_{i+1}$$

Wave Equation (Mass-Spring)

Coupled *second-order* differential equations

$$\ddot{V}_i = aV_{i-1} - bV_i + aV_{i+1}$$

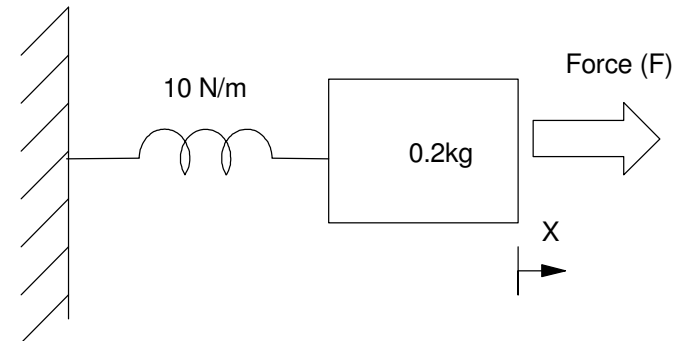
Example: Mass-Spring System

Force = Mass * Acceleration

$$F - Kx = M\ddot{x}$$

or

$$\ddot{x} = -50x + 5F$$



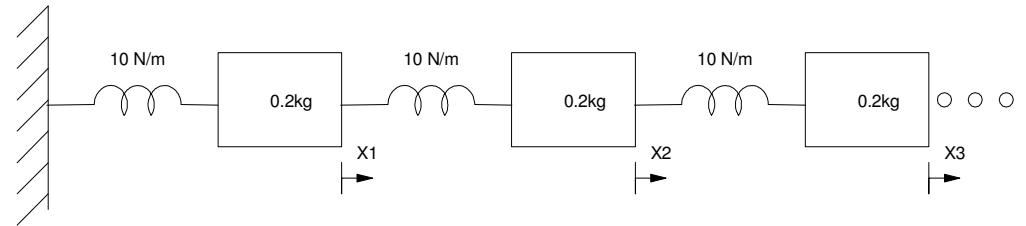
Cascade N mass-spring systems together

$$F_2 = m\ddot{x}_2$$

$$10(x_1 - x_2) + 10(x_3 - x_2) = 0.2\ddot{x}_2$$

or

$$\ddot{x}_2 = 10x_1 - 20x_2 + 10x_3$$



The same pattern holds for all N nodes.

This is an important equation as it describes

- Transmission lines galloping on windy days,
 - Cars stuck in traffic,
 - Buildings swaying during an earthquake or a wind storm,
 - Bridges oscillating during wind storms,
 - etc.
-

Wave Equation Examples: Galloping Transmission Line

- <https://www.youtube.com/watch?v=GEGbYRii1d4>



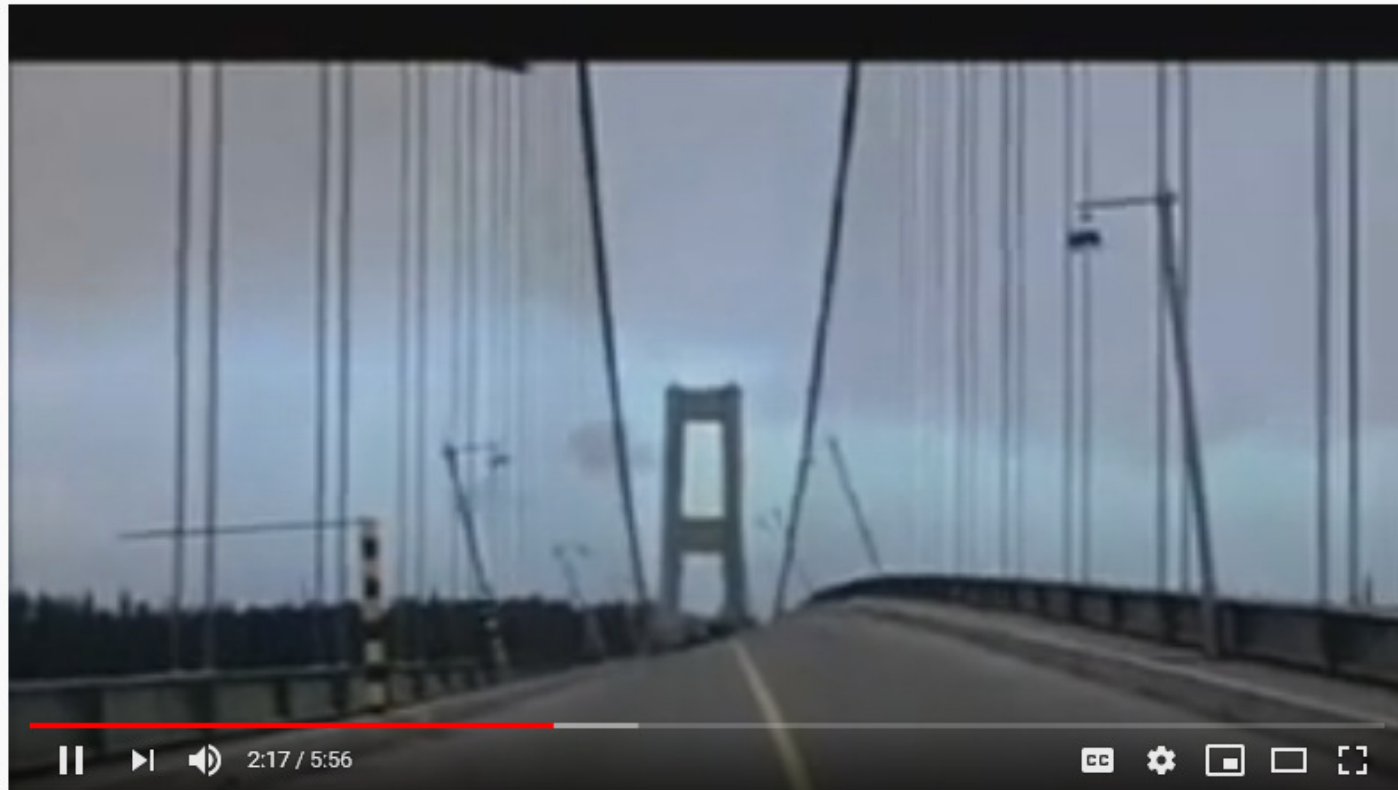
Wave Equation Example: Cars in Traffic

- <https://www.youtube.com/watch?v=19S3OdK6710&t=171s>



Wave Equation Example: Tacoma Narrows Bridge

- <https://www.youtube.com/watch?v=j-zczJXSxnw&t=136s>



Wave Equation Example: Swaying Skyscrapers

- <https://www.youtube.com/watch?v=2t2xxKMN-Ic&t=49s>



Wave Equation (LC Circuits)

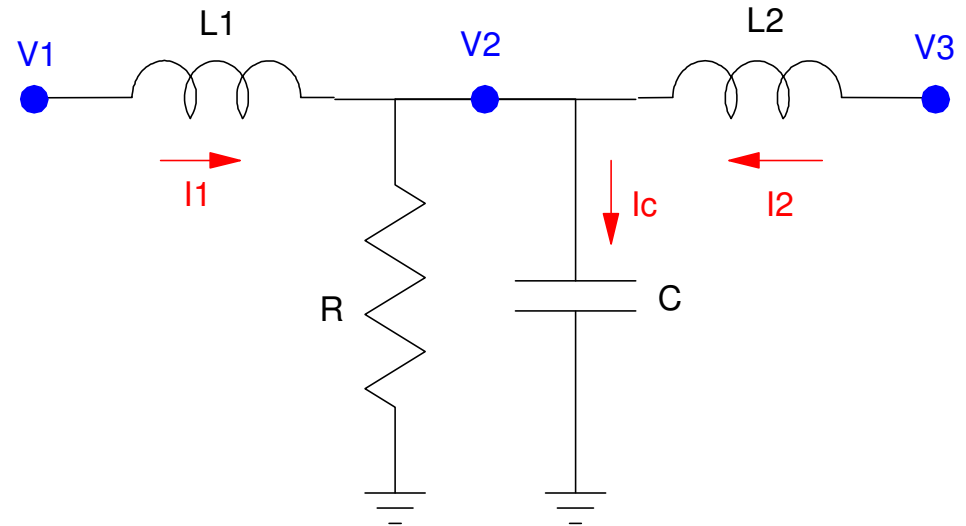
Also applies to LC Circuits

Long Transmission Lines

- L: Inductance of wires
- C: Parallel plate capacitor (wire & ground)

Circuit Boards

- L: Inductance of circuit trace
- C: Trace to ground plane



Equations:

$$L_1 \dot{I}_1 = V_1 - V_2$$

$$L_2 \dot{I}_2 = V_3 - V_2$$

$$C \dot{V}_2 = I_1 + I_2 - \frac{V_2}{R}$$

Differentiating

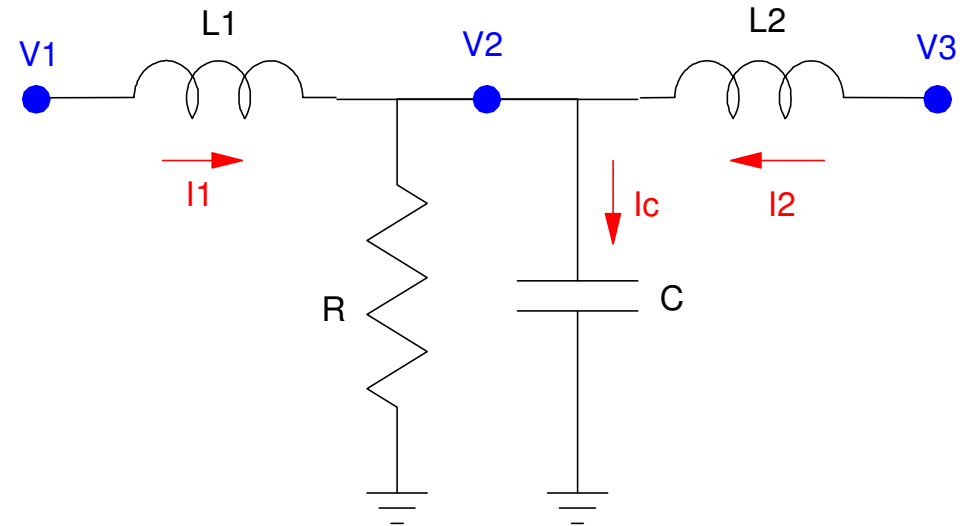
$$C \ddot{V}_2 = \dot{I}_1 + \dot{I}_2 - \frac{\dot{V}_2}{R}$$

Substituting the inductor equation

$$C \ddot{V}_2 = \left(\frac{V_1 - V_2}{L_1} \right) + \left(\frac{V_3 - V_2}{L_2} \right) - \frac{\dot{V}_2}{R}$$

or

$$C \ddot{V}_2 = \left(\frac{1}{L_1} \right) V_1 - \left(\frac{1}{L_1} + \frac{1}{L_2} \right) V_2 + \left(\frac{1}{L_2} \right) V_3 - \left(\frac{1}{R} \right) \dot{V}_2$$



Dynamic Response of the Wave Equation

$$\ddot{V}_1 = \left(\frac{1}{LC}\right) V_0 - \left(\frac{2}{LC}\right) V_1 + \left(\frac{1}{LC}\right) V_2 - \left(\frac{1}{RC}\right) \dot{V}_1$$

$$\ddot{V}_2 = \left(\frac{1}{LC}\right) V_1 - \left(\frac{2}{LC}\right) V_2 + \left(\frac{1}{LC}\right) V_3 - \left(\frac{1}{RC}\right) \dot{V}_2$$

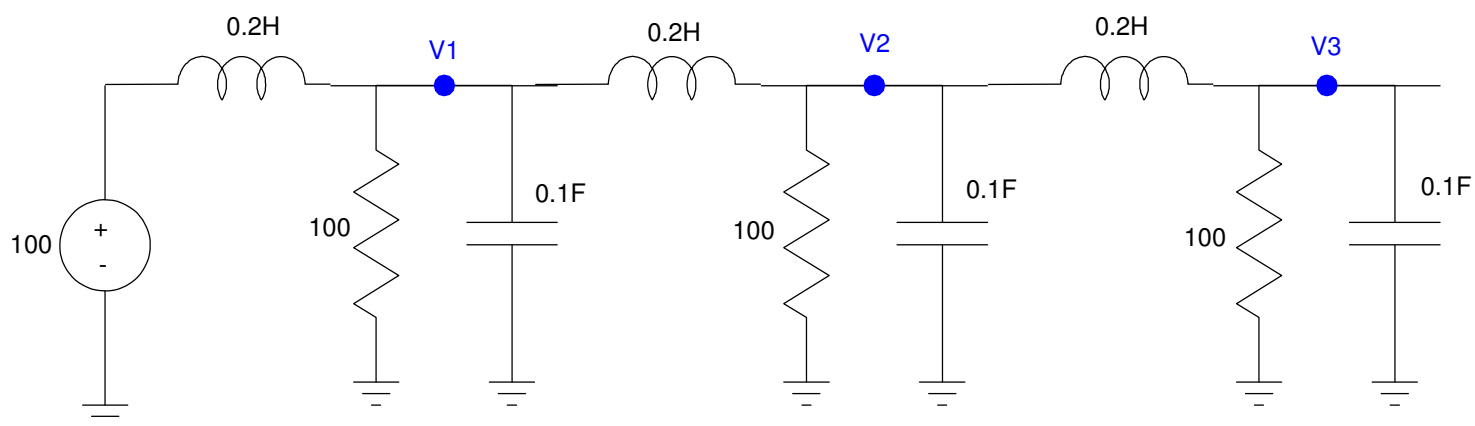
$$\ddot{V}_3 = \left(\frac{1}{LC}\right) V_2 - \left(\frac{1}{LC}\right) V_3 - \left(\frac{1}{RC}\right) \dot{V}_3$$

or

$$\ddot{V}_1 = 50V_0 - 100V_1 + 50V_2 - 0.1\dot{V}_1$$

$$\ddot{V}_2 = 50V_1 - 100V_2 + 50V_3 - 0.1\dot{V}_2$$

$$\ddot{V}_3 = 50V_2 - 50V_3 - 0.1\dot{V}_3$$



In MATLAB

```
V0 = 100;
V1 = 0;
V2 = 0;
V3 = 0;

dV1 = 0;
dV2 = 0;
dV3 = 0;

V = [];

dt = 0.01;

for i=1:300

    ddV1 = 50*V0 - 100*V1 + 50*V2 - 0.01*dV1;
    ddV2 = 50*V1 - 100*V2 + 50*V3 - 0.01*dV2;
    ddV3 = 50*V2 - 50*V3 - 0.01*dV3;

    dV1 = dV1 + ddV1*dt;
    dV2 = dV2 + ddV2*dt;
    dV3 = dV3 + ddV3*dt;

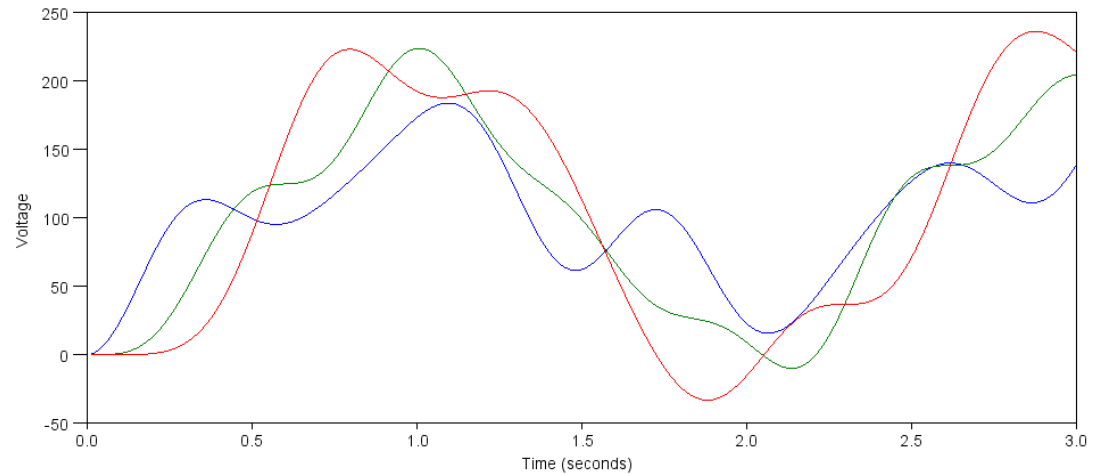
    V1 = V1 + dV1*dt;
    V2 = V2 + dV2*dt;
    V3 = V3 + dV3*dt;

    V = [V; V1, V2, V3];

end

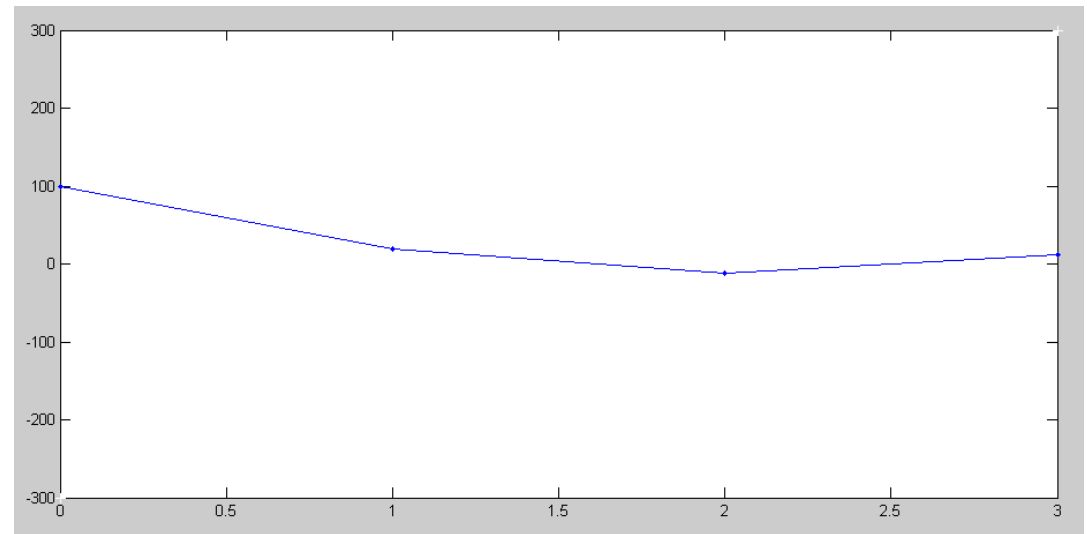
t = [1:300]' * dt;

plot(t,V);
xlabel('Time (seconds)');
ylabel('Voltage');
```



It's a little more fun to watch the voltages as the simulation runs

```
V0 = 100;  
V1 = 0;  
V2 = 0;  
V3 = 0;  
  
dV1 = 0;  
dV2 = 0;  
dV3 = 0;  
  
V = [];  
  
t = 0;  
dt = 0.01;  
  
while(t < 100)  
  
    ddV1 = 50*V0 - 100*V1 + 50*V2 - 0.01*dV1;  
    ddV2 = 50*V1 - 100*V2 + 50*V3 - 0.01*dV2;  
    ddV3 = 50*V2 - 50*V3 - 0.01*dV3;  
  
    dV1 = dV1 + ddV1*dt;  
    dV2 = dV2 + ddV2*dt;  
    dV3 = dV3 + ddV3*dt;  
  
    V1 = V1 + dV1*dt;  
    V2 = V2 + dV2*dt;  
    V3 = V3 + dV3*dt;  
  
    plot([0,1,2,3],[V0,V1,V2,V3],'.-');  
    ylim([-300,300]);  
    pause(0.01);  
  
    V = [V; V1, V2, V3];  
  
end
```



30-Stage Transmission Line:

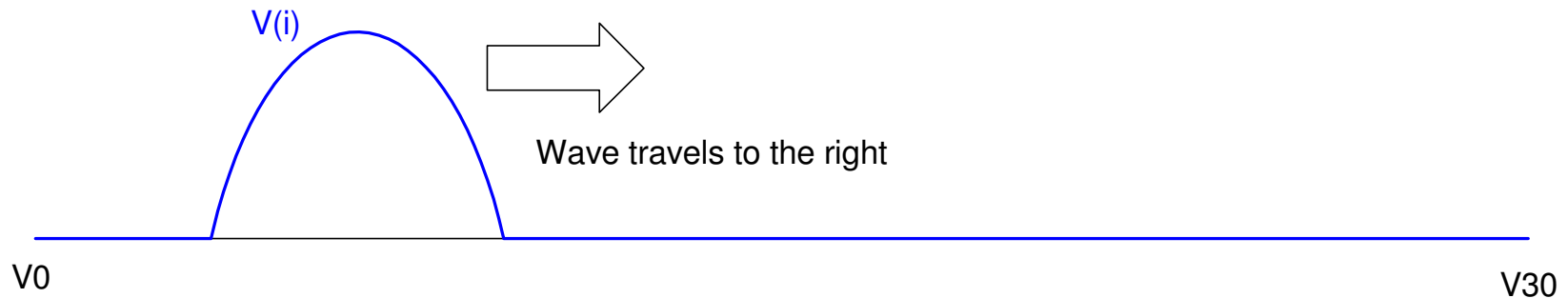
$$\ddot{V}_1 = 50V_0 - 100V_1 + 50V_2 - 0.01\dot{V}_1$$

⋮

$$\ddot{V}_{29} = 50V_{28} - 100V_{29} + 50V_{30} - 0.01\dot{V}_{29}$$

$$\ddot{V}_{30} = 50V_{29} - 50V_{30} - 0.01\dot{V}_{30}$$

Creates a traveling wave



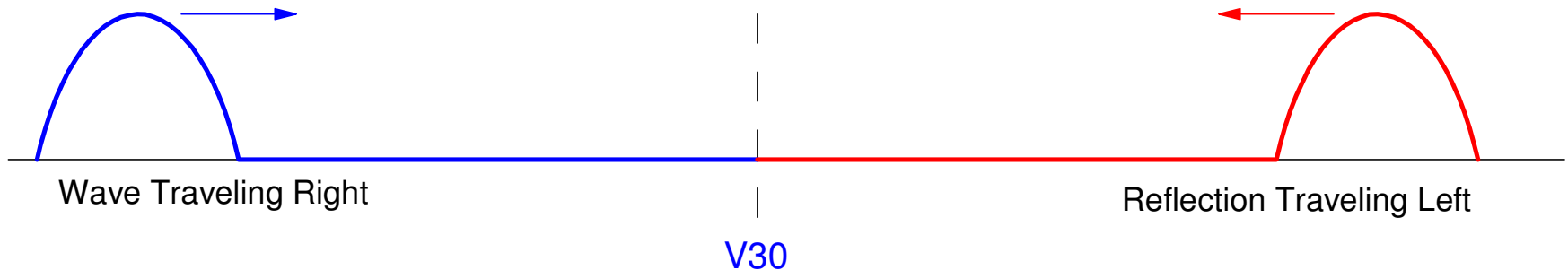
Reflection Coefficients:

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

where

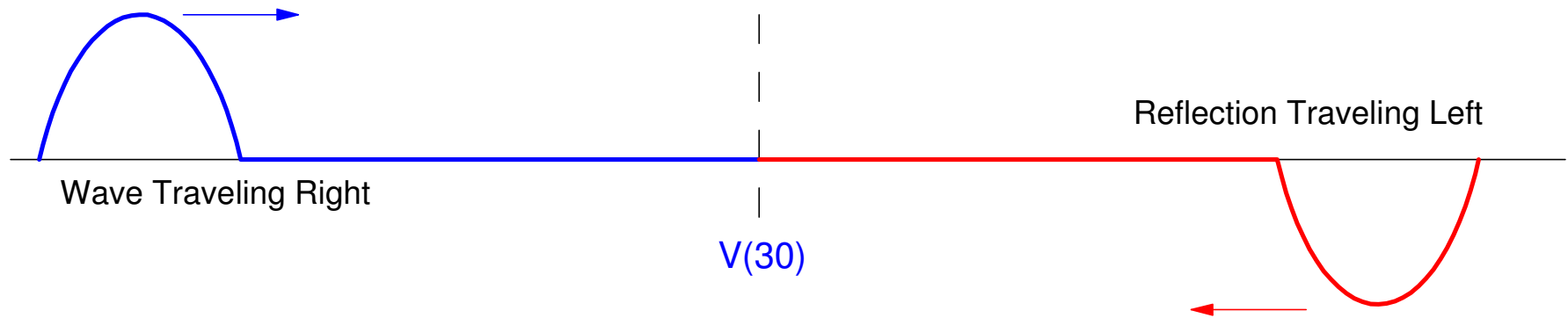
- Z_0 is the impedance of the transmission line and
- Z_1 is the impedance of the load (R_{30} in this case)

$\Gamma = +1$: When R_{30} is large (100 Ohms)



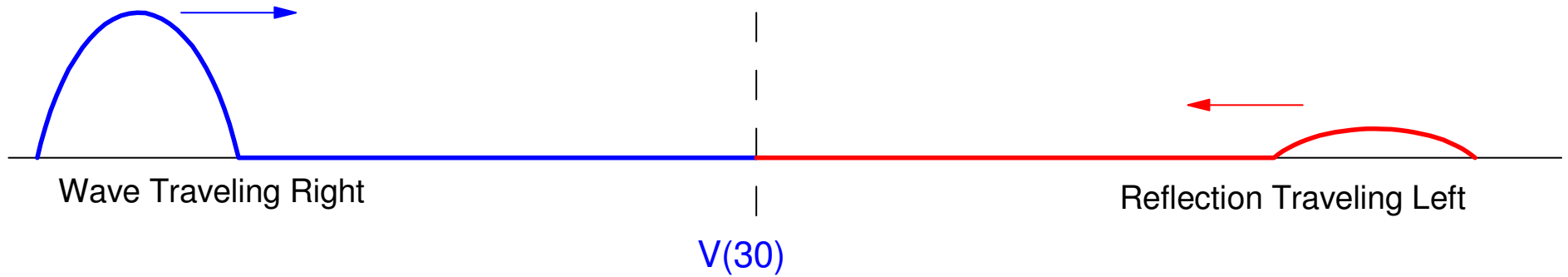
$\Gamma = -1$: R30 is small (0.01 Ohm)

- An infinite transmission line with a wave traveling to the right
- A second wave (the reflection) with the opposite amplitude traveling to the left
- When they hit, the amplitude at node 20 cancels and the point V(20) remains at zero



$\Gamma = 0$:

- Goldilocks R: "Just Right"
- No Reflection



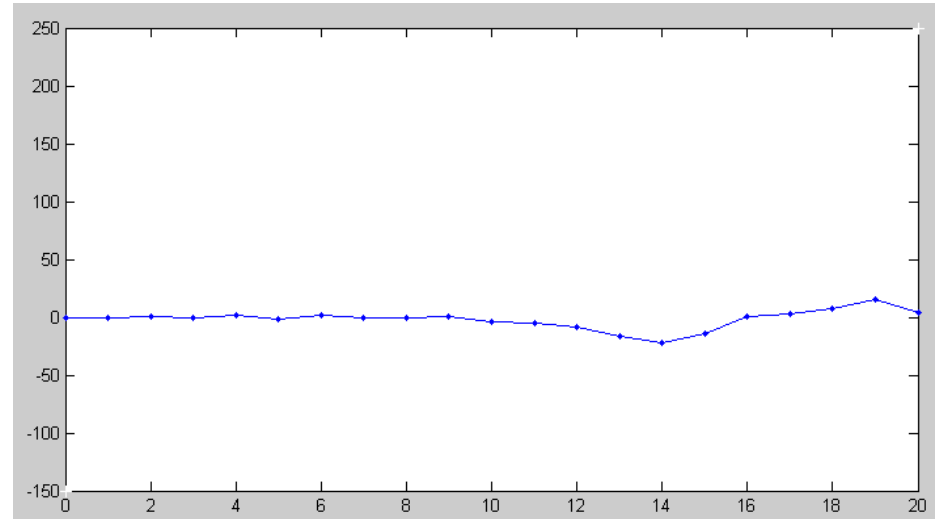
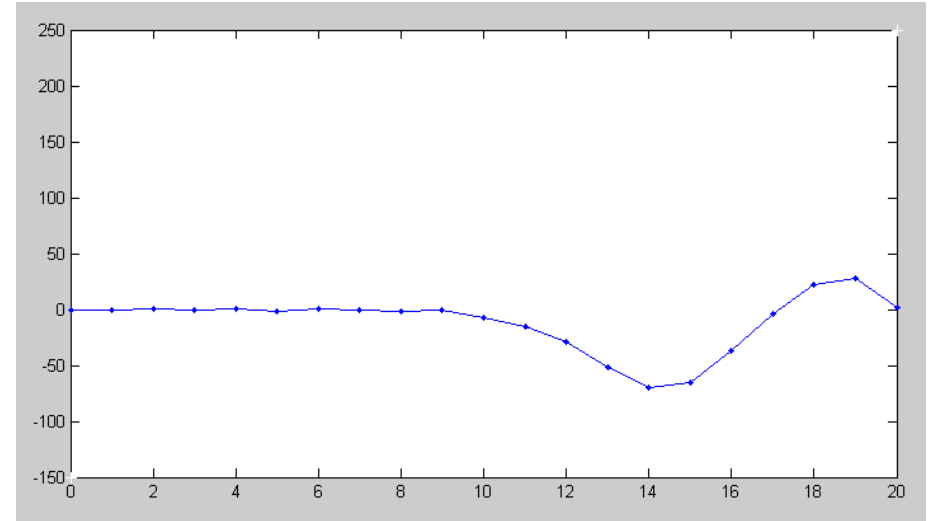
Matlab Simulation:

- Adjust R30 until the reflection is zero
- Reflection Coefficient for $R_{30} = 0.1$:
- Value in code is $1 / R_{30}C$

```
V = zeros(30,1);
dV = zeros(30,1);
t = 0;
dt = 0.01;
while(t < 1000)
    for i=1:3

        if (t < 1.5) V0 = 100* ( sin(2*t).^2);
            else V0 = 0;
            end
        ddV(1) = 50*V0 - 100*V(1) + 50*V(2) - 0.01*dV(1);
        for i=2:29
            ddV(i) = 50*V(i-1) - 100*V(i) + 50*V(i+1) - 0.01*dV(i);
            end
        ddV(30) = 50*V(29) - 50*V(30) - 100*dV(30);
        for i=1:30
            dV(i) = dV(i) + ddV(i)*dt;
            V(i) = V(i) + dV(i)*dt;
            end
        t = t + dt;
    end

    hold on
    plot([0:30],[V0;V],'.-');
    ylim([-200,200]);
    pause(0.01);
end
```



Why Reflection Coefficients are Important

Wave equation models:

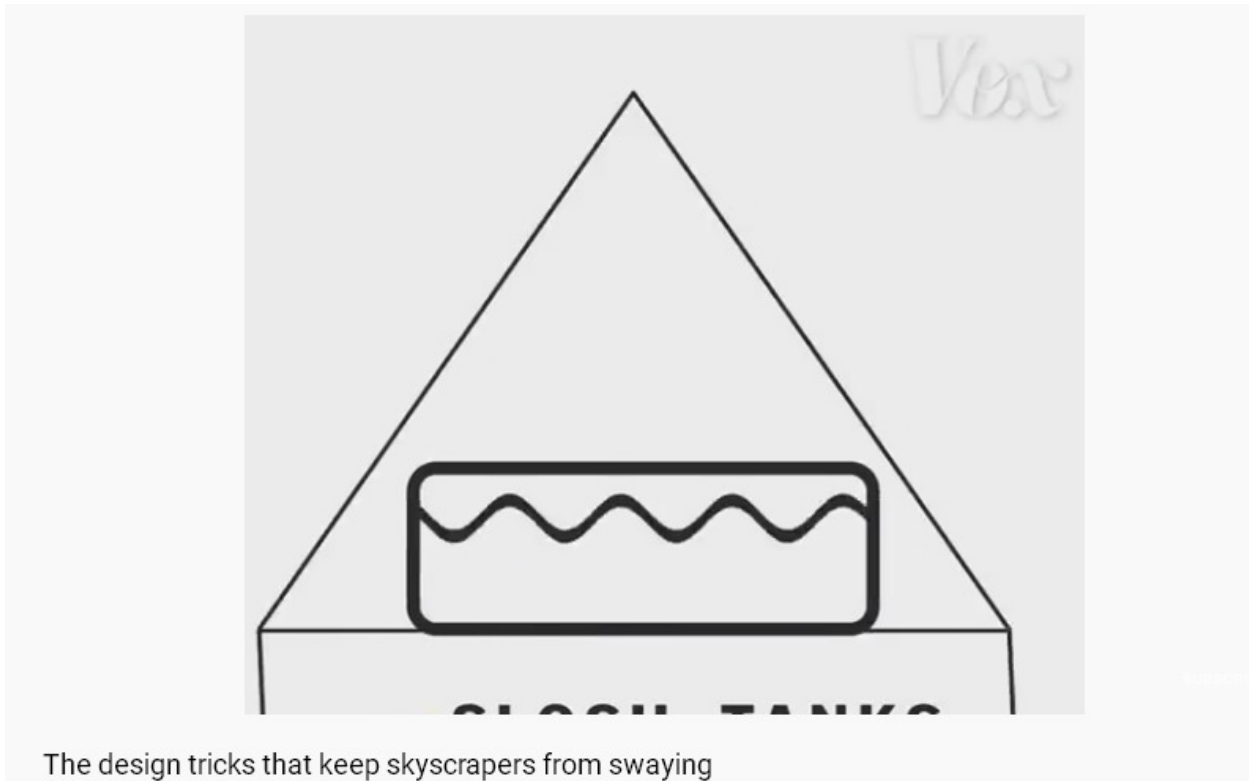
- Cars stuck in traffic
- Buildings swaying during an earthquake or a wind storm,
- Bridges oscillating during wind storms.
- Transmission lines galloping on windy days,
- Long electrical transmission lines, and
- Circuit traces

The reflections create standing waves, which result in

- Traffic coming to a standstill, clogging the highways
 - Buildings swaying and collapsing if the swaying becomes excessive
 - Bridges oscillating and then collapsing
 - Transmission lines galloping, hitting each other or snapping due to metal fatigue,
 - Large power surges on the nations high-voltage grid, and
-

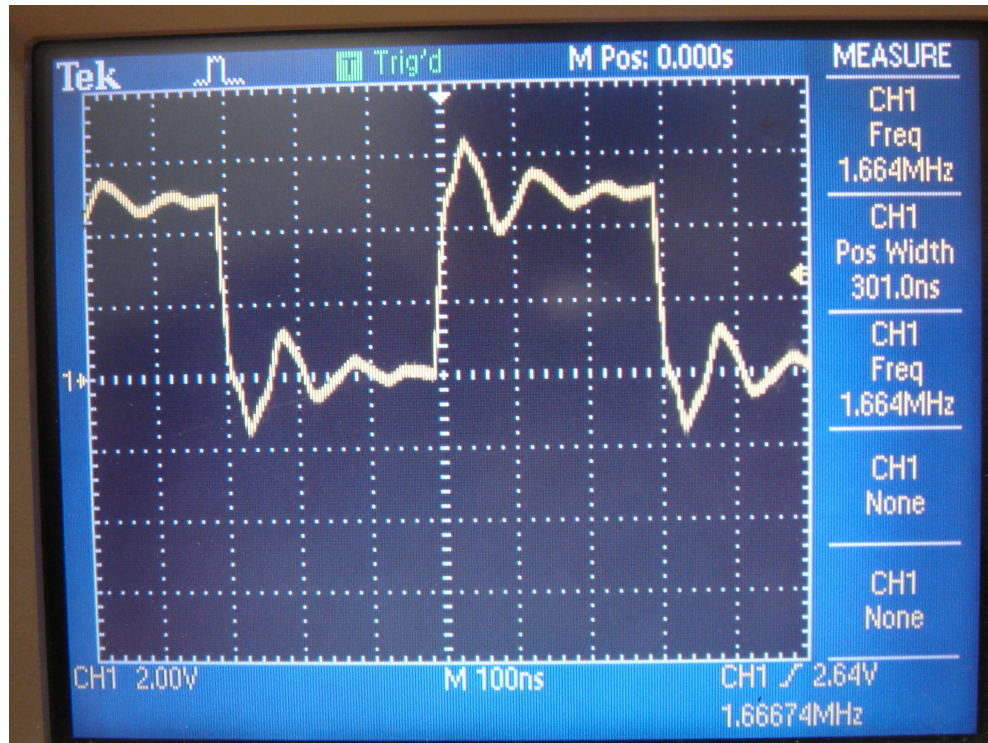
Solutions: SkyScrapers

- Add mass dampers at the roof
- <https://www.youtube.com/watch?v=ebx5Y5qOmTM>
- Go to 3:00



Circuits:

- Reflections can cause false triggers
- Example: Voltage on PIC board used in ECE 376
- Similar problem in industry: Higher clock frequency makes traces look like waveguides



Circuit Solutions:

Dr. Braaten offers two fairly unique courses at NDSU

- ECE 453. Signal Integrity.
- ECE 455. Designing for Electromagnetic Compatibility.

These courses look at how to model circuits from the standpoint of electromagnetics (i.e. waveguides) and how to design them so they do not transmit radiation or have standing waves.

NDSU NORTH DAKOTA STATE UNIVERSITY

UNIVERSITY CATALOG 2021-22

Search catalog

Home / Course Descriptions / Course Descriptions / Electrical & Computer Engineering (ECE)

PRINT OPTIONS

Department Information

Electrical & Computer Engineering (ECE)

for graduate credit - see [ECE 651](#) .

ECE 453. Signal Integrity. 3 Credits.
Topics in system level signal integrity are presented. The construction and design of passive printed circuit cards are discussed, with computer aided design software used for analysis and class presentations. Circuit card fabrication issues and case examples of applications are discussed. Prereq: [ECE 311](#), [ECE 351](#). {Also offered for graduate credit - see ECE 653.}

ECE 455. Designing for Electromagnetic Compatibility. 3 Credits.
Principles and methods concerning electronic system designs that are not sources of or susceptible to electromagnetic interference. 3 lectures. Laboratory. Prereq: [ECE 343](#), [ECE 351](#). {Also offered for graduate credit - see [ECE 655](#) .}

ECE 461. Control Systems I. 4 Credits.
Modeling and control of dynamic systems, including root-locus, Bode plots, and Nichols charts. 3 lectures, 1 two-hour laboratory. Prereq: [ECE 343](#). {Also offered for graduate credit - see [ECE 661](#) .}

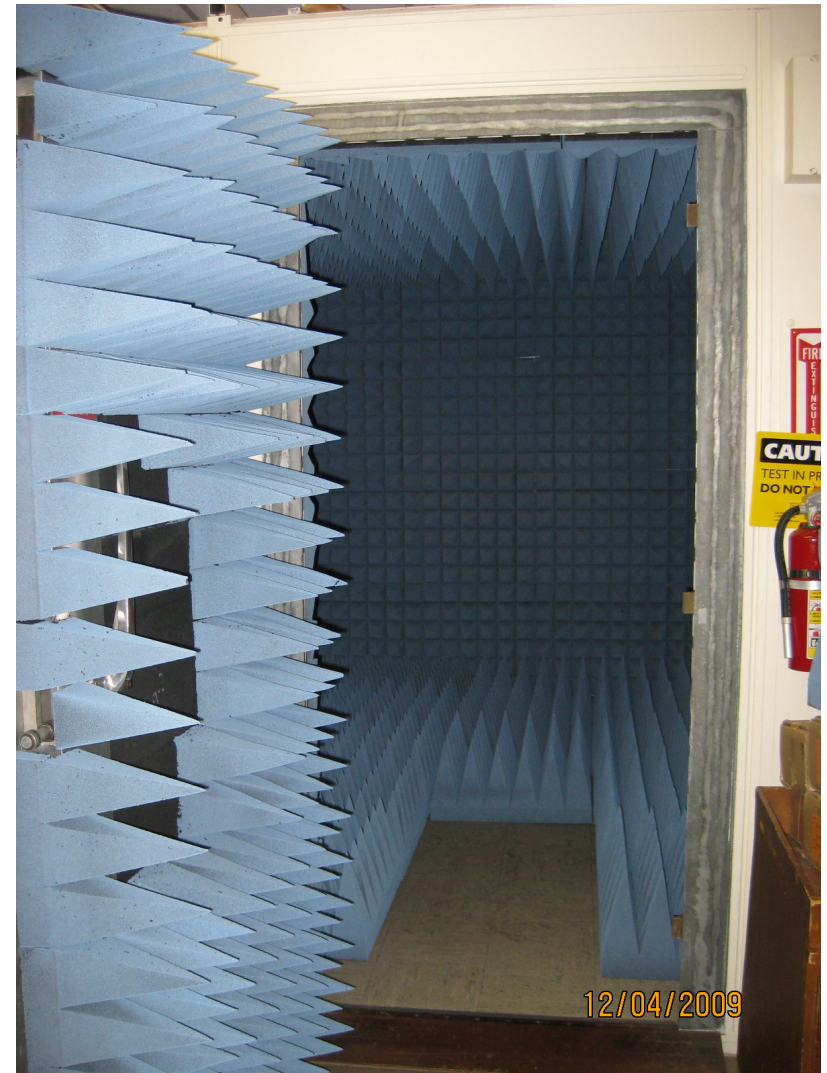
ECE 463. Modern Control. 3 Credits.
Analysis and design of controllers for linear and non-linear systems using state-space methods. Design to specifications, controllability, observability, stability, optimization, and state-estimation. 3 lectures. Prereq: [ECE 343](#). {Also offered for graduate credit - see [ECE 663](#) .}

ECE Shield Room

- From physics, if you have a perfectly conducting sphere, there should be no electric fields inside the sphere.
- Any fields you *do measure* must come from inside the sphere.

With this room,

- Measure how much your circuit radiates
- Measure how much your circuit receives
- Test before & after: how much techniques taught in these courses help



Summary

Heat Equation

- Coupled 1st-order differential equations
- RC circuits

Wave Equation

- Coupled 2nd-order differential equations
- RLC circuits

The two behave very differently

Matlab can be used to solve both types of equations

- What's important is to get the differential equations right
 - Solving using numerical methods remains about the same.
-