CHAPTER 17

Influence Lines

The structures we have considered so far have been subjected to loading systems that were stationary, i.e. the loads remained in a fixed position in relation to the structure. In many practical situations, however, structures carry loads that vary continuously. Thus a building supports a system of stationary loads which consist of its self-weight, the weight of any permanent fixtures such as partitions, machinery, etc., and also a system of imposed or 'live' loads which comprise snow loads, wind loads or any movable equipment. The structural elements of the building must then be designed to withstand the worst combination of these fixed and movable loads.

Other forms of movable load consist of vehicles and trains that cross bridges and viaducts. Again, these structures must be designed to support their self-weight, the weight of any permanent fixtures such as a road deck or railway track and also the forces produced by the passage of vehicles or trains. It is then necessary to determine the critical positions of the vehicles or trains in relation to the bridge or viaduct. Although these loads are moving loads, they are assumed to be moving or changing at such a slow rate that dynamic effects such as vibrations and oscillating stresses are absent.

The effects of loads that occupy different positions on a structure can be studied by means of *influence lines*. Influence lines give the value at a *particular* point in a structure of functions such as shear force, bending moment and displacement for *all* positions of a travelling unit load; they may also be constructed to show the variation of support reaction with the unit load position. From these influence lines the value of a function at a point can be calculated for a system of loads traversing the structure. For this we use the principle of superposition so that the structural systems we consider must be linearly elastic.

17.1 Influence lines for beams in contact with the load

We shall now investigate the construction of influence lines for support reactions and for the shear force and bending moment at a section of a beam when the travelling load is in continuous contact with the beam.

Consider the simply supported beam AB shown in Fig. 17.1(a) and suppose that we wish to construct the influence lines for the support reactions, R_A and R_B, and also for the shear force, S_K, and bending moment, M_K, at a given section K; all the influence lines are constructed by considering the passage of a unit load across the beam.
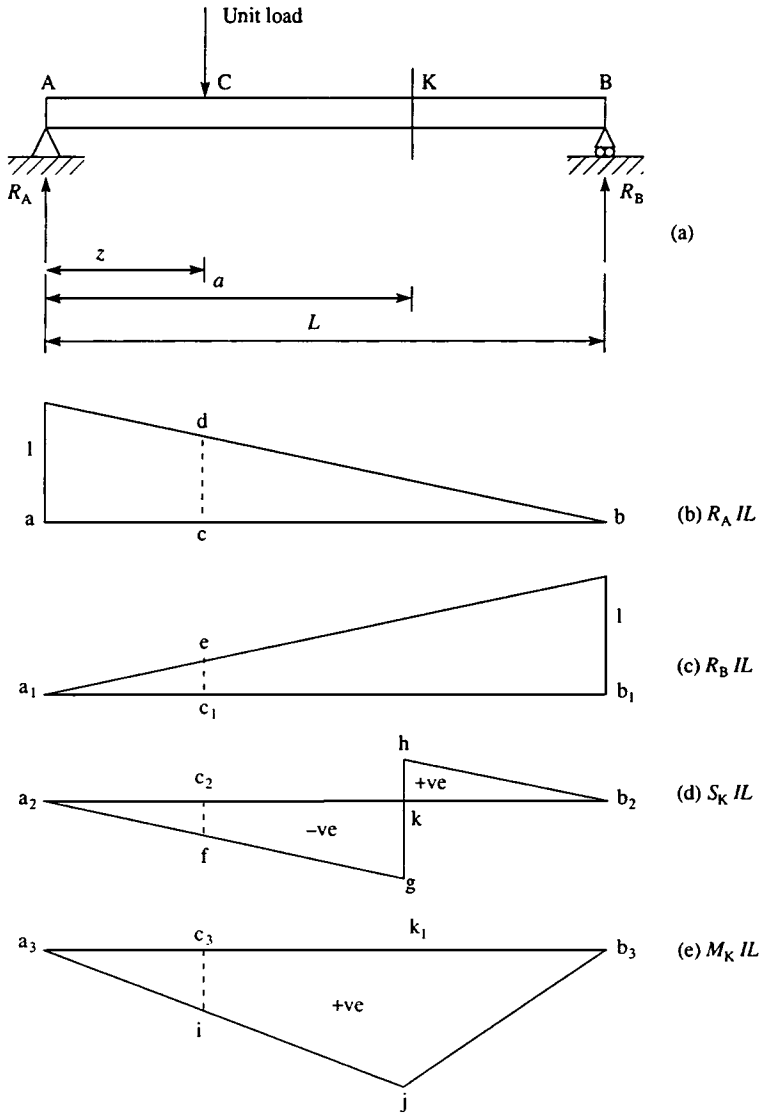
Fig. 17.1   Reaction, shear force and bending moment influence lines for a simply supported beam

R_A influence line

Suppose that the unit load has reached a position C, a distance z from A, as it travels across the beam. Then, considering the moment equilibrium of the beam about B we have

$$R_A L - 1(L - z) = 0$$

which gives
$$R_A = (L - z)/L \qquad (17.1)$$

Hence R_A is a linear function of z and when $z = 0$, $R_A = 1$ and when $z = L$, $R_A = 0$; both these results are obvious from inspection. The influence line (IL) for R_A ($R_A IL$) is then as shown in Fig. 17.1(b). Note that when the unit load is at C, the value of R_A is given by the ordinate cd in the R_A influence line.

R_B influence line

The influence line for the reaction R_B is constructed in an identical manner. Thus, taking moments about A

$$R_B L - 1z = 0$$

so that
$$R_B = z/L \tag{17.2}$$

Equation (17.2) shows that R_B is a linear function of z. Further, when $z = 0$, $R_B = 0$ and when $z = L$, $R_B = 1$, giving the influence line shown in Fig. 17.1(c). Again, with the unit load at C the value of R_B is equal to the ordinate $c_1 e$ in Fig. 17.1(c).

S_K influence line

The value of the shear force at the section K depends upon the position of the unit load, i.e. whether it is between A and K or between K and B. Suppose initially that the unit load is at the point C between A and K. Then the shear force at K is given by

$$S_K = -R_B$$

so that from Eq. (17.2)

$$S_K = -\frac{z}{L} \quad (0 \leq z \leq a) \tag{17.3}$$

The sign convention for shear force is that adopted in Section 3.2. We could have established Eq. (17.3) by expressing S_K in terms of R_A. Thus

$$S_K = R_A - 1$$

Substituting for R_A from Eq. (17.1) we obtain

$$S_K = \frac{L - z}{L} - 1 = -\frac{z}{L}$$

as before. Clearly, however, expressing S_K in the terms of R_B is the most direct approach.

We see from Eq. (17.3) that S_K varies linearly with the position of the load. Therefore, when $z = 0$, $S_K = 0$ and when $z = a$, $S_K = -a/L$, the ordinate kg in Fig. 17.1(d), and is the value of S_K with the unit load immediately to the left of K. Thus, with the load between A and K the S_K influence line is the line $a_2 g$ in Fig. 17.1(d) so that, when the unit load is at C, the value of S_K is equal to the ordinate $c_2 f$.

With the unit load between K and B the shear force at K is given by

$$S_K = +R_A \quad (\text{or } S_K = 1 - R_B)$$

Substituting for R_A from Eq. (17.1) we have

$$S_K = (L - z)/L \qquad (a \leqslant z \leqslant L) \tag{17.4}$$

Again S_K is a linear function of load position. Therefore when $z = L$, $S_K = 0$ and when $z = a$, i.e. the unit load is immediately to the right of K, $S_K = (L - a)/L$ which is the ordinate kh in Fig. 17.1(d).

From Fig. 17.1(d) we see that the gradient of the line a_2g is equal to $[(-a/L) - 0]/a = -1/L$ and that the gradient of the line hb_2 is equal to $[0 - (L - a)/L]/(L - a) = -1/L$. Thus the gradient of the S_K influence line is the same on both sides of K. Furthermore, $gh = kh + kg$ or $gh = (L - a)/L + a/L = 1$.

M_K influence line

The value of the bending moment at K also depends upon whether the unit load is to the left or right of K. With the unit load at C

$$M_K = R_B(L - a) \qquad (\text{or } M_K = R_A a - 1(a - z)$$

which, when substituting for R_B from Eq. (17.2) becomes

$$M_K = (L - a)z/L \qquad (0 \leqslant z \leqslant a) \tag{17.5}$$

From Eq. (17.5) we see that M_K varies linearly with z. Therefore, when $z = 0$, $M_K = 0$ and when $z = a$, $M_K = (L - a)a/L$, which is the ordinate k_1j in Fig. 17.1(e).

Now with the unit load between K and B

$$M_K = R_A a$$

which becomes, from Eq. (17.1)

$$M_K = [(L - z)/L]a \qquad (a \leqslant z \leqslant L) \tag{17.6}$$

Again M_K is a linear function of z so that when $z = a$, $M_K = (L - a)a/L$, the ordinate k_1j in Fig. 17.1 (e), and when $z = L$, $M_K = 0$. The complete influence line for the bending moment at K is then the line a_3jb_3 as shown in Fig. 17.1(e). Hence the bending moment at K with the unit load at C is the ordinate c_3i in Fig. 17.1(e).

In establishing the shear force and bending moment influence lines for the section K of the beam in Fig. 17.1(a) we have made use of the previously derived relationships for the support reactions, R_A and R_B. If only the influence lines for S_K and M_K had been required, the procedure would have been as follows.

With the unit load between A and K

$$S_K = -R_B$$

Now, taking moments about A

$$R_B L - 1z = 0$$

whence

$$R_B = \frac{z}{L}$$

Thus

$$S_K = -\frac{z}{L}$$

This, of course, amounts to the same procedure as before except that the calculation of R_B follows the writing down of the expression for S_K. The remaining equations for the influence lines for S_K and M_K are derived in a similar manner.

We note from Fig. 17.1 that all the influence lines are composed of straight-line segments. This is always the case for statically determinate structures. We shall therefore make use of this property when considering other beam arrangements.

Example 17.1 Draw influence lines for the shear force and bending moment at the section C of the beam shown in Fig. 17.2(a).

In this example we are not required to obtain the influence lines for the support reactions. However, the influence line for the reaction R_A has been included to illustrate the difference between this influence line and the influence line for R_A in Fig. 17.1(b); the reader should verify the R_A influence line in Fig. 17.2(b).

Since we have established that influence lines for statically determinate structures consist of linear segments, they may be constructed by placing the unit load at different positions, which will enable us to calculate the principal values.

S_C influence line

With the unit load at A
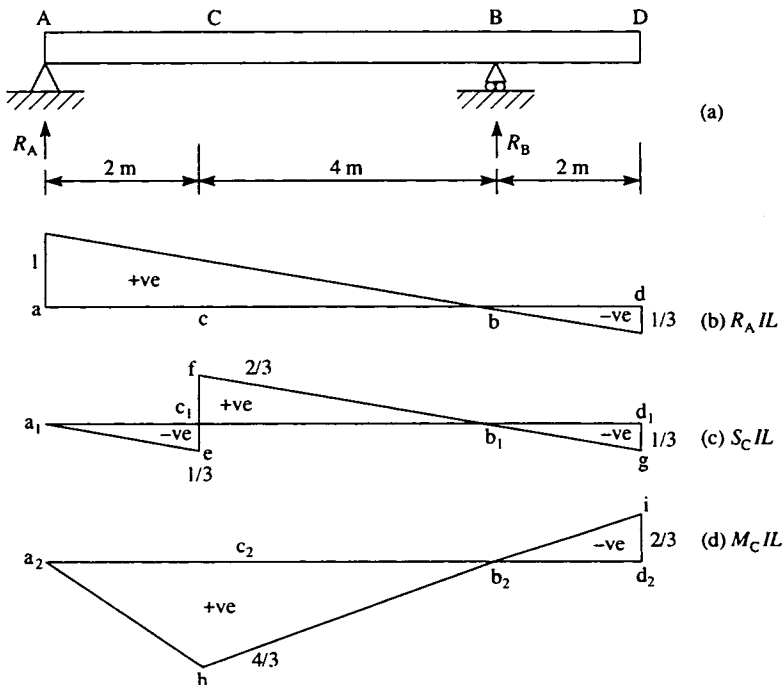
$$S_C = -R_B = 0 \qquad \text{(by inspection)}$$

Fig. 17.2 Shear force and bending moment influence lines for the beam of Ex. 17.1

With the unit load immediately to the left of C

$$S_C = -R_B \qquad \text{(i)}$$

Now taking moments about A we have

$$R_B \times 6 - 1 \times 2 = 0$$

which gives
$$R_B = \frac{1}{3}$$

Therefore, from Eq. (i)

$$S_C = -\frac{1}{3} \qquad \text{(ii)}$$

Now with the unit load immediately to the right of C

$$S_C = +R_A \qquad \text{(iii)}$$

Taking moments about B gives

$$R_A \times 6 - 1 \times 4 = 0$$

whence
$$R_A = \frac{2}{3}$$

so that, from Eq. (iii)

$$S_C = +\frac{2}{3}$$

With the unit load at B

$$S_C = +R_A = 0 \qquad \text{(by inspection)} \qquad \text{(v)}$$

Placing the unit load at D we have

$$S_C = +R_A \qquad \text{(vi)}$$

Again taking moments about B

$$R_A \times 6 + 1 \times 2 = 0$$

from which $\qquad\qquad R_A = -1/3$

Hence $\qquad\qquad S_C = -1/3 \qquad \text{(vii)}$

The complete influence line for the shear force at C is then as shown in Fig. 17.2(c). Note that the gradient of each of the lines a_1e, fb_1, and b_1g is the same.

M_C influence line

With the unit load placed at A

$$M_C = +R_B \times 4 = 0 \qquad (R_B = 0 \text{ by inspection})$$

With the unit load at C

$$M_C = +R_A \times 2 = +\frac{4}{3}$$

in which $R_A = 2/3$ with the unit load at C (see above). With the unit load at B

$$M_C = +R_A \times 2 = 0 \ (R_A = 0 \text{ by inspection})$$

Finally, with the unit load at D

$$M_C = +R_A \times 2$$

but, again from the calculation of S_C, $R_A = -1/3$. Hence

$$M_C = -\frac{2}{3}$$

The complete influence line for the bending moment at C is shown in Fig. 17.2(d). Note that the line hb$_2$i is one continuous line.

17.2   Mueller–Breslau principle

A simple and convenient method of constructing influence lines is to employ the Mueller–Breslau principle which gives the shape of an influence line without the values of its ordinates; these, however, are easily calculated for statically determinate systems from geometry.

Consider the simply supported beam, AB, shown in Fig. 17.3(a) and suppose that a unit load is crossing the beam and has reached the point C a distance z from A. Suppose also that we wish to determine the influence line for the moment at the section K, a distance a from A. We now impose a virtual displacement, v_C, at C such that internal work is done only by the moment at K, i.e. we allow a change in gradient, θ_K, at K so that the lengths AK and KB rotate as rigid links as shown in Fig. 17.3(b). Therefore, from the principle of virtual work (Chapter 15), the external virtual work done by the unit load is equal to the internal virtual work done by the moment, M_K, at K. Thus

$$1 v_C = M_K \theta_K$$

If we choose v_C so that θ_K is equal to unity

$$M_K = v_C \tag{17.7}$$

i.e. the moment at the section K due to a unit load at the point C, an arbitrary distance z from A, is equal to the magnitude of the virtual displacement at C. But, as we have seen in Section 17.1, the moment at a section K due to a unit load at a point C is the influence line for the moment at K. Therefore the M_K influence line may be constructed by introducing a hinge at K and imposing a unit change in angle at K; the displaced shape is then the influence line.

The argument may be extended to the construction of the influence line for the shear force, S_K, at the section K. Suppose now that the virtual displacement, v_C,
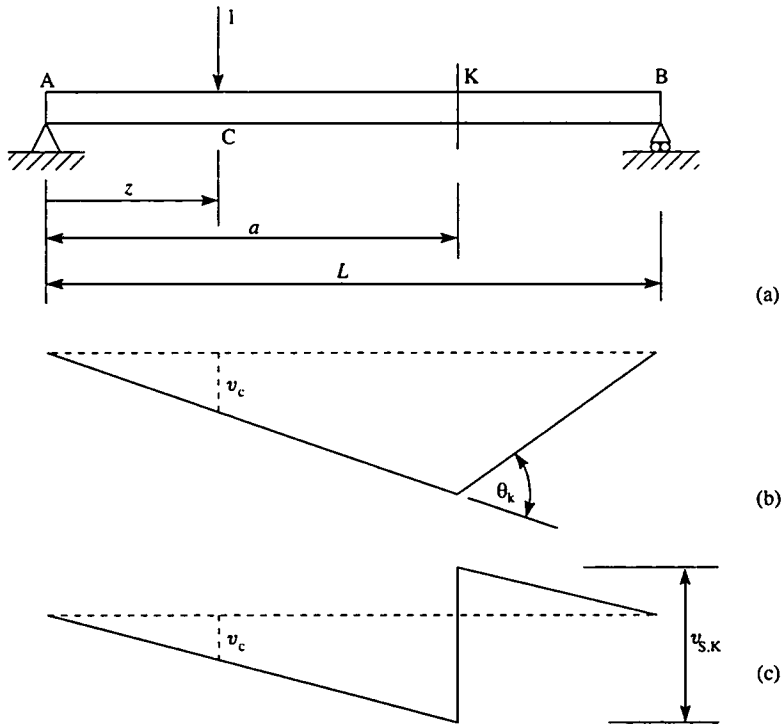
Fig. 17.3   Verification of the Mueller–Breslau principle

produces a shear displacement, $v_{S,K}$, at K as shown in Fig. 17.3(c). Again, from the principle of virtual work

$$1v_C = S_K v_{S,K}$$

If we choose v_C so that $v_{S,K} = 1$

$$S_K = v_C \qquad (17.8)$$

Hence, since the shear force at the section K due to a unit load at any point C is the influence line for the shear force at K, we see that the displaced shape in Fig. 17.3(c) is the influence line for S_K when the displacement at K produced by the virtual displacement at C is unity. A similar argument may be used to establish reaction influence lines.

The Mueller–Breslau principle demonstrated above may be stated in general terms as follows:

The shape of an influence line for a particular function (support reaction, shear force, bending moment, etc.) can be obtained by removing the resistance of the structure to that function at the section for which the influence line is required and applying an internal force corresponding to that function so that a unit displacement is produced at the section. The resulting displaced shape of the structure then represents the shape of the influence line.

Example 17.2 Use the Mueller–Breslau principle to determine the shape of the shear force and bending moment influence lines for the section C in the beam in Ex. 17.1 (Fig. 17.2(a)) and calculate the values of the principal ordinates.

In Fig. 17.4(b) we impose a unit shear displacement at the section C. In effect we are removing the resistance to shear of the beam at C by cutting the beam at C. We then apply positive shear forces to the two faces of the cut section in accordance with the sign convention of Section 3.2. Thus the beam to the right of C is displaced upwards while the beam to the left of C is displaced downwards. Since the slope of the influence line is the same on each side of C we can determine the ordinates of the influence line by geometry. Hence, in Fig. 17.4(b).

$$\frac{c_1e}{c_1a_1} = \frac{c_1f}{c_1b_1}$$

Therefore
$$c_1e = \frac{c_1a_1}{c_1b_1}\, c_1f = \frac{1}{2}c_1f$$

Further, since
$$c_1e + c_1f = 1$$

$$c_1e = \frac{1}{3}, \quad c_1f = \frac{2}{3}$$

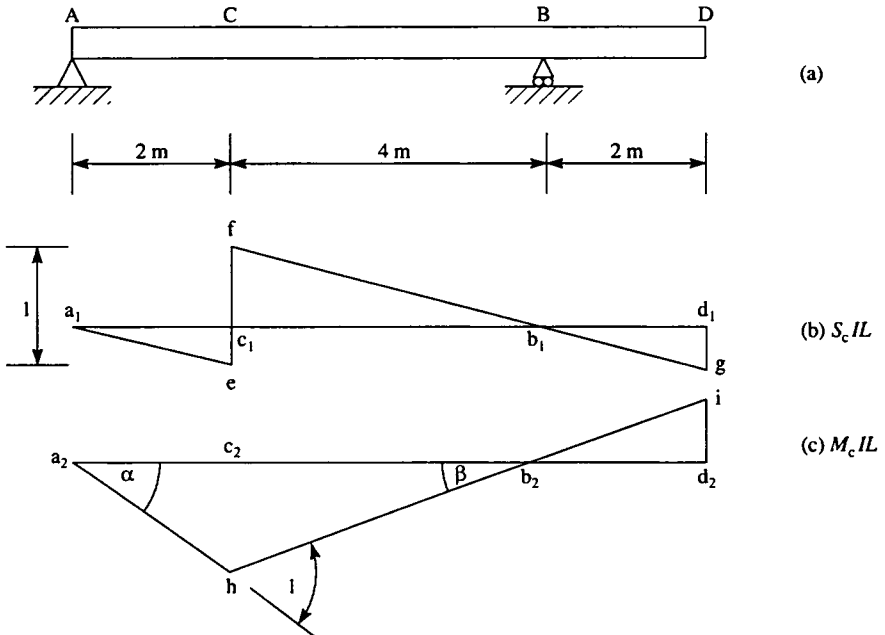as before. The ordinate $d_1g(=\tfrac{1}{3})$ follows.

Fig. 17.4 Construction of influence lines using the Mueller–Breslau principle

In Fig. 17.4(c) we have, from the geometry of a triangle,

$\alpha + \beta = 1$ (external angle = sum of opposite internal angles)

Then, assuming that the angles α and β are small so that their tangents are equal to the angles in radians

$$\frac{c_2 h}{c_2 a_2} + \frac{c_2 h}{c_2 b_2} = 1$$

or
$$c_2 h \left(\frac{1}{2} + \frac{1}{4} \right) = 1$$

whence
$$c_2 h = \frac{4}{3}$$

as in Fig. 17.2(d). The ordinate $d_2 i \left(= \frac{2}{3} \right)$ follows from similar triangles.

17.3 Systems of travelling loads

Influence lines for beams are constructed, as we have seen, by considering the passage of a unit load across a beam or by employing the Mueller–Breslau principle. Once constructed, an influence line may be used to determine the value of the particular function for shear force, bending moment, etc., at a section of a beam produced by any system of travelling loads. These may be concentrated loads, distributed loads or combinations of both. Generally we require the maximum values of a function as the loads cross the beam.

Concentrated loads

By definition the ordinate of an influence line at a point gives the value of the function at a specified section of a beam due to a unit load positioned at the point. Thus, in the beam shown in Fig. 17.1(a) the shear force at K due to a unit load at C is equal to the ordinate $c_2 f$ in Fig. 17.1(d). Since we are assuming that the system is linear it follows that the shear force at K produced by a load, W, at C is $W c_2 f$.

The argument may be extended to any number of travelling loads whose positions are fixed in relation to each other. In Fig. 17.5(a), for example, three concentrated loads, W_1, W_2 and W_3 are crossing the beam AB and are at fixed distances c and d apart. Suppose that they have reached the positions C, D and E, respectively. Let us also suppose that we require values of shear force and bending moment at the section K; the S_K and M_K influence lines are then constructed using either of the methods described in Sections 17.1 and 17.2.

Since the system is linear we can use the principle of superposition to determine the combined effects of the loads. Therefore, with the loads in the positions shown, and referring to Fig. 17.5(b),

$$S_K = -W_1 s_1 - W_2 s_2 - W_3 s_3 \qquad (17.9)$$

in which s_1, s_2 and s_3 are the ordinates under the loads in the S_K influence line.
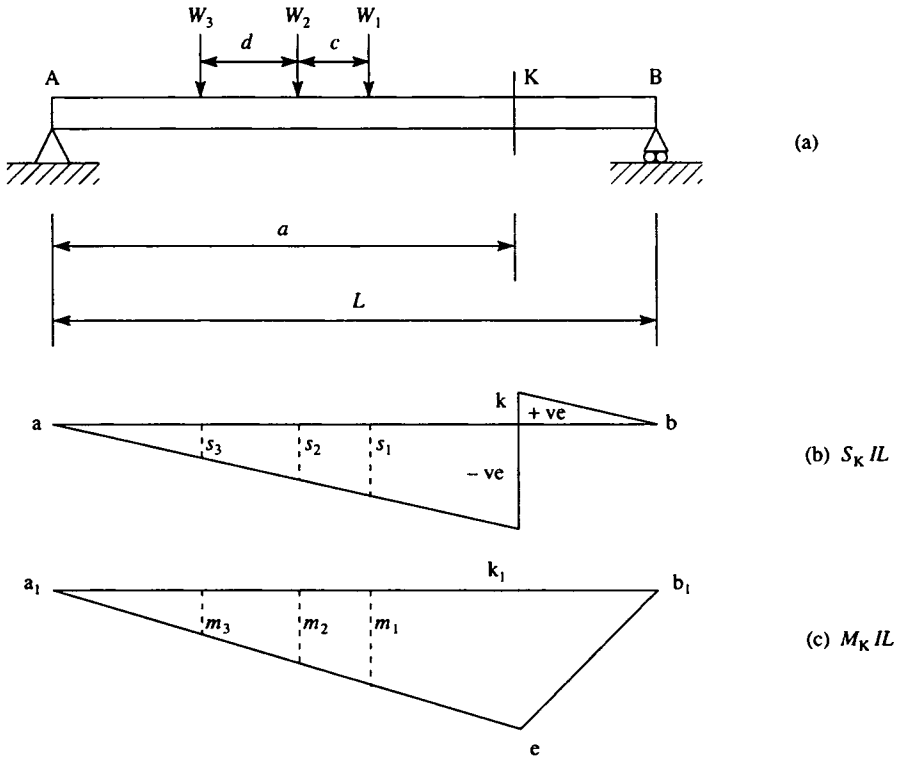
Fig. 17.5 Number of concentrated travelling loads

Similarly, from Fig. 17.5(c)

$$M_K = W_1 m_1 + W_2 m_2 + W_3 m_3 \qquad (17.10)$$

where m_1, m_2, and m_3 are the ordinates under the loads in the M_K influence line.

Maximum shear force at K

It can be seen from Fig. 17.5(b) that, as the loads W_1, W_2 and W_3 move to the right, the ordinates s_1, s_2 and s_3 increase in magnitude so that the shear force at K increases negatively to a peak value with W_1 just to the left of K. When W_1 passes to the right of K, the ordinate, s_1, becomes positive, then

$$S_K = + W_1 s_1 - W_2 s_2 - W_3 s_3$$

and the magnitude of S_K suddenly drops. As the loads move further to the right the now positive ordinate s_1 decreases in magnitude while the ordinates s_2 and s_3 increase negatively. Therefore a second peak value of S_K occurs with W_2 just to the left of K. When W_2 passes to the right of K the ordinate s_2 becomes positive and

$$S_K = + W_1 s_1 + W_2 s_2 - W_3 s_3$$

so that again there is a sudden fall in the negative value of S_K. A third peak value is

reached with W_3 just to the left of K and then, as W_3 passes to the right of K, S_K becomes completely positive. The same arguments apply for positive values of S_K as the loads travel from right to left.

Thus we see that maximum positive and negative values of shear force at a section of a beam occur when one of the loads is at that section. In some cases it is obvious which load will give the greatest value, in other cases a trial-and-error method is used.

Maximum bending moment at K

A similar situation arises when determining the position of a set of loads to give the maximum bending moment at a section of a beam although, as we shall see, a more methodical approach than trial and error may be used when the critical load position is not obvious.

With the loads W_1, W_2 and W_3 positioned as shown in Fig. 17.5(a) the bending moment, M_K, at K is given by Eq. (17.10), i.e.

$$M_K = W_1 m_1 + W_2 m_2 + W_3 m_3$$

As the loads move to the right the ordinates m_1, m_2 and m_3 increase in magnitude until W_1 passes K and m_1 begins to decrease. Thus M_K reaches a peak value with W_1 at K. Further movement of the loads to the right causes m_2 and m_3 to increase, while m_1 decreases so that a second peak value occurs with W_2 at K; similarly, a third peak value is reached with W_3 at K. Thus the maximum bending moment at K will occur with a load at K. In some cases this critical load is obvious, or it may be found by trial and error as for the maximum shear force at K. However, alternatively, the critical load may be found as follows.

Suppose that the beam in Fig. 17.5(a) carries a system of concentrated loads, W_1, $W_2, \ldots, W_j, \ldots, W_n$, and that they are in any position on the beam. Then, from Eq. (17.10)

$$M_K = \sum_{j=1}^{n} W_j m_j \tag{17.11}$$

Suppose now that the loads are given a small displacement δz. The bending moment at K then becomes $M_K + \delta M_K$ and each ordinate m becomes $m + \delta m$. Therefore, from Eq. (17.11)

$$M_K + \delta M_K = \sum_{j=1}^{n} W_j (m_j + \delta m_j)$$

or

$$M_K + \delta M_K = \sum_{j=1}^{n} W_j m_j + \sum_{j=1}^{n} W_j \delta m_j$$

whence

$$\delta M_K = \sum_{j=1}^{n} W_j \delta m_j$$

Therefore, in the limit as $\delta z \to 0$

$$\frac{dM_K}{dz} = \sum_{j=1}^{n} W_j \frac{dm_j}{dz}$$

in which dm_j/dz is the gradient of the M_K influence line. Therefore, if

$$\sum_{j=1}^{n} W_{j,L}$$

is the sum of the loads to the left of K and

$$\sum_{j=1}^{n} W_{j,R}$$

is the sum of the loads to the right of K, we have, from Eqs (17.5) and (17.6)

$$\frac{dM_K}{dz} = \sum_{j=1}^{n} W_{j,L} \left(\frac{L-a}{L} \right) + \sum_{j=1}^{n} W_{j,R} \left(-\frac{a}{L} \right)$$

For a maximum value of M_K, $dM_K/dz = 0$ so that

$$\sum_{j=1}^{n} W_{j,L} \left(\frac{L-a}{L} \right) = \sum_{j=1}^{n} W_{j,R} \frac{a}{L}$$

or

$$\frac{\sum_{j=1}^{n} W_{j,L}}{a} = \frac{\sum_{j=1}^{n} W_{j,R}}{L-a} \tag{17.12}$$

From Eq. (17.12) we see that the bending moment at K will be a maximum with one of the loads at K (from the previous argument) and when the load per unit length of beam to the left of K is equal to the load per unit length of beam to the right of K. Part of the load at K may be allocated to AK and part to KB as required to fulfil this condition.

Equation (17.12) may be extended as follows. Since

$$\sum_{j=1}^{n} W_j = \sum_{j=1}^{n} W_{j,L} + \sum_{j=1}^{n} W_{j,R}$$

then

$$\sum_{j=1}^{n} W_{j,R} = \sum_{j=1}^{n} W_j - \sum_{j=1}^{n} W_{j,L}$$

Substituting for

$$\sum_{j=1}^{n} W_{j,R}$$

in Eq. (17.12) we obtain

$$\frac{1}{a} \sum_{j=1}^{n} W_{j,\text{L}} = \left(\frac{1}{L-a}\right)\left(\sum_{j=1}^{n} W_j - \sum_{j=1}^{n} W_{j,\text{L}}\right)$$

Rearranging we have

$$\frac{L-a}{a} = \frac{\sum_{j=1}^{n} W_j - \sum_{j=1}^{n} W_{j,\text{L}}}{\sum_{j=1}^{n} W_{j,\text{L}}}$$

whence

$$\frac{\sum_{j=1}^{n} W_j}{L} = \frac{\sum_{j=1}^{n} W_{j,\text{L}}}{a} \qquad (17.13)$$

Combining Eqs (17.12) and (17.13) we have

$$\frac{\sum_{j=1}^{n} W_j}{L} = \frac{\sum_{j=1}^{n} W_{j,\text{L}}}{a} = \frac{\sum_{j=1}^{n} W_{j,\text{R}}}{L-a} \qquad (17.14)$$

Therefore, for M_K to be a maximum, there must be a load at K such that the load per unit length over the complete span is equal to the load per unit length of beam to the left of K and the load per unit length of beam to the right of K.

Example 17.3   Determine the maximum positive and negative values of shear force and the maximum value of bending moment at the section K in the simply supported beam AB shown in Fig. 17.6(a) when it is crossed by the system of loads shown in Fig. 17.6(b).

The influence lines for the shear force and bending moment at K are constructed using either of the methods described in Sections 17.1 and 17.2 as shown in Fig. 17.6(c) and (d).

Maximum negative shear force at K

It is clear from inspection that S_K will be a maximum with the 5 kN load just to the left of K, in which case the 3 kN load is off the beam and the ordinate under the 4 kN load in the S_K influence line is, from similar triangles, $-0\cdot1$. Then

$$S_\text{K}(\text{max}) = 5 \times (-0\cdot3) + 4 \times (-0\cdot1) = -1\cdot9 \text{ kN}$$

Maximum positive shear force at K

There are two possible load positions which could give the maximum positive value of shear force at K; neither can be eliminated by inspection. First we shall place the 3 kN load just to the right of K. The ordinates under the 4 kN load and 5 kN load are calculated from similar triangles and are $+0\cdot5$ and $+0\cdot3$, respectively. Then

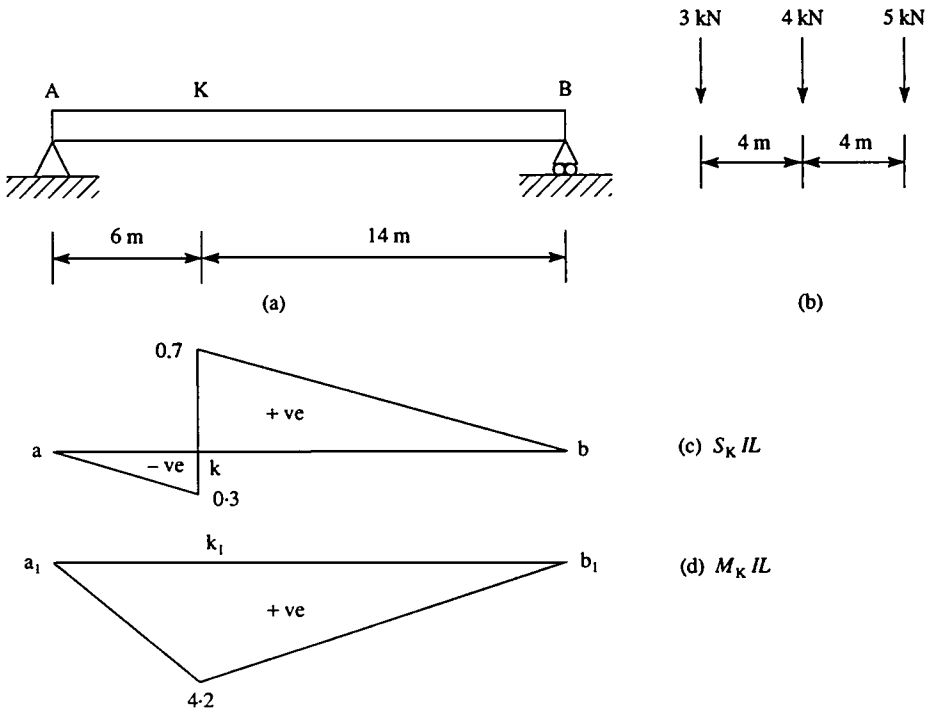$$S_\text{K} = 3 \times 0\cdot7 + 4 \times 0\cdot5 + 5 \times 0\cdot3 = 5\cdot6 \text{ kN}$$

Fig. 17.6 Determination of the maximum shear force and bending moment at a section of a beam

Now with the 4 kN load just to the right of K, the ordinates under the 3 kN load and 5 kN load are -0.1 and $+0.5$, respectively. Then

$$S_K = 3 \times (-0.1) + 4 \times 0.7 + 5 \times 0.5 = 5.0 \text{ kN}$$

Therefore the maximum positive value of S_K is 5·6 kN and occurs with the 3 kN load immediately to the right of K.

Maximum bending moment at K

We position the loads in accordance with the criterion of Eq. (17.14). The load per unit length of the complete beam is $(3+4+5)/20 = 0.6$ kN/m. Therefore if we position the 4 kN load at K and allocate 0·6 kN of the load to AK the load per unit length on AK is $(3+0.6)/6 = 0.6$ kN/m and the load per unit length on KB is $(3.4+5)/14 = 0.6$ kN/m. The maximum bending moment at K therefore occurs with the 4 kN load at K; in this example the critical load position could have been deduced by inspection.

With the loads in this position the ordinates under the 3 kN and 5 kN loads in the M_K influence line are 1·4 and 3·0, respectively. Then

$$M_K(\text{max}) = 3 \times 1.4 + 4 \times 4.2 + 5 \times 3.0 = 36.0 \text{ kNm}$$

Distributed loads

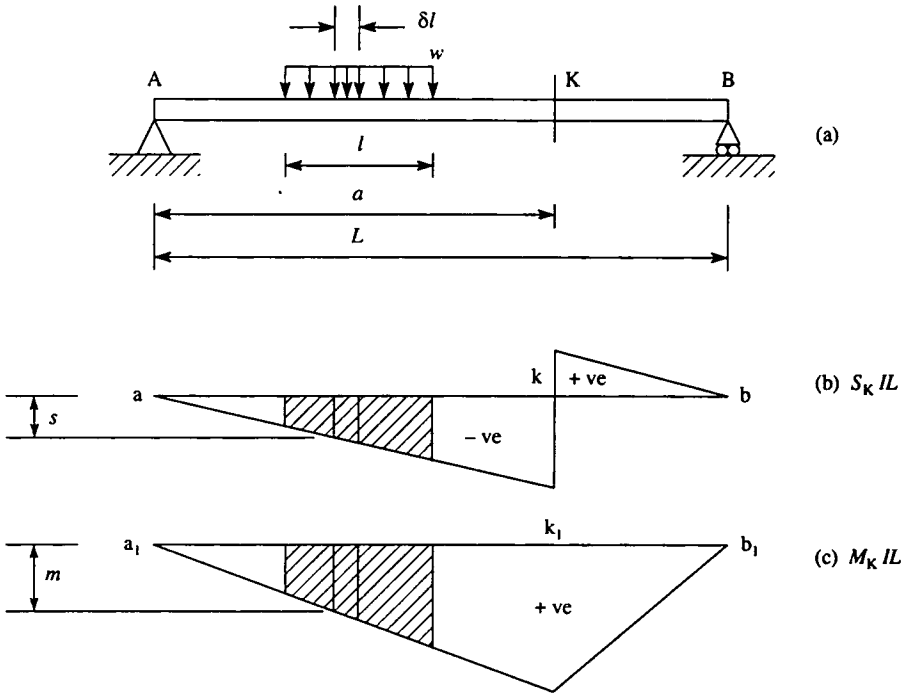Figure 17.7(a) shows a simply supported beam AB on which a uniformly distributed

Fig. 17.7   Shear force and bending moment due to a moving uniformly distributed load

load of intensity w and length l is crossing from left to right. Suppose we wish to obtain values of shear force and bending moment at the section K of the beam. Again we construct the S_K and M_K influence lines using either of the methods described in Sections 17.1 and 17.2.

If we consider an elemental length δl of the load, we may regard this as a concentrated load of magnitude $w\delta l$. The shear force, δS_K, at K produced by this elemental length of load is then, from Fig. 17.7(b),

$$\delta S_K = w\delta ls$$

The total shear force, S_K, at K due to the complete length of load is then

$$S_K = \int_0^l ws \, \mathrm{d}l$$

or, since the load is uniformly distributed

$$S_K = w \int_0^l s \, \mathrm{d}l \qquad (17.15)$$

Hence $S_K = w \times$ *area under the projection of the load in the S_K influence line.*

Similarly $\qquad\qquad\qquad M_K = w \int_0^l m \, \mathrm{d}l \qquad (17.16)$

so that $\quad M_K = w \times$ *area under the projection of the load in the M_K influence line.*

Maximum shear force at K

It is clear from Fig. 17.7(b) that the maximum negative shear force at K occurs with the head of the load at K while the maximum positive shear force at K occurs with the tail of the load at K. Note that the shear force at K would be zero if the load straddled K such that the negative area under the load in the S_K influence line was equal to the positive area under the load.

Maximum bending moment at K

If we regard the distributed load as comprising an infinite number of concentrated loads, we can apply the criterion of Eq. (17.14) to obtain the maximum value of bending moment at K. Thus the load per unit length of the complete beam is equal to the load per unit length of beam to the left of K and the load per unit length of beam to the right of K. Therefore, in Fig. 17.8, we position the load such that

$$\frac{w\,ck_1}{a_1k_1} = \frac{w\,dk_1}{k_1b_1}$$

or

$$\frac{ck_1}{a_1k_1} = \frac{dk_1}{k_1b_1} \tag{17.17}$$

From Fig. 17.8

$$\frac{fc}{hk_1} = \frac{a_1c}{a_1k_1}$$

so that

$$fc = \frac{a_1c}{a_1k_1}\,hk_1 = \left(\frac{a_1k_1 - ck_1}{a_1k_1}\right)hk_1 = \left(1 - \frac{ck_1}{a_1k_1}\right)hk_1$$

Similarly

$$dg = \left(1 - \frac{dk_1}{b_1k_1}\right)hk_1$$

Therefore, from Eq. (17.17) we see that

$$fc = dg$$

and the ordinates under the extremities of the load in the M_K influence line are equal. It may also be shown that the area under the load in the M_K influence line is a maximum when fc = dg. This is an alternative method of deducing the position of
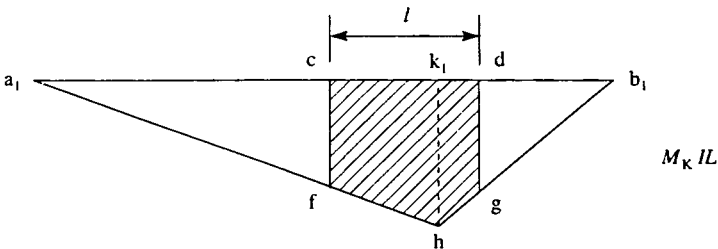

Fig. 17.8 Load position for maximum bending moment at K.

the load for maximum bending moment at K. Note that, from Eq. (17.17), K divides the load in the same ratio as it divides the span.

Example 17.4 A load of length 2 m and intensity 2 kN/m crosses the simply supported beam AB shown in Fig. 17.9(a). Calculate the maximum positive and negative values of shear force and the maximum value of bending moment at the quarter-span point.

The shear force and bending moment influence lines for the quarter-span point K are constructed in the same way as before and are shown in Fig. 17.9(b) and (c).

Maximum shear force at K

The maximum negative shear force at K occurs with the head of the load at K. In this position the ordinate under the tail of the load is $-0\cdot05$. Hence

$$S_K(\text{max. } -\text{ve}) = -2 \times \tfrac{1}{2}(0\cdot05 + 0\cdot25) \times 2 = -0\cdot6 \text{ kN}$$

The maximum positive shear force at K occurs with the tail of the load at K. With the load in this position the ordinate under the head of the load is $0\cdot55$. Thus

$$S_K(\text{max. } +\text{ve}) = 2 \times \tfrac{1}{2}(0\cdot75 + 0\cdot55) \times 2 = +2\cdot6 \text{ kN}$$
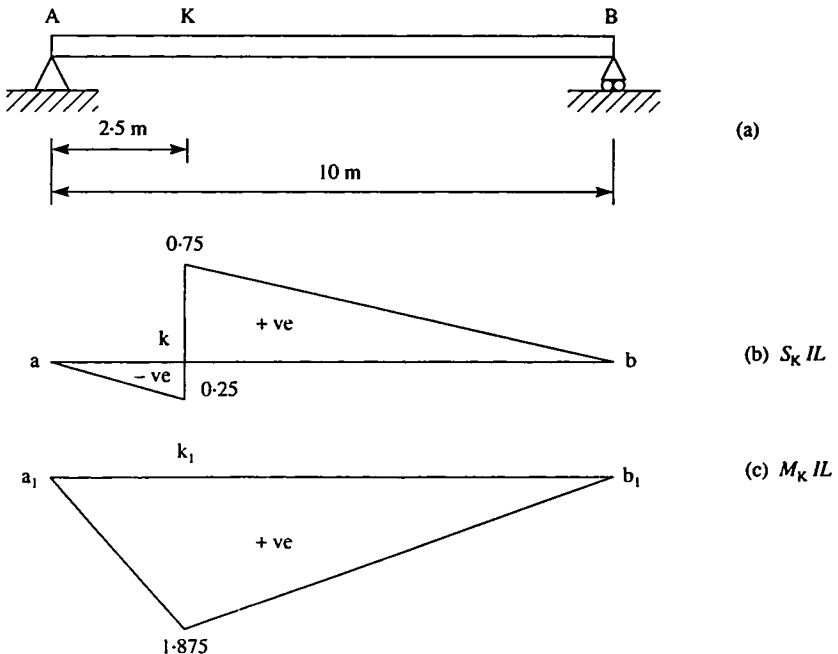

Fig. 17.9 Maximum shear force and bending moment at the quarter-span point in the beam of Ex. 17.4

Maximum bending moment at K

We position the load so that K divides the load in the same ratio that it divides the span. Therefore 0·5 m of the load is to the left of K and 1·5 m to the right of K. The ordinate in the M_K influence line under the tail of the load is then 1·5 as is the ordinate under the head of the load. The maximum value of M_K is thus given by

$$M_K(\text{max}) = 2[\tfrac{1}{2}(1·5 + 1·875) \times 0·5 + \tfrac{1}{2}(1·875 + 1·5) \times 1·5]$$

which gives

$$M_K(\text{max}) = 6·75 \text{ kN m}$$

Diagram of maximum shear force

Consider the simply supported beam shown in Fig. 17.10 (a) and suppose that a uniformly distributed load of intensity w and length $L/5$ (any fraction of L may be chosen) is crossing the beam. We can draw a series of influence lines for the sections, A, K_1, K_2, K_3, K_4 and B as shown in Fig. 17.10(b) and then determine the maximum negative and positive values of shear force at each of the sections K_1, K_2, etc., by considering first the head of the load at K_1, K_2, etc., and then the tail of the load at A, K_1, K_2, etc. These values are then plotted as shown in Fig. 17.10(c).

With the head of the load at K_1, K_2, K_3, K_4 and B the maximum negative shear force is given by $w(ak_1)s_1$, $w(k_1 k_2)s_2$, and so on, where s_1, s_2, etc., are the mid-ordinates of the areas ak_1, k_1k_2, etc. Since s_1, s_2, etc. increase linearly, the maximum negative shear force also increases linearly at all sections of the beam between K_1 and B. At a section between A and K_1, the complete length of load will not be on the beam so that the maximum value of negative shear force at this section will not
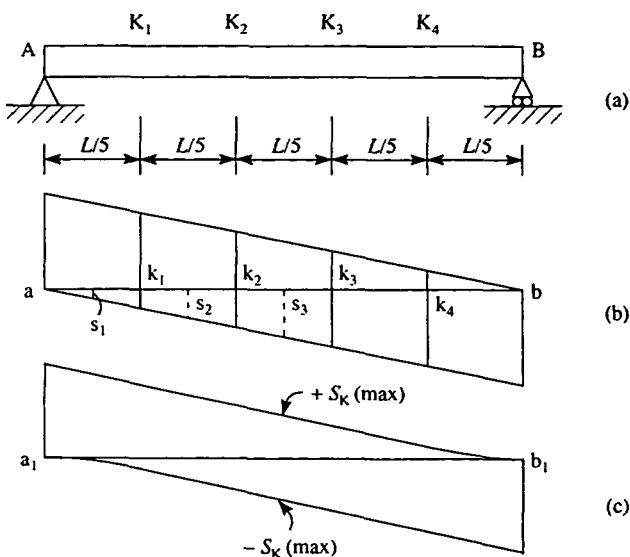

Fig. 17.10 Diagram of maximum shear force

lie on the straight line and the diagram of maximum negative shear force between A and K_1, will be curved; the maximum negative shear force should be calculated for at least one section between A and K_1.

An identical argument applies to the calculation of the maximum positive shear force which occurs with the tail of the load at a beam section. Thus, in this case, the non-linearity will occur as the load begins to leave the beam between K_4 and B.

Reversal of shear force

In some structures it is beneficial to know in which parts of the structure, if any, the maximum shear force changes sign. In Section 4.5, for example, we saw that the diagonals of a truss resist the shear forces and therefore could be in tension or compression depending upon their orientation and the sign of the shear force. If, therefore, we knew that the sign of the shear force would remain the same under the design loading in a particular part of a truss we could arrange the inclination of the diagonals so that they would always be in tension and would not be subject to instability produced by compressive forces. If, at the same time, we knew in which parts of the truss the shear force could change sign we could introduce counterbracing (see Section 17.5).

Consider the simply supported beam AB shown in Fig. 17.11(a) and suppose that it carries a uniformly distributed dead load (self-weight, etc.) of intensity w_{DL}. The shear force due to this dead load (the dead load shear (DLS)) varies linearly from $+w_{DL}L/2$ at A to $-w_{DL}L/2$ at B as shown in Fig. 17.11(b). Suppose now that a uniformly distributed live load of length less than the span AB crosses the beam. As for the beam in Fig. 17.10, we can plot diagrams of maximum positive and negative shear force produced by the live load; these are also shown in Fig. 17.11(b). Then, at any section of the beam, the maximum shear force is equal
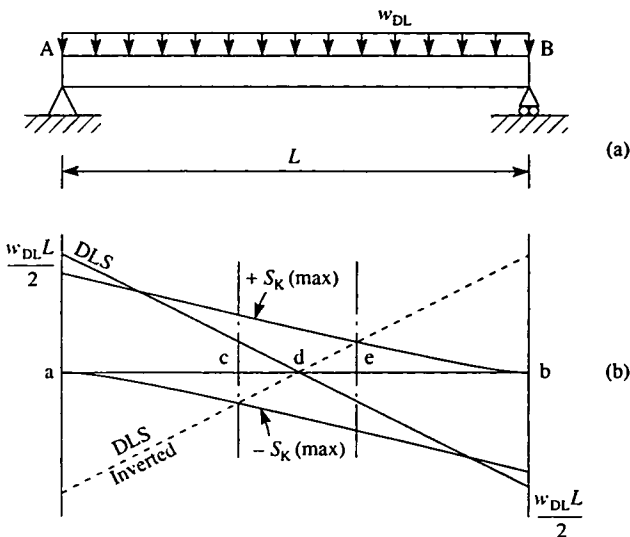

Fig. 17.11   Reversal of shear force in a beam

to the sum of the maximum positive shear force due to the live load and the dead load shear force, or the sum of the maximum negative shear force due to the live load and the dead load shear force. The variation in this maximum shear force along the length of the beam will be more easily understood if we invert the dead load shear force diagram.

Referring to Fig. 17.11(b) we see that the sum of the maximum positive shear force due to the live load and the dead load shear force is always positive between a and c. Furthermore, between a and c, the sum of the maximum negative shear force due to the live load and the dead load shear force is always positive. Similarly, between e and b the maximum shear force is always negative. However, between c and e the summation of the maximum positive shear force produced by the live load and the dead load shear force is positive, while the summation of the maximum negative shear force due to the live load and the dead load shear force is negative. Therefore the maximum shear force between c and e may be positive or negative, i.e. there is a possible *reversal* of maximum shear force in this length of the beam.

Example 17.5 A simply supported beam AB has a span of 5 m and carries a uniformly distributed dead load of 0·6 kN/m (Fig. 17.12(a)). A similarly distributed live load of length greater than 5 m and intensity 1·5 kN/m travels across the beam. Calculate the length of beam over which reversal of shear force occurs and sketch the diagram of maximum shear force for the beam.

The shear force at a section of the beam will be a maximum with the head or tail of the load at that section. Initially, before writing down an expression for shear force, we require the support reaction at A, R_A. Thus, with the head of the load at a
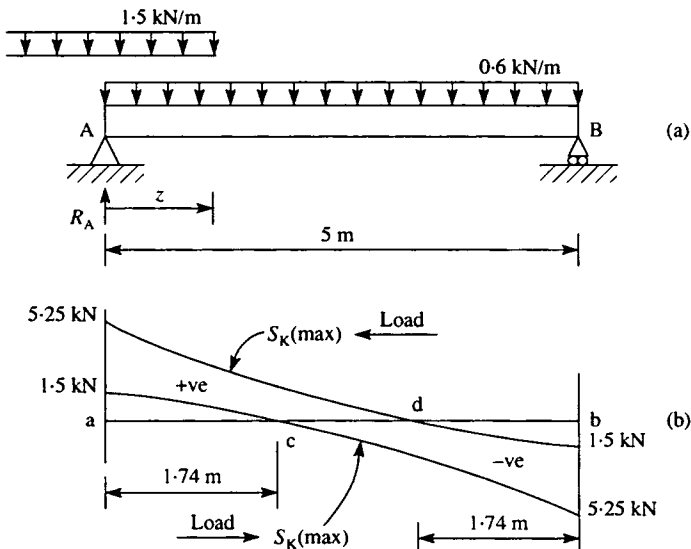

Fig. 17.12 Reversal of shear force in the beam of Ex. 17.5

section a distance z from A, the reaction, R_A, is found by taking moments about B. Thus

$$R_A \times 5 - 0 \cdot 6 \times 5 \times 2 \cdot 5 - 1 \cdot 5z(5 - z/2) = 0$$

whence $\qquad R_A = 1 \cdot 5 + 1 \cdot 5z - 0 \cdot 15z^2 \qquad$ (i)

The maximum shear force at the section is then

$$S(\max) = R_A - 0 \cdot 6z - 1 \cdot 5z \qquad \text{(ii)}$$

or, substituting in Eq. (ii) for R_A from Eq. (i)

$$S(\max) = 1 \cdot 5 - 0 \cdot 6z - 0 \cdot 15z^2 \qquad \text{(iii)}$$

Equation (iii) gives the maximum shear force at any section of the beam with the load moving from left to right. Then, when $z = 0$, $S(\max) = 1 \cdot 5$ kN and when $z = 5$ m, $S(\max) = -5 \cdot 25$ kN. Furthermore, from Eq. (iii) $S(\max) = 0$ when $z = 1 \cdot 74$ m.

The maximum shear force for the load travelling from right to left is found in a similar manner. The final diagram of maximum shear force is shown in Fig. 17.12(b) where we see that reversal of shear force may take place within the length cd of the beam; cd is sometimes called the *focal length*.

Determination of the point of maximum bending moment in a beam

Previously we have been concerned with determining the position of a set of loads on a beam that would produce the maximum bending moment at a given section of the beam. We shall now determine the section and the position of the loads for the bending moment to be the absolute maximum.

Consider a section K a distance z_1 from the mid-span of the beam in Fig. 17.13 and suppose that a set of loads having a total magnitude W_T is crossing the beam. The bending moment at K will be a maximum when one of the loads is at K; let this
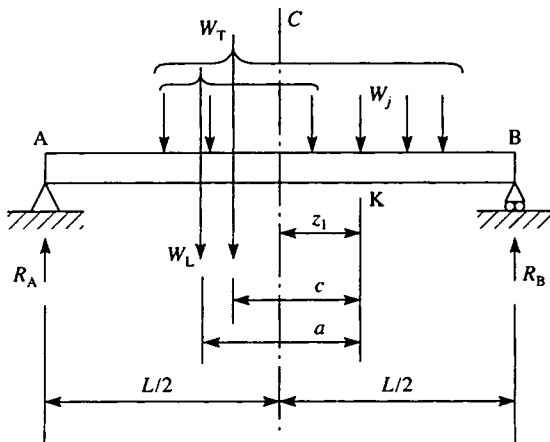

Fig. 17.13 Determination of the absolute maximum bending moment in a beam

load be W_j. Also, suppose that the centre of gravity of the complete set of loads is a distance c from the load W_j and that the total weight of all the loads to the left of W_j is W_L, acting at a distance a from W_j; a and c are fixed values for a given set of loads.

Initially we find R_A by taking moments about B.

Hence

$$R_A L - W_T \left(\frac{L}{2} - z_1 + c \right) = 0$$

which gives

$$R_A = \frac{W_T}{L} \left(\frac{L}{2} - z_1 + c \right)$$

The bending moment, M_K, at K is then given by

$$M_K = R_A \left(\frac{L}{2} + z_1 \right) - W_L a$$

or, substituting for R_A,

$$M_K = \frac{W_T}{L} \left(\frac{L}{2} - z_1 + c \right)\left(\frac{L}{2} + z_1 \right) - W_L a$$

Differentiating M_K with respect to z_1 we have

$$\frac{dM_K}{dz_1} = \frac{W_T}{L} \left[-1\left(\frac{L}{2} + z_1 \right) + 1\left(\frac{L}{2} - z_1 + c \right) \right]$$

or

$$\frac{dM_K}{dz_1} = \frac{W_T}{L} (-2z_1 + c)$$

For a maximum value of M_K, $dM_K/dz_1 = 0$ so that

$$z_1 = \frac{c}{2} \tag{17.18}$$

Therefore the maximum bending moment occurs at a section K under a load W_j such that the section K and the centre of gravity of the complete set of loads are positioned at equal distances either side of the mid-span of the beam.

To apply this rule we select one of the larger central loads and position it over a section K such that K and the centre of gravity of the set of loads are placed at equal distances on either side of the mid-span of the beam. We then check to determine whether the load per unit length to the left of K is equal to the load per unit length to the right of K. If this condition is not satisfied, another load and another section K must be selected.

Example 17.6 The set of loads shown in Fig. 17.14(b) crosses the simply supported beam AB shown in Fig. 17.14(a). Calculate the position and magnitude of the maximum bending moment in the beam.
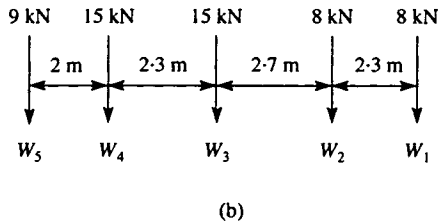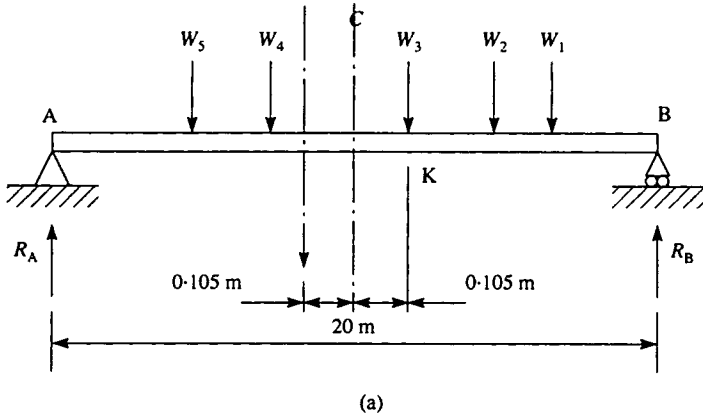
(a)

(b)

Fig. 17.14 Determination of absolute maximum bending moment in the beam of Ex. 17.6

The first step is to find the position of the centre of gravity of the set of loads. Thus, taking moments about the load W_5 we have

$$(9 + 15 + 15 + 8 + 8)\bar{z} = 15 \times 2 + 15 \times 4 \cdot 3 + 8 \times 7 \cdot 0 + 8 \times 9 \cdot 3$$

whence $\bar{z} = 4 \cdot 09$ m

Therefore the centre of gravity of the loads is $0 \cdot 21$ m to the left of the load W_3.

By inspection of Fig. 17.14(b) we see that it is probable that the maximum bending moment will occur under the load W_3. We therefore position W_3 and the centre of gravity of the set of loads at equal distances either side of the mid-span of the beam as shown in Fig. 17.14(a). We now check to determine whether this position of the loads satisfies the load per unit length condition. The load per unit length on $AB = 55/20 = 2 \cdot 75$ kN/m. Therefore the total load required on $AK = 2 \cdot 75 \times 10 \cdot 105 = 27 \cdot 79$ kN. This is satisfied by W_5, W_4 and part ($3 \cdot 79$ kN) of W_3.

Having found the load position, the bending moment at K is most easily found by direct calculation. Thus taking moments about B we have

$$R_A \times 20 - 55 \times 10 \cdot 105 = 0$$

whence $R_A = 27 \cdot 8$ kN

Hence $M_K = 27 \cdot 8 \times 10 \cdot 105 - 9 \times 4 \cdot 3 - 15 \times 2 \cdot 3 = 207 \cdot 7$ kN m

It is possible that in some load systems there may be more than one load position which satisfies both criteria for maximum bending moment but the corresponding bending moments have different values. Generally the absolute maximum bending moment will occur under one of the loads between which the centre of gravity of the system lies. If the larger of these two loads is closer to the centre of gravity than the other, then this load will be the critical load; if not then both cases must be analysed.

17.4 Influence lines for beams not in contact with the load

In many practical situations, such a bridge construction for example, the moving loads are not in direct contact with the main beam or girder. Fig. 17.15 shows a typical bridge construction in which the deck is supported by stringers that are mounted on cross beams which, in turn, are carried by the main beams or girders. The deck loads are therefore transmitted via the stringers and cross beams to the main beams. Generally, in the analysis, we assume that the segments of the stringers are simply supported at each of the cross beams. In Fig. 17.15 the portion of the main beam between the cross beams, for example FG, is called a *panel* and the points F and G are called *panel points*.

Figure 17.16 shows a simply supported main beam AB which supports a bridge deck via an arrangement of cross beams and stringers. Let us suppose that we wish to construct shear force and bending moment influence lines for the section K of the main beam within the panel CD. As before we consider the passage of a unit load; in this case, however, it crosses the bridge deck.

S_K influence line

With the unit load outside and to the left of the panel CD (position 1) the shear force, S_K, at K is given by

$$S_K = -R_B = -\frac{z_1}{L} \qquad (17.19)$$

S_K therefore varies linearly as the load moves from A to C. Thus, from Eq. (17.19), when $z_1 = 0$, $S_K = 0$ and when $z_1 = a$, $S_K = -a/L$, the ordinate cf in the S_K influence line shown in Fig. 17.16(b). Furthermore, from Fig. 17.16(a) we see that
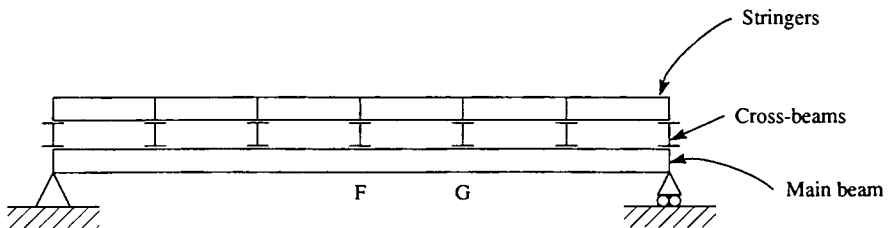

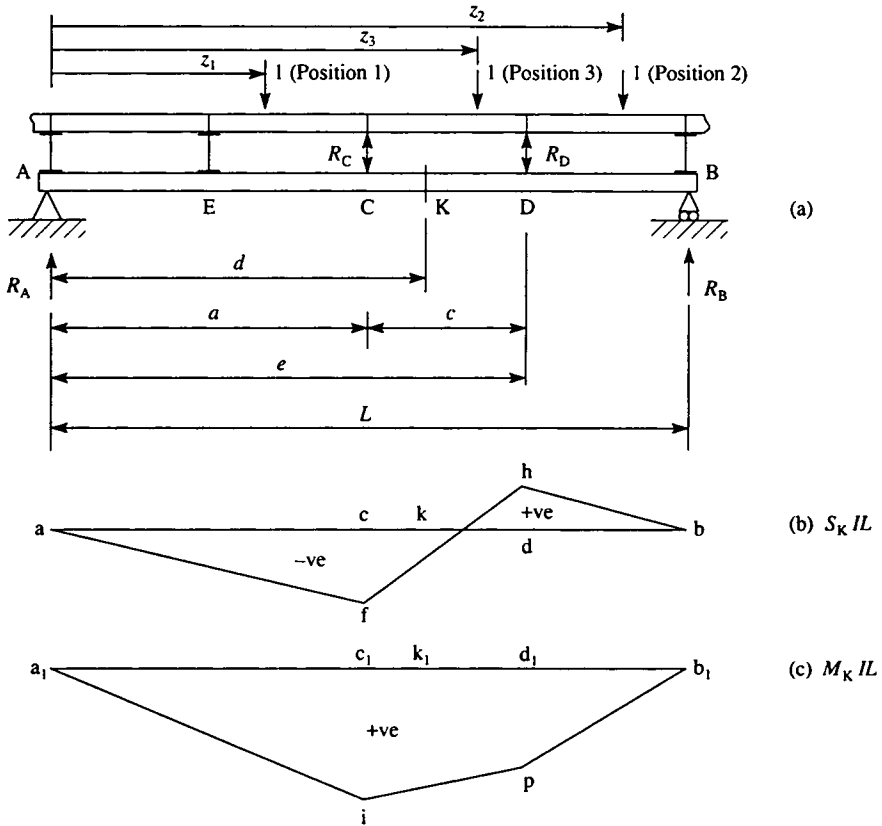**Fig. 17.15** Typical bridge construction

Fig. 17.16 Influence lines for a beam not in direct contact with the moving load

$S_K = S_C = S_D$ with the load between A and C, so that for a given position of the load the shear force in the panel CD has the same value at all sections.

Suppose now that the unit load is to the right of D between D and B (position 2). Then

$$S_K = +R_A = \frac{L - z_2}{L} \qquad (17.20)$$

and is linear. Therefore when $z_2 = L$, $S_K = 0$ and when $z_2 = e$, $S_K = (L - e)/L$, the ordinate dh in the S_K influence line. Also, with the unit load between D and B, $S_K = S_C = S_D (= +R_A)$ so that for a given position of the load, the shear force in the panel CD has the same value at all sections.

Now consider the unit load at some point between C and D (position 3). There will now be reaction forces, R_C and R_D, as shown in Fig. 17.16(a) acting on the stringer and the beam where, by considering the portion of the stringer immediately above the panel CD as a simply supported beam, we see that $R_C = (e - z_3)/c$ and $R_D = (z_3 - a)/c$. Therefore the shear force at K is given by

$$S_K = -R_B + R_D \text{ (or } S_K = +R_A - R_C)$$

so that

$$S_K = -\frac{z_3}{L} + \frac{(z_3 - a)}{c} \qquad (17.21)$$

S_K therefore varies linearly as the load moves between C and D. Furthermore, when $z_3 = a$, $S_K = -a/L$, the ordinate cf in the S_K influence line, and when $z_3 = e$, $S_K = (L - e)/L$ the ordinate dh in the S_K influence line. Note that in the calculation of the latter value, $e - a = c$.

Note also that for all positions of the unit load between C and D, $S_K = -R_B + R_D$ which is independent of the position of K. Therefore, for a given load position between C and D, the shear force is the same at all sections of the panel.

M_K influence line

With the unit load in position 1 between A and C, the bending moment, M_K, at K is given by

$$M_K = R_B(L - d) = \frac{z_1}{L}(L - d) \qquad (17.22)$$

M_K therefore varies linearly with the load position between A and C. Also, when $z_1 = 0$, $M_K = 0$ and when $z_1 = a$, $M_K = a(L - d)/L$, the ordinate c_1i in the M_K influence line in Fig. 17.16(c).

With the unit load in position 2 between D and B

$$M_K = R_A d = \frac{(L - z_2)}{L} d \qquad (17.23)$$

Again, M_K varies linearly with load position so that when $z_2 = e$, $M_K = (L - e)d/L$, the ordinate d_1p in the M_K influence line. Furthermore, when $z_2 = L$, $M_K = 0$.

When the unit load is between C and D (position 3)

$$M_K = R_B(L - d) - R_D(e - d)$$

As before we consider the stringer over the panel CD as a simply supported beam so that $R_D = (z_3 - a)/c$. Then since

$$R_B = z_3/L$$

$$M_K = \frac{z_3}{L}(L - d) - \left(\frac{z_3 - a}{c}\right)(e - d) \qquad (17.24)$$

Equation (17.24) shows that M_K varies linearly with load position between C and D. Therefore, when $z_3 = a$, $M_K = a(L - d)/L$, the ordinate c_1i in the M_K influence line, and when $z_3 = e$, $M_K = d(L - e)/L$, the ordinate d_1p in the M_K influence line. Note, that in the latter calculation, $e - a = c$.

Maximum values of S_K and M_K

In determining maximum values of shear force and bending moment at a section of a beam that is not in direct contact with the load, certain points are worthy of note.

1. When the section K coincides with a panel point (C or D, say) the S_K and M_K influence lines are identical in geometry to those for a beam that is in direct contact with the moving load; the same rules governing maximum and minimum values therefore apply.

2. The absolute maximum value of shear force will occur in an end panel, AE or DB, when the S_K influence line will be identical in form to the bending moment influence line for a section in a simply supported beam that is in direct contact with the moving load. Therefore the same criteria for load positioning may be used for determining the maximum shear force, i.e. the load per unit length of beam is equal to the load per unit length to the left of E or D and the load per unit length to the right of E or D.

3. To obtain maximum values of shear force and bending moment in a panel, a trial-and-error method is the simplest approach remembering that, for concentrated loads, a load must be placed at the point where the influence line changes slope.

17.5   Forces in the members of a truss

In some instances the main beams in a bridge are trusses, in which case the cross beams are positioned at the joints of the truss. The shear force and bending moment influence lines for a panel of the truss may then be used to determine the variation in the truss member forces as moving loads cross the bridge.
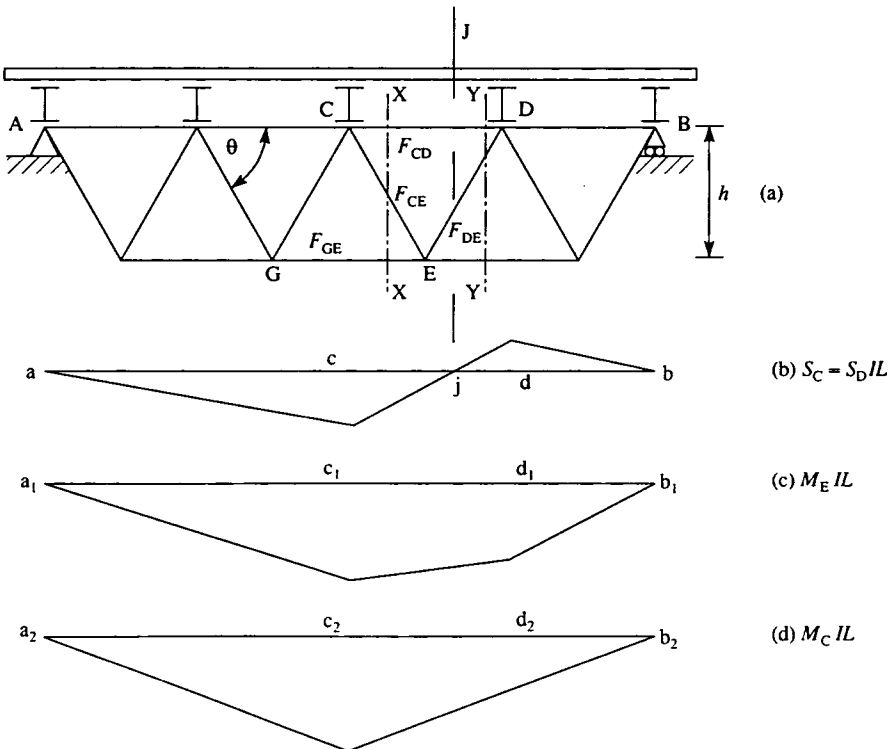

Fig. 17.17   Determination of forces in the members of a truss

Consider the simply supported Warren truss shown in Fig. 17.17(a) and suppose that it carries cross beams at its upper chord joints which, in turn, support the bridge deck. Alternatively, the truss could be inverted and the cross beams supported by the lower chord joints; the bridge deck is then the *through* type. Suppose also that we wish to determine the forces in the members CD, CE, DE and GE of the truss.

We have seen in Section 4.5 the mechanism by which a truss resists shear forces and bending moments. Thus shear forces are resisted by diagonal members, while bending moments are generally resisted by a combination of both diagonal and horizontal members. Therefore, referring to Fig. 17.17(a), we see that the forces in the members CE and DE may be determined from the shear force in the panel CD, while the forces in the members CD and GE may be found from the bending moments at E and C, respectively. We therefore construct the influence lines for the shear force in the panel CD and for the bending moment at E and C, as shown in Fig. 17.17(b), (c) and (d).

In Section 17.4 we saw that, for a given load position, the shear force in a panel such as CD is constant at all sections in the panel; we will call this shear force S_{CD}. Then, considering a section XX through CE, CD and GE, we have

$$F_{CE} \sin \theta = S_{CD}$$

so that

$$F_{CE} = \frac{S_{CD}}{\sin \theta} \qquad (17.25)$$

Similarly

$$F_{DE} = \frac{S_{CD}}{\sin \theta} \qquad (17.26)$$

From Fig. 17.17(b) we see that for a load position between A and J, S_{CD} is negative. Therefore, referring to Fig. 17.17(a), F_{CE} is compressive while F_{DE} is tensile. For a load position between J and B, S_{CD} is positive so that F_{CE} is tensile and F_{DE} is compressive. Thus F_{CE} and F_{DE} will always be of opposite sign; this may also be deduced from a consideration of the vertical equilibrium of joint E.

If we now consider the moment equilibrium of the truss at a vertical section through joint E we have

$$F_{CD} h = M_E$$

or

$$F_{CD} = \frac{M_E}{h} \qquad (17.27)$$

Since M_E is positive for all load positions (Fig. 17.17(c)), F_{CD} is compressive.

The force in the member GE is obtained from the M_C influence line in Fig. 17.17(d). Thus

$$F_{GE} h = M_C$$

which gives

$$F_{GE} = \frac{M_C}{h} \qquad (17.28)$$

F_{GE} will be tensile since M_C is positive for all load positions.

It is clear from Eqs (17.25)–(17.28) that the influence lines for the forces in the members could be constructed from the appropriate shear force and bending moment influence lines. Thus, for example, the influence line for F_{CE} would be identical in shape to the shear force influence line in Fig. 17.17(b) but would have the ordinates factored by $1/\sin\theta$. The influence line for F_{DE} would also have the S_{CD} influence line ordinates factored by $1/\sin\theta$ but, in addition, would have the signs reversed.

Example 17.17 Determine the maximum tensile and compressive forces in the member EC in the Pratt truss shown in Fig. 17.18(a) when it is crossed by a uniformly distributed load of intensity 2·5 kN/m and length 4 m; the load is applied on the bottom chord of the truss.

The vertical component of the force in the member EC resists the shear force in the panel DC. We therefore construct the shear force influence line for the panel DC as shown in Fig. 17.18(b). From Eq. (17.19) the ordinate $df = 2\times1\cdot4/(8\times1\cdot4) = 0\cdot25$ while from Eq. (17.20) the ordinate $cg = (8\times1\cdot4-3\times1\cdot4)/(8\times1\cdot4) = 0\cdot625$. Furthermore, we see that S_{DC} changes sign at the point j (Fig. 17.18(b)) where jd, from similar triangles, is 0·4.

The member EC will be in compression when the shear force in the panel DC is negative and its maximum value will occur when the head of the load is at j, thereby completely covering the length aj in the S_{DC} influence line. Therefore

$$F_{EC}\sin45° = S_{DC} = 2\cdot5\times\tfrac{1}{2}\times3\cdot2\times0\cdot25$$

whence $F_{EC} = 1\cdot41$ kN   (compression)

The force in the member EC will be tensile when the shear force in the panel DC is positive. Therefore to find the maximum tensile value of F_{EC} we must position the load within the part jb of the S_{DC} influence line such that the maximum value of S_{DC} occurs. Since the positive portion of the S_{DC} influence line is triangular, we may use the criterion previously established for maximum bending moment. Thus the load per unit length over jb must be equal to the load per unit length over jc and the load per unit length over cb. In other words, c divides the load in the same ratio that it
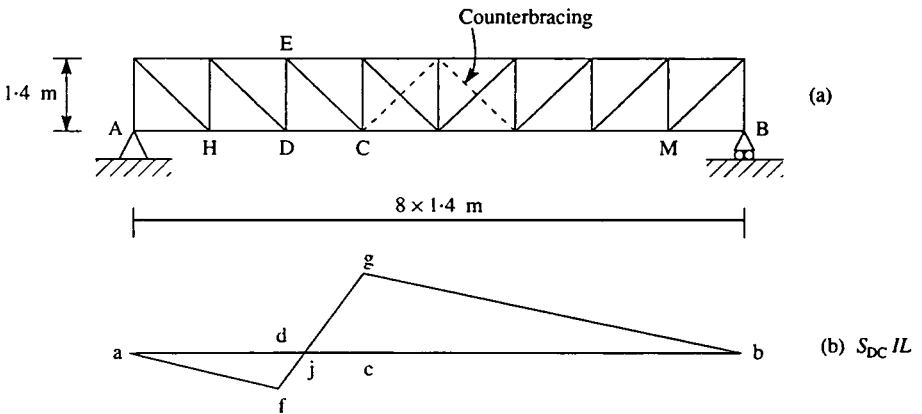

Fig. 17.18 Determination of the force in a member of the Pratt girder of Ex. 17.7

divides jb, i.e. 1:7. Therefore 0·5 m of the load is to the left of c, 3·5 m to the right. The ordinates under the extremities of the load in the S_{DC} influence line are then both 0·3125 m. Hence the maximum positive shear force in the panel CD is

$$S_{CD}(\text{max. }+\text{ve}) = 2 \cdot 5[\tfrac{1}{2}(0 \cdot 3125 + 0 \cdot 625) \times 0 \cdot 5 + \tfrac{1}{2}(0 \cdot 625 + 0 \cdot 3125) \times 3 \cdot 5]$$

which gives $\qquad\qquad S_{CD}(\text{max. }+\text{ve}) = 4 \cdot 69 \text{ kN}$

Then, since $\qquad\qquad\qquad F_{EC} \sin 45° = S_{CD}$

$$F_{EC} = 6 \cdot 63 \text{ kN}$$

which is the maximum tensile force in the member EC.

Counterbracing

A diagonal member of a Pratt truss will, as we saw for the member EC in Ex. 17.7, be in tension or compression depending on the sign of the shear force in the particular panel in which the member is placed. The exceptions are the diagonals in the end panels where, in the Pratt truss of Fig. 17.18(a), construction of the shear force influence lines for the panels AH and MB shows that the shear force in the panel AH is always positive and that the shear force in the panel MB is always negative; the diagonals in these panels are therefore always in tension.

In some situations the diagonal members are unsuitable for compressive forces so that *counterbracing* is required. This consists of diagonals inclined in the opposite direction to the original diagonals as shown in Fig. 17.18(a) for the two centre panels. The original diagonals are then assumed to be carrying zero force while the counterbracing is in tension.

It is clear from Ex. 17.17 that the shear force in all the panels, except the two outer ones, of a Pratt truss can be positive or negative so that all the diagonals in these panels could experience compression. Therefore it would appear that all the interior panels of a Pratt truss require counterbracing. However, as we saw in Section 17.3, the dead load acting on a beam has a beneficial effect in that it reduces the length of the beam subjected to shear reversal. This, in turn, will reduce the number of panels requiring counterbracing.

Example 17.8 The Pratt truss shown in Fig. 17.19(a) carries a dead load of 1·0 kN/m applied at its upper chord joints. A uniformly distributed live load, which exceeds 9 m in length, has an intensity of 1·5 kN/m and is also carried at the upper chord joints. If the diagonal members are designed to resist tension only find which panels require counterbracing.

A family of influence lines may be drawn as shown in Fig. 17.19(b) for the shear force in each of the ten panels. We begin the analysis at the centre of the truss where the dead load shear force has its least effect; initially, therefore, we consider panel 5. The shear force, S_5, in panel 5 with the head of the live load at n_5 is given by

$$S_5 = 1 \cdot 0(\text{area } n_5 gb - \text{area } n_5 qa) - 1 \cdot 5(\text{area } n_5 qa)$$

i.e. $\qquad\qquad S_5 = 1 \cdot 0 \times \text{area } n_5 gb - 2 \cdot 5 \times \text{area } n_5 qa \qquad\qquad$ (i)
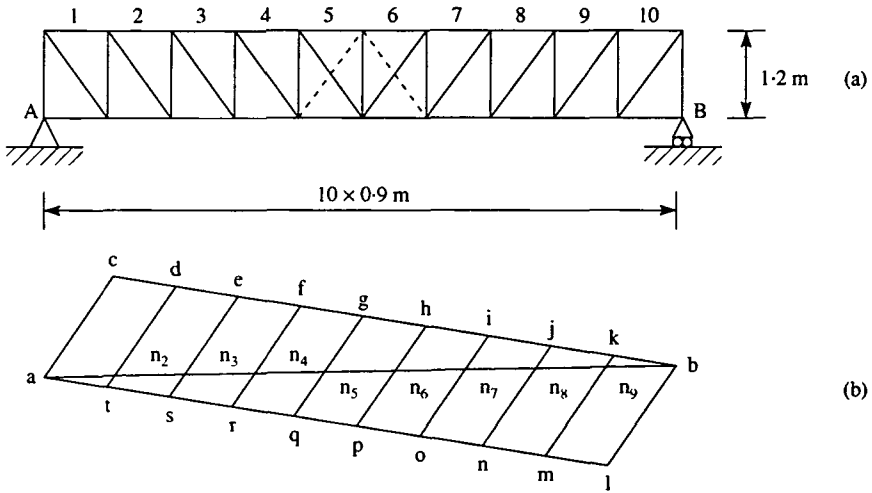
Fig. 17.19 Counterbracing in a Pratt truss

The ordinates in the S_5 influence line at g and q are found from similar triangles and are 0·5 and 0·4, respectively. Also, from similar triangles, n_5 divides the horizontal distance between q and g in the ratio 0·4:0·5. Therefore, from Eq. (i)

$$S_5 = 1·0 \times \tfrac{1}{2} \times 5·0 \times 0·5 - 2·5 \times \tfrac{1}{2} \times 4·0 \times 0·4$$

which gives $S_5 = -0·75$ kN

Therefore, since S_5 is negative, the diagonal in panel 5 will be in compression so that panel 5, and from symmetry panel 6, requires counterbracing.

Now with the head of the live load at n_4, $S_4 = 1·0$(area $n_4$fb – area $n_4$ra) – 1·5(area $n_4$ra).

The ordinates and base lengths in the triangles $n_4$fb and $n_4$ra are determined as before. Then

$$S_4 = 1·0 \times \tfrac{1}{2} \times 6·0 \times 0·6 - 2·5 \times \tfrac{1}{2} \times 3·0 \times 0·3$$

from which $S_4 = 0·675$ kN

Therefore, since S_4 is positive, panel 4, and therefore panel 7, do not require counterbracing.

Clearly the remaining panels will not require counterbracing.

Note that for a Pratt truss having an odd number of panels the net value of the dead load shear force in the central panel is zero, so that this panel will always require counterbracing.

17.6 Influence lines for continuous beams

The structures we have investigated so far in this chapter have been statically determinate so that the influence lines for the different functions have comprised straight line segments. A different situation arises for statically indeterminate structures such as continuous beams.

Consider the two-span continuous beam ABC shown in Fig. 17.20(a) and let us suppose that we wish to construct influence lines for the reaction at B, the shear force at the section D in AB and the bending moment at the section F in BC.

The shape of the influence lines may be obtained by employing the Mueller–Breslau principle described in Section 17.2. Thus, in Fig. 17.20(b) we remove the support at B and apply a unit displacement in the direction of the support reaction, R_B. The beam will bend into the shape shown since it remains pinned to the supports at A and C. This would not have been the case, of course, if the span BC did not exist for then the beam would rotate about A as a rigid link and the R_B influence line would have been straight as in Fig. 17.1(c).

To obtain the shear force influence line for the section D we 'cut' the beam at D and apply a unit shear displacement as shown in Fig. 17.20(c). Again, since the beam is attached to the support at C, the resulting displaced shape is curved. Furthermore, the gradient of the influence line must be the same on each side of D because, otherwise, it would imply the presence of a moment causing a relative rotation. This is not possible since the displacement we have specified is due solely to shear. It follows that the influence line between A and D must also be curved.

The influence line for the bending moment at F is found by inserting a hinge at F and applying a relative unit rotation as shown in Fig. 17.20(d). Again the portion
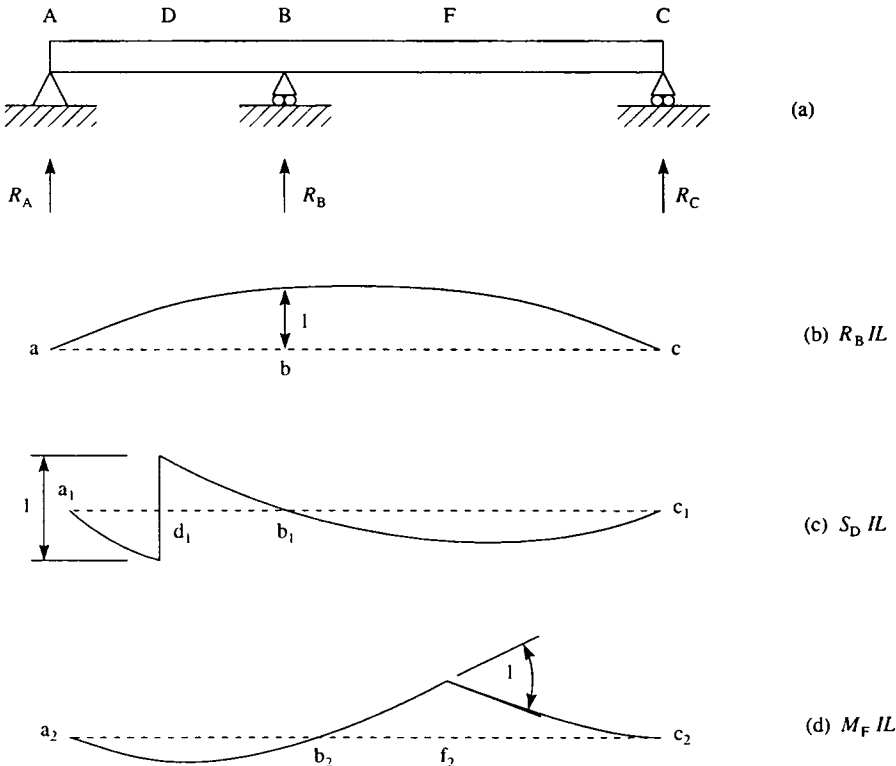

Fig. 17.20 Influence lines for a continuous beam using the Mueller–Breslau principle

ABF of the beam will be curved, as will the portion FC, since this part of the beam must rotate so that the sum of the rotations of the two portions of the beam at F is equal to unity.

Example 17.9   Construct influence lines for the reaction at B and for the shear force and bending moment at D in the two-span continuous beam shown in Fig. 17.21(a).

The shape of each influence line may be drawn using the Mueller–Breslau principle as shown in Fig. 17.21(b), (c) and (d). However, before they can be of direct use in determining maximum values, say, of the various functions due to the passage of loading systems, the ordinates must be calculated; for this, since the influence lines are comprised of curved segments, we need to derive their equations.
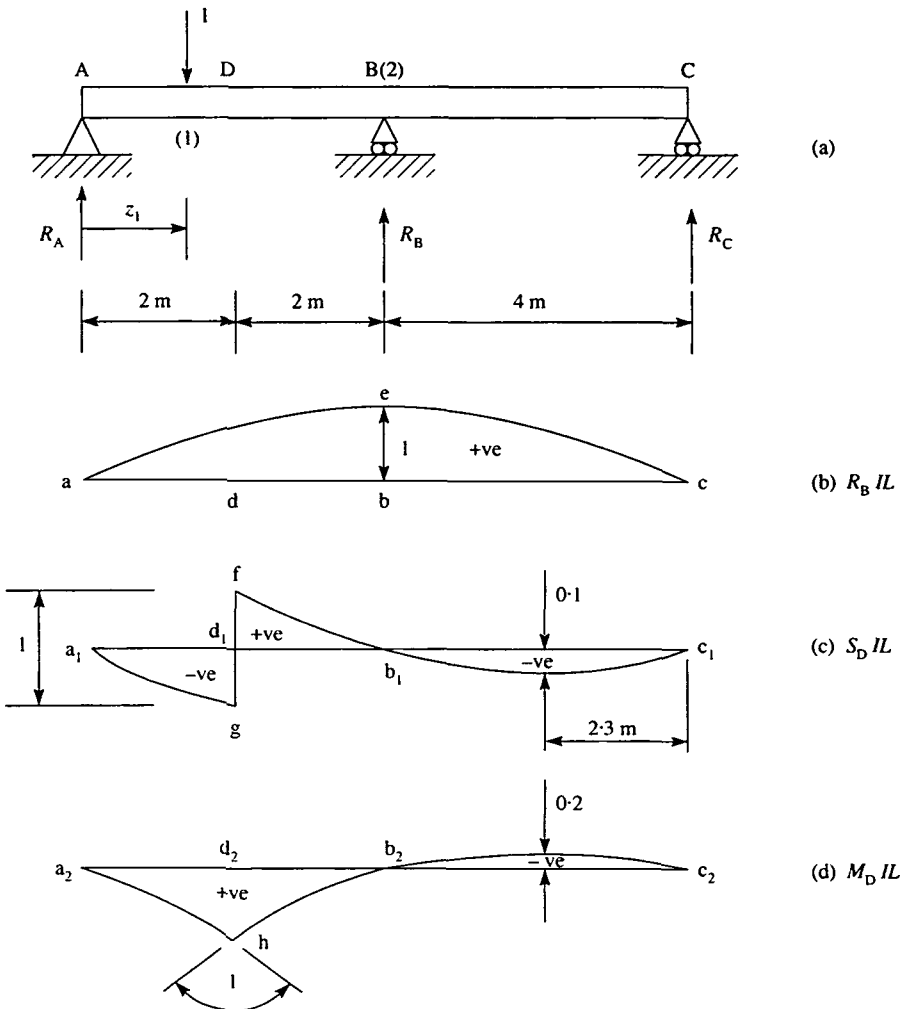
Fig. 17.21   Influence lines for the continuous beam of Ex. 17.9

However, once the influence line for a support reaction, R_B in this case, has been established, the remaining influence lines follow from statical equilibrium.

R_B influence line

Suppose initially that a unit load is a distance z_1 from A, between A and B. To determine R_B we may use the flexibility method described in Section 16.4. Thus we remove the support at B (point 2) and calculate the displacement, a_{21}, at B due to the unit load at z_1 (point 1). We then calculate the displacement, a_{22}, at B due to a vertically downward unit load at B. The total displacement at B due to the unit load at z_1 and the reaction R_B is then

$$a_{21} - a_{22}R_B = 0 \tag{i}$$

since the support at B is not displaced. In Eq. (i) the term $a_{22}R_B$ is negative since R_B is in the opposite direction to the applied unit load at B.

Both the flexibility coefficients in Eq. (i) may be obtained from a single unit load application since, from the reciprocal theorem (Section 15.4), the displacement at B due to a unit load at z_1 is equal to the displacement at z_1 due to a unit load at B. We therefore apply a vertically downward unit load at B.

The equation for the displaced shape of the beam is that for a simply supported beam carrying a central concentrated load. Therefore, from Eq. (iv) of Ex. 13.5

$$v = -\frac{1}{48EI}(4z^3 - 3L^2z) \tag{ii}$$

or, for the beam of Fig. 17.21(a)

$$v = -\frac{z}{12EI}(z^2 - 48) \tag{iii}$$

At B, when $z = 4$ m
$$v_B = \frac{32}{3EI} = a_{22} \tag{iv}$$

Furthermore, the displacement at B due to the unit load at z_1 (=displacement at z_1 due to a unit load at B) is, from Eq. (iii),

$$v_{z_1} = -\frac{z_1}{12EI}(z_1^2 - 48) = a_{21} \tag{v}$$

Substituting for a_{22} and a_{21} in Eq. (i) we have

$$-\frac{z_1}{12EI}(z_1^2 - 48) - \frac{32}{3EI}R_B = 0$$

from which
$$R_B = -\frac{z_1}{128}(z_1^2 - 48) \quad (0 \leq z_1 \leq 4{\cdot}0 \text{ m}) \tag{vi}$$

Equation (vi) gives the influence line for R_B with the unit load between A and B; the remainder of the influence line follows from symmetry. Eq. (vi) may be checked

since we know the value of R_B with the unit load at A and B. Thus from Eq. (vi), when $z_1 = 0$, $R_B = 0$ and when $z_1 = 4 \cdot 0$ m, $R_B = 1$ as expected.

If the support at B were not symmetrically positioned, the above procedure would be repeated for the unit load on the span BC. In this case the equations for the deflected shape of AB and BC would be Eqs (xiv) and (xv) in Ex. 13.6.

In this example we require the S_D influence line so that we shall, in fact, need to consider the value of R_B with the unit load on the span BC. Therefore from Eq. (xv) in Ex. 13.6

$$v_{z_1} = \frac{1}{12EI} (z_1^3 - 24z_1^2 + 144z_1 - 128) \quad (4 \cdot 0 \text{ m} \leqslant z_1 \leqslant 8 \cdot 0 \text{ m}) \tag{vii}$$

Hence from Eq. (i)

$$R_B = \frac{1}{128} (z_1^3 - 24z_1^2 + 144z_1 - 128) \quad (4 \cdot 0 \text{ m} \leqslant z_1 \leqslant 8 \cdot 0 \text{ m}) \tag{viii}$$

A check on Eq. (viii) shows that when $z_1 = 4 \cdot 0$ m, $R_B = 1$ and when $z_1 = 8 \cdot 0$ m, $R_B = 0$.

S_D influence line

With the unit load to the left of D, the shear force, S_D, at D is most simply given by

$$S_D = +R_A - 1 \tag{ix}$$

where, by taking moments about C, we have

$$R_A \times 8 - 1(8 - z_1) + R_B \times 4 = 0 \tag{x}$$

Substituting in Eq. (x) for R_B from Eq. (vi) and rearranging gives

$$R_A = \frac{1}{256} (z_1^3 - 80z_1 + 256) \tag{xi}$$

whence, from Eq. (ix)

$$S_D = \frac{1}{256} (z_1^3 - 80z_1) \quad (0 \leqslant z_1 \leqslant 2 \cdot 0 \text{ m}) \tag{xii}$$

Therefore, when $z_1 = 0$, $S_D = 0$ and when $z_1 = 2 \cdot 0$ m, $S_D = -0 \cdot 59$, the ordinate d_1g in the S_D influence line in Fig. 17.21 (c).

With the unit load between D and B

$$S_D = +R_A$$

so that, substituting for R_A from Eq. (xi)

$$S_D = \frac{1}{256} (z_1^3 - 80z_1 + 256) \quad (2 \cdot 0 \text{ m} \leqslant z_1 \leqslant 4 \cdot 0 \text{ m}) \tag{xiii}$$

Thus, when $z_1 = 2 \cdot 0$ m, $S_D = +0 \cdot 41$, the ordinate d_1f in Fig. 17.21 (c) and when $z_1 = 4 \cdot 0$ m, $S_D = 0$.

Now consider the unit load between B and C. Again

$$S_D = +R_A$$

but in this case, R_B in Eq. (x) is given by Eq. (viii). Substituting for R_B from Eq. (viii) in Eq. (x) we obtain

$$R_A = S_D = -\frac{1}{256}(z_1^3 - 24z_1^2 + 176z_1 - 384) \quad (4 \cdot 0 \text{ m} \leqslant z_1 \leqslant 8 \cdot 0 \text{ m}) \quad \text{(xiv)}$$

Therefore the S_D influence line consists of three segments, a_1g, fb_1 and b_1c_1.

M_D *influence line*

With the unit load between A and D

$$M_D = R_A \times 2 - 1(2 - z_1) \quad \text{(xv)}$$

Substituting for R_A from Eq. (xi) in Eq. (xv) and simplifying, we obtain

$$M_D = \frac{1}{128}(z_1^3 + 48z_1) \quad (0 \leqslant z_1 \leqslant 2 \cdot 0 \text{ m}) \quad \text{(xvi)}$$

When $z_1 = 0$, $M_D = 0$ and when $z_1 = 2 \cdot 0$ m, $M_D = 0 \cdot 81$, the ordinate d_2h in the M_D influence line in Fig. 17.21 (d).

Now with the unit load between D and B

$$M_D = R_A \times 2 \quad \text{(xvii)}$$

Therefore, substituting for R_A from Eq. (xi) we have

$$M_D = \frac{1}{128}(z_1^3 - 80z_1 + 256) \quad (2 \cdot 0 \text{ m} \leqslant z_1 \leqslant 4 \cdot 0 \text{ m}) \quad \text{(xviii)}$$

From Eq. (xviii) we see that when $z_1 = 2 \cdot 0$ m, $M_D = 0 \cdot 81$, again the ordinate d_2h in Fig. 17.21 (d). Also, when $z_1 = 4 \cdot 0$ m, $M_D = 0$.

Finally, with the unit load between B and C, M_D is again given by Eq. (xvii) but in which R_A is given by Eq. (xiv). Hence

$$M_D = -\frac{1}{128}(z_1^3 - 24z_1^2 + 176z_1 - 384) \quad (4 \cdot 0 \text{ m} \leqslant z_1 \leqslant 8 \cdot 0 \text{ m}) \quad \text{(xix)}$$

The maximum ordinates in the S_D and M_D influence lines for the span BC may be found by differentiating Eqs (xiv) and (xix) with respect to z_1, equating to zero and then substituting the resulting values of z_1 back in the equations. Thus, for example, from Eq. (xiv),

$$\frac{dS_D}{dz_1} = -\frac{1}{256}(3z_1^2 - 48z_1 + 176) = 0$$

from which $z_1 = 5 \cdot 7$ m. Hence

$$S_D(\text{max}) = -0 \cdot 1$$

Similarly $M_D(\text{max}) = -0 \cdot 2$ at $z_1 = 5 \cdot 7$ m.

In this chapter we have constructed influence lines for beams, trusses and continuous beams. Clearly influence lines can be drawn for a wide variety of structures that carry moving loads. Their construction, whatever the structure, is based on considering the passage of a unit load across the structure.

Problems

P.17.1 Construct influence lines for the support reaction at A in the beams shown in Fig. P.17.1(a),(b) and (c).

Ans. (a) Unit load at C, $R_A = 1 \cdot 25$.
     (b) Unit load at C, $R_A = 1 \cdot 25$; at D, $R_A = -0 \cdot 25$.
     (c) Unit load between A and B, $R_A = 1$; at C, $R_A = 0$.

(a)   (b)

(c)
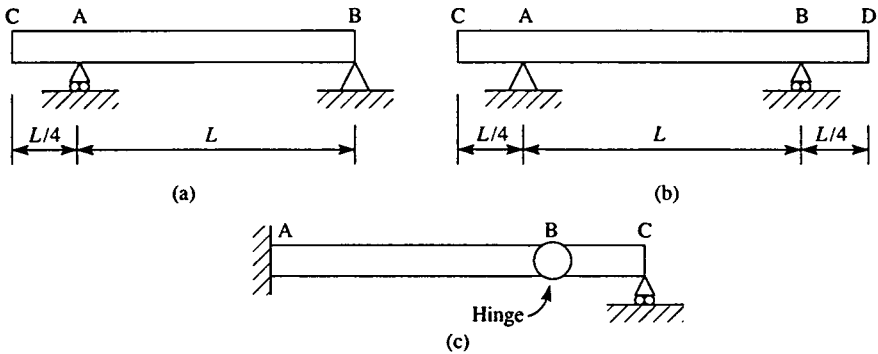
Fig. P.17.1

P.17.2 Draw influence lines for the shear force at C in the beams shown in Figs P.17.2(a) and (b).

Ans. Influence line ordinates
     (a) D = 0·25, A = 0, C = ±0·5, B = 0.
     (b) D = 0·25, A = B = 0, C = ±0·5, E = −0·25.

(a)   (b)
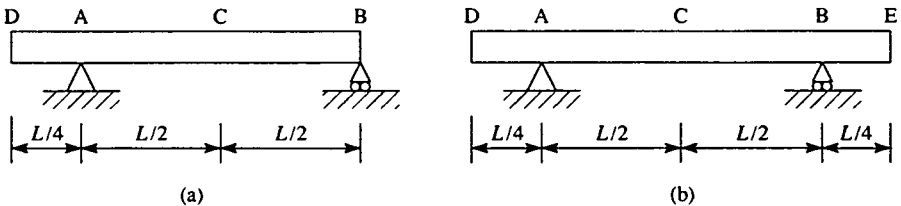
Fig. P.17.2

P.17.3 Draw influence lines for the bending moment at C in the beams shown in Fig. P.17.2(a) and (b).

Ans. Influence line ordinates
     (a) D = −0·125L, A = B = 0, C = 0·25L.
     (b) D = E = −0·125L, A = B = 0, C = 0·25L.

P.17.4 The simply supported beam shown in Fig. P.17.4 carries a uniformly distributed travelling load of length 10 m and intensity 20 kN/m. Calculate the maximum positive and negative values of shear force and bending moment at the section C of the beam.

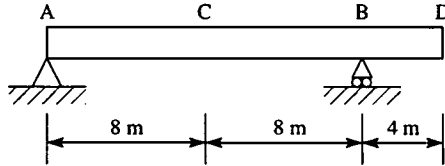*Ans.* $S_C = +37 \cdot 5$ kN, $-40 \cdot 0$ kN, $M_C = +550$ kN m, -80 kN m.

Fig. P.17.4

P.17.5 The beam shown in Fig. P.17.5(a) is crossed by the train of four loads shown in Fig. P.17.5(b). For a section at mid-span, determine the maximum sagging and hogging bending moments.

Ans. $+161 \cdot 3$ kN m, $-77 \cdot 5$ kN m.
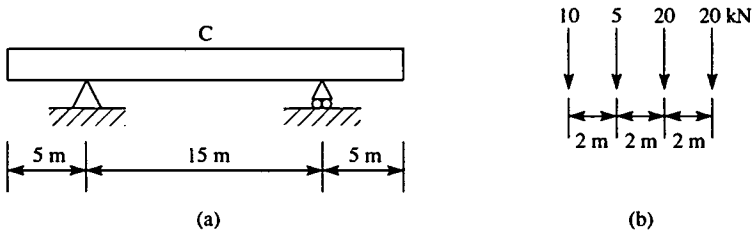
Fig. P.17.5

P.17.6 A simply supported beam AB of span 20 m is crossed by the train of loads shown in Fig. P.17.6. Determine the position and magnitude of the absolute maximum bending moment on the beam and also the maximum values of positive and negative shear force anywhere on the beam.

Ans. $M(\text{max}) = 466 \cdot 7$ kN m under a central load $10 \cdot 5$ m from A.
$S(\text{max} +\text{ve}) = 104$ kN at A, $S(\text{max} -\text{ve}) = -97 \cdot 5$ kN at B.
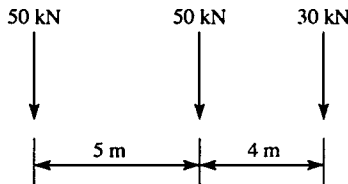
Fig. P.17.6

P.17.7 The three-span beam shown in Fig. P.17.7 has hinges at C and E in its central span. Construct influence lines for the reaction at B and for the shear force and bending moment at the sections K and D.

Ans. Influence line ordinates

R_B; A = 0, B = 1, C = 1·25, E = F = G = 0.
S_K; A = 0, K = ±0·5, B = 0, C = −0·25, E = 0.
S_D; A = B = 0, D = 1·0, C = 1·0, E = F = G = 0.
M_K; A = B = 0, K = 1·0, C = −0·5, E = F = G = 0.
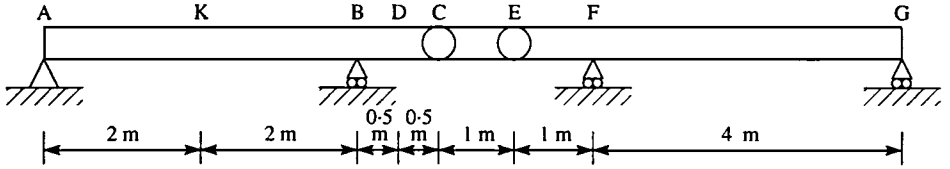M_D; A = B = D = 0, C = −0·5, E = F = G = 0.

Fig. P.17.7

P.17.8   Draw influence lines for the reactions at A and C and for the bending moment at E in the beam system shown in Fig. P.17.8. Note that the beam AB is supported on the lower beam at D by a roller.

If two 10 kN loads, 5 m apart, cross the upper beam AB, determine the maximum values of the reactions at A and C and the bending moment at E.

Ans. R_A(max) = 16·7 kN, R_C(max) = 17·5 kN, M_E(max) = 58·3 kN m.
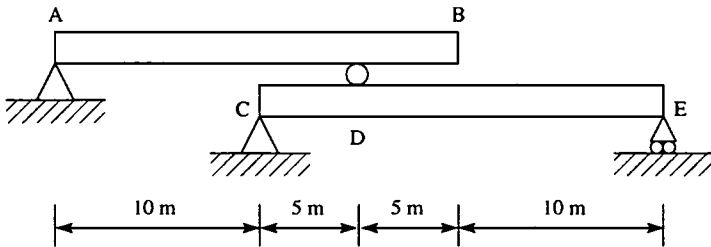
Fig. P.17.8

P.17.9   A simply supported beam having a span of 5 m has a self-weight of 0·5 kN/m and carries a travelling uniformly distributed load of intensity 1·2 kN/m and length 1 m. Calculate the length of beam over which shear reversal occurs.

Ans. The central 1·3 m (graphical solution).

P.17.10   Construct an influence line for the force in the member CD of the truss shown in Fig. P.17.10 and calculate the force in the member produced by the loads positioned at C, D and E.

Ans. 28·1 kN (compression).

P.17.11   The truss shown in Fig. P.17.11 carries a train of loads consisting of, left to right, 40 kN, 70 kN, 70 kN and 60 kN spaced at 2 m, 3 m and 3 m, respectively. If the self-weight of the truss is 15 kN/m, calculate the maximum force in each of the members CG, HD and FE.

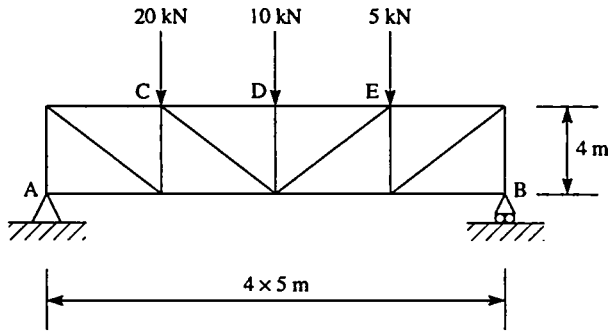*Ans.* CG = 763 kN, HD = −724 kN, FE = −326 kN.
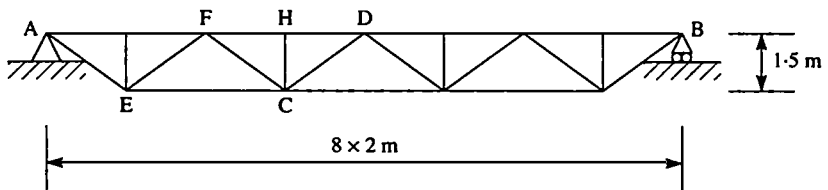
Fig. P.17.10

Fig. P.17.11

P.17.12 One of the main girders of a bridge is the truss shown in Fig. P.17.12. Loads are transmitted to the truss through cross beams attached at the lower panel points. The self-weight of the truss is 30 kN/m and it carries a live load of intensity 15 kN/m and of length greater than the span. Draw influence lines for the force in each of the members CE and DE and determine their maximum values.

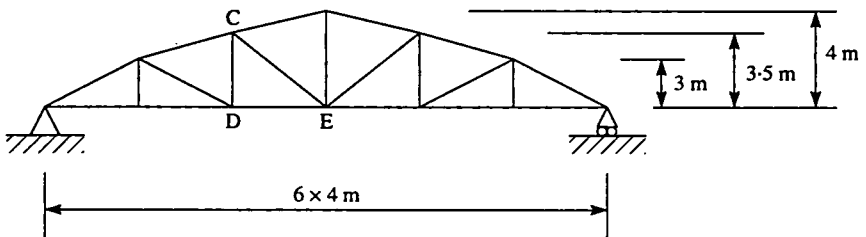*Ans.* CE = +37·3 kN, −65·3 kN, DE = +96·1 kN.
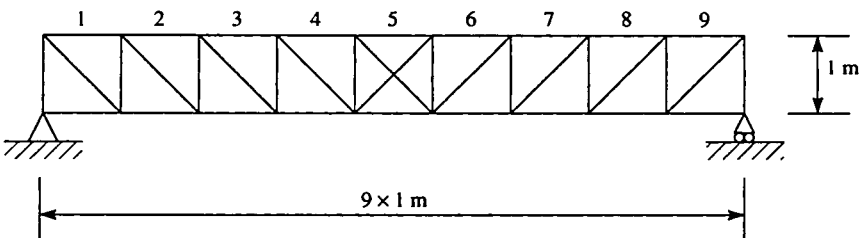

Fig. P.17.12

Fig. P.17.13

P.17.13   The Pratt truss shown in Fig. P.17.13 has a self-weight of $1\cdot2$ kN/m and carries a uniformly distributed live load longer than the span of intensity $2\cdot8$ kN/m, both being applied at the upper chord joints. If the diagonal members are designed to resist tension only, determine which panels require counterbracing.

Ans. Panels 4, 5 and 6.

P.17.14   Using the Mueller–Breslau principle sketch the shape of the influence lines for the support reactions at A and B, and the shear force and bending moment at E in the continuous beam shown in Fig. P.17.14.
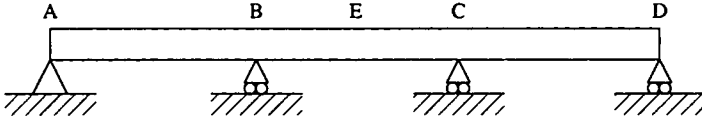

Fig. P.17.14

P.17.15   Construct influence lines for the reaction at A, the shear force at D and the bending moment at B in the continuous beam shown in Fig. P.17.15.

Ans.

$$R_A = (z_1{}^3 - 20z_1 + 32)/32 \qquad (0 \leqslant z_1 \leqslant 2\cdot0 \text{ m})$$
$$R_A = (-z_1{}^3 + 12z_1{}^2 - 44z_1 + 48)/32 \qquad (2\cdot0 \text{ m} \leqslant z_1 \leqslant 4\cdot0 \text{ m})$$
$$S_D = (z_1{}^3 - 20z_1)/32 \qquad (0 \leqslant z_1 \leqslant 1\cdot5 \text{ m})$$
$$S_D = (z_1{}^3 - 20z_1 + 32)/32 \qquad (1\cdot5 \text{ m} \leqslant z_1 \leqslant 2\cdot0 \text{ m})$$
$$M_B = (z_1{}^3 - 4z_1)/16 \qquad (0 \leqslant z_1 \leqslant 2\cdot0 \text{ m})$$
$$M_B = (-z_1{}^3 + 12z_1{}^2 - 44z_1 + 48)/16 \qquad (2\cdot0 \text{ m} \leqslant z_1 \leqslant 4\cdot0 \text{ m})$$

Fig. P.17.15