Informational Black Holes in Financial Markets

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ABSTRACT

We study how well primary financial markets allocate capital when information about investment opportunities is dispersed across market participants. Paradoxically, the fact that information is valuable for real investment decisions destroys the efficiency of the market. To add to the paradox, as the number of market participants with useful information increases, a growing share of them fall into an "informational black hole," making markets even less efficient. Contrary to the predictions of standard theory, investment inefficiencies and the cost of capital to firms seeking financing can increase with the size of the market.

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The main role of primary financial markets is to channel resources to firms with worthwhile projects. This process requires information about demand, technological feasibility, management, and current industry and macroeconomic conditions, as well as views on how to interpret such information. The efficiency of the capital allocation process depends on how well markets aggregate all this information. Today, a large and growing number of professional investors such as business angels, venture capitalists, and private equity firms alongside traditional commercial banks compete to invest in firms with good investment opportunities.

One might expect that when a larger number of experts are active in the market in which a firm is seeking financing, investment decisions should improve and the cost of capital for the firm should go down. There are two compelling reasons from economic theory to support these expectations. First, increased competition between investors should reduce their informational rents and drive down the cost of capital. Second, when investors as an aggregate possess more information about the viability of a project, investment decisions should become more efficient—which should further decrease the cost of capital. Yet, the fact that periods in which a record number of investors are present in a subsector of the financial market often coincides with episodes of large misallocation of capital, such as in the dot-com bubble and the financial crisis of 2007-2008, has led many observers to question whether increasing the size of financial markets is socially useful.

In this paper, we develop a model of information aggregation and capital allocation in primary financial markets and identify a new economic mechanism that leads to a trade-off between competition and informational efficiency. We show that larger and more competitive markets can lead to worse information aggregation, and therefore to less efficient investment decisions and a higher cost of capital. Our results have normative implications for how issuing firms should maximize revenues that drastically contrast with common wisdom. We show that policies restricting competition and allowing collusion among investors may lead to higher social surplus and higher revenues to firms.

In our model, informed investors compete for the right to finance a new project of a firm and only few of them take a stake in the firm in return for providing financing. The stakes can be in the form of debt, equity, convertible debt, or any of the other securities that are used in practice. Also, competition between investors can take many forms, ranging from structured auctions to informal negotiations. Our results hold for any competitive capital raising process in which investors take stakes that are neither risk free nor profit when the firm does badly. There are few primary financial markets that do not satisfy these assumptions.

The important departure from the existing literature is that in our setup the information generated in a financing mechanism is useful for subsequent investment decisions, and in particular, for the decision whether to start the project or not. In our setting, any investor with sufficiently pessimistic information who wins the right to finance the project would conclude that the project is negative NPV and not worth investing in. Relatively pessimistic investors therefore abstain from bidding.[1] As a result, all their information is pooled together and lost—they

[1] Investors are free to submit negative bids, but never do so in equilibrium.

fall into an "informational black hole". This loss of information is costly, and leads to investment mistakes—some projects that would have been worth pursuing had all market information been utilized do not get financed, while some that are not worth pursuing get financed.

The problem is exacerbated as the market grows larger, because of the winner's curse. In a larger market, even an investor with somewhat favorable information will conclude that the project is not worth investing in if he wins, since winning implies that all other investors are more pessimistic. Hence, the informational black hole and the amount of information destroyed grow with the size of the market. As a result, the investment mistakes continue to persist even in large financial markets with many experts and large amounts of information. We show that in many cases, social surplus as well as the expected revenues to the firm can actually decrease with the size of the market.

It should be stressed that the winner's curse alone cannot explain our results. It is the interplay between the winner's curse and the fact that information generated in the fundraising process can affect the decision whether to start the project or not that is necessary for our results. The winner's curse is present is any standard auction. Yet, as Bali and Jackson (2002) show under very general assumptions about values, revenues approach their maximum as the number of bidders goes to infinity when standard assets are auctioned. This is not necessarily true in our setting. Thus, our results may help explain the phenomenon of "proprietary transactions" in venture capital and private equity in which entrepreneurs appear to voluntarily restrict competition when seeking financing. Similarly, they provide support for the common practice in acquisition procedures for investment banks to restrict the set of invited bidders, and for the results of Boone and Mulherin (2007) who show that there is no evidence that this practice reduces seller revenues.

When firms cannot commit to restrict the number of investors[2], we show that the equilibrium size of the market may be inefficiently large. This happens because the marginal investor does not internalize the negative externality he imposes on allocational efficiency when he enters the market. We show that social welfare can decrease with a decrease in the cost of setting up an informed intermediary, and that policies aimed at restricting the market size can lead to Pareto improvements.

In our setting, efficiency can be improved by committing to give a stake in the project to a sufficiently large number of investors if this is practically feasible. This is in contrast to the standard setting, where revenues are maximized by concentrating the allocation to the highest bidder. In a multi-unit auction where the number of units grows with the number of bidders, a loser's curse balances out the winner's curse (as shown in Pesendorfer and Swinkels (1997) for standard multi-unit auctions) which in our setting leads to higher participation and better information aggregation, and hence a higher surplus. Thus, our findings may provide one rationale for crowd-funding, in which start-ups seek financing on a platform that looks very much like a multi-unit auction, and may also help explain rationing in IPO allocations since rationing increases the number of winning investors.

[2]To commit to restrict the number of investors, a firm needs to commit not to consider unsolicited offers, because ex post it is always optimal to consider all offers.

A related solution is to allow multiple investors to form syndicates and submit joint "club bids" in the fundraising process. Club bids and syndicates are common practice among both angel investors, venture capitalists, and private equity firms, and have been the subject of investigation by competition authorities for creating anti-competitive collusion. Indeed, in a standard auction setting, club bids reduce the expected revenues of the seller. But in our setting, the opposite may hold—because club bids reduce the winner's curse problem, they encourage participation, which increases the efficiency of the market.

Another prescription of our theory for improving efficiency which is markedly different from that of the standard auctions concerns the timing of information release. According to the famous "linkage principle" of Milgrom and Weber (1982) any value-relevant information that can be revealed before an auction should be revealed in order to lower the informational rent of bidders. In our setting, to the contrary, it is often better to attempt financing of the project *before* some value-relevant information is revealed. The reason is that residual uncertainty adds an option value to the project which makes less optimistic investors participate, which in turn improves the information aggregation properties of the market and leads to higher social surplus. This prediction of our theory squares well with practice whereby firms up for sale or engaged in capital raising often release information to investors in stages. In the first stage, only some general information is shared, and only serious investors, who advance to the second stage, get access to full information.

A driver of our results is the difficulty of profiting from negative information in primary markets, where there are no existing assets to short. We show that efficiency can be improved by creating a shorting market where a derivative contract that pays off if the entrepreneur secures financing but opts not to pursue the project. Such a market allows pessimistic investors to express their views, which can lead to more efficient investment decisions. A number of critical features point to the difficulty of creating such a market. First, the shorting market needs to be subsidized—there are no gains from trade between third parties taking opposite positions in the shorting market. Since the entrepreneur has no resources of her own, the subsidy must come from the participants in the regular financing market. Second, since the key economic role of the shorting market is to produce information that helps a marginal investor avoid bad projects, the contract must pay off when the project is not started. Hence, it cannot be a standard derivative or short position that is contingent on the value of an existing asset. Third, to prevent conflicts of interest from distorting the investment decision, the agent taking the decision should have no stake on either side of the shorting market.

We obtain most of our results in a setting with common values, where the number of potential investors is known, and the entrepreneur has no assets in place. In the extension section, we show that our results are robust to the presence of private values and assets in place, and hold when the number of investors is stochastic. Furthermore, we show that uncertainty about market size often leads to less efficient outcomes.

More generally, our results have implications for different architectures of primary financial markets. This is an area in which there is currently much market experimentation. Traditional venture capital and small business lending markets operate as relatively opaque search markets, with frictions that tend to limit competition. New innovation such as peer-to-peer lending and crowdfunding platforms create a more transparent and competitive market architecture. When is it useful to have more competition? When is it useful to spread out the allocation, and should this be done through the platform or by endogenous syndication? Our framework can be used to answer these questions.

Our paper is related to several different strands of literature. A few papers in auction theory show that restricting the number of bidders can be optimal. Samuelson (1985) and Levin and Smith (1994) consider auctions with participation costs and show that it may be optimal to restrict entry to reduce wasteful expenditures in equilibrium. In both papers, efficiency increases as the costs decrease. Furthermore, the optimal size of the market goes to infinity as costs go to zero. In contrast, we show optimal market size can be finite even with zero costs and that lowering costs can lead to a decrease in social surplus. Thus, both the economics mechanism and implications of Samuelson (1985) and Levin and Smith (1994) are very different from those in our paper. Similar to our paper the winner's curse is also important for the results of Bulow and Klemperer (2002) and Parlour and Rajan (2005) who argue that rationing in IPO can lead to higher revenues. However, in both papers information has no value for real investment decisions. Therefore, the economic role of the winner's curse in Bulow and Klemperer (2002) nor Parlour and Rajan (2005) is very different from that in our paper.

At a more general level, our paper is also linked to the literature on the social value and optimal size of financial markets. Several papers have argued that gains associated with purely speculative trading or rent-seeking activities can attract too many entrants into financial markets (see, e.g., Murphy, Shleifer and Vishny (1991) and Bolton, Santos and Scheinkman (2016)). We provide an alternative mechanism in which each market participant possesses valuable information for guiding real production, but competition inhibits the effective use of information.

Our paper is also connected to the literature on how well prices aggregate information in auctions. This literature Wilson (1977), Milgrom (1979), and Milgrom (1981) show that in first-price and second-price auctions the price aggregates information only under special assumptions about the signal distribution. In contrast, Kremer (2002) and Han and Shum (2004) show that the price in ascending-price auctions always aggregates information. For multi-unit auctions, Pesendorfer and Swinkels (1997) show that the price converges to the true value of the asset in uniform-price auctions if the number of units sold also grows sufficiently large. Atakan and Ekmekci (2014) show that information aggregation of prices can fail in a large uniform-price auction if the buyer of each object can make a separate decision about how to use it.

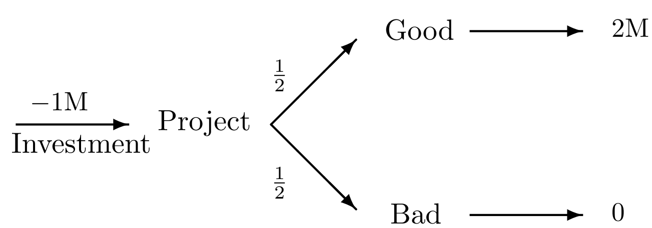
Unlike the above literature, we allow the decision maker to observe all equilibrium actions and messages in a general set of competitive mechanisms. In all of the above settings, observing equilibrium actions would lead to full information aggregation in large markets. In contrast, we show that information aggregation can still fail when information is valuable for productive decisions. For example, the ascending-price auction no longer aggregates information in our setting. Furthermore, we show that not only might markets not aggregate information as the number of investors grows large, but informational efficiency may decrease with market size.

More generally, the link between the informativeness of financial markets (such as stock markets) and real decisions by firms or governments is studied in the "feed back" literature (for a summary of this literature, see Bond, Edmans and Goldstein (2012)). The closest to our work in this literature are the papers by Bond and Eraslan (2010), Bond and Goldstein (2014) and Goldstein, Ozdenoren, Yuan (2011) who show that when an economic actor takes real decisions based on the information in asset prices, they affect the incentives to trade on this information in an endogenous way that may destroy the informational efficiency of the market. None of these papers analyze the effect of market size on efficiency, which is one of our main objectives. Furthermore, our paper shows that informational and allocational efficiency can fail even in the primary market for capital, where investors directly bear the consequences of their actions.

Finally, like us, Broecker (1990) studies a project financing setting. He considers a special case of our model when the financing mechanism is the first-price auction, signals are binary, and investors who provide financing do not have the option to cancel a project after an offer is accepted. Broecker (1990) does not study information aggregation and surplus specifically and does not consider the effect of reducing the number of investors, releasing information, revealing bids, or allowing investors to endogenously decide on the investment after the auction is over.

Example

We start with an example to convey the main idea of the paper in the simplest possible setting. A prospective entrepreneur has an idea for a startup that requires a 1M investment. She is uncertain whether it is worth it—there is an equal probability that the project is good (G), in which case it would return 2M, or bad (B) in which case it would return nothing. The unconditional net present value is therefore zero, and as it stands she weakly prefers to stay in her current job.



To test her idea and potentially arrange financing, the entrepreneur sends her business plan to a venture capitalist (VC), who is an expert at evaluating startups. The VC can get a high or a low signal about the project, with $\Pr(H|G) = 1, \Pr(H|B) = 1/2$. If the VC gets a low signal, he learns that the project is bad, since good projects never generate low signals. Therefore, he will not finance the startup, and the entrepreneur stays in her old job. If the VC gets a high signal, he updates the probability that the project is good to $2/3$:

$$\Pr(G|H) = \frac{\Pr(H|G)\Pr(G)}{\Pr(H|G)\Pr(G) + \Pr(H|B)\Pr(B)} = \frac{2}{3}.$$

Therefore, conditional on a high signal, the project is positive NPV:

$$V^H = \frac{2}{3} \times 1M - \frac{1}{3} \times 1M = \frac{1M}{3},$$

and the expected surplus is

$$\Pr(H) \times V^H = \frac{3}{4} \times \frac{1M}{3} = \frac{1M}{4}.$$

The VC and the entrepreneur split this surplus in some way during bargaining, and the business is started. The existence of an informed investor has increased both social surplus and the value to the entrepreneur by making the investment decision more efficient.

Now suppose the entrepreneur sends her business plan to two competing VCs instead. She argues that inviting more VCs to join a bargaining process with her will both increase the informativeness of her decision and the share of surplus she can keep due to increased competition. Both VCs get informative signals that are drawn independently conditional on the project type. If either signal is low the project is sure to be bad, while if both signals are high the project is good with probability 4/5 :

$$\Pr(G|HH) = \frac{\Pr(H|G)^2 \Pr(G)}{\Pr(H|G)^2 \Pr(G) + \Pr(H|B)^2 \Pr(B)} = \frac{4}{5}.$$

Hence, if the information of the two VCs is used efficiently, the project is started if and only if both get high signals. Conditional on two high signals, the NPV of the project is now:

$$V^{HH} = \frac{4}{5} \times 1M - \frac{1}{5} \times 1M = \frac{3M}{5}.$$

Therefore, the expected surplus is

$$\Pr(HH) \times V^{HH} = \frac{5}{8}\frac{3M}{5} = \frac{3M}{8} > \frac{1M}{4}.$$

But this is not what happens. Suppose that VCs have some minimal hassle cost $\varepsilon > 0$ of entering the bargaining process, and hence never bother to participate if they receive low signals. Suppose a VC with a high signal enters with some probability $0 \leq \mu \leq 1$ in a symmetric equilibrium. If both enter, it becomes common knowledge that they both have high signals, so bargaining is done under symmetric information. The entrepreneur will have the VCs bid against each other, and competition will drive VC's share of surplus to zero.

Hence, a VC can only break even on his small participation cost when he is alone bargaining with the entrepreneur. But when VC_1 is the only participant, VC_2 is likely to have had a low signal—in fact, if VCs only stay out when they get low signals ($\mu = 1$), he must have had a low signal, implying that the project is negative NPV. A VC can then never break even on his participation cost in any state: In states where the project is likely to be good, competition drives his profits to zero, and in states where competition is absent, a winner's curse makes the project unattractive.

The winner's curse is worse the more likely it is that a VC with a good signal participates. Hence, in a symmetric equilibrium, the participation rate μ for a VC with a high signal must be low enough such that his competitor can profitably finance the project when he participates alone.

How does the surplus in such an equilibrium compare to the one where the entrepreneur deals exclusively with one VC? For states where the original VC still enters, there is no change since the project is still financed. We therefore compare investment efficiency on the set of projects the original VC now passes on, which consists of all projects he gets a low signal on and a fraction $(1-\mu)$ of projects he gets a high signal on.

Projects from this pool will now be financed if and only if the new VC gets a high signal and enters. He therefore finances a good project from the pool with probability $\mu \Pr(H|G) = \mu$, and a bad project with probability $\mu \Pr(H|B) = \mu \times 1/2$. Since the project at least weakly breaks even when the investor has a high signal, the surplus created on the pool is no higher than if he had always entered—i.e., if he invested in all good projects and a fraction 1/2 of bad projects.

The original VC, when he was the only invited investor, financed all good projects out of the pool. Of the bad projects in the pool, he invested in the ones where he erroneously received a high signal, which consists of a fraction

$$\frac{\Pr(H|B)(1-\mu)}{\Pr(H|B)(1-\mu)+\Pr(L|B)} = \frac{\frac{1}{2}(1-\mu)}{\frac{1}{2}(1-\mu)+\frac{1}{2}} = \frac{1-\mu}{2-\mu},$$

which is lower than 1/2. Therefore, the screening of the original VC on this pool when there was no competition was more efficient, so surplus goes down with two VCs. If the entrepreneur has enough bargaining power, her revenues also goes down.

What went wrong? The extra competition from adding a VC led to no rents to VCs in the best states, and a winner's curse made it hard to break even when bargaining alone with the entrepreneur. This led to lower participation by the original VC, and the valuable information he had when not participating was lost. The extra information of the added investor was not enough to compensate for the lost information, so that investment decisions were *less* informed than before. Even though the share of surplus captured by the entrepreneur increased with added competition, the negative effect on total surplus made her welfare go down.

The problem gets even worse with more VCs, even though the market as an aggregate has more information. As the number of investors increases, it becomes harder and harder to break even in marginal states due to the winner's curse, so that the non-participation region $1-\mu$ increases. We call this region the *information black hole* since the information of non-participating investors in this region is lost.

Figure 1 shows potential social surplus if all information is used efficiently, and the actual equilibrium surplus as a function of the number of VCs in the market. Potential surplus increases and approaches the first best outcome where only good projects are financed, while market surplus declines.

The example has a simple investment decision (start or abandon), binary signals, no assets in place, and a particular market structure. The remainder of the paper generalizes this example to allow for general investment policy choices for both mature and new firms, general information structures, financing contracts, and modes of competition between investors.

1 Setup

The model has two sets of risk-neutral agents: A firm seeking financing to start a new project, and a set $\{1, ..., N\}$ of potential investors who have private information about the prospects of the project. In our main analysis, we will refer to the firm as the "entrepreneur" and assume that the firm has no other assets or financial resources—we show in Section 4.4 that our results apply equally well to mature firms with assets in place. Each investor gets a private signal S_i that is informative about whether the project type θ is good ($\theta = G$) or bad ($\theta = B$). The ex ante probability that the project is good is π_0.

1.1 Signals

Conditional on the project type θ, signals are drawn identically and independently on $[0, 1]$ with continuous conditional densities $f_G(s)$ and $f_B(s)$ satisfying the strict maximum likelihood ratio property:

Assumption 1 *Strict MLRP:*

$$\forall s > s', \quad \frac{f_G(s)}{f_B(s)} > \frac{f_G(s')}{f_B(s')}.$$

Assumption 1 ensures that higher signals are better news than lower signals.[3] We also assume that $f_B(1) > 0$, and that the likelihood ratio $f_G(1)/f_B(1)$ at the most optimistic signal realization $s = 1$ is bounded. These assumptions ensure that the observation of a single signal can never rule out the possibility of the project being bad, while an observer of all signals will be able to learn the project type perfectly as the number of investors goes to infinity.

1.2 Project

The information of investors is valuable for deciding the investment policy of the firm. The investment policy a is picked from a compact choice set A and leads to a random surplus V_a net of any investments and opportunity costs. We allow for the possibility that the project value V_a is a function of variables other than the type θ, but we assume that V_a is independent of investor signals conditional on the project type. Hence, given information $\boldsymbol{\omega}$ learned during a

[3] As we show in the working paper version of this paper, all results go through under the weaker assumptions that $f_G(s)$ and $f_B(s)$ are left-continuous and have right limits everywhere, and signals satisfy weak MLRP: $\forall s \geq s'$, $f_G(s)/f_B(s) \geq f_G(s')/f_B(s')$. The discrete signal distribution in the motivating example can be represented with left-continuous densities satisfying weak MLRP: $f_B(s) = 1$ for all s, $f_G(s) = 0$ for $s \leq 1/2$, and $f_G(s) = 2$ for $s > 1/2$.

financing process, the expected net present value depends only on the updated probability of the type, $\pi(\boldsymbol{\omega}) = \Pr(G|\boldsymbol{\omega})$, and the expected payoffs given the type:

$$E(V_a|\boldsymbol{\omega}) = \pi(\boldsymbol{\omega})E(V_a|G) + (1 - \pi(\boldsymbol{\omega}))E(V_a|B). \tag{1}$$

The choice set always includes the option $a = 0$ of abandoning the project opportunity without cost ($V_0 \equiv 0$), while any policy $a \neq 0$ requires financing and leads to strictly positive net present value for good project and strictly negative net present value for bad projects:

Assumption 2

$$E(V_a|G) > 0 > E(V_a|B) \quad \forall a \neq 0. \tag{2}$$

For applications where the number of possible policy choices are infinite—e.g., when the project scale can be chosen from a continuum—we also assume that $E(V_a|\theta)$ is a continuous function of a for each type θ and that there is no "minimal scale" version of the project which is positive NPV even with an arbitrarily small probability of the project being good:

Assumption 3

$$\max_{a \neq 0} \frac{E(V_a|G)}{|E(V_a|B)|} < \infty. \tag{3}$$

We show in Lemma 1 that assumptions 2 and 3 together imply that the project should be abandoned if sufficiently negative information is learned during a financing process.

Our assumptions put few restrictions on a project. The project can be scalable, have fixed or marginal costs, and the choice set can include any set of risk-return profiles that do not have a guaranteed positive NPV. The choice can also be over different dynamic investment strategies if new information is expected to arrive over time—for example, whether to invest immediately, keep the option to invest alive while waiting for resolution of uncertainty, or completely abandon the project (see Section 5 for more details). The critical assumption driving our results is that there is a non-trivial choice at the extensive margin between the status quo action $a = 0$ requiring no financing and any other choice $a \neq 0$. Most of our results can therefore be derived in the special case with a binary choice set $a \in \{0, 1\}$ as in our motivating example, where $a = 1$ means starting the project.

Given information $\boldsymbol{\omega}$ the optimal investment policy that delivers the maximal expected payoff solves:

$$a(\boldsymbol{\omega}) = \arg\max_{a \in A} \pi(\boldsymbol{\omega})E(V_a|G) + (1 - \pi(\boldsymbol{\omega}))E(V_a|B). \tag{4}$$

From equation (4) it is clear that the optimal investment policy $a(\boldsymbol{\omega})$ depends only on the updated probability of the type, $\pi(\boldsymbol{\omega})$. Hence, social surplus is given by $v(\pi(\boldsymbol{\omega}))$, where

$$v(\pi) = \max_{a \in A} \pi E(V_a|G) + (1 - \pi)E(V_a|B). \tag{5}$$

The following lemma provides a first-best benchmark for social surplus when all market information is available:

Lemma 1

(i) There is a $\pi^* > 0$ *such that* $a(\pi) = 0$ *and* $v(\pi) = 0$ *for all* $\pi \in [0, \pi^*]$*, and* $v(\pi) > 0$ *for all* $\pi > \pi^*$*.* $v(\pi)$ *is strictly increasing for* $\pi > \pi^*$ *and convex.*

(ii) When $\boldsymbol{\omega} = \mathbf{S} \equiv \{S_i\}_{i=1}^N$*, social surplus increases with* N *and approaches the first-best in the limit:*

$$\lim_{N\to\infty} E(v(\pi(\mathbf{S}))) = E\left(\max_a E(V_a|\theta)\right).$$

Proof: See the Appendix.

The lemma shows the value of investor information. Although the net present value under a given policy a is linear in the expected type π of the project, the value under the optimal investment policy is convex. As in any real option setting with convex payoffs, more information allows for better fine-tuning of the investment decision and increases the value. As the number of investor signals grows without bound, an observer of all signals would avoid all bad projects and invest optimally in all good projects.

1.3 Fundraising market

Investors compete over what financing contract, if any, to supply to the firm. We first describe the set of financing contracts we allow as feasible outcomes, and then describe the market equilibrium conditions.

Any market outcome can be described as an investment policy a and a financing contract $\mathbf{w}_a = \{w_{i,a}\}_{i=1}^N$ that implements a, where $w_{i,a}$ specifies investor i's net payoff. A typical example is where an investor i contributes capital I_i to the firm in exchange for a security z_i backed by the post-investment cash flows $V_a + \sum_j I_j$ of the firm, so that $w_{i,a} = z_i(V_a + \sum_j I_j) - I_i$. The entrepreneur retains $V_a - \sum_i w_{i,a}$. We allow the payoffs to depend both on the realized value of the project and investor signals, but omit this dependence in the notation whenever possible to avoid cluttering the exposition.

Given an investment policy a, we allow for any financing contract $\mathbf{w}_a$ that satisfies the following feasibility restrictions:

Assumption 4 *Limited liability:* $V_a - \sum_i w_{i,a} \geq 0$

Assumption 5 *Monotonicity: For all* $w_{i,a}$*,* $E(w_{i,a}|B) \leq \min(E(w_{i,a}|G), 0)$*.*

The limited liability condition simply requires that the penniless entrepreneur should be able to implement the contracted investment policy. We relax the assumption in Section 4.4, where we allow the firm to have existing assets that can back securities.

The monotonicity condition is automatically satisfied if the firm has only one class of investors (where $w_{i,a} = q_i w_a$ for some constant $q_i \geq 0$ and contract w_a).[4] For the case where the firm has multiple classes of investors, the monotonicity condition says that investors should be sufficiently

[4] This follows from Assumption 2 and limited liability; with only one class of securities, investors cannot make profits on bad projects, and so must make some profits on good projects in order to break even.

aligned, in the sense that they all prefer the project to do well rather than poorly. This is a standard condition in the security design literature (see e.g., DeMarzo, Kremer, and Skrzypacz (2005)) that can be microfounded as a way to prevent ex post moral hazard between a firm's active investors, or between the entrepreneur and some set of investors. The monotonicity condition also says that no investor should receive a claim that has strictly positive profits regardless of the state of the project—that is, there is no "free lunch" ex post.[5]

1.3.1 Market equilibrium

We are interested in how efficiently the equilibrium investment policy a reflects the market information contained in investor signals when investors compete to finance the firm. To put an upper bound on efficiency, we allow for any type of information extraction from investors who participate in the market. As in the motivating example, the loss of information will be due to investors who decide not to participate after observing their signal.

We restrict attention to symmetric equilibria in which investors with signals above some threshold $\hat{s} \in [0, 1]$ participate. As we will see (Proposition 1), it is without loss of generality to focus on threshold participation strategies when the monotonicity condition holds. This is because investors can only make profits if the project is good, so that more optimistic investors always expect to break even when less optimistic investors do. Non-participating investors leave the game, so equilibrium allocations can only depend on information in the censored signal vector $\mathbf{S}_{\geq \hat{s}} \equiv \{S_i \times 1_{\{s_i \geq \hat{s}\}}\}_{i=1}^N$.

As we want the model to be applicable to a wide set of primary capital market, we do not specify a particular way in which investors compete. However, as is clear from our motivating example, competition plays an important role for our result. We will allow any market structure that is mildly competitive, in that a "maximally pessimistic" participant is outcompeted and cannot make profits when there are more aggressive participants with enough resources:

Assumption 6 *Competitive market: If there are $n > K$ participants where the resources of K investors are enough to finance the firm, an investor with a signal just above the equilibrium participation threshold $\hat{s}$ makes vanishing profits:*

$$\lim_{s \downarrow \hat{s}} E(w_{i,a} | S_i = s, n > K \text{ participants}) \leq 0. \tag{6}$$

This is a generalized version of the assumption about competition we made in the motivating example. In the example, all participants have equally optimistic signals H, so any participant

[5] Although we know of no primary financial market where the monotonicity condition is violated, it is not without loss of generality—it rules out shorting markets, and more generally, non-monotone surplus-extraction mechanisms as in McAfee, McMillan and Reny (1989) that rely on a set of side transfers $t_i(\mathbf{s})$ between investors based purely on messages sent in the financing process. Such contracts are not ex post rational, which may explain why they are seldom—if ever—used in practice. We show in Section 4.3 how the introduction of a carefully designed shorting market can improve informational efficiency if the monotonicity condition is relaxed, but the market has to be subsidized by the firm's investors, and participants will have ex post incentives to either renege or influence the firm to take inefficient actions.

is marginal. When both VCs enter, their signals are common knowledge and competition drive their rents to zero.

With a more general signal structure, the analogous logic is as follows. The lowest possible investor type that participates in equilibrium has no informational rents, since his reservation value is the lowest possible value compatible with an observed set of participating investors. Hence, competitors can always outbid him, and an entrepreneur can safely hold out until the reservation price of the lowest type is reached. The competitive assumption holds in any non-collusive, non-rationed auction or bargaining setting such as uniform or discriminatory price auctions, irrespective of whether investors make their bids before or after they have observed the number of competitors. In particular, the assumption is automatically satisfied when only the investors with the K highest signals get an allocation in equilibrium.

In our main analysis, to make the exposition simpler, we assume that all investors have deep pockets so that $K = 1$. All our results hold for any fixed K that does not grow with the size of the market. We study the case of $K > 1$ as well as collusion and rationing in Section 4.

The final requirement we put on an equilibrium is that it is informationally robust. Since we assume that participation is free, there are typically a multitude of equilibria with different levels of efficiency depending on whether indifferent investors participate and share their information or not. Our robustness criterion rules out equilibria that rely critically on information from indifferent investors who would never participate in any possible financing mechanism if there were even an arbitrarily small participation cost $\varepsilon > 0$.

We start with the definition of an equilibrium with an arbitrary participation cost, and then give the formal definition of a robust equilibrium:

Definition 1 *$\{\hat{s}_\varepsilon, a_\varepsilon, \mathbf{w}_{a_\varepsilon}\}$ is a symmetric, competitive equilibrium with a participation cost ε if:*

(i) $\mathbf{w}_{a_\varepsilon}$ is a feasible contract, that is, Assumptions 4 and 5 hold.

(ii) a_ε and $\mathbf{w}_{a_\varepsilon}$ are measurable with respect to $\mathbf{S}_{\geq \hat{s}_\varepsilon}$.

(iii) Assumption 6 holds.

(iv) Incentive compatibility holds: For any $s_i \in [0, 1]$

$$s_i \in \arg\max_{s' \in [0,1]} 1_{\{s' \geq \hat{s}\}}(E(w_{i,a_\varepsilon}(\mathbf{S^{-i}}, s')|S_i = s_i) - \varepsilon), \tag{7}$$

where $w_{i,a_\varepsilon}(\mathbf{S^{-i}}, s')$ is the payoff to an investor with signal s_i if he acts as if he has signal s'.

Definition 2 *$\{\hat{s}, a, \mathbf{w}_a\}$ is a robust symmetric, competitive equilibrium with zero participation cost if for every δ, there is a symmetric, competitive equilibrium of a financing mechanism with participation cost $\varepsilon > 0$ such that $|\hat{s} - \hat{s}_\varepsilon| < \delta$ and allocations are within δ:*

$$E(|V_a - V_{a_\varepsilon}|) + \sum_i E(|w_i - w_{i,a_\varepsilon}|) < \delta. \tag{8}$$

To make clear the role of the robustness criterion, recall our motivating example. In the example, if participation costs are zero, there is a continuum of non-robust equilibria with different degrees of participation. The most efficient of these is an equilibrium where VCs always participate when they get a high signal ($\mu = 1$), but the project is only financed if all VCs participate. This equilibrium fully aggregates information and achieves the first best. However, in this equilibrium all investors always make exactly zero profits, and so even an arbitrarily small participation cost destroys the equilibrium.

Our equilibrium definition allows for a very wide class of market structures and financial contracts, and is as unrestrictive as possible in order to put a lower bound on investment inefficiencies. We have not required that the outcome is renegotiation proof, or that the prescribed equilibrium investment policy is incentive compatible for the final decision maker. We show in Section 2.1 that a straight equity financing contract issued to investors with the highest willingness to pay achieves maximal efficiency, is renegotiation proof, and induces the entrepreneur and investors to agree on the ex post surplus-maximizing action.

2 Analysis: Informational black holes

In this section we study how well fundraising markets incorporate information into investment decisions. We derive a maximal set $(\hat{s}, 1]$ of participating investors from which information can be learned, and show that even with maximum information there are significant investment inefficiencies relative to the first best.

To derive a lower bound on the participation threshold, we focus on a marginal participating investor with signal s just above the participation threshold $\hat{s}_\varepsilon$ when there is an arbitrarily small participation cost $\varepsilon > 0$. From the competitiveness assumption, if s is sufficiently close to the threshold he will make vanishingly small profits if any other investors participate, and hence cannot break even on his participation cost. A necessary condition for him to participate is therefore that he breaks even when noone else participates:

$$E(w_i | S_i = s, \max_j S_j \leq \hat{s}_\varepsilon) \geq \varepsilon. \tag{9}$$

When this investor is the only participant, limited liability implies that $w_i \leq V_a$ so that the investor cannot get more than the available surplus. Hence, a necessary condition for participation is:

$$\max_a E(V_a | S_i = s, \max_j S_j \leq \hat{s}_\varepsilon) \geq \varepsilon. \tag{10}$$

Hence, for a marginal investor with a non-zero participation cost, it must be worth starting the project based only on the information that other investors are not participating. The winner's curse for a marginal investor is worse the lower the participation threshold is, which puts a lower bound on the threshold in any robust equilibrium:

Proposition 1 *In any robust symmetric equilibrium, the participation threshold is no smaller than the smallest value* $\hat{s}_N$ *such that*

$$\max_{a \neq 0} E(V_a | \max_i S_i = \hat{s}_N) \geq 0, \tag{11}$$

and, given a set of participants $\kappa_N \equiv \sum 1_{\{S_i \geq \hat{s}\}}$*, no investment policy is more efficient than the policy* $a(\mathbf{s}_{\geq \hat{\mathbf{s}}})$ *given by:*

$$a(\mathbf{s}_{\geq \hat{\mathbf{s}}}) = \max_{a \neq 0} E(V_a | \mathbf{S}_{\geq \hat{\mathbf{s}}} = \mathbf{s}_{\geq \hat{\mathbf{s}}}) \text{ if } \kappa_N > 0 \text{ and } a(\mathbf{s}_{\geq \hat{\mathbf{s}}}) = 0 \text{ if } \kappa_N = 0. \tag{12}$$

Proof. See the Appendix.

Whenever $\hat{s}_N > 0$ we call the minimal non-participation region $[0, \hat{s}_N]$ the *informational black hole*, since the investment decision $a(\mathbf{S}_{\geq \hat{\mathbf{s}}})$ cannot vary with signals below the threshold—all signals below $\hat{s}_N$ are pooled together and lost.

Proposition 1 implies that even in a constrained efficient robust equilibrium, the most important investment decision—whether to start the project or not—cannot rely on any more information than what is contained in the highest signal among investors. To see this, note that if the highest signal is below the threshold, there is no participation and the project is abandoned. Conversely, whenever the highest signal is above the threshold, it is always efficient to start the project. This is so since even in the least optimistic such scenario, where a marginal participant wins alone, starting the project is efficient. Hence, the project is started if and only if any investor participates.

The existence of the informational black hole leads to inevitable investment inefficiencies. For example, when all signals are in the informational black hole but close to the participation threshold, the project will not be undertaken even though it can be positive NPV. In contrast, even if all but one investor have the most negative signal possible so that the project is strictly unprofitable, the constrained efficient policy is to start the project whenever one investor participates.

In what follows, we will focus on the case where there is at least *some* participation in a robust equilibrium:

Assumption 7 *There is an action* a *such that*

$$E(V_a | S_i = 1) > 0. \tag{13}$$

Assumption 7 says that a single investor who observes the highest possible signal will find it worthwhile to start the project. When this assumption is violated, Proposition 1 implies that there is never any participation, so that the financing market breaks down completely.

Given Assumption 7, the minimal participation threshold $\hat{s}_N$ is always strictly below one. However, as the market grows larger, $\hat{s}_N$ will increase due to the winner's curse, so that more and more investors end up in the informational black hole. The next proposition shows that as the number of investors N and hence the amount of information in the market grows, the amount of information lost in the informational black hole grows correspondingly so that substantial uncertainty remains even in arbitrarily large markets:

Proposition 2 *As $N \to \infty$, the informational black hole grows with N such that the number of participants κ_N converges to a poisson-distributed random variable κ with*

$$\Pr(\kappa = k|B) = e^{-\tau}\frac{\tau^k}{k!}, \quad k = 0, 1, \ldots, \tag{14}$$

$$\Pr(\kappa = k|G) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \tag{15}$$

where $\lambda = f_G(1)/f_B(1)$. The arrival rate $\tau > 0$ of participants is the unique solution to the marginal investor's break-even condition:

$$\frac{\Pr(G|\kappa = 1)}{\Pr(B|\kappa = 1)} = \frac{\pi^*}{1-\pi^*} \Leftrightarrow \lambda\frac{e^{-\lambda\tau}}{e^{-\tau}}\frac{\pi_0}{1-\pi_0} = \frac{\pi^*}{1-\pi^*}, \tag{16}$$

where π^* is the break-even probability for the project defined in Lemma 1. The number of participants κ becomes a sufficient statistic for equilibrium information in the limit:

$$\frac{\pi(\mathbf{S}_{\geq \hat{s}_N})}{1-\pi(\mathbf{S}_{\geq \hat{s}_N})} \to \lambda^{\kappa-1}\frac{\pi^*}{1-\pi^*}. \tag{17}$$

$$\tag{18}$$

Proof. See the Appendix.

Corollary 1 *Information never aggregates, and both over- and under-investment happens with positive probability as N goes to infinity:*

$$\lim_{N\to\infty} \Pr(a = 0|G) = e^{-\lambda\tau}, \tag{19}$$

$$\lim_{N\to\infty} \Pr(a \neq 0|B) = 1 - e^{-\tau}. \tag{20}$$

Proposition 2 shows that the participation cut-off $\hat{s}_N$ increases with the number of investors. The reason is the winners curse: In a larger market, winning with the same signal is bad news because winning implies that all other investors are more pessimistic. Assumption 7 guarantees that equation (16) has a nonnegative solution. Equations (14) and (15) show that as long as the likelihood ratio λ at the top signal is finite, the number of investors who actually participate in a financing process converges to the Poisson distribution with parameter τ if the project is bad and the Poisson distribution with parameter $\lambda\tau$ if the project is good. This limited participation impedes the inference of project type. As a result, sizable investment mistakes persist for any market size.

Propositions 1 and 2 illustrate the critical importance of Assumption 2, which is our main departure from standard theories of information aggregation. Assumption 2 says that the decision about whether to start the project or not is non-trivial, because starting the project leads to losses if the project is bad. If Assumption 2 is violated and there is an action a such that the project is strictly positive NPV even in the bad state, Equation 11 implies that $\hat{s}_N = 0$ for any N. Hence, everyone participates, and a constrained efficient robust mechanism can achieve the first best. A larger market then always produces more information. For example, in standard auction theory where there is no action choice but an existing asset is sold, an ascending price auction reveals all information in the market. Of course, in that setting, information has no social value—once information has social value and a non-trivial decision has to me made, Proposition 1 shows that, paradoxically, the market can no longer aggregate information.

Propositions 1 and 2 show that the recoverable market information depends on three things: The size of the market, the signal distribution, and how "in the money" the project is as summarized by the difference between the prior π_0 and the break-even probability π^*. We will discuss the effect of market size and how it interacts with the signal distribution in detail in Section 3. In the asymptotic limit described in Proposition 2, the amount of available information is conveniently summarized in the expected number of participants $\lambda\tau$ when the project is good:

$$\lambda\tau = \frac{\lambda}{\lambda - 1}\left(\ln \lambda + \left[\ln \frac{\pi_0}{1-\pi_0} - \ln \frac{\pi^*}{1-\pi^*}\right]\right). \tag{21}$$

When this arrival rate is high, more market information is recovered and the posterior for the decision maker is more informative. In large markets, the effect of the signal distribution is summarized by the top likelihood ratio λ, and not surprisingly, more is learned when signals are more informative. In Section 4.2.1, we discuss implications of this when investors can affect λ by forming bidding clubs or coming together in a partnership firm.

The "moneyness" of the project is higher when the prior π_0 is high and the break-even probability π^* is low. A project that is more likely to be positive NPV will generate more participation, and therefore more information. This market outcome is often inefficient—projects where the extensive margin decision is less important, so that information is less socially useful, will generate the most information!

The moneyness depends on project characteristics through π^*, which in turn is lower the more real option value the project is expected to have after fundraising. We use this fact in Section 4 where we show that social surplus is increased if fundraising is done before public resolution of uncertainty.

Remarkably, the available equilibrium information does not depend on any other project parameter than the break-even rate π^*, which is determined solely by the payoffs to the unique optimal policy $a(\pi^*)$ for a marginal participant. Hence, any value from being able to pick other infra-marginal policies when $\pi(\mathbf{s}_{\geq \hat{s}_N}) > \pi^*$ has no effect on equilibrium information, despite the fact that more information is extra valuable when there is a richer policy set to pick from. This fact holds for any number of investors N, not only asymptotically, and is a negative informational externality imposed by the marginal participant on more optimistic investors.

When there is a lot of value from picking the right infra-marginal action, a government policy of subsidizing investors may therefore improve surplus by lowering the participation threshold. Thus, our model provides a new justification for government programs that give tax breaks or credit guarantees to direct investors in private companies. There is a cost to such subsidies, however—the marginal participant will finance socially wasteful negative NPV projects. If the action choice is binary, such a policy would destroy surplus, as infra-marginal information then has no value.

Before describing the detailed implications of the informational black hole for surplus and revenues, we show in the next section that a straightforward financing process closely resembling most real-world primary markets implements the constrained efficient investment policy.

2.1 Implementation with straight equity financing

We show that the constrained efficient investment policy can be implemented by selling straight equity to the highest bidder in a standard auction. An equity auction proceeds as follows. The entrepreneur first sets a target I of capital to be raised into the firm through a new equity issue, and a fraction $\beta \in [0,1]$ of her own shares to be sold for cash. This captures both total buyouts, where $\beta = 1$, and venture capital settings where the entrepreneur remains fully invested, where $\beta = 0$. The capital budget I is set so that *any* investment policy can be implemented, i.e., $V_a + I \geq 0$ for all $a \in A$.

Normalizing the number of shares of the firm before capital raising to one, the price per share (or "pre-money valuation") p at which equity is sold is set in a second- or ascending price auction:

1. Second-price auction: Bids $b_i(s_i)$ are submitted simultaneously, the highest bidder h wins, and $p = \max_{i \neq h} b_i$.
2. Ascending-price auction: Bidding starts at price $p = 0$, and the price is gradually increased until all but one bidder have dropped out.

If the entrepreneur gets any bids, the winner of the auction pays $I + \beta p$ for a fraction $\frac{I+\beta p}{I+p}$ of shares backed by the post-money cash flows $V_a + I$, so that the winner's pay off function is given by:

$$w_h = \frac{I + \beta p}{I + p} (V_a + I) - (I + \beta p) = \frac{I + \beta p}{I + p} (V_a - p) . \tag{22}$$

The entrepreneur keeps $(1-\beta)$ shares and gets βp in cash for her remaining shares. The winning investor and the entrepreneur then jointly decide on the investment policy, taking into account all the information learned from the bidding behavior of other investors.

If bids of participating investors are strictly increasing in signals, bids will perfectly reveal their signals. Since the investor and the entrepreneur are perfectly aligned ex post with payoffs that are linear in the surplus V_a, they will therefore agree to take the surplus maximizing action conditional on the information contained in the bids:

$$a(\mathbf{s}_{\geq \hat{\mathbf{s}}}) = \arg\max_a E\left(V_a | \mathbf{S}_{\geq \hat{s}} = \mathbf{s}_{\geq \hat{\mathbf{s}}}\right) . \tag{23}$$

If the participation threshold $\hat{s}$ is at the lower bound $\hat{s}_N$ in a robust equilibrium, the equity auction achieves the upper bound on efficiency. We show that this is the case in the following proposition:

Proposition 3 *There is a robust symmetric equilibrium in the second-price and ascending-price equity auction. Each format delivers the maximum possible social surplus. Investor i participates in the auction if and only if $S_i \geq \hat{s}_N$, where $\hat{s}_N$ is defined by equation (11).*

Proof. See the Appendix.

Because of perfect alignment, the equity auction is robust to ex post moral hazard and renegotiation. Neither the investor nor the entrepreneur has an incentive to distort any information before the investment policy choice, and it is immaterial whether the action a is observable or contractible. Straight equity financing is unique in this regard, as any other security structure will create conflicts of interest for some specification of possible action choices.

We conclude this section with a discussion of the requirement for an equilibrium to be robust. In general, there can be nonrobust equilibria with participation thresholds below $\hat{s}_N$. To see this, note that social surplus decreases in the participation cut-off $\hat{s}$. When an investor expects others to participate over a larger signal interval so that the informational black hole is smaller, he expects surplus to be larger because of the extra information, which justifies bidding higher and participating for lower signal realizations. Hence, the expectation of the size of an informational black hole can be self-fulfilling and lead to multiple equilibria. As an example, suppose that each investor participates if and only if his signal is above the cut-off $\underline{s}_N$ defined as the smallest value such that

$$\max_{a \neq 0} E(V_a | S_1 = S_2 = \ldots S_N = \underline{s}_N) \geq 0. \tag{24}$$

The cut-off $\underline{s}_N$ is defined such that the project just breaks even conditional on *all* investors having this signal. Therefore, when an investor with signal s just above $\underline{s}_N$ wins the right to finance the project he only starts the project if all $N-1$ competitors participate and have their signals between $\underline{s}_N$ and s. As s gets closer to $\underline{s}_N$ the probability of such an event goes to zero, so that the marginal participating investor never makes any profit. Hence, this marginal investor would not participate in the fundrasing mechanism in the presence of an arbitrarily small participation costs. By not participating, he makes it impossible to break even for the investor with a signal just above him. As a result, the whole participation process unravels and stops only when the participation cut-off reaches $\hat{s}_N$. Thus, all equilibria with cut-offs below $\hat{s}_N$ are fragile.[6]

3 Smaller versus larger markets

We now study how surplus and revenue change with the number of investors N, which we have loosely referred to as market size. When the issuing firm does not actively restrict participation,

[6] In the working paper version of this paper we showed that these equilibria also no longer exist when participation costs are zero, but bids are made in arbitrarily small increments.

we think of N as the number of all potential investors that may conceivably be interested in financing the firm. For example, for a start-up this could be the number of VC firms and angel investors active within the region and industry, and for a mature firm it could be the set of potential financial and strategic investors. If the issuing firm can restrict participation, for example by engaging in a proprietary transaction or a targeted auction, N is the number of invited investors.

To the extent that the set of potential investors N_j relevant for a particular firm j comoves with the overall set of investors in primary capital markets, our model also has aggregate implications. We think this is a realistic assumption; for example, entry of investors into both private equity and venture capital tend to comove strongly across all subsectors, and indeed with the overall size of financial markets.

As we show, most of our results will be driven by the extensive margin decision between the status quo action $a = 0$ and any other choice $a \neq 0$. We will therefore focus our main analysis on the case of only two actions $a = 0$ and $a = 1$, and show that the results are robust to having more than two action choices in Section 5.1. With binary actions, the investment policy is completely determined by the realization of the highest signal. We assume that a financing mechanism ensures participation at the lowest possible participation threshold $\hat{s}_N$. Thus, the project is started whenever the highest signal is above $\hat{s}_N$. Therefore, social surplus is equal to

$$\pi_0 \Pr(\max_i S_i \geq \hat{s}_N|G)E(V_1|G) + (1-\pi_0)\Pr(\max_i S_i \geq \hat{s}_N|B)E(V_1|B). \tag{25}$$

In our motivating example we showed that social surplus is maximized with just one investor. In this section we show that this case is not an isolated example. Because the informational black hole grows with the number of investors, the investment mistakes can be increasing in the number of investors as well. As a result, markets with a large number of investors can lead to strictly worse social surplus and revenues for the entrepreneur. Below we provide necessary and sufficient conditions on the distribution functions F_B and F_G for social surplus to be increasing or decreasing with the number of investors.

Adding an investor to the market changes the set of started projects, and hence social surplus, in two ways. First, the participation threshold increases due to the stronger winner's curse in a larger market. This implies that some projects that were previously marginally approved are now dropped if the signal of the new investor is not above the participation threshold. However, because these marginal projects are close to zero NPV due to investors' participation optimization, the effect on social surplus from dropping them is small—as we show in the proof of Proposition 4, the effect vanishes when we add investors in a continuous way and use the envelope theorem. In our motivating example, for example, when an extra VC is invited, the original VC withdraws from offering financing after a high signal with probability $(1-\mu)$. When the extra VC does not participate, this project is dropped. But equilibrium participation is set such that this project is zero NPV anyway—the positive information in the original VC's high signal is exactly offset by the information that the new VC does not participate.

The second effect is that some projects that would previously not have received financing now get started if the added investor has a sufficiently optimistic signal. In our motivating example, a project on which a single invited VC has a low signal does not get financed. This is a bad project for sure, but with an extra VC it gets financed with probability $\Pr(H|B)\mu > 0$, so surplus goes down. The extra high signal cannot compensate for the very bad information of the original VC.

For the general case, the effect of adding an extra investor can go either way depending on how informative an extra high signal is. Suppose that a project is not financed in a market with N investors, which happens when no one participates so that $\max_{i\leq N} S_i < \hat{s}_N$. If another investor is added to the market, such a project will be financed if $S_{N+1} > \hat{s}_{N+1}$. For this change to increase surplus, the project must be positive NPV when the extra investor has the highest possible signal $S_{N+1} = 1$:

$$\Pr(G|\max_{i\leq N} S_i \leq \hat{s}_N, S_{N+1} = 1) \geq \pi^*, \tag{26}$$

where π^* is the break-even probability defined in Lemma 1. To see under what conditions Equation 26 holds, we compare this event with the break-even condition for a marginal participant in a market with N investors, which from the definition of the threshold $\hat{s}_N$ is given by:[7]

$$\Pr(G|\max_{i\leq N-1} S_i \leq \hat{s}_N, S_N = \hat{s}_N) = \pi^*, \tag{27}$$

i.e., when there is one marginal participant with signal $\hat{s}_N$, and all other $N-1$ investors have signals below the threshold, the project just breaks even. This event differs from the event in Equation 26 by having one bidder at the threshold rather than a combination of one bidder below the threshold and one bidder with a top signal. The signal of an extra investor can therefore never make a previously rejected project positive NPV if

$$\Pr(G|S_{N+1} = 1, S_N \leq \hat{s}_N) < \Pr(G|S_N = \hat{s}_N), \tag{28}$$

which, using Bayes' law and rewriting, can be written as

$$\frac{f_G(1)}{f_B(1)} \leq \frac{f_G(\hat{s}_N)}{F_G(\hat{s}_N)} \Big/ \frac{f_B(\hat{s}_N)}{F_B(\hat{s}_N)}. \tag{29}$$

The right-hand side of Equation 29 is the likelihood ratio at the top of the informational black hole threshold, conditional on still being in the informational black hole. It is a measure of how efficiently a market of size N screens projects. It is large when the normal likelihood ratio f_G/f_B is high at the break-even threshold $\hat{s}_N$ relative to signals below, so that there is a big difference in quality between accepted and rejected projects. As the size of the market grows and $\hat{s}_N$ goes to one, this conditional likelihood ratio goes to $f_G(1)/f_B(1) = \lambda$. Hence, if the conditional likelihood ratio is decreasing, increasing market size leads to less informative screening and lower surplus: [8]

[7] This assumes the participation threshold has an interior solution—if $\hat{s}_N = 0$ it is always efficient to increase the market size. For N sufficiently large, there is always an interior solution.

[8] The argument here has skirted over the fact that adding an extra investor is a discrete change and so changes

Proposition 4 *If* $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ *is an increasing function on* $[\hat{s}_1, 1]$ *then social surplus (25) increases with the number of investors. If there exists* N *such that* $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ *is a decreasing function on* $[\hat{s}_N, 1]$ *then maximal social surplus is achieved with no more than* N *investors.*

Proof: See the Appendix.

The conditional likelihood ratio $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ is the ratio of the likelihood ratio $f_G(s)/f_B(s)$ and the probability ratio $F_G(s)/F_B(s)$ of an original signal. Both ratios increase with s because of MLPR. Therefore, the conditional likelihood ratio decreases with s if and only if the likelihood ratio of an original signal is sufficiently flat at the top signals. The most extreme example is the case of discrete signals where the likelihood ratio is constant over some interval $[a, 1]$. In this case

$$\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)} = \lambda \frac{F_B(s)}{F_G(s)}$$

for some λ, which decreases with s.

We next consider entrepreneurial revenues as a function of market size. If the entrepreneur has the power to pick the number of investors, he will do so in order to maximize revenues rather than surplus. The private optimum may differ from the social optimum if the entrepreneur captures only part of the surplus. As in standard common-value auction, the extra competition from added investors tends to drive their share of surplus down, and for competitive mechanisms the entrepreneur captures all the surplus as the number of investor grows without bound. Hence, if surplus increases with N, there is no conflict between the private and social optimum—the entrepreneur will prefer the maximal number of investors.

The non-trivial case is when surplus decreases with N. Will the entrepreneur find it optimal to restrict the number of investors even though this may entail surrendering a higher fraction of the surplus to investors? Our answer is a qualified "Yes". The next proposition gives a sufficient condition for when this is the case.

Proposition 5 *Suppose financing is done by using straight equity in any of the standard format auction. For any* $s^* < 1$ *there exist* $\delta > 0$ *and* $N^* < \infty$ *such that for any* f_G *and* f_B *satisfying*

$$\frac{f_G(1)}{f_B(1)} - \frac{f_G(s^*)}{f_B(s^*)} < \delta$$

entrepreneur's revenue strictly decreases with N *for* $N > N^*$.

Proof: See the Appendix.

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the participation threshold discretely rather than continuously. Adding investors in a continuous way corresponds to adding an "informationally small" signal to the market with distribution function $F_\theta(s)^\varepsilon$ for vanishingly small ε. This signal has likelihood ratio $(f_G(s)/F_G(s))/(f_B(s)/F_B(s))$, so that the change in surplus when the added signal is above the participation threshold and leads to extra investment is negative if

$$\frac{f_G(\hat{s}_N)}{F_G(\hat{s}_N)}/\frac{f_B(\hat{s}_N)}{F_B(\hat{s}_N)} \geq \frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}, \tag{30}$$

which holds if the conditional likelihood ratio decreases above the cut-off.

To understand this result, recall our motivating example. There, investors with high signals are equally informed, and therefore are unable to earn any profits beyond their vanishing participation cost. Hence, the entrepreneur captures all the surplus, so revenues go down in tandem with surplus. With a general signal distribution, participating investors who are not marginal capture some informational rents. But if the likelihood ratio at the top of the signal distribution is relatively flat, participating investors in large markets are informationally close to each other. Therefore, these investors capture little informational rent. As a result, increasing N beyond a certain level has little effect on the split of revenues but a large negative effect on surplus.

Figure 2 Panel A plots both social surplus and profit for a modified version of our motivating example, where signals are continuous and satisfy strict MLRP: F_B is a uniform $[0, 1]$ distribution, and F_G is the normal distribution with mean 1 and standard deviation 1, truncated to $[0, 1]$. Competition is assumed take place in a second-price equity auction. We can see that social surplus is maximized at a market size of two. In contrast, the entrepreneur's revenues are maximized at a substantially larger market size of 13. The entrepreneur prefers a larger market size than what maximizes social surplus because increased competition between investors reduces their share of the surplus. One can construct other examples where the entrepreneur's revenue is maximized for any given number of investors by appropriately choosing the signal distribution. In all these examples, the entrepreneur is better off if he restricts the number of investors who participate in the fundraising process.

Our result that small markets may be optimal for firms provides a new explanation for the phenomenon of "proprietary transactions" in venture capital and private equity, or "targeted auctions" in the sale of firms in which only a select set of acquirers are invited to submit bids. The results we derive for the case of stochastic bidders (see Section 5.4) identifies a further value of small markets—as we show, uncertainty about the number of bidders often leads to less efficient outcomes than when the number of bidders is known, and a targeted auction where the number of participants is made public removes this uncertainty.

3.1 Can financial markets be too big?

In the previous section we established that small markets may be preferable both from the entrepreneur's and from a social surplus perspective. In this section we show that the equilibrium size of the market can be too large relative to both the social and the entrepreneurial optimum, and can be Pareto inferior relative to a market with one less investor.

If the entrepreneur can commit to seek financing from a restricted set of investors, the market can obviously never be larger than what is optimal for the entrepreneur. However, restricting the set of potential investors may be difficult in practice because it is ex post optimal for the entrepreneur to consider any offer he receives, even if the offer is unsolicited. In this section we therefore assume no commitment so that investors can enter any auction.

So far, we have assumed that investors observe signals for free to make our results on the failure of information aggregation in large markets as striking as possible. In order to have a non-trivial equilibrium market size, we now assume that investors face some costs of gathering information. Assume that each potential investors i has a cost c_i of gathering information about the project, and that c_i is strictly increasing. We focus on the case where $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ is a decreasing function at $s = 1$ so that social surplus (gross of investor costs) is maximized at a finite market size. The socially optimal market size net of costs is then even smaller.

Proposition 6 *Suppose that* $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ *is a decreasing function at* $s = 1$*. Then, there is* $c > 0$ *such that if sufficiently many investors have costs of gathering information below* c*, the equilibrium size of the market is larger than the socially optimal size. Lowering information gathering costs proportionally for all investors can lead to a decrease in both net and gross of fees social surplus.*

Proof: See the Appendix.

The proposition shows that there is no reason to believe that markets will become more efficient as information technology improves. This is in contrast to the predictions of Samuelson (1985) and Levin and Smith (1994) who study information costs in an otherwise standard auction theory setting. In both papers, the optimal size of the market goes to infinity as costs go to zero. Proposition 6 shows that there can be too much entry in equilibrium relative to the social optimum.

Let us continue with our motivating example, modified to have signals from the truncated normal distribution when the project is good. Figure 2 Panel B shows expected gross profits to investors from participating in a second-price equity auction as a function of market size, as well as a particular specification for the cost c_i of information gathering for each investor. In equilibrium, investors will enter as long as expected profits cover their cost, so that for the specific costs drawn in the figure the first five investors will enter in equilibrium with the 15^{th} investor indifferent between entering and staying out. Recall that the entrepreneur's revenue is maximized at the market size of 13. Hence, the equilibrium market size is larger than both the social optimum and the entrepreneur's optimum.

Now suppose that every investor's cost was just slightly larger. This would be the case if, for example, tax rates on venture capitalist profits are increased slightly. The equilibrium market size would drop to 14, which would constitute a Pareto improvement. Participating investors would make higher profits because of both reduced competition and more efficient investment decisions. The entrepreneur's revenues would increase because the increased surplus from more efficient investment outweighs the loss from reduced competition. Finally, the investor who drops out of the market is no worse off since he was just breaking even before.

We have restricted the analysis in this section to the extensive-margin binary action case. For the case with an arbitrary action space, the informational black hole and the extensive margin decision of whether to start the project or not remains exactly the same–the project is started if and only if anyone participates. For the binary action case, no other information affects surplus, which simplifies the analysis. When the firm can choose from different investment policies conditional on starting the project, the infra-marginal information of participating bidders has value, so that surplus depends on the number and distribution of signals above the threshold.

This is reflected in the fact that the surplus $v(\pi)$ under the optimal investment policy is a convex function of π for $\pi \geq \pi^*$ in the general case, while it is linear in the case of binary action. Provided that $v(\pi)$ is not too convex our result that small markets dominate large markets continue to hold; see Section 5.1 for an example with scalable investments.

4 Remedies for the informational black hole

The source of inefficiency in our model is the effect the winner's curse has on the participation of pessimistic investors, an effect that becomes stronger as the market grows larger. In this section we discuss a number of other market characteristics that affect the size of the informational black hole, and possible remedies that can help reduce it. First, we show that it may be beneficial to raise capital before important information is learnt in order to increase the option value embedded in the project. Second, we show that allowing a larger set of investors to co-finance the project helps reduce the informational black hole but does not eliminate it. Third, we show that if the monotonicity condition is relaxed, a carefully designed shorting market can in theory eliminate the informational black hole, but its elaborate construction points to its fragility. Finally, we show that informational black holes continue to exist for mature firms with assets in place.

4.1 Choosing when to finance and the linkage principle

The size of the informational black hole depends on public information just before fundraising, as summarized by the common prior π_0, as well as any extra public information after fundraising that can influence the investment policy choice.[9] So far, we have taken the informational environment around the time of fundraising as exogenous. However, the entrepreneur can often influence this environment. If there is some public resolution of uncertainty over time, the entrepreneur can choose whether to start fundraising before or after resolution of uncertainty. Alternatively, if the entrepreneur has private information that can be credibly communicated, she can choose how much and when to release this information. We now show the implications of our model for these types of choices.

Suppose that there is some exogenous signal X that helps predict the value of the project. This could be a signal about demand conditions for the products the project is meant to create, or in general, any relevant information the entrepreneur might have that can be credibly communicated to the investors. We assume that X satisfies the following properties:

Assumption 8

(i) Investors' signals S_i are symmetric and conditionally independent given X and the project type θ.

[9] Any public information that becomes available after fundraising is implicitly captured in the policy choice set A, which can contain flexible strategies that react to information—see our example in Section 5.2.

(ii) The conditional density $\Pr(S_i = s|X = x, \theta) \equiv f_\theta(s|x)$ *exists and for any* x *satisfies the strict MLRP:*

$$\forall s > s', \quad \frac{f_G(s|x)}{f_B(s|x)} > \frac{f_G(s'|x)}{f_B(s'|x)}.$$

Assumption 8 guarantees that if fundraising is done after the public realization of X, the participation policy is still of threshold type. Any signal X which is independent of investor signals conditional on project type trivially satisfies Assumption 8.

Note that it is always optimal to make all possible information available at the time of the investment policy choice, since firm value is maximized when the policy is more informed. Our question is whether it is better to raise funds before or after the information is released.

For a standard setting in which an existing asset is sold, the linkage principle of Milgrom and Weber (1982) suggests that it is better to raise funds after all value-relevant information is realized in order to lower the informational asymmetry among investors. However, in our setting there is a countervailing effect. Any signal which is revealed after the funds are raised but before investments are made adds extra option value to the project. Proposition 7 shows that this option value prompts investors with lower signals to participate in the hope that the project turns out to be positive NPV. As a result, the participation cut-off when the signal is released after the funds are raised is always lower compared to that when the signal is released before the fundraising process. As a consequence, social surplus is higher if funds are raised before the signal is released:

Proposition 7 *Suppose there is a signal* X *that satisfies Assumption 8. The maximal surplus is always higher if* X *is released after funds are raised but before the investment policy choice is made, rather than releasing information before fundraising.*

Proof: See the Appendix.

Social surplus depends only on the participation cut-off, and the participation cut-off depends only on whether there exists some scenario in which a marginal investor can break even—not on how likely that scenario is. Hence, the participation threshold when funds are raised before the release of information is the minimal threshold that can occur if fundraising is done after release of information.

Of course, the entrepreneur may care more about her revenues than social surplus. Once investors have made their participation decision, our setting is similar to the standard setting. Any information released to participants during bidding will tend to lower informational rents and increase revenues. Hence, our model suggests that information should be released in stages. In the first stage, to increase participation and get as much information as possible from investors, the entrepreneur can reveal only some general information. Then, in the second stage, after serious investors are identified and before the bidding starts, she can reveal full information to make bidding more competitive. This multi-stage process is indeed the typical procedure in private equity and M&A transactions (see e.g., Zeisberger, Prahl, and White (2017)).

4.2 Co-financing

In the previous sections we assumed that only one investor ends up with a stake in the project. In this section we allow for the possibility that $K > 1$ investors can co-finance the project. Allowing for more investors to receive an allocation weakens the winner's curse and hence encourages more investors to submit non-zero bids, which has a positive effect on efficiency.

We establish three results. First, we show that our results on the failure of information aggregation are robust to having multiple investors in the capital structure, and that a uniform-price equity auction is constrained efficient. In a K-unit uniform-price equity auction, investors bid their pre-money valuation. If less than K investors participate, fundraising fails. Otherwise, the price p is the $K+1^{st}$ highest bid (or zero if there are only K participants). The K highest bidders share the investment costs and get the same number of shares, so that their payoff is

$$w_{i,a} = \frac{1}{K}\frac{I+\beta p}{I+p}\left(V_a - p\right).$$

Second, we show that if the entrepreneur can commit to ration allocations so that the number of investors who receive an allocation grows proportionately with N, a uniform-price equity auction delivers the first-best surplus in the limit. Finally, in Section 4.2.1, we study the case where the competitiveness assumption is violated because investors on the buy side can collude through the formation of bidding clubs or syndicates. We show that co-financing through collusion can, maybe surprisingly, increase surplus and revenues.

Assume first that the number of possible co-investors the firm can have is bounded by a constant $K > 1$. This could be because K investors have enough aggregate resources to finance the firm and the competitiveness assumption prevents a marginal participant to make profits when there are K other participants. Alternatively, as is typical in venture capital or private equity situations, there may be an upper limit on the number of investors a firm can have in its capital structure to avoid excess costs of coordinating the exercise of control rights.

We will restrict attention to financing mechanisms in which the decision to start the project is ex post efficient based on the information of all investors who receive an allocation:

Assumption 9 *A financing mechanism is ex post efficient if given information $\boldsymbol{\omega}$ available for the investment decision, with $S_i \in \boldsymbol{\omega}$ if $w_i > 0$, the project is only started when it is efficient to do so:*

$$a = 0 \text{ if } \arg\max_a E\left(V_a|\boldsymbol{\omega}\right) = 0.$$

This restriction simplifies our proof by ruling out mechanisms in which investors hold different securities, and a marginal investor holds a stake that makes it profitable to hide negative information so that negative net present value projects get started. The restriction is sufficient but not necessary for our results.

Denote the order statistics of the N signals received by investors by $Y_{1,N}, ..., Y_{N,N}$ so that $Y_{1,N}$ represents the highest signal, $Y_{2,N}$ represents the second-highest signal, et cetera. We have:

Proposition 8 *In any robust ex post efficient equilibrium where no more than a finite number K investors can get an allocation, the participation threshold is no smaller than the smallest value $\hat{s}_{K,N}$ such that*

$$\max_{a \neq 0} E(V_a | Y_{K,N} = \hat{s}_{K,N}, Y_{K-1,N} ... = Y_{1,N} = 1) \geq 0. \tag{31}$$

Information is not aggregated and the limiting social surplus as $N \to \infty$ is strictly lower than the first-best social surplus. There is a robust symmetric equilibrium in the K-unit equity auction that delivers maximum possible social surplus given the cut-off $\hat{s}_{K,N}$.

Proof. See the Appendix.

Next, we show that our result on the behavior of surplus in large markets also generalizes to the case of multiple security holders:

Proposition 9 *Suppose that the project choice is binary. For any $s^* < 1$ there exist $\delta > 0$ and $N^* < \infty$ such that for any f_G and f_B satisfying*

$$\frac{f_G(1)}{f_B(1)} - \frac{f_G(s^*)}{f_B(s^*)} < \delta$$

social surplus and the entrepreneur's revenue in the K-unit equity auction strictly decrease with N for $N > N^*$.

Proof: See the Appendix.

These results show that when the feasible number of security holders has an upper bound, whether due to competitiveness or some other reason, the inefficiency created by the informational black hole is unavoidable for any market size, and may grow larger with the market. Nevertheless, since the winner's curse and hence the informational black hole decrease with K, efficiency is improved when allocations are more dispersed. We next show that if the entrepreneur can commit ex ante to ration the allocation so that a non-vanishing fraction K/N of investors receive an allocation as N grows large, the K-unit auction fully aggregates information as $N \to \infty$.

Proposition 10 *Suppose there exists $\lim_{N\to\infty} K/N = \alpha$, $\alpha \in (0,1)$. Then the K-unit auction delivers the first-best social surplus in the limit.*

Proof: See the Appendix.

These results provide one rationale for crowd-funding, in which start-ups seek financing on a platform that looks very much like a multi-unit auction, and may also help explain rationing in IPO allocations. Rationing in IPOs, where there is excess demand at the issue price and some investors do not get their desired allocation even if they are willing to pay a higher price, have been extensively documented and studied in the academic literature (see e.g., Cornelli and Goldreich (2001) and Ritter and Welch (2002)). Such rationing also occurs more generally in fundraising processes—top PE- and VC funds are typically oversubscribed at the issue price. The puzzle is why issuers leave money on the table instead of increasing prices until the market clears. Several explanations for this phenomenon have been proposed (Rock (1986), Benveniste and Spindt (1989), Parlour and Rajan (2005)). We propose a new explanation—rationing increases participation, which improves information aggregation and productive efficiency.

Note that rationing requires commitment power on the side of the entrepreneur, as it is typically ex post optimal to increase prices—hence, this type of rationing violates our competitiveness assumption. A prediction of our model is therefore that rationing and dispersion of allocations should be more commonly observed in settings where the issuer has more commitment power. This prediction is consistent with the observation that oversubscribed issues are more commonly observed for repeat issuers in fundraising markets, such as established, reputable venture capital and private equity firms. For one-time issuers such as entrepreneurs in startups or firms going public, who have little commitment power on their own, our results suggests a new rationale for the use of intermediaries such as underwriters and crowdfunding platforms. These intermediaries can substitute for the lack of commitment power by being repeat players themselves, and by having payoffs that are less strongly linked to the issue price.

4.2.1 Co-financing through buy-side collusion: syndicates and club bids

Another way to increase participation is through buy-side collusion, in which investors form consortia and submit joint bids. Joint bids can facilitate profitable participation of less optimistic investor, if other members of the consortium have sufficiently positive information to justify a competitive bid. A full analysis of club bidding is challenging for several reasons. First, club formation is an endogenous process which may result in clubs of different size, and therefore requires analysis of financing mechanisms with asymmetric bidders. Second, there may be incentive problems within the club that prevent full sharing of information among club members.

Dealing with these issues is beyond the scope of our paper and we therefore consider a simplified setting where we assume clubs are of equal and exogenously given size, the number of investors is large, and information is freely shared within the club. We assume that there are $N \times M$ investors in the market. We will contrast two market settings. In the first, there is no collusion among investors and everyone submits bids independently. In the second, investors are randomly allocated to N symmetric clubs each consisting of M investors, whereupon each club submits a joint bid in one of the standard auction formats.

In the standard setting, where the asset for sale is already in place, surplus is always the same so collusion among investors tends to lower seller revenues (see e.g, Axelson (2008)). There are two countervailing forces favoring club bidding in our setting. First, club bidding reduces the effective number of bidders, which is beneficial when markets are inefficiently large, even if the club would submit a bid based on the signal of only one member. Second, signals become more informative whenever there is some information sharing within the club. When these effects outweigh the reduced competition, the entrepreneur gains. Proposition 11 shows that for large N, the information sharing channel always dominates the competition effect even when signal structure favors having as large markets as possible.

Proposition 11 *For large enough N, the entrepreneur's revenue in any of the standard auction formats among N clubs consisting of M investors is higher than social surplus with $N \times M$ individual investors.*

Proof: See the Appendix.

Proposition 11 provides a benign rationale for the prevalent use of club bids in private equity and the use of syndicates in venture capital that has come under scrutiny by competition authorities.[10] Our theory predicts that some arrangement for sharing/selling of signals via co-ownership should develop when feasible. In general, the extent of co-investment/syndication depends on the trade-off between increased informational efficiency and the extra costs of adding more investors to the capital structure. In VC markets, practitioners view the cost of adding members to a syndicate as quite large. The main costs come from the difficulty of coordinating the exercise of control rights, free-riding problems in the governance of the firm, and the cost to the lead VC of sharing some of the surplus with other investors. As a result, when syndicates do exist, they tend to be of very limited size relative to the set of potential investors. In comparison, bank syndicates in loan markets tend to be of larger size, possibly reflecting the more limited role banks play in the governance of the firm.

We can also interpret a club as the boundary of a firm. For example, a VC partnership is a club in which expert investors share linearly in the profits of deals and have a long term relationship, so that incentive compatible information sharing is easier than through market transactions. Proposition 11 then says that there is social value to forming such partnerships, at least if the number of expert investors in the market is large enough.

4.3 Relaxing monotonicity: Shorting markets

The driving force behind our results is the combined effect of competitiveness and monotonicity on marginal participants. A marginal participant needs to break even in the state where no competitors are trying to finance the firm, which is not only the state in which expected surplus conditional on winning is the lowest, but also the state in which no other market information is learned. The fact that the marginal participant must be happy to always start the project based on this minimal information is what creates sizable inefficiencies regardless of market size.

We now show that if the monotonicity condition is relaxed, a carefully designed shorting market can improve efficiency by increasing the information learned by a winning marginal investor. Our goal in this section is not to describe a realistic solution to the information problem, but rather to illustrate the driving forces behind our results—in fact, the elaborate construction needed for a shorting market to function points to its fragility. Contracts, decision rights, and information sharing arrangements need to be carefully balanced, and the outcome is not ex post rational—there are no ex post gains from trade between a holder of a shorting contract and the firm and its investors, so one side would always prefer to renege. Conflicts of interests about the preferred investment policy also makes the market outcome vulnerable to manipulation.

[10] See Bailey (2007) for further discussion.

The way we construct the shorting market is as follows. The entrepreneur runs two simultaneous second-price auction markets, the financing market and the shorting market. In the financing market, she runs an equity auction to raise $I + c_1$ where I is enough to finance the project and $c_1 > 0$ are extra proceeds used to subsidize the shorting market. In the shorting market, she runs an auction of a shorting derivative contract $\{c_1, c_2\}$ that pays $c_1 > 0$ if the firm receives financing but does not start the project, and loses $c_2 > 0$ if the firm receives financing and starts the project. If the firm is financed, the entrepreneur pockets the sales proceeds from the shorting market. If there are bids for the shorting contract but the firm does not manage to raise financing, the contract is canceled and no bids are paid. The investor who financed the firm decides on the investment policy after getting to observe all bids in both markets *except* the winning bid in the shorting market, and the shorting contract is settled.

There are a number of critical features of this construction that point to the difficulty of creating such a market. First, the shorting market needs to be subsidized—there are no gains from trade between third parties taking opposite positions in the shorting market. Since the entrepreneur has no resources of her own, the subsidy must come from the participants in the regular financing market. Second, since the key economic role of the shorting market is to produce information that helps a marginal investor avoid bad projects, the contract must pay off when the project is not started. Hence, it cannot be a standard derivative or short position that is contingent on the value of an existing asset. As in Edmans, Goldstein, Jiang (2017), because such a contract would not pay off unless the project is started, it would not attract sufficient participation. Third, to prevent conflicts of interest from distorting the investment decision, the agent taking the decision should have no stake on either side of the shorting market. In our construction, the entrepreneur acts as a budget breaker that takes the opposite side in the shorting contract, and the winning investor has all the decision rights.

We show that every participation threshold $\hat{s} \in [\underline{\hat{s}}_N, \hat{s}_N]$ in the financing market can be supported by constructing an appropriate shorting contract, where $\underline{\hat{s}}_N$ is defined in Equation 24.

For ease of exposition, we assume that $f_G(0) > 0$ and N is sufficiently large so that even if two investors have the lowest possible signal, the project can still break even if other investor signals are sufficiently optimistic.

Assumption 10 *$f_G(0) > 0$ and N is large enough so that*

$$\text{Max}_{a \neq 0} E(V_a | Y_{1,N} = ... = Y_{N,N-2} = 1, Y_{N,N-1} = Y_{N,N} = 0) > 0.$$

We have the following result:

Proposition 12 *For any $\hat{s} \in [\underline{\hat{s}}_N, \hat{s}_N)$, define $\underline{s}(\hat{s})$ as the solution to*

$$\text{Max}_{a \neq 0} E(V_a | Y_{1,N} = \hat{s}, Y_{N,N} = \underline{s}(\hat{s})) = 0,$$

where $0 < \underline{s}(\hat{s}) \leq \underline{\hat{s}}_N \leq \overline{s}$ and $\underline{s}(\hat{s})$ is strictly decreasing with $\underline{s}(\underline{\hat{s}}_N) = \underline{\hat{s}}_N$.

There exists a symmetric robust equilibrium with a shorting contract $\{c_1(\hat{s}), c_2(\hat{s})\}$ *such that all investors with signals in* $(\hat{s}, 1]$ *participate in the financing market with strictly increasing bids, and all investors with signals in* $[0, \underline{s}(\hat{s}))$ *participate in the shorting market with strictly decreasing bids. Surplus and entrepreneurial revenues are strictly higher than without a shorting market, and increase with higher participation (lower* $\hat{s}$*).*

For $\hat{s} = \underline{\hat{s}}_N$*, all investors participate,* $c_1 = c_2 = 0$*, and* $a(\mathbf{S}) = \text{Max}_a E(V_a | \text{Y}_{1,N} ..., Y_{N,N-1})$*, so that all signals except the lowest are used efficiently.*

Proof. See the Appendix.

The last part of the proposition may seem surprising—when the monotonicity condition is relaxed, there exists a robust equilibrium with full participation where the monotonicity condition holds! The reason for this is that there is no "close" equilibrium with a small participation cost where the monotonicity condition holds, but there is such a close equilibrium with a non-monotonic shorting contract.

4.4 Relaxing limited liability: Assets in place

The limited liability constraint 4 stems from the assumption that the entrepreneur cannot pledge any assets other than the incremental value of the new project when entering into a financing contract. Although this is a natural assumption for start-up firms, where assets consists mainly of human capital and the new project idea, it is less appropriate for more mature firms with substantial assets in place. We now show that when the firm does have assets in place, our results continue to hold if we introduce the realistic assumption that the firm cannot commit to a financing contract that, based on the information learned during fund raising, makes firm insiders worse off.

We introduce assets in place by assuming that the firm value V_0 when the project is abandoned is positive and correlated with the type of the project, so that $V_0 \geq 0$ and $E(V_0|G) > E(V_0|B) > 0$. For policy $a \neq 0$, the firm now has value $V_a + V_0$. We assume that feasible contracts can now be contingent on firm cash flows $V_a + V_0$, and have to satisfy the extended version of the limited liability constraint:

Assumption 11 *Limited Liability:* $V_a + V_0 - w_a \geq 0$*,*

and

Assumption 12 *Ex post participation constraint of the entrepreneur:*

$$E\left(V_a|\boldsymbol{\omega}\right) \geq E\left(w_a|\boldsymbol{\omega}\right). \tag{32}$$

Recall that $\boldsymbol{\omega}$ is information learned during a financing process. We assume that the minimal information the entrepreneur can learn is the contract w_a and the number of participating investors κ_N. Assumption 12 states that the entrepreneur cannot commit not to back out of a contract that makes her worse off based on the information learned in the fundraising process. This lack of commitment power is a reasonable assumption in most unstructured financing environments involving start-ups and small businesses, and is also in line with the explicit fiduciary duty of the board of directors in larger companies to look out for the best interest of shareholders when voting on corporate decisions.

The following result shows that having assets in place backing financing contracts does not help in alleviating investment inefficiencies when firms cannot commit ex ante:

Proposition 13 *With assets in place, the maximal surplus achievable as $N \to \infty$ in a robust symmetric equilibrium satisfying the ex post participation constraint is no higher than without assets in place. Any fundraising equilibrium in which the proposed action a is efficient given the information of participating investors and in which the entrepreneur learns the proposed action has a participation threshold no smaller than $\hat{s}_N$.*

Proof. See the Appendix.

5 Extensions and robustness

The main goal of this section is to demonstrate the general robustness of our results. We relax some of the assumptions made in the main text and show that our results continue to hold. We first show that our base setup covers the case of scalable investments.

5.1 Scalable investments

Suppose that the project's production function is state-dependent and equal to

$$k_\theta \ln(1+I) - I,$$

where $I \geq 0$ is an investment, and $k_G > 1 > k_B$. Taking the first-order conditions it is straightforward to see that the investment is positive if only if

$$\pi k_G + (1-\pi)k_B > 1.$$

Hence, the critical value π^* belows which the project should be abandoned is

$$\pi^* = (1-k_B)/(k_G-k_B) > 0.$$

Figure 3 plots social surplus as a function of the number of investors when $\pi_0 = 1/2$, and signals are discrete as in our motivating example. We can see that social surplus is maximized with just two investors.

5.2 Real options

We next show that our model incorporates situations where the firm has real options available after fundraising. Consider a typical Dixit-Pindyck type model (Dixit and Pindyck (1994)).

Suppose that right after the fundraising process, investors can observe news about the project type. The news process evolves according to

$$dX_t = \mu_\theta dt + \sigma dB_t,$$

where B_t is a standard Brownian motion, $\mu_\theta = \mu_G$ if the project is good, and $\mu_\theta = \mu_B$ otherwise. At each time t, the entire history of news $\{X_s\}_{0 \le s \le t}$ is observable. The parameters μ_G, μ_B and σ are common knowledge. Without loss of generality, we set $\mu_G - \mu_B \ge 0$. Define the signal-to-noise ratio $\varphi = (\mu_G - \mu_B)/\sigma$. When $\varphi = 0$, the news is completely uninformative. Larger values of φ imply more informative news. In what follows, we assume $\varphi > 0$.

At each time t, the winner of the fundraising process faces the following decision tree. He can either start the project, postpone it, incur the cost $c > 0$ per unit of time and observe the news, or completely abandon the project. We have the following standard result:

Lemma 2 *There exist probabilities $\pi^* > 0$ and $\pi^{**} > \pi^*$. For $\pi < \pi^*$ the project is abandoned, for $\pi \ge \pi^{**}$ the project is started immediately, and for $\pi^* \le \pi \le \pi^{**}$ the project is postponed.*

Proof. See the Appendix.

The action set A for this setting can be specified as a set of barrier pairs $\{\underline{X}(\boldsymbol{\omega}), \overline{X}(\boldsymbol{\omega})\}$ such that the project is abandoned if $X_t \le \underline{X}(\boldsymbol{\omega})$, started if $X_t \ge \overline{X}(\boldsymbol{\omega})$, and postponed otherwise. As a numerical example, consider a simple case where a bad project, if started, has negative NPV of $-1/2$, and a good project when started has positive NPV of $1/2$. Assume the following parameter values: $(\mu_G - \mu_B) = 10\%$, $\sigma = 20\%$. The table below shows values π^* and π^{**} for different values of cost c.

c	π^*	π^{**}
0.025	0.25	0.75
0.05	0.35	0.65
0.1	0.42	0.58
0.2	0.46	0.54

5.3 Private values

One could also imagine a more general model in which along with a common value component there is a private value one. For example, suppose that the NPV of the project for investor i is $V_a + \alpha S_i$, where as before, V_a is the common value component, and αS_i is an extra private value component, which is perfectly correlated with investor's signal S_i. Provided that there exists a $\pi^* > 0$ such that for all $\pi < \pi^*$ and all $a \ne 0$,

$$\pi E(V_a|G) + (1 - \pi) E(V_a|B) + \alpha < 0,$$

the informational black hole and investment inefficiencies will continue to exist. In particular, the participation threshold will still solve

$$P(G|\max_i S_i = \hat{s}_N) = \pi^*.$$

Propositions 5 and 6, which show that small markets can be more efficient and can create higher revenue than large markets, go through when the private value component α is not too large—increasing the size of the market now has the benefit that there is more likely to be an investor with a high private value, which acts as a countervailing force to the investment inefficiencies in large market.[11]

5.4 Stochastic number of investors

In this section, we extend our theory to a stochastic number of investors. We show that for a wide class of distributions of the number of investors, informational black holes not only continue to exist but lead to even less efficient investment decisions than in the deterministic case (Proposition 14). We also show that under some conditions, informational black holes can disappear and full information can be achieved (Proposition 15). Overall, our results suggest that the existence of informational black holes is a robust phenomenon.

Consider the following extension of our main case. Consider a sequence of markets indexed by $N = 1, 2, \ldots$, and assume that each investor thinks that the number of other investors in market N is $N\nu$, where ν is a non-negative random variable with a cumulative distribution function F over $[0, \infty)$. Investors know N and F but not the realization of ν. If ν is one with probability one we are back to the deterministic case considered in main part of the paper.[12] We make the following assumptions about distribution F:

Assumption 13 *F has a continuous density at zero.*

Assumption 14 *ν is smaller in the likelihood ratio ordering than $\lambda\nu$.*

Assumption 13 implies that the probability that the market is populated by any finite number of investors as N goes to infinity goes to zero, which is necessary for markets to ever become fully efficient in the limit. Assumption 14 is not important for our results on when markets feature informational black holes and investment inefficiencies. The main role of this assumption is to ensure the uniqueness of the informational black hole equilibrium. Without it, there could potentially be multiple black hole equilibria. The assumption is satisfied for many distributions. Examples include the uniform and exponential distributions.

Proposition 14 *Suppose that Assumptions 13 and 14 hold. Suppose that either (i) $\pi_0 < \pi^*$, or (ii) there is an $\hat{\nu} > 0$ such that $F(\hat{\nu}) = 0$. Then for large enough N, in each market N, there*

[11]One can also show that all our results are robust to investors having a private value component which is independent of their common value component.

[12]We allow the number of investors $N\nu$ to be non-integer. Our results would not change if we round $N\nu$ to the nearest integer, but formulas become cumbersome.

exists a robust equilibrium in any standard auction format. Any robust equilibrium has the same participation threshold $\hat{s}_N$. The threshold $\hat{s}_N$ goes to one with N, and is a unique solution to Equation (33):

$$\frac{Ee^{-\lambda\tau\nu}}{Ee^{-\tau\nu}} = \frac{1}{\lambda}\frac{\pi^*}{1-\pi^*}\frac{1-\pi_0}{\pi_0}. \tag{33}$$

where $\lambda = f_G(1)/f_B(1)$. Furthermore, there exist limits

$$\lim_{N\to\infty} \Pr(\textit{Project is started } |B) = 1 - Ee^{-\tau\nu} > 0, \tag{34}$$

$$\lim_{N\to\infty} \Pr(\textit{Project is not started } |G) = Ee^{-\lambda\tau\nu} > 0. \tag{35}$$

When the action space is binary, the limiting social surplus with stochastic number of bidders is strictly lower than limiting social surplus with a known number of bidders.

Proof: See the Appendix.

Proposition 14 shows that the existence of informational black holes is robust to having a stochastic number of investors. Investors with sufficiently negative information will not want to participate, and the informational black hole grows with the expected size of the market, even when there is a possibility that the actual number of investors is small. Investment efficiency is lower than in the deterministic case because the inference of a winning investor about project quality is confounded with inference about the size of the market.

The randomness in the number of investors makes the winner's curse weaker, because winning with a low bid is a signal that there may be fewer potential investors in the market. As we show in Proposition 14, this effect is not strong enough to eliminate the informational black hole if the unconditional NPV of the project is negative, or if there is a lower bound on the potential number of investors which grows with N. The following Proposition shows conditions under which the markets can aggregate information:

Proposition 15 *Suppose that $\pi_0 > \pi^*$, that $F(\nu)$ has a strictly positive continuous density at zero, and that $f_B(s)$ and $f_G(s)$ are continuously differentiable. Then there exists a robust equilibrium that leads to full efficiency as $N \to \infty$ if bids are revealed ex post.*

Proof: See the Appendix.

The economics behind Proposition 15 are as follows. When F has a strictly positive continuous density at zero, it may be possible to sustain equilibria with a participation threshold that does not go to one. In such an equilibrium, when the marginal investor wins the auction, he concludes that he is in a market with very few potential investors independent of how large N is. If the unconditional NPV of the project is positive, he can then break even. Because the participation threshold is bounded away from one, the set of observed bids generates a lot of information and investment inefficiencies are eliminated as $N \to \infty$.

Propositions 14 and 15 are derived under the assumption that the number of potential investors $N\nu$ does not depend on the quality of the project. There may be settings where it is more natural to assume that the expected number of potential investors is larger when the project is good. For example, this would be the case if VCs do a quick check initially and only acquire a serious signal if the initial check is positive. Our results can be easily extended to such a case by assuming that ν is drawn from a distribution $F^G(\nu)$ when the project is good and from a distribution $F^B(\nu)$ when the project is bad. One can show that informational black hole equilibria are then even easier to sustain, because the winner's curse gets stronger. Winning with a low bid signals that the market is small, but this is now negative information for the NPV of the project.

6 Conclusion

Our paper studies how well primary financial markets allocate capital when information is dispersed among market participants, and how the efficiency of the market is affected by market size. We show that markets fail to aggregate information once information has real value for guiding investment decisions, and that the resulting investment inefficiencies can grow larger with the size of the market. Our analysis shows that several intuitive prescriptions from standard theory need to be reexamined when information has a real allocational role: a more competitive, larger financial market may reduce welfare and increase a firm's cost of capital, early releases of information may be suboptimal, and rationing allocations and allowing collusion among investors may be beneficial for a firm seeking financing.

Our framework is sufficiently general to be usefully adopted in applied work across a range of firm investment settings. Our theory could also be extended in several fruitful directions. For example, we have not investigated the implications of our model for how a profit-maximizing firm should optimally design the cash flow and control rights of securities. As a second example, a more fully developed general equilibrium framework, in which investors act as intermediaries who must raise capital and enter endogenously, would give richer predictions about the aggregate consequences of informational black holes and would allow for a more nuanced welfare analysis. We think these topics would be interesting areas for future research.

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Figures

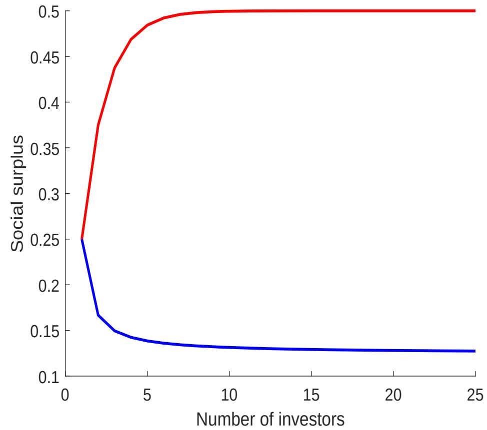
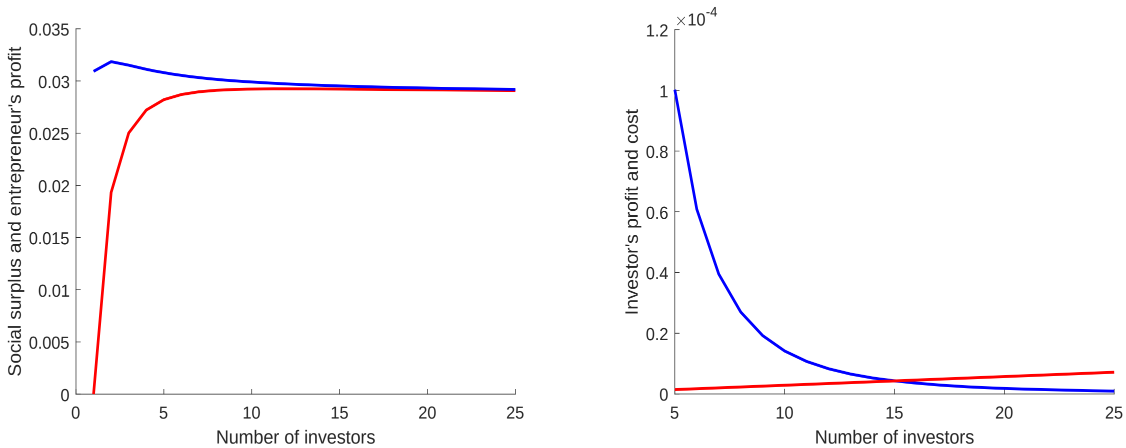


Figure 1. The blue line plots social surplus as a function of number of investors in the setting of the Example, where $E(V|G) = -E(V|B) = 1$ and investors get a high or a low signal about the project, with $\Pr(H|G) = 1$, $\Pr(H|B) = 1/2$. The red line plots maximum possible surplus if all signals are used efficiently.



Panel A

Panel B

Figure 2. Equilibrium market size. Panel A of Figure 2 shows social surplus gross of investor costs and the expected revenues to the entrepreneur as a function of the size of the market. Panel B shows expected gross profits to investors from participating in the auction as a function of market size, as well as a particular specification for the cost c_i of information gathering for each investor. The parameters are as follows: The project is good or bad with equal probabilities. A good project has net present value of 1 and a bad project has net present value of -1; $f_B(s) \equiv 1$; $f_G(s)$ is the normal distribution with unit mean and standard deviation truncated to the interval $[0, 1]$.

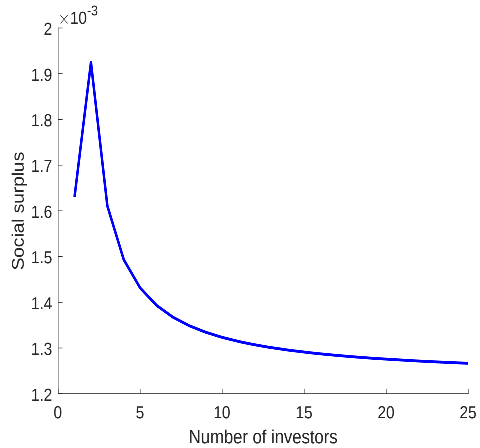


Figure 3. Figuree 3 plots social surplus as a function of number of investors in the setting where investors get a high or a low signal about the project, with $\Pr(H|G) = 1$, $\Pr(H|B) = 1/2$. The project's production function is as in Section 5.1.

Internet Appendix. Proofs

Proof of Lemma 1: To prove (i), note that for each a,

$$v_a = \pi E(V_a|G) + (1-\pi)E(V_a|B) \tag{A1}$$

is an increasing and convex function of π, therefore, the pointwise maximum, $\max_a v_a(\pi)$, is also an increasing and convex function of π.

Next, let $\pi^* = 1/(1+k)$, where

$$k = \sup_{a\neq 0}\left|\frac{E(V_a|G)}{E(V_a|B)}\right| < \infty.$$

Suppose there exists $\pi \leq \pi^*$ such that $v(\pi) > 0$. Then there is an action a such that

$$\pi E(V_a|G) + (1-\pi)E(V_a|B) > 0. \tag{A2}$$

Equation (A2) implies that

$$\left|\frac{E(V_a|G)}{E(V_a|B)}\right| > \frac{1-\pi}{\pi} \geq \frac{1-\pi^*}{\pi^*} = k^*.$$

Thus, we arrive at contradiction. The proof that $v(\pi) = 0$ for $\pi \in [0, \pi^*]$ is similar.

Finally, that $E(v(\pi(\mathbf{S}))$ is strictly increasing in N follows from Jensen's inequality. As N goes to ∞, $\pi(\mathbf{s})$ converges to one if the project is good and to zero if the project is bad. Hence,

$$\lim_{N\to\infty} E(v(\pi(\mathbf{S})) = E(\max_a E(V_a|\theta)).$$

Q.E.D.

Proof of Proposition 1: We first prove that participation decisions are monotone, that is, if an investor with signal $\hat{s}$ participates in the financing mechanism with participation cost $\varepsilon > 0$ then any investor with signal above $\hat{s}$ will also choose to participate. Denote the region of signals where investors do not participate as $\mathbf{B} \subseteq [0,1]$. Note that by individual rationality the expected profit of the investor with signal $\hat{s}$ must cover his participation cost:

$$E\left[w_{i,a_\varepsilon(\mathbf{S^{-i}},\hat{s})}(\mathbf{S^{-i}},\hat{s})|S_i = \hat{s}\right] \geq \varepsilon > 0. \tag{A3}$$

Because signals are conditionally independent we can rewrite the expected profit as

$$\sum_{\theta\in\{G,B\}} \Pr(\theta|S_i = \hat{s})E\left[w_{i,a_\varepsilon(\mathbf{S^{-i}},\hat{s})}(\mathbf{S^{-i}},\hat{s})|\theta\right]. \tag{A4}$$

Note that by the monotonicity condition (Assumption 5)

$$E\left[w_{i,a_\varepsilon(\mathbf{S^{-i}},\hat{s})}(\mathbf{S^{-i}},\hat{s})|B\right] \leq 0$$

Hence, it must be that

$$E\left[w_{i,a_\varepsilon(\mathbf{S^{-i}},\hat{s})}(\mathbf{S^{-i}},\hat{s})|G\right] > 0.$$

Consider now an investor with signal $s > \hat{s}$. From strict MLRP, $\Pr(\theta|S_i = s) > \Pr(\theta|S_i = \hat{s})$. Therefore, if this investor plays a strategy of the investor with signal $\hat{s}$ his expected payoff is strictly larger than that of the investor with signal $\hat{s}$. From the incentive compatibility condition (7) it then follows that such an investor will participate in the financing mechanism.

Next, we show that a participation cut-off cannot be lower than $\hat{s}_N$ defined in (11). Suppose, on the contrary, that there is a financing mechanism with a robust symmetric competitive equilibrium where the participation cut-off is strictly less than $\hat{s}_N$. Hence, there is a symmetric competitive equilibrium with a participation cost $\varepsilon > 0$ and a participation cut-off $\hat{s} < \hat{s}_N$.

Consider an investor with a signal s' such that $\hat{s} \leq s' < \hat{s}_N$ just above the participation cut-off. In a competitive mechanism (Assumption 6) such an investor is always outbid by other investors whenever other investors participate. Hence, the only time when such an investor expects to make profit is when he is the only participant. In this case, he only learns that his signals is the highest of all investors. Thus, it must be that

$$E\left[w_{i,a_\varepsilon(\mathbf{S^{-i}},s')}(\mathbf{S^{-i}},s')|\max_i S_i = s'\right] > 0.$$

Since other investors do not participate $w_{j,a_\varepsilon} = 0$ for $j \neq i$. Therefore, by the limited liability Assumption 4 $w_{i,a_\varepsilon} \leq V_{a_\varepsilon}$. Thus, it must be that

$$E\left[V_{a_\varepsilon}|\max_i S_i = s'\right] > 0.$$

From MLRP condition,

$$E\left[V_{a_\varepsilon}|\max_i S_i = \hat{s}_N\right] \geq E\left[V_{a_\varepsilon}|\max_i S_i = s'\right] > 0.$$

The above condition, however, contradicts the definition of $\hat{s}_N$. Thus, we arrived at contradiction. *Q.E.D.*

Proof of Proposition 2: Equation (11) implies that $\hat{s}_N$ is the smallest value such that

$$\left(\frac{F_G(\hat{s}_N)}{F_B(\hat{s}_N)}\right)^{N-1}\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)} \geq \frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)}. \quad \text{(A5)}$$

The MLRP implies that for any $s < 1$, the ratio $F_G(s)/F_B(s)$ is strictly less than one and is increasing in s. Thus, the left-hand side of equation (A5) strictly increases in s. It is equal to $\lambda = f_G(1)/f_B(1)$ at $s = 1$ and $(f_G(0)/f_B(0))^N$ at $s = 0$. Assumption 7 guarantees that the right-hand side of equation (A5) is less than λ. By assumption the densities f_G and f_B are continuous functions. Therefore, for large enough N, equation (A5) has a unique solution $\hat{s}_N > 0$.

For any fixed $\hat{s}_N < 1$, the left-hand side of equation (A5) goes to zero as N goes to infinity.

Therefore, it must be that $\lim_{N\to\infty} \hat{s}_N = 1$. Taking the logarithm of both parts of equation (A5) we have

$$\lim_{N\to\infty}(N-1)\ln\left(\frac{F_G(\hat{s}_N)}{F_B(\hat{s}_N)}\right) = \ln\left(\frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)}\right) - \ln\lambda. \tag{A6}$$

Since both f_G and f_B are continuous functions there exist limits

$$\lim_{s\to 1}\frac{1-F_G(s)}{1-s} = f_G(1),$$
$$\lim_{s\to 1}\frac{1-F_B(s)}{1-s} = f_B(1).$$

Hence, there exist limits

$$\lim_{N\to\infty} -(N-1)\ln(F_B(\hat{s}_N)) = \tau,$$
$$\lim_{N\to\infty} -(N-1)\ln(F_G(\hat{s}_N)) = \lambda\tau,$$

where $\lambda = f_G(1)/f_B(1)$. From equation (A6) then τ solves

$$(\lambda-1)\tau = \ln\lambda - \ln\left(\frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)}\right). \tag{A7}$$

By Theorem 4.2.1 of Embrechts, Klüppelberg and Mikosch (2012), for $k = 0, 1, \ldots,$

$$\lim_{N\to\infty}\Pr(\kappa_N = k|B) = e^{-\tau}\frac{\tau^k}{k!},$$
$$\lim_{N\to\infty}\Pr(\kappa_N = k|G) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}.$$

Since $\hat{s}_N$ goes to one, then conditional on $\kappa_N = k$

$$\frac{\pi(\mathbf{S}_{\geq\hat{s}_N})}{1-\pi(\mathbf{S}_{\geq\hat{s}_N})} \xrightarrow{d} \frac{\pi_0}{1-\pi_0}\lambda^k\left(\frac{F_G(\hat{s}_N)}{F_B(\hat{s}_N)}\right)^{N-k}. \tag{A8}$$

Equation (A5) implies that for large N

$$\frac{\pi_0}{1-\pi_0}\lambda\left(\frac{F_G(\hat{s}_N)}{F_B(\hat{s}_N)}\right)^{N-1} = \frac{\pi^*}{(1-\pi^*)}. \tag{A9}$$

Therefore, equation (A8) implies that

$$\frac{\pi(\mathbf{S}_{\geq\hat{s}_N})}{1-\pi(\mathbf{S}_{\geq\hat{s}_N})} \xrightarrow{d} \lambda^{\kappa_N-1}\frac{\pi^*}{1-\pi^*},$$

which completes the proof of the proposition. *Q.E.D.*

Proof of Proposition 3: Consider a sequence of participation cost $\varepsilon_m > 0$ converging to zero as $m \to \infty$. For each $\varepsilon_m > 0$ let $\hat{s}_{\varepsilon_m,N}$ be the smallest value such that

$$\Pr(\max_{j\neq i} S_j \leq \hat{s}_{\varepsilon_m,N} | S_i = \hat{s}_{\varepsilon_m,N}) \max_{a\neq 0} E(V_a | S_i = \hat{s}_{\varepsilon_m,N}, \max_{j\neq i} S_j \leq \hat{s}_{\varepsilon_m,N}) \geq \varepsilon_m. \tag{A10}$$

Clearly, $\hat{s}_{\varepsilon_m,N} \to \hat{s}_N$ as $\varepsilon_m \to 0$. Following similar steps as in the standard setting of Milgrom and Weber (1982) one can verify that it is a dominant strategy for each investor who participates in a second-price auction to bid his pre-money valuation of the project conditional on marginally winning the auction:

$$b_i(s_i) = \max_{a\neq 0} E(V_a | S_i = s_i, \max_{j\neq i} S_j = s_i). \tag{A11}$$

From (A11) it is clear that $b_i(s_i)$ increases in s_i for $s_i \geq \hat{s}_{\varepsilon_m,N}$. Also, since the function

$$\Pr(\max_{j\neq i} S_j \leq s | S_i = s) \max_{a\neq 0} E(V_a | S_i = s, \max_{j\neq i} S_j \leq s). \tag{A12}$$

increases in s each investor will participate in the auction if and only if his signal is above the cut-off $\hat{s}_{\varepsilon_m,N}$.

The proof for ascending-price auction is similar. Since both auction formats have the same limiting cut-off $\hat{s}_N$ and post-auction information available for investment decisions is the same each auction format delivers the same maximal social surplus. *Q.E.D.*

Proof of Proposition 4: We can write social surplus (25) as

$$\begin{aligned} U(\hat{s}_N, N) &= \pi_0 E(V_1|G) \Pr(\max_i S_i > \hat{s}_N | G) + (1-\pi_0) E(V_1|B) \Pr(\max_i S_i > \hat{s}_N | B) \\ &= \pi_0 E(V_1|G)(1 - F_G(\hat{s}_N)^N) + (1-\pi_0) E(V_1|B)(1 - F_B(\hat{s}_N)^N) \\ &= \pi_0 E(V_1|G) \left((1 - F_G(\hat{s}_N)^N) - \frac{(1-\pi_0)}{\pi_0} \frac{\pi^*}{(1-\pi^*)} (1 - F_B(\hat{s}_N)^N) \right). \end{aligned} \tag{A13}$$

Recall that $\hat{s}_N$ solves

$$\frac{F_G(\hat{s}_N)^{N-1} f_G(\hat{s}_N)}{F_B(\hat{s}_N)^{N-1} f_B(\hat{s}_N)} = \frac{(1-\pi_0)}{\pi_0} \frac{\pi^*}{(1-\pi^*)}. \tag{A14}$$

Note that one can extend $U(\hat{s}_N, N)$ to real N using the above formulas. Equation (A14) implies that $\frac{\partial}{\partial \hat{s}_N} U(\hat{s}_N, N) = 0$. Therefore, we have

$$\frac{d}{dN} U(\hat{s}_N, N) = \frac{\partial}{\partial \hat{s}_N} U(\hat{s}_N, N) \frac{d\hat{s}_N}{dN} + \frac{\partial}{\partial N} U(\hat{s}_N, N) = \frac{\partial}{\partial N} U(\hat{s}_N, N). \tag{A15}$$

From equation (A13),

$$\frac{\frac{\partial}{\partial N} U(\hat{s}_N, N)}{\pi_0 E(V_1|G)} = \frac{(1-\pi_0)}{\pi_0} \frac{\pi^*}{(1-\pi^*)} \ln(F_B(\hat{s}_N)) F_B(\hat{s}_N)^N - \ln(F_G(\hat{s}_N)) F_G(\hat{s}_N)^N. \tag{A16}$$

Using equation (A14), we can rewrite equation (A16) as

$$\frac{\frac{\partial}{\partial N}U(\hat{s}_N, N)}{\pi_0 E(V_1|G)} = f_G(\hat{s}_N)F_G(\hat{s}_N)^{N-1}\left(\ln(F_B(\hat{s}_N))\frac{F_B(\hat{s}_N)}{f_B(\hat{s}_N)} - \ln(F_G(\hat{s}_N))\frac{F_G(\hat{s}_N)}{f_G(\hat{s}_N)}\right). \tag{A17}$$

$$\begin{aligned}\left(\frac{\ln(F_G(\hat{s}_N))}{\ln(F_B(\hat{s}_N))}\right)' &= \frac{f_G(\hat{s}_N)}{F_G(\hat{s}_N)\ln(F_B(\hat{s}_N))} - \frac{\ln(F_G(\hat{s}_N))f_B(\hat{s}_N)}{F_B(\hat{s}_N)\ln(F_B(\hat{s}_N))^2}\\ &= \frac{f_G(\hat{s}_N)f_B(\hat{s}_N)}{F_G(\hat{s}_N)F_B(\hat{s}_N)\ln(F_B(\hat{s}_N))^2}\left(\ln(F_B(\hat{s}_N))\frac{F_B(\hat{s}_N)}{f_B(\hat{s}_N)} - \ln(F_G(\hat{s}_N))\frac{F_G(\hat{s}_N)}{f_G(\hat{s}_N)}\right).\end{aligned}$$

Thus, the surplus $U(\hat{s}_N, N)$ increases(decreases) with N whenever $\ln(F_G(s))/\ln(F_B(s))$ is an increasing(decreasing) function at $\hat{s}_N$. It is straightforward to verify that $\ln(F_G(s))/\ln(F_B(s))$ is an increasing(decreasing) function on $(x, 1]$ if $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ is an increasing(decreasing) function on $(x, 1]$. *Q.E.D.*

Proof of Proposition 5: Denote the space of continuous functions on $[0, 1]$ equipped with the uniform norm as $\mathcal{C}$. For each N, social surplus (25) defines a continuous nonlinear functional $\mathcal{S}_N$ that maps $(f_G, f_B) \in \mathcal{C} \times \mathcal{C}$ to $\mathbf{R}$. Note that by making if necessary a change of variables $t = F_B(s)$ we can always ensure that $f_B \equiv 1$. Therefore, without loss of generality we assume that $f_B \equiv 1$. To simplify the notation we write $\mathcal{S}_N(f_G, 1)$ simply as $\mathcal{S}_N(f_G)$.

Consider first-price cash auction. Following similar steps as in the standard setting of Milgrom and Weber (1982) one can show that an equilibrium bid in the first-price auction, $b^I(s; \hat{s}_N)$, is an average of the bids $b^{II}(s; \hat{s}_N)$ investors with lower signals would have submitted in the second-price auction:

$$b^I(s; \hat{s}_N) = \int_{\hat{s}_N}^{s} b^{II}(s'; \hat{s}_N)dL(s'|s), \tag{A18}$$

where

$$b^{II}(s; \hat{s}_N) = E[V_1|Y_{1,N} = Y_{2,N} = s], \tag{A19}$$

and

$$L(s'|s) = \exp\left(\int_{s'}^{s} \frac{h(t|s)}{H(t|s)}dt\right).$$

The function $H(\cdot|s)$ is the distribution of $Y_{2,N}$ conditional on $Y_{1,N} = s$ and $h(\cdot|s)$ is the associated conditional density function. It can be verified that the expected revenue in the first-price auction is also a continuous functional $\mathcal{R}_N$ that maps $(f_G, f_B) \in \mathcal{C} \times \mathcal{C}$ to $\mathbf{R}$. As before, we write $\mathcal{R}_N(f_G, 1)$ simply as $\mathcal{R}_N(f_G)$.

Consider first the case in which $f_B(s) = 1$ for all $s \in [0, 1]$ and $f_G(s) = \lambda$ for $s \in (s^*, 1]$, $s^* > 1 - 1/\lambda$. By Proposition 3 there exists large enough N^* such that for all $N > N^*$ the participation threshold $\hat{s}_N \in (s^*, 1)$. For such N, $\hat{s}_N$ satisfies the first-order condition:

$$\left(\frac{1 - \lambda(1 - \hat{s}_N)}{\hat{s}_N}\right)^{N-1} = -\frac{1}{\lambda}\frac{1 - \pi_0}{\pi_0}\frac{E(V_1|B)}{E(V_1|G)}. \tag{A20}$$

Let

$$\xi = -\frac{1}{\lambda}\frac{1-\pi_0}{\pi_0}\frac{E(V_1|B)}{E(V_1|G)}.$$

Assumption 7 guarantees that $\xi < 1$. Therefore, (A20) has a unique solution. Solving (A20) for $\hat{s}_N$ we have

$$\hat{s}_N = \frac{1 - 1/\lambda}{1 - (1/\lambda)\xi^{\frac{1}{N-1}}}. \tag{A21}$$

Plugging (A21) into (25) we obtain an explicit expression for social surplus:

$$\chi_N = \pi_0 E(V_1|G) + (1-\pi_0)E(V_1|B) - (1-\pi_0)E(V_1|B)\frac{(1-1/\lambda)^N}{\left(1-(1/\lambda)\xi^{\frac{1}{N-1}}\right)^{N-1}}. \tag{A22}$$

It can be verified that χ_N strictly decreases with N. Let $\Delta_N = \chi_N - \chi_{N+1}$.

Consider $N > N^*$. By continuity of ψ_N, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any f_G satisfying $f_G(1) - f_G(s^*) < \delta$ we have $|\chi_N - \psi_N(f_G)| < \varepsilon$. In particular, we can always choose δ small enough so that $\psi_N(f_G)$ decreases with N and $\psi_N(f_G) - \psi_{N+1}(f_G) > \Delta_N/2$. Furthermore, we can also ensure that the participation threshold $\hat{s}_N$ is greater than s^*.

Note that if $f_G(s) = \lambda$ for $s \in (s^*, 1]$ then for $N > N^*$ $\mathcal{R}_N(f_G) = \chi_N(f_G)$. In other words, the entrepreneur captures all social surplus. This follows from the fact that all investor with signals in the interval $(s^*, 1]$ must make the same expected profits because their signals have the same informational content. Investors with signals in $[s^*, \hat{s}_N]$ do not participate and hence make zero profits, which implies that all investors make zero profits. Therefore, by continuity of $\mathcal{R}_N$, there exists $\delta' > 0$ such that for any f_G satisfying $f_G(1) - f_G(s^*) < \delta'$ we have $|\chi_N - \mathcal{R}_N(f_G)| < \Delta_N/4$. Hence, the expected revenue with N investors is strictly larger than that with $N' > N$ investors.

Finally, note that the above proof implies that the proposition also holds for equity auctions, and second-price and ascending-price auction because they deliver larger expected profit to the entrepreneur. *Q.E.D.*

Proof of Proposition 6: Suppose all costs are zero. Then by Proposition 4 there is an N^* such that surplus decreases with $N > N^*$. Thus, the optimal size of the market cannot be larger than N^*. Because the MLRP holds strictly all investors earn strictly positive expected profit. Fix any $N > N^*$. Let p_N be the expected profit of an individual investor in the market with N investors. It is clear then that if $c < p_N$ than the size of the market will be larger than socially optimal size N^*.

To show that a proportional decrease in costs for all investors can lead to a decrease in social surplus consider the following situation. Fix $N > N^*$ and let Δ be the difference in social surplus in the market with N and $N+1$ investors. Since $N > N^*$ this difference is positive. Suppose that gathering costs are such that $c_N < p_{N+1}$ and $c_{N+1} > p_{N+1}$. In this case, the market size is N. Let $C = \sum_{i=1}^{N+1} c_i$. Suppose $\gamma > 0$ is small enough so that $\gamma C < \Delta$ and $\gamma C < C_{N+1}$, and that $c_{N+1}(1-\gamma) < p_{N+1}$. Then a proportional decrease in all costs by a factor of γ will lead to a new new market size of $N+1$ and a overall reduction in both net and gross of fees social surplus. *Q.E.D.*

Proof of Proposition 7: As before, if X is released before the fundraising process than the participation threshold $\hat{s}_N(x)$ solves as

$$\frac{\Pr(G|Y_{1,N} = \hat{s}_N(x), X = x)}{\Pr(B|Y_{1,N} = \hat{s}_N(x), X = x)} = \frac{\pi^*}{1 - \pi^*}. \tag{A23}$$

If X is released after the fundraising process than investors will bid a positive amount as long as there is a nonzero probability that after X is released the updated probability that the project is good is above π^*. Hence, the participation threshold $\hat{s}_N$ solves

$$\frac{\Pr(G|Y_{1,N} = \hat{s}_N)}{\Pr(B|Y_{1,N} = \hat{s}_N)} \times \operatorname*{ess\,sup}_x \frac{\Pr(X = x|G, Y_{1,N} = \hat{s}_N)}{\Pr(X = x|B, Y_{1,N} = \hat{s}_N)} = \frac{\pi^*}{1 - \pi^*}. \tag{A24}$$

Note that

$$\frac{\Pr(G|Y_{1,N} = \hat{s}_N(x), X = x)}{\Pr(B|Y_{1,N} = \hat{s}_N(x), X = x)} = \frac{\Pr(G|Y_{1,N} = \hat{s}_N)}{\Pr(B|Y_{1,N} = \hat{s}_N)} \times \frac{\Pr(X = x|G, Y_{1,N} = \hat{s}_N)}{\Pr(X = x|B, Y_{1,N} = \hat{s}_N)}.$$

Therefore, we can write Equation (A24) as

$$\operatorname*{ess\,sup}_x \frac{\Pr(G|Y_{1,N} = \hat{s}_N, X = x)}{\Pr(B|Y_{1,N} = \hat{s}_N, X = x)} = \frac{\pi^*}{1 - \pi^*}. \tag{A25}$$

By Assumption 8 the likelihood ratio

$$\frac{\Pr(G|Y_{1,N} = \hat{s}_N, X = x)}{\Pr(B|Y_{1,N} = \hat{s}_N, X = x)}$$

is increasing in $\hat{s}_N$. Therefore, for any x, $\hat{s}_N \leq \hat{s}_N(x)$. *Q.E.D.*

Proof of Proposition 8: Proposition 1 establishes that participation decisions are monotone. The proof is general and does not depend on a particular value of K. Thus, we have to show that a participation cut-off cannot be lower than $\hat{s}_{K,N}$ defined in (31). Suppose, on the contrary, that there is a financing mechanism with a robust symmetric competitive equilibrium where the participation cut-off is strictly less than $\hat{s}_{K,N}$. Hence, there is a symmetric competitive equilibrium with a participation cost $\varepsilon > 0$ and a participation cut-off $\hat{s} < \hat{s}_{K,N}$.

Consider an investor with a signal s' such that $\hat{s} \leq s' < \hat{s}_{K,N}$ just above the participation cut-off. In a competitive mechanism (Assumption 6) such an investor is always outbid by other investors whenever at least K other investors participate. Hence, the only time when such an investor expects to make profit is when there are no more than $K - 1$ other participating investors. In this case, by the definition of $\hat{s}_{K,N}$ the project is negative NPV and therefore, is not started. Hence, the investor makes profit irrespective of whether the project is good or bad, which contradicts the monotonicity Assumption 5. Thus, we arrived at contradiction.

Next, consider any $\hat{s} > \hat{s}_{K,N}$. Note there is a sufficiently small participation cost $\varepsilon_m > 0$ and $\varepsilon > 0$ such that

$$\Pr(Y_{K,N} = \hat{s}, Y_{K-1,N} > 1-\varepsilon | S_i = \hat{s}) \max_{a\neq 0} E(V_a | S_i = Y_{K,N} = \hat{s}, Y_{K-1,N} > 1-\varepsilon) \geq \varepsilon_m.$$

Since the above function increases in $\hat{s}$ any investor with signal above $\hat{s}$ will participate in the auction. Because, $\hat{s}$ can be arbitrary close to $\hat{s}_{K,N}$ this proves that that there is a robust symmetric ex post efficient equilibrium in the K-unit auction with a participation cut-off $\hat{s}_{K,N}$.

Equation (31) implies that for large enough N $\hat{s}_{K,N}$ solves

$$\frac{F_G(\hat{s}_{K,N})^{N-K}}{F_B(\hat{s}_{K,N})^{N-K}} \frac{f_G(\hat{s}_{K,N})}{f_B(\hat{s}_{K,N})} = \frac{1}{\lambda^{K-1}} \frac{(1-\pi_0)}{\pi_0} \frac{\pi^*}{(1-\pi^*)}. \tag{A26}$$

Following similar steps as in the proof of Proposition 3 one can show that there exist limits

$$\begin{aligned} \lim_{N\to\infty} -(N-K)\ln(F_B(\hat{s}_{K,N})) &= \tau, \\ \lim_{N\to\infty} -(N-K)\ln(F_G(\hat{s}_{K,N})) &= \lambda\tau. \end{aligned}$$

where $\tau > 0$ is unique solution the following equation:

$$(\lambda-1)\tau = K\ln\lambda - \ln\left(\frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)}\right). \tag{A27}$$

By Theorem 4.2.3 of Embrechts, Klüppelberg and Mikosch (2012)

$$\lim_{N\to\infty} \Pr(Y_{K,N} \leq \hat{s}_{K,N}|B) = e^{-\tau}\sum_{i=0}^{K-1}\frac{\tau^i}{i!},$$

$$\lim_{N\to\infty} \Pr(Y_{K,N} \leq \hat{s}_{K,N}|G) = e^{-\lambda\tau}\sum_{i=0}^{K-1}\frac{(\lambda\tau)^i}{i!}.$$

Since $\tau > 0$ some bad projects are financed with positive probability and some good projects are not financed. Thus, social surplus is strictly lower than the first-best social surplus. *Q.E.D.*

Proof of Proposition 9: As in the proof of Proposition 5 consider first the case in which $f_B(s) = 1$ for all $s \in [0,1]$ and $f_G(s) = \lambda$ for $s \in (s^*, 1]$, $s^* > 1 - 1/\lambda$. Equation (A26) implies that for large enough N, the participation cut-off $\hat{s}_{K,N} \in (s^*, 1)$ and solves

$$\left(\frac{1-\lambda(1-\hat{s}_{K,N})}{\hat{s}_{K,N}}\right)^{N-K} = -\frac{1}{\lambda^K}\frac{1-\pi_0}{\pi_0}\frac{E(V_1|B)}{E(V_1|G)}. \tag{A28}$$

Let

$$\xi = -\frac{1}{\lambda^K}\frac{1-\pi_0}{\pi_0}\frac{E(V_1|B)}{E(V_1|G)}.$$

Assumption 7 guarantees that $\xi < 1$. Therefore, (A28) has a unique solution. Solving (A28) for $\hat{s}_{K,N}$ we have

$$\hat{s}_{K,N} = \frac{1 - 1/\lambda}{1 - (1/\lambda)\xi^{\frac{1}{N-K}}}. \tag{A29}$$

Since the project is started if and only if the K-highest signal is above the participation cut-off $\hat{s}_{K,N}$ social surplus is equal to

$$\chi_{K,N} = \pi_0 \Pr(Y_{K,N} \geq \hat{s}_{K,N}|G)E(V_1|G) + (1 - \pi_0) \Pr(Y_{K,N} \geq \hat{s}_{K,N}|B)E(V_1|B). \tag{A30}$$

Recall that

$$\Pr(Y_{K,N} \leq s) = \sum_{i=N-K+1}^{N} C_N^i F(s)^i (1 - F(s))^{N-i}, \tag{A31}$$

where $F(s)$ is a cumulative distribution function of a signal. Using (A31) and (A28) we obtain an explicit expression for social surplus (A30):

$$\chi_{K,N} = \pi_0 E(V_1|G) + (1 - \pi_0)E(V_1|B) - \tag{A32}$$

$$(1 - \pi_0)E(V_1|B) \sum_{i=N-K+1}^{N} C_N^i \hat{s}_{K,N}^i (1 - \hat{s}_{K,N})^{N-i} \left(1 - \left((1/\lambda)\xi^{\frac{1}{N-K}}\right)^{i-N+K}\right). \tag{A33}$$

Direct but tedious calculations show that $\chi_{K,N}$ strictly decreases with N for any finite K. The rest of the proof follows similar lines as the proof of Proposition 5 and therefore, is omitted. *Q.E.D.*

Proof of Proposition 10: As $N \to \infty$ and $K/N \to 1 - \alpha$ the k-order statistics $Y_{K,N}$ becomes an α^{th} sample quantile. It is well-known that

$$\sqrt{N}(Y_{K,N} - s_\alpha) \xrightarrow{d} N(0, \alpha(1 - \alpha)/f(s_\alpha)^2),$$

where $f(x)$ and $F(x)$ are pdf and cdf of observations and $F(s_\alpha) = \alpha$. Let $s_{\alpha,G}$ and $s_{\alpha,B}$ be such that $F_G(s_{\alpha,G}) = \alpha$ and $F_B(s_{\alpha,B}) = \alpha$. Because of the MLRP, $s_{\alpha,B} < s_{\alpha,G}$.

We showed in the proof of Proposition 8 that the participation cut-off $\hat{s}_{K,N}$ is the smallest value such that

$$\lambda^{K-1} \frac{F_G(\hat{s}_{K,N})^{N-K}}{F_B(\hat{s}_{K,N})^{N-K}} \frac{f_G(\hat{s}_{K,N})}{f_B(\hat{s}_{K,N})} \geq \frac{(1 - \pi_0)}{\pi_0} \frac{\pi^*}{(1 - \pi^*)}. \tag{A34}$$

If $K/N = 1 - \alpha$ then we can write equation (A34) as

$$\left(\lambda^{1-\alpha} \frac{F_G(\hat{s}_{K,N})^\alpha}{F_B(\hat{s}_{K,N})^\alpha}\right)^N \frac{f_G(\hat{s}_{K,N})}{f_B(\hat{s}_{K,N})} \geq \frac{1}{\lambda} \frac{(1 - \pi_0)}{\pi_0} \frac{\pi^*}{(1 - \pi^*)}.$$

Note that for any s

$$\frac{1 - F_G(s)}{1 - F_B(s)} \leq \lambda.$$

Therefore,

$$\lambda^{1-\alpha}\frac{F_G(\hat{s}_{K,N})^\alpha}{F_B(\hat{s}_{K,N})^\alpha} \geq \frac{F_G(\hat{s}_{K,N})^\alpha(1-F_G(\hat{s}_{K,N}))^{1-\alpha}}{F_B(\hat{s}_{K,N})^\alpha(1-F_B(\hat{s}_{K,N}))^{1-\alpha}}. \tag{A35}$$

Consider equation

$$F_G(s_\alpha)^\alpha(1-F_G(s_\alpha))^{1-\alpha} = F_B(s_\alpha)^\alpha(1-F_B(s_\alpha))^{1-\alpha}. \tag{A36}$$

Notice that $x^\alpha(1-x)^{1-\alpha}$ is a single-peaked function that reaches its maximum at $x = \alpha$. Therefore, $s_{\alpha,B} < s_\alpha < s_{\alpha,G}$. Equation (A35) implies that $\hat{s}_{K,N} \leq s_\alpha$. Thus, $\hat{s}_{K,N} \leq s_{\alpha,G}$. Therefore, if the project is good the probability that at least K investors will participate in the auction goes to one. Since $\hat{s}_{K,N}$ is bounded away from one, observing the censored vector signal $\mathbf{S}_{\geq \hat{s}_{K,N}}$ allows one to deduce the project type with probability one. *Q.E.D.*

Proof of Proposition 11: In general, auctions with multi-dimensional signals are notoriously difficult to analyze. What makes our model amenable for analysis is that one can reduce a multi-dimensional signal to a one-dimensional one. To see this, for each club $i = 1, ..., N$ partition the signal space $[0,1]^M$ into a collection of $(M-1)$-dimensional hyperplanes such that on each hyperplane, the likelihood ratio of M club signals is constant:

$$L_i(s) = \{s_{i1}, ..., s_{iM} \in [0,1]^M : \prod_{j=1}^{M} \frac{f_G(s_{ij})}{f_B(s_{ij})} = \left(\frac{f_G(s)}{f_B(s)}\right)^M\}, \quad s \in [0,1]. \tag{A37}$$

Note that because signals are conditionally independent L_i is a sufficient statistics to update the likelihood that the project is good given club $i's$ investor signals $S_{i1}, ..., S_{iM}$. Requiring then club bids to be measurable with respect to the partition induced by hyperplanes L_i reduces a fundraising process with multi-dimensional signals to a fundraising process with a one-dimensional signal, to which all theory we have developed in this paper applies.

For each $s \in [0,1]$ define

$$\hat{F}_G(s) = \Pr(\prod_{j=1}^{M} \frac{f_G(S_j)}{f_B(S_j)} \leq \left(\frac{f_G(s)}{f_B(s)}\right)^M |G),$$
$$\hat{F}_B(s) = \Pr(\prod_{j=1}^{M} \frac{f_G(S_j)}{f_B(S_j)} \leq \left(\frac{f_G(s)}{f_B(s)}\right)^M |B).$$

It is straightforward to verify that functions $\hat{F}_G$ and $\hat{F}_B$ satisfy

$$\lim_{s\to 1} \hat{F}_G(s) = 1 - cf_G(1)^M(1-s)^M,$$
$$\lim_{s\to 1} \hat{F}_B(s) = 1 - cf_B(1)^M(1-s)^M,$$

where c is some constant. Let $\hat{s}_{M,N}$ be a participation threshold with N clubs where each club consists of M investors. Following similar steps as in the proof of Proposition 2 one can show that $\hat{s}_{M,N}$ satisfies

$$\frac{\hat{F}_G(\hat{s}_{M,N})^{N-1}}{\hat{F}_B(\hat{s}_{M,N})^{N-1}}\frac{f_G(\hat{s}_{M,N})^M}{f_B(\hat{s}_{M,N})^M} = \frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)}, \tag{A38}$$

and that there exist limits

$$\begin{aligned} \lim_{N\to\infty} -(N-1)\ln(\hat{F}_B(\hat{s}_{M,N})) &= \tau, \\ \lim_{N\to\infty} -(N-1)\ln(\hat{F}_G(\hat{s}_{M,N})) &= \lambda^M\tau. \end{aligned}$$

From Equation (A38) then τ must solve

$$(\lambda^M - 1)\tau = M\ln\lambda - \ln\left(\frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)}\right). \tag{A39}$$

Thus, the limiting participation threshold with clubs is the same as the participation threshold with individual investors where the informativeness of the top signal is λ^M. Proposition 2 imply that the asymptotic social surplus when the number of investor is large depends only on the informativeness of the top signal λ, and is equal to

$$\pi_0 E(V_1|G)\left(1 - \frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{1-\pi^*} + (\lambda-1)\left(\frac{1}{\lambda}\frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)}\right)^{\frac{\lambda}{\lambda-1}}\right). \tag{A40}$$

It can be directly verified that (A40) is increasing in λ. Since the top club signal is more informative than a top signal of a single investor social surplus with club bids is higher than that with individual investors for large N. Therefore, for large N, the entrepreneur will obtain higher revenues with club bids in any competitive fundraising mechanism where her share of the total surplus goes to one with the number of investors. *Q.E.D.*

Proof of Proposition 12: Consider any $\hat{s} \in (\underline{\hat{s}}_N, \hat{s}_N]$. We will show that there is a contract $\{c_1, c_2\}$ such that there is a robust equilibrium as described in the proposition. Denote the equilibrium bidding function in the shorting market by $b(s)$, where we set $b(s) = 0$ if $s \geq \underline{s}(\hat{s})$. Let $\Upsilon(b_i)$ be the event such that investor i wins the shorting auction with a bid $b_i > 0$ and the project is financed:

$$\Upsilon(b_i) \equiv \{\mathbf{S^{-i}} : Y_1^{-i} > \hat{s},\ b(Y_{N-1}^{-i}) \leq b_i\}$$

where Y_k^{-i} is the k^{th} highest signal, or order statistic, amongst bidder i's $N-1$ competitiors.

Denote by $a(\mathbf{S^{-i}})$ the equilibrium action taken by a winning investor in the primary market when investor i wins the shorting auction given other investor signals $\mathbf{S^{-i}}$. Note that this action is independent of both the bid and the true signal of investor i as the winning bid in the shorting market is not revealed to the winning investor in the primary market. Let W be the payoff of the shorting contract to investor i when the firm is financed. We have

$$W(\mathbf{S^{-i}}, c_1, c_2) \equiv 1_{\{a(\mathbf{S^{-i}})=0\}}(c_1 + c_2) - c_2.$$

Lemma 3 establishes the following intermediate result, which will lead to standard second-price auction equilibrium bidding in the shorting market:

Lemma 3 *$W(\mathbf{s^{-i}}, c_1, c_2)$ is a decreasing function. If $E\left[W(\mathbf{S^{-i}}, c_1, c_2)|\Upsilon(0), S_i = \underline{s}\right] = 0$, then*

$$E\left[W(\mathbf{S^{-i}}, c_1, c_2)|\Upsilon(b_i), S_i = s\right]$$

is strictly increasing in b_i for $b_i \geq 0$ and strictly decreasing in s for $s \leq \underline{s}$, and

$$E\left[W(\mathbf{S^{-i}}, c_1, c_2)|\Upsilon(b_i), b(Y_{N-1}^{-i}) = q, \theta\right]$$

is strictly increasing in q and b_i for $0 < q < b_i$ and strictly decreasing in θ.

Proof: Let $A(\mathbf{s^{-i}})$ be the information set available to the winning investor in the primary market given the realization of other signal $\mathbf{S^{-i}} = \mathbf{s^{-i}}$. The decision maker will set $a(\mathbf{s^{-i}}) = 0$ if

$$\Pr(G|A(\mathbf{s^{-i}})) \leq \pi^*,$$

or equivalently if

$$\mathcal{L}(\mathbf{s^{-i}}) \equiv \frac{\Pr(A(\mathbf{s^{-i}})|G)}{\Pr(A(\mathbf{s^{-i}})|B)} \leq \frac{\pi^*}{1-\pi^*}\frac{1-\pi_0}{\pi_0}.$$

From the definition of $\hat{s}$ and $\underline{s}$ we have

$$\mathcal{L}(A(\mathbf{s^{-i}})) = \frac{F_G(\min(\mathbf{s^{-i}}, \underline{s})\Pi_{s_j^{-i} \notin [\underline{s}, \hat{s}]} f_G(s_j^{-i})\Pi_{s_j^{-i} \in [\underline{s}, \hat{s}]}(F_G(\overline{s}) - F_G(\underline{s}))}{F_B(\min(\mathbf{s^{-i}}, \underline{s})\Pi_{s_j^{-i} \notin [\underline{s}, \hat{s}]} f_B(s_j^{-i})\Pi_{s_j^{-i} \in [\underline{s}, \hat{s}]}(F_B(\overline{s}) - F_B(\underline{s}))},$$

which is strictly increasing in all $s_j^{-i} \notin [\underline{s}, \hat{s}]$. Hence,

$$1_{\{a(\mathbf{s^{-i}})=0\}} = 1_{\{\mathcal{L}(A(\mathbf{s^{-i}})) \leq \frac{\pi^*}{1-\pi^*}\frac{1-\pi_0}{\pi_0}\}}$$

is a decreasing function, and so is W. From the definition of $\hat{s}$ and $\underline{s}$ and by Assumption 10,

$$0 < \Pr(a(\mathbf{S^{-i}}) = 0|\theta, b(Y_{N-1}^{-i}) = q, \Upsilon(b_i)) < 1$$

for any $\theta, q \geq 0, b_i \geq 0$. Thus, W is not constant over any of the conditioning sets in the lemma. Since $\mathbf{S^{-i}}$ is strictly affiliated with S_i and θ when MLRP holds strictly, the results in the Lemma then follows from Milgrom and Weber (1982) Theorems 1-5. *Q.E.D.*

From Lemma 3, it then follows from standard arguments (see, e.g., Milgrom and Weber (1982)) that if $E\left[W(\mathbf{S^{-i}}, c_1, c_2)|\Upsilon(0), S_i = \underline{s}\right] = 0$, the equilibrium bidding function $b(s)$ for $s \leq \underline{s}$ is given by

$$b(s) = E(W(\mathbf{S^{-i}}, c_1, c_2)|S_i = Y_{N-1}^{-i} = s, Y_1^{-i} > \hat{s}),$$

that is, participants bid their valuation of the shorting contract contingent on marginally winning and conditional on the project being financed ($Y_1^{-i} > \hat{s}$). For N large enough so that

Assumption 10 holds, $b(s)$ is a strictly decreasing function of s.

It remains to show that $\underline{s}$ and $\hat{s}$ are indeed participation cut-offs in the shorting and primary markets. To ensure that $\underline{s}$ is a participation cut-off in the shorting market, set c_2/c_1 so that

$$\frac{c_2}{c_1} = \frac{E(1_{\{a(\mathbf{S^{-i}})=\mathbf{0}\}}|Y_1^{-i} > \hat{s}, Y_{N-1}^{-i} \geq \underline{s}, S_i = \underline{s})}{1 - E(1_{\{a(\mathbf{S^{-i}})=\mathbf{0}\}}|Y_1^{-i} > \hat{s}, Y_{N-1}^{-i} \geq \underline{s}, S_i = \underline{s})}. \tag{A41}$$

Note that from the definition of cut-offs, for any $c_1 > 0$, there is always such a c_2/c_1.

To ensure that $\hat{s}$ is a participation threshold in the primary market, set

$$E(V_{a(\mathbf{S})}|S_i = \hat{s}, Y_1^{-i} \leq \hat{s}, Y_{N-1}^{-i} \geq \underline{s})P(Y_{N-1}^{-i} \geq \underline{s}|S_i = \hat{s}, Y_1^{-i} \leq \hat{s}) = c_1.$$

Note that for $\hat{s} > \underline{\hat{s}}_N$, there is always such a $c_1 > 0$. Showing that bidding in the financing market is strictly increasing for $s > \hat{s}$ follows the same steps as for the financing market without a shorting market present.

Hence, we have shown that there is an equilibrium, but it remains to show that it is robust. We show that we can always implement the same cut-offs for a sufficiently small participation cost $\varepsilon > 0$ with $c_{1\varepsilon}, c_{2\varepsilon} > 0$ that converge to c_1, c_2 as $\varepsilon \to 0$. Note that bidding for participants in both the shorting and primary market is unchanged, after replacing c_1, c_2 with $c_{1\varepsilon}, c_{2\varepsilon} > 0$ in the expression for W. Let us set $c_{1\varepsilon}$ and $c_{2\varepsilon}$ so that the following equations hold.

$$c_{1\varepsilon} = E(V_{a(\mathbf{S})}|S_i = \hat{s}, Y_1^{-i} \leq \hat{s}, Y_{N-1}^{-i} \geq \underline{s}) \Pr(Y_{N-1}^{-i} \geq \underline{s}|S_i = \hat{s}, Y_1^{-i} \leq \hat{s}) - \frac{\varepsilon}{\Pr(Y_1^{-i} \leq \hat{s}|S_i = \hat{s})},$$

$$c_{2\varepsilon} = \frac{c_{1\varepsilon}E(1_{a(\mathbf{S^{-i}})=\mathbf{0}}|Y_1^{-i} > \hat{s}, Y_{N-1}^{-i} \geq \underline{s}, S_i = \underline{s}) - \frac{\varepsilon}{\Pr(Y_{N-1}^{-i} \geq \underline{s},\ Y_1^{-i} > \hat{s}|S_i = \underline{s})}}{1 - E(1_{a(\mathbf{S^{-i}})=\mathbf{0}}|Y_1^{-i} > \hat{s}, Y_{N-1}^{-i} \geq \underline{s}, S_i = \underline{s})}.$$

The above values of $c_{1\varepsilon}$ and $c_{2\varepsilon}$ ensure that investors with signals $\overline{s}$ and $\underline{s}$ are marginal in the financing and shorting markets respectfully. To conclude the proof we need to show that participation in both markets is monotone in the signal. The proof of monotone participation in the financing market is straightforward and follows same steps as the proof of Proposition 1.

To show that participation in the shorting market is monotone consider the expected payoff of the shorting contract for an investor with $S_i = s$ with a bid of b_i

$$\Pr(G|s)\Pr(\Upsilon(b_i)|G)E\left[W(\mathbf{S^{-i}}, c_{1\varepsilon}, c_{2\varepsilon})|\Upsilon(b_i), G\right] + \Pr(B|s)\Pr(\Upsilon(b_i)|B)E\left[W(\mathbf{S^{-i}}, c_{1\varepsilon}, c_{2\varepsilon})|\Upsilon(b_i), B\right] - \varepsilon. \tag{A42}$$

Note that equation (A41) implies that

$$E\left[W(\mathbf{S^{-i}}, c_1, c_2)|\Upsilon(0), G\right] < 0,$$
$$E\left[W(\mathbf{S^{-i}}, c_1, c_2)|\Upsilon(0), B\right] > 0.$$

By setting ε sufficiently small, we can have $c_{1\varepsilon}$ and $c_{2\varepsilon}$ sufficiently close to c_1 and c_2 so that

$$E\left[W(\mathbf{S^{-i}}, c_{1\varepsilon}, c_{2\varepsilon})|\Upsilon(0), G\right] < 0, \tag{A43}$$

$$E\left[W(\mathbf{S^{-i}}, c_{1\varepsilon}, c_{2\varepsilon})|\Upsilon(0), B\right] > 0. \tag{A44}$$

Since by the strict MLRP $\Pr(G|s)$ increases in s equation (A42) together with (A43) and (A44) imply that any investor with a signal $s < \underline{s}$ will find it profitable to participate in the shorting market. *Q.E.D.*

Finally, to show that we can have $\hat{s} = \underline{\hat{s}}_N$ in a robust equilibrium, for any $\delta > 0$, take $\overline{s}_\delta > \overline{s}$ such that allocations with $\varepsilon = 0$ with this new cut-off are within $\delta/2$. Then, find $\varepsilon(\delta) > 0$ such that allocations with cut-off $\overline{s}_\delta$ and contracts $c_{1,\varepsilon(\delta)}$, $c_{2\varepsilon(\delta)}$ are within $\delta/2$ of allocations with $\varepsilon = 0$ and cut-off $\overline{s}_\delta$. *Q.E.D.*

Proof of Proposition 13: We showed in the proof of Proposition 2 that limiting surplus is completely determined by the value of

$$\tau = \lim_{N\to\infty} -(N-1)\ln(F_B(\hat{s}_N)).$$

Hence, in order for limiting surplus to be improved, it is necessary to have a sequence of cut-off signals $s'_N < \hat{s}_N$ such that

$$\lim_{N\to\infty} -(N-1)\ln(F_B(s'_N)) = \tau' > \tau. \tag{A45}$$

Note that for such cut-offs when N is sufficiently large, the expected present value of the project with only one participating bidder under any other action than abandoning is strictly negative:

$$\lim_{N\to\infty} P(G|Y_{1,N} = 1, Y_{2,N} \le s'_N) < \pi^*. \tag{A46}$$

This follows from the fact that

$$\frac{\lim_{N\to\infty} P(G|Y_{1,N} = 1, Y_{2,N} \le s'_N)}{\lim_{N\to\infty} P(B|Y_{1,N} = 1, Y_{2,N} \le s'_N)} = \frac{\lambda\pi_0}{1-\pi_0}e^{-(\lambda-1)\tau'} < \frac{\lambda\pi_0}{1-\pi_0}e^{-(\lambda-1)\tau} = \frac{\pi^*}{1-\pi^*}. \tag{A47}$$

Suppose there is such a sequence of equilibrium cut-offs s'_N that a marginal investor with $S_i = s'_N$ participates. This investor can only make profit when he is the only participating bidder, so that $\kappa_N = 1$. From (A46) for large enough N, when the marginal investor wins in equilibrium we have

$$E\left(V_a|\kappa_N = 1\right) \le 0, \tag{A48}$$

and hence, from the ex post participation constraint 12 of the entrepreneur,

$$E\left(w_a|\kappa_N = 1\right) \le 0, \tag{A49}$$

so that this marginal investor can never recoup his participation cost. Hence, such s'_N cannot be the equilibrium cut-off.

To prove the last part of the proposition, suppose that the equilibrium cut-off is some $s'_N < \hat{s}_N$. By the definition of $\hat{s}_N$, for a marginal participating investor with signal $S = s'_N$, the efficient action contingent on winning is to abandon the project. But when the investor learns that $a = 0$ is efficient, the ex post participation constraint reduces to

$$0 \geq E\left(V_a|\omega\right) \geq E\left(w_a|\omega\right), \tag{A50}$$

so that the investor can never break even at any positive participation cost. Hence, the equilibrium cut-off cannot be smaller than $\hat{s}_N$.*Q.E.D.*

Proof of Proposition 14: Recall that the threshold $\hat{s}_N$ is defined as the highest signal such that if a investor with signal $\hat{s}_N$ wins the auction he concludes that the project is zero NPV. When the number of potential investors is stochastic, $\hat{s}_N$ solves

$$\Pr(G|Y_{1,N\nu+1} = \hat{s}_N) = \pi^*, \tag{A51}$$

where $Y_{1,N\nu+1}$ is the first-order statistic from a random sample of size $N\nu + 1$. Equation (A51) can be written explicitly as

$$\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)} \frac{\int_0^\infty F_G(\hat{s}_N)^{Nv} dF(\nu)}{\int_0^\infty F_B(\hat{s}_N)^{Nv} dF(\nu)} = \frac{(1-\pi_0)}{\pi_0} \frac{\pi^*}{(1-\pi^*)}. \tag{A52}$$

Let $\phi(\tau)$ denote the Laplace transform of the random variable ν :

$$\phi(\tau) = \int_0^\infty e^{-\tau\nu} dF(\nu).$$

Let

$$\tau_{G,N}(\hat{s}_N) = -N \ln(F_G(\hat{s}_N)),$$
$$\tau_{B,N}(\hat{s}_N) = -N \ln(F_B(\hat{s}_N)).$$

Equation (A52) can be written as

$$\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)} \frac{\phi(\tau_{G,N}(\hat{s}_N))}{\phi(\tau_{B,N}(\hat{s}_N))} = \frac{(1-\pi_0)}{\pi_0} \frac{\pi^*}{(1-\pi^*)}. \tag{A53}$$

Consider a sequence of $\tau_{B,N}(\hat{s}_N)$ as N goes to infinity. There can be two cases: either $\tau_{B,N}(\hat{s}_N)$ stays bounded or it goes to infinity. We will show that the latter case cannot realize. Suppose that $\tau_{B,N}(\hat{s}_N)$ goes to infinity. Note that by the MLRP $F_B(s) \geq F_G(s)$ for any s. Therefore, if $\tau_{B,N}(\hat{s}_N)$ goes to infinity then $\tau_{G,N}(\hat{s}_N)$ also goes to infinity.

Suppose first that $F(\hat{\nu}) > 0$ for any $\hat{\nu} > 0$ and that $\pi_0 < \pi^*$. Since ν has a continuous density at zero

$$F(t) \sim t^{\rho} L\left(\frac{1}{t}\right), \quad t \to 0,$$

where L is some positive function varying slowly at ∞ (see e.g., Feller (1971), ch 8 for a definition), and $\rho \geq 1$. Therefore, for large N, by theorems 2 and 3 in Feller (1971), ch 8.5, we have

$$\frac{\phi(\tau_{G,N}(\hat{s}_N))}{\phi(\tau_{B,N}(\hat{s}_N))} \sim \left(\frac{\tau_{B,N}(\hat{s}_N)}{\tau_{G,N}(\hat{s}_N)}\right)^{\rho} = \left(\frac{\ln(F_B(\hat{s}_N))}{\ln(F_G(\hat{s}_N))}\right)^{\rho}.$$

The condition $\pi_0 < \pi^*$ implies that

$$\frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)} > 1.$$

Hence, if $\tau_{B,N}(\hat{s}_N) \to \infty$ with N the limit of $\hat{s}_N$ must solve

$$\lim_{N\to\infty} \frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)} \left(\frac{\ln(F_B(\hat{s}_N))}{\ln(F_G(\hat{s}_N))}\right)^{\rho} = \frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{(1-\pi^*)}. \tag{A54}$$

The above equation, however, has no solution. To see this, note that for $0 \leq x \leq y \leq 1$

$$\frac{\ln y}{\ln x} \leq \frac{1-y}{1-x}.$$

Therefore,

$$\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)} \left(\frac{\ln(F_B(\hat{s}_N))}{\ln(F_G(\hat{s}_N))}\right)^{\rho} \leq \frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)} \left(\frac{1-F_B(\hat{s}_N)}{1-F_G(\hat{s}_N)}\right)^{\rho} \leq 1,$$

where the last inequality follows from the MLRP.

Suppose now that there exists $\hat{\nu} > 0$ such that $F(\hat{\nu}) = 0$. Then as $\tau_{B,N}(\hat{s}_N)$ and $\tau_{G,N}(\hat{s}_N)$ go to infinity

$$\frac{\phi(\tau_{G,N}(\hat{s}_N))}{\phi(\tau_{B,N}(\hat{s}_N))} \to 0.$$

Hence, equation (A53) cannot have a solution.

Thus, we arrive at a contradiction and it must be that $\tau_{B,N}(\hat{s}_N)$ stays bounded as N goes to infinity. Hence, there is a subsequence N_k such that there is a finite limit

$$\tau = \lim_{N_k\to\infty} \tau_{B,N_k}(\hat{s}_{N_k}).$$

Since $\tau_{B,N}(\hat{s}_N) = -N\ln(F_B(\hat{s}_N))$ has a finite limit, $\hat{s}_{N_k}$ must go 1 with N_k. Since

$$\begin{aligned}
&F_G(s) \sim 1 - f_G(1)(1-s), \quad s \to 1\\
&F_B(s) \sim 1 - f_B(1)(1-s), \quad s \to 1\\
&f_G(1)/f_B(1) = \lambda,
\end{aligned}$$

there also exists a limit

$$\lambda\tau = \lim_{N_k\to\infty} \tau_{G,N_k}(\hat{s}_{N_k}).$$

Thus, τ must solve equation (33).

Note that the lhs of equation (33) is a continuous function of τ. Since $\phi(\tau) \to 1$ as τ goes to 0, the lhs of Equation (33) goes to one as τ goes to 0. Suppose again that $F(\hat{\nu}) > 0$ for any $\hat{\nu} > 0$ and that $\pi_0 < \pi^*$. As s_N goes to 1 and τ goes to ∞ the lhs of equation (33)

$$\lim_{N_k \to \infty} \frac{\phi(\tau_{G,N}(\hat{s}_{N_k}))}{\phi(\tau_{B,N}(\hat{s}_{N_k}))} = \lim_{N_k \to \infty} \left(\frac{\ln(F_B(\hat{s}_{N_k}))}{\ln(F_G(\hat{s}_{N_k}))} \right)^{\rho} = \lambda^{-\rho}.$$

If there exists $\hat{\nu} > 0$ such that $F(\hat{\nu}) = 0$ then the lhs of equation (33) goes to zero with N_k. By Assumption 7

$$\frac{1}{\lambda} \frac{(1-\pi_0)}{\pi_0} \frac{\pi^*}{(1-\pi^*)} < 1.$$

The condition $\pi_0 < \pi^*$ implies that

$$\frac{1}{\lambda} \frac{(1-\pi_0)}{\pi_0} \frac{\pi^*}{(1-\pi^*)} > \frac{1}{\lambda}.$$

Therefore, equation (33) has a solution.

By Assumption 14 the lhs of equation (33) is a strictly decreasing function of τ. Therefore, Equation (33) has, in fact, a unique solution. As a result, $\tau_{B,N}(\hat{s}_N)$ converges to τ with N. The last fact implies

$$\hat{s}_N = 1 - \frac{\tau}{N} + o(\frac{1}{N}) \tag{A55}$$

By Theorem 4.3.2 of Embrechts, Klüppelberg and Mikosch (2012)

$$\lim_{N \to \infty} \Pr(Y_{1,N\nu+1} \geq \hat{s}_N | B) = 1 - E e^{-\tau \nu},$$
$$\lim_{N \to \infty} \Pr(Y_{1,N\nu+1} \leq \hat{s}_N | G) = E e^{-\lambda \tau \nu},$$

which are the same as equations (34) and (35). Therefore, to prove (34) and (35) it remains to show that if an investor with signal above $\hat{s}_N$ wins the auction he always starts the project even after observing bids of other investors. We showed Section 2 that this is always the case when the number of investors is known. When the number of investors is stochastic this may not necessarily be the case because by observing the number of investors above the participation cut-off the winner may conclude that were many potential investors, and so his signal is actually not so good.

Assumption 14, however, ensures that if an investor with signal above $\hat{s}_N$ wins the auction he always starts the project even after observing bids of other investors. Intuitively, under Assumption 14, the likelihood of the project being good increases with the observed number of investors. Formally, suppose an investor with signal $\hat{s} > \hat{s}_N$ wins the auction. Fix k signal thresholds

$$s_{(1)} = \hat{s} > s_{(2)} > s_{(3)} > \cdots > s_{(k-1)} > s_{(k)} = \hat{s}_N.$$

Define the variables

$$B_i = \sum_{j=1}^{N\nu} \mathbf{1}_{S_j > s_{(i)}}, \quad i = 1, \dots, k,$$

which count the number of investors with signals above the thresholds $s_{(i)}$, $i = 1, \dots, k$. Define numbers τ_i, $i = 1, \dots, k$ as

$$s_{(k)} = 1 - \frac{\tau_k}{N}.$$

By Theorem 4.3.1 of Embrechts, Klüppelberg and Mikosch (2012), for all integers $l_i \geq 0$, $i = 1, \dots, k$,

$$\lim_{N\to\infty} \Pr\left(B_1 = l_1, B_2 = l_1 + l_2, \dots, B_k = l_1 + \dots + l_k | B\right)$$
$$= E\left[\frac{(\nu\tau_1)^{l_1}}{l_1!}\frac{(\nu(\tau_2-\tau_1))^{l_2}}{l_2!}\frac{(\nu(\tau-\tau_{k-1}))^{l_k}}{l_k!}e^{-\nu\tau}\right],$$

and

$$\lim_{N\to\infty} \Pr\left(B_1 = l_1, B_2 = l_1 + l_2, \dots, B_k = l_1 + \dots + l_k | G\right)$$
$$= E\left[\frac{(\lambda\nu\tau_1)^{l_1}}{l_1!}\frac{(\lambda\nu(\tau_2-\tau_1))^{l_2}}{l_2!}\frac{(\lambda\nu(\tau-\tau_{k-1}))^{l_k}}{l_k!}e^{-\lambda\nu\tau}\right].$$

Therefore,

$$\lim_{N\to\infty} \frac{\Pr\left(B_1 = l_1, B_2 = l_1 + l_2, \dots, B_k = l_1 + \dots + l_k | G\right)}{\Pr\left(B_1 = l_1, B_2 = l_1 + l_2, \dots, B_k = l_1 + \dots + l_k | B\right)}$$

$$\frac{E\left[\frac{(\lambda\nu\tau_1)^{l_1}}{l_1!}\frac{(\lambda\nu(\tau_2-\tau_1))^{l_2}}{l_2!}\frac{(\lambda\nu(\tau-\tau_{k-1}))^{l_k}}{l_k!}e^{-\lambda\nu\tau}\right]}{E\left[\frac{(\nu\tau_1)^{l_1}}{l_1!}\frac{(\nu(\tau_2-\tau_1))^{l_2}}{l_2!}\frac{(\nu(\tau-\tau_{k-1}))^{l_k}}{l_k!}e^{-\nu\tau}\right]} = \frac{E\left[\lambda^L\nu^L e^{-\lambda\nu\tau}\right]}{E\left[\nu^L e^{-\nu\tau}\right]} = \frac{\lambda^L\phi^{(L)}(\lambda\tau)}{\phi^{(L)}(\tau)}, \tag{A56}$$

where $L = l_1 + \dots + l_k$, and $\psi^{(L)}$ is the L-order derivative of ψ. By theorem 2.1 of Jarrahiferiz, Borzadaran and Roknabadi (2016), for all $L \geq 0$, $\phi^{(L)}(\lambda\tau)/\phi^{(L)}(\tau)$ is decreasing in τ. Therefore,

$$\frac{\lambda^L\phi^{(L)}(\lambda\tau)}{\phi^{(L)}(\tau)} \geq \frac{\lambda^{L-1}\phi^{(L-1)}(\lambda\tau)}{\phi^{(L-1)}(\tau)} \geq \dots \geq \frac{\phi(\lambda\tau)}{\phi(\tau)}.$$

Thus, we have shown that the likelihood of the project being good increases with the observed number of investors. Therefore, if an investor with signal above $\hat{s}_N$ wins the auction he always starts the project. Thus, social surplus is

$$\pi_0 \Pr(Y_{1,N\nu} \geq \hat{s}_N|G)E(V_1|G) + (1-\pi_0)\Pr(Y_{1,N\nu} \geq \hat{s}_N|B)E(V_1|B). \tag{A57}$$

Using formulas (34) and (35) we can write the limiting social surplus as

$$\pi_0 E(V_1|G)\left(1 - \frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{1-\pi^*} + (\lambda-1)Ee^{-\lambda\tau\nu}\right).$$

Therefore, to find ν that maximizes the limiting social surplus one needs to solve

$$\sup_{\nu} Ee^{-\lambda\tau\nu} \tag{A58}$$

$$s.t. \quad E(e^{-\lambda\tau\nu} - \xi e^{-\tau\nu}) = 0. \tag{A59}$$

where

$$\xi = \frac{1}{\lambda}\frac{(1-\pi_0)}{\pi_0}\frac{\pi^*}{1-\pi^*}.$$

Let us introduce a new random variable $v = e^{-\lambda\tau\nu}$. We can rewrite the above problem as

$$\sup_{v\in[0,1]} \; Ev \tag{A60}$$

$$s.t. \quad Eg(v) = 0, \tag{A61}$$

where

$$g(x) = (x - \xi x^{\frac{1}{\lambda}}).$$

Note that $g(x)$ is a convex function. Therefore, by the Jensen's inequality

$$0 = Eg(v) \geq g(Ev).$$

Hence, the maximum possible value of Ev is $\xi^{\frac{\lambda}{\lambda-1}}$ and is achieved when v takes value $\xi^{\frac{\lambda}{\lambda-1}}$ with probability one. Thus, the limiting social surplus is maximized when there is no uncertainty about the number of investors. *Q.E.D.*

Proof of Proposition 15: Consider the following equation:

$$\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)}\frac{\ln(F_B(\hat{s}_N))}{\ln(F_G(\hat{s}_N))} = \frac{\pi^*}{1-\pi^*}\frac{1-\pi_0}{\pi_0}. \tag{A62}$$

The condition $\pi_0 > \pi^*$ implies that

$$\frac{\pi^*}{1-\pi^*}\frac{1-\pi_0}{\pi_0} < 1.$$

Suppose

$$\frac{f_G(0)}{f_B(0)} < \frac{\pi^*}{1-\pi^*}\frac{1-\pi_0}{\pi_0}.$$

Then,

$$\lim_{\hat{s}_N\to 0}\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)}\frac{\ln(F_B(\hat{s}_N))}{\ln(F_G(\hat{s}_N))} = \frac{f_G(0)}{f_B(0)} < \frac{\pi^*}{1-\pi^*}\frac{1-\pi_0}{\pi_0}.$$

We showed in the proof of Proposition 14 that

$$\lim_{\hat{s}_N\to 1}\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)}\frac{\ln(F_B(\hat{s}_N))}{\ln(F_G(\hat{s}_N))} = 1 > \frac{\pi^*}{1-\pi^*}\frac{1-\pi_0}{\pi_0}.$$

Since

$$\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)}\frac{\ln(F_B(\hat{s}_N))}{\ln(F_G(\hat{s}_N))}$$

is a continuous function of $\hat{s}_N$, Equation (A62) has a solution. Let $s^* \in (0,1)$ be the largest solution to Equation (A62) such that

$$\left(\frac{f_G(\hat{s}_N)}{f_B(\hat{s}_N)}\frac{\ln(F_B(\hat{s}_N))}{\ln(F_G(\hat{s}_N))}\right)'_{|\hat{s}_N=s^*} \neq 0.$$

Fix a small neighborhood $B(s^*)$ of s^*. For any $\hat{s} \in B(s^*)$ define

$$\tau_{G,N}(\hat{s}) = -N\ln(F_G(\hat{s})),$$
$$\tau_{B,N}(\hat{s}) = -N\ln(F_B(\hat{s})).$$

We showed in the proof of Proposition 14 that

$$\lim_{N\to\infty}\frac{\phi(\tau_{G,N}(\hat{s}))}{\phi(\tau_{B,N}(\hat{s}))} = \frac{\ln(F_B(\hat{s}))}{\ln(F_G(\hat{s}))},$$

where we use the fact that if F has a strictly positive density at zero then $\rho = 1$. By the implicit function theorem, for large enough N, there is a solution $\hat{s}_N$ to Equation (A51):

$$\Pr(G|Y_{1,N\nu+1} = \hat{s}_N) = \pi^*.$$

Furthermore,

$$\lim_{N\to\infty}\hat{s}_N = s^*.$$

Since $s^* < 1$ as N goes to infinity the number of signals above $\hat{s}_N$ goes to ∞ with N. Since bids are strictly increasing observing them leads to perfect knowledge of the project type in the limit. *Q.E.D.*

Proof of Lemma 2: Let π_t be the probability that the project is good given the history of news up to the moment t. It is well-known (see e.g., Liptser and Shiryaev (1978)) that π_t evolves according to

$$d\pi_t = \varphi\pi_t(1-\pi_t)d\hat{B}_t, \quad \pi_0 = \hat{\pi}$$

where

$$\hat{B}_t = (\mu_\theta/\sigma)t - \varphi\int_0^t \pi_s ds + X_t$$

is a standard Brownian motion under investors' filtration. Since π_t is a Markov process we can write the value of the project (which now includes the real option value to wait) as $V(\pi_t)$. The standard result then is that in the continuation region (see e.g., Dixit and Pindyck (1994)) the value function satisfies the Bellman equation:

$$\frac{1}{2}\varphi^2\pi^2(1-\pi)^2V''(\pi) - c = 0.$$

A general solution to the above equation is given by

$$V(\pi) = C_1 + C_2\pi - \frac{c}{\varphi^2}(1 - 2\pi)\ln\left(\frac{\pi}{1-\pi}\right), \tag{A63}$$

where constants C_1 and C_2 are determined from the boundary conditions:

$$V(\pi^*) = 0, \tag{A64}$$

$$V'(\pi^*) = 0, \tag{A65}$$

$$V(\pi^{**}) = v(\pi^{**}), \tag{A66}$$

$$V'(\pi^{**}) = v'(\pi^{**}). \tag{A67}$$

Note that if $c > 0$, $V(\pi)$ in Equation (A63) goes to $-\infty$ as $\pi \to 0$. Therefore, it must be that $\pi^* > 0$. *Q.E.D.*