Innovative Uses of Excel in Linear Algebra

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Abstract: *Over the years many new mathematical applications have been developed for spreadsheets such as Microsoft Excel. This paper examines some new illustrations of the use of Excel in linear algebra. Here Excel is used to reinforce definitions and concepts, as well as to carry out computations and to produce eye-catching interactive graphics. Three categories of interest are highlighted – solving systems of linear equations, creating illustrations of lines and planes, and investigating eigenvalues and eigenvectors.*

It has become increasingly clear that a spreadsheet such as Excel is an excellent tool for the study and application of mathematics, especially for applications to modeling, curve sketching, and the implementation of algorithms. However, Excel is also a valuable tool for teaching and visualizing a diverse range of concepts in mathematics. Here we examine some examples from the author's experience in teaching linear algebra in universities in several countries.

Using dynamic Excel models and graphs proved to be an effective approach for communicating concepts visually to well-prepared students in classes taught in English in Korea and to mathematics majors in the United States. Using Excel for computations and graphing helped to clarify concepts, techniques, and definitions in a more basic linear algebra class in Papua New Guinea, and provided a working tool for the students with which they were comfortable.

While acknowledging the strengths and capabilities of powerful mathematical programs like Mathlab, Maple, or Mathematica for teaching linear algebra, Excel also offers some nice advantages.

- Excel is readily available, which is a special advantage in developing nations or when the funding that is needed for special purpose software is limited.
- While its mathematical undergirding may not seem as sophisticated as some software, Excel nonetheless provides very good interactive computational and graphic capabilities.
- In using Excel, students gain skills and experience in using the principal mathematical tool of the workplace, a tool that they will continue to use after their university years.
- Most students are familiar with Excel, reducing the time needed to learn the use of supporting software.
- The spreadsheet format often closely matches the way that we do and view mathematics.
- Frequently we can use Excel to develop mathematical ideas directly, reversing the usual approach of first using formal mathematics to develop topics, and only then implementing the ideas on a computer.
- Using Excel shows that we can do substantial mathematics without needing specialized software.

Solving Systems of Linear Equations

Gaussian Elimination. The primary method of solving a system of linear equations is Gaussian elimination, or pivoting. Although this process is relatively straight-forward to carry out by hand in small examples, that approach is time consuming and prone to errors. Moreover, writing a formal computer program to do this can be quite daunting for many students. One way in which we can use Excel to carry out the pivoting is by using elementary matrices. We illustrate the process in Figure 1, using the system

$$2x_1 + 2x_2 - 3x_3 = 3, \quad 2x_1 - 6x_2 + x_3 = -1, \quad -x_1 + 4x_2 + x_3 = 5$$

In the display of Figure 1a we first enter the system coefficients in Block E4:H6. We then generate repeated copies of the identity matrix in Columns A:C, with blank cells separating successive matrices. We next use Excel's built-in matrix multiplication in Block E8:H10 to multiply the initial system matrix by the first identity matrix. To do this we use the mouse to highlight the Block E8:H10, use the keyboard and mouse to create the equation =MMULT(A4:C6,E4:H6), and press the CTRL-SHIFT-ENTER combination. We then copy the resulting block down Columns E:H, again leaving empty cells between successive matrices.

The first step in the pivoting process is to divide the first equation by the entry in Cell E4, if it is not zero. We do this by entering the formula =1/E4 in Cell A4 of the first identity matrix. This generates 1 in the first position of the first row of the resulting matrix product. The next step is to generate 0's in the first column of the other rows by multiplying the top row by first entry of each successive row and subtracting the multiplied row from the successive rows. We do this for Row 2 by entering the formula =-E9 in Cell A9 in the second identity matrix. We use the same approach for Row 3 by entering =-E10 in Cell A10. We see the resulting output in Figure 1b.

We then repeat this process in successive rows, k, first dividing Row k by the entry in the pivot location of Row k, Column k, and then making the other entries in the column to be 0. We eventually arrive at a place where we can read off the solution as $x = 3, \quad y = 1.5, \quad z = 2$. There are a few additional aspects to consider in certain cases. One additional step occurs if the proposed pivot entry is 0. Then we must first interchange that row with a following row. We do this by modifying the identity matrix appropriately. For example, multiplying by the following matrix, in which we have interchanged Rows 2 and 3 of the identity matrix, will interchange Rows 2 and 3 of the current system matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

	A	B	C	D	E	F	G	H
2	elem matrices				system of equations			
3					x_1	x_2	x_3	c
4	1	0	0		2	2	-3	3
5	0	1	0		2	-6	1	-1
6	0	0	1		-1	4	1	5
7								
8	1	0	0		2	2	-3	3
9	0	1	0		2	-6	1	-1
10	0	0	1		-1	4	1	5
11								
12	1	0	0		2	2	-3	3
13	0	1	0		2	-6	1	-1
14	0	0	1		-1	4	1	5
27	…						…	
28					2	2	-3	3
29					2	-6	1	-1
30					-1	4	1	5

	A	B	C	D	E	F	G	H
2	elem matrices				system of equations			
3					x_1	x_2	x_3	c
4	0.5	0	0		2	2	-3	3
5	0	1	0		2	-6	1	-1
6	0	0	1		-1	4	1	5
7								
8	1	0	0		1	1	-1.5	1.5
9	-2	1	0		2	-6	1	-1
10	1	0	1		-1	4	1	5
11								
12	1	0	0		1	1	-1.5	1.5
13	0	-0.13	0		0	-8	4	-4
14	0	0	1		0	5	-0.5	6.5
27	…						…	
28					1	0	0	3
29					0	1	0	1.5
30					0	0	1	2

Figure 1 Gaussian Pivoting via Elementary Matrices

There are many additional topics that we and our students can pursue using the same approach: putting the matrices of $m \times n$ systems into canonical form, finding the inverse of a square matrix A by replacing the column of constants by the columns of an identity matrix, using the pivoting matrices to factor the matrices A and A^{-1}, and to obtain the determinant of A. We can also use Excel's matrix multiplication function to check our answers.

Those students who had weaker skills in traditional programming languages found this approach much easier to understand and implement than other methods, and students in general found it easy to understand how to factor matrices and their inverses into products of elementary matrices.

Solving $n \times n$ Systems Using Inverses. If the determinant of the matrix coefficients, $|A|$, of an $n \times n$ system, $AX = C$, is non-zero, then we can use the inverse of A to solve the system $AX = C$ by computing $X = A^{-1}C$. In the display below we find the determinant of A as =MDETERM(A3:C5) in Cell G2. After entering the coefficients of the system in the Block A3:D5, we use the mouse to highlight the solution cells G3:G5, and then enter the formula =MMULT(MINVERSE(A3:C5),D3:D5), and press CTRL-SHIFT-ENTER to complete the process. We can also check our result by using matrix multiplication.

	A	B	C	D	E	F	G
2	x_1	x_2	x_3	c		det	-32
3	2	2	-3	3		x_1	3
4	2	-6	1	-1		x_2	1.5
5	-1	4	1	5		x_3	2

Figure 2 Solving a Linear System Using Matrix Inverse

Using Excel's Solver. In this model we can find the solution to a linear system by using Excel's Solver tool. Although our approach here is not generally used in a linear algebra class, it is used in many other areas of mathematics, including statistics, and provides students with an illustration of the use of this powerful Excel tool.

In Cells G3:G5 of Figure 3a we enter an estimate for the solution of the system. In Column I we enter a formula to compute the value that results when we substitute our estimates into each equation. In Column J we compute the square of the error between the result in Column I and the equation constant in Column D. We find the sum of these cells in J6. If the value of J6 is 0, then each of the squared errors will be 0, and we will have found the solution. To ensure that this happens, we activate Excel's Solver command, set the value in Cell J6 to 0 by changing the entries in Cells G3:G5. When we click on the Solve button, Excel adjusts our estimate to produce the result in Figure 3b. The solutions appear in Column G.

	A	B	C	D	E	F	G	H	I	J
2	x_1	x_2	x_3	c			est		sub	err^2
3	2	2	-3	3		x_1	1		1	4
4	2	-6	1	-1		x_2	1		-3	4
5	-1	4	1	5		x_3	1		4	1
6									error	9

	A	B	C	D	E	F	G	H	I	J
2	x_1	x_2	x_3	c			est		sub	err^2
3	2	2	-3	3		x_1	3		3	4E-10
4	2	-6	1	-1		x_2	1.5		-1	3E-10
5	-1	4	1	5		x_3	2		5	1E-10
6									error	8E-10

Figure 3 Solving a Linear System with Excel's Solver

We can use the Solver to pursue a series of similar activities, such as solving non-linear equations and systems, determining the regression line in statistics, and finding the extreme values of a function in calculus.

Cramer's Rule. Cramer's Rule uses determinants to solve an $n \times n$ system $Ax = C$ for $|A| \neq 0$, by $x_i = |A_i|/|A|$, for $i = 1, \ldots, n$ where A_i is the matrix obtained by replacing Column i of A by the constant column. In Figure 4, we compute the entries for A_i in Block F3:H5 by entering the equation =IF(A$1=$G$1,$D3,A3) into Cell F3 and copying it to F3:H5. We use Excel's MDETERM function in Cells J3 and J4, and divide the two determinants in Cell J5. We use text functions to generate the label identifying the component in Cell I5.

	A	B	C	D	E	F	G	H	I	J
1	1	2	3			var	1			
2	x_1	x_2	x_3	c		numerator				
3	2	2	-3	3		3	2	-3	det n	-96
4	2	-6	1	-1		-1	-6	1	det d	-32
5	-1	4	1	5		5	4	1	x1 =	3

Figure 4 Solving a Linear System with Cramer's Rule

Iterative Algorithms. In numerical analysis we study computational algorithms for solving large systems for which our earlier techniques are computationally inadequate. One of these is Jacobi's algorithm for solving linear systems, which is also easy to implement in Excel. To use the algorithm, we first rewrite the equations to solve the i^{th} equation for x_i. Thus,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1,\quad a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2,\quad a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

$$x_1 = c_1/a_{11} - a_{12}x_2/a_{11} - a_{13}x_3/a_{11}, x_2 = c/a_{22} - a_{21}x_1/a_{22} - a_{23}x_3/a_{22}, x_3 = c_3/a_{33} - a_{31}x_1/a_{33} - a_{32}x_2/a_{33}$$

In the Jacobi algorithm shown in Figure 5a, we enter the original system coefficients in Cells A3:D5, and formulas for the adjusted equations in Cells A9:D11. We note that the system used in this example is different from our initial example. We next enter an initial estimate of the solution in Cells H3:J3, and substitute those entries into the adjusted equations to obtain a second estimate in Cells H4:J4. We then copy these expressions down their columns and see that the process converges to the solution. However, the Jacobi algorithm does not always converge, although there are conditions under which it does. For example, it always converges for a diagonally dominant system, such as the one of Figure 5. The xy-chart of Figure 5b illustrates the convergence.

	A	B	C	D	E	F	G	H	I	J
1	original equations						iterated approximations			
2	x_1	x_2	x_3	c		n	con	x_1	x_2	x_3
3	5	2	-3	3		0	1	1	1	1
4	2	-6	1	1		1	1	0.8	0.333	2.5
5	-1	1	2	5		2	1	1.967	0.517	2.733
6						3	1	2.033	0.944	3.225
7	rearranged equations					4	1	2.157	1.049	3.044
8	c	x_1	x_2	x_3		5	1	2.007	1.06	3.054
9	0.6	0	-0.4	0.6		6	1	2.009	1.011	2.974
10	-0.17	0.333	0	0.167		7	1	1.98	0.999	2.999
11	2.5	0.5	-0.5	0		8	1	2	0.993	2.991
12						9	1	1.997	0.998	3.003
13						10	1	2.003	1	2.999
14						11	1	2	1.001	3.002
15						12	1	2.001	1	2.999
16						13	1	2	1	3

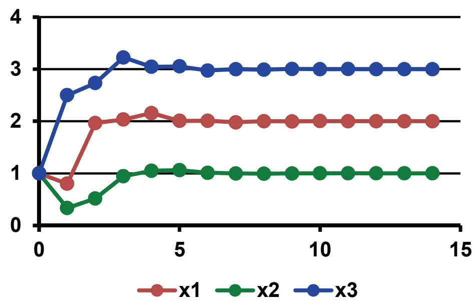

Figure 5 Jacobi Algorithm

Numerical analysis texts provide additional iterative algorithms to solve systems. The Gauss-Seidel algorithm is similar to Jacobi, except that to find the new value of x_2, it uses the new value (same row) of x_1 together with the previous value of x_3. We treat other variables similarly. Using these illustrations provides students with examples of techniques that are used to approximate the solutions of much larger systems that appear in applied fields. The Excel models are also handy for comparing the two algorithms. Students can discover that while Gauss-Seidel appears to be more efficient than Jacobi, there are examples in which it diverges while Jacobi converges, and vice versa. Students can also encounter the need in some cases to provide good initial estimates of the solution.

1. Lines, Planes, and Bases

In studying the structures and algorithms of linear algebra, it often helps to provide pictures that illustrate the ideas involved. This is most easily done in 2- or 3-dimensional space. In this section we indicate a few of the things that we can do.

Lines in $\mathbb{R}^2$. There are a variety of ways in which we can write the equations of lines. One that is particularly useful in linear algebra is illustrated in the following model. Suppose that we start with two points in the *xy*-plane, $\mathbb{R}^2$. In Figure 6a we enter their coordinates $(x_1, y_1) = (1,2)$ and $(x_2, y_2) = (2,3)$ in Cells A3:B3, A4:B4. Then in Cells C3:D3 we compute the components of the vector $[u,v]$ drawn from the first point to the second. Other points on the line are found by adding multiples of that vector to the vector from the origin to (x_1, y_1) as $[x_1, y_1] + t[u,v]$. This gives us parametric equations for the line as $x = x_1 + tu$, $y = y_1 + tv$. We produce line segments in the graph of Figure 6b from the various columns. Next, we create a scroll bar to vary the value t (see [3] for a discussion of creating scroll bars) which allows students to see a picture of what this represents. Thus, $t = 0$ gives us the first point, $t = 1$ the second point, and $t = 2$ another point further along the line, and so on. The distance between points is established by the length of the vector $[u,v]$. Positive values of t produce points in the direction of the vector, while negative values give points in the opposite direction.

Because scroll bars can take on only non-negative integer values, we create an integer value in Cell F3, and link it to the scroll bar. We then compute the t value in Cell F5 as =(F3-40)/10. If we allow the scroll bar to vary from 0 to 80 in steps of size 1, then t varies from -4.0 to 4.0 in steps of size 0.1. As we move the slider on the scroll bar, we see the line being traced out.

	A	B	C	D	E	F	G	H	I
1	points		vector						
2	x	y	x	y					
3	1	2	1	1		60			
4	2	3							
5					t	2	t = 2.0		
6	points		from origin		line segment			marked line	
7	x	y	x	y	x	y	t	x	y
8	**1**	**2**	0	0	1	2	-4	-3	-2
9	2	3	1	2	3	4	-3	-2	-1
10							-2	-1	0
11	coords	(1,2)					-1	0	1
12		(2,3)					0	1	2
13							1	2	3
14							2	3	4
15							3	4	5
16							4	5	6

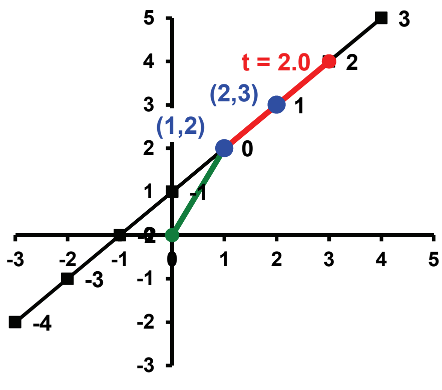

Figure 6 Interactive Graph of a Line in 2 Dimensions

Graphs such as the one in Figure 6 help students to get a feeling for the roles of the various components that arise in deriving the parametric equations of a line. Students generally appreciate having an interactive graph through which they can understand concepts that are new to them, rather than just the static picture in a text, or one that is created and modified on the board by the instructor.

Planes and Lines in 3-space. While lines and planes in $\mathbb{R}^3$ are more difficult to create in Excel, we can still do this. Because of the large size of our models, here we present only portions of a few of them to show what can do in Excel. Although Excel does provide some 3-dimensional charts, we prefer to use xy-charts to create perspective drawings in $\mathbb{R}^3$. In Figure 7 we enter the coordinates of points as (x, y, z), and then convert them into two dimensional points (X, Y) to plot by using $X = y - x/\sqrt{2}$ and $Y = z - x/\sqrt{2}$.

	A	B	C	D	E	F	G	H	I	J	K
1	coordinate axes						unit cube				
2	x	y	z	X	Y		x	y	z	X	Y
3	0	0	0	0	0		0	0	1	0	1
4	0	0	1	0	1		1	0	1	-0.7	0.29
5	0	0	2	0	2		1	0	0	-0.7	-0.7
6							1	1	0	0.29	-0.7
7	0	0	0	0	0		0	1	0	1	0
8	0	1	0	1	0		0	1	1	1	1
9	0	2	0	2	0		0	0	1	0	1
10											
11	0	0	0	0	0		1	1	0	0.29	-0.7
12	1	0	0	-0.7	-0.7		1	1	1	0.29	0.29
13	2	0	0	-1.4	-1.4						
14							1	0	1	-0.7	0.29
15							1	1	1	0.29	0.29
16							0	1	1	1	1

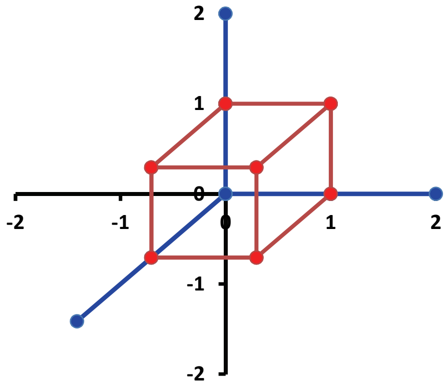

Figure 7 Perspective Drawings in 3 Dimensions

In Figure 8a we have created a line in $\mathbb{R}^3$ much as we did for $\mathbb{R}^2$. To better envision 3-dimensional aspects, we shade the coordinate planes (see [1] for a discussion of the technique used) and supply dashed lines to help us to see directions. The picture in Figure 8b illustrates how we can start with two vectors that are not co-linear and use them to create a plane in $\mathbb{R}^3$.

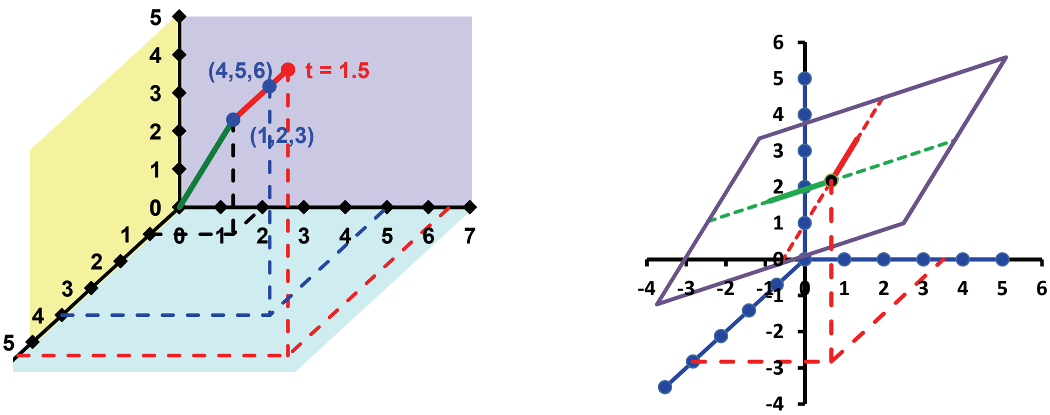

Figure 8 Perspective Drawings of Planes

We can also effectively use the techniques described here in calculus. Figure 9 shows tangent vectors to a surface, $z = f(x, y)$. These vectors are found using the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$. We create scroll bars to move the point and the tangent vectors. Calculus provides us with a great variety of places to use Excel – to illustrate theorems such as the Mean Value Theorem, to implement Newton's Method, and many others.

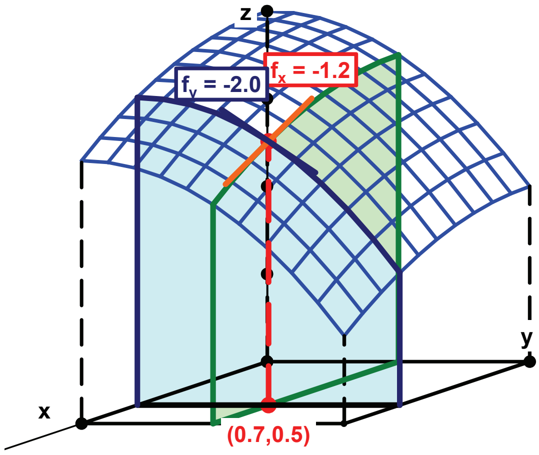

Figure 9 Vectors and Planes in Calculus of Surfaces

Change of Bases. One of the most challenging topics for many students of linear algebra is that of a basis. In 2 and 3 dimensions we can create the gridlines determined by a basis to help to explain the coordinates of a point with respect to the usual basis and another one. In Figure 10a we first enter two vectors of a basis in Cells A2:A3 (blue) and B2:B3 (red). These column vectors allow us to do our calculations as we usually do, but in graphing in Excel it is often easier to use row vectors as in Rows 7-8 of Columns A:B and C:D. With a little work we generate the red and blue grid lines.

Next, in Cells F2:F3 we enter the standard coordinates $\begin{bmatrix} x_s \\ y_s \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ of the green vector with respect usual axes. From Figure 10b we can see that with respect to the new basis the point's coordinates would be $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We observe that $\begin{bmatrix} x_s \\ y_s \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. It follows that if we let $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ be the matrix formed from the new basis, and let $\begin{bmatrix} x_u \\ y_u \end{bmatrix}$, $\begin{bmatrix} x_n \\ y_n \end{bmatrix}$ be the components of a vector with respect to the standard and new bases, then $\begin{bmatrix} x_s \\ y_s \end{bmatrix} = A\begin{bmatrix} x_n \\ y_n \end{bmatrix}$ and

$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^{-1} \begin{bmatrix} x_s \\ y_s \end{bmatrix}$. In our model we enter the standard components in Cells F2:F3, compute the new components in Cells G2:G3, and use that to re-compute the standard components in Cells H2:H3.

	A	B	C	D	E	F	G	H	I	J	K
1	new basis		inverse mat			v_s	v_n	v_s			
2	2	1	0.429	-0.14		4	1	4		[4.0,5.0]	
3	-1	3	0.143	0.286		5	2	5		[1.0,2.0]	
4											
5	blue vector		red vector			red parallels		blue parallels		green vec	
6	x	y	x	y	n	x	y	x	y	x	y
7	0	0	0	0	-6	-6	24	6	-24	0	0
8	2	-1	1	3	-6	-18	-12	-18	-12	4	5
9										extra label	
10					-5	-4	23	7	-21	x	y
11					-5	-16	-13	-17	-9	4	5
12											
13					-4	-2	22	8	-18		
14					-4	-14	-14	-16	-6		

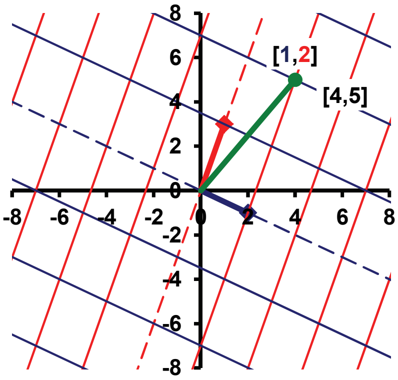

Figure 10 Illustrating Change of Basis

It has been the author's experience that displaying the grid of an alternate basis in this way is an effective means for helping students to come to a better understanding of what a basis is. They come to see and better understand how a vector can have more than one representation, and to see how the components of the representations are related to each other. Having taught linear algebra in Papua New Guinea in 1975 when no computational tools were available, and in 2010 when Excel was used, there was a marked increase in the students' overall conceptualization and understanding of these and other ideas of linear algebra.

2. Eigenvalues and Eigenvectors

Definition of Eigenvalues and Eigenvectors. Much of linear algebra focuses on the topic of eigenvalues and eigenvectors. We recall that a real number, λ, is an eigenvalue of an $n \times n$ matrix, A, if there is a non-zero vector v so that $Av = \lambda v$. The vector v is called an eigenvector corresponding to λ. In Figure 11 we look at the geometric interpretation of this definition. Thus, if λ is an eigenvalue of A, then Av and λv go in the same (or directly opposite) direction.

In Figure 11 we enter a 2×2 matrix A into Cells C2:D3, and generate a unit vector, v, in the direction of α degrees measured anticlockwise from the x-axis in Cells G2:G3, using the formula $v = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, where t is the radian measure of α. We compute the vector Av in Cells H2:H3 and its length, κ, in Cell H4.

We create our graphic image in Figure 12 by drawing in the vectors v and Av together with circles of radius 1 (length of v) and κ (length of Av). We use a scroll bar to vary the angle of vector, v. As we move the slider in the scroll bar, the vectors change. When they align, we have located an eigenvalue. The eigenvalue is the radius of the circle whose radius is the length of Av.

	A	B	C	D	E	F	G	H	I	J
1						angle	v	Av		
2		matrix	-1	3	deg	76	0.242	2.669		
3			2	0	rad	1.326	0.97	0.484		
4		angle				length	1	2.71	e v	2
5							[0.2,1.0]		e v	-3
6			in circle		out circle		in vector v		out vec Av	
7	n	t	x	y	x	y	x	y	x	y
8	0	0	1	0	2.712	0	0	0	0	0
9	1	0.017	1	0.017	2.712	0.047	0.242	0.97	2.669	0.484
10	2	0.035	0.999	0.035	2.711	0.095				
11	3	0.052	0.999	0.052	2.709	0.142				

Figure 11 Visualizing Definition of Eigenvectors and Eigenvalues (layout)

In Figure 12a we see a blue vector which is not an eigenvector, while in Figure 12b we see that 2 is an eigenvalue, with the eigenvector [0.707,0.707]. Actually, this is $\left[1/\sqrt{2},1/\sqrt{2}\right]$. This example has served as an effective demonstration in helping students to understand the definition of eigenvalues and eigenvectors.

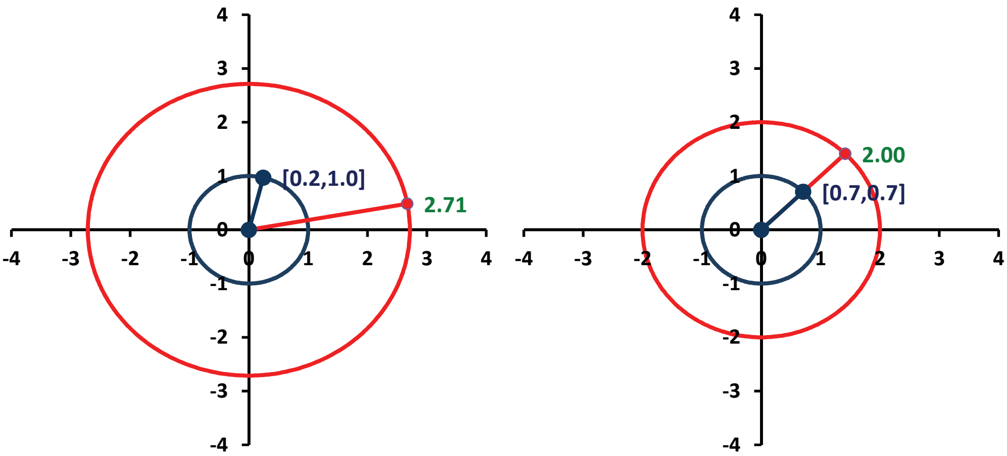

Figure 12 Visualizing Definition of Eigenvectors and Eigenvalues (graphs)

Using Excel's Solver for Eigenvalues and Eigenvectors. There are many other ways to investigate eigenvalues and eigenvectors in Excel. Here we present one more. We start with the 4×4 matrix, A, given in Cells B3:E6. In linear algebra (see [2], [4]) we find that λ is an eigenvalue of A if and only if $|A-\lambda I|=0$. There are many mathematical and computational methods that are used to find the values of λ that satisfy this condition. Here we will use the Excel Solver. We enter an estimate of an eigenvalue in Cell B1 and enter the MDETERM function in Cell E to find the value of $|A-\lambda I|$. We show this in Figure 13a.

We now use the Solver to set the value of Cell E1 to 0 by changing Cell B1. This produces the output of Figure 13b with $\lambda \approx 3.61$. We next enter an estimate for a corresponding eigenvector in the Cells H3:H6. We compute the length of that eigenvector in Cell H1 as the square root of the squares of the components of the vector. We now want to make sure that $Av = \lambda v$. We do this by computing the components of Av and λv in Columns I and J. We then find the square of the difference (i.e. the errors) of the paired components in Column K, and compute the sum of the errors in Cell K8. If the value of Cell K8 is 0, then each of the error terms must be 0, making the vectors equal. Next, we use the Solver again, this time setting Cell K8 to 0 by changing Cells H3:H6. Before doing this, we first add a constraint to ensure that the length of the vector is 1. This will prevent us from obtaining the trivial zero vector $[0,0,0,0]$. The result of our work appears in Figure 13c.

	A	B	C	D	E
1	λ	3		|A–λI|	-38
2					
3	A	1	5	1	1
4		1	-2	1	0
5		1	1	0	1
6		1	1	1	1
7					
8	A–λI	-2	5	1	1
9		1	-5	1	0
10		1	1	-3	1
11		1	1	1	-2

	A	B	C	D	E	F	G	H	I	J	K
1	λ	3.62		|A–λI|	0		len	2			
2								v	λv	Av	err^2
3	A	1	5	1	1		x_1	1	3.62	8	19.2
4		1	-2	1	0		x_2	1	3.62	0	13.1
5		1	1	0	1		x_3	1	3.62	3	0.38
6		1	1	1	1		x_4	1	3.62	4	0.15
7											
8	A–λI	-2.6	5	1	1					sum err^2	32.8
9		1	-5.6	1	0						
10		1	1	-3.6	1						
11		1	1	1	-2.6						

	G	H	I	J	K
1	len	1			
2		v	λv	Av	err^2
3	x_1	0.73	2.65	2.65	0
4	x_2	0.2	0.73	0.73	0
5	x_3	0.4	1.45	1.45	0
6	x_4	0.51	1.85	1.85	0
7					
8				sum err^2	0
9					
10					
11					

Figure 13 Approximation of Eigenvalues and Eigenvectors Using Solver

Quadratic Forms. We conclude with a traditional geometric application. The equation $5x^2 - 4xy + 5y^2 = 16$ is called a quadratic form. We can write this using matrices as $[x, y]\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 16$. If we rewrite vectors in terms of the eigenvectors and eigenvalues of the matrix we obtain the equation $3x^{*2} + 7y^{*2} = 16$ to get the curve of Figure 14a. Similarly, the curve on the right comes from the quadratic form $x^2 - 8xy - 5y^2 = 16$ (see [2], pp. 401-405).

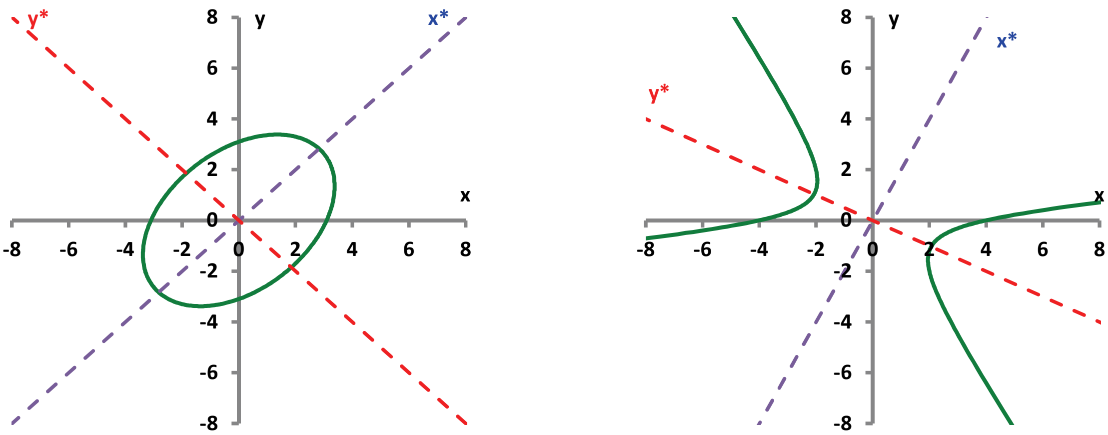

Figure 14 Curves and Quadratic Forms

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