Inroduction in Theoretical Physics
Part I: Electrodynamics

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Lecture notes partly taken from scripts by Prof. Claudius Gros, (University of Frankfurt), Prof. W. Cassing, (University of Giessen), Prof. Dr .G. Soff, (TU Dresden) Prof. E. Schachinger (TU Graz) with kind permission.

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To be found under:

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→ Introduction in Theoretical Physics

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Kapitel 0

Introduction in electrodynamics

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0.1 Electric charge

While the *mass* takes a central role in classical mechanics, the starting point of electrodynamics is the *charge*. It displays a series of fundamental characteristics, which have been confirmed by various experimental measurements:

1.) There are two kinds of **charges:** *positive and negative* ones. Charges with the same sign repel each other while the ones with opposite sign attract each other.

2.) The total charge of a system of mass points is the algebraic sum of the separate charges; *The charge is a scalar.*

3.) The *total charge* of a closed system is *constant* and its numerical value is independent of the motion of the system.

4.) The charge only shows up as a multiple of an *elementary charge* e (charge of an electron),

$$q = ne; \qquad n = 0, \pm 1, \pm 2, \pm 3, ...$$

A classical evidence for the *quantisation of the charge* is the *Millikan experiment*. Though one-third-valued charges occur in the elementary particles *Quarks*, e.g. $q = \pm(\frac{1}{3})e$, or $q = \pm(\frac{2}{3})e$, these Quarks are not observable as free particles in the energy region of interest.

0.2 Electrostatics

The most simple problem of electrodynamics is the case of charges at rest, which is called electrostatics. If a *test charge* q is placed in the vicinity of one (or more) point charges, a force K acts on this test charge, which is generally dependent on its location r:

$$\mathbf{K} = \mathbf{K}(\mathbf{r}) \ .$$

If q is replaced by another test charge q', then the force K' acting on q' reads:

$$\mathbf{K}'/q' \; = \; \mathbf{K}/q \; .$$

Electric field

This observation suggests to introduce the concept of electric field:

$$\mathbf{E}(\mathbf{r}) \; = \; \frac{1}{q}\mathbf{K}(\mathbf{r}) \tag{1}$$

This field, which is generated by the static point charges, assigns a triple of real numbers, which transforms like a vector, to every point in space $\mathbf{r}$.

The task of electrostatics is to find the general connection between charge distribution $\rho(\mathbf{r})$ and electric field $\mathbf{E}(\mathbf{r})$ and to calculate the field for a given charge distribution (e.g. for a homogenous sphere).

0.3 Magnetostatics

Moving charges in the form of stationary currents produce magnetostatic fields, which we introduce in analogy to the electrostatic fields. We have the following experimental observation: If a test charge q is placed in the vicinity of a conductor in which a stationary current flows, the force acting on q in the position $\mathbf{r}$ can be written as

$$\mathbf{K}(\mathbf{r}) \; = \; q\left(\mathbf{v} \times \mathbf{B}(\mathbf{r})\right) \; .$$

Here, $\mathbf{v}$ is the velocity of the test charge and $\mathbf{B}(\mathbf{r})$ a vector field (independent of $\mathbf{v}$), the **magnetic induction**, produced by the stationary current.

The task of magnetostatics is to find the general relation between a stationary current distribution $\mathbf{j}(\mathbf{r})$ and the magnetic field $\mathbf{B}(\mathbf{r})$, i.e. to evaluate the field for a given current distribution (e.g. for a stationary circular current).

0.4 Literature

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Teil I

Electrostatics

Kapitel 1

Coulomb's law

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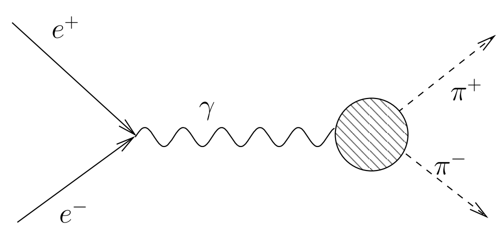
1.1 Charge conservation and charge invariance

In the introduction we have briefly summarized the fundamental properties of electric charge. For an experimental verification of these properties, we first need a measure prescription for the charge. Such a prescription will be presented in the next subsection. First we present some additional issues about *charge conservation and charge invariance.*

Pair production

Especially impressive evidences for charge conservation are pair production and pair annihilation processes. For example, an electron (e^-) and a positron (e^+) annihilate into a high-energy, massive photon (γ-quantum), which has no charge. On the other hand, in pair production processes the same amount of positive and negative charge is generated (e.g. π^+, π^- mesons). .



$$+e + (-e) = 0 \qquad q = 0 \qquad + e + (-e) = 0$$

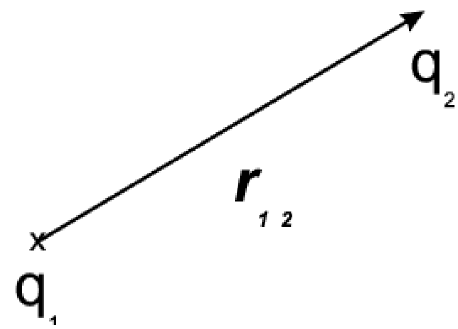
Charge invariance can be verified, for example, by the fact that atoms and molecules are neutral, even though the state of motion of protons and electrons is very different. This is especially clear in the case of the helium atom (^4He) and the deutrium molecule (D_2). Both consist of 2 protons, 2 neutrons and 2 electrons, which makes them electrically neutral, even though the state of motion of the protons in the nucleus of the helium atom is very different from the one in the D_2 molecule: The *ratio between their kinetic energies is about* 10^6. The average distance between the protons in the D_2 molecule is of the order of 10^{-8} cm, while it is of the order of 10^{-13} in the He-nucleus.

1.2 Coulomb force

As an experimentally established starting point for electrostatics we consider Coulomb's force law between two point charges:

$$\mathbf{K}_{12} = \Gamma_e \frac{q_1 q_2}{r_{12}^3} \mathbf{r}_{12} \tag{1.1}$$

is the force acted upon charge q_2 from charge q_1. Here $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1, r_{12} = |\mathbf{r}_{12}|$ and Γ_e a not yet defined proportionality constant.



Properties:

1.) Attraction (repulsion) for charges with opposite (equal) sign.

2.) $\mathbf{K}_{12} = -\mathbf{K}_{21}$: Actio = Reactio. As a consequence, the momentum of the two particles is conserved.

3.) Central force: Since a point charge (described by the scalar quantities m, q) has no distinguished direction in space. ($\rightarrow$ conservation of angular momentum)

Note: (1.1) does not hold any more for fast charges. The electromagnetic field has to be taken into account in the momentum- and angular-momentum-balance.
(1.1) has to be complemented with the *superposition principle*:

$$\mathbf{K}_1 = \mathbf{K}_{21} + \mathbf{K}_{31} \tag{1.2}$$

for the force that two point charges q_2 and q_3 act upon q_1.

Measure prescription for the charge
Two point charges q, q' can be compared by looking at the force acted upon them by a given charge Q. According to (1.1), we have:

$$\frac{q}{q'} = \frac{K}{K'} . \tag{1.3}$$

In this way one can determine the ratio between charges by measuring forces. After a unit charge (charge of the electron or positron) is chosen, one can measure charges relative to this unit charge.

Units systems
There are two possibilities to choose the proportionality constant Γ_e:

i) Gaussian cgs-system: Here Γ_e is chosen as a dimensionless constant, specifically

$$\Gamma_e = 1 , \tag{1.4}$$

then the units for the charge is determined according to (1.1) as

$$[q] = [\text{ Force}]^{1/2}[\text{Length}] = \text{dyn}^{1/2} \times \text{ cm} . \tag{1.5}$$

The electrostatic unit charge is then that charge which acts upon another equal charge with the force of 1 dyn at a distance of 1 cm. The Gaussian cgs-system is preferred in fundamental physics.

ii) **MKSA**-System.

In addition to the mechanic units (meter, kilogramm, second) one has to introduce a unit for the charge **Coulomb** = ampere× second. 1 ampere is defined as the electric current that separates 1.118 mg silver per second out of a silver nitrate solution. If we write

$$\Gamma_e = \frac{1}{4\pi\epsilon_0} , \tag{1.6}$$

then the constant ϵ_0 takes the value

$$\epsilon_0 = 8.854 \cdot 10^{-12} \frac{\text{Coulomb}^2}{\text{Newton} \cdot \text{ Meter}^2} \tag{1.7}$$

The MKSA-system has established itself in applied electrodynamics (e.g. electrical engineering).

1.3 The electric field of a system of point charges

The force produced by N static point charges q_i at positions $\mathbf{r}_i$ acting on a test charge q at position $\mathbf{r}$ is according to (1.1) and (1.2):

$$\mathbf{K} = \frac{q}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} = q\mathbf{E}(\mathbf{r}) , \tag{1.8}$$

where we denote

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \frac{(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \tag{1.9}$$

as the (static) **electric field**, produced by the point charges q_i at position $\mathbf{r}$. According to (1.8) $\mathbf{E}$ is a vector field, since q is a scalar. For given charge q, (1) (or (1.8)) shows how an electric field can be experimentally measured. In doing so, one must make sure that the test charge is so small that its influence on the measured field can be neglected.

Electrostatic potential
in analogy to the case of gravitational theory in mechanics, we can extract the vector-function $\mathbf{E}(\mathbf{r})$ from the electostatic potential of a scalar function through differentiation:

$$\phi(\mathbf{r}) \;=\; \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}-\mathbf{r}_i|} \;, \tag{1.10}$$

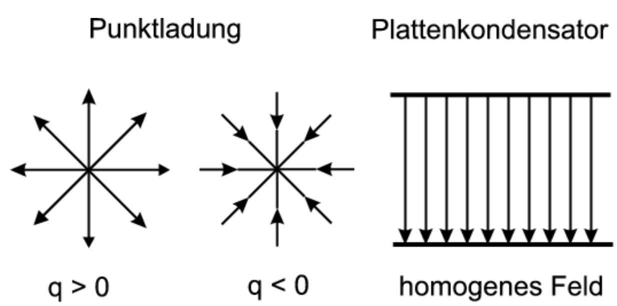
$$\mathbf{E} \;=\; -\nabla\phi \;. \tag{1.11}$$

The (potential) energy of static mass points with charges q_i is therefore

$$U \;=\; \frac{1}{2}\sum_{i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}_i-\mathbf{r}_j|} \;=\; \frac{1}{2}\sum_{i=1}^{N} q_i \phi(\mathbf{r}_i) \;, \tag{1.12}$$

where $\phi(\mathbf{r}_i)$ is the potential in $\mathbf{r_i}$ produced by the other charges The factor (1/2) on the right hand side of (1.12) corrects the double counting of the contributions in the sum $\sum_{i\neq j}$.
Note: In (1.12), strictly speaking, the (infinite) *self-energy* for $i = j$ must be substracted from the term on the right side.

Examples:



1.4 Continuous charge distributions

We replace

$$\sum_i q_i \;.... \quad \rightarrow \quad \int dV \rho(\mathbf{r}) \;... \;, \tag{1.13}$$

where $\rho(\mathbf{r})$ is the charge density in position $\mathbf{r}$, with the normalisation

$$Q \;=\; \sum_i q_i \;=\; \int dV \rho(\mathbf{r}) \;. \tag{1.14}$$

Therefore, we have instead of (1.9), (1.10), (1.12):

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0} \int dV' \rho(\mathbf{r}') \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \;, \tag{1.15}$$

$$\phi(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\int dV'\rho(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{1.16}$$

and

$$U \;=\; \frac{1}{2}\int dV\rho(\mathbf{r})\phi(\mathbf{r})\;. \tag{1.17}$$

Example: homogenously charged sphere

$$\rho(\mathbf{r}) \;=\; \rho_0 \quad \text{für} \quad |\mathbf{r}| \le R; \quad \rho(\mathbf{r}) \;=\; 0 \quad \text{sonst.} \tag{1.18}$$

The integration in (1.16) reads (with $r \equiv |\mathbf{r}|$ and $c = \cos\theta$):

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\;2\;\pi\rho_0\int_0^R r'^2\;d\;r'\int_{-1}^1 d\;c\;(r^2 + r'^{\;2} - 2r\;r'\;c)^{-1/2}$$

and gives:

$$\phi(\mathbf{r}) \;=\; \frac{Q}{4\pi\epsilon_0|\mathbf{r}|} \quad \text{for} \quad r \ge R; \quad \phi(\mathbf{r}) \;=\; \frac{\rho_0}{\epsilon_0}(\frac{R^2}{2} - \frac{r^2}{6}) \quad \text{for} \quad r \le R \tag{1.19}$$

wih

$$Q \;=\; \int dV\rho(\mathbf{r}) \;=\; \frac{4\pi}{3}\rho_0 R^3\;. \tag{1.20}$$

From (1.11) it then follows for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3} \quad \text{für} \quad r \ge R; \quad \mathbf{E}(\mathbf{r}) \;=\; \frac{\rho_0}{3\epsilon_0}\mathbf{r} \quad \text{für} \quad r \le R\;. \tag{1.21}$$

For the energy U we find with (1.17) and (1.19):

$$U \;=\; \frac{\rho_0}{2}\int dV\;\phi(\mathbf{r}) \;=\; \frac{4\pi\rho_0^2}{2\epsilon_0}\int_0^R r^2 dr\;(\frac{R^2}{2} - \frac{r^2}{6}) \;=\; 2\pi\frac{\rho_0^2}{\epsilon_0}\frac{2R^5}{15} = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0}\frac{1}{R}\;. \tag{1.22}$$

Application: Determination of the classic electron radius

According to (1.22) the **self energy** of a point particle ($R \to 0$) becomes infinite. Now, according to the theory of relativity, the energy of a particle at rest, e.g. of an electron, is connected with its rest mass m_0 via

$$E_0 \;=\; m_0 c^2\;. \tag{1.23}$$

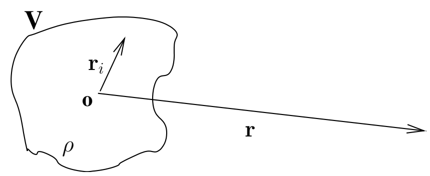
A strictly point-like (charged) particle would then have an infinite large rest mass, according to (1.22)! If we on the other hand associate the total (finite) rest mass of an electron to its electrostatic energy, we must assign to the electron a finite radius R_0, the *classical electron radius*,

$$R_0 \;=\; \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 m_0 c^2} \;\approx\; 10^{-13}\,\text{cm} \;=\; 1\,\text{fm} \;=\; 10^{-5}\,\mathring{A}\;. \tag{1.24}$$

For sizes $< 10^{-13}$ cm we should expect in the case of electrons deviation from Coulomb's law.

1.5 Electric dipole

Let's assume a *charge distribution limited to a volume* V (discreet or continuous) and analyze the potential ϕ in a point P far from the volume V.



The *point of origin* 0 *lies within* V; we can for example assume that 0 is the center of mass of all charge carriers, defined by

$$\mathbf{r}_q = \frac{\sum_i |q_i| \mathbf{r}_i}{\sum_i |q_i|} . \tag{1.25}$$

As long as $r_i \ll r$, we can represent (1.10) as a *Taylor-expansion*,

$$\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \ldots , \tag{1.26}$$

if we use

$$\frac{1}{|\mathbf{r}-\mathbf{r}_i|} = \frac{1}{\sqrt{r^2 - 2\,\mathbf{r}\cdot\mathbf{r}_i + r_i^2}} = \frac{1}{r}\,\frac{1}{\sqrt{1-\left(2\,\mathbf{r}\cdot\mathbf{r}_i - r_i^2\right)/r^2}} \tag{1.27}$$

and expand in $\left(2\,\mathbf{r}\cdot\mathbf{r}_i - r_i^2\right)/r^2$. With $(1-x)^{-1/2} = 1 + x/2 + O(x^2)$ the result is correct until order $O(r_i/r)^2$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}_i|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}_i}{r^3} + \ldots . \tag{1.28}$$

For the electrostatic potential $\phi(\mathbf{r})$ we therefore find

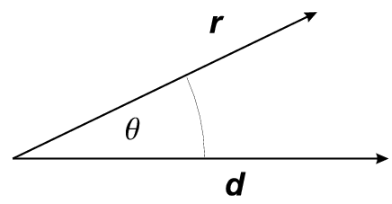
$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + \ldots . \tag{1.29}$$

The terms mean:

1.) Monopole contribution

$$\phi_0(\mathbf{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r} . \tag{1.30}$$

The total charge (or **monopole moment**) $\boxed{Q = \sum_i q_i}$ created in 0. As an approximation a field, which behaves like the field of a point charge located in 0 for sufficiently large distances.



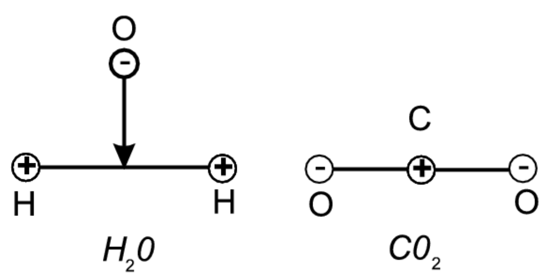
2.) Dipole contribution

$$\phi_1(\mathbf{r}) \;=\; \frac{\mathbf{d}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3} \;=\; \frac{d\ cos\theta}{4\pi\epsilon_0 r^2}\ . \tag{1.31}$$

The **dipole moment d** is $\boxed{\mathbf{d} \;=\; \sum_i q_i \mathbf{r}_i}$. The angle θ is the angle between **r** and **d**:

Construction instruction: Determine the center of mass of the positive and negative charge carriers. If those share the same center of mass, then is $\mathbf{d} = \sum_i q_i \mathbf{r}_i = 0$. Otherwise their connecting line sets the direction of **d**; their distance is furthermore a measure for the absolute value of **d**.

Example: molecules.



The next term, neglected in (1.29), is the **quadrupole moment.**

E-field of a dipole

The **E**-field of a dipole can be calculated using (1.31) and

$$\mathbf{E}_1(\mathbf{r}) = -\boldsymbol{\nabla}\phi_1(\mathbf{r}) = -\boldsymbol{\nabla}\frac{\mathbf{d}\cdot\mathbf{r}}{4\pi\epsilon_0\ r^3}\ . \tag{1.32}$$

Using the product rule we obtain

$$\begin{aligned} & -\frac{1}{4\pi\epsilon_0}\left(\mathbf{d}\cdot\mathbf{r}\boldsymbol{\nabla}r^{-3} + r^{-3}\ \boldsymbol{\nabla}\mathbf{d}\cdot\mathbf{r}\right) \\ &= \frac{1}{4\pi\epsilon_0}\left(\mathbf{d}\cdot\mathbf{r}\ 3\ r^{-4}\ \frac{\mathbf{r}}{r} - r^{-3}\ \mathbf{d}\right) \\ &= \frac{1}{4\pi\epsilon_0\ r^5}\left(3(\mathbf{d}\cdot\mathbf{r})\ \mathbf{r} - r^2\ \mathbf{d}\right)\ . \end{aligned} \tag{1.33}$$

Kapitel 2

Fundaments of electrostatics

2.1 Flux of a vector field

We now want to look for an equivalent formulation of Coulomb's law . To achieve that we introduce the concept of the flux of a vector-field.

A vector-field $\mathbf{A}(\mathbf{r})$ is defined on a surface F. F is *measurable* and *two-sides*, that means F must have a finite area and top- and down side of F are well-defined via the vector normal to the surface $\mathbf{n}$. **Counterexample**: Möbiu's strip.

Then we define the flux Φ_F of a vector-field $\mathbf{A}$ through the area F via the surface integral

$$\Phi_F \;=\; \int_F \mathbf{A} \cdot d\mathbf{f} \;=\; \int_F A_n df \;, \tag{2.1}$$

where $A_n = \mathbf{A} \cdot \mathbf{n}$ is the component of $\mathbf{A}$ in the direction of the surface normal $\mathbf{n}$. This directional surface element $d\mathbf{f}$ is parallel to $\mathbf{n}$, $df \equiv |d\mathbf{f}|$.

For the interpretation of (2.1), we consider a fluid flowing with the velocity $\mathbf{v}(\mathbf{r})$ and the density $\rho(\mathbf{r})$. If we choose

$$\mathbf{A}(\mathbf{r}) \;=\; \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) \;, \tag{2.2}$$

then

$$\int_F \mathbf{A} \cdot d\mathbf{f} \;=\; \int_F \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) \cdot d\mathbf{f} \tag{2.3}$$

corresponds to the amount of fluid flowing through F per unit time. (2.3) shows that only the surface perpendicular to the current contributes.

2.2 Gauss' law: Application to electrostatics

We now take for $\mathbf{A}$ in (2.3) the electrostatic field $\mathbf{E}$ and take a closed area F enclosing the volume V_F with the above characteristics. Then the **flux of the electric field** is

$$\Phi_F = \oint_F \mathbf{E} \cdot d\mathbf{f} \tag{2.4}$$

Gauss's theorem states that Φ_F is equal to the volume integral of the divergence of $\mathbf{E}$ in V_F:

$$\Phi_F = \int_{V_F} \nabla \cdot \mathbf{E}(\mathbf{r})\; dV \; . \tag{2.5}$$

We use for $\mathbf{E}$ the expression (1.15) together with the important relation for electrodynamics (see (A.19), Sec.A.5.5)

$$\nabla \cdot \frac{\mathbf{r}}{|\mathbf{r}|^3} = 4\pi\delta^3(\mathbf{r}) \; , \tag{2.6}$$

this gives for the divergence of the electric field [mehr in B.1.2] :

$$\nabla \cdot \mathbf{E}(\mathbf{r}) \;=\; \frac{\rho(\mathbf{r})}{\epsilon_0} \; . \tag{2.7}$$

We use (2.7) in (2.5) und obtain

$$\Phi_F = \frac{1}{\epsilon_0} \int_{V_F} dV \rho(\mathbf{r}) = \frac{Q}{\epsilon_0} \; , \tag{2.8}$$

where Q is the total charge contained in the volume V. From (2.8) we can see that the total flux of the electric field through a closed area F is produced by the charge contained in it. The charge is therefore a "source" for the electric field lines.

2.3 Applications of Gauss's theorem

For symmetric charge distributionss (2.8) offers the possibility to calculate the field intensity with less effort. We consider two examples:

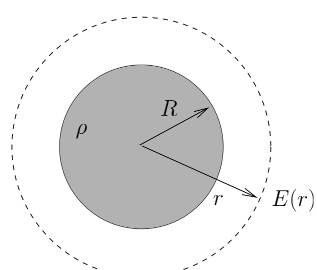
1.) Field of a homogenously charged sphere
Take

$$\rho(\mathbf{r}) \;=\; \rho(r) \quad \text{for} \quad r \le R, \quad \rho(\mathbf{r}) \;=\; 0 \quad \text{otherwise.} \tag{2.9}$$

Due to the spherical symmetry, $\mathbf{E}$ points in the radial direction so that

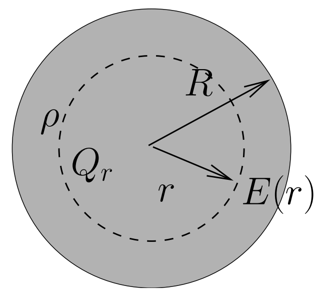
$$\Phi_F \;=\; 4\pi r^2 E(r) \;=\; \frac{Q_r}{\epsilon_0} \; , \tag{2.10}$$

where Q_r is the charge contained in a concentric (imaginary) sphere with radius r.



$Q_r = Q$ is tht total charge for points within $r \geq R$ and from (2.10) it follows:

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2} \quad \text{for} \quad r \geq R. \tag{2.11}$$



For $r \leq R$ the result depends on the special form of $\rho(r)$. As an example we choose

$$\rho(r) = \rho_0 = \text{const}, \tag{2.12}$$

then we have:

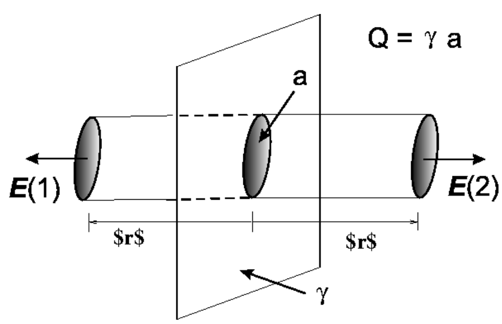
$$Q_r = \frac{4\pi}{3} r^3 \rho_0 \ , \tag{2.13}$$

therefore, as in (1.21):

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_r}{r^2} = \frac{\rho_0}{3\epsilon_0} r. \tag{2.14}$$

Compare the effort here with the one required by (1.15)!

2.) Homogenously charged infinite plane



For symmetry reasons, **E** is perpendicular to the plane, the absolute value E is the same points 1 and 2, which have the same distance r from the plane. Gauss' theorem then yields:

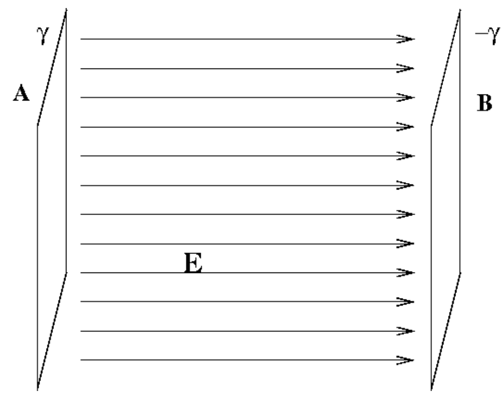
$$\Phi_F = \oint_F \mathbf{E} \cdot d\mathbf{f} = aE(1) + aE(2) = \frac{Q}{\epsilon_0} = \frac{\gamma a}{\epsilon_0} \ , \tag{2.15}$$

where a is the cylinder basis and γ the surface charge density. There is no contribution from the cylinder barrel, since **E** has no component in the direction of the normal on the cylinder barrel. Result:

$$E = \frac{\gamma}{2\epsilon_0} \tag{2.16}$$

independent of r.

A **Plate condensator** consists of two parallel plates with opposite surface charge density γ and $-\gamma$ so that according to (2.16) the **E**-field vanishes outside of the plates. Between the plates the total field (in absolute value) is $E = \frac{\gamma}{\epsilon_0}$. The potential between the



plates is determined by $\boldsymbol{\nabla}\phi(\mathbf{r}) = -\mathbf{E} \Rightarrow \phi(\mathbf{r}) = -\mathbf{E}\cdot\mathbf{r}$. The potential difference (Voltage) between the two layers is, thus, $V = \phi(A) - \phi(B) = E\ d$, where d is the distance between the plates.

The capacity of a condensator with the total area F, storing a total charge $Q = F\gamma$, is defined as

$$C \equiv \frac{Q}{V} \tag{2.17}$$

and equals

$$C = \frac{F\gamma}{d\gamma/\epsilon_0} = \frac{\epsilon_0 F}{d} \tag{2.18}$$

and only depens on the geometry of the condensator.

2.4 Differential equations for the electric field and potential

Alternatively, the relation (2.8) between divergence of **E** and the total charge can be expressed in the differential form (2.7):

$$\nabla\cdot\mathbf{E}(\mathbf{r}) \;=\; \frac{\rho(\mathbf{r})}{\epsilon_0}\;. \tag{2.19}$$

So it is possible to use (2.19) as basis for electrostatics ("postulate") instead of Coulomb's law (1.15). In fact, (2.19) is one of the *Maxwell equations* for the electric field.

So if we start from (2.19) we see that this equation does not change, if an arbitrary divergenceless vector function $\mathbf{E}'$ is added to $\mathbf{E}$. Therefore, equation (2.19) is not sufficient to determine the electric field.

Irrotationality of the electric field

We obtain a second differential relation for **E** from the fact that **E** is a conservative field (see (1.11)):

$$\mathbf{E} = -\nabla\phi, \tag{2.20}$$

whereby ϕ is the electocstatic potential. Via the vector identity

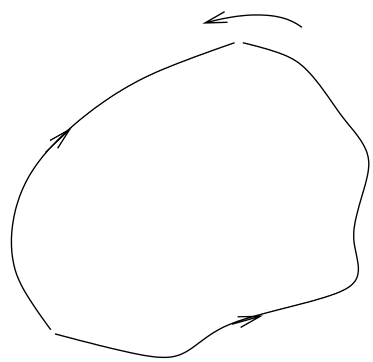
$$\nabla \times (\nabla f) = 0 \,, \tag{2.21}$$

equation (2.20) is equivalent to the *irrotationality* of the electric field

$$\boxed{\nabla \times \mathbf{E} = 0} \tag{2.22}$$

For a given **E**-field, the potential can be obtained from the inversion of (2.20) (apart for a constant), namely:

$$\phi(\mathbf{r}) = \phi(\mathbf{r}_0) - \int_{C(\mathbf{r}_0 \to \mathbf{r})} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}' \,, \tag{2.23}$$



i.e. with a curve integral of the **E**-field from a given starting point $\mathbf{r}_0$ to the observation point $\mathbf{r}$. Since **E** is conservative, the integral in (2.23) only depends from the start- and end-points, but not on the choice of connection curve $C(\mathbf{r}_0 \to \mathbf{r})$.
This can also be proven with the help of Stoke's theorem. We evaluate the difference between the integrals (2.23) on two different curves C_1 and C_2:

$$\left(\int_{C_1(\mathbf{r}_0 \to \mathbf{r})} - \int_{C_2(\mathbf{r}_0 \to \mathbf{r})}\right) \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}' = \oint_D \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}' \tag{2.24}$$

where D is the closed curve consisting of C_1 and C_2. From Stoke's theorem we have

$$\oint_D \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}' = \int_F \boldsymbol{\nabla} \times \mathbf{E} \cdot d\mathbf{f} \tag{2.25}$$

which vanishes due to $\boldsymbol{\nabla} \times \mathbf{E} = 0$. This result does not hold in electrodynamics anymore. In that case the **E**-field is no longer conservative and (2.23) does not apply anymore, since the integral dependens on the connection curve. In principle, Eqs. (2.19) and (2.22) are sufficient to determine the electrostatic field **E** for given boundary conditions (see Chap. 3).

Poisson's equation

In practice one goes one step further from the field $\mathbf{E}$ to the potential ϕ, out of which $\mathbf{E}$ can be determined via differentiation according to (2.20). If we insert (2.20) into (2.19) we obtain:

$$\nabla \cdot (\nabla \phi) \; = \; \boxed{\nabla^2 \phi \; = \; -\frac{\rho}{\epsilon_0}} \tag{2.26}$$

i.e., *Poisson's equation. Here one uses the abbrevation*

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.27}$$

for the Laplace operator ∇^2 *(often also denoted via Δ).*

Laplace equation

Once the solution of (2.26) *is found, an arbitrary solution of the homogenous equation, the* Laplace equation,

$$\nabla^2 \phi \; = \; 0 \tag{2.28}$$

can be added to it and one obtains a new solution of (2.26). This ambiguity can be eliminated with specification of boundary conditions. For further discussion, see chapter 3.

Kapitel 3

Boundary value problems in electrostatics

3.1 Uniqueness theorem

In the following subchapter we will show that the Poisson equation and the Laplace equation have a unique solution if one of the following boundary conditions apply:

(i) Dirichlet condition

$$\phi \text{ is given on a closed area } F, \textit{ or} \tag{3.1}$$

(ii) von Neumann condition

$$\mathbf{n}\cdot\nabla\phi(\equiv \frac{\partial\phi}{\partial n}) \text{ is given on a closed area } F\ , \tag{3.2}$$

$\mathbf{n}$ is the normal vector to the area F.
Proof
We assume that there are 2 solutions ϕ_1 and ϕ_2 of

$$\nabla^2\phi(\mathbf{r}) \;=\; -\frac{\rho(\mathbf{r})}{\epsilon_0} \tag{3.3}$$

with the same boundary conditions, (3.1) or (3.2). In that case the difference of those solutions is $U = \phi_1 - \phi_2$:

$$\nabla^2 U \;=\; 0 \tag{3.4}$$

in the volume V enclosed by the area F. Furthermore is due to the existing boundary conditions

$$U \;=\; 0 \quad \text{on } F \tag{3.5}$$

or

$$\mathbf{n}\cdot\nabla U \;=\; 0 \quad \text{on } F\ . \tag{3.6}$$

With the identity

$$\nabla\cdot(U\nabla U) \;=\; (\nabla U)^2 + U\nabla^2 U \tag{3.7}$$

and (3.4) it can be expressed as follows:

$$\int_V (\nabla U)^2 \, dV \;=\; \int_V \left(\nabla \cdot (U \nabla U) \;-\; U \underbrace{\nabla^2 U}_{=0} \right) dV \;=\; \oint_F \underbrace{U \, \nabla U \cdot d\mathbf{f}}_{=0} \;=\; 0 \qquad (3.8)$$

under application of Gauss's theorem, for the case that one of the two conditions (3.5) or (3.6) hold. Also:

$$\int_V (\nabla U)^2 \, dV \;=\; 0 \;, \qquad (3.9)$$

meaning in V is:

$$\nabla U \;=\; 0 \;, \qquad (3.10)$$

since $(\nabla U)^2 \geq 0$. This results to

$$U \;=\; \text{const} \qquad (3.11)$$

and ϕ_1 as well as ϕ_2 only differ by a constant, which has no effect on the electric field $\mathbf{E}$.

Special case $V \to \infty$

While V represents the entire R_3 the solution to Poisson's equation is unique if ρ is limited to a finite range and $\phi(\mathbf{r})$ decays asymptotically fast enough that:

$$r^2 \phi(\mathbf{r}) \frac{\partial \phi(\mathbf{r})}{\partial n} \;\to\; 0 \qquad \text{for} \;\; r \;\to\; \infty, \qquad (3.12)$$

whereas $\partial \phi / \partial n$ denotes the normal derivative of ϕ. The proof above transfers directly if we consider that the surface of an enclosed volume scales with r^2.

3.1.1 Physical application: metals

Why are we interested in problems with boundary conditions? Let's take a look at the properties of an *ideal* metal (conductor) *in the static case*. An ideal metal is an object which contains only freely movable charge carries (usually electrons). If one introduces such an object into an electric field forces act on the charge carriers redistributing them until an equilibrium state is reached. The $\mathbf{E}$-field of the redistributed charge carriers is added to the external $\mathbf{E}$-field. The resulting $\mathbf{E}$-field counts.

The equilibrium conditions are:

(i) $\mathbf{E} = 0$ *inside* the entire metal. If this weren't the case the charge carriers would experience forces which would redistribute them until $\mathbf{E} = 0$.

(ii) because of (i) the potential ϕ **is constant inside the entire metal.**

(iii) because of (ii) and Poisson's equation (2.26) **the charge density vanishes inside the metal.** In an ideal metal only an infinitesimally thin layer of charge carriers (surface charge) exists on the surface of the object.

The surface charge density γ was already introduced in eq. (2.15). In a metal γ generally depends on the position vector $\mathbf{r}$ on the surface of the metal: $\gamma(\mathbf{r}) =$*(charge per area.* In the following subchapter we will take a look at the components of the electric field $\mathbf{E}$

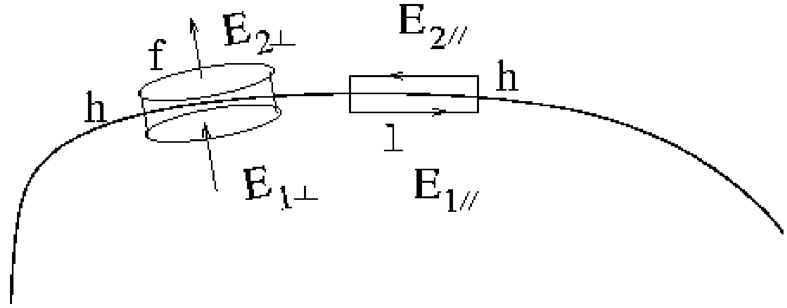
parallel ($\mathbf{E}_{\|}$) and perpendicular ($\mathbf{E}\cdot\mathbf{n}$, where $\mathbf{n}$ donates the normal vector) to the metal surface.
(iv) Let $\mathbf{E}_2$ be the $\mathbf{E}$-field on the outside and $\mathbf{E}_1$ on the inside of the metal [there is actually no electric field inside a metal as stated in (i) with $\mathbf{E}_1 = 0$, however let's just generally assume one for now]. Next we want to show that:

$$\mathbf{E}_{1\|} = \mathbf{E}_{2\|} \; , \tag{3.13}$$

meaning the parallel component is continuous through the surface, as well as

$$\mathbf{n}\cdot(\mathbf{E}_2 - \mathbf{E}_1) = \frac{\gamma(\mathbf{r})}{\epsilon_0} \; , \tag{3.14}$$

meaning **the normal component has a discontinuity which is proportional to γ.**



In order to prove this we will use the integral form of Gauss's theorem as well as the information that the curl of a stationary electric field $\mathbf{E}$ vanishes (2.22).

We first construct a volume element $V = f\ h$ from two small surface elements (with area f) which are parallel to the surface of the metal (sketch). The distance h between the two surface elements is selected (vanishingly) small ($h^2 \ll f$). According to Gauss's theorem the total charge $q = f\ \gamma$ inside the volume element is proportional to the flux Φ_V of the $\mathbf{E}$-field through the surface around V.

$$\Phi_V = \frac{q}{\epsilon_0} = f\ \frac{\gamma(\mathbf{r})}{\epsilon_0} \quad , \tag{3.15}$$

For a vanishingly small h we can now neglect the flux through the sides and we get

$$\Phi_V = \mathbf{n}\cdot(\mathbf{E}_2 - \mathbf{E}_1)\ f \; . \tag{3.16}$$

This yields (3.14).

Next we will use two small line elements parallel to the metal surface of length l with a (vanishingly small) distance h to construct a surface $F = h\ l$.We then get from $\nabla\times\mathbf{E} = 0$ using Stokes' theorem:

$$0 = \oint_{\partial F} d\mathbf{s}\cdot\mathbf{E} = l\ (E_{2\|} - E_{1\|}) \; , \tag{3.17}$$

whereas the contribution to the line integral along h can be neglected. wobei der Beitrag zum Linienintegral entlang h vernachlässigt werden kann. Which is evident from (3.13).

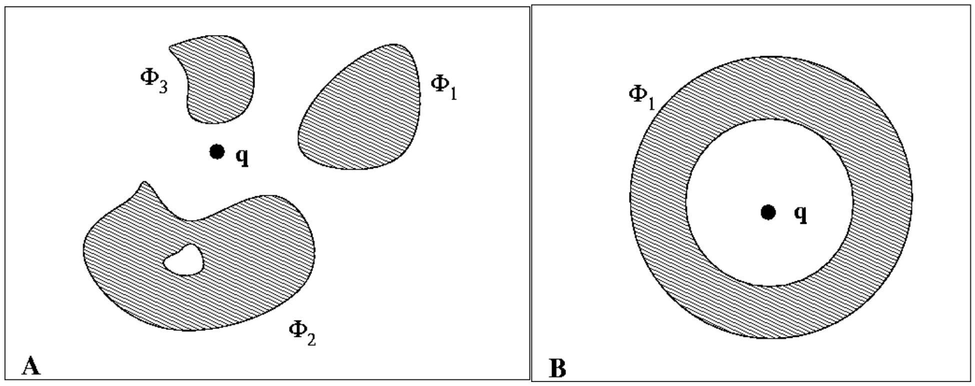


Abbildung 3.1: {metall}

In a metal in which $\mathbf{E}_1 = 0$ we can therefore obtain from (3.13) and (3.14)

$$\mathbf{n} \cdot \mathbf{E}_2 = \frac{\gamma(\mathbf{r})}{\epsilon_0} \qquad \mathbf{E}_{2\|} = 0 \; . \tag{3.18}$$

Typical Dirichlet-problems are problems with a given charge density in the vicinity of one or several metallic objects (Fig. 3.1) (A), which possibly surround the physical space (B).

A Von Neumann problem one can imagine as a problem in which a defined surface charge density is given (together with $\mathbf{E} \cdot \mathbf{n}$ from (3.14)).

We will study methods which allow finding *one* solution of Poisson's equation for given boundary conditions and charge distribution. *The Uniqueness theorem (Sec. 3.1) then ensures that this solution is the only solution.*

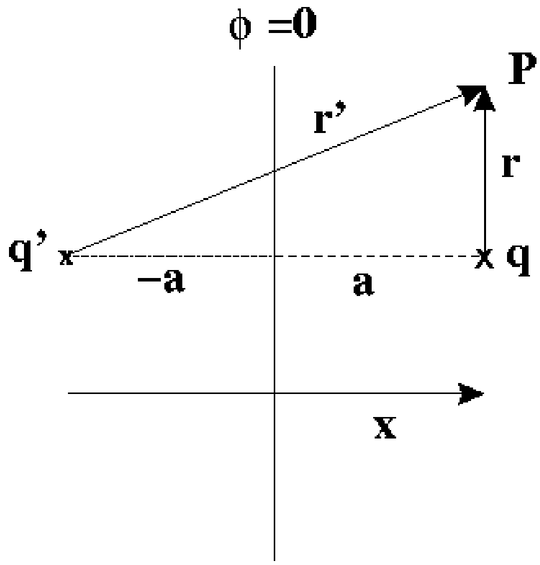
3.2 Image charge

This method for finding a solution to the boundary value problem consists in the placement of so called image charges of appropriate charge and location *outside of the investigated volume V* in a way that the existing boundary conditions are satisfied.

This method is allowed because one can always add a solution of the homogeneous Laplace equation *(in V)* to the inhomogeneous Poisson equation(cf. chapter 2.4). By using the Image charge method the particular solution of Laplace's equation is selected which satisfies the given boundary conditions when added to the selected (known) particular solution of Poisson's equation (for a single charge).

3.2.1 Point charge in front of a conducting plane

As a simple example we will consider a point charge q with a distance a to a conducting plane which is grounded (meaning $\phi = 0$ on the plane). We will here place the Image charge q' mirror-symmetric to q with respect to the plane (sketch).



The potential at the point P is then:

$$(4\pi\epsilon_0)\,\phi(P) \;=\; \frac{q}{r} + \frac{q'}{r'} \tag{3.19}$$

and we obtain as requested $\phi = 0$ for all points on the conducting plane $x = 0$ if we choose:

$$q' \;=\; -q\ . \tag{3.20}$$

In the (here interesting) range $x > 0$ is $q/(4\pi\epsilon_0 r)$ a particular solution to Poisson's equation, $q'/(4\pi\epsilon_0 r')$ a solution to Laplace's equation which actually ensures that the boundary condition for $x = 0$ is satisfied.

Electric field and induced electric charge

The x-component of an electric field $\mathbf{E}$ can be derived from (3.19) and (3.20) [mehr in B.1.1] :

$$E_x(P) \;=\; -\frac{\partial\phi}{\partial x} \;=\; \frac{q}{4\pi\epsilon_0}\left(\frac{x-a}{r^3} - \frac{x+a}{r'^3}\right)\ , \tag{3.21}$$

therefore the x-component for the plane $x = 0$ is

$$E_x(x=0) \;=\; -\frac{2qa}{4\pi\epsilon_0 r^3}\ . \tag{3.22}$$

The parallel components of $\mathbf{E}$ on the plane $x = 0$ vanish due to (3.13) and since $\mathbf{E}_1 = 0$. Eq. (3.22) means according to (3.14) that a charge with the location-dependent surface density

$$\gamma \;=\; \epsilon_0 E_x(x=0) \;=\; -\frac{qa}{2\pi r^3} \tag{3.23}$$

is *induced* on the plane $x = 0$ by the charge q (induced electric charge). This is the actual charge that creates the potential (3.19).

3.3 Overview electrostatics

⋮

1.) Basis: **Coulomb's law**

$$\mathbf{K} = q\mathbf{E} \quad \text{mit} \quad \mathbf{E}(\mathbf{r}) = \sum_i \frac{q_i(\mathbf{r}-\mathbf{r}_i)}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_i|^3} = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\, dV'$$

2.) Field equations (Maxwell's equations for electrostatics):

a) integral: $\oint_S \mathbf{E}\cdot d\mathbf{s} = 0$ $\quad$ $\oint_F \mathbf{E}\cdot d\mathbf{f} = \frac{Q}{\epsilon_0}$

$\Updownarrow$ $\quad$ $\Updownarrow$

b) differental: $\nabla\times\mathbf{E} = 0$ $\quad$ $\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$

3.) Electrostatic potential:

$$\mathbf{E} = -\nabla\phi \quad \to \quad \nabla^2\phi = -\frac{\rho}{\epsilon_0} \quad : \qquad \text{Poisson's equation}$$

5.) Ideal metal:

- inside: $\phi = \text{const.}$, $\mathbf{E} = 0$
- surface: $\mathbf{E}\cdot\mathbf{n} = \frac{\gamma}{\epsilon_0}$, $\mathbf{E}\times\mathbf{n} = 0.$

6.) Nützliche Formeln:

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$\nabla\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi\delta(\mathbf{r}-\mathbf{r}')$$

Teil II

Magnetostatik

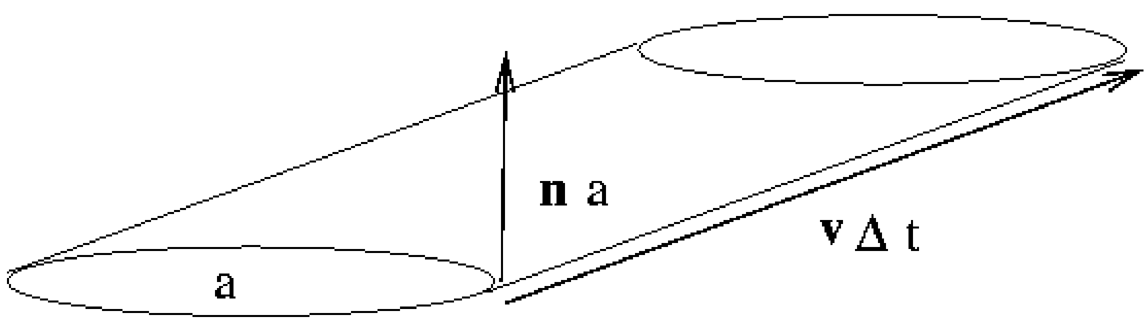
Kapitel 4

Ampere's Law

4.1 Electric current and conservation of charge

Electric currents induced by the movement of charged particles. Those charged particles can be for example charged ions in a particle accelerator, an electrolyte or a gas, electrons in a metal etc. The primary cause for the movement of charges are electric fields, however also material transport of charged particles can induce movement of charge. The electric current is defined as the amount of charge flowing through the cross section of a conductor per unit time.

Current density



We will now consider the simplest case of charge carriers of the same *charge* q and *velocity* $\mathbf{v}$. Let $\mathbf{a}$ *be the vector perpendicular to the cross section of the conducting medium* (sketch) whereas the absolute value a represents the cross sectional area and n *the charge density.* The charge carriers in the *volume* $\Delta V = (\mathbf{a} \cdot \mathbf{v})\Delta t$ then pass the cross section of the conductor in the *time interval* Δt, in total yielding an amount of charge carriers of $n(\mathbf{a} \cdot \mathbf{v})\Delta t$. Thereof one can calculate the electrical current

$$I(a) = \frac{nq(\mathbf{a} \cdot \mathbf{v})\Delta t}{\Delta t} = nq(\mathbf{a} \cdot \mathbf{v}). \qquad (4.1)$$

If we now assume per unit volume n_i charge carriers q_i with the velocity $\mathbf{v}_i$ we get:

$$I(a) \;=\; \mathbf{a}\cdot\underbrace{(\sum_i n_i q_i \mathbf{v}_i)}_{\equiv \mathbf{j}}\,. \tag{4.2}$$

Equations (4.1) and (4.2) now suggest the introduction of the *current density* $\mathbf{j}$ as

$$\boxed{\mathbf{j} \;=\; \sum_i n_i q_i \mathbf{v}_i} \tag{4.3}$$

so that for an arbitrary infinitesimal (oriented) area $\mathbf{\Delta a}$ the current can be expressed as

$$\Delta I = \mathbf{\Delta a}\cdot\mathbf{j}\,.$$

For several particles of the same charge $q_i = q$ one can calculate $\mathbf{j}$ with the mean velocity

$$<\mathbf{v}> = \frac{1}{n}\sum_i n_i \mathbf{v}_i \tag{4.4}$$

to:

$$\mathbf{j} \;=\; nq <\mathbf{v}> = \;\rho <\mathbf{v}>\,. \tag{4.5}$$

Equation (4.5) makes clear that high absolute velocities (4.5) of the charge carriers don't necessarily account for a strong electric current as only the mean of the velocities contributes to an effective current. For example if the velocities of the charge carriers are equally distributed over all directions then there is no resulting current $<\mathbf{v}>= 0$ and therefore also $\mathbf{j} = 0$.

In a general case are ρ and $<\mathbf{v}>$ space and time dependent, therefore

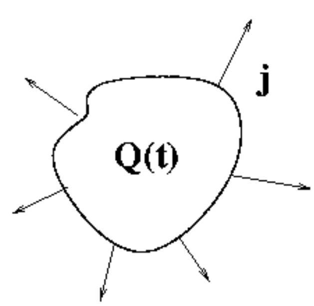
$$\mathbf{j} \;=\; \mathbf{j}(\mathbf{r}, t). \tag{4.6}$$

The total current through an area F is also given by

$$I = \int_F \mathbf{j}(\mathbf{r}, t)\cdot d\mathbf{f} \tag{4.7}$$

Continuity equation

We can express the conservation law of charge using the charge and current density as follows: From the conservation of charge follows:
We assume an arbitrary and finite volume V with the surface F.



Let the charge within this volume be $Q = Q(t)$. If V is time independent, the change in charge within the volume V per unit time is:

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV. \tag{4.8}$$

Since charge can neither be created nor annihilated, the decrease (increase) of charge within the volume V must be equal to the charge flowing through the surface F of the volume in the observed time interval. The latter is given through the surface integral of the current density which can be transformed into a volume integral according to Gauss's theorem:

$$-\frac{dQ}{dt} = \oint_F \mathbf{j} \cdot d\mathbf{f} = \int_V \nabla \cdot \mathbf{j} \, dV. \tag{4.9}$$

Therefore is the charge balance:

$$-\int_V \frac{\partial \rho}{\partial t} \, dV = \int_V \nabla \cdot \mathbf{j} \, dV \tag{4.10}$$

or since the volume V can be chosed arbitrarily, we obtain the *Continuity equation*:

$$\boxed{\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.} \tag{4.11}$$

While (4.9) describes the conservation of charge in the *integral* form, (4.11) describes the conservation of charge in the *differential* form.

Special cases

(i) electrostatics: stationary charges

$$\mathbf{j} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} = 0 \quad \rightarrow \quad \rho = \rho(\mathbf{r}) \tag{4.12}$$

(ii) magnetostatics: stationary currents

$$\mathbf{j} = \mathbf{j}(\mathbf{r}) \quad \text{and} \quad \nabla \cdot \mathbf{j} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} = 0. \tag{4.13}$$

For a stationary current is $\mathbf{j}$ (and therefore also $\nabla \cdot \mathbf{j}$) constant with respect to time. $\nabla \cdot \mathbf{j}$ must be zero at every point in space otherwise the charge density would increase unboundedly.

4.2 Ampere's law

Consider a given stationary current density $\mathbf{j} = \mathbf{j}(\mathbf{r})$. In order to eliminate electrostatics effects we will assume that the density of the moving charge carriers which induce the electric current is compensated by the charge carriers at rest with the opposite value of charge (e.g. moving conduction electrons and the ionic cores at rest in a metallic conductor). On a moving charge q then acts a - non electrostatic - force in the proximity of a current-carrying conductor, for which one can experimentally find:

$$\boxed{\mathbf{K} \;=\; q\,[\mathbf{v}\times\mathbf{B}]} \tag{4.14}$$

with

$$\boxed{\mathbf{B}(\mathbf{r}) \;=\; \Gamma_m \int_V \frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dV'} \tag{4.15}$$

as the field of the *magnetic induction* (short: **B**-field). The equations (4.14) and (4.15) are as fundamental principles of magnetostatics equally verified as

$$\mathbf{K} = q\,\mathbf{E} \tag{4.16}$$

and (1.15) in electrostatics. Concluding, (4.14) and (1.8) yield the total force (**Lorentz's force**) acting on a charge q.

$$\boxed{\mathbf{K} \;=\; q\;(\mathbf{E}+\mathbf{v}\times\mathbf{B})\;.} \tag{4.17}$$

(4.17) provides a precise measurement instruction for the electrostatic field **E** and for the magnetic induction **B**. The contributions to (4.17) from both components can be identified independently by first measuring the acting force on a charge at rest ($\mathbf{v} = 0$). This force is due to (4.17) solely caused by the electric field **E**. Then the charge is set in motion. From the resulting force one can subtract the electrostatic contribution. The calculated difference is then equivalent to the magnetic contribution to the total acting force.

Systems of measurement

Is Γ_e a set value, meaning one has defined the unit charge (compare section 1.2) then all occuring quantities in (4.14) and (4.15) are uniquely specified with respect to their unit. Therefore Γ_m can no longer be freely chosen:

(ii) **cgs** system:

$$\Gamma_e \;=\; 1, \qquad\qquad \Gamma_m \;=\; \frac{1}{c} \tag{4.18}$$

with the speed of light c. Note that in the cgs measurement system Lorentz's force has a different form:

$$\mathbf{K} \;=\; q\,\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\;.$$

So that the **E** and **B** field have the same dimension.

(i) **MKSA** system:

$$\Gamma_e \;=\; \frac{1}{4\pi\epsilon_0}, \qquad\qquad \Gamma_m \;=\; \frac{\mu_0}{4\pi} \tag{4.19}$$

with

$$\epsilon_0 \;=\; 8.854\cdot 10^{-12}\,\frac{\mathrm{Cb}^2}{\mathrm{N\,m}^2}, \qquad\qquad \mu_0 \;=\; 4\pi\cdot 10^{-7}\,\frac{\mathrm{m\,kg}}{\mathrm{Cb}^2}. \tag{4.20}$$

μ_0 is the *magnetic permeability*.

Principle of superposition

Equation (4.15) contains - as in (1.8) - the principle of superposition: the fields of two current distributions $\mathbf{j}_1$ and $\mathbf{j}_2$ superpose linearly: $\mathbf{B}(\mathbf{j}_1 + \mathbf{j}_2) = \mathbf{B}(\mathbf{j}_1) + \mathbf{B}(\mathbf{j}_2)$.

Connection to relativity

The ratio Γ_m/Γ_e is independent of the choice of unit charge (therefore of ϵ_0) since the ratio of (4.14) to (1.8) is dimensionless. From (4.17) has the ratio of $|\mathbf{B}|/|\mathbf{E}|$ the dimension of an inverse velocity and from (4.15) to (1.8) has the ratio of $\Gamma_m/\Gamma_e = \epsilon_0\,\mu_0$ the dimension of an inverse velocity squared.

With (4.19) we obtain the relation

$$\boxed{\epsilon_0\,\mu_0 \;=\; \frac{1}{c^2}} \qquad (4.21)$$

This fundamental relation already indicates a connection to the special theory of relativity. Indeed one can transform (4.14) and (4.15) to (4.16) and (1.15) using a Lorentz transformation .

4.3 Biot-Savart law

In this subchapter we will calculate the vector field $\mathbf{B}(\mathbf{r})$ for different simple current densities.

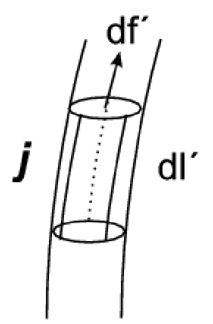
For a *thin* conductor we can immediately integrate over the cross section f of the conductor [meaning $\int \mathbf{j}\; d\,V \to I \int d\mathbf{l}$] and therefore we obtain from (4.15)

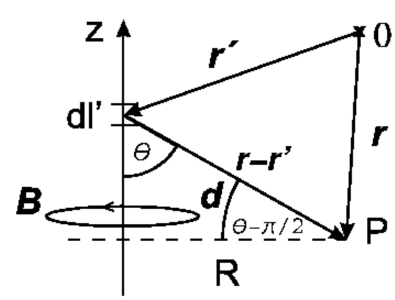
$$\boxed{\mathbf{B}(\mathbf{r}) \;=\; \frac{\mu_0 I}{4\pi} \int_L \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}} \qquad (4.22)$$

with

$$I \;=\; \int_F \mathbf{j} \cdot d\mathbf{f}' \qquad (4.23)$$

as the electrical current (compare sketch).





If the conductor is furthermore straight we can derive from (4.22) or also (4.15) that the field lines of $\mathbf{B}$ are concentric around the conductor. One then only has to calculate the absolute value of B since all contributions to the integral (4.22) for a small **straight conductor** have the same direction. From the sketch we see that:

$$B(P) \;=\; \frac{\mu_0 I}{4\pi} \int_L \frac{\sin\theta}{|\mathbf{r} - \mathbf{r}'|^2} \, dz \tag{4.24}$$

We will evaluate the remaining integral for an infinitely long conductor: With

$$R \;=\; |\mathbf{r} - \mathbf{r}'| \;\; \sin\theta; \qquad z \;=\; \frac{R}{\tan\theta} \qquad \rightarrow \qquad dz \;=\; \frac{-R}{\sin^2\theta} \, d\theta \tag{4.25}$$

we obtain for the field intensity in point P:

$$B(P) \;=\; \frac{\mu_0 I}{4\pi R} \int_{-1}^{1} d(\;\cos\theta) \;=\; \frac{\mu_0 I}{2\pi R}. \tag{4.26}$$

This is known as the Biot-Savart law for a thin, straight and infinitely long conductor.

Kapitel 5

Fundamental equations of magnetostatics

Like for the electrostatic field we now want to write down the equations for the **B**-field in differential form.

5.1 Divergence of the magnetic induction

We rewrite (4.15) as follows:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi}\int_V \mathbf{j}(\mathbf{r}') \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} dV'$$

$$= -\frac{\mu_o}{4\pi}\int_V \mathbf{j}(\mathbf{r}') \times \nabla_{\mathbf{r}} |\mathbf{r}-\mathbf{r}'|^{-1} dV'$$

since $\nabla_{\mathbf{r}}$ only affects $\mathbf{r}$ we obtain:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \left(\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV' \right) , \tag{5.1}$$

According to (5.1), **B** can be written in the form

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{5.2}$$

with

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV'}. \tag{5.3}$$

$\mathbf{A}(\mathbf{r})$ denotes the so called vector potential. Since $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$, we obtain

$$\boxed{\nabla \cdot \mathbf{B} = 0}\ . \tag{5.4}$$

Equation (5.4) should be compared to $\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$ to see that *magnetic charges* do not exist. If we form the integral corresponding to (5.4) we get:

$$\int_V \nabla \cdot \mathbf{B}\, dV = \oint_F \mathbf{B} \cdot d\mathbf{f} = 0, \tag{5.5}$$

and can therefore see that the magnetic flux through a closed surface F vanishes.

5.2 Curl of B

In electrostatics we have found

$$\nabla \times \mathbf{E} \;=\; 0 \tag{5.6}$$

or equivalently

$$\oint_S \mathbf{E} \cdot d\mathbf{s} \;=\; 0 \tag{5.7}$$

according to Stoke's theorem. For the **B**-field we derived from (5.2)

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A} \; . \tag{5.8}$$

Next we will show that

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \; . \tag{5.9}$$

Proof: from (5.3):

$$\boldsymbol{\nabla}_{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \mathbf{j}(\mathbf{r}') \cdot \boldsymbol{\nabla}_{\mathbf{r}} |\mathbf{r} - \mathbf{r}'|^{-1} \; dV' = -\frac{\mu_0}{4\pi} \int_V \mathbf{j}(\mathbf{r}') \cdot \boldsymbol{\nabla}_{\mathbf{r}'} |\mathbf{r} - \mathbf{r}'|^{-1} \; dV' \; . \tag{5.10}$$

Up next we need a

Lemma: General properties of a stationary current density bounded in space

We now use the following relation **valid for a stationary** $\nabla \mathbf{j} = 0$ **current density, which vanishes outside a volume** V **(as well as on its surface, or vanishing „rapidly enough“ in infinity)**.

With the condition given above and for arbitrary (sufficiently differentiable) scalar fields g, f holds: [1]

$$\int_V (f \; \mathbf{j} \cdot \nabla g + g \; \mathbf{j} \cdot \nabla f) \; dV = 0 \; . \tag{5.11}$$

Using (5.11) with $f(\mathbf{r}') = 1$ and $g(\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}$, yields the last expression 0.

Now we use a formula from electrostatics

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

If we apply the Laplace operator to (5.3) we then obtain

$$\boldsymbol{\nabla}^2 \mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \int_V \mathbf{j}(\mathbf{r}') 4\pi\delta(\mathbf{r} - \mathbf{r}') \; dV' = -\mu_0 \; \mathbf{j}(\mathbf{r}) \; . \tag{5.12}$$

Together with (5.8) and (5.9) we get:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \; \mathbf{j} \; .} \tag{5.13}$$

The integral form of (5.13) can then be derived using Stoke's theorem:

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{s} \;=\; \mu_0 I} \; , \tag{5.14}$$

[1] $\nabla \cdot (g \; f \; \mathbf{j}) = (f \; \mathbf{j} \cdot \nabla g + g \; \mathbf{j} \cdot \nabla f)$, since $\nabla \cdot \mathbf{j} = 0$. Then we integrate over V, where the first term is transformed into an integral over the surface using Gauss's theorem and it therefore vanishes.

where I denotes the current enclosed by S. **Annotation:**
In praxis (e.g. coil) S can circulate the current several times. In the case of S circulating the current n-fold, one must substitute I with nI.
(5.13) and (5.4) form the fundamental equations (Maxwell's equations) of magnetostatics. Contrary to the electrostatic field $\mathbf{E}$ with $\nabla\times\mathbf{E}=0$ is the $\mathbf{B}$-field therefore *not irrotational*

5.3 Vector potential and gauge

In chapter 5.1 we have introduced a auxiliary quantity, the so called *vector potential* $\mathbf{A}$. From this vector potential one can derive the magnetic induction by differentiation (similarly to the electrostatic potential ϕ). But unlike the electrostatic potential ϕ is $\mathbf{A}$ a vector field and $\mathbf{B}$ is given by

$$\mathbf{B} \;=\; \nabla\times\mathbf{A}\,. \tag{5.15}$$

In (5.12) we have found a differential equation for the vector potential $\mathbf{A}$, from which $\mathbf{A}$ can be determined for a given current distribution $\mathbf{j}$.
Coulomb gauge
However, one has to consider that the relation (5.15) does not uniquely specify the vector potential for a given $\mathbf{B}$-field. This is due to the fact that the $\mathbf{B}$-field does not change when performing a so called *gauge transformation*

$$\boxed{\mathbf{A} \quad\Rightarrow\quad \mathbf{A}' \;=\; \mathbf{A}+\nabla\chi} \tag{5.16}$$

where χ denotes an arbitrary (however at least twofold partially differentiable) scalar function. Since:

$$\nabla\times\mathbf{A}' \;=\; \nabla\times\mathbf{A}+\nabla\times(\nabla\chi) \;=\; \nabla\times\mathbf{A}+0. \tag{5.17}$$

(5.16) provides additional freedom in the selection of $\mathbf{A}$.
(5.3) is therefore not the only possible expression for $\mathbf{A}$. With (5.3) is (compare (5.9)) $\mathbf{A}$ divergence free. A choice for $\mathbf{A}$, which satisfies (5.9) is generally called *Coulomb gauge.* Since physics only depends on $\mathbf{B}$ and not on $\mathbf{A}$, there are no special limitations to the choice of a gauge. In Coulomb gauge (5.9) satisfies $\mathbf{A}$ [compare (5.8), (5.13) and (5.9)] the differential equation:

$$\boxed{\boldsymbol{\nabla}^2\mathbf{A} \;=\; -\mu_0\,\mathbf{j}}_{\text{ Coulomb gauge}}\,. \tag{5.18}$$

The vectorial equation (5.18) can be split up into 3 components which are mathematically speaking of the same type as the already introduced Poisson equation (2.26).

However, if one starts from a vector potential $\mathbf{A}$ which doesn't satisfy the Coulomb gauge condition

$$\nabla\cdot\mathbf{A} \;\neq\; 0, \tag{5.19}$$

one can choose the *gauge potential* χ in a way that

$$\nabla\cdot\mathbf{A}' \;=\; \nabla\cdot\mathbf{A}+\nabla\cdot(\nabla\chi) \;=\; 0\,, \tag{5.20}$$

$\mathbf{A}$ then satisfies the Coulomb gauge. One can determine χ by solving a differential equation of the type (2.28):

$$\boldsymbol{\nabla}^2\chi = -\boldsymbol{\nabla}\cdot\mathbf{A}, \qquad (5.21)$$

where $-\nabla\cdot\mathbf{A}$ must be considered a given inhomogeneity.

5.4 Overview of magnetostatics

1.) Basis: **Ampere's law**

$$\boxed{\mathbf{K} \;=\; q(\mathbf{v}\times\mathbf{B}) \qquad \text{mit} \qquad \mathbf{B} \;=\; \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dV'}$$

for stationary currents, where $\nabla\cdot\mathbf{j} = -\partial\rho/\partial t = 0$.

2.) **Field equations**: from

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \qquad \text{mit} \qquad \mathbf{A} \;=\; \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV'$$

one can derive

a) **differential**:

$$\boxed{\nabla\cdot\mathbf{B} \;=\; 0; \qquad\qquad \nabla\times\mathbf{B} \;=\; \mu_0\,\mathbf{j}}$$

b) **integral**

$$\boxed{\text{flux:} \;\oint_F \mathbf{B}\cdot d\mathbf{f} \;=\; 0; \qquad \text{circulation:} \qquad \oint_S \mathbf{B}\cdot d\mathbf{s} \;=\; \mu_0 I}$$

3.) **Vector potential**:

$$\nabla\times(\nabla\times\mathbf{A}) = \mu_0\mathbf{j} \quad\rightarrow\quad \nabla^2\mathbf{A} = -\mu_0\,\mathbf{j}$$

for $\nabla\cdot\mathbf{A} = 0$ (Coulomb gauge).

Static Maxwell's equations

$(A)\quad \boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$	$(B)\quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0$
$(C)\quad \boldsymbol{\nabla}\times\mathbf{E} = 0$	$(D)\quad \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\,\mathbf{j}$

(5.22)

Lorentz force

$$\boxed{\mathbf{K} = q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \Rightarrow \int (\rho\,\mathbf{E}+\mathbf{j}\times\mathbf{B})\;d\,V} \tag{5.23}$$

Teil III

Fundamentals of elektrodynamics

Kapitel 6

Maxwell's equations

6.1 Concept of the electromagnetic field

The following chapter will explain the fundamental equations for the electric field $\mathbf{E}(\mathbf{r}, t)$ and for the magnetic induction $\mathbf{B}(\mathbf{r}, t)$ for the case of arbitrary *time dependent* charge and current distributions:

$$\rho \;=\; \rho(\mathbf{r}, t); \quad \mathbf{j} \;=\; \mathbf{j}(\mathbf{r}, t) \tag{6.1}$$

As a definition of the fields we will use Lorentz's force

(**Lorentz's force**)
$$\mathbf{K} \;=\; q\,[\,\mathbf{E} + (\mathbf{v} \times \mathbf{B})\,]\;. \tag{6.2}$$

Since ρ and $\mathbf{j}$ are now interconnected by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} \;=\; 0 \tag{6.3}$$

it is clear that the electric and magnetic fields can no longer be treated independently: The *Maxwell's equations* are a system of coupled differential equations for the fields $\mathbf{E}$ and $\mathbf{B}$.

6.2 Incompleteness of the static Maxwell's equations

Once more the **static** Maxwell's equations (in vacuum) which we derived so far. These equations only hold for time independent fields.

Static Maxwell's equations

$(A) \quad \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$(B) \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$
$(C) \quad \boldsymbol{\nabla} \times \mathbf{E} = 0$	$(D) \quad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0\, \mathbf{j}$

(6.4)

In these equations the **E** and **B** field can be treated separately, however as has been mentioned above this is not correct for general fields. There are two reasons why the **E** and **B** field must be coupled:

(1) By the use of different inertial reference frames **E** and **B** are mixing with each other.

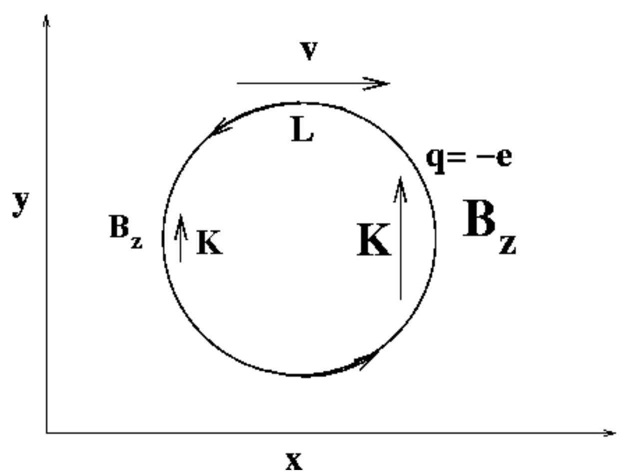
(2) Because of the continuity equation are ρ and $\mathbf{j}$ and therefore also **E** and **B** coupled.

We will see that an exact analysis of these two reasons will yield the appropriate generalization of (6.4).

6.3 Faraday's law of induction

Electromotive force

Let us consider a conductor ring (circular shape) R, which travels at the speed $\mathbf{v}$ in a non-homogeneous **B** field. As an example we will now assume a **B** -field oriented in the z-direction, which increases along the x-direction (sketch). The ring lies on the $x-y$-plane and moves in the x-direction.



Due to the **B** -field the free charge carriers $q = -e < 0$ on the conductor ring now experience a force $\mathbf{K} = q\ \mathbf{v} \times \mathbf{B}$ in the $+y$-direction. This force, however, is larger on the right side of the ring where the **B** -field is stronger. Because of this effect the charge carriers are more strongly accelerated counter-clockwise around the ring. Mathematically this means that the circulation of the force along R does not vanish.

Let's assume to be in the inertial reference frame of the conductor ring with velocity $\mathbf{v}$. In this system the charges are immobile, however, they experience the same force (in the non-relativistic case) as they do in the laboratory system. But since the charges are immobile those forces cannot originate from the **B** -field but rather from an electric field $\mathbf{E} = \mathbf{K}/q$ (Lorentz's force). Just like the force **K** has **E** also a non-vanishing circulation which contradicts (6.4)(C). In the reference frame of the movable conductor therefore exists a **E** -field which is no longer non-rotational as well as a time-dependent **B** -field.

We now want to find the relation between these two fields, meaning we want to mathematically derive the suitable generalization of (6.4)(C). We start by using the index "R" for the fields in the reference frame of the conductor ring and no index for the laboratory system. The $\mathbf{B}_R$-field is the same as $\mathbf{B}$ just spatially displaced by $\mathbf{v}t$.
By comparing the forces in both reference frames we obtain $\mathbf{E}_R$:

$$\mathbf{K} = q\ \mathbf{v} \times \mathbf{B} = \mathbf{K}_R = q\ \mathbf{E}_R\ . \tag{6.5}$$

While the ring moves through the distance $\delta \mathbf{r}$ in the time interval δt, $\mathbf{B}_R$ changes by

$$\delta \mathbf{B}_R = (\delta \mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{B}_R\ , \tag{6.6}$$

the time derivative of the $\mathbf{B}_R$-field in the system R is therefore given by

$$\frac{\partial\ \mathbf{B}_R}{\partial\ t} = (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B}_R\ . \tag{6.7}$$

The last term can be rewritten using the relation (valid for homogeneous $\mathbf{v}$)

$$\boldsymbol{\nabla} \times (\mathbf{B} \times \mathbf{v}) = (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B} - \mathbf{v}\underbrace{(\boldsymbol{\nabla} \cdot \mathbf{B})}_{=0} = (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B}\ . \tag{6.8}$$

Leading to

$$\boxed{-\frac{\partial\ \mathbf{B}_R}{\partial\ t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}_R) = \boldsymbol{\nabla} \times \mathbf{E}_R}\ , \tag{6.9}$$

where we have used (6.5). This equation defines a relationship between the time derivative of $\mathbf{B}$ and the curl of $\mathbf{E}$. (6.9) is therefore the appropriate generalization of the Maxwell's equation (6.4)(C).

In order to study the effects on the conductor ring let us take a look at the integral form of (6.9) using Stokes' theorem. The circulation of $\mathbf{E}$ along R is equivalent to an induction voltage V_{ind} and is given by (we can omit the index R)

$$V_{ind} = \oint_R \mathbf{E}\ d\mathbf{l} = -k\ \frac{d}{d\ t} \int_S \mathbf{B} \cdot d\ \mathbf{f} \qquad (k = 1)\ . \tag{6.10}$$

This is known as Faraday's law of induction. (6.10) means that a change in magnetic flux through a closed conductor ring induces a voltage in this circuit.

Annotations

- (6.10) and its differential form

  $$\boxed{\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial\ \mathbf{B}}{\partial\ t} = 0} \tag{6.11}$$

  are universally valid: The change in flux can either be caused by moving the conductor ring (as discussed above) or by a time dependence of the $\mathbf{B}$ -field.

- Equation (6.10) is valid even if the conductor ring doesn't actually exist, the ring only facilitates the measurement of the induced field.

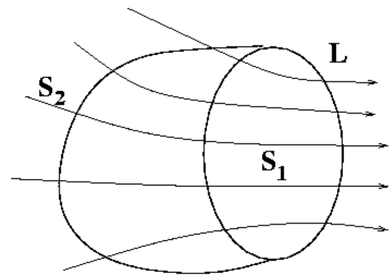
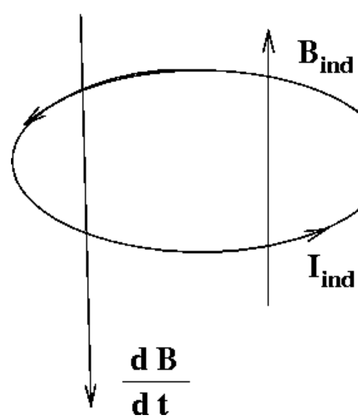


Abbildung 6.1: {flux} The flux of **B** through the surfaces S_1 and S_2 delimited by R is the same, according to (6.10). Therefore is the total flux through the surface $S_1 + S_2$ equal to zero.

- By calculating the divergence of (6.11) one obtains $\frac{\partial\, \nabla \mathbf{B}}{\partial\ t} = 0$, meaning (6.4)(B) remains valid. Also in (6.10) it is implicit that different surfaces delimited by R have the same magnetic flux. This means that the magnetic flux through a closed surface vanishes (Fig. **??**).

- The sign in (6.10) reflects **Lenz's law**. This means that the voltage, induced by the time dependent **B** -field, and therefore current, induces (due to (4.22)) a magnetic field of the opposite orientation with respect to the (growing) **B** -field (sketch).



- In the **cgs**-system is the constant k in (6.10) $k = 1/c$.

Applications:
Betatron: In the electric field, induced by a time dependent **B**-field, charge carriers are being accelerated.
Alternating current generator: A rotating coil in a constant **B** -field experiences an induced voltage due to the time dependence of the magnetic flux in the coil.

6.4 Extension of Ampère's law

Ampère's law of magnetostatics (6.4)(D)

$$\nabla \times \mathbf{B} \;=\; \mu_0\, \mathbf{j} \tag{6.12}$$

is only valid for stationary currents. If one evaluates

$$\nabla \cdot (\nabla \times \mathbf{B}) \;=\; \mu_0\, \nabla \cdot \mathbf{j}, \tag{6.13}$$

one obtains, using the identity

$$\nabla \cdot (\nabla \times \mathbf{a}) \;=\; 0, \tag{6.14}$$

that $\nabla \cdot \mathbf{j} = 0$, meaning stationary currents. The continuity equation, however, is generally valid

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} \;=\; 0, \tag{6.15}$$

so that (6.12) has to be modified for non-stationary currents.

What this modification has to look like will be instantly clear when one uses Gauss's law of electrostatics ((6.4)(A)):

$$\nabla \cdot \mathbf{E} \;=\; \frac{\rho}{\epsilon_0} \tag{6.16}$$

which is supported by the concept of charge invariance postulated in chapter 1. If one combines the continuity equation (6.15) and (6.16), one obtains:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} \;=\; \nabla \cdot \left(\mathbf{j} + \epsilon_0\, \frac{\partial \mathbf{E}}{\partial t} \right) \;=\; 0. \tag{6.17}$$

Therefore by replacing

$$\mathbf{j} \quad \rightarrow \quad \mathbf{j}_M \equiv \mathbf{j} + \epsilon_0\, \frac{\partial \mathbf{E}}{\partial t}, \tag{6.18}$$

one yet again obtains the „current density" with a vanishing divergence, just like in magnetostatics.

This „current density" $\mathbf{j}_M$ is known as Maxwell's displacement current. In accordance with the conservation of charge we therefore extend (6.12) as follows:

$$\boxed{\nabla \times \mathbf{B} \;= \mu_0\; \mathbf{j}_M = \;\mu_0\, \mathbf{j} + \mu_0 \epsilon_0\, \frac{\partial \mathbf{E}}{\partial t}}\;. \tag{6.19}$$

Ampère's law (6.19) finds its experimental confirmation in the existence of electro-magnetic waves (chapter IV).

Self-induction

A current-carrying conductor induces a $\mathbf{B}$ (and $\mathbf{E}$) field in it's surroundings according to (6.19). If the flux of the $\mathbf{B}$ -field through the conductor ring changes, an induction current is induced in the ring (self-induction), which as per (6.10) has the opposite orientation as the initial current (Lenz's law). The self-induction depends on the geometry of the conductor. For a quantitative formulation one can conveniently use electro-magnetic potentials (chapter 7).

6.5 Overview of Maxwell's equations

Homogeneous equations

$$\boxed{\nabla \cdot \mathbf{B} \;=\; 0}, \tag{6.20}$$

which corresponds to the absence of magnetic monopols.

$$\boxed{\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \;=\; 0}, \tag{6.21}$$

which corresponds to Faraday's law of induction.

Inhomogeneous equations

$$\boxed{\nabla \cdot \mathbf{E} \;=\; \frac{\rho}{\epsilon_0}}, \tag{6.22}$$

which corresponds to Gauss's law.

$$\boxed{\nabla \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \;=\; \mu_0 \mathbf{j}}\;, \tag{6.23}$$

which corresponds to Ampère-Maxwell's equation.

In (6.22) and (6.23) is the conservation of charge (6.15) contained implicitly. (6.21) and (6.23) show, that a time-dependent magnetic field $\mathbf{B}$ induces an electric field $\mathbf{E}$ and vice versa. The equations (6.20) – (6.23) together with Lorentz's force

$$\mathbf{K} \;=\; q\,[\,\mathbf{E} + (\mathbf{v} \times \mathbf{B})\,]\;\;. \tag{6.24}$$

fully describe the electro-magnetic interaction of charged particles in the framework of classical physics.

Maxwell's equations in vacuum

$(A) \quad \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$(B) \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$
$(C) \quad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$(D) \quad \boldsymbol{\nabla} \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0\, \mathbf{j}$

(6.25)

Kapitel 7

Electromagnetic potentials

7.1 Scalar potential and vector potential

In order to solve the coupled differential equations (6.20) - (6.23) for $\mathbf{E}$ and $\mathbf{B}$ it is often convenient - analogous to the procedure in electro and magnetostatics - to introduce electromagnetic potentials.

Since

$$\nabla \cdot \mathbf{B} = 0, \tag{7.1}$$

holds also for dynamic fields, we can continue to introduce a vector potential $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$ using the relation

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}\ . \tag{7.2}$$

The law of induction (6.25)(C) then rewrites as

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad . \tag{7.3}$$

$\mathbf{E}$ *itself* is no longer irrotational and *can therefore no longer be written as the gradient of a potential.* However, the vector field in (7.3) can be written as the gradient of a scalar function $\phi = \phi(\mathbf{r}, t)$:

$$\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = -\nabla \phi, \tag{7.4}$$

or

$$\boxed{\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi}. \tag{7.5}$$

With (7.2) and (7.5) are $\mathbf{E}$ and $\mathbf{B}$ determined by the vector potential $\mathbf{A}$ and the scalar potential ϕ.

Equations for A and ϕ

We will now construct the differential equations from which $\mathbf{A}$ and ϕ can be calculated for a given ρ and $\mathbf{j}$. For this task we will use the inhomogeneous equations (6.22) and (6.23). From (6.22) one can obtain the following formula using $\mathbf{E}$ from (7.5):

$$\boldsymbol{\nabla}^2 \phi + \nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t} \right) = -\frac{\rho}{\epsilon_0} \tag{7.6}$$

and from (6.23) with (7.2):

$$\nabla \times (\nabla \times \mathbf{A}) + \mu_0\epsilon_0 \left(\nabla\frac{\partial \phi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = \mu_0 \mathbf{j}. \tag{7.7}$$

With the identity

$$\nabla \times (\nabla \times \mathbf{a}) = -\boldsymbol{\nabla}^2 \mathbf{a} + \nabla(\nabla \cdot \mathbf{a}) \tag{7.8}$$

(7.7) transforms into:

$$\boldsymbol{\nabla}^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j} + \nabla\left(\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right) \tag{7.9}$$

With this procedure we successfully transformed the 8 Maxwell's equations for **E** and **B** into 4 equations ((7.6) and (7.9)) for the potentials **A** and ϕ, which, however, are coupled to each other.

Gauge invariance

In order to decouple ((7.6) and (7.9)) we will make use of the so-called gauge invariance. According to this invariance the physical fields (7.2) and (7.5) (and therefore also Maxwell's equations) are invariant under the following gauge transformations:

$$\boxed{\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi}, \tag{7.10}$$

$$\boxed{\phi \to \phi' = \phi - \frac{\partial\chi}{\partial t}} \tag{7.11}$$

Hereby $\chi(\mathbf{r}, t)$ is an arbitrary (twofold continuously differentiable) function.
Proof:

$$\mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{A} + \boldsymbol{\nabla}\chi) = \boldsymbol{\nabla} \times \mathbf{A} \tag{7.12}$$

$$-\mathbf{E} = \frac{\partial\ \mathbf{A}}{\partial t} + \frac{\partial\ \boldsymbol{\nabla}\chi}{\partial t} + \boldsymbol{\nabla}\phi - \boldsymbol{\nabla}\frac{\partial\chi}{\partial t} = \frac{\partial\ \mathbf{A}}{\partial t} + \boldsymbol{\nabla}\phi \tag{7.13}$$

7.2 Lorenz gauge

[1]

Equation (7.9) suggests to choose χ in a way that

$$\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t} = 0, \tag{7.14}$$

which corresponds to the **Lorenz-convention**. (That this is always possible will be shown shortly).

[1] By Danish physicist Ludvig Lorenz. It is often incorrectly (also in earlier versions of this script) spelled Lorentz gauge and attributed to the Dutch physicist Hendrik Antoon Lorentz.

One can obtain decoupled equations from (7.9), (7.6) and (7.14):

$$\boldsymbol{\nabla}^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}. \tag{7.15}$$

$$\boldsymbol{\nabla}^2\phi - \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \tag{7.16}$$

both sharing the identical mathematical structure. For time-independent fields these formulae simplify to (2.26) and (5.18) known from electro and magnetostatics. The Lorenz gauge (7.14) is used in the relativistic formulation of electrodynamics under application of

$$\boxed{\mu_0\epsilon_0 = c^{-2}}\ . \tag{7.17}$$

Construction of χ

In the case that

$$\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t} \equiv f(\mathbf{r},t) \neq 0\ , \tag{7.18}$$

we can perform a gauge transformation ((7.10),(7.11)) and demand for the transformed fields (7.22) (7.23) that:

$$\nabla\cdot\mathbf{A}' + \mu_0\epsilon_0\frac{\partial\phi'}{\partial t} = \nabla\cdot\mathbf{A} + \boldsymbol{\nabla}^2\chi + \mu_0\epsilon_0\frac{\partial\phi}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\chi}{\partial t^2} = 0. \tag{7.19}$$

Equation (7.19) is an inhomogeneous, partial differential equation of 2nd order of the form

$$\boldsymbol{\nabla}^2\chi - \mu_0\epsilon_0\frac{\partial^2\chi}{\partial t^2} = -f(\mathbf{r},t). \tag{7.20}$$

For a given inhomogeneity $-f(\mathbf{r},t)$ is the solution not unique, since to the solution of (7.20) one can always add a solution to the homogeneous equation

$$\boldsymbol{\nabla}^2\chi - \mu_0\epsilon_0\frac{\partial^2\chi}{\partial t^2} = 0\ . \tag{7.21}$$

7.3 Coulomb gauge

In atomic and nuclear physics χ is chosen in a way that

$$\nabla\cdot\mathbf{A} = 0. \tag{7.22}$$

(7.6) then transforms into

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho}{\epsilon_0}, \tag{7.23}$$

with the already known (particular) solution:[2]

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dV'\ . \tag{7.24}$$

[2](7.24) seems to be a *long-distance effect* on the potential ϕ. However, it turns out that the effect on the **B** and **E** fields is always a proximity effect.

Kapitel 8

Energy of the electromagnetic field

In this chapter we want to construct the energy balance for an arbitrary electromagnetic field.

First let's assume a point charge q traveling at the speed $\mathbf{v}$ in an electromagnetic field $\{\mathbf{E}, \mathbf{B}\}$. The work performed by the field on the charge is given by:

$$\boxed{\frac{dW_M}{dt} \;=\; \mathbf{K} \cdot \mathbf{v} \;=\; q\,[\,\mathbf{E} + (\mathbf{v} \times \mathbf{B})\,] \cdot \mathbf{v} \;=\; q\mathbf{E} \cdot \mathbf{v}}\,, \tag{8.1}$$

since the magnetic field does not perform any work. Accordingly holds for a current with the current density $\mathbf{j}$:

$$\frac{dW_M}{dt} \;=\; \int_V \mathbf{E} \cdot \mathbf{j}\; dV. \tag{8.2}$$

The work performed on the charge by the field is accounted for by the energy of the electromagnetic field, for which we will derive an explicit relation.

As in the static case we want to look upon the potential energies of the charges as field energies. Therefore the charges and currents must be eliminated using Maxwell's equations.

In (8.2) we will start by eliminating the current density $\mathbf{j}$ regarding the moving points of mass using Ampère-Maxwell's law (6.23):

$$\int_V \mathbf{E} \cdot \mathbf{j}\; dV \;=\; \int_V \left(\frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}\right)\; dV\;. \tag{8.3}$$

This expression , which only contains the fields $\mathbf{E}$ and $\mathbf{B}$, can be symmetrized with respect to $\mathbf{E}$ and $\mathbf{B}$. We use

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) \;=\; \partial_i\, \epsilon_{ijk}\, E_j B_k \;=\; \epsilon_{ijk}\, (\partial_i E_j)\, B_k + \epsilon_{ijk}\, E_j\, (\partial_i B_k) \;=\; = \mathbf{B}\cdot(\nabla \times \mathbf{E}) - \mathbf{E}\cdot(\nabla \times \mathbf{B})$$

as well as the law of induction $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$ and find:

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) \;=\; \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \;=\; -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \tag{8.4}$$

If we put (8.4) in formula (8.3), we obtain:

$$\frac{dW_M}{dt} = \int_V \mathbf{E}\cdot\mathbf{j}\,dV = -\int_V \left(\underbrace{\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t} + \frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t}}_{\partial\omega_F/\partial t} + \underbrace{\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})}_{\boldsymbol{\nabla}\cdot\mathbf{S}} \right) dV. \qquad (8.5)$$

Interpretation

We write (8.5) as the sum of three contributions:

$$\frac{dW_M}{dt} + \frac{dU_F}{dt} + \int_V \boldsymbol{\nabla}\cdot\mathbf{S}\; d\,V = 0;\,, \qquad (8.6)$$

with the **field energy**

$$\boxed{U_F = \int_V \left(\frac{1}{2\mu_0}B^2 + \frac{\epsilon_0}{2}E^2\right) dV}\,, \qquad (8.7)$$

and the **Poynting-vector**

$$\boxed{\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})}\,. \qquad (8.8)$$

In (8.7) we can now introduce the **energy density of the electromagnetic field**

$$\omega_F = \frac{1}{2\mu_0}B^2 + \frac{\epsilon_0}{2}E^2\,, \qquad (8.9)$$

which consists of an electric contribution

$$\omega_{el} = \frac{\epsilon_0}{2}E^2 \qquad (8.10)$$

and a magnetic contribution

$$\omega_{mag} = \frac{1}{2\mu_0}B^2\,. \qquad (8.11)$$

Using Gauss's theorem we can now transform the divergence in (8.6) into a surface integral:

$$\boxed{\frac{dU_F}{dt} = -\frac{dW_M}{dt} - \int_{\partial V}\mathbf{S}\cdot d\mathbf{f}}\,. \qquad (8.12)$$

(8.12) therefore describes the following energy balance: The field energy U_F in a volume V can change

(a) when the electromagnetic field performs work ($\frac{dW_M}{dt}$) on charges (8.2) (U_F is therefore transformed into kinetic energy of the charges)

(b) by radiating energy in the form of electromagnetic radiation through the volume V (also into the volume from the outside in the case that the surface integral in (8.12) is negative). Analogous to the conservation of charge (section 4.1) we identify the Poynting-vector $\mathbf{S}$ as the energy current density.

The energy balance shows that the energy of a closed system (point charges plus the electromagnetic field) is a conserved quantity.

The energy balance (8.12) can be rewritten in the differential form as

$$\mathbf{E} \cdot \mathbf{j} + \frac{\partial \omega_F}{\partial t} + \nabla \cdot \mathbf{S} \;=\; 0 \,. \tag{8.13}$$

In the case of an infinite volume and that the fields **E** and **B** decay asymptotically fast enough (meaning that the product of the fields decays faster than $1/r^2$), the surface integral vanishes. The energy balance (8.12) is then given by the work and the field energy.

While the condition given above is fulfilled for static fields, it is not for radiation fields. As we will see, energy is so to speak sent off "to infinity" by radiation fields.

Similar to (8.6), there are also conservation laws for the momentum and for the angular momentum of electromagnetic fields.

Teil IV

Electromagnetic radiation in vacuum

$$\vdots$$

Kapitel 9

The electromagnetic field in vacuum

9.1 Homogeneous wave equation

Maxwell's equations in vacuum ($\rho = 0$; $\mathbf{j} = 0$) are defined as

$$\nabla \cdot \mathbf{E} = 0; \quad \nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \quad \nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{9.1}$$

In order to *decouple* **E** and **B** we construct

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}) &= \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \\ &= \epsilon_0 \mu_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E} = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}. \end{aligned} \tag{9.2}$$

This results in a homogeneous wave equation

$$\boxed{\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{B} = 0}\,; \qquad \frac{1}{c^2} = \epsilon_0 \mu_0. \tag{9.3}$$

Analogous we can proceed with the **E**-field. With the abbreviation

$$\Box \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 \tag{9.4}$$

for the **d'Alembert-operator** $\Box$ we obtain instead of (9.1)

$$\begin{aligned} \Box \mathbf{B} &= 0; \qquad \nabla \cdot \mathbf{B} = 0 \\ \Box \mathbf{E} &= 0; \qquad \nabla \cdot \mathbf{E} = 0 \end{aligned} \tag{9.5}$$

The corresponding potentials can be found in chapter 7:

$$\Box \mathbf{A} = 0; \quad \nabla \cdot \mathbf{A} = 0 \tag{9.6}$$

$$\phi = 0 \tag{9.7}$$

in the so-called Coulomb-gauge .

We therefore have to solve a differential equation of the type

$$\Box f(\mathbf{r}, t) \;=\; 0 \;, \tag{9.8}$$

where f stands for any component of $\mathbf{E}, \mathbf{B}$ or $\mathbf{A}$. The solutions for $\mathbf{E}, \mathbf{B}$ and $\mathbf{A}$ are then furthermore limited by the secondary condition that the divergence must vanish (transversality condition).

9.2 Monochromatic plane waves

In order to solve (9.8) we start off with the approach

$$f = f_0 \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)). \tag{9.9}$$

(9.8) yields

$$(-\mathbf{k^2} + \frac{\omega^{\mathbf{2}}}{\mathbf{c^2}})\mathbf{f} = \mathbf{0} \;. \tag{9.10}$$

(9.9) is therefore a solution to (9.8) provided that the **dispersion relation**

$$\boxed{\omega^2 \;=\; k^2 c^2} \tag{9.11}$$

is valid.

For the electric field strength and the magnetic induction we obtain:

$$\mathbf{E} \;= \mathbf{E}_0 \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)), \tag{9.12}$$

$$\mathbf{B} \;= \mathbf{B}_0 \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t)), \tag{9.13}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant vectors that furthermore have to satisfy several conditions (see below). The solution for $\mathbf{A}$ is analogous.

Complex vs. real fields

$\mathbf{E}, \mathbf{A}$ and $\mathbf{B}$ are as measurable quantities real-valued vector fields. As arranged should the complex notation in equations (9.9,9.12) be interpreted in a way that only the *real part* corresponds to the actual physical vector field. The complex notation is at times (for example when differentiating) more convenient than the real one; there arise no problems as long as only linear operations are performed.

Caution is required when calculating physical quantities as for instance the energy current density. In this case products of vector fields occur, as for example

$$\mathbf{E^2} \;. \tag{9.14}$$

Here has to be taken the real part of the fields **before** the product is evaluated:

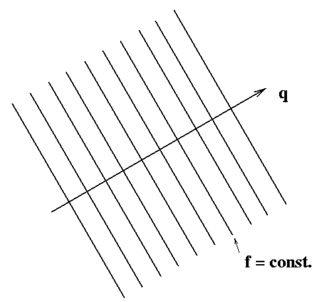
$$\begin{aligned} \mathbf{E}^2 \;&=\; (Re\;\mathbf{E}_0 \exp(i(\mathbf{k} \cdot \mathbf{r} \mp \omega t)))^2 \\ &\neq\; Re\;(\mathbf{E}_0 \exp(i(\mathbf{k} \cdot \mathbf{r} \mp \omega t)))^2 \quad \textit{wrong !} \end{aligned} \tag{9.15}$$

By taking a look at the convention that the real part is being taken at the end of linear operations, expressions like (9.12) can be easily differentiated. One can easily show (compare sec. 9.3) that the differentiation can be replaced as follows:

$$\boxed{\boldsymbol{\nabla} \cdots \to i\mathbf{k} \qquad \frac{\partial}{\partial\, t} \to -i\omega} \;. \tag{9.16}$$

Properties of the solution

i) Plane waves



Functions of type (9.12) describe plane waves, meaning their wave fronts are planes: The points $\mathbf{r}$ in which $f(\mathbf{r}, t)$ has the same function value at a fixed point in time t lie on a plane (Hesse normal form)

$$\mathbf{k} \cdot \mathbf{r} \;=\; \text{const} \;, \tag{9.17}$$

which is perpendicular to $\mathbf{k}$. $\mathbf{k}$ denotes the **direction of propagation**. Depending on the sign of ω one obtains waves running in the $\pm\mathbf{k}$-direction.

ii) Transversality of electromagnetic waves

From $\nabla \cdot \mathbf{B} = 0$ using (9.16) we can derive

$$i\mathbf{k} \cdot \mathbf{B} = 0 \to \mathbf{k} \cdot \mathbf{B}_0 = 0 \tag{9.18}$$

and similarly for $\mathbf{E}$:

$$\mathbf{k} \cdot \mathbf{E} = 0 \;, \tag{9.19}$$

meaning the fields are **transversal** with respect to the direction of propagation. The same holds for $\mathbf{A}$.

iii) Orthogonality of $\mathbf{E}$ and $\mathbf{B}$

From the Maxwell's equation

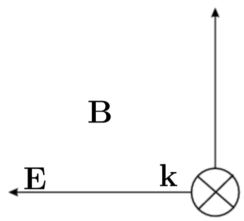
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9.20}$$

using (9.16) can be derived that

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B} \;. \tag{9.21}$$

therefore $\mathbf{E} \perp \mathbf{B}$. $\mathbf{E}, \mathbf{B}$ and $\mathbf{k}$ also construct an orthogonal trihedron (compare sketch). (9.21) with (9.11) furthermore determines the **absolute values** of $\mathbf{B}$ and $\mathbf{E}$, namely:

$$|\mathbf{B}| = |\mathbf{E}|/c \,. \tag{9.22}$$



Annotations

1.) Apart from plane waves are for example also spherical waves solutions to (9.8); they have the form:

$$\boxed{\frac{f(r-ct)}{r}}, \tag{9.23}$$

where f denotes an arbitrary (at least twofold differentiable) function.

2.) The existence of electromagnetic waves (e.g. light waves, radio waves, micro waves, γ-radiation etc.) proves the correctness of the relation $\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \partial \mathbf{E}/\partial t$ in vacuum, which was crucial for the derivation of the wave equation. It represents the experimental confirmation for Maxwell-Ampère's law (6.19).

Terminology

wave vector	$\mathbf{k}$	
wave number	k	$k = \lvert\mathbf{k}\rvert$
angular frequency	ω	$\omega = \pm c\,k$
frequency	ν	$\nu = \omega/(2\pi)$
wavelength	λ	$\lambda = (2\pi)/k$
oscillating period	τ	$\tau = (2\pi)/\omega$

In (9.12) we can see, that τ describes the temporal periodicity of the wave at a fixed point in space $\mathbf{r}$,

$$\exp(i\omega(t+\tau)) = \exp(i\omega t + 2\pi i) = \exp(i\omega t); \tag{9.24}$$

analogously describes the wavelength λ the spatial periodicity

$$\exp(ik(z+\lambda)) = \exp(ikz + 2\pi i) = \exp(ikz) \tag{9.25}$$

for a wave oriented in z-direction at a fixed point in time t.

Phase velocity

The quantity

$$\psi \;=\; \mathbf{k}\cdot\mathbf{r} - \omega t \tag{9.26}$$

is referred to as the *phase* of the wave. The phase velocity v_{ph} describes the velocity at which a wave point with a fixed phase travels. In order to determine v_{ph} we will now again take a look at the plane wave in z-direction and construct the total differential of $\psi(z,t)$:

$$d\psi \;=\; kdz - \omega dt. \tag{9.27}$$

For $\psi = \text{const.}$ then follows that:

$$v_{ph} \;=\; \frac{dz}{dt} \;=\; \frac{\omega}{k} \;=\; c; \tag{9.28}$$

the phase velocity is therefore equal to the speed of light c.

9.3 Addendum: Differential operators for plane waves

In this section we want to derive results like (9.16) in more detail and provide suitable examples to demonstrate their validity.

Let's assume a **monochromatic plane wave**

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\ \widetilde{\mathbf{E}}(\mathbf{r},t)\ . \tag{9.29}$$

$\mathbf{E}$ is the physical (real) field, while

$$\widetilde{\mathbf{E}}(\mathbf{r},t) = \mathbf{u}\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\ t)} \tag{9.30}$$

is a conveniently introduced complex field. The formation of the real part for linear operations, however, is not an issue: The derivations regarding real variables such as $\boldsymbol{\nabla}$, $\partial/\partial t$ „commute" with Re, since

$$\frac{\partial}{\partial\ r_i}\mathrm{Re}\ f(\mathbf{r}) = \mathrm{Re}\ \frac{\partial}{\partial\ r_i}\ f(\mathbf{r})\ , \tag{9.31}$$

as one can verify using the definition of derivation. This means that when applying a linear operator to (9.29), one can simply apply it to the more convenient form (9.30) first and *then form the real part at the end of the calculation.* This, however, is *only valid for linear operators.*

Let's assume for example

$$\boldsymbol{\nabla}\times\widetilde{\mathbf{E}}(\mathbf{r},t) = i\ \mathbf{k}\times\mathbf{u}\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\ t)}\ , \tag{9.32}$$

or

$$\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t) = \mathbf{k}\times\mathrm{Re}\ i\ \mathbf{u}\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\ t)}\ . \tag{9.33}$$

It is practical to keep $\mathbf{u}$ „right" of the real part, since generally $\mathbf{u}$ can be complex. A complex $\mathbf{u}$ describes different polarization states .

As shown in (9.16), it is therefore possible to replace differential operators with multiplicative factors for monochromatic plane waves (9.29), which can be written as the real part of (9.30). *With the convention, that the real part is taken at the end of the calculation.*
Trivial annotation: Plane waves potentially carry a $+\omega$ instead of $-\omega$ in the exponent of (9.30). In that case the sign of ω flips accordingly in (9.16). Similarly this can occur with the sign of $\mathbf{k}$: Caution is required!

Absolute value of r

Not so simple is the case of a vector field of the form (e.g.) (not a plane wave!)

$$\mathbf{E}(\mathbf{r},t) = \mathbf{u}\ f(q|\mathbf{r}|+c\ t)\ , \tag{9.34}$$

or something like

$$\mathbf{E}(\mathbf{r},t) = \mathbf{u}\ \frac{f(q|\mathbf{r}|+c\ t)}{|\mathbf{r}|}\ , \tag{9.35}$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{r}\ \frac{f(q|\mathbf{r}| + c\ t)}{|\mathbf{r}|}\ , \tag{9.36}$$

or

$$\mathbf{E}(\mathbf{r},t) = \mathbf{u} \times \mathbf{r}\ \frac{f(q|\mathbf{r}| + c\ t)}{|\mathbf{r}|}\ . \tag{9.37}$$

Let's assume the simplest case (9.34). The chain rule yields

$$\boldsymbol{\nabla}\ f(q|\mathbf{r}| + c\ t) = f'(q|\mathbf{r}| + c\ t)\boldsymbol{\nabla}(q|\mathbf{r}| + c\ t) = f'(q|\mathbf{r}| + c\ t)q\frac{\mathbf{r}}{|\mathbf{r}|}\ . \tag{9.38}$$

In the case of (9.34) we therefore have the „rule“

$$\boldsymbol{\nabla} \cdots \to q\frac{\mathbf{r}}{|\mathbf{r}|}\frac{\partial}{\partial \xi} \cdots \qquad (\xi = q|\mathbf{r}| + c\ t)\ . \tag{9.39}$$

We have already seen something similar before in the exercise tasks, namely

$$\boldsymbol{\nabla} f(|\mathbf{r}|) = \frac{\mathbf{r}}{|\mathbf{r}|} f'(|\mathbf{r}|)\ . \tag{9.40}$$

The time derivative is simpler.

For formulas like (9.35), (9.36), (9.37) one can apply the product rule and then apply $\boldsymbol{\nabla}$ to each term separately.

For example:

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|}\ f(q|\mathbf{r}| + c\ t)\right) = \frac{f(\xi)}{|\mathbf{r}|}\boldsymbol{\nabla} \cdot \mathbf{r} + \mathbf{r} \cdot \boldsymbol{\nabla}\frac{f(\xi)}{|\mathbf{r}|} = \\
&= 3\ \frac{f(\xi)}{|\mathbf{r}|} + (\mathbf{r} \cdot \frac{\mathbf{r}}{|\mathbf{r}|})\frac{\partial}{\partial |\mathbf{r}|}\frac{f(\xi)}{|\mathbf{r}|} = \\
&= 3\ \frac{f(\xi)}{|\mathbf{r}|} + \frac{f'(\xi)\ q|\mathbf{r}| - f(\xi)}{|\mathbf{r}|} \\
&(\xi = q|\mathbf{r}| + c\ t)
\end{aligned} \tag{9.41}$$

Teil V

Quellen elektromagnetischer Strahlung

$\vdots$

Kapitel 10

Lösungen der inhomogenen Wellengleichungen

10.1 Problemstellung

Bei Anwesenheit von Ladungen haben wir die inhomogenen Gleichungen (vgl. (7.15), (7.16))

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} \;=\; -\mu \mathbf{j}, \tag{10.1}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi \;=\; -\frac{\rho}{\epsilon} \tag{10.2}$$

mit der Nebenbedingung (Lorenz-Eichung)

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} \;=\; 0 \tag{10.3}$$

zu lösen. Das Problem ist also die Lösung einer inhomogenen Wellengleichung

$$\Box \Psi(\mathbf{r}, t) \;=\; \gamma(\mathbf{r}, t)\ , \tag{10.4}$$

wo Ψ für ϕ, A_i und γ für ρ/ϵ, μj_i steht.

Green'schen Funktionen

Die allgemeine Lösung von (10.4) setzt sich aus der (in Abschnitt 9 diskutierten) allgemeinen Lösung der homogenen Wellengleichung (9.9) und einer speziellen Lösung der inhomogenen Wellengleichung zusammen. Zur Konstruktion einer speziellen Lösung von (10.4) benutzen wir die Methode der *Green'schen Funktionen*: Mit der Definition der Green'schen Funktion:

$$\boxed{\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, \mathbf{r}'; t, t') \;=\; -\delta(\mathbf{r} - \mathbf{r}')\ \delta(t - t')} \tag{10.5}$$

können wir als (formale) Lösung angeben:

$$\boxed{\Psi(\mathbf{r}, t) \;=\; \int G(\mathbf{r}, \mathbf{r}'; t, t')\ \gamma(\mathbf{r}', t')\ d^3r' dt'}\ , \tag{10.6}$$

wie man durch Einsetzen von (10.6) in (10.4) direkt bestätigt. Dabei wird ohne (die an sich nötigen) Skrupel die Reihenfolge von Integration bzgl. $\mathbf{r}', t'$ und Differentiation bzgl. $\mathbf{r}, t$ vertauscht.

10.2 Konstruktion von G

Die Green'sche Funktion hat zwei fundamentale Eigenschaften:

$$G(\mathbf{r}, \mathbf{r}'; t, t') \;=\; G(\mathbf{r} - \mathbf{r}'; t - t') \tag{10.7}$$

aufgrund der Invarianz von (10.5) gegen Raum- und Zeit-Translationen;

$$G(\mathbf{r} - \mathbf{r}'; t - t') \;=\; 0 \quad \text{für } t < t' \tag{10.8}$$

wegen des Kausalitätsprinzips.

Wir transformieren wegen (10.7) $\mathbf{r} - \mathbf{r}' \to \mathbf{r}$ und $t - t' \to t$ und suchen eine Lösung der Differentialgleichung

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t) = -\delta^3(\mathbf{r})\delta(t) \;. \tag{10.9}$$

Dazu führen wir eine Fouriertransformation bezüglich der Zeitkoordinate durch:

$$G(\mathbf{r}, t) = \frac{1}{2\pi}\int G(\mathbf{r}, \omega) e^{-i\omega t}\; d\omega \;. \tag{10.10}$$

Mit Hilfe von

$$\delta(t) = \frac{1}{2\pi}\int e^{-i\omega t}\; d\omega \;, \tag{10.11}$$

wird (10.9) zu

$$\frac{1}{2\pi}\int d\omega\; e^{-i\omega t}\; \left[\left(\boldsymbol{\nabla}^2 + \frac{\omega^2}{c^2}\right) G(\mathbf{r}, \omega) + \delta(\mathbf{r})\right] = 0 \;. \tag{10.12}$$

Da diese Gleichung für alle t verschwinden muß, müssen auch alle Fourierkoeffizienten verschwinden:

$$\left(\boldsymbol{\nabla}^2 + \frac{\omega^2}{c^2}\right) G(\mathbf{r}, \omega) = -\delta(\mathbf{r}) \;. \tag{10.13}$$

Zur Lösung dieser Gleichung benutzen wir die Eigenschaft (die wir im Absch.10.2.1 beweisen werden) gültig für eine beliebige, ausreichend (auch im Ursprung) differentierbare Funktion $f(r)$ des Betrages $r = |\mathbf{r}|$:

$$\boldsymbol{\nabla}^2 \frac{f(r)}{r} = \frac{1}{r}\frac{\partial^2 f(r)}{\partial r^2} - 4\pi f(0)\delta(\mathbf{r}) \;. \tag{10.14}$$

Als Ansatz für $G(\mathbf{r}, \omega)$ nehmen wir genau diese Form

$$G(\mathbf{r}, \omega) = \frac{f_\omega(r)}{r} \;, \tag{10.15}$$

dann gibt (10.13)

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) G(\mathbf{r},\omega) = \frac{1}{r}\left(f''_\omega(r) + \frac{\omega^2}{c^2} f_\omega(r)\right) - 4\pi f_\omega(0)\delta(\mathbf{r}) = -\delta(\mathbf{r}) \;, \tag{10.16}$$

was die Lösungen

$$f_\omega(r) = \frac{1}{4\pi} e^{\pm i\omega r/c} \tag{10.17}$$

hat. Durch Fouriertransformation (10.10) erhält man mit Hilfe von (10.17),(10.15)

$$G(\mathbf{r},t) = \frac{1}{4\pi r}\frac{1}{2\pi}\int e^{-i\omega(t\mp r/c)} d\omega = \frac{1}{4\pi r}\delta(t \mp \frac{r}{c}) \;. \tag{10.18}$$

Das *Kausalitätsprinzip* (10.8) zwingt uns, die Lösung mit dem oberen $(-)$ Vorzeichen zu wählen. Diese Greensfunktion ist die sogenannte *„retardierte"* Greensfunktion, weil diese eine *verzögerte* Antwort ($t = \frac{r}{c}$) beschreibt. Die Funktion mit dem $+$ Vorzeichen heißt *„avancierte"* Greensfunktion, weil in diesem Fall die Antwort *vor* deren Ursache stattfindet. Die *retardierte* Greensfunktion für die Wellengleichung lautet also (wir führen die relativen Koordinaten $\mathbf{r} - \mathbf{r}'$ $t - t'$ wieder ein):

$$\boxed{G(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c})} \;. \tag{10.19}$$

Interpretation von G

Die Inhomogenität in (10.5) stellt eine punktförmige Quelle dar, welche zur Zeit t' am Ort $\mathbf{r}'$ für eine (infinitesimal) kurze Zeit angeschaltet wird. Die von dieser Quelle hervorgerufene Störung breitet sich als Kugelwelle mit der Geschwindigkeit c aus. Die **retardierte** Greensfunktion (10.19) erfüllt also folgende physikalische Erwartungen:

i) Die Kugelwelle G muss für $t < t'$ nach dem Kausalitätsprinzip verschwinden.

ii) Sie muss am Ort $\mathbf{r}$ zur Zeit $t = t' + |\mathbf{r} - \mathbf{r}'|/c$ ankommen, da elektromagnetische Wellen sich mit der (endlichen) Lichtgeschwindigkeit c im Vakuum ausbreiten.

Gleichung (10.6) zeigt, wie man die Potentiale $\mathbf{A}, \phi$ zu gegebener Quellen-Verteilung $\rho, \mathbf{j}$ aus den Beiträgen für punktförmige Quellen aufbauen kann.

10.2.1 Beweis von (10.14)

(10.14) kennen wir bereits aus der Elektrostatik für den Fall $f(r) = 1$. Schon in diesem Fall führt die naive Anwendung des Laplace-Operators in Polarkoordinaten (hier brauchen wir nur den r-Anteil)

$$\nabla^2 g(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r g(r)) \tag{10.20}$$

zum **falschen** Ergebnis $\nabla^2 1/r = 0,$ was die ganze Elektro-statik und -dynamik entkräften würde. Der Grund ist, dass (10.20) nicht in der Nähe von $r = 0$ angewandt werden kann, wo die Funktion singulär ist. Es ist dagegen klar, dass für $r > 0$, (10.20) korrekt sein muss. $\boldsymbol{\nabla}^2 f(r)/r$ besteht also aus einem regulären Anteil $\boldsymbol{\nabla}^2_{reg}$ (gegeben in (10.20)), der für jedes $r > 0$ gültig ist und einem irregulären Anteil bei $r = 0$. Aus (10.20) ist es klar, dass

$$\boldsymbol{\nabla}^2_{reg} \frac{f(r)}{r} = \frac{1}{r} f''(r) \ . \tag{10.21}$$

Um den irregulären Anteil zu bestimmen integrieren wir $\boldsymbol{\nabla}^2 f(r)/r$ in einer *Kugel $K(a)$ mit Radius a:*

$$\begin{aligned} &\int_{K(a)} \boldsymbol{\nabla}^2 \frac{f(r)}{r} d^3\mathbf{r} = \int_{K(a)} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \frac{f(r)}{r}) d^3\mathbf{r} = \\ &\oint_{\partial K(a)} (\boldsymbol{\nabla} \frac{f(r)}{r}) \cdot \mathbf{n} \ df = 4\pi a^2 \left(\frac{f'(a)}{a} - \frac{f(a)}{a^2} \right) \ , \end{aligned} \tag{10.22}$$

wo wir das Gauß'sche Theorem, sowie

$$\boldsymbol{\nabla} \frac{f(r)}{r} = \frac{\mathbf{r}}{r} \left(\frac{f'(r)}{r} - \frac{f(r)}{r^2} \right) \tag{10.23}$$

benutzt haben. Den irregulären Anteil bekommen wir indem wir den Limes von (10.22) für $a \to 0$ nehmen:

$$\lim_{a \to 0} \int_{K(a)} \boldsymbol{\nabla}^2 \frac{f(r)}{r} d^3\mathbf{r} = -4\pi f(0) \tag{10.24}$$

was bedeutet, dass in der Nähe von $r = 0$

$$\boldsymbol{\nabla}^2_{irreg} \frac{f(r)}{r} = -4\pi \delta(\mathbf{r}) \ f(0) \ . \tag{10.25}$$

Insgesamt also haben wir (10.14)

$$(\boldsymbol{\nabla}^2_{irreg} + \boldsymbol{\nabla}^2_{reg}) \frac{f(r)}{r} = \frac{f''(r)}{r} - 4\pi \delta(\mathbf{r}) f(0) \ . \tag{10.26}$$

10.3 Lösung der Wellengleichung und retardierte Potentiale

Die (asymptotisch verschwindende) Lösung der Wellengleichung (10.4) von (10.6) ist also mit (10.19) gegeben durch:

$$\Psi(\mathbf{r}, t) \quad = \int d^3\mathbf{r}' \int dt' \ \frac{\delta(t - t' - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi |\mathbf{r} - \mathbf{r}'|} \gamma(\mathbf{r}', t') = \int d^3\mathbf{r}' \ \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \gamma(\mathbf{r}', t'_{ret}) \tag{10.27}$$

wo wir im letzten Term die Integration über t' mit Hilfe der δ-Funktion durchgeführt haben und die retardierte Zeit

$$t'_{ret} = t - |\mathbf{r} - \mathbf{r}'|/c \tag{10.28}$$

eingeführt haben. Die Differenz $t - t_{ret}$ ist die Zeit, die ein Lichtstrahl braucht, um von $\mathbf{r}'$ den Aufpunkt $\mathbf{r}$ zu erreichen. Die Interpretation von (10.27) ist, dass ein Quellenelement

$\gamma(\mathbf{r}', t')$ im Punkt $\mathbf{r}'$ bei der Zeit t' das Potential im Aufpunkt $\mathbf{r}$ erst *bei einer späteren Zeit* $t = t' + |\mathbf{r} - \mathbf{r}'|/c$ beeinflusst, d.h. die Information bewegt sich mit Lichtgeschwindigkeit.

Die Allgemeine Lösung der Wellengleichung ist gegeben durch die partikuläre Lösung (10.27) **plus eine beliebige Lösung der homogenen Wellengleichung, d.h. eine beliebige Linearkombination monochromatischer, ebenen Wellen wie in** (9.9)**.**

Angewandt auf Ladungen und Ströme gibt (10.27) die *retardierten Potentiale*

$$\boxed{\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon} \int \frac{\rho(\mathbf{r}', t')\delta(t - t' - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'dt' = \frac{1}{4\pi\epsilon} \int \frac{\rho(\mathbf{r}', t'_{ret})}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}'} \qquad (10.29)$$

und

$$\boxed{\mathbf{A}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t')\delta(t - t' - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'dt' = \frac{\mu}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t'_{ret})}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \, .} \qquad (10.30)$$

Die Lösungen (10.29) und (10.30) sind über (10.3) bzw. die Ladungserhaltung (6.3) miteinander verknüpft.

Teil VI

Mathematik

$\vdots$

Anhang A

Summary of mathematical concepts

[1]

A.1 Vector algebra

With the help of the δ_{ij} (Kronecker-delta) and ϵ_{ijk} (Ricci-tensor) and of the Einstein sum convention **(always implicit!)** we have:

A.1.1 Scalar produkt

$$\mathbf{A} \cdot \mathbf{B} = A_i B_j \delta_{ij} = A_i B_i \qquad (\sum_i \cdots \text{implicit!}) \tag{A.1}$$

A.1.2 Vector product (or cross product)

$$\mathbf{A} \times \mathbf{B} = A_i B_j \epsilon_{ijk} \hat{\mathbf{e}}_k \qquad (\text{Äquivalent: } (\mathbf{A} \times \mathbf{B})_k = A_i B_j \epsilon_{ijk}) \tag{A.2}$$

where $\hat{\mathbf{e}}_k$ is the unit vector in the direction k ($k = 1, 2, 3$ or, equivalently x, y, z) in cartesian coordinates.

Properties

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad \Longleftrightarrow \quad \epsilon_{ijk} = -\epsilon_{jik} \tag{A.3}$$

$$(\text{z.B. } \epsilon_{iik} = 0 \Leftrightarrow \mathbf{A} \times \mathbf{A} = 0)$$

The exchange relation (A.3) together with $\epsilon_{123} = 1$ define ϵ_{ijk} completely.

A.1.3 Scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \quad \Longleftrightarrow \quad \epsilon_{ijk} = \epsilon_{kij} \tag{A.4}$$

(cyclic permutation, resulting from two exchanges from Eq. (A.3))

[1]Herzlichen Dank an Herrn G. Huhs für das Tippen!

Properties
We additionally need

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}\,(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}\,(\mathbf{A}\cdot\mathbf{B}) \quad\Longleftrightarrow\quad \epsilon_{mil}\;\epsilon_{jkl}=\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij}\;. \tag{A.5}$$

Here, we used the Einstein convention to sum over the index l, which occurs twice. From this, one can derive additional relations.

A.2 Nabla-"operator"

Taking into account the fact that this is a differential operator (product rule, possibly chain rule: it must be clear on which terms it acts), the Nabla operator $\boldsymbol{\nabla}$ can be treated as a vector

$$\boldsymbol{\nabla}=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\equiv(\partial_1,\partial_2,\partial_3) \tag{A.6}$$

$$\boldsymbol{\nabla}\ldots=\hat{\mathbf{e}}_i\;\partial_i\ldots=\partial_i\;\hat{\mathbf{e}}_i\ldots \tag{A.7}$$

(Careful: $\boldsymbol{\nabla}$ needs to be applied to something)

Remark::
Position coordinates are typically denoted as $\mathbf{x}$ or $\mathbf{r}$, which also corresponds to (x, y, z) or (x_1, x_2, x_3).

A.2.1 Divergence ⟷ „Scalar produkt"

Divergence of a Vevtor field $\mathbf{v}$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v} &= \partial_i\;v_i \qquad \text{(Sum convention!)} \\ &= \frac{\partial}{\partial x}v_x+\frac{\partial}{\partial y}v_y+\frac{\partial}{\partial z}v_z\end{aligned} \tag{A.8}$$

A.2.2 Curl ⟷ „Vector product"

Curl of a Vector field $\mathbf{v}$

$$\boldsymbol{\nabla}\times\mathbf{v}=\epsilon_{ijk}\;\partial_i\;v_j\;\hat{\mathbf{e}}_k \tag{A.9}$$

A.2.3 Gradient ⟷ „Product with a scalar"

Gradient of a scalar field ϕ

$$\boldsymbol{\nabla}\cdot\phi=\hat{\mathbf{e}}_i\;\partial_i\;\phi \tag{A.10}$$

A.3 „Nabla-calculus"

A.3.1 Examples

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = ?$$

2 methods:

1.
$$\begin{aligned}
[\nabla \times (\mathbf{A} \times \mathbf{B})]_k &= \partial_i (\mathbf{A} \times \mathbf{B})_j \; \epsilon_{ijk} \\
&\quad \text{Careful! don't change the order} \\
&= \partial_i \; A_n \; B_m \; \underbrace{\epsilon_{nmj} \; \epsilon_{ijk}} \\
&\qquad \text{(Application of (A.5))} = \epsilon_{nmj} \; \epsilon_{kij} \\
&\qquad\qquad\qquad\qquad\quad = \delta_{nk}\delta_{mi} - \delta_{ni}\delta_{mk} \\
&= \partial_m \; A_k \; B_m - \partial_n \; A_n \; B_k \\
&= A_k \partial_m \; B_m + B_m \partial_m \; A_k - B_k \partial_n \; A_n - A_n \partial_n \; B_k \\
= \big[\mathbf{A}(\nabla \cdot \mathbf{B}) &+ (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B}\big]_k
\end{aligned}$$

The first and the last vector in parentheses ar therefore identical. The convention is that ∇ applies to everything which is on its right-hand side (unless otherwise specified, see below).

2. Using
$$\begin{aligned}
\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{C} \cdot \mathbf{A})
\end{aligned}$$
Replace $\mathbf{C} \to \nabla$!
But careful!
∇ applies to $\mathbf{A}$ and $\mathbf{B}$
$\Rightarrow$ mark terms to which ∇ applies (product rule)
$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \nabla \times (\overset{\downarrow}{\mathbf{A}} \times \mathbf{B}) + \nabla \times (\mathbf{A} \times \overset{\downarrow}{\mathbf{B}}) \\
&= \overset{\downarrow}{\mathbf{A}}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \overset{\downarrow}{\mathbf{A}}) + \mathbf{A}(\nabla \cdot \overset{\downarrow}{\mathbf{B}}) - \overset{\downarrow}{\mathbf{B}}(\nabla \cdot \mathbf{A}) \\
&\quad \text{Using the convention „}\nabla \text{ applies to the right"} \\
&= \underbrace{(\mathbf{B} \cdot \nabla)\mathbf{A}}_{\mathbf{v}} - \underbrace{\mathbf{B}(\nabla \cdot \mathbf{A})}_{\mathbf{u}} + \mathbf{A}(\nabla \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\mathbf{B} \\
\text{mit } \; v_i &= B_j \; \partial_j \; A_i \\
u_i &= B_i \; \partial_j \; A_j
\end{aligned}$$

A.3.2 Using the chain rule

$f(|\mathbf{r}|)$ is a scalar function of the modulus of the coordinates (one often uses the notation r, without boldface or arrow to denote $|\mathbf{r}|$).

$$\begin{aligned}
\nabla \; f(|\mathbf{r}|) &= f'(|\mathbf{r}|)\, \nabla \; |\mathbf{r}| \qquad \text{(Chain rule)} \\
(\nabla \; |\mathbf{r}|)_i &= \partial_i \sqrt{{r_1}^2 + {r_2}^2 + {r_3}^2} = \frac{r_i}{|\mathbf{r}|} = (\hat{\mathbf{e}}_r)_i \\
\nabla \; |\mathbf{r}| &= \hat{\mathbf{e}}_r = \frac{\mathbf{r}}{|\mathbf{r}|} \quad \text{A formula often used in electrodynamics} \qquad \text{(A.11)}
\end{aligned}$$

also

$$\boldsymbol{\nabla}\, f\left(|\mathbf{r}|\right) = f'\left(|\mathbf{r}|\right)\hat{\mathbf{e}}_r$$

A.3.3 Further important aspects

1. **Linearity**

$$\begin{aligned} \text{z.B.}\quad \boldsymbol{\nabla}\cdot(\mathbf{A}+\mathbf{B}) &= \boldsymbol{\nabla}\cdot\mathbf{A}+\boldsymbol{\nabla}\cdot\mathbf{B} \\ \boldsymbol{\nabla}\times(\mathbf{A}+\mathbf{B}) &= \boldsymbol{\nabla}\times\mathbf{A}+\boldsymbol{\nabla}\times\mathbf{B} \end{aligned}$$

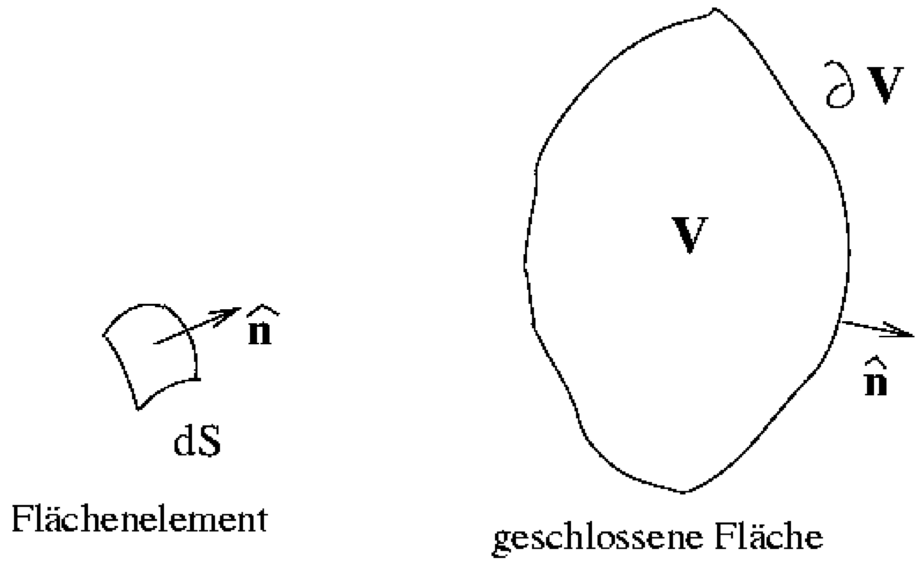
2. $\boldsymbol{\nabla}$ does not act to **constante fields**

$$\text{z.B. } \boldsymbol{\nabla}\cdot\hat{\mathbf{e}}_i\,\phi = \phi\,\underbrace{\boldsymbol{\nabla}\cdot\hat{\mathbf{e}}_i}_{=0} + \hat{\mathbf{e}}_i\cdot\boldsymbol{\nabla}\,\phi = \hat{\mathbf{e}}_i\cdot\boldsymbol{\nabla}\,\phi$$

   In the case of unit vectors, one should consider that cartesian unit vectors $\hat{\mathbf{e}}_i$, $\hat{\mathbf{e}}_x$, ... are constant, while $\hat{\mathbf{e}}_r$ **not!**

A.4 Gauss' law

1. **Surface element (oriented)** (see Fig.)



$$d\mathbf{S} = dS\ \hat{\mathbf{n}} \tag{A.12}$$

   where $\hat{\mathbf{n}}$ is the vector perpendicular to the surface

2. **Flux Φ of a vector field v** through a closed surface ∂V, enclosing a volume V (see Fig.)

$$\Phi\left(\partial V\right) = \int\limits_{\partial V} \mathbf{v} \cdot d\mathbf{S} \tag{A.13}$$

3. *is equal to* the **volum integral** of the divergence of $\mathbf{v}$ in V

$$\Phi\left(\partial V\right) = \int\limits_{V} \boldsymbol{\nabla} \cdot \mathbf{v}\ d^3r \tag{A.14}$$

A.4.1 Interpretation

div$\mathbf{v}$ = $\boldsymbol{\nabla} \cdot \mathbf{v}$ = flux (to the outside) per unit volume of the vector field $\mathbf{v}$ through the boundary surface of an infinitesimal volume

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\delta\Phi\left(\partial\delta V\right)}{\delta V}$$

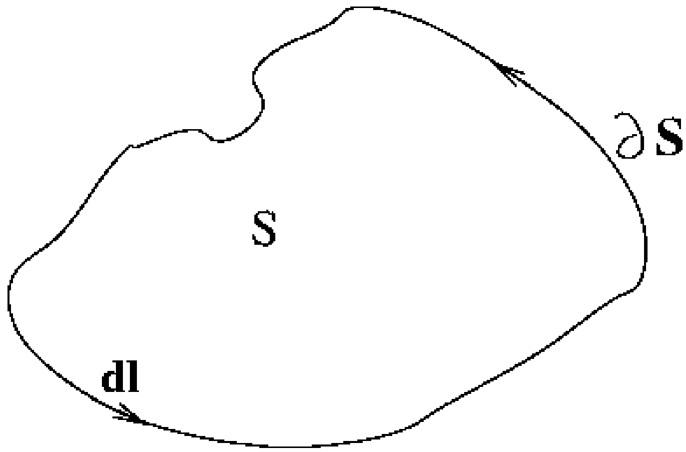
Whenever $\boldsymbol{\nabla} \cdot \mathbf{v} \neq 0$ additional field lines „arise“: **source**

A.4.2 Application

- In **Electrostatics**
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
Charge density if the source of the electric field

- **Continuity equation**
$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}$$
A variation of charge in time can only originate from an incoming or outgoing current flux

A.5 Stokes' law



1. **Curve** in three-dimensional space

$$d\mathbf{l} = \text{oriented curve element}$$

2. **Circulation** Z = line integral of a vector field $\mathbf{v}$ along a closed curve ∂S, which bounds a surface S (∂S does not uniquely define S!) (see Fig.).
   The orientation of the surface can be determined from the „thumb rule“.

$$Z(\partial S) = \int_{\partial S} \mathbf{v} \cdot d\mathbf{l} \tag{A.15}$$

3. **= Fluc of the curl** of $\mathbf{v}$ through S

$$Z(\partial S) = \int_{S} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} \tag{A.16}$$

   [The fact that the flux is independent of S for given ∂S is a consequence of $\nabla \cdot (\nabla \times \mathbf{v}) = 0$]

A.5.1 Interpretation

$(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}} = \mathrm{rot}\mathbf{v} \cdot \hat{\mathbf{n}}$ = Circulation per unit surface of $\mathbf{v}$ around an infinitesimal surface oriented in the direction $\hat{\mathbf{n}}$

$$(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}} = \frac{\delta Z(\partial \delta S)}{\delta S}$$

A.5.2 Applications

In **Electrostatics**

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

A.5.3 Multidimensional δ-function

$$\delta^3(\mathbf{r}-\mathbf{r}_0) = \delta(x-x_0)\ \delta(y-y_0)\ \delta(z-z_0) \tag{A.17}$$

A.5.4 Fourier-representation

$$\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iq(x-x_0)}dq \tag{A.18}$$

A.5.5 Important result for electrodynamics

$$\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{|\mathbf{r}|^3} = 4\pi\ \delta(\mathbf{r}) \tag{A.19}$$

Proof:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{|\mathbf{r}|^3} &= \frac{1}{|\mathbf{r}|^3}\underbrace{\boldsymbol{\nabla}\cdot\mathbf{r}}_{=3}+\mathbf{r}\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{r}|^3}\\
&= \frac{3}{|\mathbf{r}|^3}+\underbrace{\mathbf{r}\cdot\hat{\mathbf{e}}_r}_{|\mathbf{r}|}\frac{-3}{|\mathbf{r}|^4}\\
&= 0 \qquad \text{Valid for } |\mathbf{r}|\neq 0
\end{aligned}$$

$$\text{However: } \int\limits_{\partial V}\frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot d\mathbf{S} = 4\pi \qquad (V \text{ is a sphere of radius } R)$$

$$\Rightarrow \int\limits_{V}\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{|\mathbf{r}|^3}dV = 4\pi$$

from which in total (A.19) follows

Anhang B

A few details

⋮

B.1

B.1.1

⋮

Let (x, y, z) be the coordinates of point P.
Now assume two point charges q and q' with their own set of coordinates.
Determine r and r' from these coordinates and apply them to formula (3.19)

B.1.2

⋮

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int dV'\rho(\mathbf{r}')\boldsymbol{\nabla}\cdot\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\ , \\ &= \frac{1}{4\pi\epsilon_0}\int dV'\rho(\mathbf{r}')4\pi\delta^3(\mathbf{r}-\mathbf{r}')\ , \\ &= \rho(\mathbf{r})/\epsilon_0\end{aligned}$$

B.1.3

⋮

One cannot choose $\nabla G(\mathbf{r}',\mathbf{r})\cdot\mathbf{n}=0$, since

$$\oint_F \nabla G(\mathbf{r}',\mathbf{r})\cdot\mathbf{n}\ dS = 0 = \int_V \nabla^2 G(\mathbf{r}',\mathbf{r})\ d\ V = -\frac{1}{\epsilon_0}$$

B.1.4

⋮

On both sides one can furthermore apply

$$\frac{2}{y_0}\int_0^{y_0}\sin(k_m y)\ d\ y\ .$$

We can then use the *orthogonality properties of the sine function*:

$$\frac{2}{y_0}\int_0^{y_0}\sin(k_m y)\sin(k_n y)\ d\ y = \delta_{nm}$$

B.1.5

⋮

Alternatively we can also approach this with „nabla-calculus“:
$\mathbf{E}\,(\nabla \cdot \mathbf{E}) - \frac{1}{2}\boldsymbol{\nabla}\mathbf{E}^2 + (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} = \boldsymbol{\nabla} \cdot \mathbf{E}\mathbf{E} - \frac{1}{2}\boldsymbol{\nabla}\mathbf{E}^2$

B.1.6

⋮

Let us consider a "photon" occupying a (small) volume $\Delta V = \Delta F \Delta l$ propagating perpendicularly through the area ΔF with velocity c. The photon requires the time interval $\Delta t = \Delta l/c$ to pass the plane. The total energy carried by the photon is therefore $\Delta E = S\ \Delta F\ \Delta t$. Its momentum is furthermore $\Delta P = \pi_F \Delta V = \pi_F \Delta F\ c\ \Delta t$. Therefore is the dispersion relation

$$\frac{\Delta E}{\Delta P} = \frac{S}{c\pi_F} = c\ .$$

This is the dispersion relation of a relativistic particle with velocity c and rest mass 0, compare (**??**).

B.1.7

⋮

One can verify this by performing a Fourier transformation for all three components one by one:

$$g(\mathbf{r}) = g(r1, r2, r3) = \frac{1}{(2\pi)^{1/2}} \int \bar{g}(k1, r2, r3) \exp(ik_1\ r_1)\ dk_1$$

$$= \frac{1}{(2\pi)^{2/2}} \int \int \bar{\bar{g}}(k1, k2, r3) \exp(ik_2\ r_2)\ dk_2 \exp(ik_1\ r_1)\ dk_1 =$$

$$\frac{1}{(2\pi)^{3/2}} \int \int \int \tilde{g}(k1, k2, k3) \exp(ik_3\ r_3)\ dk_3 \exp(ik_2\ r_2)\ dk_2 \exp(ik_1\ r_1)\ dk_1$$

$$= \frac{1}{(2\pi)^{3/2}} \int \tilde{g}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r})\ d^3\mathbf{k}$$